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ABSTRACT

The Proceedings of PME-XVII has been published in three volumes because of the large number of papers presented at the conference. Volume I contains a brief Plenary Panel report, 4 full-scale Plenary Addresses, the brief reports of 10 Working Groups and 4 Discussion Groups, and a total of 23 Research Reports grouped under 4 themes. Volume II contains 37 Research Reports grouped under 7 themes. Volume III contains 28 Research Reports grouped under 5 themes, 25 Oral Communications, and 19 Poster Presentations. In summary, the 3 volumes contain 88 full-scale Research Reports, 4 full-scale Plenary Addresses, and 59 briefer reports. Conference subject matter can be conveyed through a listing of the 15 themes under which Research Reports were grouped: Advanced Mathematical Thinking; Algebraic Thinking; Assessment and Evaluation; Pupil's Beliefs and Teacher's Beliefs; Computers and Calculators; Early Number Learning; Functions and Graphs; Geometrical and Spatial Thinking; Imagery and Visualization; Language and Mathematics; Epistemology, Metacognition, and Social Construction; Probability, Statistics, and Combinatorics; Problem Solving; Methods of Proof; Rational Numbers and Proportions; Social Factors and Cultural Factors. Each volume contains an author index covering all three volumes. (MKR)

SE

Psychology of Mathematics Education

PME XVII

JULY 18-23, 1993

PME 17

Proceedings of the Seventeenth International Conference

UNIVERSITY OF TSUKUBA
TSUKUBA, IBARAKI, JAPAN

Psychology of Mathematics Education

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Proceedings of the Seventeenth International Conference

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Volume I

Proceedings of the Seventeenth International Conference for the Psychology of Mathematics Education

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Editors: Ichiei Hirabayashi, Nobuhiko Nohda, Keiichi Shigematsu and Fou-Lai Lin

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PME 17 PROCEEDINGS
Edited by Ichiei Hirabayashi, Nobuhiko Nohda, Keiichi Shigematsu and Fou-Lai Lin

PREFACE

The first meeting of PME took place in Karlsruhe, Germany in 1976. Thereafter different countries (Netherlands, Germany, U.K., U.S.A., Belgium, Israel, Australia, Canada, Hungary, Mexico, Italy) hosted the conference. In 1993, the PME conference will be held in Japan for the first time. The conference will take place at the University of Tsukuba, in Tsukuba city. The university is now twenty years old. It is organized into three Clusters and two Institutes. There are about 11,000 students and 1,500 faculty members. The Institute of Education at the University of Tsukuba has a strong commitment to mathematics education.

The academic program of PME 17 includes:

- 88 research reports (1 from an honorary member)
- 4 plenary addresses
- 1 plenary panel
- 11 working groups
- 4 discussion groups
- 25 short oral presentations
- 19 poster presentations

The review process

The Program Committee received a total of 102 research proposals that encompassed a wide variety of themes and approaches. After the proposers' research category sheets had been matched with those provided by potential reviewers, each research report was submitted to three outside reviewers who were knowledgeable in the specific research area. Papers which received acceptances from at least two external reviewers were automatically accepted. Those which failed to do so were then reviewed by two members of the International Program Committee. In the event of a tie (which sometimes occurred, for example, when only two external reviewers returned their evaluations), a third member of the Program Committee read the paper. Papers which received at least two decisions "against" acceptance, that is a greater number of decisions "against" acceptance than "for", were rejected. If a reviewer submitted written comments they were forwarded to the author(s) along with the Program Committee's decision. All oral communications and poster proposals were reviewed by the International Program Committee.

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We also wish to express our heartful thanks to the following local committee and local supporters who contributed to the success of this conference:

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HISTORY AND AIMS OF THE P.M.E. GROUP

At the Third International Congress on Mathematical Education (ICME 3, Karlsruhe, 1976) Professor E. Fischbein of the Tel Aviv University, Israel, instituted a study group bringing together people working in the area of the psychology of mathematics education. PME is affiliated with the International Commission for Mathematical Instruction (ICMI). Its past presidents have been Prof. Efraim Fischbein, Prof. Richard R. Skemp of the University of Warwick, Dr. Gerard Vergnaud of the Centre National de la Recherche Scientifique (C.N.R.S.) in Paris, Prof. Kevin F. Collis of the University of Tasmania, Prof. Pearla Nesher of the University of Haifa, Dr. Nicolas Balacheff, C.N.R.S. - Lyon.

The major goals of the Group are:

- To promote international contacts and the exchange of scientific information in the psychology of mathematics education;
- To promote and stimulate interdisciplinary research in the aforesaid area with the cooperation of psychologists, mathematicians and mathematics teachers;
- To further a deeper and better understanding of the psychological aspects of teaching and learning mathematics and the implications thereof.

Membership

Membership is open to people involved in active research consistent with the Group's aims, or professionally interested in the results of such research.

Membership is open on an annual basis and depends on payment of the subscription for the current year (January to December).

The subscription can be paid together with the conference fee.

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Plenary panel

HOW TO LINK AFFECTIVE AND COGNITIVE ASPECTS IN MATHEMATICS EDUCATION

Fou-Lai Lin (Chair),
Claude Comiti., Nobuhiko Nohda,
Thomas L. Schroeder, Dina Tirosh

Worldwide mathematics learning and teaching are very often not satisfactory. Many children continue to experience a great deal of difficulty with mathematics. Mathematics educators the world over are continually searching for ways to remedy this situation.

"Maybe it is not useful to them" so inquiries take place to identify the needs.

"Maybe the children don't enjoy it" it is said, so there are strategies to make mathematics fun.

"Maybe it's too hard for them", so attempts are made to create easier experiences.

"Maybe they don't see the point of it", so the search is on for 'relevance'.

"Maybe it is biased towards merely finding the answer" so the focus is towards the process.

It seems that mathematics educators has been taken both affective and cognitive variables such as attitude

motivation, feeling, belief, understanding, abilities ... etc, into account in their researches. However, most of researches were studied either affective or cognitive affect separately. Students who perform well in mathematics may dislike mathematics. The situation of mathematics learning and teaching is still not satisfied in almost all societies. Thus, the topic : How to Link Affective and Cognitive Aspects in Mathematics Education, is proposed to be the theme of this panel.

N.Nohda shall show one example of teaching in linking both affective and cognitive aspects in mathematics classroom.

C. Comiti shall present a research work from practical teaching and learning mathematics point of view to show the linkage of affective and cognitive aspects.

D. Tirosh shall present some variables in the affective domain and describe some major research findings related to the relationship between affective variables and understanding mathematics, particularly from students' perspectives.

T.L.Schroeder shall speak on Affective and Cognitive variables: Teachers' perspectives.

After opening statements by the panelists, there will be opportunities for them to react to each other's viewpoints as well as to comments and questions from the floor.

PRACTICAL TEACHING/LEARNING MATHS : AFFECTIVE AND COGNITIVE ASPECTS

Claude Comiti, I.U.F.M. de Grenoble et L.S.D.2, Université Joseph Fourier

In her study about intelligence and affectivity in young children T.GOUIN DECARIE shows the necessity to take into account cognitive processes when studying affective phenomena and affective modalities when studying intellectual phenomena. She concludes that those two aspects of the personality are inseparable. It may be asked whether it is not the same thing for the maths teacher.

In the first step of our research work (IMAT) we studied maths teachers'beliefs about their discipline, the type of mathematics they have to teach and other phenomena such as the conditions under which knowledge is transmitted and learning is acquired by their students. We also asked them what image they had of their maths teacher when they themselves were students. The analysis of their answers send back the image of somebody vanishing behind an abstract subject :" *He had the power that confers knowledge. He was untouchable, we were in two separate universes, he was precise just and rigourous, he taught a noble subject*". The choice of the job of maths teacher is explained by the encounter with a subject in which the personality seems not to be concerned :"*I am convinced that maths are a means not to expose your personality.*" There is even an obliteration of the teacher as a subject since it is the validity of the theories that commands in classroom situations :" *A job in which you are not involved personaly since you set forth theories that have been found by others*" One chooses this job for the sake of mathematics. We rejoin here the signification of the disciplinary choice that J.NIMIER (1983) has described as a possibility of isolation, of an individual's step back due to the exteriority of the validity of his teaching.

Yet, when asked which part of their training was the most helpful for them, nearly half of those teachers mention the biological process of parenthood as the most direct means of understanding what a child actually is :" *6th grade students (11 years'old) were all question-marks for me ...The good recipe consists in having children of one's one first and to see their way of reasoning* ". " *As I have my two sons, I realize that some things worked well with the first and not with the other. You have a personality haven't you . Your explanation are good for some and bad for others. You have the same mode of thinking, or you do'nt. You don't always understand the difficulties of every one, there is no universal recipe.*"

What is apparent here is an illuminating analogy : you have to experience in your own life what is the true nature of a child in order to better understand what a student is! A disturbing recognition of a lack of preparation where good will is not enough : if there is in both cases a relationship between an adult and children, it is not interchangeable, according whether you are the teacher or the father or the mother! Also a recognition of deep affective aspects hidden behind the job of the maths teacher, since only through becoming a parent can one have access to the understanding of students. A recognition of the link between the job and the desire which underlies the teacher-student relation. That further confirms the connection between the very personal history of the subject and his role as a teacher (G.LAPIERRE).

When we asked teachers about their beliefs on the ways students learn, a lot of answers sent back to students' abilities first : learning is possible only if students have a background, intuition, if they are able to listen to the teacher, to organize their work, if they understand what they are doing and if they work hard enough. But a mathematical background capacities for doing maths and hard work are not enough : students may be prompted by internal dynamics described in terms of motivation, striving, desire, pleasure :*"It all depends whether the students are motivated or not"*. Teachers also attach great importance to students' self-confidence and to the quality of students' communication with themselves.... We can see here that if they denie the importance of affective factors for themselves they concede it for students.

In the last few years a lot of studies have stressed that " The teacher is a person" (A.ABRAHAM) As the teacher doesn't only talks about mathematics to his students, he just talks : in each mathematical utterance he his present, he conveys something of himself. The teacher interacts with his students through his tone, through his choice of the moment for bringing some information (too soon, too late or at the right time for the student), of the proposed methods (working groups or collectif work...), of the focus of his attention (results, reasoning, presentation...), through the atmosphere that he creates (serious, humourous, playful, steadily dramatric) . His own fantasies projected on mathematics, his desire to use this object for one point or for another are involved in this communicative process .

C.BLANCHARD-LAVILLE shows that the teacher is subjected to compulsive internal forces restraining themselves - internal constraints - which unconsciously prompt him towards some behaviours and some decisions he had not planned when preparing the lesson.

The researcher's problem is to determine among the situations in which the cognitive the affective and the social interact the ones that can be acted upon. Therefore he will attempt " to understand how the subject turns the mathematical object into an object that can be used in his own psychic dynamics thus giving rise to love or hate towards it among other feelings." (english translation of J.NIMIER, 1988).

Let us take the example of errors. For the teacher the error is the result of an inadequate action of the student - lack of attention, false reasoning, calculating error -. Trying to make the causes of each of those types of error explicit quite naturally refers to the authors of such actions. In his explanations the teacher seems to apprehend teaching problems by putting forward elements other than the characteristics of mathematical knowledge - for instance, students'motivation, capacity for work, absent-mindedness...- in other words he doesn't take into account the specificity of the subject he teaches. Let us consider students, starting from an analysis of 16 to 17 year'old students'remarks (11th grade).

" In mathematics, there are too many ways of being mistaken, too many places where you can make mistakes, sometimes only because of lack of attention." "You may have studied very well and still make lots of mistakes!" "In maths a mistake conjures up something grave because one error can make a result false even if your reasoning is right." "Sometimes a mistake on a sign is enough to make a whole reasoning false, it's unfair!" Maths are perceived by students as a subject in which it is very difficult to prevent errors. Students refer to mistakes in maths as to contagious diseases. This fact generates a feeling of fatality of powerlessness and of injustice.

"A mistake is never pleasant to admit because your self-esteem is hurted when the teacher pronounces the word." "Often we're afraid becausc teachers are not so passive as they claim sometimes they lose their self-control they shout." "Very often when we make a mistake for teachers we're just daft even if it is a small mistake !" For the students the teacher's image seems very remote from that of a being taking refuge behind his subject. As far as pointing to errors is concerned, it is not felt as neutral by the student who perceives it as directly linked to the teacher's emotion, in term of loss of identity of loss of power of incompetence mingled with fear.

" Mistakes : I have a bad mark and it's terrible !" And so, the error does not exist in relation to a field of knowledge, but as an element of social relationships - a judgement, a punishment - and with social consequences - a bad end-of-term reports can affect orientation -.

"I stop short I am stuck I do not dare to go on. I'm sure I'm going to make a mistake again !" Making maths is a dangerous situation you are frightened you are at a loss.

"What appears on paper is my stupidity." The student adds his own judgement to the teacher's.

Many students account for their errors in terms of lack of attention or absent-mindedness - by definition something that escaped them and thus that is beyond reason -.

"When I have made a mistake I feel bad even guilty." If errors are inherently irrational how can you prevent it from occurring again unless you blame yourself : incompetent or guilty !

That example shows that something which seems to be objective, like detecting errors or giving marks refers to echos full of affectivity. The teacher is caught in a process of proclaiming errors. Affectivity interfere with the serene analysis that would be required for students to improve.

The teacher has to create conditions for the students to be in a position to learn. No doubt it is necessary for the teacher not only to master mathematics hut also to control his own emotions. This is a necessary condition for him to become conscious of the value of " the compromise achieved in himself between the way he personifies his role and his deep inclinations. ... The role is not an "En-moi" which would be dictated by institutional requirements but a compromise between the requirements of the task the expectations of the group and the emotions of the individual which he reveals and conceals at the same time." This quotation from A.MOYNE will end my contribution to this plenary panel.

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I-8

HOW TO LINK AFFECTIVE AND COGNITIVE ASPECTS IN MATHEMATICS CLASS

Comparison of Two Teaching Trials on Problem Solving --

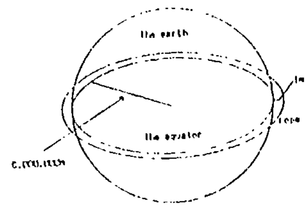
Dr. Nobuhiko Nohda
University of TSUKUBA

For about a hundred years, computational skills in mathematics have been highly emphasized in the elementary and secondary school mathematics curriculum in Japan. Computational skills have been emphasized in classroom lessons and solutions by paper pencil, abacus and mental computation have been used to solve problems in the classroom activities. Japanese students have good skills in computational problems. It was a well known fact that the Japanese students got higher scores in the international tests during the first and second IEA study of mathematics, and recently, International Mathematical Olympiad. However, there are still many Japanese students in junior and senior high school who do not like mathematics and felt lots of anxiety about it.

We would like to assert some tentative conclusions on the effectual link between affective and cognitive aspects in the use of computational tools, in particular, the pocket calculator. Pocket calculators seem to be a more efficient way bringing student's thinking into the mathematical problem than paper pencil. This report on the use pocket calculator in the problem solving has some effectual outcomes in the teaching methods with affective and cognitive aspects.

We conclude in asserting the value in the use of pocket calculator in the process of understanding, solving and retension of the effect. This research suggests that higher order thinking of mathematical problem solving is developed especially by the use of the pocket calculator. It is easier for students to operate on the complex number given using the pocket calculator since it is transformed into a more simple one. The value in the use of calculator process on understanding, solving and extending the problem lies in its usefulness for the students to get the structure of the problem more easily. Also, it is useful for the students to forecast solutions and to make similar and general problems.

Problem used in the lessons
Imagine the earth as a big globe.
It's radius is about 6378.136Km.
If we tie a rope one meter above
the equator of the earth, what is
the difference between the original
circumference and the length of that rope?



Studebts' impressions on the lesson: At the start of the lesson, lots of students in the paper pencil class have good feeling and few students in the computation class are satisfied with the task. During the lesson: i.e. during the problem solving processes, all classes were in confusion. At the end of the lesson, the Calculator class has the excellent feeling to the lesson but Paper pencil class was not satisfied with the lesson. (See item (6) in Check List)

(4) As soon as the lesson was finished, the teacher distributed the Check List of Feeling about the Task problem and the lesson to students.
Check list and responses were as follows:

Check list of Student's Feeling after Lesson

Please read the following sentences, then mark the choice fitting your feeling.

	Yes	5		3		1	No
At the beginning of the lesson,							
(1) Do you like mathematics lesson everyday?:	Yes	P		C			No
After reading the task,							
(2) Did you understand the task?:	Yes		P	C			No
(3) Did you it interesting?:	Yes		P	C			No
(4) Do you think you can solve the problem?:	Yes	C		P			No
(5) Can you forecast the answer?	Yes	C		P			No

While you are solving the task,

(6) Please mark the item fitting your impression of the following: (Ommited)
(a) You can solve the task the same way as your forecast
(b) You changed the way of task solving while solving it.
(c) You failed to solve the task the same way as your forecast, hence have a hard time doing it.
(d) others: please write, briefly.

	Yes	5		3		1	No
At the end of the teaching,							
(7) Do you have many ways of solving it?:	Yes	C		P			No
(8) Are you satisfied with the lesson?:	Yes	C	P				No
(9) Can you understand the explanations by the teacher:	Yes	C			P		No
(10) What kinds of mathematics do you like in lesson? If you have some ideas or topics of the problem, please write it briefly.	Yes	C			P		No

Practical trials to teach the lessons in problem solving context

(1) Teacher explained the earth problem using a big terrestrial globe.

Students were asked to forecast the difference of the lengths. Their forecasts are shown in Table 1 Table 2 shows students answer after computation. There were 36 students in each class. (Male 18, Female 18)

Table 1. Student's forecast (%)

	Calculator	Paper Pencil
a) about 6000Km	8.6	5.7
b) about 1000Km	22.9	31.4
c) about 1000m	31.4	14.3
# d) about 6m	37.1	48.6
e) We can't guess	0	0

Table 2. Students computed answer (%)

	Calculator	Paper Pencil
f) expression	60.0	25.7
g) calculation	57.1	14.3
# h) correct ans.	68.6	11.4

Students' forecast in the paper pencil class were good but others did not do well (See Table 1.)

(2) After the students showed their forecast in each classroom, they were asked to solve the problem. This time, they recalled the formula:

C 2 × Π × R (C: Circumference, R: Radius, Π: 3.14).

Calculator class got good scores in items a) to h) The Paper-Pencil class did not understand the problem, hence they did not get the expression, calculation and correct answer to the problem. (See Table 2.)

The teachers are the ones who determine the correct answer including both the expression and calculation of problem solving in Japan. They emphasized the expression of equations of the problems. Students' expressions were as follows:

Table 3 Characterization of student's expression(%)

Student's expression	Calculator	Paper Pencil
i) synthesis ex.	51.4	11.4
j) analysis ex.	5.7	8.6
k) ( ), X expre.	2.9	2.9
l) special num.	0	2.9

Note: Synthesis expression: (6,000,000 + 1) ×2 ×3.14 6,000,000×2 ×3.14

Analysis expression: 6,000,001×2 ×3.14 37,680,006.28

6,000,000×2 ×3.14 37,680,000

37,680,006 28 37,680,000 6.28 A. 6.28m

letter used expression: (X + 1) ×2 ×3.14 X×2 ×3.14 6.28

Special number/10m (10 + 1) ×2 ×3.14 10×2 ×3.14 6.28

AFFECTIVE AND COGNITIVE VARIABLES: STUDENTS' PERSPECTIVES

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Research in mathematics education has shown that students' performance on mathematical tasks does not solely depend on cognitive variables such as intelligence, memory, intuition, learning style and knowledge of requisite content. Other variables, including affective ones (e.g., confidence in one's ability to perform well in mathematics, motivation to learn mathematics, emotion during work on mathematical tasks, beliefs held about the nature of mathematics) also influence students' performance. It is impossible to completely separate the effects of these two types of factors.

In the last decades, considerable attention was paid by sociologists, psychologists and mathematics educators, to the issue of affect in connection with mathematics education. These researchers suggested different definitions for the affective domain and its various aspects (see for instance Hart's 1989 review on this issue), described alternative theoretical frameworks for research on affect (e.g., Fenemma, 1989; Mandler, 1984; Mcleod, 1992), and analyzed the relationship between socio-economic variables, gender, affect and achievement (e.g., Hembree, 1990; Leder, 1987; Lester, Garofalo, & Kroll, 1989; Schoenfeld, 1985). Growing efforts have also been directed towards developing quantitative as well as quantitative research methods to be used in such research (e.g., Fenemma & Sherman, 1976; Mcleod, Craviotto, & Ortega, 1990).

In the panel, I shall present some definitions of the affective domain, discuss the importance of affect in mathematics education and describe some major research findings related to the relationship between affective variables and understanding mathematics. Then several suggestions for themes to be discussed in the panel will be made. One of these themes is briefly discussed below.

<u>The nature, role and interrelations of specific and global affect in learning mathematics</u>. Learning mathematics at school involves studying different mathematical topics (e.g., algebra, geometry, calculus), addressing different types of tasks (e.g., computations, non-routine problems), dealing with various related activities (e.g., preparing routine assignments, independent projects), experiencing different instructional methods, and exposure to different conceptions of the nature of mathematics and ways by which knowledge is acquired.

It is most probable that a student develops differential beliefs towards various topics ("I believe geometry is important but calculus is not") and different feelings as to his ability to perform different types of tasks ("I'm quite confident I'll succeed in computing, but I'm not sure about word-problems; probably I'll fail them"). It is possible to describe this particular, student's beliefs, attitudes and emotions toward mathematics in terms of "specific affects", namely, his reactions and feelings toward specific mathematical topics, specific tasks and specific activities. However, when it comes to more global attitudes students have with regard to mathematics, questions like the following, naturally arise: Would it be meaningful to discuss this student's beliefs, attitudes and emotions towards mathematics in general term? How can these be measured? Would his responses be mainly determined by his positive feelings or experience, or, on the other hand, by their negative counterparts?

This issue will be discussed in light of research findings that show that students often report different, specific feelings toward different mathematical topics and different tasks (see, for instance, Brown, Carpenter, Kouba, Lindquist, Silver & Swafford, 1988; Corbitt, 1984). Interview data will be presented in which secondary school students claimed they could not seriously answer global questions such as "I enjoy mathematics", "I am good at mathematics" because "the answer depends on the specific topic, and the specific task."

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Plenary Addresses

CONFIDENCE IN SUCCESS

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It is a tradition of P.M.E. that the retiring President be asked to give a plenary lecture. The 17th annual meeting has the theme of linking the cognitive and affective domains, so I will seek to write with that in mind.

I have been involved in mathematics education research, either as a researcher or a user, for many years and I take this opportunity to describe my own work over that time and to express some opinions on the place of researchers in mathematics education generally.

My most recent experience has been to produce mathematics material for teachers and children, aged 11-16 years, (of all attainment levels). Wherever possible the content, progression and pedagogy have been informed by research results, so I have been considering very seriously the question of what our research can give to teachers.

Teachers in Training

In Great Britain, during my entire career, there has been a shortage of mathematically qualified school teachers. In the secondary schools, the intention has been to employ mathematics graduates who have been trained as teachers. There has been the need, however, to employ others who are qualified in allied subjects, or who simply show an interest. At present 30 per cent of those employed as secondary mathematics teachers in Great Britain have no discernible qualifications in mathematics.

In primary schools (from age 5 to 11) we have generalist teachers whose training involved less than a year's course of mathematics and teaching methods in that subject. There was, until 1980, no requirement that they pass the school-leaving certificate in mathematics (at age 16). The majority of the members of P.M.E. work with teachers, during their research, as teacher-trainers or on in-service courses. They will be able to compare the mathematical qualifications of *their* country's teachers.

Some teachers will learn a great deal of mathematics whilst they are teaching it; from the books they are using with the children, from their colleagues and from in-service courses of various lengths. These are the facts. Most of our teachers of mathematics have not studied a great deal of mathematics. This does not mean *all* mathematics teachers, of course, but we must not be led to generalise from the exceptional members of the profession. The fact that these are able to display certain attainment and expertise does not mean that others in their profession are similarly capable. Added to this, is the important question of whether the

attainment and expertise leads to more children learning. The basic premise from which we must start is that: schools are for educating children and mathematics lessons are for educating children in mathematics.

Student Teachers with a Fear of Mathematics

Future teachers say they have always had a mental block against mathematics

My first piece of research concerned the attitude towards mathematics, of future teachers. Amongst the intake of a College of Education in 1970, training for a teaching career in a subject other than mathematics, but nevertheless qualified to teach mathematics in a junior school, were 40 young people with no discernible school-leavers' qualification in mathematics. They had advanced school qualifications in other subjects but not even an ordinary pass in mathematics. They described themselves as having "a mental block towards mathematics". This is an expression more socially acceptable than "failure in mathematics". Of more interest however was the fact that they felt this block had always been there from their first days at school. In response to a questionnaire (Hart, 1973), given to all students three months after their arrival at College, (n=239 returns), 99 students felt that they had been weak at mathematics at some time during their school career, shown in figure 1.

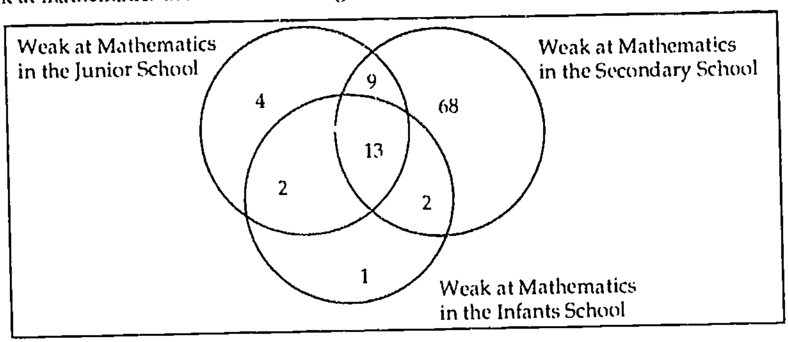

Figure 1: Answers to mathematics attainment questions

In 1973, an attitude questionnaire involving statements concerning a) the liking of the subject of mathematics, b) attitude to teachers of mathematics and c) attitude to the teaching of mathematics was given to 154 of these students who had entered in 1970. By now they had chosen their main area of study, had taught in schools and those who had opted definitely to teach children younger than 11 years had so declared. The answers were scored on a Likert scale of 1 to 5 and there was no significant differences between the scores of the 77 who

intended teaching in a junior school, the 77 who did not and 17 practising junior schools teachers from local schools.

To pursue the point of whether it was possible for a seven-year-old to display a mental block against mathematics, the students and teachers were asked to express their opinions on this in the questionnaire. The teachers from schools agreed overwhelmingly that such a phenomenon existed. I therefore sought out such children by visiting seven schools in South-West London. The teachers of seven-year olds in each school were asked to identify any child who was 'good' at reading and 'poor' at mathematics. (Attainment records kept for children of this age predominately concern performance in English and Mathematics.) Twenty three children, aged seven years, from six schools were chosen by their class teachers as fitting this category. Each group within each class was compared with a randomly selected group of ten students from the same class, on a standardised mathematics test. In no case did the chosen group have a significantly worse mathematics score than the random group. When the performances on different types of mathematics were compared, for example on computation, there were three differences of note. In two cases the 'poor' group was significantly better than the comparison group. Individual scores showed six children with below average mathematics scores but they were also only average readers. Only one child could be classed as a very good reader and poor at mathematics and she was very often absent from school.

Two years later, 21 of the 23 were interviewed twice on a) their attitude to learning mathematics and b) the amount of time they spent on the subject. Their replies showed marked similarities. All but one (Sarah) liked reading and most liked it very much; they read at home and belonged to the library. All, except Sarah, stated that they had disliked mathematics at some time in their school career. All mentioned that mathematics was 'hard'. All equated liking mathematics with being 'good at it', unlike other subjects, for example Simon said that his favourite subject was games even though he was not 'good at it'. Most of those interviewed at nine years of age now disliked mathematics. The children were not convinced that mathematics would be useful to them. Asked how useful to the world an adult who was good at mathematics would be, the only suggestion was that he could help those who were not good at it, particularly his children. They were asked if they would support a law to abolish mathematics in schools. There was no support, although Ellen thought future pop-stars should be exempt from the subject.

Although 'out of school' time was spent on reading, very little leisure time was given to mathematics. Almost all the child's mathematics was done in school. When the school careers of the children in the chosen group were analysed, 13 of the 23 had been in that school

for three years education, when I saw them first. But five of the sample had changed schools and another four had been promoted to higher classes, even though they had been in education for less than three years.

The great majority of these children were leaders in their classes and enjoyed a degree of superiority because of their ability in English. Although at the age of seven they had not been poor at mathematics, by the time the children were nine years old they felt they had been, or in many cases, still were, poor at mathematics and they certainly did not like the subject

Improving Young Children's Attitudes

Make it more exciting and they will like it!

In 1975, the vogue was very much 'discovery' learning and research which involved the measurement of a child's liking of mathematics was beginning (Aitken, 1969). We were starting to differentiate among different types of attitude to mathematics, for example, to the subject, to the teacher and to school in general, or among thinking the subject was useful, a favourite topic in school or one in which the child thinks he is successful. There was conflicting evidence 20 years ago on the correlation between attitude and achievement although we all *hoped* that children who like mathematics would automatically achieve at the subject. In 1975 (Hart, 1975), I made a study of the effects of mathematical games on the achievement and attitude of 149 nine-ten year olds who had attended school for five years. Four schools and six classes were used. In each class, children were randomly assigned to three groups a) those who were given mathematical games and puzzles, b) those who were given computational practice and c) those who were given non-mathematical puzzles -- anagrams, opposites and punctuation exercises. The materials, supplied by the researcher, were regarded as leisure-time activities and not part of the timetabled mathematics lessons. 'Leisure time' for the experiment was defined as those times during the school day when the teacher was occupied (usually with administration) and wanted the children to occupy themselves in an unsupervised way. All the children in a class spent the same amount of time with the materials. The experiment lasted five weeks and the amount of time spent on the material varied from a total of 3.5 hours to 7.5 hours. Normally during such spare-time the pupils would have been expected to read a book.

All comparisons were done within classes as the influence of the teacher was considered to be important in the formation of attitudes. The attitude variables considered in the study were a) general attitude towards mathematics, b) self-concept in mathematics and c) interest in mathematics. A typical statement of type a), to which the child was asked to agree, disagree or omit was, "Most mathematics is too concerned with ideas to be really useful". Whereas a

'self-concept' statement was "I like to be asked questions in Maths lessons". 'Interest in mathematics' was measured by statements such as "I wish we spent longer each day on maths". At the pre-test stage, the correlation coefficient between achievement and all three measures of attitude was significant at better than the .001 level. By looking at the values of r^2, however, it could be seen that a very small part (13-15 per cent) of the variance of attitude, self-concept and interest could be predicted from the variance of achievement. The incidence of a child with a low rating on any of three attitude variables being a high achiever was very small. Analysis of covariance was used to test improvements after the experiment with the covariants being the pretest scores on the variable for which the comparison was made. After the experiment it was seen that the different treatment groups compared classwise and altogether did not differ in measures of achievement, 'attitude', 'self concept' or 'interest'. The materials had had no obvious effect. The only significance recorded was that high achievers attempted more computations than others, but all classes of attainment attempted the games. Nobody disliked the games and puzzles because they were easy, but six liked them for that reason. Three liked these activities because they were 'hard' and three disliked them for the same reason.

Interviews with Older Children

Are methods invented by children necessarily good?

The aims of the project 'Concepts in Secondary Mathematics and Science' (CSMS) were to 1) give information to teachers on the levels of difficulty in several secondary school mathematics topics and 2) provide them with forms of assessment. During the interviews we were able to identify and describe methods children used to solve word problems. Very often these methods were invented by the child or came from some out of school source. In some cases, they had been learned many years previously and were insufficiently sophisticated for the level of problem being set. In many cases (particularly in Ratio and Proportion) these methods greatly limited what the child could accomplish. They were, however, the means by which the child had success on at least *some* of the problems he had been asked to solve.

For at least twenty years, there has been a tendency for teacher-trainers/educationalists to decry the use of algorithms and formal methods in mathematics classes. Teachers were asked to encourage children to invent methods. This may be useful if a teacher has the time and sees the need to keep track of the child's methods and to help the move to greater sophistication, but consider the effort expended here by Peter, who is told that the volume of a cuboid is $42cm^3$ and two sides of it are 3 cm and 2 cm. He is asked to find the third side.

(I: Interviewer; P: Peter)

P: I think that would be 5 and 3, so, um, ... three fives are fifteen.
I: True.
P: And two fifteen's, um... make 30 so that wouldn't be really the answer.
I: Why wouldn't it be the answer?
P: Because that's 42 centimetres, it should make up to 42.
I: Oh I see, so what are you going to do?
P: Oh, I'll just, you see these 2 make 30, so I'm trying to get higher than that, so .. um, I think that's 11.
I: Well, why don't you try that?
⋮
P: Well it can't be 11 because if you had two 32s makes 64, so it can't be the answer, could it?
I: 32 ... where did 32 come in?
P: Well, um ... I got 11 and three 11s.
I: Ah, three 11s are ...
P: 32.
I: I see, and two of them, you said ...
P: 64, I think
I: Yes.
P: And well, I don't think that could be the answer.
I: Why not?
P: Because it's 42 centimetres
I: I see, well it's somewhere then in between which numbers?
P: So it must be about six.

You might say the immediate problem could be solved if he had been given a calculator but is that really the solution? Skills without meaning are no more preferable. In 1928, Brownell was condemning this approach. However, possessing inadequate skills does not engender confidence.

CSMS provided us with written data in the form of completed tests for 10,000 children and interview data on 300. From these it was possible to identify errors to questions, which were widespread within the British secondary school population. Some, such as the 'incorrect addition strategy' had been previously identified in other populations (Piaget, Karplus). This error occurs when the child adds a fixed amount in order to enlarge a figure such as in figure 2, in which the unknown side of the enlargement would be 4 units because 2 + 3 = 5 and 2 + 2 = 4.

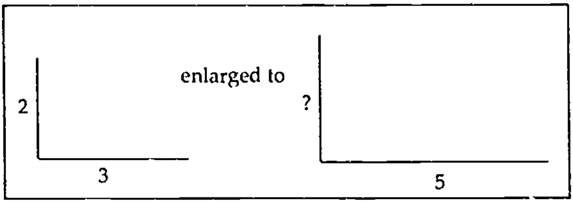

Figure 2: Incorrect addition strategy

A longitudinal survey in which 200 children were tested at age 13, 14 and 15 years using the same questions showed that the incorrect addition strategy was persistent and likely to be used on difficult questions each time the child was tested. The class teachers were informed when this error appeared but we have no information on whether they intervened at all. The study of identified widespread errors and the formation of possible strategies for their eradication were the bases of the research 'Strategies and Errors in Secondary Mathematics', (Booth 1984, Hart 1984, Kerslake 1986). The Algebra research (Booth, 1984) identified different types of error, some of which were more susceptible to treatment than others. In each case the structured intervention proposed produced better performance. This is the same story as that obtained by Swan and Bell (1985) in their diagnostic teaching experiments when the teacher used an erroneous answer to instigate conflict and discussion. The teachers in all these studies had a particular part to play and were not observers or simply managers.

The Teachers

Do teachers take notice of research findings?

It is a continually lamented fact that research findings do not find their way to teachers. Are teachers interested in research findings? Who interprets and disseminates research findings so that they can be of use to teachers? Do teacher-trainers have a responsibility to keep abreast of research and then to pass it on to young teachers in training? In the USA the Standards (1989) produced by NCTM exhort teachers to change their attitude to mathematics and their ways of teaching.

> This constructive, active view of the learning process must be reflected in the way much of mathematics is taught. Thus, instruction should vary and include opportunities for -
>
> - appropriate project work;
> - group and individual assignments;
> - discussion between teacher and students and among students;
> - practice on mathematical methods;
> - exposition by the teacher.

> Our ideas about problem situations and learning are reflected in the verbs we use to describe student actions (e.g., to investigate, to formulate, to find, to verify) throughout the *Standards*.
>
> (From Standards, p.10, 1989, Reston: NCTM)

In the UK, mathematics educators and the government seem to be giving opposite and conflicting advice on what change is needed. Very little evidence is produced by any of these agencies on why teachers should change. Why should they change when the reasons given by mathematics educators are seldom based on increased child performance or on greater numbers volunteering to study mathematics at a high level. For example, the merits of group discussion in classrooms has long been extolled by those giving advice to teachers. There is some British research which showed very increased performance measures in primary school mathematics classes when the focus of the lessons was discussion *but* an extra teacher was hired to specifically listen and to instigate discussion in a classroom. Compare this with the situation when one teacher is trying to encourage and initiate discussion. As Desforges (1989) says:

> Discussions are generally considered to be an important part of a child's mathematical experience and particularly so in respect of the development of skills associated with reflection and applications. Despite the exhortation of decades, fruitful discussions are rarely seen in primary classrooms. This is so even in the classrooms of teachers who recognise and endorse their value. When discussions are seen, they are mainly teacher dominated, brief and quickly routinized.
>
> It has been suggested that the sheer information load a teacher must process in order to manage her class and conduct a discussion makes the successful appearance of such work an unlikely prospect. Whilst there have been reported sightings of this rare event, its existence remains to be confirmed.
>
> What are the implications of this argument? Certainly there are few implications for the teacher. It would be futile to write out advice to the effect that teachers should try harder. Nor will teachers be convinced by demonstrations of discussion given on a one-off basis by a lavishly prepared and equipped virtuoso and involving half a dozen children. Even when these extravaganzas are impressive as performances, they say little about what the children learned.
>
> The implications of this chapter arise in the main for those who dictate how classrooms are resourced both intellectually and materially. If discussions are to be standard practice, then quite clearly the time will have to be made to conduct them in an undistracted fashion. This entails reducing drastically the breadth of the curriculum that teachers feel obliged to rush over. Secondly, it entails having more adults in classrooms to ease the teachers' management concerns. These are the very minimum requirements.
>
> (Mathematics Teaching: The State of the Art, p.149, P. Ernest (Ed), 1989, Falmer Press)

Researchers should tease out the variables which produce the change rather than give advice which is a generalisation of a minor piece of research or in some cases just 'a good idea'.

Using Manipulatives

Sums is sums and bricks is bricks

Mathematics educators who are interested in cognition have been greatly influenced by the work of Piaget. His theories have been criticised, interpreted, generalised and often the result has little to do with the original theory. One such adaption of the theory has been the fostering of the 'cult of the concrete' or 'manipulatives are good'. Interpreted in the Cockcroft Report (1982)

> For most children a very great deal of practical exploration and experience is needed before the underlying mathematical ideas become assimilated into their thinking. We emphasise again that discussion both with the teacher and with other pupils is a necessary part of this process. A few children pass through the various stages of mathematical development in rapid succession and need to advance to more challenging and more abstract work before they leave primary school. For the majority, however, the transition from the use of concrete materials to abstract thinking takes place slowly and gradually; and even those children to whom abstract thinking appears to come easily often need to undertake practical exploration at the beginning of a new topic.
>
> (Mathematics Counts, Committee of Inquiry into the Teaching of Mathematics in Schools, p.87, 1982)

This and other similar advice (Standards, HMI reports for example) led one to believe that the tactile aspect is the most important. The suitability of the material to model the mathematics, its limitations and whether the use facilitates the leaning of mathematics is as important. The issue is really much more subtle than 'Concrete is Good'. In 'Children's Mathematical Frameworks' (Johnson, editor, 1989) we investigated one use of concrete materials. Teachers in Britain have been advised for many years that an effective introduction to some formal pieces of school mathematics (algorithms, rules, generalisations) is through experience with concrete materials. Thus a teacher who wishes the pupils to understand and use the formula for the area of a rectangle might provide the experiences shown in table 1 (from Nuffield Maths 3 Teachers' Handbook, Latham and Truelove, 1989).

Table 1: Steps in Teaching Area

- Fitting shapes together: 3-D and 2-D shapes
- covering surfaces (irregular and regular) with 3-D objects leading onto 2-D objects so there are no gaps
- Developing conservational aspects of area through for example tangram activities.

- Comparing surfaces (irregular and regular) leading to notion of need for same sized unit to be used for covering surfaces.
- Recognizing need for a standard unit of measure of surface and for this to be based on unit of length leading to introduction of square centimetre.
- Measuring area of irregular and regular shapes by counting squares.
- Using grids of squares for measuring area.
- Begin to calculate area of a rectangle on the basis of the number of squares in a row and the number of rows.

Advice for teachers often as here is accompanied by expressions such as "Now children realise that ..." Our research was with mathematics teachers of children aged 8 - 13 years. The teachers were pursuing a master's degree part-time. The research involved interviews of teacher and children together with observation of the teachers in their classes carrying out their lesson plans, as shown in table 2:

Table 2: Research Methodology

writing of a scheme of work or teaching plan, in some detail by the teacher (these were discussed with the researchers);

interviewing of six children in the class to be taught, before the specific teaching of the topic took place

reporting by the teacher of any change to the lesson plan and the provision to the researcher of work cards, assignment sheets, etc., used in class;

teaching of the topic

interviewing of the six children immediately before the formalisation experience was planned to take place;

observation and tape recording by the researcher of the formalisation lesson(s);

interviewing of the same children, three months later

Table 3: Topics Explored

Topic	*Nature of Formalisation*
Place value: subtraction	Decomposition (changing the values in units, tens, etc.)
Equivalent fractions	Able to generate a set of equivalent fractions (ascertain whether two fractions are equivalent)
Circumference of a circle	pi as a constant ratio and the formula $C = \pi d$ or $2\pi r$
Volume of a cuboid	The formula $V = l \times b \times h$
Area of a rectangle	The formula $A = l \times \omega$
Equations	Method of solution - 'doing the same to both sides'
Ratio: enlargement	Using a method for producing an enlargement (using scale factor, centre of enlargement, and 'pattern lines')

Notable results were:

a) Success rates in understanding and remembering the formalisations were low.

b) The provision of materials and the subsequent lessons did not produce the same effect of learning with all the children being taught, even in a class of eight!

c) Some three months after the practical experience only one child remembered that it led up to the formalisation, most of the participants did not connect the two experiences (practical and formal).

d) On some occasions, although the teacher's intention was to lead to the formalisation the material given to the children did not facilitate this. For example, the best way to use bricks which are easily separated from each other (such as Unifix) to solve 65 - 29, is to remove the two 'tens' then another, break off one unit and count what is left.

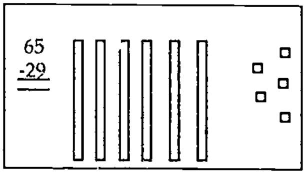

Figure 3: 65-29 with bricks

This is completely at variance with the algorithm which was the objective of the scheme. In this the 'Ones' column is dealt with first and when there are insufficient 'ones' for removal to take place, a 'ten' is broken down into 'ones'.

Before we recommend to teachers that they use manipulatives we should advise them to view the appropriateness and limitations of the materials for the purpose of leading to and authenticating a part of formal mathematics.

e) The time spent on the scheme, sometimes as long as nine weeks, was often heavily weighted to the practical work and very little time was given to the articulation of the formula or rule.

f) Sometimes, with children who had difficulties in applying the rules, the teacher would advise them to return to the concrete aids. This advice presupposed that they could remembers the connection and that they could re-invent the model. This is extremely difficult and in some cases, such as the demonstration of equivalent fractions $\frac{3}{5} = \frac{15}{25}$, you need to know the rule before you can select the appropriate amount of material.

We need to research when manipulatives are appropriate as well as the balance of time given to different activities within the same scheme.

An interested observer might have asked of each teacher "Why are you doing this?". A participant child might have asked the same question. No teacher explained why one might prefer to have a generalisable method for solving problems. Only two teachers gave partial reasons 1) "You do not want to carry bricks around all your life" and 2) "Let us find the volume of this room. Can we fill it with cubes?" Do we, as adults, embark on large commitments of time without having a reason for so doing? Do we continue to attend classes, (language, car-driving, ski-ing or pottery), if we do not know the intended outcome? Do we not leave the class if we are unsuccessful?

A follow-up piece of research (Hart and Sinkinson, 1988) also involved observation of teachers in the same situation as described in CMF. However, in this additional research the researcher and teacher discussed and planned a possible 'bridging' activity to forge a link between the concrete materials and the formalisation. This activity was to be of a distinctly different type from those used in the concrete and formalisations stages. It could be a link with graphs, tabulation or diagrams, or through child discussion. Again the sample was composed of teachers studying for a master's degree in mathematics education; the methods used for collecting data were from classroom observation and interviews with both teachers and children. The analysis showed that the concrete nature of the materials was not taken seriously by the teachers - for example, using pieces of wood but suggesting a) they could be cut and b) that the children pretend a piece was 5 units long, then 17 units. Children know that rods in the mathematics classroom cannot be broken and are made in multiples of other rods so cannot be 5 units and then 17. Of great importance was the lack of pre-requisite knowledge apparent in the pre-test of many child participants. The time taken for the 'bridge' activity was very brief and in one case was set as homework. It seems that more work on useful 'bridging' activities needs to be done.

Putting Research Into the Teachers' Hands

Most teachers use textbooks most of the time

There are few pieces of research concerning how teachers use textbooks or how they choose the ones they will use. The research community has tended to look at other activities in the classroom rather than at the activity which predominates. Is this because we feel there is nothing we can do about textbooks or that they should not be used anyway? A popular belief in many countries is that teachers should write the materials they use in class and that they need to be working towards this goal. Some teachers do write some material and its

effectiveness very often depends on them being in the room to explain and amplify the mathematics contained in it. Nuffield Secondary Mathematics (Hart, 1992) was a curriculum development project which was to produce materials, using available research, for the age range 11-16 years. The initial plan was that teachers should write it in their spare time! Fifty teachers sent material as examples of their work and expressed an interest in writing for the project. The work of seven was coherent, clearly explained and useful, even without the author's presence to teach it. If a teacher's job is to teach, possibly we should not expect them to be writers as well? All materials were tried by teachers in their own classrooms and it is here that their expertise is so valuable. There were three sorts of Nuffield material, a) topic books on five different mathematical strands (Number, Algebra, Space, Probability and Statistics and Measurement), 2) books of problems, projects and investigative situations and 3) guides for teachers. The intention was to match the topic book to the attainment level (in that topic) of the child and so enable progression to be made going from one success to another. It was thus of great importance that one should not overestimate the child's knowledge and provide him with material on which he was bound to fail. Teachers were provided with questions for interviewing pupils, written questions and clues on what seemed to be essential prerequisites, so that they could make a sensitive assessment before assigning a book. They were asked to group together the children using the same book so that the task of teaching did not become impossible and so that they could introduce new content and respond to problems. I discounted suggestions that the children should be always 'stretched' since this implies that they will always find mathematics hard and often a cause of failure. The expression 'stretch' reminds one of the theory that the brain is a muscle which must be constantly exercised to keep it in good repair. A theory destroyed by the research evidence of Thorndike in the nineteen-twenties.

The intention was that the investigative work (in the other set of books), be attempted by everybody and therefore in many tasks there were problems requiring a minimum of computational skills as well as those assuming a higher order of these. Investigative tasks are designed so that children can demonstrate thinking skills and be involved with processes rather than facts. How does the child come to know that this is needed? We assume often that 'the child will come to realise that' there is a pattern to be found or an hypothesis to be formed. In our experience children could work on eight such tasks and not realise that 'looking for a pattern' or 'isolating and listing the variables' were general, useful strategies. Why not encourage teachers to point these out? Building the child's confidence so that he/she can reflect on her own resources and plan an approach to solving a problem using mathematics is of paramount importance. Similarly providing sufficient reference material in

each classroom for the child to feel supported, is surely worthwhile. That is the way adults work.

Confidence

I dare suggest that most mathematics teachers have studied mathematics at a high level because they considered they 'were good at it' for most of their school careers. In Britain, teachers do not encourage children to pursue the study of mathematics after the age of 16 unless they are considered to be very successful in learning it. At University, those who survive the course are mathematically talented.

Making decisions based on a personal estimate of success and expected future success requires confidence. Confidence is based on feeling and good experiences. It is not dependent on constant challenge and likely failure.

According to the Oxford dictionary, confident means *firm trust; assured expectation; self-reliance; boldness; impudence.*

When discussing attitudes and achievement, or the affective and cognitive domains we should be aware of the part played by confidence. You do not become confident through liking, but you do like a task because you are confident that you are in control of it. My opinion is that in teaching mathematics to children one should strive to provide them with enough mathematics so that they confidently apply and use what they know. I say 'provide' because it is the teacher's responsibility to foster this confidence and to see that the learner meets with success. This is not to say that the child is always given tasks which are found very easy, but that whatever he is set to do is within his powers. We must not set teachers and children tasks on which they will fail. This is cruel and time-wasting. The child has only so many years in which to prepare for adulthood and a lifetime of work. What he does in school is of vital importance. What his teacher does plays a large part. Is his teacher trying to put into practice theories which seldom work as intended and which are adjusted (in many different ways) to suit the circumstances?

P.M.E., in my opinion, should be in the forefront of providing evidence for teachers on effective ways of learning, on misconceptions children have, and we must start acquiring data to support the theories we espouse.

Remember, if you are being fed by a 'plain cook' it might be best to have the food he can cook, everyday, rather than starve until he is able to produce a single gourmet dinner.

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Practical Requirements to Psychological Research in Mathematics Education

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Résumé

The main purpose of this lecture is to ask more attention to the long-lasting effects of mathematics education and to stimulate those psychological researches which could contribute to produce such educational efficiencies. Because the co-operation between psychologists and mathematics educators of this country is not so close and most results from psychological laboratories have not been so useful to class-room practice as they might be, the more we wish to ask psychologists to co-operate with us and to make more efforts in promoting such researches. Especially we wish to develop the research from the viewpoints of constructivism rather than formalism, because it may be believed that the former might be more promising to produce long-lasting educational effects in the cultural formation of pupils.

1. Some of My Concerns to Psychology

At the beginning of my lecture, I should like to talk about my relation to PME. In the conference of PME in Sydney 1984, I happened to be elected as a member of international committee and since then I have had some concerns to academical activities of this international study group.

Among others it was happy and exciting for me to have many new acquaintances of many countries and in various professional fields: psychologists, mathematicians, educators and teachers in each school level. In this group, all of them could be gathering in a room to discuss common problems and each could have something useful to develop his/her own theme. In this country there is not yet established such a place like this where not only mathematicians or mathematics educators but also

professional standpoints.

Since long before I have sincerely expected various supports from psychology to problems practical and theoretical in mathematical education. I myself had some of psychologist friend, but when I asked to one of them to work with me on the problem of psychological nature in our field, he coldly refused my propose only saying that he was too busy in his job to think over other's problem and more than that, because he was weak in mathematics. And at that time he argued that mathematics educators should be psychologist and develop their own problems by themselves from the available psychological viewpoints and using psychological methods if they need.

Since then I have studied some of psychology by myself, but still now I wonder if I have become psychologist enough to be able to develop the psychological problems in the area of mathematics education by myself.

Such being the case, I was very happy to be a member of PME and could enjoy the opportunities to have many instructions from there. But still I often wonder if I really understand the aims of the establishment of this study group at the first time by Prof. Efraim Fischbein in 1977. Some years before I had an opportunity to read the proceedings of the earlier international conferences of this group, but I could not find more detailed description on the requirements of its foundation than what we can read in the constitution of PME.

One of my psychologist friend, who has a sharp tongue, once cited a famous Japanese word originated from olden Chinese to express the nature of PME. That was DOUSHOU-IMU which is literally translated as "different dreams in the same bed". He meant that in PME mathematics educators would expect something useful to their teaching, while psychologists wish to find something suitable to demonstrate their psychological theories using mathematics as an experimental material because mathematics is very simple and easy to treat with.

To my regret, the connotation of this Chinese word is not so good and often it is used to underestimate other's behavior or achievement. But in our case of PME, I should like to give it a good meaning. We are in the

same bed and never mind if we dream different dreams if the dreams are happy and encourage us in our scientific activities.

2. A Brief Historical Survey

Roughly divided, class-room practice depends on two sorts of philosophy: formalism and constructivism, both of which we can trace its historical origin in this country.

The aims of this lecture is to have a grasp of interrelations between class-room practice and psychology, and to ascertain the place where our co-operation is required and to develop a new horizon of the research in the teaching of mathematics.

Before going into the main consideration, it would be advisable to give a brief information of the history of education of this country in focusing on the methodological philosophy in teaching arithmetic because we have inherited and still now we have maintained at least some parts of these traditions.

As Prof. von Glasersfeld[6] said in one of his articles, "changes of thinking and of attitude one has to make are formidable" much like a change of the physical habits, every teacher in each generation taught pupils as he/she was taught by his/her teacher. Even if a teacher admits in the mind a new thought as a promising one, it is scarcely possible to change his/her method of teaching actually along with that thought. Under these circumstances, some old methodological thoughts are still surviving even in today's schools, and it would be admitted and reasonable to retrospect the old state of affairs of class-rooms.

Soon after the beginning of Westernization (1867), in the new school system, the modern methods of teaching were introduced by the strong leadership of the new government. It was called 'developmental mode of teaching' which was imported from the United States of America and, as widely known, it was originated in Pestalozzi's educational thoughts. But this mode of teaching, perhaps contrarily to the founder's original intention, was badly formalized in the class-room practice, and curiously enough, there are some signs that this was theoretically systematized by the English

empiricism, philosophy of *tabula rasa*. I believe this was the origin of formalism as the methodological thought of class-room.

Afterward, around 1990, though this method was replaced by the "formal-stages method" of Herbart, the nature of formalism itself was still more strongly fostered in the mind of teachers. Even today we can often clearly recognize traces of Herbartian formalism.

Contrary to a notorious philosophy of formalism, the origin of constructivism of this country is not so clearly identified even among Japanese teachers. But I dare say that it was J. Dewey's book "Psychology of Number" (1) written with his colleague J. A. McLellan in 1895. This book was introduced around 1900 but it was severely abused and rejected by the then top-leader of mathematics education Prof. R. Fujisawa as a careless and faulty book.

Some years ago, I happened to read this book and was very much impressed to see there the fundamental conceptions of J. Piaget's epistemology. I still clearly remember some passages that struck me intensely; it was almost as follows:

"Number is not a property of the objects which can be realized through the mere use of the senses, ..., but objects are not number. Nor does the bare perception constitute number." (p. 24)

"Number is a product of the way in which the mind deals with objects in the operation of making a vague whole definite." (p. 32)

I had also become acquainted at that time with some of Piaget's books. But it was felt to me a little strange that Piaget didn't mention any reference to Dewey so far as I read. As is well-known, Piaget was the author of books of enormous volumes and in somewhere he might have referred to Dewey. I should be very much obliged if someone knows the place and tell me.

I was heard that this Dewey's book was highly evaluated as the origin of his 'instrumentalism' a unique aspect of his magnificent philosophy. However, the direct influence of this book to our country was very small, while all his thoughts was strongly and widely influenced as a general

philosophy of education.

Including Dewey's thoughts, many varieties of educational and methodological philosophy were introduced around 1920 from Europe and America under the name of New Education, and the main features of them were, I could say, constructivistic and stood against Herbartian formalism which had been long reigning with authorized theory of teaching.

Under these historical circumstances, various kinds of educational philosophy have been permuting in the school practice and become to be main features of today's class-room. But I should like to divide them roughly into two sorts: formalism and constructivism. But rather than to say so, it may be more appropriate to say that we can have a line of spectrum of methodological philosophy, one edge of which is formalism and another is constructivism and in some position between these two edges each philosophy may be posed. In the extremity of one edge is situated the so-called 'cramming system', and in the other may be the current 'radical constructivism'.

I would not assent to vanish other philosophy in favor of one; every philosophy has both merits and demerits, and for the practitioners a matter of importance is not the theoretical consequence but the practical efficiency. And it is my requirement from practical viewpoints for psychological research to examine more closely the actual teaching and learning phenomena and proposing teachers with psychologically sound suggestions so that they could improve their lessons and produce more effective and valuable educational results.

3. Some Examples in Class-room

Now in the following, I will show some examples of class-room events which I myself have experienced. The aims of this is to expose some psychological implications which may be concealed in these examples and to have the suggestions on the possible improvement of the ways of teaching from there.

I have read the late Prof. H. Freudenthal[3] in some of his papers saying that, it's important in the research of mathematics education to examine 'paradigmatic' examples and scrutinize them carefully before generalization.

I cannot say whether the following examples are really paradigmatic or not, but they were very much impressing or even shocked to me, and I hope they will stimulate some of your interests.

I have begun my professional life as a mathematics teacher in a secondary school. One of the examples which I wish to show you, happened at that time. I had taught how to solve 'linear equation' in the 8th grade class and gave pupils some of exercise problems. A pupil, who was rather able in mathematics, was noticed to me, because he seemed not to do exercise diligently but was doing something wrong to his neighbour friends. I went to him and asked "Did you finish all the exercises?". "Yes sir, already." he replied and certainly he had done all. Then I asked again "Did you examine the answer by substitution?", but he was somewhat hesitating, so I urged him saying: "Substitute your answer into the left side of the equation and calculate." He did it correctly and speedy. "And next, to the right side.", I said. He obediently did the same thing about the right side. He had solved the equation coreectly and his numerical calculations were all correct, and so it is natural the answers of the two sides were equal. Then I said "So you see that the two sides are equal, don't you?" At this moment, this pupil cried "*Arah, honto da*!" in Japanese. I don't know how to translate this Japanese into the suitable English, and asked my friend English teacher. She put it like this: "Oh! marvellous, I've not come up with it." He seemed as if he did not expect, or didn't know that the two sides come necessarily to equal, though he solved the equation correctly.

This young teacher, that was I, was also very shocked, far more than that pupil cried. Certainly he taught how to solve the equation, and all his pupils came to solve equation correctly, but he did have forgotten to teach what does it mean to solve equation. Never the less, it would be sure that this pupil succeed in the entrance examination to the upper school, because for the paper test it is enough for pupils to have an ability more or less to solve equation or the like correctly, even if he didn't know what it mean to solve it.

In this country, the competition in the entrance examination is very severe and most of it is done through the paper test, but it will be very

difficult for this kind of test to measure the degree of understanding; in this kind of tests it is only possible to estimate overt abilities and for the learning of these the formalistic method of teaching will be the most promising. And here is the reason why the 'cramming system' is welcomed by not only pupils but also parents and often by teachers themselves. But this is rather a superficial problem. The more serious problem of psychological nature is that pupils are seemed to be intrinsically fond of learning under the formalistic method of teaching.

If it is the case, it may not be a good policy of teaching to recommend exclusively constructive mode of teaching for the real understanding of mathematics. It seems that the constructivism is the main current mode of teaching and formalism is often imprudently rejected, but the good compromise of them or 'complimentalization' of them would be the most real resolution, which is expected not only for the practical but also for psychological research.

Another example that I wish to show you is very simple. It also surprised the young teacher in teaching 'simultaneous equations' in the same class as above. He noticed a girl who was in trouble to solve an equation, and I went beside her and suggested: "You had better vanish x instead of y." But to his great surprise she vanished x by rubber eraser.

I am not talking to you a comic short story. I seriously beg your notice to the fact that in the formalistic teaching, mathematical object is often treated as if it were physical object which is possible to move, carry, connect, separate or vanish. In mathematics we express mathematical ideas or operations in words analogically to physical things or actions. And we should notice that a word is the most formalistic object convenient to treat as if it were physical objects, and often be taken as its denotation itself.

It may be also psychological problem to research how mathematical notations or technical terms are 'really' interpreted in the mind of pupils and what role they are playing in pupils' mathematical thinking.

Like examples that I mentioned above, in the class-room we can often take a quick look of the phenomenon which deserve more detailed psychological investigation. I believe that to catch it quickly belongs to the

sensibility of teacher or even to the quality of teacher. And psychologists would be able to have many challenging problems from such teachers.

4. Aspects of Mathematics Classroom

It may be an unique scene of this country, I believe, that the research of mathematics education has a close relation with teaching practice of daily mathematics class-room. And, many of leading teachers are actively joining in the research activities and they have their own national or regional meeting including some of university professors.

However, perhaps because of immaturity in methodological technique and the complexity of problems, most of class-room informations coming from these meeting have been left without scientifically sound inspections especially from the psychological viewpoints. In order to organize these enormous accumulation of informations, I believe, it will be the most necessary to classify them into some categories. As such categories I wish propose a tentative division of teaching scenes which is referred as "aspects of class-room lesson". This idea comes to me in regarding mathematical thinking as a kind of "model thinking". They are as follows:

1) Understanding Aspect

Here pupils use models to understand new mathematical knowledges and skills and these models are usually given by teacher from among what seemed to be well-known to pupils; especially in primary schools, they are often topics from daily life familiar to children: playing, buying, games, school or social events, etc..

2) Application-exercise Aspect

After they have well understood mathematical contents, pupils go into the exercise using that mathematics as the model to solve new problems; in case of arithmetic these problems are often called 'application-problem', 'practical problem' or 'real problem'.

Here we should notice that in this aspect the learned mathematical knowledge come to be model and the new problem is so-called "prototype"

which is to be solved using this mathematical "model". In the first aspect mathematics was prototype and daily topic was model, while in this second aspect the role of the two is reversed. I have often observed that the teacher didn't well understand this important reversal, especially in arithmetic classroom, because there appears seemingly the same daily topic as a model in one aspect and as a prototype in another.

3) Problem-solving Aspect

The most characteristic feature of this aspect is that pupils should make a mathematical model by themselves which is suitable for solving the problem they are confronted with, instead of only using the given models. Often pupils are required to combine several models that they have learned or to invent a new model, but anyhow in this aspect there is a fundamental difference from the second in that the models themselves should be made by pupils.

Fundamentally and psychologically, it should be the most serious problem to reflect "what is model and what role does it play in scientific thinking?". Exclusively on this problem, I know, an international colloquium(2) was once held and as such this problem is too comprehensive to discuss hastily in this place. But it should be important, though it's a matter of course, to understand clearly that "model", whatever it may be, is a well-known something while "prototype" is an unknown thing to be investigated, and our thinking goes from the known to the unknown.

Among these three aspects, the second and the third were actively investigated. The second aspect (application-exercise) is the counterpart of calculation exercise and has been closely studied since Thorndike and his colleagues, mainly from the formalistic standpoints. The third aspect (problem-solving) is the hottest field of research in today's mathematics education and is developing mainly from constructivistic standpoints. But the first aspect is the most crucial especially from the constructivistic viewpoints and is still left without conclusive means of teaching. So, I will mention some examples of this aspect which show how difficult it is to

select models suitable to pupils' well understanding of mathematical concepts.

The first example happened in the class of the 6-th grade in a primary school. The teacher wished to introduce the concept of 'ratio' for the first time. At the beginning of the lesson, he asked pupils:

"In which they feel more narrow between 5 men in 6 mats-room and 7 men in 8-mats room?"

Perhaps as you know, in this country we often express the width of a room by the number of mats called 'tatami' and this is a well-known situation for pupils. A pupil immediately hold up his hand, and answered:

" It's the same, because in both cases everyone has one mat each and one mat is left."

A very much reasonable answer it was, but for the teacher it was a very much troublesome answer which disturbed him to introduce the concept of ratio.

The next example was in the same grade of another school. Soon after learned the addition of two fractions, a pupil came to the teacher to argue that " 3/4 + 5/6 is equal to 8/10 ". The reason which this pupil explained was almost as follows:

"In one basket there 4 apples, 3 of which are red, that is, 3/4 of which is red. In another basket there are 6 apples, 5 of which are red, that is, 5/6 of which is red. If we add together these two baskets of apples and make a new basket, we have 10 apples in it, 8 of which are red , that is, 8/10 of the new basket is red."

These two examples give us a common advice to teaching: in the first aspect when we introduce a new concept, we should deliberately select an appropriate model. In reality "appropriateness" of a model is the most crucial in the first aspect, most of which research are committed to the co-operation of practitioners and researchers including psychologists.

5 . Residue of Mathematics Education

In our research we are apt to turn our attention only to short-term results instead of the long lasting. For instance we often say we have had a good results in mathematics education on the base of the children's good

marks in the test immediately after the teaching. But the real achievements of education as such should be estimated fairly long after leaving schools.

In connection to this, I am tempted to talk about an anecdote which I myself experienced. Many years ago I was working as a lecturer in a women's college to teach girl students majoring in home domestic science some technique of statistics. At the first lesson I wished to know how much they were keeping high school mathematics and give them some written questions on it. Among them I asked to write down Pythagorean Theorem, not the proof of it, only its description. The ratio of right answer to this question was only 1/4 or 1/5, I remember. One of the students wrote to me: "I am sorry I've completely forgotten because in these days I didn't do math." Another student wrote only a formula: $a^2 + b^2 = c^2$. I called her to me and asked what a, b,c mean. She only replied "I don't know." as if it is a matter of no concern to the answer.

It was only three months after they had left high school and their mathematical abilities might be considered to have decreased to one-fifth of the beginning. If, at this rate, the mathematical knowledge disappears from their heads, it would be the most probable that it rapidly tends to zero soon after leaving from school and going out into the world. It may not be the psychological problem, but it is certainly a serious political problem: why should we pay so much money to mathematics education which produces only such a poor result.

Someone said that it will be sure that the residue of mathematics education seems to be very small but it is on the superficial knowledge and skill; we have surely achieved much in cultivating mathematical ways of thinking or attitude. We will be happy, if it is true, but how could we prove that it is? Is it possible to prove it psychologically so as to persuade politician to issue more financial support, and give a good psychological suggestion to teachers so as to promote much more result in this domain?

In schools of this country, perhaps in the same way as other countries, we keep a record of each pupil on his/her achievement of each school subject and improvement of behavior, which is called "Student Record". There we can read items of "evaluation view-points" of each school subject.

It was newly revised last year but in arithmetic section of the former edition they were four as follows:

1. knowledge and understanding
2. skill
3. mathematical thinking
4. interest in and attitude to quantity and figure

In these four items, the former two (1. and 2.) seem to correspond to the short-term achievement, and the latter two (3. and 4.) have comparatively long lasting effect and need a long time for their formation.

In connection of this division, I should like to refer to an article of Dr. Kratz[4] in Germany, who divided mathematical contents into two sorts: "Aufbaustoff" (constructing materials) and "Entfaltungsstoff" (fostering materials). And I believe this division corresponds respectively to that of our evaluation items mentioned above. The former may be constitution or frame of mathematics and the latter may be said to be flesh or muscle of mathematics. The former is easy to teach but easily forgotten, while the latter is seemed to be not so easy to form but have a long duration. To which should we give the prominence of education?

The moderate answer would be "to both", but we should notice that every material has more or less both two characters in each and it is difficult to cut them out and to favor the one exclusively to the other.

Nonetheless until now in this country, we had to make a great favorite of the former (1., 2.) because teachers have an urgent requirement to train pupils for the entrance examination in which mathematical achievements are mainly evaluated in knowledges and skills. But on the other hand, it was true that various educationally ill effects appeared.

I am not sure it's because of these circumstances or not, however, the evaluation viewpoints of the Student Record were changed in the new revision last year as follows:

1. interest, willingness in and attitude to arithmetic
2. mathematical thinking
3. representation and processing of quantity and figure
4. knowledge and understanding of quantity and figure

An attention should be paid to the order of description of the items, though the order may not be the order of importance. Roughly speaking, the order of first couple (1. and 2.) and the last couple (3. and 4.) of the former edition seems to have changed in this new revision. In fact some of members who concerned themselves with this revision said that they had intended to show the change of importance of items in changing their description order, and it is happy if it was the case, I think.

During my lecture, some of you would remember the famous distinction of mathematical understanding according to Dr. Skemp: instrumental and relational understanding[5]. I think this is also the same division in understanding as I showed above in other researching areas of mathematics education. Dr. Skemp also emphasizes the educational importance of relational understanding for its duration in one's life.

6. Concluding Remarks

In this lecture, perhaps some of you are aware that I am always standing on a fairly loose dualism which may be a common conception in the division of formalism and constructivism as the educational thought, Dr. Kratz's division of teaching materials, two kinds of evaluation-viewpoints of Student Record in this country and Dr. Skemp's division of understandings.

In the last I like to introduce here one more division of the same kind which is implied in a Japanese word KYOUIKU which means 'education' in English. This word is originated in Chinese and composed of two words: KYOU and IKU, which means 'teaching' and 'fostering' respectively. Education is indeed composed of this two activities, and I am very proud of the implication which is shown in this Japanese word KYOUIKU.

These two activities in education, have an opposite feature: the one is teacher-centered and the other is children-centered; the one has a nuance of physical movement, something like cramming things into a box, the other has a biological analogy, like watering and fertilizing to plants; the one may produce an immediate results, the other needs a long time to see its real effects.

In the feature of 'teaching' we have been given already much

information from psychological researches since Thorndike, but in the 'fostering' feature we have still now rather few. As a requirement from the practice of mathematics education, we intensely ask to psychological researchers to give us more abundant information to the ways of fostering young pupils in learning mathematics as their real properties, and show us more promising method with the persuasive proof so that not only teachers but also parents and politicians cannot but recognize the necessity of mathematical education as having essential and good effects in their children's future lives.

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RECONCILING AFFECTIVE AND COGNITIVE ASPECTS OF MATHEMATICS LEARNING: REALITY OR A PIOUS HOPE?

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INTRODUCTION

> *I hated doing that maths test.... I was so tense about the whole thing I can't even remember the questions.... Feeling dreadful I tried to put things into perspective when I got home.* (Rosemary, a mature student)

Affect is a term used to denote 'a wide range of concepts and phenomena including feelings, emotion, moods, motivation, and certain drives and instincts' (Corsini, 1984, p. 32). Much has been written about the distinctions between attitudes, beliefs, moods and emotions. Useful discussions with reference to this issue in mathematics education can be found in Hart (1989) and McLeod (1987; 1992). The former, for example, designated beliefs 'to reflect certain types of judgements about a set of objects' (Hart, 1989, p. 44), attitudes 'to refer to emotional reactions to the object, behavior toward the object, and beliefs about the object' (p. 44), and emotion 'to represent a hot gut-level reaction' (p. 44). In this paper I do not wish to explore these distinctions, despite the importance of ensuring that terminology facilitates, rather than hinders or clouds, debate and discussion. Reference to a commonly used definition of attitudes serves as a partial justification for this decision.

> Attitudes involve what people *think* about, *feel* about, and how they would *like to behave* toward an attitude object. Behavior is not only determined by what people *would like* to do but also by what they think they should do, that is, social *norms*, by what they have usually done, that is *habits*, and the *expected consequences of behavior*. (Triandis, 1971, p. 14)

This conceptualisation of attitudes suggests that an individual's attitudes towards mathematics can be inferred from her/his emotional reaction to mathematics, avoidance or selection of mathematical activities, and beliefs about mathematics. Previous patterns of behaviour and anticipated consequences of the path considered may also influence the course of action ultimately

undertaken. A journal entry illustrates the different factors highlighted.

> Thinking back to those first few lectures way back in September, ... I thought of math as a series of steps that followed one after the other. If the steps were taught well, math was easy. If a teacher skipped some steps then math was hard. I had a very narrow view about math and my own personal fear further restricted that view. I always felt that a person could either do math well or couldn't do it at all and that when you did math it was either right or wrong. This course certainly changed my mind!
>
> ...
>
> My first attempts at working from the book *Thinking Mathematically* were disastrous and frustrating. 'I can't do this' was my common complaint and I began to experience again the agony of math classes. It wasn't until well into the course that I began to put one and one together ... By personally attacking the problems it became clear that there were no right or wrong methods. Math was personal and I could use which ever approach suited me best. Often problems ... involved some thinking, figuring out, and reattacking the problem.... (Brandau, 1988, p. 197)[1]

In this paper I trace the work on affect, in its broadest sense, presented at PME conferences in the last ten years, and examine its themes, strengths and weaknesses. A brief comparison is also made between this work and research activities and trends reported beyond PME. Finally, I describe some of my own work involving a new approach to the measurement of attitudes and related factors towards mathematics.

AFFECT AND MATHEMATICS EDUCATION: A DECADE IN PME

The major goals of the Group are:

> *. To promote international contacts and the exchange of scientific information in the psychology of mathematics education;*
>
> *. To promote and stimulate interdisciplinary research in the aforesaid area with the cooperation of psychologists, mathematicians and mathematics teachers;*
>
> *. To further a deeper and better understanding of the psychological aspects of teaching and learning mathematics and the implications thereof.* (Extract from the History and Aims of the PME group)

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[1]Articles which appeared in PME proceedings are not referenced separately

That research reports on affective issues were presented at each PME conference held between 1983 and 1992 is no doubt a reflection of the increasingly widespread recognition among mathematics educators that understanding the nature of mathematics learning requires exploration of affective as well as cognitive factors. The substantial body of research reported over the years is not readily categorised. The space restrictions rigidly enforced in the Conference Proceedings severely limit the amount of material and information able to be presented by authors, and increase the possibility of misinterpretation of their work. As McLeod (1987) has noted: 'It seems to me that short papers like these may constitute a form of projective test; readers are likely to see in the paper reflections of their own interests' (p. 170). Undoubtedly, my own interests have affected the choice of articles reviewed and the material highlighted and summarised. For ease of presentation and discussion I have selected the following groupings: measurement of affective factors - with gender being a further variable of interest in some of these studies, descriptive studies, and comparisons of affective and cognitive variables. It scarcely needs to be said that these divisions are somewhat simplistic, overlapping and certainly not unique.

Measurement of affective factors

An interesting feature of the research that falls most readily into this category is the variety of settings, educational and geographic, in which the work reported was carried out. Most of the studies used primarily self-report measures. A small number compared self-report measures with observational or physiological data. Details of the samples and instruments used in selected studies, as well as the affective variable(s) of primary interest, are given in Table 1.

Table 1: Overview: measurement of affective factors[1]

Author(s)	Year	Sample	Instruments (Affective variables)
Abreu, Bishop, & Pompeu	1992	8-16 year old students; teachers, Brazil	interviews, questionnaires *(beliefs, attitudes)*
Amit	1988	university students, Israel	self-report: Likert scales *(attributions for success and failure)*
Barboza	1984	grade 10 students, Australia	self-report: Likert scales *(attitudes - six scales)*
Beharie & Naidoo	1989	25 students in grades 7, 8 and 9 at a multicultural school in South Africa	interviews *(attitudes - maths, significant others, school)*
Bishop & Pompeu	1989	teachers, Brazil	questionnaire items: rankings *(attitudes)*
Carmeli, Ben-Chaim, & Fresko	1989	junior high school classes and teachers, Israel	self-report: 4 point Likert scale, questionnaire items *(perceptions of classroom learning environment)*
Ernest	1988	pre-service primary school teachers, the U.K.	self-report: Likert scale and observations *(attitude, liking, confidence)*
Evans	1987	adults, U.K.	self-report: 7 point Likert scale; interview *(maths anxiety)*
Evans	1991	mature students, U.K.	questionnaire items, interview *(maths anxiety)*
Fresko & Ben-Chaim	1985	elementary teachers, Israel	self-report: 4 point Likert scales *(confidence)*

[1]The entries in this table are not included in the reference list.

Author(s)	Year	Sample	Instruments & Affective variable(s)
Gentry & Underhill	1987	liberal arts students, the U.S.A.	self-report, paper-and-pencil questionnaire and electromyograph *(math anxiety)*
Glencross	1984	high school students, Zimbabwe	self-report: semantic differential and sets of true-false statements *(attitudes)*
Hadass & Movshovitz-Hadar	1989	high school students, Israel	self-report: 4 point Likert scale *(test anxiety)*
Kuyper & Otten	1989	secondary students, the Netherlands	self-report: 6 point Likert scales *(satisfaction with academic choice)*
Kuyper & van der Werf	1990	teachers and secondary school students, the Netherlands	self-report: Likert scale items and observations *(expectations; attitudes)*
Legault	1987	6th grade students, Canada	projective technique: Rorschach and Thematic Apperception Test *(math anxiety)*
Leder	1989	students in grades 3 and 6, Australia	self-report: interview *(attitudes)*
Leder	1992	grade 7 students, Australia	self-report measures (various), observations *(attitudes)*
Lucock	1987	junior high school students, U.K.	self-report: interview; observations *(attitudes)*
McLeod, Metzger, & Craviotto	1989	undergraduate and research mathematicians, U.S.A.	interview *(emotions)*
McLeod, Craviotto & Ortega	1990	undergraduate students, U.S.A.	self-report: interview and graph *(emotions)*

Author(s)	Year	Sample	Instruments & Affective variable(s)
Miller	1987	students in grades 1 to 12, the U.S.A.	self-report: Likert scale; a second self-report scale; interviews *(attitudes)*
Minato & Kamada	1992	students, Japan	self-report measures: semantic differential and Likert scales
Moreira	1991	primary teachers, Brazil & the U.K.	self-report: 5 point Likert scale *(attitudes)*
Mukuni	1987	high school students in Kenyan schools	self-report: Likert scale *(attitudes)*
Noss & Hoyles	1991	secondary maths teachers, U.K.	interviews, written work, observations *(attitudes)*
Oprea & Stone-water	1987	experienced teachers, the U.S.A.	self-report: Likert scale; open ended items *(belief systems)*
Otten & Kuyper	1988	14-15 year old high school students, the Netherlands	self-report questionnaire items *(attitudes)*
Ponte	1990	mathematics teachers, Portugal	questionnaire items *(attitudes, particularly to aspects of computers)*
Ponte et al	1992	7th and 10th grade students, Portugal	documental analysis, interviews, observations *(attitudes)*
Relich	1984	grade 6 students, Australia	self-report questionnaire items *(learned helplessness)*
Relich	1992	teachers, Australia	self-report: 8 point Likert scale, interviews *(self-concept*
Schifter	1990	mathematics teachers, the U.S.A.	self-report: journal entries *(feelings)*

Author(s)	Year	Sample	Instruments & Affective variable(s)
Schroeder	1991	grade 4 students, teachers	self-report: questionnaire items - some Likert scale *(attitudes and beliefs)*
Scott-Hodges & Lerman	1990	practising mathematicians and math educators	interviews *(attitudes)*
Underhill	1990	district and school staff, the U.S.A.	interviews (?) *(beliefs)*
Visser	1988	7th and 9th grade students & their parents, South Africa	self-report: Likert scales *(attitudes - 11 scales)*
Yackel, Cobb, & Wood	1989	2nd grade students, the U.S.A.	observations *(emotions)*

The entries in Table 1 illustrate the range of affective variables explored and terminology used. Those studies which included gender as an important independent variable (e.g. Amit, 1988; Evans, 1987; 1991; Jackson and Coutts, 1987; Kuyper & Otten, 1989; Kuyper & van der Werf, 1990; Leder, 1989; Lucock, 1987; Mukuni, 1987; Otten and Kuyper, 1988; Visser, 1988) generally reported more functional responses from males than females.

The data in Table 1 also confirm the popularity of Likert scales as a tool for eliciting attitudes towards and about mathematics. Interviews, journal extracts, and observational data have become more prevalent in recent research. While multiple measures were generally seen as an opportunity to optimise data gathering, contradictory findings obtained from different techniques have also been reported. Thus in the one study in which self-report and physiological data were compared, the authors concluded that

> there was insufficient evidence to indicate that a linear relationship exists between paper-and-pencil (MAI) and (EMG) physiological measures of mathematics anxiety, implying that the two instruments may be tapping different dimensions of the mathematics anxiety construct. (Gentry & Underhill, 1987, p. 104)

Descriptive studies

Among the research reports that included affective factors as variables of interest were a number which presented limited data on this aspect in their report of more extensive work carried out. Of those who included teacher factors, Grouws et al (1990) noted that teachers generally did not expect students to perform well on problem solving tasks and planned their teaching accordingly, Nolder (1990) described difficulties perceived or experienced by teachers wishing to introduce innovative practices, while Noss, Hoyles, and Sutherland (1990) examined teachers' views and attitudes about mathematics, mathematics teaching, and computers before and after special in-service activities. Turning to those more concerned with students, Bassarear (1987) speculated whether students' attitudes towards learning mathematics might be obstacles to constructivist instruction. The influence of significant others (peers, teachers, ...) on student motivation and self-perceptions was emphasised by Bishop (1985), while Eisenberg (1991) described exemplar methods for building students' self-confidence in mathematics. Lacasse and Gattuso (1987; 1988; 1989) attempted to overcome negative attitudes towards mathematics through carefully planned instructional activities. Klein and Habermann (1988) reviewed relevant Hungarian work. Lester and Kroll (1990) concluded that 'willingness to be reflective about one's problem solving is closely related to one's attitudes and beliefs' (p. 157), Mevarech (1985) found that students involved in Computer Assisted Instruction showed lower mathematics anxiety than those in other programs, and Sierpinska (1989) hypothesised that children's attitudes towards the rules of mathematics influence the acquisition of new concepts.

Results from studies with more detailed information about affective and performance variables, and their interaction, warrant closer attention.

Affective and cognitive variables

The considerable variety in the ages of the samples and settings used in the research reported was inevitably reflected in the diversity of instruments selected as indicators of cognitive

performance. In line with findings reported elsewhere, the influence of affective variables on achievement scores - frequently assessed through correlational techniques - was typically weak and positive. One large scale study revealed :

> Detailed analyses of correlations between students' achievement and their attitudes, beliefs, and opinions have shown that students with positive attitudes (e.g. those responding that mathematics is important, or easy, or enjoyable) generally scored 5% to 10% higher than students with negative attitudes. (Schroeder, 1991, p. 244)

Compatible trends were described in a small case study carried out at Potsdam College, an institution located close to the Canadian border in New York state and recognised for its success in attracting large numbers of students into rigorous mathematics programs:

> The predominant characteristic of this environment is its culture of success. Students ... are more concerned about whether they will do well enough to achieve high honours in a course rather than whether they will fail it. They expect to do well and they do.... There is a strong belief in the students' ability to master difficult ideas in mathematics and this is communicated to the students who in turn come to believe in themselves. (Rogers, 1988, p. 539)

Acceptance of the link between affective and performance variables is implicit in a curriculum initiative such as that described by Movshovitz-Hadar and Reiner (1983). Video-programs, aimed at improving motivation and decreasing fear and dislike of mathematics were used to introduce grade 4 students to a variety of mathematical topics. Whether or not the anticipated performance benefits were realised was not revealed in the report included in the Proceedings.

Extensive theoretical papers are beyond the space constraints imposed on the written contributions for the PME proceedings. Nevertheless, there were reports of fruitful and constructive attempts to integrate affective and cognitive factors. Goldin's (1988) inclusion of 'affective states' in his model of competence in problem solving is noteworthy in this context, as were the richer portrayals of the role of affect in mathematics learning and problem solving depicted through student reflections in journals or interviews (see entries in Table 1). Other studies

that went beyond comparisons or correlations of test data in an effort to identify interactions between affective and cognitive variables were those which drew substantially on observational data (e.g., Coutts & Jackson, 1987; Jackson & Coutts, 1987; Kuyper & van der Werf, 1990; Leder, 1992; Lucock, 1987; Noss & Hoyles, 1991; Yackel, Cobb, & Wood, 1989). An important contribution was also made by McLeod who presented a theoretical framework for research on affect and problem solving (McLeod, 1985; 1987). Subsequent research (McLeod et al, 1989; 1990) reflects their admonition that 'research on affect should include the use of individual observations, clinical interviews, and teaching experiments' (McLeod, 1987, p. 138), techniques that have increased our understanding of cognitive development.

BEYOND PME

> *Piaget saw affect as a kind of motor that propelled but did not shape intellectual development. Most motivational theorists adopt a similar perspective: the nature of mathematics is given, the role of motivational theory is to understand the conditions under which children will like it enough to learn it.* (Papert, 1986, p. 57)

In common with trends observed in many other countries, formal Australian curriculum documents now frequently refer to the importance of students' attitudes towards school and learning. The influential *National Statement on Mathematics for Australian Schools* (Australian Education Council, 1991) is a good example:

> An important aim of mathematics education is to develop in students positive attitudes towards mathematics.... The notion of having a positive attitude towards mathematics encompasses both liking mathematics and feeling good about one's own capacity to deal with situations in which mathematics is involved. (p. 31)

Yet when outcomes to be achieved are enumerated in some detail they are generally confined to cognitive behaviours. Drafts of the *National Curriculum Profiles* (Australian Educational Council, 1992) which operationalises the more general outlines given in the *National Statement* prescribe that students should be able to 'pose questions', 'justify conjectures', use appropriate problem solving strategies, 'identify assumptions', 'verify solutions', and 'give logical accounts of their mathematical work'. There is

no explicit reference in the *Profiles* to attitudinal goals. That this is not an isolated example of a lack of attention to affective factors can be inferred from comments made by McLeod (1992) and Schoenfeld (1992), for example.

> Research on affect has been voluminous, but not particularly powerful in influencing the field of mathematics education. It seems that research on instruction in most cases goes on without any particular attention to affective issues. (McLeod, 1992, p. 590)

Excluded from 'most cases' are those concerned with gender related differences in mathematics learning, whose work McLeod acknowledges as having made an impact on the development and delivery of the mathematics curriculum and other selected programs.

Measurement, rather than instructional issues, are an integral part of Schoenfeld's concerns. Measurement issues are also the focus of the final part of this paper.

> The arena of beliefs and affects needs concentrated attention. It is basically underconceptualized, and it stands in need of new methodologies and new explanatory frames. The older measurement tools and concepts found in the affective literature are simply inadequate; they are not at a level of mechanism and most often tell us that *something* happens without offering good suggestions as to how or why.... Despite some theoretical advances in recent years and increasing interest in the topic, we are still a long way from a unified perspective that allows for the meaningful integration of cognition and affect or, if such unification is not possible, from understanding why it is not. (Schoenfeld, 1992, p. 364)

It is beyond the scope of this paper to discuss other areas of concern. Recognised clearly, inside and outside the PME community is the need to go beyond self-report questionnaires and simple performance test measures if an integrated perspective between affective and cognitive variables is to be achieved (see, for example, McLeod & Adams, 1989; McLeod, 1992). Ensuring that the behaviours sampled reflect those encountered in a realistic classroom setting imposes a further challenge. A promising approach for interpreting and describing students' cognitive and affective engagement in mathematical tasks, tackled during regular m hematics lessons, is the topic for the remainder of

this paper.

A NEW APPROACH TO MEASURING COGNITIVE AND AFFECTIVE BEHAVIOURS IN THE MATHEMATICS CLASSROOM[3]

> *I cannot give any scientist of any age better advice than this: the intensity of the conviction that a hypothesis is true has no bearing on whether it is true or not.* (Medawar, 1979, p. 39)

A number of important self-imposed requirements shaped the design and methodology of the study:

**the setting was to be the regular mathematics classroom*
**the task to be set should be realistic and challenging*
**students were to be observed working in a collaborative group setting*
**affective behaviours were to be described through self-report and observational measures*
**specific variables included in models hypothesised to explain gender differences warranted particular attention*
**a period of sustained observation was needed if high as well as low inferences were to be drawn*

The task and setting

Students in a grade 7 class were asked to determine the feasibility of building a new canteen in the school. Recognising increased pressures on existing facilities, the school administration was, in fact, exploring this option. It was consistent with the school's philosophy to allow students to put forward their views. The school 'parliament' was a particularly convenient mechanism for this.

The 28 students in the class were arranged by the teacher into six groups of four or five. Eight 45 minute lessons were allotted to the prescribed task. (Because of a time-table clash, one of these lessons was not observed.)

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[3]I wish to acknowledge the contribution made by Helen Forgasz to the work described in this section

The sample

The group of particular interest consisted of five students: three females and two males. Illness or participation in other activities - e.g., music - resulted in the absence of some of the group for a lesson, or parts of a lesson.

Procedure

Data gathering involved videotaping each lesson, transcribing the tapes, keeping field notes, monitoring students' contributions and reactions to the work done each day, obtaining self-report measures of students' attitudes and beliefs about mathematics and themselves as learners of mathematics. Low and high inference analyses were applied to these data.

Variables of particular interest

Models of mathematics learning which included gender as an important variable are summarised in Table 2. Their elements shaped the selection and content of the self-report measures which were administered and the foci of the observations.

Table 2: Selected models of mathematics learning

Author(s)	Year	Relevant components in the model
Deaux & Major	1987	Beliefs about the target, about self, social expectations, effects of context, response to action selected
Eccles	1985	Persistence, self-concept of ability, attitudes, expectations, attributions
Ethington	1992	Self-concept, expectations for success, stereotyping of mathematics, difficulty of mathematics
Fennema & Peterson	1985	Confidence, willingness to work independently, sex-role congruency, attributional style, engagement in high cognitive tasks
Leder	1990	Confidence, attributional style, learned helplessness, mastery orientation, sex-role congruency
Reyes & Stanic	1988	Societal influences, teacher attitudes, student attitudes

Low inference data

(1) *Observational*

An overview of the lessons monitored is given in Table 3. The students who were present and a brief description of the work given is shown for each lesson. The data provide a useful context for more detailed descriptions of other events.

Table 3: Overview of lessons monitored[4]

Date	Students present	Main activities
Mon. 26/8	Ch, C, B, M Absent: J	- DM introduced task; class brainstormed ideas - Ch nominated as group co-ordinator - discussion about questions to include in survey; which classes to survey; who will conduct the survey
Tues. 27/8	Ch, M, C (came late), Ben (put in by DM - sent back to own group during lesson) Absent: J & B	- DM handed out survey sheets for class to complete - these comprised the Year 7 sample. Explained how survey had been compiled and completed by one Year 5 and one Year 6 class. - Much time wasted as group waited for a set of survey sheets to arrive for data to be extracted - Group devised own means for extracting and recording data
Wed. 28/8	J, C, B, M Absent: Ch	- completed data extraction - decided how data to be plotted on graphs - responses from the three year levels combined - began preparing graph sheets and plotting bar graphs
Thurs. 29/8	J, C, Ch, B, M	- girls coloured bar graphs, labelled graphs and axes - boys carefully plotted graphs - discussed further data needed and calculations required to further argument eg. teachers' views, projected additional monies spent by students, projected profits

[4]B, C, Ch, J, and M are students; DM is the teacher

Date	Students present	Main activities
Fri. 30/8		Lesson missed owing to timetable clash. J. wrote summary - boys "arranged and argued over various mathematical results"; girls "began writing up our report".
Mon. 2/9	J, C, Ch, B, M	- DM discussed assessment of task - girls worked on writing up report - boys worked on mathematical calculations related to projected profits of new t/s etc.
Tues. 3/9	J (arrived late following music lesson), C, Ch, B, M.	- typed up report on computer - DM suggested the need to show working for calculated figures - boys worked on this; they told girls how to integrate figures into report
Wed. 4/9	J, C, Ch, B Absent: M	- J and Ch finished off presentation of report - C and B did work from blackboard set by DM

(2) *Self-report*

One of the self-report instruments administered at the end of each lesson asked students to indicate their feelings about, and understanding of, the work just completed. An indication of the information obtained in this way from two students is given by the excerpts summarised in Table 4.

Table 4: Student's reflections about mathematics lessons

Student	Lesson	Feel about lesson?	Point of lesson?	Understood?
B	3	Pleased	graph work, how to cooperate	'I understood'
J		*Pleased*	*'seeing whether as a group we could successfully establish a hypothesis based on our survey results'*	*learnt to deduce facts from graphs. Easy to understand*

Student	Lesson	Feel about lesson?	Point of lesson?	Understood?
B	4	Neutral	Graph work, working with figures	Not sure, more figures would have helped
J		*Frustrating*	*Sorting out what was messed up the previous lesson*	*Learnt nothing, but the work was easy*
		...	...	
B	8	Pleased	Organisation and cooperation	Understood now it's finished. 'It was a very good project which was difficult
J		*Pleased*	*Finishing the project: 'frantic but fun'*	*work was easy: 'doesn't require much thought'*

Despite working collaboratively in the same group and on the same task, these brief excerpts indicate that the two students differed in important ways in their perceptions of and attitudes towards the lessons. (Consistent comments were made with respect to the other lessons and in interviews conducted at a later date.) J seemed preoccupied primarily with the organisation, discussion and writing up of the work. Graph work and using the calculator were further singled out by B. J seemed more confident than B that she had understood the work. The observational data gathered, and reported next, can be used to reinforce or challenge the students' comments.

Low/medium inference data

The nature of the students' activities is captured by a description of their behaviours during two time intervals. The first reflects what took place some six minutes into the lesson, the second what took place about half an hour later. Because of space constraints, the emphasis is on the activities of the two students already identified.

Excerpt 1

Description/low inference

(1) B and M discuss method for drawing graph. M is still concerned that work done previously needs to be altered. B and M discuss a sensible scale for the graph.

(2) Ch and J continue to colour in the graphs. J organises labelling of completed sheets. The two discuss whether to use pencils or textas and whose pencils to use.

Synthesis/medium inference

B and M work on labelling the axes and plotting the graph. J and Ch colour in the graph, exchange pencils. Some off task discussion takes place between J and Ch to which C contributes.

The only 'cross group' exchange occurs when M comments on J's pencil case.

Excerpt 2

Description/low inference

When DM (the teacher) joins the group his help is enlisted by J to obtain the information on profits. DM outlines the most significant relevant findings. J, B, and M interact with DM. While J is vocal and engages a lot of DM's attention her contributions are often a repetition of what he has just said or an irrelevant comment: 'Well, I just know a lot of them'; 'Why not, that's what politicians do.' Ch does not seem to be involved in this exchange. She asks M advice about the presentation format.

Synthesis/medium inference

While highly visible in the exchange with DM, J's suggestions are irrelevant rather than incisive. M and B attend to DM but do not interact with him directly. Nevertheless, B suddenly makes a constructive (high cognitive) suggestion which seems to indicate that he has been following DM's train of reasoning. From M's contribution it is difficult to assess what he is thinking. Ch continues with her transcription task...

These two excerpts reflect the different tasks taken on by the students in the group. J had a very high profile. She organised the others, spent a lot of time colouring in, but also participated in discussions about mathematics with DM and B and M in particular. Her mathematical reasoning seemed at a low rather than high cognitive level. B spent most of the lessons at mathematically relevant tasks: working out data to plot the graph, organising new material into a suitable format. Where possible, he used mathematical reasoning to check the suggestions of others and substantiate his own answers.

The division of labour was reflected in the students' own assessments of their engagement during the lesson. The nature of the tasks on which they worked clarifies why J generally found the work easy while B considered it more challenging.

A higher inference, fine grained analysis of the students' conduct and discussion again allows both cognitive and affective behaviours to be described. The context provided by the earlier descriptions and the self-report data should increase the validity of the inferences made.

Medium/high inference data

A detailed examination of excerpts from the lesson considered above allows the students' behaviours to be described and interpreted in considerably more detail.

Excerpt

Time: 35.49 - 40.18

* J: DM, is there a copy of the profits of the tuckshop? 'Cos we're going to nave to find out whether they're going to make a profit with the new tuckshop

J: confidence (Affect (Aff), High Inference HI)) - initiates discussion with DM by the question (but on topic already discussed among group members). May be giving DM impression of High Level (HL) thinking.

* DM: And they said they make, that they spend on their stock, each week, $2000
J: $2000 a week
M: (writing it down) I got it
DM: And they make $500 profit on it
M: So they make...
J: So they make $2500 roughly
M: (to DM) So they sell $2500 worth
DM: Yes
M: ...and they spend $2000
DM: Yes
J: Right, so...

M: affirmation from important other (DM): positive feedback accompanying independent thinking (cog, HL).

* B: Does that include electricity as well?
DM: No, no, that doesn't include their costs for labour...
C: What does it cost?
DM: ...and it doesn't include their costs for electricity, and it doesn't include their costs for equipment
B: I would have thought they'd lose money
DM: What they do, no, they don't lose money and they don't

make money, they break even. OK?

B: independent thinking (cog, HL)

C: no affirmation (from DM, teacher): ignored (aff, HI) by DM accompanying question which reflects independent thinking (cog, HL). C reacts by tuning out.

*	DM:	Well, how do they make the profit? How is the profit made?
	J:	(M begins to speak simultaneously but stops) By people buying things and people said they'd buy...
	DM:	And, by buying things at a cost greater than what they pay for it, right?
	J:	Mmm
	DM:	Now, how do you increase your profit? You either increase the cost, increase your price or...
	B:	or increase the number of people that come in
	DM:	(nodding) or increase the number of people, good, increase the number of buyers in the...

B: affirmation from DM (aff, HI) for B accompanying independent thinking (cog, HL, HI)

This brief excerpt illustrates the way in which an integrated description of students' cognitive and affective behaviours can be achieved. Subtle responses and reactions were also able to be captured. The approach adopted allows the nature of the tasks undertaken, the quality and cognitive level of the mathematical contributions made by students, as well as their affective involvement to be reported in detail and globally.

Summary

It was a conscious decision to obtain as rich a context as possible for the high inference analyses of events and incidents that occurred during the lesson. Drawing on the full sequence of lessons, gathering information about the students' perceptions of the lessons and their attitudes and beliefs about themselves and mathematics were all part of a deliberate strategy that questions the validity of the 'snapshot' approach to the measurement of attitudes. The different sets of data gathered were informative in their own right and were essential prerequisites for the integrated, detailed and high inference analyses undertaken. The multi-layered approach adopted to the measurement of attitudes and related affective behaviours illustrates vividly that the adequacy, subtleties and richness of the descriptions obtained are directly proportional to the

effort and attention to detail expended.

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ON MATHEMATICAL PROBLEM POSING[1]

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Problem posing is an important aspect of mathematical activity and intellectual inquiry. It has received attention in mathematics education, both as a means to attain other curricular goals and as a goal itself. Despite this interest, however, there is no coherent, comprehensive account of problem posing as a part of mathematics curriculum and instruction, nor has there been systematic research on mathematical problem posing. This paper identifies various types of activities and cognitive processes that have been referred to as problem posing, and discusses several perspectives from which one can view the role and place of problem posing in mathematics education; relevant related research is also reviewed. Special attention is called to the potential that inquiry into problem posing offers as a way to examine both cognitive and affective dimensions of mathematics learning and performance.

In mathematics classes at all levels of schooling in all countries of the world, students can be observed solving problems. The quality and authenticity of these mathematics problems has been the subject of many discussions and debates in recent years. Much of this attention has resulted in a richer, more diverse collection of problems being incorporated into school mathematics curricula. Although the problems themselves have received much scrutiny, less attention has been paid to diversifying the sources for the problems that students are asked to consider in school. Students are almost always asked to solve only the problems that have been presented by a teacher or a textbook. Students are rarely, if ever, given opportunities to pose in some public way their own mathematics problems. Traditional transmission/reception models of mathematics instruction and learning, which emphasized students passively receiving knowledge as a result of transmission teaching, were compatible with a pedagogy that placed the responsibility for problem posing exclusively in the hands of teachers and textbook authors. On the other hand, contemporary constructivist theories of teaching and learning require that we acknowledge the importance of student-generated problem posing as a component of instructional activity.

Problem posing has been identified by some distinguished leaders in mathematics and mathematics education as an important aspect of mathematics education (e.g., Freudenthal, 1973; Polya, 1954). And problem posing has recently begun to receive increased attention

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in the literature on curricular and pedagogical innovation in mathematics education. In the United States, for example, recent reports, such as the *Curriculum and Evaluation Standards for School Mathematics* (NCTM, 1989) and the *Professional Standards for Teaching Mathematics* (NCTM, 1991), have called for an increase in the use of problem-posing activities in the mathematics classroom. Both reports have suggested the inclusion of activities emphasizing *student-generated* problems in addition to having students solve pre-formulated problems, as is clearly illustrated in the following excerpt from the *Professional Standards for Teaching Mathematics:*

> Teaching mathematics from a problem-solving perspective entails more than solving nonroutine but often isolated problems or typical textbook types of problems. It involves the notion that the very essence of studying mathematics is itself an exercise in exploring, conjecturing, examining, and testing--all aspects of problem solving. Tasks should be created and presented that are accessible to students and extend their knowledge of mathematics and problem solving. Students should be given opportunities to formulate problems from given situations and create new problems by modifying the conditions of a given problem." (NCTM, 1991, p. 95)

Despite this interest, however, there is no coherent, comprehensive account of problem posing as a part of mathematics curriculum and instruction nor has there been systematic research on mathematical problem posing (Kilpatrick, 1987). For the past several years, I have been working with colleagues and students on a number of investigations into various aspects of problem posing.[2] Our experiences in studying mathematical problem posing, and our reading of the work of others interested in this area have formed the basis for this paper.

This paper begins with a brief introduction to the types of activities and cognitive processes that have been referred to as problem posing, and then identifies and discusses various perspectives from which one can view the role and place of problem posing in the school mathematics curriculum. Kilpatrick has argued that "problem formulating should be viewed not only as a goal of instruction but also as means of instruction" (1987, p. 123), and both views of problem posing will be evident in this paper. The nature and findings of some research related to mathematical problem posing is also discussed in order to characterize some of the available research evidence associated with each of the perspectives discussed and to suggest some important issues in need of further investigation. To illustrate the

2 I would like to acknowledge the contributions to the work and to my thinking of Joanna Mamona, Lora Shapiro, and Patricia Kenney, who have worked with me as Postdoctoral Associates. I also acknowledge the valuable assistance of several graduate students who have participated in ongoing discussions regarding mathematical problem posing and who have worked on particular studies: Cengiz Alacaci, Mary Lee Burkett, Jinfa Cai, Susan Leung, Barbara Moskal, and Melanie Parker. Relatively recent additions to our group include Cathy Schloemer and Edward McDonald, both of whom have made contributions to our discussions. My colleague Jose Mestre is acknowledged for pointing out how problem-posing ideas can also be applied to studying the learning of physics

international interest in mathematical problem posing, examples of research and opinion from around the world are discussed.

<u>What Is Mathematical Problem Posing?</u>

Problem posing refers to both the generation of new problems and the re-formulation of given problems. Thus, posing can occur before, during or after the solution of a problem.

One kind of problem posing, usually referred to as problem formulation or re-formulation, occurs within the process of problem solving. When solving a nontrivial problem, a solver engages in this form of problem posing by recreating a given problem in some ways to make it more accessible for solution. Problem formulation represents a kind of problem posing process because the solver transforms a given statement of a problem into a new version that becomes the focus of solving. Problem formulation is related to planning, since it may involve posing problems that represent subgoals for the larger problem. Polya's heuristic advice, "Think of a related, more accessible problem," suggests another way in which problem formulation involves problem posing. If the source of the original problem is outside the solver, the problem posing occurs as the given problem is reformulated and "personalized" through the process of re-formulation. The operative question that stimulates this form of posing is: How can I formulate this problem so that it can be solved?

At least since Duncker's (1945) observation that problem solving consists of successive re-formulations of an initial problem, problem formulation has been extensively studied by psychologists interested in understanding complex problem solving. According to contemporary information-processing models of complex problem solving, a problem is solved by establishing a series of successively more refined problem representations which incorporate relationships between the given information and the desired goal, and into which new information is added as subgoals are satisfied. One of the major findings of an extensive body of research on the differences between experts and novices in a variety of complex task domains is that experts tend to spend considerable time engaging in problem formulation and re-formulation, usually engaging in qualitative rather than quantitative analysis, in contrast to novices who spend relatively little time in formulation and re-formulation (Silver & Marshall, 1989). For relatively simple problems, problem formulation may occur primarily in the early stages of problem solving, but in extended [illegible]atical investigations, "problem formulation and problem solution go hand in hand, each eliciting the other as the investigation progresses" (Davis , 1985, p. 23).

Not all problem posing occurs within the process of solving a complex problem. Problem posing can also occur at times when the goal is not the *solution* of a given problem but the

creation of a new problem from a situation or experience. Such problem posing can occur *prior to* any problem solving, as would be the case if problems were generated from a given contrived or naturalistic situation. This type of problem generation is also sometimes referred to as problem formulation, but the process being described here is different from that described above.[3] Problem posing can also occur *after* having solved a particular problem, when one might examine the conditions of the problem to generate alternative related problems. This kind of problem posing is associated with the "Looking Back" phase of problem solving discussed by Polya (1957). Brown and Walter (1983) have written extensively about a version of this type of problem posing, in which problem conditions and constraints are examined and freely changed through a process they refer to as "What-if?" and "What-if-not?". The operative question that drives these kinds of problem posing is: What new problems are suggested by this situation, problem or experience?

Perspectives on Mathematical Problem Posing

Having discussed briefly the nature of mathematical problem posing, we now turn our attention to several perspectives from which to view the importance and role of mathematical problem posing as an object of pedagogical and research attention. The purpose here is not to focus on sharp distinctions among these perspectives, since they are not mutually exclusive, but rather to use these perspectives as lenses through which to view various research studies and instructional interventions that have been undertaken.

Problem Posing as a Feature of Creative Activity or Exceptional Mathematical Ability

Problem posing has long been viewed as a characteristic of creative activity or exceptional talent. For example, Hadamard (1945) identified the ability to find key research questions as an indicator of exceptional mathematical talent. Related observations have been made about professionals in various science fields (e.g., Mansfield & Busse, 1981). Similarly, Getzels and Csikszentmihalyi (1976) studied artistic creativity and characterized problem finding as being central to the creative artistic experience.

The apparent link between posing and creativity is clear from the fact that posing tasks have been included in tests designed to identify creative individuals. For example, Getzels and Jackson (1962) developed a battery of tests to measure creativity, of which one task asked subjects to pose mathematical problems that could be answered using information provided in a set of stories about real-world situations. Getzels and Jackson scored the subjects'

[3]It is worth noting that teachers and textbook authors engage in this kind of posing when they pose questions for students to solve. Posing problems so as to be evocative of good mathematical thinking has received some attention in the literature (e.g., Butts, 1980). In this paper, however, the primary focus is on problems that are generated by students.

problems according to the complexity of the procedures that would be used to obtain a solution (i. e., the number and type of arithmetic operations used), and they used the results as a measure of creativity. Balka (1974) also asked subjects to pose mathematical problems that could be answered on the basis of information provided in a set of stories about real-world situations. Analysis of the responses attended to three aspects: fluency, flexibility, and originality. Fluency refers to the number of problems posed or questions generated, flexibility to the number of different categories of problems generated, and originality to how rare the response is in the set of all responses. This analytic scheme closely parallels that used in many approaches to measuring creativity (Torrance,1966)

The relationship of problem posing to exceptional mathematical ability has also been explored. For example, Krutetskii (1976) and Ellerton (1986) each contrasted the problem posing of subjects with different ability levels in mathematics. In his study of mathematical "giftedness", Krutetskii (1976) used as one of his measures of exceptional talent a problem-posing task in which students were presented with problems in which there was an unstated question (e.g., A pupil bought 2x notebooks in one store, and in another bought 1.5 times as many.), for which the student was required to pose and then answer a question on the basis of the given information. Krutetskii argued that there was a problem that "naturally followed" from the given information, and he found that high ability subjects were able to "see" this problem and pose it directly; whereas, students of lesser ability either required hints or were unable to pose the question. In Ellerton's (1986) study, students were asked to pose a mathematics problem that would be difficult for a friend to solve. She found that the "more able" students posed problems of greater computational difficulty (i. e., more complex numbers and requiring more operations for solution) than did their "less able" peers.

Since problem posing has been embedded in the assessment of creativity or mathematical talent, it is reasonable to assume that there is some link between posing and creativity. In fact, creativity has been associated both with novel productions (Newell, Shaw & Simon, 1962) and with ill-structured problem solving (Voss & Means, 1989), so a relationship to problem posing seems clear. On the other hand, the general relationship between creativity and problem posing is unclear. Haylock (1987) reviewed a set of studies that examined the relation between creativity and various aspects of mathematics, and he found an incomplete basis for asserting a relationship. More recently, Leung (1993) studied the relationship between the problems posed by a group of preservice elementary school teachers and their performance on tests of creativity and mathematical knowledge. After rating the posed problems along several dimensions of cognitive and mathematical complexity, she found essentially no relationship with their scores on the test of creativity. On the other hand, Leung

did report a strong relationship between the mathematical knowledge of the subjects and the quality of the problems they posed.

Because of the association of problem posing with the identification of persons with exceptional creativity or talent, one might infer that instruction related to problem posing would be appropriate only for "gifted" students. However, Leung's findings suggest that mathematical problem posing is an activity that need not be reserved for use only with students identified as exceptionally talented or creative. In fact, problem posing is a salient feature of broad-based, inquiry-oriented approaches to education that are discussed next.

Problem Posing as a Feature of Inquiry-oriented Instruction

In classrooms where children are encouraged to be autonomous learners, problem posing would be a natural and frequent occurrence. According to Ernest (1991), unlike inquiry approaches that emphasize discovery or problem solving, an investigatory approach to inquiry-oriented mathematics teaching is characterized by having responsibility for problem formulation and solution rest as much with the students as with the teacher. In the *Curriculum and Evaluation Standards*, one finds expressions of support for this view of problem posing in the mathematics classroom, as providing situations in which "mathematical ideas have originated with the children rather than the teacher" (NCTM, 1989, p. 24). This sentiment is further illustrated in the following excerpt from the same document:

> Experiences designed to foster continued intellectual curiosity and increasing independence should encourage students to become self-directed learners who routinely engage in constructing, symbolizing, applying, and generalizing mathematical ideas." (NCTM, 1989, p. 128)

Collins (1986) has identified three different general goals for inquiry-oriented teaching: (a) to help students construct general rules, theories or principles that are already known and match an expert's understanding of a domain, (b) to help students construct genuinely novel theories or principles that emerge from their inquiry, and (c) to teach students how to solve problems through the use of self-questioning and self-regulatory techniques and metacognitive skills. Collins identified Beberman's *discovery* teaching as an example of mathematics teaching directed at the first goal, and he cited Schoenfeld's (1985) problem-solving instruction as an example of instruction directed toward the third goal. Although Collins did not give an example of mathematics instruction directed toward the second goal, he would have been justified in citing the work of Brown and Walter.

Brown and Walter (1983) have written extensively about their incorporation of problem posing in the teaching of mathematics at the college and precollege levels. Their

instructional approach emphasizes the generation of new problems from previously solved problems by varying the conditions or goals of the original problem. The essence of Brown and Walter's "What-if-Not?" process is the systematic variation of problem conditions or goals. A rationale for this type of posing-oriented instruction, which is closely aligned with the general inquiry-oriented philosophy discussed above, is presented by Brown (1984).

Some form of inquiry-oriented instruction has long been offered to students from social and economic elite groups, but it has generally been denied to those who come from less privileged backgrounds. Despite this historical pattern, inquiry-oriented instruction can be seen to have close connections to arguments for emancipatory education for all students (e.g., Freire, 1970; Gerdes, 1985). Ernest (1991) provides a fairly complete summary of this view by showing how an inquiry-oriented pedagogy, with an emphasis on problem solving and problem posing, can be used to challenge the rigid hierarchies associated with conventional conceptions of mathematics, mathematics curriculum and mathematical ability. Through such pedagogy, Ernest argues that mathematics can be empowering for all learners and not just for those who are privileged by the current social, political and economic arrangements. One version of this approach is being implemented in the United States through the QUASAR project, which provides mathematics instructional programs aimed at high-level thinking, reasoning and inquiry to students (grades 6-8) from economically disadvantaged communities (Silver, Smith & Nelson, in press). Authors writing from a feminist perspective (e.g., Noddings, 1984) have also shown that inquiry-oriented instruction can be used in ways that honor alternative ways of knowing and solving problems.

Problem posing has figured prominently in some some inquiry-oriented instruction that has freed students and teachers from the textbook as the main source of wisdom and problems in a school mathematics course. Several authors have written about instructional experiments in which students have written a mathematics textbook for themselves or for children who will be in the class at a later time. Van den Brink (1987) reported such an experiment with children in first grade in the Netherlands. They were each asked to write and illustrate a page of arithmetic sums for children who would be entering first grade during the following year. Streefland (1987, 1991) has also employed similar authorship experiences for students as part of his "realistic mathematics education" instruction in the Netherlands. In the United States, Healy (1993) has used a similar approach with secondary school students studying geometry. In Healy's "Build-a-book" approach students do not use a commercial textbook but create their own book of important findings based on their geometric investigations.

Another example is drawn from Australia, where a primary grade teacher has written of her experiences in using problem posing as a central feature of her mathematics instruction involving a group of children over two and one-half years, spanning grades K-2 (Skinner, 1991). In her teaching, Skinner had her students engage in an extensive amount of problem posing. They shared their posed problems with each other, and these formed the basis for much of the problem-solving activity in the class. Skinner also incorporated larger investigation-oriented work that provided ill-structured problems which engaged the students for relatively long periods of time.

Winograd (1991) has provided another example of mathematics instruction emphasizing problem posing. He provided fifth-grade students with a year-long experience in which they wrote, shared with classmates, and solved original story problems. Winograd did not have a comparison group, but he reported a generally positive impact of the problem authorship experience on students' achievement, and especially on their disposition toward mathematics. Similarly, problem posing has been a prominent feature of geometry instruction based on the use of the *Geometric Supposers* (Yerushalmy, Chazan & Gordon, 1993) and also of the video-based, inquiry-oriented instruction developed by Bransford and his colleagues (Bransford, Hasselbring, Barron, Kulewicz, Littlefield & Goin, 1988), and positive claims about student outcomes have been made in that work.

In general, inquiry-oriented instructional activity has not been subjected to serious scrutiny, either with respect to the role of the problem posing in the instruction or to the long-term impact of the instruction on the students. The authors have provided some description of the instruction and of the students' responses or work, but there has been little or no systematic analysis of the nature of the problem posing and inquiry that occurred or of the impact that these experiences had on students' mathematical performance. Inquiry-oriented instruction can be closely tied to mathematics or it can be based more on a general framework. In the next section, we consider approaches that would be closely tied to mathematical activity.

Problem Posing as a Prominent Feature of Mathematical Activity

One argument for focusing curricular or research attention on the generative process of problem posing is that it is central to the discipline of mathematics and the nature of mathematical thinking. When mathematicians engage in the intellectual work of the discipline, it can be argued that self-directed problem posing is an important characteristic (Polya, 1954). Mathematicians may solve some problems that have been posed for them by others or may work on problems that have been identified as important problems in the literature, but it is more common for them to formulate their own problems, based on their personal experience and interests (Poincare, 1948). Rather than being presented for

solution by an outside source, mathematical problems often arise out of attempts to generalize a known result, or they represent tentative conjectures for working hypotheses, or they appear as subproblems embedded in the search for the solution to some larger problem. Thus, it has been argued that professional mathematicians, whether working in pure or applied mathematics, frequently encounter ill-structured problems and situations which require problem posing and conjecturing, and their intellectual goal is often the generation of novel conjectures or results (Pollak, 1987).

The relation of this view to the inclusion of problem posing in the curriculum is evident in the following excerpt from the *Curriculum and Evaluation Standards for School Mathematics*: "'Students in grades 9-12 should also have some experience recognizing and formulating their own problems, an activity that is at the heart of doing mathematics" (NCTM, 1989, p. 138). Such views are compatible with the emerging view that to understand what mathematics is, one needs to understand the activities or practice of persons who are makers of mathematics. A view of mathematical knowing as a practice (in the sense of professional practice) comes from analyses of the history and philosophy of mathematics (e.g., Lakatos, 1976; Kitcher, 1984), which highlight important social aspects of mathematics that remained hidden from view in classical logical analyses. Those who view the purpose of mathematics education as providing students with authentic experiences like those that characterize the activity of professional mathematicians would identify problem posing as an important component because of its apparently central role in the creation of mathematics.

It has been argued that ill-structured problem situations are often encountered by those who create or apply mathematics in professional activity and that such situations serve as a major source of problem posing done by professionals in the field of mathematics. Moreover, Hadamard (1945) identified the ability to find key research questions as an important characteristic of talented mathematicians. Nevertheless, beyond anecdotal accounts, little direct evidence of problem posing by mathematicians has been produced. Ill-structured problem solving of the sort done by mathematicians has not been systematically investigated, but it has been studied in some other professional domains. For example, Reitman (1965) examined the processes utilized by artists and composers in large-scale ill-structured problem settings, like musical composition. He argued that the observation of persons solving ill-structured problems exposed many more differences in the memory structures of respective solvers than became exposed when they solved well-structured problems. Simon (1973) extended Reitman's analysis of ill-structured task domains and suggested that although there was little difference in the processes required to solve well-structured or ill-structured problems, ill-structured problems required a wider range of processes in formulating and solving the problem and in recognizing the solution when it was obtained.

and that much of the cognitive activity in such problem solving is directed at structuring the task. Thus, ill-structured problem provide a rich arena in which to study complex cognitive activity, such as problem posing.

Some research has considered the application of mathematics to ill-structured problems. For example, Lesh and colleagues (Lesh, 1981; Lesh, Landau, & Hamilton, 1983) characterized the processes used by young adolescents as they solved applied mathematical problems embedded in real world, meaningful contexts. The findings of this research suggested that the processes used in solving applied problems were somewhat different from the processes observed when the same students solved well-structured school mathematics problems. In particular, in applied problem solving more cognitive attention was devoted to the processes of formulation and re-formulation during problem solving. Thus, significant mathematical problem posing activity occurs not only in the creation of mathematics by professional mathematicians but also in the thoughtful application of mathematics by students. Therefore, problem posing would also be a salient feature of instruction designed by those who view the purpose of mathematics education as being less about introducing students to the culture of professional mathematics and more about assisting students to learn the ways of thinking and reasoning employed by those who apply mathematics and quantitative reasoning effectively to solve real-world problems. Since most students will not become professional mathematicians, an education that prepares them to be intelligent users of mathematics in order to solve problems of importance or interest to them may be better suited for them than one which is based on the activity of professional mathematics, and extensive experience in problem posing would be an important component of instruction aimed at such a goal (Blum & Niss, 1991).

The research discussed above provides a foundation on which to build, but further study of the posing and solving processes involved in the solution of ill-structured, applied problems is needed. The general connection between problem posing and many forms of problem solving is further discussed in the next section.

Problem Posing as a Means to Improving Students' Problem Solving

Probably the most frequently cited motivation for curricular and instructional interest in problem posing is its perceived potential value in assisting students to become better problem solvers. In fact, a perceived connection between mathematical problem posing and curricular goals related to problem solving permeates the NCTM *Professional Teaching Standards*. Interestingly, although problem posing has not been a common feature of mathematics instruction, advocacy for problem posing as a means of improving students' problem-solving performance is not a new idea. For example, Connor and Hawkins (1936)

argued that having students generate their own problems improved their ability to apply arithmetic concepts and skills in solving problems. Twenty years later, Koenker (1958) included problem posing as one of 20 ways to help students improve their problem solving.

Problem posing has been incorporated as a feature of some Japanese experimental teaching which employs problem posing as a means of assisting students to analyze problems more completely, thereby enhancing students' problem-solving competence. Several authors (Shimada, 1977; Hashimoto & Sawada, 1984; Nohda, 1986) have described various versions of a style of teaching, known as "open approach teaching" or teaching with "open-end or open-ended problems." Their descriptions, and those of others, suggest various ways in which problem posing is embedded in the instruction. For example, Hashimoto (1987), has described and provided a transcript of a lesson in which students pose mathematical problems on the basis of one solved the previous day.

Another interesting analysis of problem posing has been done by Sweller and his colleagues in Australia (e.g., Sweller, Mawer, & Ward, 1983; Owen & Sweller, 1985). Some of Sweller's studies have involved ill-structured mathematics problems from the domain of geometry and trigonometry. In general, these studies have demonstrated that subjects are far more likely to use means-ends analysis on goal-specific problems (given an angle in a figure, find the value of a particular other angle in the figure) than on non-goal-specific problems (given an angle in a figure, find the measure of as many other angles as you can). Moreover, Sweller's results show that, although means-ends analysis is a powerful problem-solving strategy, the unavailability of means-ends analysis in non-goal-specific problems may lead subjects not only to use more expert-like, forward-directed problem-solving behavior but also to develop powerful problem-solving schemas, thereby positively affecting students' learning. This work suggests that students' engagement with problem posing and conjecture formulation activities, in the context of solving ill-structured mathematics problems, can have a positive effect on their subsequent knowledge and problem solving.

A few experimental or quasi-experimental studies have been conducted in the United States, in which students receiving a form of mathematics instruction in which problem posing has been embedded are contrasted with students who have comparable instruction without the posing experience. Keil (1965) found that sixth-grade students who had experience writing and then solving their own mathematics problems in response to a situation did better on tests of mathematics achievement than students who simply solved textbook story problems. Perez (1985) found similar results with college students studying remedial mathematics, and he also reported that the experimental treatment, which involved some writing and some rewriting of story problems, had a positive effect on students' attitudes toward mathematics.

Unfortunately, these studies did not examine the direct impact of the instructional experience on students' problem generation itself.

Despite the interest in problem posing because of its potential to improve problem solving, no clear, simple link has been established between competence in posing and solving. Silver & Cai (1993) examined the responses of middle school students (grades 6 and 7) to a task asking them to generate three problems on the basis of a brief story (Jerome, Elliott, and Arturo took turns driving home from a trip. Arturo drove 80 miles more than Elliott. Elliott drove twice as many miles as Jerome. Jerome drove 50 miles.). The student-generated problems were classified according to mathematical complexity (number of operations required for solution), and this measure of problem posing was compared to students' performance in solving eight open-ended mathematical problems. Silver and Cai found a strong positive relationship between posing and solving performance. On the other hand, Silver & Mamona (1989) found no overt link between the problem posing of middle school mathematics teachers and their problem solving. In that study, Silver and Mamona asked the teachers to pose problems in the context of a task environment (or microworld) called Billiard Ball Mathematics, consisting of an idealized rectangular billiard ball table with pockets only at the corners and on which a single ball is hit from the lower left corner at an angle of 45° to the sides. Problem posing occurred prior to and immediately after the teachers solved a specific problem concerning the relationship between the dimensions of the table and the final destination of the ball. Except for the fact that subjects' post-solution posing was influenced by their problem-solving experience (i. e., in post-solution posing, they posed more problems like the one they solved than they had in the pre-solution posing), there was no other relationship between posing and solving that could be detected. Clearly, there is a need for further research that examines the complex relationship between problem posing and problem solving. In addition, there is also interest in exploring the relation of posing to other aspects of mathematical knowing and mathematical performance.

Problem Posing as a Window into Students' Mathematical Understanding

Interest in problem posing as a means of helping students become sensitive to facts and relations embedded in situations has been evident for a long time. For example, Brueckner (1932) advocated the use of student-generated problems as a means of helping students to develop a sense of number relations and to generalize number concepts. In generating problems based on the mathematical ideas and relations embedded in situations, students engage in "mathematizing" those situations. Such experience may assist them to overcome the well-documented tendency of students to fail to connect mathematics sensibly to situations when they are asked to solve pre-formulated problems (Silver & Shapiro, 1992). The following excerpt from the *Professional Teaching Standards* illustrates this point of view:

"Writing stories to go with division sentences may help students to focus on the meaning of the procedure" (NCTM, 1991, p. 29).

In England, Hart (1981) used problem posing as one research technique to examine students' understanding of important mathematical concepts. By providing answers or equations and asking students to generate problem situations that would correspond to the given answer or equation, Hart showed that one could open a window through which to view children's thinking. More recently, Greer and McCann (1991) used Hart's approach by providing multiplication and division calculations to students (ages 9-15) in Northern Ireland and asking them to generate story problems that matched a given calculation. A similar approach has been used by Simon (1993) and by Silver and Burkett (1993) in studies with preservice elementary school teachers' understanding of division. A variation on this approach was used by Ellerton (1986), who, without presenting any additional context or stimulus, simply asked Australian students (ages 11-13) to create a problem that would be difficult for a friend to solve. Based on the children's choice of numbers in the problems (e.g., fractions that did or did not permit cancellation), Ellerton made inferences about some aspects of the children's mathematical knowledge. Another technique, discussed above in reference to Krutetskii's (1976) study of mathematically talented students, is the use of problems with an unstated question. As these brief descriptions suggest, some researchers have found problem posing to have potential as a means of exploring the nature of students' understanding of mathematical ideas.

Regrettably, the research cited above has generally found a fairly weak connection between real life situations and mathematical ideas or symbols. For example, Greer and McCann found that some students used a fraction to represent a number of people in a posed problem; this finding is similar to the finding that many students will solve problems by providing answers that have weak connection to the real world setting described in the problem (Silver & Shapiro, 1992). Unfortunately, a lack of concern about sensible connection to real world settings has been reported in studies of posing by preservice elementary school teachers (Silver & Burkett, 1993; Simon, 1993). Moreover, Ellerton found that students' conceptions of difficulty seemed to be linked almost entirely to computational complexity rather than to situational or semantic complexity. These findings are consistent with conventional mathematics instruction, which tends not to relate mathematics to real world settings in any systematic manner, and they appear to be closely related to van den Brink's observation concerning the arithmetic books constructed by first-grade children in his study: "A striking aspect of the books was that arithmetic as applicable knowledge only appeared in the class book when it had been learned that way" (1987, p. 47). Thus, it appears that problem posing provides not only a window through which to view students'

understandings of mathematics but also a mirror which reflects the content and character of their school mathematics experience. Opening the problem posing window also affords an opportunity to view aspects of students' attitudes and dispositions toward mathematics.

Problem Posing as a Means of Improving Student Disposition toward Mathematics

There are several different aspects of problem posing that are thought to have important relationships to student disposition toward mathematics. For example, posing offers a means of connecting mathematics to students' interests. As the *Curriculum and Evaluation Standards* suggests: "Students should have opportunities to formulate problems and questions that stem from their own interests" (NCTM, 1989, p. 67). Nevertheless, personal interest is not the sole motivation for posing problems. Within a classroom community, students could be encouraged to pose problems that others in the class might find interesting or novel. In a study of one such instructional experiment, Winograd (1991) reported that the fifth-grade students in his study appeared to be highly motivated to pose problems that their classmates would find interesting or difficult. He also noted that students' personal interest was sustained in his study through a process of sharing problems with others.

There is also a reciprocal expectation regarding problem posing, since engagement with problem generation is also thought to stimulate student interest in mathematics. Students who have difficulty with mathematics are sometimes characterized by a syndrome of fear and avoidance known as mathematics anxiety. Some have claimed that mathematics anxiety can be reduced through problem posing (Moses, Bjork & Goldenberg, 1990), since student participation in problem posing makes mathematics seem less "intimidating" (Brown & Walter, 1983). In fact, Perez (1985) taught college-age students studying remedial mathematics, and therefore likely to have mathematics anxiety and poor attitudes toward mathematics, using a problem-posing approach. He reported improvement in the students' attitudes toward mathematics, as well as in their achievement.

In general, reports of problem-posing instruction do not discuss instances in which students have rejected or reacted negatively to this instructional approach. Nevertheless, it seems plausible that some students, perhaps especially those who have been successful for a long period of time in school settings characterized by didactic, teacher-directed instruction, would react negatively to a style of teaching that was less directive and placed on them more responsibility for learning. For these students, there may be little desire or motivation to alter the existing power relations in the classroom, or to alter the hierarchical assumptions underlying current conceptions of mathematical performance. There is evidence from other sources that students can sometimes resist changes in classroom instruction that require

them to deal with higher levels of uncertainty about expectations or higher levels of responsibility for their own learning (e.g., Davis & McKnight, 1976; Doyle & Carter, 1982) .

Some evidence of the plausibility that some students might reject or resist mathematics instruction based on posing comes from two recent non-instructional studies. Comments made by a few of the middle school teachers who were part of the study by Silver and Mamona (1989) indicated hostility toward the task requirement to pose their own mathematical problems (e.g., "This is stupid!", "Why are we being asked to do this?"). Similarly, Silver and Cai (1993) found that some students in grades 6-7, when asked to pose three problems on the basis of a story situation, expressed profound dismay at being asked to do this (e.g., "This is unfair", "My teacher didn't teach us how to do this"). Understanding how students, especially those who have been successful in less inquiry-oriented classrooms, do or do not make a transition to participation in problem posing and acceptance of posing-oriented instruction is an important research topic.

Healy (1993) provides an example that illustrates how an emphasis on student-generated problem posing can humanize and personalize mathematics learning and instruction in profound ways. For many of his students, mathematics became something other than a neutral body of knowledge filled with abstract ideas and symbolism that others had created and which was accessible only through imitation and memorization. Instead, many students became passionately concerned about mathematical issues that they were investigating because of personal interest and commitment. Clearly, the affective dimension of such an instructional experience is significant, as is exemplified in the following quote from one of Healy's students after three months of the course: "In this class we make enemies out of friends arguing over things we couldn't have cared less about last summer" (1993, p. x). Even in a less competitive setting than the one implied by this student's comment, one would expect the passionate, personal engagement of students with mathematical ideas to produce learning situations in which affective and cognitive issues would both have great import.

Conclusion

In their historical account of the treatment of problem solving in the mathematics curriculum, Stanic and Kilpatrick (1988) argued that problem solving could be viewed as means to teach desired curricular material or it could itself be viewed as an educational end or goal. Similarly, in this paper, problem posing has been discussed in ways that correspond to it being viewed as a means to achieve other curricular or instructional ends or as an educational goal itself. Many purposes for which problem posing might be included as a feature of school mathematics have been considered, as has research evidence associated with these purposes. From the fairly unsystematic collection of findings that characterizes

the research literature on mathematical problem posing, many studies were cited and some of their findings and approaches organized for presentation in this paper.

From the perspective of research, three major conclusions seem warranted from this review. First, it is clear that problem-posing tasks can provide researchers with both a window through which to view students' mathematical thinking and a mirror in which to see a reflection of students' mathematical experiences. Second, problem-posing experiences provide a potentially rich arena in which to explore the interplay between the cognitive and affective dimensions of students' mathematical learning. Finally, much more systematic research is needed on the impact of problem-posing experiences on students' problem posing, problem solving, mathematical understanding and disposition toward mathematics.

Coda

Problem-posing experiences can afford students opportunities to develop personal relationships with mathematics. The process of personalizing and humanizing mathematics for students through the use of open-ended problem-posing tasks invites them to express their lived experiences, and this can have important consequences for teachers and for researchers. For example, if allowed to do so, students may pose problems different from the ones that the teacher or researcher had in mind. In all of the problem-posing studies I have conducted, at least some of the subjects have given responses or engaged in behavior that was entirely unexpected. For example, in studies involving the Billiard Ball Mathematics task (e.g., Silver & Mamona, 1989), many subjects have not only imported aspects of their experience in playing billiards to pose problems but also generated problems that are not the "standard" problems that have neat mathematical solutions. When one poses a problem, one may not know whether or not the problem will have a simple solution, or any solution at all.

Another consequence of personalizing and humanizing mathematics through problem posing is that students can and will respond in ways that reflect their personal commitments and values. In some cases, this personal attachment can have positive consequences, as in the work reported by van den Brink (1987), in which he reported that the young children in his study made almost no mistakes in their self-constructed arithmetic books. This stands in sharp contrast to the commonly observed carelessness of students and their tolerance of mistakes in mathematics classrooms in which they feel no personal ownership of mathematics. In one recent study of problem posing (Silver & Cai, 1993), students posed problems on the basis of a story about three persons driving in a car, and their responses suggested that issues of morality, justice and human relationships may have been as important to some students as issues of formal mathematics. For example, students revealed an apparent concern about an equitable distribution of driving responsibilities when they

posed the following kinds of questions: "If they each drive an equal amount, how many miles would each person drive?", "Why does Arturo drive so long?" and "Why did Elliott drive twice as far as Jerome?". From the perspective of the research being conducted in that study, which focused on the semantic and syntactic complexity of the problems generated by the students, some of these responses were treated as being of marginal interest. Viewed from a broader perspective, however, these responses suggest not only the power of problem posing as an experience in which people express themselves with respect to mathematical situations or ideas but also the complexity of the educational and research challenges connected with understanding what the posed problems themselves represent as products of human activity. For the reasons discussed here, such problem-posing experiences are likely to be both especially important and especially problematic in teaching or research settings involving culturally diverse groups of students.

It would be easy for instructional developers and psychological researchers to overemphasize the role of problem posing as a means to accomplish other aims, such as improving the learning or study of problem solving. Our orientation toward problem solving in mathematics is so strong that we could miss the value of problem posing for its own sake. As we proceed with an agenda of instruction and research related to mathematical problem posing, let us be mindful not only of the potential that posing may offer for accomplishing other goals but also that the unsolved questions themselves offer great promise to us and to our students. We would be wise to heed the advice of the poet, Rainer Maria Rilke:

Be patient toward all that is unsolved
in your heart
Try to love the questions themselves.

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Working Groups

ADVANCED MATHEMATICAL THINKING

Organisers: David Tall (U.K.), Gontran Ervynck (Belgium)

The group is concerned with all kinds of mathematical thinking of students beyond the age of 16, extending and developing theories of the psychology of Mathematics Education that cover development of mathematics over the full age range.

SESSION I: **Proof** (Gary Davis, Men Fon Huang, John Selden).

How can students learn to construct proof and what kind of background knowledge is needed ? Proof construction requires a knowledge of the style in which proofs are written and an ability to check their validity. When proving a theorem, a mathematician usually devises a tentative global strategy and finds a number of plausible statements together with their justifications. These are then assembled like the pieces of a puzzle, except that the pieces are created, instead of being given. Can anything be said about the way students might find such plausible statements/justifications ? Should students learn to find them before starting to construct larger, more complex proofs ? (excerpt from a question of John Selden at PME-16).

In session we want to focus on some empirical research and some development of an appropriate theory.

SESSION II: **Mathematical Knowledge in the Information Age** (John Selden).

Increasing availability of computing equipment means that in a near future most of what is now taught has to be revised and that many university curricula will need assessment. Hence two questions: (1) What kind of knowledge and skills should students of AMT learn ? (2) What is the meaning of understanding mathematical knowledge ?

SESSION III: **Inconsistencies of students in AMT** (Dina Tirosh, Shlomo Vinner).

What kind of mistakes do students in AMT make ? As they often give incompatible arguments, the emphasis in teaching should be on the promotion of consistency. A discussion of inconsistencies needs to take place within a theory of epistemological growth which sets the inconsistency in context.

SESSION IV: **Future plans of the group.**

The group should pay attention to a long term planning; the organisers insist that plans be set in the context of a long term growth in working group activity. In session IV suggestions for PME-18 (1994, Portugal) will be discussed, together with two proposals to extend the activities of the Group. (1) It has been suggested that the AMT Group could avantageously organise joint sessions with an other Working Group, in particular with the group on *Representations*. (2) To stimulate young researchers interested in the domain of Math Ed for students from 16+ on to participate actively in the work of the AMT group.

PME Working Group
Algebraic Processes and Structure

Coordinator: Rosamund Sutherland

This Group was set up in PME XIV with the aim of characterising the shifts that appear to be involved in developing an algebraic mode of thinking and to investigate the role of symbolising in this development. The group is also concerned with the implications of their collective research for classroom practice. The discussion at PME XVI centred around the following questions: What is the relationship between mental and written transformations and why is it so difficult to transform algebraic expressions mentally?; Can you build notions of structure if you start with an empirical approach to algebra?; How does the learning of algebra relate to the learning of natural language?; What algebraic notions do pupils develop from working in a spreadsheet environment?

In PME XVII the working group will focus on the following themes a) What are the distinctive characteristics of algebraic reasoning and what is the role of structure as opposed to empirical algebra? b) What conceptions of symbol/variable/unknown do pupils develop in computer-based and paper-based settings and what is the relationship between the two? c) What is the role of symbols/language in thinking algebraically? d) What type of classroom activities do we need to be developing?

A number of regular participants will be invited to make short presentations in order to provoke discussion centred around the above themes. The group is working towards a collection of chapters which they aim to produce in 1994.

Classroom Research

The purpose of this group is to examine issues and techniques associated with classroom research and the impact of such research on educational conditions.

The focus of discussion at PME XVII will be on the use of children's drawings as a research tool. Participants are invited to present short reports related to this theme. Discussions will center on examining the nature of the information that can be collected, the types of interpretations that can be made, and the strengths and limitations of such research.

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WORKING GROUP ON

CULTURAL ASPECTS IN THE LEARNING OF MATHEMATICS

This Working group met for the first time last year to consider questions relating to *the meaning of culture in the learning of mathematics*.

During the work of an earlier Discussion Group, a number of interests had been expressed and three directions were identified.

Hence, at PME XVI, questions relating to the following were discussed:

1. Informal education and formal mathematical knowledge
2. The effects of cultural environment, including language, on the mental representations of students and teachers
3. Cognitive processes in learning mathematics, using a comparative approach for different cultures.

With the above work as background, this year, we intend to continue to build our specific *problématique*.

At PME XVII, following is the program outline for the sessions.

We shall try to take into account the cultural context of the country in which PME is held.

Session 1

Invited speakers will address issues identified from the deliberations of last year's group.

A specific introduction to the Japanese cultural context will also be provided, taking into account the three above-mentioned directions.

Session 2

Sub-groups, following from the research presented in session 1, will be formed.

Discussion and elaboration of research in preparation for PME XVIII will be expected.

Session 3

Plenary session with reports from each sub-group.

Summary and plans for subsequent years.

The objectives of our working group include contributing to the formulation of a methodological and theoretical framework by presenting original research at PME conferences and identifying the areas relevant to this approach. These contributions may be interdisciplinary in nature, possibly by making use of the fields of psychology, mathematics education, art education, and didactics of geography.

Bernadette Denys

GEOMETRY WORKING GROUP

The Geometry Working Group is the meeting place for those P.M.E. members who are interested in research related to the teaching-learning of Geometry. We understand Geometry in a broad sense, which includes: The various geometrical topics taught in any educational level; formal reasoning and proof in geometrical contexts; visualization; learning environments, instruments and technology; and the different theories apliable to the teaching-learning of Geometry; and others.

In last years the interest of the P.M.E. community in Geometry has grown, as shown by the wider diversity of themes of interest arisen during the last Group's meetings. It seems to us that the 1993 meeting would be a suitable moment for starting an overview of the main problems of teaching and learning Geometry, identifying the most interesting areas of Geometry, and defining research question that should be approached in the near future.

The first session of the 1993 meeting of the Geometry Working Group will be devoted to reflect on that point: Which areas of Geometry need more research from the P.M.E. view? Which problems should be investigated in each of such areas? The conclusions from this session shall influence the contents of our meetings in next years and, hopefully, shall provide us with ideas for new research projects.

The remaining sessions shall be devoted to discussion on two psychological components of the process of teaching-learning Geometry that likely shall be present in the Group's debate about most areas of Geometry:

a) Reasoning processes in Geometry: Which characteristics have these processes? Which ones are specific of Geometry, different from those in other areas of Mathematics?

b) Geometry problem solving: What kinds of problems are useful for the current ways of teaching Geometry? Which variables are relevant for discriminating geometry problems? Which ones for discriminating problem solvers?

When planing the activities for this year, we have had in mind that the meeting will take place in a country with a fascinating, but some times unknown, mathematical and educational culture. With the help of some Japanese colleages, we want to profit this opportunity for having a better knowledge of the research on Geometry Education carried out in Japan.

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RESEARCH ON THE PSYCHOLOGY OF MATHEMATICS TEACHER DEVELOPMENT

The Working Group *Research on the Psychology of Mathematics Teacher Development* was first convened as a Discussion Group at PME X in London in 1986, and continued in this format until the Working Group was formed in 1990. This year, at PME 17, we hope to build on the foundation of shared understandings that have developed over the last six years.

Aims of the Working Group

- The development, communication and examination of paradigms and frameworks for research in the psychology of mathematics teacher development.
- The collection, development, discussion and critiquing of tools and methodologies for conducting naturalistic and intervention research studies on the development of mathematics teachers' knowledge, beliefs, actions and thinking.
- The implementation of collaborative research projects.
- The fostering and development of communication between participants.
- The production of a joint publication on research frameworks and methodological issues within this research domain.

Research Questions

In 1992, strong cross-country bonds were established between those participating in the Working Group sessions, and discussion focused on participating members' research interests and projects which related to the theme of the Working Group. Through this discussion, examples of the practice of mathematics teachers and teacher educators that inform our notions of what constitutes good pedagogy in general, and the role of assessment, in particular were shared. In 1993, one of the aims of the Working Group is to share and discuss examples of different methodologies that are appropriate for research into the psychology of mathematics teacher development.

Proposed Outcomes of the Working Group at PME 17

1. ***Presentation and Discussion of Appropriate Research Methodologies:*** Different methodologies appropriate for researching the psychology of mathematics teacher development will be shared and critiqued. It is hoped that, through this approach, all researchers in the Group can enrich their understanding of different methodologies applied in different contexts.
2. ***Collaborative Research Projects:*** Members of the Working Group have overlapping research interests, and it is hoped that (a) collaborative research projects can be mounted, and (b) a continuing dialogue can be established between members of the Working Group.
3. ***Publication of an Edited Collection of Research Papers:*** The Working Group plans to publish a collection of research papers. The proposed title is: *Effective Methods for Achieving Change in Mathematics Teacher Development: An International Research Perspective.*

Nerida Ellerton, Convenor

PME Working Group on Ratio and Proportion

Organizers: Fou-Lai Lin, Kathlern M.Hart and Robert Hunting

Session 1. Learning and Teaching of Begining Fractions/Ratio

Each participant of this group shall present briefly his/her research interesting at the begining of this session. Then the discussion will focus on the learning and teaching of begining fractions/ratio.

Session 2. Adult's Understanding of Fractions, Ratio and Proportion

This session will focus on the understanding of senior high school students, pre-service teachers and in-service teachers on fractions, ratio and proportion.

Session 3. Recognization of Ratio and Inverse Ratio

This session will focus on recognization of ratio and inverse ratio situations, particularly, the analogies in science and mathematics.

The IGPME Working Group on Representations

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The Working Group on Representations of the International Group for the Psychology of Mathematics Education will meet again in Japan in 1993. I took on the responsibility of coordinating it at the end of the 1989 meeting in Paris, and Claude Janvier (Montreal, Canada) provided valuable assistance. At the 1990 meeting in Mexico our focus broadened substantially, to include a number of different interpretations that have been given to the term "representation" in connection with mathematics learning, teaching, and development. These discussions continued during the 1991 Italy meeting and the 1992 meeting in the USA. After the 1993 meeting, the group will be coordinated by Claude Janvier and Gérard Vergnaud.

A special issue of the *Journal of Mathematical Behavior*, based on the work of our group, is presently being planned.

Some of the interpretations we are giving to "representations" have included the following

(a) *External physical embodiments (including computer environments):* An external, structured physical situation or set of situations that can be described mathematically or seen as embodying a mathematical idea. (b) *Linguistic embodiments:* Verbal, syntactic, and related semantic aspects of the language in which problems are posed and mathematics is discussed. (c) *Formal mathematical constructs:* A different meaning of "representation", still with emphasis on a problem environment external to the individual, is that of a formal structural or mathematical analysis of a situation or set of situations. (d) *Internal cognitive representations:* Very important emphases include students' internal, individual representation(s) for mathematical ideas such as "area", "functions", etc., as well as *systems* of cognitive representation in a broader sense that can describe the processes of human learning and problem solving in mathematics. Distinctions between static and dynamic representations, temporal and spatial, etc. are of great interest.

The scope of the Working Group on Representations includes many related questions. The following have been discussed in previous years:

What are the consequences of creating and manipulating particular external representations of mathematical concepts? How can we develop new external systems of representation that foster more effective learning and problem solving? How can we describe in detail the internal cognitive representations of learners and problem solvers? What is the nature of the interaction between external and internal representations? How do we *infer* internal representations by observing external behaviors? How do individuals construct internal representations from their experience of external environments? What can *theories* based on cognitive representation tell us about making mathematics education more effective? What is the role of metaphor in cognitive representation in mathematics? Such questions are addressed not only as general, abstract considerations, but in the particular contexts of mathematical activity in specified domains, and with various populations of students, teachers, mathematicians, and so forth.

PME-XVII-01

WORKING GROUP ON SOCIAL PSYCHOLOGY OF MATHEMATICS EDUCATION

JEFF EVANS, Middlesex University, London, UK (Enfield, Middx. EN3 4SF)
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Four 90-min. sessions:

1. Introd. Organisation and re-organisation of national curricula.
2. Assessment for control and/or empowerment
3. Alternative curricula appropriate for democratic citizenship
4. Looking ahead.

Session 1. Introduction

The contexts and determinants of the mathematics curriculum as a consequence of the social, political and educational changes taking place in most countries. These changes manifest themselves broadly in two aspects:
(a) re-structuring of the control of the maths curriculum; and
(b) re-organisation of tne content and assessment of the curriculum.
The motivations for change are not only educational, but also political. The changes in different societies may have far-reaching and diverse consequences.

In this first session, participants will have a chance to introduce themselves and to report on the situation in their countries, as proposed at PME-XVI.

Session 2. Assessment

Assessment provides a tool that may be used in several ways - for political or administrative control of learning, or for teacher and pupil empowerment (peer assessment, self-assessment). Participants are encouraged to bring examples of each from their own experience.

Session 3. Alternative curricula for democratic citizenship

Many teachers of mathematics aspire to a curriculum which provides the basis for "democracy", "responsible citizenship", etc. What would such curricula look like: would there be any real maths? Or would it more resemble statistics? What problems might arise: for example, this approach seems likely, at the very least, to lead to problems in the "transfer" of school learning to applications in civic life, which are not too dissimilar to problems of applying "vocational" curricula at work.

Session 4. Looking ahead.

What might usefully be done between now and next PME in Portugal? Links with the Cultural Aspects working group.
Proposal for a name change; one possibility: "Social and Political Aspects of Mathematics Education".

Working Group: Teachers as Researchers in Mathematics Education

co-convenors: Stephen Lerman and Judy Mousley

The group has been meeting annually since 1988 and as a working group since 1990. The aims of the group are to review the issues surrounding the theme of teachers as researchers in mathematics education, and to engage in collaborative research.

The stimulus for the notion that classroom teachers can and should carry out research whilst concerned with the practice of teaching mathematics comes from a number of sources, including: teachers as reflective practitioners; empowerment; teaching as a continuous learning process; the nature of the theory/practice interface; the problems of dissemination of research when it is centred in colleges; research problems being generated in the classroom, and finding solutions within the context in which the questions arise. These themes are seen to be equally relevant to the teacher education situation. and provide a focus for the reflective activities of ourselves as teacher-educators.

Since the meeting in Assissi in 1991, we have established a network amongst members and circulated papers, ideas and questions. The programme in Tsukuba will centre around the issues that were raised in New Hampshire, namely research on our own practice as teacher educators, and/or teachers of mathematics. We will also review the work of present and new members in this field and report on research carried out during the year.

Discussion Groups

I-101

The Enhancement of Under Presented Countries in South-East Asia and East Europe

South - East Asia Countries arranged
by Du Wey,
Takahiro Kunioka
East Europe Countries arranged
by Hideki Iwasaki,
Katsuhiko Shimizu

The exchange of knowledge and facts about mathematics education from selected South-East Asia and East European countries with presented countries is aimed in this Discussion Group.

Researchers from above countries will report the intended, implemented and learned curriculum in each counties.

We will present the cross-cultural perspectives for the research on the above theme.

We will discuss shared methods to resolve common problems on mathematics education in those countries where participants can clarify them.

The mathematics educators from "presented countries" are also welcomed.

Interactive Geometry with Cabri-Géomètre

Jean-Marie LABORDE, Bernadette DENYS

IMAG-LSD2 Université Joseph Fourier, Grenoble, FRANCE

This group is open to people concerned with the use of software for the teaching and the learning of geometry.

In particular, this group will discuss the following:

1. the new categories of geometry problems which can be investigated by Cabri-Géomètre

2. the use of the instrument Cabri-Géomètre as a tool of research for studying problems connected with the learning of geometry

3. the evolution of geometry software towards "microworlds under direct manipulation" i.e.providing direct access to the geometrical objects constituting a figure, to change them and to make them interact.

Discussion Group

PHILOSOPHY OF MATHEMATICS EDUCATION

Paul Ernest (UK)

In mathematics education epistemological and philosophical issues are gaining in importance as theories of learning become more epistemologically orientated (e.g. constructivism). Various areas of inquiry in PME, including problem solving, teacher beliefs, applications of the Perry Theory, and ethnomathematics, all relate directly to the philosophy of mathematics and epistemological issues. Currently psychological researchers are bringing social perspectives to bear on their research problems, drawing on the work of Vygotsky, activity theorists, situated cognition, poststructuralists, hermeneutics, social constructionism, and other traditions of thought. Some of these influences are shown in the work of Bartolini-Bussi, Bauersfeld, Boero, Cobb, Nunes, Saxe, Steiner, Walkerdine, Wood, and others. These new perspectives all raise deep epistemological and philosophical issues for the psychology of mathematics education.

A central philosophical issue concerning the nature of mathematics which needs to be addressed at PME is the move by some philosophers and researchers to abandon the absolutist paradigm of mathematical knowledge. Publications by Lakatos, Davis, Hersh, Kitcher, Kline, and Tymoczko, for example, suggest a new fallibilist paradigm with profound implications for the psychology of mathematics education. If mathematics itself is no longer seen as a fixed, hierarchical body of objective knowledge, then the status of hierarchical theories of mathematical learning or of subjective knowledge of mathematics is open to question. One outcome is that they can no longer claim indisputably to be representing the logical structure of mathematics.

Researchers are becoming increasingly aware of the epistemological assumptions and foundations of their inquiries, since any inquiry into the learning and teaching of mathematics depends upon the nature of mathematics, and teachers' and researchers' philosophical assumptions about it. Many of these issues recur at PME, and the opportunity for an open and continuing discussion has proved to be both useful and timely. The group met at PME-16 in New Hampshire, and this meeting will continue the lively and well supported discussion begun there.

The aim is to offer a forum for a discussion of issues including the implications for theory and research in PME of:

* Social views of mathematics and mathematics learning;
* Interdisciplinary theoretical & philosophical developments;
* Epistemological bases of research paradigms & methodologies;
* Other relevant issues raised by participants.

USING OPEN-ENDED PROBLEMS IN CLASSROOM

Erkki Pehkonen, Dept Teacher Education, University of Helsinki (Finland)

The method of using open-ended problems in classroom for promoting mathematical discussion, the so called "open-approach" method, was developed in Japan in the 1970's (Shimada 1977, Nohda 1988). About at the same time in England, the use of investigations, a kind of open-ended problems, in mathematics teaching became popular (Mason 1991), and the idea was spread more by Cockcroft-report (1982). In the 1980's, the idea to use open problems (or open-ended problems) in classroom spread all over the world (see Pehkonen 1991), and research of the possibilities to use open problems is especially now very vivid in many countries (e.g. Pehkonen 1989, Silver & Mamona 1989, Nohda 1991, Zimmermann 1991, Clarke & Sullivan 1992).

In some countries, the idea of using open-ended problems in mathematics teaching has also been written in one form or other into the curriculum. E.g. in the new mathematics curriculum for the comprehensive school in Hamburg (Germany), about one fifth of the teaching time is left content-free, in order to encourage the use of mathematical activities (Anon. 1990). In California, they are suggesting open-ended problems to be used beside the ordinary multiple-choice tests (Anon. 1991).

In the discussion group, the following questions will be dealt with: What are "open-ended problems"? Why use open-ended problems? How to use open-ended problems? For structuring the discussion, there will be two or three short presentations (about 10–15 min) for a time slot from different parts of the world.

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For other literature notes see the list of references in the paper of Pehkonen (elsewhere in the Proceedings).

Research Reports

INTERPRETATIONS OF DOUBLE MODALITIES

Ole Björkqvist
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Abstract. *The personal conceptions of logical necessity and possibility of upper secondary school students were studied. The students were asked to interpret sentences involving two modalities by ranking given alternative formulations according to logical equivalence. The results gave evidence of influence from everyday conceptions in the form of systematic personal preferences. They also indicated dependence on key elements in the structure of the sentence, most clearly if the first element of the sentence is a positive necessity or if the sentence contains two negations. The study has a connection to the validation of formal systems of modal logic.*

Introduction.

A basic cultural tradition of mathematical logic is the dominance of two-valued logic. From the earliest school years, the mathematical thinking of children is affected by ideas which suggest that truth and falsehood are the only relevant properties of propositions in mathematics. For an understanding of the development of mathematics, in which the exclusion of a third alternative has played an important role, this educational emphasis is a natural thing. There are, however, branches of mathematical logic that have considered the consequences of more than two values for propositions. A special case is *modal logic* (Chellas, 1980), which is concerned with logical *necessity* and *possibility* in addition to truth and falsehood.

Necessity and possibility are concepts that are used frequently in everyday life and they thus carry meanings that influence the apparent validity of a specific variant of modal logic. An example is the fact that the two propositions "it is not necessary that P is true" and "it is possible that P is not true" might be considered equivalent in everyday life. A specific variant of modal logic that does not agree with this general use would then be less applicable than another.

Modal logic can be developed from a variety of axioms, and it would seem reasonable to choose the axioms in accordance with the prevalent conceptions of necessity and possibility.

Modal logic, just like ordinary two-valued logic, is concerned with sentences built up from elementary propositions. In some cases, a complicated sentence can be *reduced* to a simpler form using the axioms.

The fact that modal propositions are not very common in school mathematics while they are in regular use in everyday life make them an interesting object for the study of the relationships between concepts in and out of school. In school, the most natural time to investigate necessity and possibility is in connection with probability theory.

The research problem.

The purpose of the research reported here was to investigate the *personal conceptions of necessity and possibility* of upper secondary school students. Specifically, the interest was focussed on the personal interpretations of sentences involving two modal expressions, *double modalities.* Such a sentence has the general structure "it is (not) necessary/possible that it is (not) necessary/possible that P is true". Either modality can thus be negated. There are 16 structurally different sentences of this type. In each sentence, the first modality refers to the rest of the sentence whereas the second modality refers to P being true.

Such rather complicated sentences were chosen in order to investigate *preferences in the interpretations* that might indicate natural starting points for a system of modal logic. This would involve finding easily acceptable substitutions of expressions for other expressions, reduction of sentences, etc. It was also anticipated that the position in the sentence might make a difference for the interpretation of a modality.

It was further anticipated that different categories of interpretations exist and that these might show similarities with the normal interpretations of necessity and possibility in everyday life.

Methods.

There seems to be no previous research on the psychological aspects of double modalities. Similar research concerning other mathematical conceptions favorizes structured interviews. In this case, it was judged that the unfamiliarity and degree of abstraction of the problem would require plenty of time for the subjects to reflect on details in the modal sentences. It was therefore decided to present each doubly modal sentence in writing followed by a set of alternative sentences (not necessarily simpler in structure) that might be considered logically equivalent with the doubly modal sentence by some of the subjects, and to instruct the subjects to rank those alternatives with respect to logical equivalence. Alternatives that were not considered acceptable could be totally eliminated.

To make possible many enough alternatives (five for each doubly modal sentence) it was decided to study just those doubly modal sentences in which both necessity and possibility occurred, leaving out sentences without either (e.g., "it is necessary that it is not necessary that P is true"). Thus the study involved eight items - eight doubly modal sentences with five pseudo-logical equivalents.

The setting of each sentence was the same. It was concerned with drawing a ball from a bag of black and/or white balls. In the first item, the doubly modal sentence

"it is necessary that it is possible that the ball is white"

was followed by the five alternatives (translated into English)

(1) One ought to see to it that there are white balls in the bag.

(2) It is not possible that there are no white balls in the bag.

(3) It is a logical must that there are white balls in the bag.

(4) The contents of the bag must be such as to make it logically thinkable that the ball is white.

(5) The ball may be white.

The structure and order of the alternatives was exactly the same for the other items.

Alternative 1 expresses *finality* - in everyday life it is quite normal that something is necessary or, to a lesser extent, possible in order for something else to occur. A check with mathematicians shows that this alternative is considered rather non-mathematical, but it was included as a quite likely logical equivalent for secondary school students.

Alternative 2 expresses a necessity as a possibility and vice versa. This change is made in the first modality, employing systematically the substitutions (necessary → not possible that not) and (possible → not necessary that not) and cancelling any two consecutive negations.

Alternatives 3 and 4 attempt to examine in which position abstract logical necessity/possibility is preferred if one modality is emphasized as logical while the other modality is given a context-dependent interpretation. A systematic preference of alternative 3 to alternative 4 would indicate that abstract logic primarily enters in the first position.

Alternative 5 is an alternative that reduces the sentence to a simpler one. It has the character of deduction from the information given in the doubly modal sentence.

The investigation was directed to all the students (ages 16-18) who were

enrolled in the more extensive mathematics program in one single secondary school in Finland. The students were told to take home the papers and fill in their answers independently after due consideration of all the alternatives. This procedure seemed necessary to minimize the risk for random responses. Likewise, there was no pressure put on the students to return the papers at all. The return percentage was 40, providing 32 answer sets from students belonging primarily to the top of their respective classes.

Results.

An analysis of the results shows subject-independent as well as subject-dependent patterns of answers. Some of the findings are listed below.

The alternative ranked as number one in preference varies so that there exists no over-all preference. Neither is there any over-all avoidance of any of the alternatives. However, for six of the eight items there is a clear preference (at least one third of the number one rankings) for a specific alternative. There is thus evidence for dependence on the exact wording - on the occurrence of negations and/or the order of the necessity and the possibility involved.

Alternative 1 (which expresses finality) is the number one preference most often in the two cases where the first modality is a necessity without a negation, "it is necessary that it is (not) possible that the ball is white". Alternative 1 is the alternative that is not accepted at all most often, in four of the eight cases.

Alternative 2 is chosen as the number one preference 13 times out of 32 for the item "it is not necessary that it is possible that the ball is white". The alternative then eliminates the introductory negation, "it is possible that there are no white balls in the bag". Over-all, alternative 2 is chosen as number one or number two preference twice as often if the item starts with a negation as it is otherwise.

The comparison of the relative rankings of alternative 3 and alternative 4 shows no clear-cut tendency for either to dominate. There are two exceptions, both in

favor of alternative 3, so that alternative 3 is preferred at least twice as often as alternative 4. A first look at those two items, "it is necessary that it is not possible that the ball is white" and "it is not possible that is not necessary that the ball is white" shows no immediate similarity. Only if you replace (not possible that not) by (necessary) in alternative 4 is there an indication that the preference for the logical emphasis on the first modality may require that its meaning is that of a necessity. This interpretation is at least not contradicted by the other items.

Alternative 5 (which involves a reduction of the doubly modal sentence) is the number one preference most often in two cases. The tendency is clear - "it is not necessary that it is not possible that the ball is white" reduces to "the ball may be white", and "it is not possible that it is not necessary that the ball is white" reduces to "the ball must be white". A double negation, even if separated by a modality, seems to invite to a reduction of complexity.

Even if the number of subjects is small in this study, the answer patterns of individual students may be categorized or given individual interpretations. Students who favor alternative 1 tend to do so for several of the items - the final interpretation of the modal constructions seems to be connected with the individual rather than the specific form of the sentence. A particularly clear case is a student who favors alternative 1 for six items. The other two items result in a preference for alternative 5, and those are exactly the items that contain double negations, as described above.

There are also cases with preference for other alternatives, e.g., to reduce the sentences as much as possible in accordance with alternative 5. One subject favors alternative 4 for all the items.

Finally, there are considerable individual differences in the number of alternatives rejected. Some of the students do not use that possibility at all, ranking all the alternatives in order. Others seem to be more critical and rank just one or two alternatives for most of the items.

Conclusions.

The interpretation of doubly modal sentences is a real challenge for secondary school students. The task was experienced as difficult, but interesting, by those who completed it.

The results give indications of considerable individual differences in the personal conceptions of necessity and possibility. These differences appear to reflect the everyday use of the corresponding words.

The particular structure of the doubly modal sentence influences the interpretation. If the sentence starts with a positive necessity it is likely to be interpreted as having a final meaning. If it starts with a negation, there is a tendency to prefer a reformulation without the negation. If there are two negations, there is a strong tendency to a reduction of the sentence to a sentence containing just one modality.

The two cases with clear preferences for reduction give some support for the validity of those systems of modal logic that employ the principle "if it necessary that a proposition is true, then it is true" as an axiom (Chellas, 1980). The items can be reformulated as "it is possible that it is possible that the ball is white" and "it is necessary that it is necessary that the ball is white", respectively. Using the axiom mentioned, these sentences can be reduced to "it is possible that the ball is white" and "it is necessary that the ball is white", respectively. The latter sentences constitute the reduced alternatives in the present study.

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Constructing the instructional hierarchy through analysis of students' error pattern and strategies used in solving calculus problems.

Hyun Sung Shin

The purpose of study was to investigate the students' error patterns and thinking strategies in solving calculus problems and decide the instructional hierarchy in advanced topics(limit, derivative and integration). Datas were collected on strategies, error and performance of 120 12th graders and 30 college freshmen. 45 written problems were given to those samples and over 30 students among them were interviewed during the school year 91-92. This study showed that research method including datas in study could be rich sources to clear the students' structure of understand and to construct the instructional hierarchy on those topics.

During new math. in korea, it was hoped that a properly structured course would not only prepare students better for higher math., but would also prove more accessible. Although these are admirable and certainly to be valued, there is little empirical evidence to support them. Many teachers thought that we ignored students' learning patterns. Certainly, we should consider seriously not only the complexity of math. understanding, but the difference between teachers' understanding and that which we wish learners to gain.

In this study, one research method which we can investigate the students' cognitive structures (or, structure of understanding) is implicitly suggested and students' error patterns including strategies in advanced mathematical thinking are also discussed, which are the

essential resources that construct the instructional hierarchy in the advanced topic-limit, derivative, integration.

REVIEW OF THE LITERATURE

Over the past years many studies have focussed on the study of structure. It was assumed that students could grasp rather complex math. topic, provided they were presented in form appropriate to the students' level of intellectual development. About this, Resnick(1981) stated that "those arise in the teaching-learning situation and are connected to the field of students' cognitive structure". Piaget(1960) was explicitly concerned with the process and development of thinking. By providing example "sum of inner angles in triangle", he conveyed his notion of cognitive structure. His structure was something actively constructed by the human organism. The understanding that come from activity bears a direct relation to what one might call the subject-matter structure, but it's really a personal creation. From the review of study, the author could know that nobody clearly suggested the meaning of cognitive structure and relationship between subject matter structure and cognitive structure. Meanwhile, Skemp(1971) said that "To understand something means to assimilate it into schema". This explain the subjective nature of understaning. Ginsburg(1990) attempted to situate understanding and his view pictured it as sense-making procedures. Through the reviews of understand, we could find that there was no discrete points when we can say that it is present or absent Hart(1981) studied the test construction according to the level of understanding on

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elementary math. topics. she suggested teaching implication based on errors and performance. Some part of this study was extention of Hart's results, specially, constructiong the test based on the level of understanding.

RESEARCH QUESTION

This research aimed at finding answers to the following questions.

(1) How can we possibly use error patterns in the students' ideas and strategies to clear the structure of understanding?

(2) How can we construct the content structure by datas in (1)?

(3) How can we construct a better instructional hierarchy based on the content structure on limit, derivative and integration?

Content structure described here means the structure of topics which is reflected by the students' structure of understanding.

D E S I G N

Sample : During the school year 1991-1992, 120 12th graders(group A, B) from 2 classes of two high schools and 30 college freshmen(group c) participated in the study. Each group had some characteristics : Group A consisted of average 60 students from the standardized hish school 3rd grade and group B excellent 60 students from selected high school 2nd grade. Finally, group C had 30 freshmen from teachers' college who were studying mathematics.

Test construction : During 3 test period, each student was administered 45 items which consisted of the written forms and oral question forms. Some topics on limit, derivative and integration used

written form and others oral question form(total 15 items) for each interview. Those items fell into 5 level of understanding extended from Hart's scheme (table 1).

Procedure : All students were asked to perform the written items, which they required the students to explain and justify the solution strategy. The students who finished the written form continued to answer the oral guestion from, but those who did not give clear answer on the paper were asked to interview. These interviews were conducted by the well trained teachers including the author All interviews were tape-recorded and completely transcribed afterwards. The oral forms were the test items that the topics could not be easy to represent its meaning on the writhen test form and need a variety of answers from them. The typical one would be "when $\lim_{x\to 1} f(x)=3$, $f(x)=x+2$, find the sequence $\{f(x_n)\}$ corresponding to sequence { ··2.5, 2.4, 2.3, ··· 1.02, ··· 1.00001 ···} of x values."

table1 level of understanding for test construction (eg. derivative)

Level	Criteria	items
1	prerquisite. mode is informal items have numerical value and simple strategy	$\frac{(3+0.1)^2-9}{0.1}$, $y=x^2$ graph
2	prerequisite but increased complexity Partial math. symbol. partial generality	$\lim_{h\to 0}\frac{(3+h)^2-9}{h}$ h: number values on table
3	not prerequisite. mode is informal. use generalized patterns.	$\lim_{h\to 0}\frac{(3+h)^2-9}{h} = f'(3)$
4	formal (written symbol). use generalized patterns by abstract symbols.	$\lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = f'(x)$
5.	applicecution and counter example	$f(x)=\lvert x\rvert$

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RESULT

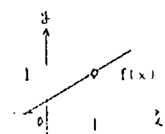
The following discussion is only focused on the critical problems which have favorable influence upon math. Teaching on error patterns, strategies, constructing the instructional hierarchy, and constructing content structure. The inconsistent error patterns were omitted.

(1) student's idea strategy and error pattern.

Limit of seguence : The major difficulty from group A, B seemed to have in understanding the limit process and numerical construction of seguence { $f(x_n)$} including $x \to 1$ in $\lim_{x \to 1} f(x) = 3$ (error 41%).

Other serious lack of understanding in group A, B was $\lim_{x \to a^+} f(x)$, $\lim_{x \to a^-} f(x)$, $\lim_{x \to a} f(x)$: f(x) discontinuity at x=a(over 46% errors).

The student did not connect understanding of limit of numerical sequence {a_n} to acquire the limit process of function. Specially, they showed limit concept by substituting x=a into f(x) and guessing the limit process on the figure. In figure, many students from group A, B answered that there was no limit because f(x) not continue at x=1 (error 46%). Probably, approximating process should be introduced much before formal representation $\lim_{x \to 1} (x^2+1) = 2$



Derivatives : There was a lack of understanding to present the rate of change including $\triangle x$, $\triangle y$ (over 30% for group A, B, 10% for group C), $\frac{f(b)-f(a)}{b-a}$ (37% B), difference $\frac{f(3+h)-f(3)}{h}$ and $\lim_{h \to 0} \frac{f(3+h)-f(3)}{h}$ (45%, A, B).

Since they showed a serious Lack of rate problem (specially, in physical situation), they could not connected f'(x) to limiting process, and answered "f'(2) f'(x)"(error 20% for group B, performance

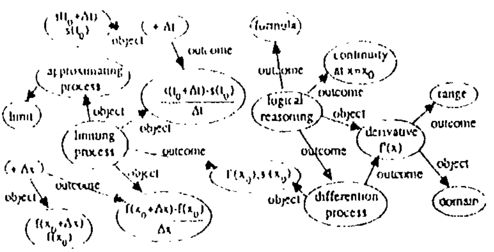
23% for group A).

Integration - The following mathematical ideas showed a serious lack of understanding Partitioning on [a, b] (performance 42% for group B), approximating area enclosed by $f(x)$, [a, b], and x a, x b (performance 20% for group A, 35% for group B), connecting approx. area ($\sum_{i=1}^{n} f(x_i^*) \triangle x$) to $\lim_{\triangle x \to 0} \sum_{i=1}^{n} f(x_i^*) \triangle x$. The smooth connection from approximating area $\sum_{i=1}^{n} f(x_i^*) \triangle x$ to $\lim_{\triangle x \to 0} \sum_{i=1}^{n} f(x_i^*) \triangle x$ was a big problem in designing the instructional hierarchy. Many students from group A, B (performance 20%) could not clearly state the meaning of $f(x^*_i) \triangle x$ independent choice $x^*_i \in [x_{i-1}, x_i]$ and $\lim \sum$.

(2) content structure

A next progress of the study was to construct the content structure based on analysis of test results above.

The following figure is representation of content structure about the concept of derivative. A format of representation came from Greeno's(1978) knowledge structure Model



The content structure about integration would be omitted.

(3) constucting the instructional hierarchy

All informations described above give wealty ideas to design the instructional hierarchy. The following is one example of the instructional hierarchy about the concept of derivative.

concept of increase (decrease) → average rate of change $\frac{f(x_0+\Delta x)-f(x)}{\Delta x}$ → approx. $\frac{f(x_0+\Delta x)-f(x_0)}{\Delta x}$ → Derivative at a point (numerical limit)

→ correspondence $x \to f'(x)$ → diff. process $\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$ → derivation open interval → derivative on physical sit

CONCLUSION AND DISCUSSION

The research reported here suggests that content structure given by textbook writter (or math. teacher) should be modified through a structure of understanding(or learner's cognitive structure) in students' mind. Since it was not easy to find a appropriate method to clear the students' cognitive structure, this study used the analysis method of students' error patterns, strategy used, and performance of The level of understanding in math. concepts (limit, derivative, intgration). Perhaps the point of greatest interest arising from the present research is indication that many students in Korean high school do not seem to copy a formal representation of the method that textbook suggests. Indeed, they may use more informal procedures of their own. These merits further investigation to clear the mathematical activity in mathematics teaching

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ANALOGIES IN SCIENCE AND MATHEMATICS: CONTINUITY VERSUS DISCRETENESS

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Abstract

Analogies play a significant role in the development and acquisition of scientific concepts. A critical question is: What causes people to perceive situations as analogical? In an attempt to refer to this central issue, we chose to examine situations which are not considered analogical from the scientific point of view, but have similar perceptual properties and might thus be perceived as analogical by students. Students in 7th to 12th grades were presented with problems related to: (1) successive division of physical and geometrical objects, and (2) comparison problems related to physical and geometrical objects. Our data suggest that external features of the problems (e.g., similar structure and figural similarity), as well as factors related to the solver (e.g., age and instruction) largely influence students' responses to the problems. The theoretical framework, however, is not the most influential factor in determining students' responses.

A critical issue in instruction is what analogies and examples could be used to promote problem solving abilities. An attempt to answer this question should include an examination of factors which makes students perceive a given, unfamiliar problem as analogical to a familiar one. Generally two types of factors could be considered: Factors related to the problem itself (e.g., its structure, its visual aspects), and factors related to the solver (e.g., age, instruction).

As a way of exploring this issue, we have chosen to concentrate on examining students' responses to problems that are not considered as analogical from the scientific point of view but which due to similarity in external features students tend to perceive as analogical. In our previous studies (Stavy & Tirosh, 1991; Tirosh & Stavy, 1992) we concentrated on successive division problems. These problems shared the same structure and process, namely:

Consider an object. Divide it into two equal parts. Divide one half into two equal parts.
Continue dividing in the same way. Will this process come to an end?

Clearly, the answer to this problem depends on the nature of the object. If it is a geometrical object (for example, a line segment, a square or a cube) the answer is that this process is endless. Geometrical objects are ideal objects and such objects can be infinity divided. If the object is a material one (for example, water, copper wire) the situation is different: The process of successive division comes to a halt when reaching the molecular or the atomic level as beyond it the material ceases to exist as such.

Two hundred upper middle class students from grade levels seven, eight, ten and twelve (50 from each grade level) participated in our previous studies. They were presented with three

problems which describe successive division processes. One problem related to successive division of a line segment whereas the other two problems related to successive division of water and a copper wire. It was found that a substantial number of students (about 52%) provided the same answer and the same explanation to these problems. While the majority (51%) of the seventh and eighth grade level students stated that these processes are finite and thus showed a concordant, finite pattern of response, the frequent concordant pattern of response among the tenth and twelfth grade students (who received extensive instruction in mathematics and had studied the particulate theory of matter) was an infinite one. Only 22% of the students provided correct, discordant responses to these problems.

Two questions that naturally arise from observing these results are: What causes students to react in the same way to these two essentially different problems? Why is there a substantial change in the nature of responses from concordant, finite ones at the lower grades, to concordant, infinite patterns at the higher grades?

The tendency to provide the same answers to these three problems suggests that the identity of the process is an important factor in students' tendency to perceive successive division problems as analogical regardless of the theoretical domains in which they are embedded. The problem related to successive division of water enables us to also study the contribution of visual similarity to students' tendency to perceive successive division problems as analogical, as this problem is clearly visually different from the objects in the other two problems. Our data showed that visual similarity affected students' responses, yet its effect was less dominant than that of the identity of the process (Tirosh & Stavy, 1992).

These two factors (the identity of the process and the visual aspects of the objects in the problems) could not, however, explain the observed change in the nature of the students' responses from finite responses to infinite ones. This change might be attributed to factors related to the solver such as age and instruction. The first part of this paper refers to the effects of solver-related factors on students' responses to successive division problems.

Effects of factors related to the solver (age and instruction) on students' perceptions of successive division problems as analogical

As stated above, an examination of the nature of students' concordant response patterns reveals that while many of the younger students (seventh and eighth grades) gave finite responses to all these problems, the older students (grades 10 and 12) came up with infinite responses. Thus, there is a shift, with grade level, in students' concordant responses from finite to infinite. This shift could result either from a developmental trend (age), or from instruction or from both. Though it is impossible to determine the differential effects of age and instruction, it is possible that instruction plays a role in this shift since the upper grade students in our studies received rather extensive instruction in mathematics (Euclidean geometry, infinite series, limits and integrals).

This instruction, which emphasizes infinite processes, could contribute to the infinite response patterns to all sub-division problems.

The assumption that instruction plays a role in determining the nature of students' responses is supported by findings from our second study. Forty-six preservice science and mathematics secondary teachers in their third year of studies in a teachers' college responded to two of the successive division problems (line segment and copper wire). Each problem was given on a different sheet of paper along with other, irrelevant, questions related to the same theoretical framework. To counter-balance the effect of the order of presentation of the problems, half of the preservice teachers first received the mathematics problems, while the other half received the science ones. Each sheet was collected after the preservice teacher had responded.

As can be seen from Table 1, preservice teachers majoring in mathematics tend to give infinite responses to successive division problems to a much larger extent than do preservice teachers majoring in science. It is, however, notable that infinite patterns of response were frequent even among the science preservice teachers.

Table 1: Preservice Teachers' Response Patterns to the Successive Division Problems (in %)

Segment	Wire		Science N=27	Math N=19	Total N=46
I Concordant patterns		- Total	51	84	64
Finite	Finite		15	10	12
Infinite	Infinite		36	74	52
II Discordant patterns		- Total			
*Infinite	Finite		49	16	36
Finite	Infinite		0	0	0

* Correct response

Our discussion thus far has been limited to problems dealing with successive division of mathematical and material objects. It was found that in this case, the identity of the process, the visual aspects, age of the students and the extensive instruction in mathematics affected students' perceptions of successive division problems as analogical. Since so far we examined only problems that dealt with successive division, it seems important to examine the effects of these factors on a different set of problems. Thus, the second part of this paper deals with comparison problems.

Comparison problems related to geometrical objects and to physical ones

Let us refer to the following problems:

- Consider two line segments, the first of which is longer than the second one. Is the number of points in the first line segment equal to the number of points in the second? Explain your answer.

- Consider two pieces of copper wire, the first of which is longer than the second one. Is the number of atoms in the first copper wire equal to the number of atoms in the second? Explain your answer.

These two problems share the same structure. The answer to each problem is determined by the theoretical framework in which it is embedded. According to Cantorian set theory, the number of points in the longer segment is equal to the number of points in the shorter one (this will be regarded as infinite response). However, in the case of the two copper wires, the longer wire contains more atoms than the smaller one (this answer will be regarded as finite response).

The 200 upper middle class students (50 from each of the 7th, 8th, 10th and 12th grade levels) who participated in the first study, were presented with the two problems related to comparison of objects. The procedure of administering these problems was identical to the one described on the previous page.

It turns out that most of the students who participated in our study (about 64%) provided the same answer, either finite or infinite, to both problems. A substantial number of students at all grade levels (about 60%) stated that the number of "basic units" (points/atoms) is larger in the longer object, only about 4% of the students stated that the number of basic units in the two objects are equal (see Table 2).

Table 2: Response Patterns to the Comparison Problems (in %)

Segment	Wire						Total
I Concordant patterns		- Total	70	67	64	57	64
Finite	Finite		68	65	54	53	60
Infinite	Infinite		2	2	10	4	4
II Discordant patterns		- Total	30	33	36	43	36
*Infinite	Finite		15	22	20	36	24
Finite	Infinite		15	11	16	7	12

* Correct response

Almost all students provided the same explanations to their answers. The frequent explanation to the finite answers was that "The longer line/wire contains more points/atoms". Some students refer to the similarity between the problems. For instance, David (grade 8) explained that "Atoms are like points. The longer segment has more points".

These findings indicate that students' tendency to respond in the same way to structurally similar problems is found not only with regard to successive division problems but also with other, structurally similar problems that are related to infinity and to the particulate theory of matter.

These two problems exposed students to visual information related to one quantity of the objects (length) and they were asked to judge the relationship in respect to another quantity (number of atoms/points). In one problem (copper wires) there is a direct relationship between the two quantities (the longer - the more). In the other problem (line segments) according to the Cantorian set theory, there is no connection between the two quantities as the number of points in any line segment is ALEF (ℵ). In the case of comparison problems, the visual dissimilarity in one quantity (length) probably imposed itself on students' judgment. It seems that the requirement to compare the number of atoms/points is affected by the visual dissimilarity of the other quantity (length) and thus directed the students towards giving a judgment that is in line with the visual information. As a result, the judgment given to both problems by most students, including those in the upper grades, is in accordance with the particulate theory of matter.

It is notable that in contrast with the previous case, there is no significant change in students' responses with age. The older students had studied in geometry that geometrical figures contain infinite number of points (and even used this knowledge when responding to the successive division problems). However, when responding to the comparison problems, they did not use this knowledge and argued that "the longer - the more". In this case, the extensive instruction in high school mathematics neither served to help the older students perceive each problem according to its theoretical framework, nor did it lead to concordant infinite responses.

In order to further examine the effect of higher level mathematics instruction, which directly refers to comparing infinite sets, 58 preservice secondary mathematics teachers who studied Cantorian set theory were asked to respond to our questionnaire. It was found that many of them (76%) came up with correct responses to the comparison problems. It is notable that, unlike the case of successive division, in the case of comparison, the introduction of the idea of actual infinity did not cause students to respond to the science problem in the same way. Thus, in this case, the instruction in mathematics did not lead the students to view all comparison problems as analogical. This can be attributed to the counter-intuitive character of actual infinity (Russell, 1956; Hahn, 1956; Tall, 1981).

Discussion

In this paper, we attempted to examine factors that make students view a problem as analogical to another one. The situation chosen for this attempt consisted of presenting students with problems which are considered non-analogical from the scientific point of view, but due to similarity in external features, students tend to conceive them as analogical. Two cases were examined: Successive division and comparison. Our data suggest that similarity in the structure of problems is a major factor which determines whether students view a given problem as analogical to another one. Although in both cases, the students were presented with problems embedded in different theoretical frameworks (particulate nature of matter and mathematical infinity) which deserved different responses, many students gave the same response to the problems. It seems that salient, external features of the problem and not the theoretical framework, largely influence students' responses to the problems.

Our results are in accordance with the findings of other studies (Carey, 1985; Chi, Fletvich and Glaser, 1981; Larkin and Rainhard, 1984) who examined differences in scientific problem solving (in the same theoretical framework) between novice and expert solvers. These studies reveal that inexperienced solvers tend to mentally represent a given problem according to surface features, whereas experienced solvers refer to scientific concepts and principles.

Although responses were identical on the individual level, many of the younger students provided finite responses to all problems, while the older students provided different responses in each of the cases: Infinite responses to all successive division problems and finite responses to all comparison problems. In the higher grades, successive division problems led to infinite responses while comparison problems led to finite responses and thus the nature of the response is determined by the structure of the problem. It seems that the identity in the process has a coercive effect on students' responses and encourages them to view problems that involve the same process as identical. This interpretation is supported by findings from other studies, all of which indicate that identity in the process described in a set of problems leads students to give the same responses (Stavy, 1981; Stavy, 1991; Tirosh, 1991).

Another factor that affected students' perceptions of problems as analogical was visual aspects of the problems. When similarity in process was accompanied by visual similarity of objects described in the problems (i.e., line segment and copper wire), the tendency to give the same response to the given problems slightly increased (in the case of successive division). Other studies (e.g. Stavy, 1991) also show that when similarity in the process is accompanied by the visual similarity of the objects, students' tendency to give the same response to problems increases: Students who studied qualitative inverse relation in the context of dissolving salt in water responded correctly to directly analogical, unlearned situations involving inverse relation which were visually similar. Many were able to correctly answer the problem that dealt with dissolving sugar in water, a substantial number of students correctly responded to the visually similar problem

of cooling water by ice. However, the problem of spreading chocolate cream on bread, which involves the same process but is visually different, was regarded as analogical to a lesser extent.

The second case, which dealt with comparing the number of points/atoms when two objects differing in length were visually presented, also indicated the impact of visual components on students' responses. In this case, students were asked to compare one quantity when visual information about another quantity was given. It seems that students' responses in this case were largely affected by the intuitive rule according to which "the longer - the more" or "the whole is greater than its parts". These rules are adequate in many cases (for instance, there are more atoms in the longer copper wire), yet not in other cases (e.g. the number of points in two differently-sized line segments). The results of this study, as well as those of others, suggest that students tend to apply these intuitive rules when solving structurally similar problems which relate to visually different quantities even in situations where, due to the nature of the quantity, these rules are not applicable (Density - Megged, 1978; Concentration - Stavy, Strauss, Orpas, and Carmi, 1982; Temperature - Strauss and Stavy, 1983; Infinity - Tirosh, 1985; Area and parameter - Hart, 1982).

To conclude, it seems that students' perception of problems as analogical is largely determined by the interaction of different external factors related to the problem, such as its structure, the process involved, visual and numerical characteristics of the objects to which the problem relates, as well as factors related to the solver, such as age and instruction. As stated above, analogy serves as an important didactic tool in both science and mathematics education. Though it is too early at this stage, for specific educational implications related to using analogy in mathematics and science education to be drawn from these findings, we feel confident enough to make several general suggestions. In many cases, instruction of problem solving in mathematics and in science is done by presenting the students first with some examples and then with problems which are scientifically similar. This type of instruction is guided by the implicit assumption that due to the similarity in the scientific nature of the problems, students will perceive them as analogical. Our findings question this assumption and suggest that attention should also be paid to factors which affect students' perceptions of problems as analogical. In addition, the choice of anchoring and bridging examples (Clement, 1987) should also take these factors into consideration.

Our findings show that students form inappropriate analogies between different theoretical frameworks. It is important that teachers of both science and mathematics will be aware of this phenomenon and of the factors which cause students to view problems embedded in different theoretical frameworks as analogical. It is, thus, important that teachers in both these domains will present students with structurally similar problems and discuss the validity of the analogies they make in light of the theoretical frameworks in which they are embedded.

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LEARNING THE LANGUAGE OF ALGEBRA

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ABSTRACT

Distinguishing between algebraic activities and algebraic language, the study focuses on reading and writing algebra and on recognising valid transformations. Tests and interviews with students from five classes, aged 13-15 years, and with adult experts show the total rejection of algebraic representation of a situation to diminish from 20% of students at 13 to zero at 15, the persistence of reversals and other notational errors in up to 10% of students at all these ages, very low levels of success in making correct transformations of formulae and in reading formulae for functional relationships, and a striking change in the reported mode of thinking used to recognise valid transformations, from checking with numbers at ages 13-15 to visualisation of symbol movements among the experts.

Introduction

> "... substantial evidence exists to indicate that the learning of algebra is addressed by many students as a problem of learning to manipulate symbols in accordance with certain transformation rules (i.e syntactically) without reference to the meaning of the expressions of transformations (i.e the semantics). This is of course not surprising, since most algebra syllabuses in the past have paid considerable attention to the syntactic aspects of algebra, precisely because of the central role that symbolic representation plays in algebra work, because power over such representation is crucial to successful performance in algebra, and because the symbolism is both new to the students and an obvious feature of this area of study". (Booth, 1989, p.58)

The situation is, in fact, more complex than this quotation suggests. Experts do manipulate algebraic expressions syntactically and we do not know what awareness of meaning they retain, or can recover, during this process. In an extensive study of the solving of numerical and literal equations by college students (Carry, Lewis and Bernard, 1980), the authors state, with regard to monitoring and checking work,

> "There seem to be two ways students might test the validity of their ideas, *use of principles* and *checking*. *Principles* refers to the basic mathematical properties of the arithmetic operations that underlie the manipulations of algebra: that multiplication distributes over addition, that multiplication and division are inverses, and so on. A student who knew these principles well could reject operations that could not be derived from them. Unfortunately, the basic principles are almost as numerous and complex as the algebraic operators based on them, so students will be doubtful about them. Further, the testing of an operation by use of the principles is an exercise in proof, and may require some creativity. It is not transparent how cross multiplication is related to the basic principles, for example, because the operation as it is performed suppresses the underlying multiplications." (p. 126).

Moreover, the manipulative aspect of algebra bears a similar relation to the whole range of algebraic activity as does formal calculation to the whole field of uses of number. In both fields, while some skill aspects may be performed mechanically, the flexible and

appropriate use of them demands knowledge of concepts and of general strategies. Typically a full episode of algebraic activity will involve the expression of a situation in algebraic language, the transformation of the algebra in some purposeful direction, and the re-interpretation of the resulting expression in the original situation. Among the variety of mathematical purposes which might govern such an activity, three are particularly characteristic - the solving of complex problems by forming and solving equations, the expression and proof of generalisations, and work with functional and other relationships expressed by formulae. Suitable classroom examples of these characteristically algebraic activities, with discussion of how the use and manipulation of the algebraic language can be developed within them, are given in *Purpose in School Algebra* (Bell, 1992). The work presented in this paper should be seen within this context.

This study focuses on *reading* and *writing* algebra, and on *recognising valid transformations* of simple formulae. Some questions also look for the ability to *make* transformations, mainly in simple cases. We are asking what are the steps by which pupils may come to use the algebraic language with some degree of mastery; and, in particular, what monitoring processes are used by students and by experts to guide correct manipulation and to recognise and avoid or correct possible error.

137 pupils of five classes drawn from years 8(1 class), 9(2) and 10(2) in an Australian senior high school took a one-hour written test (in two parts, A and B). Interviews were conducted with two samples of 10 pupils, one beforehand to guide the test design, and one afterwards, to clarify the methods being used.

Some of the tasks may be illustrated by the following:

> Which of the following formulae do you think is most likely to be correct? Underline your answer. Give reasons:
>
> Pull of the earth on a satellite at height h.
>
> $P = kh$ $\qquad$ $P = kh^2$ $\qquad$ $P = k/h$ $\qquad$ $P = k/h^2$

This involves *reading* the formulae and *recognising* the functional relationships expressed, (a mathematical task), so that it is then possible to consider which of the relations is most likely to fit the physical quantities (this involves some awareness of physical principles).

A different type of task was to read the formula $V = \pi r^2 h$ and to extract from it the various functional relationships - such as the relation between V and r, if h is fixed.

The target *transformation* task was to transform a formula such as $V = \pi r^2 h$ or $S = 2\pi r(r+h)$ to give h. Such manipulations are typically performed by the (implicit) application of the transformations of $A + B = C$ (to $A = C - B$ or $B = C - A$), or of $PQ = R$ (to $P = R/Q$ or $Q = R/P$); where the components may be numbers, single letters, or 'chunks' such as πr^2 or $(r + h)$. Other questions are focused on the basic multiplicative transformations of $PQ = R$, looking both at the numerical case and at formulae in specific contexts (speed, distance and time and current voltage and resistance). These questions asked for *recognition* of valid and invalid transformations.

In translating between verbal and algebraic representations, particular known hazards relate to false commuting or reversing of subtraction or division, and the reversal of order exposed by the students and professors problem, in which six students to every professor is symbolised as $6S = P$. This last has been shown to arise in some cases

from over-literal translation of the verbal statement, and in others from a perception of S being the big number, by association with the 6, with a degenerate reading of the = sign as indicating a correspondence rather than an equality of numbers (Clement, 1982, MacGregor, 1991).

The test questions include some designed to expose these known errors. We shall discuss first the questions demanding reading and writing of algebra, then those concerning transformations.

Reading and writing algebra

Questions covered (1) the translation of a simple verbally described situation involving adding and subtracting, into an algebraic formula, (2) testing for possible false commuting of subtraction, for the 'equals is makes' error and for the 'students and professors' reversal in additive cases; (3) forming an expression for perimeter containing both letters and numbers from an annotated diagram; (4) reading functional relationships from $V = \pi r^2 h$, and (5) choosing plausible formulae for physical situations, as illustrated above.

The first formula construction question was:

1a Clare has earned D dollars. She spent C dollars for clothes and F dollars for food, and has L dollars left. Complete this expression for L, using C, D and F.

L = ____________

(Tick one box)

Are you sure? | | | |

No Fairly Yes

1b Also write a formula to give F in terms of C, D and L.

F = ____________

(Tick one box)

Are you sure? | | | |

No Fairly Yes

(Pupils degrees of sureness were elicited as shown; these will be analysed later.)

Success rates for the five classes (Classes 8, 9L, 9T, 10L, 10U) were 30, 57, 50, 69, 70% - showing an increase over the three years. A further 10% of responses were correct except for the omission of brackets, and a steady 5 - 8% each year reversed subtraction, for example writing L = (C + F) - D instead of L = D - (C + F). In classes 8 and 9L, 6-7% gave numerical responses, assuming some actual possible amounts of dollars. The 20% or so each year of unclassified responses included D - C,F or D - C and F, D + C,F, and there were omissions declining from about 20% in Class 8 to zero in class 10U.

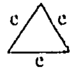
Thus we may tentatively conclude that almost 30% of students in year 8 (omissions + numerical responses) reject the possibility of representing this fairly simple and familiar money situation by using letters and signs. A further 30-40% accept the possibility of representation but make notational errors (analogous to errors of spelling or grammar in verbal language). These include reversals of the subtraction notation and using C, F or 'C and F' instead of 'C + F'. The total rejection of representation declines to zero as age increases to year 10, but the notational errors do not fall below about 30%. This

suggests a curriculum gap rather than conceptual obstacle; there is probably insufficient exposure and discussion of students' attempts at making such representations.

We note that in the Concepts in Secondary Mathematics and Science study of Number Operations (Brown, 1981), correct identification of the subtraction operation was made by 50-70% of students aged 12-15, with 15-6% giving reversals.

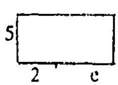
The second part of the question required a more difficult transformation of the data. In the first part, the formula could be constructed in the order of events; we start with dollars earned, subtract what was spent and finish with the amount left over. The second part requires extracting what was spent on one of the two items, and counting what was left over in the same way as the other spending item. In the year 8 class, the success rate is still 30%, but the omission rate doubles, to 40%; in the other classes, success falls sharply, to 43, 23, 42, 40% respectively.The mental reorganisation of the data involved here should not be beyond the capacity of most students of these ages. One can only conclude again that such activity coupled with the making and the discussion of its symbolic representation is a reglected area of teaching.

Question B2 asks for the perimeters of two annotated figures (a) a triangle with sides marked x cm, y cm, 8cm, and (b) a parallelogram with sides x cm, 5 cm, x cm 5 cm. The error of juxtaposition (xy8) was seen in a few cases (less than 10%). More noticeable was the large minority (20-25% in certain classes (8, 9L, 10L) who gave numerical answers, by assuming values for the lengths x and y. These pupils appeared not to accept the possibility of an expression as an answer in this case. These were somewhat larger percentages than those failing to give a formula in the previously discussed question - it appears that the mixed expression x + y + 8 is less acceptable than the wholly literal one D - C - F. One may note that x + y + 8 demands recognition of x and y as placeholders for numbers, whereas D - C - F may be regarded merely as a shorthand form of the corresponding verbal statement. They may be compared with items from Kuchemann (1981):



perimeter = ____ (94% correct at age 14)

and



area = ____ (12% correct at age 14)

Commutativity and equals-is-makes

A question asking for the value of x in 8 - 5 = x - 8 showed the false commutative response (5) in 10-15% of cases, with no decrease with increasing age. This compares with the steady 5-8% of reversals in Question 1 above. About 10% gave the 'equals is makes' response (3) which we discuss under the modelling reversals below. These misconceptions clearly do not get treated adequately in the curriculum. In a question asking for an expression for the time taken by a bus travelling 28km at 65 km per hour, the correct answer 28 ÷ 65, was given by substantially fewer pupils than gave 65 ÷ 28; in interviews, where pupils were directed to estimate the expected size of the answer, correction often took place, though a few pupils continued to assert the equivalence of these expressions. This is a case where knowledge of the sizes of the numbers involved may act as a distractor from the correct operational perception.

The students-and-professors reversal in additive problems

Two questions had the potential to expose this error. The first was:

'I have m dollars and you have k dollars. I have $6 more than you.' Which equation must be true? (Underline one of the answers below.)

$6k = m$ $\quad$ $6m = k$ $\quad$ $k + 6 = m$ $\quad$ $m + 6 = k$ $\quad$ $6 \cdot m = k$

Correct responses averaged about 50%, and the reversal error about 25%. The second was:

We are told that a and b are numbers and $a = 28 + b$; which of the following must be true? (Underline one of the answers below)

a a is larger than b $\quad$ c You can't tell which number is larger

b b is larger than a $\quad$ d a is equal to 28

Success rates here were about 60% (a is greater than b) with very few reversals, but there were some 15% each of 'a = 28' (equals is makes) and of 'you cannot tell'. (MacGregor (1991) also found this error dependent on the form of presentation).

This suggests that failures to work correctly with the = sign are broader and deeper than has previously been suggested. The 'attraction' theory proposes that students feel that the large number should be associated with the numerical multiplier or addend; so that with a large number for S, 6S = P seems correct. This appears not to operate strongly here since few reversals are shown. On the other hand, other, possibly less well defined meanings for = as 'makes' or as a general correspondence or association are supported by the fairly substantial numbers choosing a = 28 and 'you can't tell'. Dealing with these misconceptions clearly demands more explicit discussion of perceived meanings.

Reading a complex formula

Extracting the functional relations from $V = \pi r^2 h$ produced very low levels of success, except for the first question (r constant, what happens to V when h is doubled); the correct response here (doubled) was given by 50-60% - but it could be argued that this is the obvious response. There were hardly any correct responses to the remaining questions (h fixed, r doubled, V?; V fixed, r doubled, h?; transform the formula to give h.) One of the obstacles here is the kind of acceptance of lack of closure required to envisage a doubling of a general unknown value, and its effect.

Reading and judging functional relations

These were the questions asking which was the most likely correct function for each of four physical situations. These are shown.

The pull of the earth on a satellite at height h

$kh \quad kh^2 \quad \frac{k}{h} \quad \frac{k}{h^2}$

Force needed on pedals to ride bike at speed v

$kv \quad kv^2 \quad \frac{k}{v} \quad \frac{k}{v^2}$

These questions were found very hard; almost no acceptable reasons were given, and there was a majority of omissions. Ignoring the subtleties of linear or square, thus considering only the direction of the relationships, the satellite questions collected, in all, 43 (incorrect) choices of k/h or k/h^2, and 20 of k/h, k/h^2. The bike question showed 45 choices of the correct direction of relationship (kv, kv^2) and 32 incorrect ones. In the absence of intelligible reasons it is not possible to distinguish errors in the algebraic reading from incorrect understanding of the physical relationships. This is an important aspect of algebraic reading and understanding for applications and deserves fuller study.

Transformations

Attention was focused mainly on the very commonly occurring three-term multiplicative formulae. The Resistance, Current and Voltage relations, Speed, Distance and Time, and the similar relation between three numbers A, B and C were used. The last appeared twice, once using division signs, (as the contextual questions), and once using the fraction bar to denote division. One example is given.

> The current, voltage and resistance in an electric circuit are connected by the formula:
> $R = V \div C$
>
> Complete the formula for finding the voltage, given the current and the resistance.
>
> V = ______________
>
> Which of these are correct formulae? Put √ or x.
>
> a. $R = C \div V$ ____ b. $V = R \div C$ ____ c. $C = V \div R$ ____

The age and class trend in the results was of a modest steady increase through classes 8 9L 9T 10 L, and a drop in class 10U to a level close to that of class 8. Combining the four question items and the five classes gives overall percentage success rates as follows: for D S T 51%; C V R 51%; A B C with ÷ 49%; A B C with fractional bar, 45%. (The drop in the last figure comes mainly from classes 9T, 10L). It was hypothesized that the DST context would be more familiar, and hence more supportive, than CVR; indeed, it was expected that the latter result would be depressed compared with the numerical case, ABC. However, the results refute these hypotheses, suggesting that the fraction bar notation may be *less* easy to handle than the division sign, and both contexts equally supportive.

Mode of thinking

The interviews suggested that four modes of thinking were used to effect these transformations; and in the written test pupils were asked about these.

Look back at the question on C, V, R (no. 2). Tick the one or two of the following statements which is closest to the way you were thinking.

a	I thought which would be the biggest number, so the others would be divided into it	
b	I tried some actual numbers in my head	
c	I just remembered the formulae	
d	I thought the one on top on the right hand side would go underneath on the other side	

The last mode was shown in interviews only in the use of cross multiplication (with insertion of 1 where necessary, e.g., to make $A = \frac{B}{C}$ into $\frac{A}{1} = \frac{B}{C}$. However, it was conjectured that this visualisation of physical movement of the symbols was the dominant mode for experts, and we were particularly interested in its appearance in our sample.

The relative frequency of responses of four modes (with no great differences between the classes) was 27%, 41%, 21%, 11%. Interviews also confirmed the tendency to consider sizes and actual numbers; and this result relates to the relatively less successful transformation of the fraction-bar version of the ABC question discussed above.

It is also important to consider whether there is any relation between the mode of thinking and success rates. Tentative partial figures, (subject to further analysis) are, for modes *a,b,c,d* in the CVR question, 2.2, 2.0, 1.8, 1.6 (mean items correct out of 3); and for the ABC fraction bar question 0.9, 1.3, 1.6, 1.7 (these are pupils who said they used this mode for the CVR question; they were not asked about the ABC questions). These differences are not significant, but the matter is worthy of fuller investigation.

We may compare these responses with those of a sample of ten 'experts' - teachers of science and mathematical enrolled on an in-service Masters' degree course. These responded to the ABC fraction bar question, (all correctly, except for one error by one person), and to the 'mode of thinking' question. Of the ten, seven reported using mode *d*, two mode *c* and one 'none of these'. To check how far this difference in response was likely to be due to the difference in presentation of the two questions, a further small sample of nine novice mathematics teachers was given both the CVR and ABC fraction bar questions, and after doing each, was asked the 'mode of thinking' question. In all cases, they reported the same mode of thinking for the two questions; four mode *d*, four *c* and one both *c* and *d*. Some of these subjects also reported imagining the CVR relation written with a fraction bar (so that they could use mode *d*).

Questions which arise from this study are the following. How far is the eventual adoption of visual symbol-moving methods associated with a loss of the ability to recover the awareness of the underlying mathematical operations? Conversely, should the move towards visual methods be encouraged in school, or guided in such a way as to preserve the links?

We hope to explore these in a more extensive project.

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COGNITIVE PROCESSES IN ALGEBRAIC THINKING: TOWARDS A THEORETICAL FRAMEWORK.

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Abstract
This report deals with some difficulties students usually encounter in learning Algebra. In view of understanding such difficulties, the authors try here to contruct a theoretical frame (algebraic thinking as a game of interpretation), by using key-concepts taken from linguistics and psychology.
Such theoretical frame is shown in action by the analysis of the solution of an algebraic problem given by a university student.

Introduction

Recent research studies on algebraic learning have given evidence of students' difficulties in giving meaning to formal expressions and constructing, interpreting and transforming formulas (Kieran,89; Linchevski and Sfard,91; Drouhard,92; Sfard,92).
To solve these difficulties, the investigation of semantics underlying algebraic language is needed, as well as the analysis of related cognitive processes. Such analysis may allow an explanation of some difficulties peculiar to Algebra learning; at the same time, implications for teaching may be derived as a consequence.
Paragraph 1 deals with some difficulties in Algebra, as pointed out by existing literature, and related classification .
In paragraph 2 a theoretical model aiming at understanding and monitoring some aspects of algebraic thinking is outlined.
Finally, in paragraph 3 such model is applied to a case study.
We note that implications for teaching are part of the authors' ongoing research, as well as the study of how the model can be applied to the socalled "pre-algebraic thinking". However, such aspects are beyond the purposes of this report. We will make some allusion to educational implications only at the end of paragraph 2.

1.Posing the research problem: some difficulties in Algebra learning.

The symbolic language of Algebra is a powerful means of approaching and solving problems. It plays a crucial role in increasing the learner's potential for thinking, reasononing and

communicating. However there is strong evidence of students' difficulties in relating symbolic expressions to their meaning. In many cases, as research studies point out (Kieran, 89; Drouhard,92; Linchevski and Sfard,91; Sfard,92), when students are requested to interpret formulas, they cut off semantic aspects and emphasize only syntactical ones. For example, the so called pseudostructural students (Linchevski and Sfard, 91), being unable to imagine the intangible entities (functions, truth sets) to be manipulated, use pictures and symbols as a substitute: thus, each sign turns into a thing in itself, not standing for anything else. The semantic polarity of algebraic thinking is completely neglected.
In other cases, students who are not very familiar with algebraic language apply some surrogate in a more or less efficient way. The so called syncopated students (Harper, 87) do not master well the relational aspects of algebraic expressions.Thus, they use and produce symbolic expressions which do not constitute effective thinking tools. Formulas are static storage of superficial knowledge, lacking any semantic control.

In our view, there are three main obstacles to using symbolic expressions as thinking tools.

a) The first reason is of a cognitive nature: the algebraic way of thinking is often less natural than intutive models of oral arithmetic, where the stream of reasoning can be sustained by natural language and the meaning of things is easily monitored. So, novices when facing simple word problems, firstly solve them by means of oral arithmetic and successively try to translate their strategy into algebraic equations (Harper,87; Kieran,89). Many students reveal a gap between the procedural knowledge typical of arithmetic and the more relational knowledge required at the algebraic level (Chiappini and Lemut,91); so, while they can manage easily the former aspect, the only way for overcoming the latter one seems to be by means of purely mechanical rules. For those students, managing a problem in algebraic language is a difficult task as far as they suffer for a sort of "einstellung effect" (Luchins,42) in their strategy for solving problem: namely they use either arithmetic strategy or algebraic symbolism, the latter as a pure syntax without any meaning.

b) The second reason is mainly of a didactical nature. As Chevallard (89) has pointed out, the traditional way of teaching Algebra in schools "narcotizes" peculiarly novel features with respect to arithmetic: namely, the use of letters both as variables and parameters. For example, problems which can be expressed by means of equations like $ax+b=0$ (where a and b are fixed numbers) do not stimulate algebraic thinking insofar as they allow a faithful translation of mental arithmetic computation (it is not the same for equations like $ax+b=cx+d$, as shown by Filloy and Rojano,89). School algebra often does not improve a genuine use of algebraic symbolism: the basic component of algebraic thinking which is related to the capacity of anticipation can be inhibited and the "einstellung effect" reinforced.

c) The third reason regards the so-called process of putting things into formulas. It is a complex process, which involves a wide set of activities, ranging from the choice of variables to the writing of formulas. Here the gap between novice and expert is quite evident: the expert is able to choose the letters suitably for the problem from the very beginning, showing a preliminary understanding of the main relations involved in the problem itself. This process of "naming" typically involves anticipatory thinking and is crucial in the whole process of algebraic

solution. Unlike the expert, the novice chooses the variable randomly and goes into the process of naming in a weaker and more superficial way, often influenced by rigid stereotypes.

Cases a, b and c show that basic questions in algebraic learning are strongly connected with problems of interpretation.
This is evident in case a: algebra is seen as a foreign language in which the student has to translate the problem and the solution, after they have been expressed in a more familiar language. In case b we remark that doing algebra means dealing with $\forall\exists$ formulas, that is formulas like $\forall a, \forall b, \exists x \exists y / ax+by=0$ and not only formulas like $\exists x/ 3x+5=0$ or $\forall a, 2a=a+a$. Of course, formulas including only one type of quantifiers are easier to deal with, while the interpretation of $\forall\exists$ formulas is more difficult.
Finally, as far as point c is concerned, the process of naming consists of an interpretation of the text, which is aimed at identifying the variables and thus doing a mental elaboration of the text itself.
On the ground of such considerations, a more detailed analysis of the meaning of symbolic expressions in algebra is needed, together with an investigation of related cognitive processes.

2. Algebraic thinking as a game of interpretation: a theoretical frame.

In mathematics there are expressions with the same "denotation" but a different "sense", if we mean that "denotation" is concerned with the "extensional" aspects of the expression and "sense" with the "intensional" ones. These words are here used according to Frege and their meaning will be clarified by means of examples later on. For instance, the expressions 4x+2 and 2(2x+1) express two different procedures but denote the same function.
Similarly, the following equations $(x+5)^2=x$ and $x^2+x+1=0$ when to be solved in R, denote the same object (the empty set), but have different sense.
Natural language and mathematical language are both rich in expressions with different sense and the same denotation.
Algebraic transformations produce different expressions which can have different sense and the same denotation; on the other hand, given two different expressions with the same denotation, it is not always possible to transform one in the other by means of algebraic transformations (see, for example, the two equations mentioned above). Algebraic transformations do not change denotation, related to extension, but they can change sense, related to intension.
This fact is often neglected by students (Drouhard, 92; Kieran, 89) who recognize a rigid one-to-one correspondence between sense and denotation: sense and denotation are identified. For those students, a symbolic expression denotes only itself: algebra becomes a pure syntax.

The capacity of mastering sense and denotation is fundamental in algebraic thinking. However, the interplay of these two components give us only flashes and not the entire movie of the cognitive process in action. To complete our analysis, we need to introduce the notion of frame, as it is used in Artificial Intelligence (Minsky,75). A frame is a structure of data that is able to represent generic concepts or stereotyped situations and is stored in memory. Such structure

of knowledge can represent very different concepts (i.e. objects, events, social situations, etc.). In Semiotics the notion of frame is widely applied to the process of text's comprehension: understanding a given text means activating a frame, making it actual in accordance with the text's values and infering information accordingly. In our context, frames that are activated as virtual texts while interpreting a text (for example the text of a problem), allow to grasp inner relationships according to the context and circumstances expressed by the text itself (textual cooperation).

In our opinion, to interpret algebraic processes it is necessary to take into account:
i) the activation of frames and the passage from one to another;
ii)the dynamic relationship between sense and denotation, which is activated by frames.
From this point of view, *doing algebra is a process of giving sense to denotation and denotation to sense and of changing senses and denotations.*
During such processes, the student interprets texts and his/her textual cooperation is fundamental in activating one or more frames. A chain of interpreters is generated: each text interprets the preceding text and, at the same time, it is interpreted by the text which follows in the chain.

In each step a change of frames can occur, i.e. a change in interpreting a given expression. Thus doing algebra appears to be a game of interpretation (Arzarello, Bazzini and Chiappini, 92).
One interprets a text, expressed in a semiotic system (for example a problem expressed in natural language) in a text framed in another system (for example an equation); in other cases, one interprets (or transforms) a text in a system (for example an expression) in another text in the same system (for example another expression).
When a student approaches a problem, that is starts an interpretative action, he/she activates one or more frames, depending on context, circumstances and the text's peculiarities.
Once a frame is activated, the student produces a text as a result of his/her interpretation: in this domain the process of problem solving consists of successive transformations of the text, which does not exclude the production of new texts, in accordance with the frames in action. Two major types of transformations can be identified: one connected with semantic creativity (mastering of symbolic function of algebraic language) and one connected with algorithmic efficiency (mastering the algorithmic function of algebraic language). The whole process is ruled by the student's aims.

From an educational point of view, we observe that the chain of interpreters which can be built as the result of the game of interpretation is very fruitful for the study and improvement of learning Algebra. Mental representations of learners are activated by interpreters. For example, intermediate interpreters like numerical tables, spreadsheets, programming language, etc. can be useful, since they may facilitate making connections between arithmetic procedure and algebraic expressions. Similarly, verbal interpreters of algebraic expressions can stimulate reflection, social interaction in the classroom, etc.; all these activities may help students to get metacognition of the process by which they have produced certain algebraic interpreters. Last but not least, the habit of constructing different interpreters may develop forms of flexibility in semantic control and lead to the production of an upper-level knowledge (process of condensation according to Arzarello, 91).

In such a way, formulas are not used acritically as passive knowledge containers, but, since constructed in the mental space of ideal experiments, they become real thinking tools, do not close but keep open the problematic situation and do not provoke loss of semantic control.

3. Algebraic thinking as a game of interpretation: a case study.

We now take into account the solution of an algebraic problem given by an undergraduate student of Mathematics (third year, 22 year old).

Problem

Prove that the number $(p-1)(q^2-1)/8$ is an even number, in case that p and q are odd primes.

Protocol of Anna.

Episode 1.

Anna develops the formula and write the words "even" and "odd" on the paper near the formula

$$(p-1)(q^2-1)/8 = (p-1)(q+1)(q-1)/8$$

Anna points at the components of the formula and says:
"even, even, even...hmm...the remaining number is not even..."

Episode 2.

Anna develops the formula under the words even, odd:

$$(p-1)(q^2-1)/8 = (pq^2-q^2-p+1)/8$$

she makes some oral calculations of the type "odd times odd is odd", then says: "...hmm..it does not work!"

Episode 3.

It is like the previous one, but with calculations of the type "odd times odd is odd" referred to the factors $(p-1)$, (q^2-1); then Anna says: "there must be some formula for primes to use!".

Episode 4.

Anna draws some scribbling on the formulas of preceding episodes and starts verifying the formula with some primes: data are collected into a table:

p	q
3	5
5	7
3	7

$$\frac{(3-1)(25-1)}{8} = \frac{2 \times \overset{3}{\cancel{24}}}{\cancel{8}}$$

$$\frac{(5-1)(49-1)}{8} = \frac{4 \times \overset{6}{\cancel{48}}}{\cancel{8}}$$

$$\frac{(3-1)(49-1)}{8} = \frac{2 \times \overset{6}{\cancel{48}}}{\cancel{8}}$$

Anna comments: "So, it is already q square minus one that is a multiple of eight!".

(between episode 4 and 5 there is no solution of continuity in time)

Episode 5.
Anna changes the sheet of paper and writes down as follows:

$$\frac{(p-1)\,(q^2-1)}{8}$$

multiple of 2 multiple of 8

Then Anna writes the following formulas:

$$(2h+1-1)\,[(2k+1)^2-1]/8 = 2h(4k^2+4k+1-1)/8 = 2h\;4k(k+1)/8$$

(the line of fraction is only under 4k(k+1) and 8 is under it)
and says: "...if k is even, four (times) k is a multiple of 8 (Anna points at 8 in the formula), so it remains a multiple of 2 (she points at 2 in the formula), and we are O.K. If k is odd......
(Anna reduces 8 with 4 in the usual written form, by writing 2 near 8; then she reduces 2 with 2, coefficient of h).
If k is odd, it does not work...... No! if k is odd then k plus one (she points at k+1) is even and we are over!".

Episode 6.
Anna looks again at the text of the problem and says: "But primes are got nothing to do with this! Odd numbers are enough!".

Discussion

The first phase (episodes 1,2,3) is marked by stereotyped syntactical transformations: the frame even-odd numbers guides them and semantic control is very feeble. Formulas are used as such in the frame and do not really live within it as thinking objects; they incorporate the frame only in a shallow way. Indeed, the frame even-odd numbers is used only to test the formula, which continues to be manipulated in a stereotyped manner. The idea that some magic formula must be used culminates in the end of episode 3, which marks a change of frames (prime numbers) and a deep change of approach. Here the reasoning becomes arithmetic and semantics is strong: the frame is inhabited by numerical expressions, as thinking tools, which in the long run activate hypothetical reasoning in a new precise frame (multiples), which overlaps partially the first one.
The resolutive moment consists in the sudden change of strategy from episode 4 to episode 5, which corresponds to a change of frame.
Now (episode 5) Anna can read the old formula with new eyes: it is written in such a way to incorporate the old frame (even-odd) and moreover is manipulated according to anticipative thinking related to the new frame. It is not any longer the way that the formula is written which suggests standard manipulations to see what happens, but the formula is written and manipulated in a certain fashion in order to prove a conjecture. In other words, in the new frame the relationship sense-denotation of the formula does appear as a thinking tool to test a hypothesis: intensional aspects are guided and built by extensional ones and conversely (episode 5, 1st part). The second part of episode 5 shows also some stiffness of syntactical aspects: the formula incorporates the fact that $4k(k+1)$ is a multiple of 8 in a transparent way only when k is even. Anna experiences some trouble with the case that k is odd: the formal aspects force her to simplify in a certain manner and for a while the (conjectured) denotation of the formula (namely its being even after semplifications) does not deal with the right sense ($k+1$ is even if k is odd). To grasp this, it is not required to manipulate the formula but to activate a new part of the symbolic expression according to its denotation. The shift in denotation does make it possible. The formal expression does not change but its sense changes, insofar as we have looked at its denotation in a new way, shifting from the frame "multiples" to the frame "even-odd".
In summary, one way of thinking with a formula is to transform (part of) its intension manipulating it according to its extension; the second way is to discover a new intension , without doing formal manipulations, but looking at a new extension (in a possibly new frame).

In both cases such changes are activated because the intension and the extension of the formula are embedded in one or more frames.

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Our aim is to study the cognitive behavior of pupils (7th and8th-grade) in the solving of algebraic equations starting from an analysis of errors. Such an analysis rests on the relationship established between each error and the correponding violated mathematical property We have then classified different types of errors into more general categories. the method of groupings resulted in the classification of all errors recorded over the duration of the experiment in only five categories. Grouping errors according to their conceptual origins makes it possible to identify the operational invariants in the solving of equations. The disregard of an operational invariant is responsible for an entire category of errors. The complete group of operational invariants makes it possible to construct a model of the cognitive process of the pupil: the "schème" which governs the solution of algebraic equations.

ANALYSIS OF ERRORS AND A COGNITIVE MODEL IN THE SOLVING OF EQUATIONS

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INTRODUCTION

We believe that learning algebra represents a significant epistemological jump for secondary school pupils. We mean by this that the pupil must suddenly shift from one state of mathematical knowledge to another while rapidly assimilating new notions and procedures based on previously aquired knowledge but calling for an entirely new way of reasoning. The step from elementary arithmetic to algebra, for example, requires pupils to substitute for the arithmetical treatment of equations the transformation of equations according to explicit rules (a process which gives a sequence of equivalent equations). Several authors have approached the rupture between arithmetic and algebra, among them: Filloy and Rojano(1984), Filloy E.(1990). In the international publications devoted to the mathematical education many authors are concerned with the solution of mathematical equations, Kieran(1990) reviews their studies. What becomes clear is that each author adresses a particular difficulty but that a systhesis of the results is contradictory

The analysis of errors made in solving mathematical equations was for a long time been a subject of study for cognitive psychology. In a recent article Payne and Squibb(1990) observe that the relationship between errors and the mathematical concepts involved seem much closer and complex that the relationship studied in current models Now, this last remark is exactly in accordance with the fundamental approach of this present article which is to establish a relationship between the errors and the corresponding mathematical property violated. The goal of the present article is to analyse the course followed by pupils (7th and 8th-grade) across almost all the conceptual difficulties which they meet when solving mathematical equations Our theoretical framework is the "Conceptual Field Theorie" of Gerard Vergnaud(1990)

THE CONCEPTUAL FIELD IN THE SOLVING OF EQUATIONS

The theory of conceptual fields is a cognitive theory which aims to provide a coherent framework and some basic principles for the study of the development and the acquisition of complex competences Its chief goal is to provide a framework which makes it possible to understand the filiations and ruptures of concepts The cognitive functioning of the subject is modelled in terms of schème Gerard Vergnaud defines a conceptual field as a whole set of situations which can be analised from a cognitive point of view, as a whole set of tasks of which it is essential to know the exact nature and difficulties as well as the concepts and theorems which make it possible to analyse these situations as mathematical tasks (Vergnaud 1990) In order to analyse the conceptual field on which the

solution of equations depends, it is necessary to classify the exercices and to identify the mathematical theorems and concepts used by pupils (including implicit ones).

a - First degree equations with one unknown; classification of situations: the equation $ax+b=c$ is solved, with a few exceptions by dealing first with the independent term b and then with coefficient a (dealing first with coefficient a leads to errors and this approach is soon abandoned). The first algebraic transformation on both sides of the equation is a subtraction if b is positive or an addition if b is negative: $ax+b-b=c-b$ or $ax-/b/+/b/=c+/b/$; $ax=d$. The calculation of the number d leads to the calculation of the four types of additions dealing with directed numbers (for example. 17+3, 15-5 or -12+22; 13-33 or -11+7; -13-5; where the last two situations turn out to be the most difficult) The second algebraic transformation is the division by the coefficient a of both sides of the equation: $ax=o$; $ax/a=d/a$; $x=e$ The coefficient a may be either positive or negative From the four previous situations we thus obtain eight different situations for solving $ax+b=c$. Division by a negative number give rise to problems and requires the use of 'rule of signs

By adding other terms the equation $ax+b=c$ can be made more complicated We can distinguish three situations

1 - Reduction of numerical terms for example $2x+85-3-90=102$. $2x-8=102$

2 - Factorisation: reduction of terms with the unknown

For example: $52x+5x-6.6x+68=558$. $x(52+5\ 6\ 6)+68=558$

3 - Distributivity of multiplication over addition and subtraction multiplying out factors

For example $(-4)(5x+44-2x)+866=955$

4 *The treatment of the equations ($ax+b=cx+d$) in which the unknown appears on both sides of the equation* This introduces a new algebraic transformation gathering terms with the unknown on one side of the equation

Obviously, the reason for this classification is to make pupils meet all possible situations (and obstacles) encountered in the treatment of directed numbers This classification may be extended to other criteria the magnitude of numbers, their nature (decimals fractions) of which we give a summary survey

b - The "schème" which governs the solution of equations: The concept of scheme was introduced by Piaget and, later, further elaborated by G Vergnaud in order to find a model of the learning process involved in the acquisition of complex knowledge in particular of scientific knowledge According to Vergnaud(1990): 'we call **schème** the invariant organisation of behavior patterns for a class of given situations It is in the schemes' that the knowledge in-action of the subject can be found especially the cognitive elements (operational invariants) which allow the subject's action to become operational We wil study the development of the scheme which governs the solution of algebraic equations Automaticity is of course one of the most visible manifestations of the functioning of the schème . this does not prevent the subject from remaining in control of the conditions under which a given algebraic transformation is appropriate or not The 'schème' which governs the algebraic solution of equation functions in the following way relevant algebraic transformations are selected one by one in order to arrive at a solution (written $x=$) This selection is guided by two types of considerations

- To suspend the algebraic operation which cannot be carried out on an unknown numbers to obtain a number immediately from the letter and its coefficient

- To carry out relevant algebraic transformations of the unknown and the number from among all possible transformations

The generalisation of the schème to all situations of the conceptual field meets with obstacles which are at the origin of many errors The efficiency of the schème in dealing with new situations depends on the conceptualisation of more general operational invariants the 'schème' aims at adaptation

THE ERRORS IN THE SOLVING OF EQUATIONS

The first step in the analysis of errors has consisted in relating eacn error to the corresponding violated mathematical property. This process of classification has resulted in grouping all the errors observed into only five categories This allowed us to identify the operational invarian.s of the conceptual field which are the basis for the correct functioning of the "schème" The categories of errors are:

1 - Errors concerning the concepts of the mathematical unknown and equation.
2 - Errors made in algebraic transformations which are identical on both sides of the equation: the equality is not conserved.
3 - Errors made in the choice of the operations which are to be given priority.
4 - Errors in writing a new equation. failure to check.
5 - Errors in numerical calculations.

1 - Errors related to the concepts of the mathematical unknown and the equation: the concept of the equation and the unknown belong to the "schème" of the algebraic solution of equations. Theses concepts are explored by the pupils at the beginning of the experiment This introductory phase is very complex; it has been discussed in Cortes, Kavafian and Vergnaud(1990). Writing the equation which models the problem gives rise to errors which will not be analysed here: we shall limit ourselves to the case where the equation is given.

In the post-test: many pupils are held in check by a written form in which the unknown is preceded by the "-" sign -w-10.66 for example The unknown has been conceptualised as an unknown number: a number. however, is different from its opposite (7 is different from -7): many pupils give to -w the status of the unknown

2 - Errors made in algebraic transformations identically on both sides of the equation : the equality is not conserved.

Transformations identically made on both sides of the equation can be considered as a shift of terms from one side to the other The pupils have been given two rules which sum up these transformations
Rule 1: the same number may be added or subtracted from each side of the equation a new equation is then obtained which is equivalent.
Rule 2: each side of the equation may be multiplied or divided by the same number. an equivalent equation is then obtained.

These two rules contain the most importants aspects of the algebraic method of solving equations: algebraic transformations on the equation and the sequence of equivalent equations which maintain the solution invariant. These two rules also comprise part of the algebraic transformations which can be applied to the unknown. in particular adding or subtracting a unknown number on both sides of the equation. From a congnitive point of view these two rules can function as a frame of reference what is allowable in building an apropriate strategy to solve each case In the begining this reference to the rules is explicit. it quickly becomes implicit

2a - Errors in additive tansformations.

2a.1 - Only one side of the equation is transformed: the strategy to find a solution here fonctions as "remove one term": 25=3w-5 : 25=3w-5+5 : 25=3w
This error reveals a conceptual dificulty in the begining of the learning process In the post-test it is certainly due to a failure to check work: some pupils work fast. this error is closely tied to the rules and notation which are used

2a 2 - Different transformations are made on each side of the equation. In most cases one term is subtracted on one side and added on the other.

$12z+45=21$ $12z+45-45=45-21$

In general the wrong rule can be stated as: "the result of the transformation is one number minus the other (c-b or b-c)", in most cases a positive result is sought. Other errors may show a failure to check work: $39-3x=125$; $-3x-39+39=125+39$

In the post-test errors are very likely due to the absence of a rigorous check of the soundness of the written transformation. $6w+32=-32$; $32-32+6w=-32+32$
The symmetry of the notation which is used obstruct the checking process: the pupil must refer to the previous equation.

2a.3 - Algebraic transformations are not written down (pupils mentally anticipate the result and abbreviate written steps). The sign of the term is not changed when it is taken from one side of the equation to the other:

$95=55.7x-150$; $-150+95=55.7x$. It is possible to see in this example the use of the same wrong rule we saw previously. Then, one example which can be interpreted as a shift of the negative sign from one term to the other: $39-3x=15$; $3x=125+39$
In general, when pupils carry out several transformations simultaniously the risk of error increases even if they are applying the correct rule: $5t-50=125$; $t=125-50/5$
We note that this error is frequent in the post-test. Pupils who will succeed in solving the last equations which are conceptually more difficult will fail to solve rather simple equations at the begining of the test because of the absence of written work (which makes more difficult to check). At the very start of the learning process this error may be conceptual (wrong rule) but very soon it is due to a **failure to check work**: the subsequent check of a transformation which has not been made explicit implies the mental reconstruction of the whole process based nonetheless on what has been written down.

2b - Errors in the application of the second rule: division of both sides of the equation by a number.

2b 1 - Some pupils divide only one term of the equation

$11x+30=14x$; $11x/11+30=14x$

2b 2 - *Some pupils perform different operations on either side of the equation:* $45=3/2y$; $3/2\cdot45=3/2y\cdot3/2$; $3/2\cdot45=y$

b3) Some shifts in meaning appear when division by the coefficient of the unknown is not noted on both sides of the equation. The pupils must anticipate the resulting quotient. *when calculating the value of the unknown they invert the numerator and the denominator of the resulting quotient* $84y=-35$; $y=84/(-35)$

Conclusion: any algebraic transformation must conserve the equality. Hence there is an operational invariant in the tasks given to our pupils: the consevation of the **equality**. The mental construction of this operational invariant makes it possible for the pupils to identify which algebraic transformations are allowed and then to decide which one is relevant among all those which are possible. This decision is a mental process which precedes any written work. The conservation of the equality stays implicit in the case of algebraic transformations (factorisation, multiplying out...) which are carried out only on one side of the equation. We have seen that when pupils use the right rule; a large number of errors are due to a deficiency in checking the validity of the transformation. We may thus consider that there is also an operational invariant of procedure: **checking the transformation**. This allows us to explain the high number of errors observed in the post-test.

3 - Errors in establishing which operation should be given priority

Each time the pupil performs identical algebraic transformations on both sides of the equation he also needs to perform numerical operations (+,-,*,/). Similarly each time the pupil multiplies factors or considers to reduce terms he must perform

numerical operations. Identifying the operation to be given priority therefore becomes an **operational invariant** in solving equations.

3a - Errors connected with disregarding the priority of multiplication and division over addition and subtraction. In this type of errors, mlultiplication is treated as an operation which is commuting and associating with the addition. Presumably, rules of commutativity and associativity used in the treatment of algebraic sums or the product of factors. ("The permutation of terms (factors) or the grouping of calculation does not alter the result") are extended to include the treatment of the two operations.

3a.1 - *The coefficient of the unknown is added to an independent term* (in reducing one side of the equation): 40=5t+22 ; 40=27t

3a 2 - *In solving equations containing a product of factors some pupils treat a term inside the brackets as an independent term:*

3(2x+22)+34=110 ; 3(2x+22-22)+34=110-22

Most of our pupils are able to multiply this product of factors. The origin of this error thus lies in the integration of the multiplication of factors into the strategy for obtaining a solution This error desappears in the post-test.

3a.3)- The multiplication of factors is implicitly treated as an addition of terms (for exemple ax is treated as a(+)x):

- in the treatment of the coefficient of the unknown

  5t-50=125 ; 5t-5-50=125-5 ; t-50=12

- in the solution of equations in which the unknown appears on both sides:

  11x-11x+30=14x-11x , x+30=3x

- in the solution of equations containing a product of factors

  3(2x+22)=76 ; 3(2x+22)-3=76-3 . 2x+22=73

3b - ***Errors connected with the failure to respect priority of addition and subtraction over multiplication.*** In a product of factors, for example (a)(bx+c), the addition of the terms bx+c has priority over the multiplication by (a). Now, this priority of the addition is more often introduced in the teaching of mathematics as a property of multiplication the distributivity of multiplication over addition and subtraction.

3b.1 - Wrong rule applied when multiplying out factors: only the fist term inside the brackets is multiplied out: 3 (2x-2+24)+34=110 ; 3*2x+22+34=110

We note that this wrong rule rests on a conceptual construction: the priority of multiplication over addition and subtraction (established when solving all the other equations none of which contains a product of factors). The difficulty thus lies in articulating the distributivity of multiplication (established before the experiment) in the context of solving equations.

3b.2 - Factorisation is not recognised. In the last exercice in the post-test, the pupils are faced with a special product of factors: the unknown multiplies a sum of numbers Almost all the pupils of all the classes multiply out these factors and reduce the resulting terms in the following way: x(3-3.5+2.5) becomes 3x-3.5x+2.5x and then 2x This procedure is not an error nevertheless it shows that the pupils consider that multiplication has priority over addition and therefore the factorisation is not recognised. We can assume that most pupils reduce the expression 3x-3.5x+2.5x using the rule. the counting of the xs, in which x tends to signify an object or a unit: something non-active since coefficients are treated exclusively. This rule is algorithmic in character. which means that it is not easily justifiable as its mathematical justification requires factorisation.

3c - Errors connected with desregarding the priority of addition and subtraction over division. Some pupils apply the second rule at a point in the calculation when this rule is not relevant They divide by a number only one term on each side (instead of dividing each side of the equation). in various cases:

- in solving equations containing negative or fractional coefficients:

  57=3/2y+12 ,(2/3) 57 =3/2 y 2/3 +12

- in solving equations where the unknown appears on both sides:

109y=25y-35 ; 109y/25=25y/25 -35

- in solving equations which contains a product of factors:

60=(-4)(-9m)+90 ; 60/(-4) = (-4)/(-4)(-9m) +90

One could say that these pupils treat division as an operation which has priority over addition and subtraction of terms according to a rule "first I divide then I add".

General observation: In almost all the errors observed which concern the priority of operations the pupils recognise multiplication thus they are not errors made in reading algebraic notation. These errors are frequent during the learning process and disappear in the post-test. They are essentially conceptual errors.

4 - Errors in writing a new equation: failure to check

Once the correct algebraic transformation has been determined and the numerical operations have been performed, it is then necessary to write down a new equation. Some errors are due to a failure to check when writing a new equation. A new kind of algebraic checking is in question here: the pupil checks thoroughly neither the transfer of terms from one equation to the next nor the correspondence between the written numerical results (in the new equation) and the numerical operations in the previous equation

4a - Omission or incorrect repetition of a term in the next equation.

- A term has been omited or changed in the next equation:

57-3/2y+12 ; 57=1.5y or -37=6v+65 ; 6v+65=-65

- A numerical operation has been written down but not performed one of the two terms has been omitted in the next equation: z+20-4z-20=-56-20 ; z-4z=-56

In general, of the two operations which have been written down, only the one defined by the strategy for obtaining a solution is performed (cancel out one term in one member of the equation).

4b - Omission of the "-" sign in writing a new equation. The frequency of this error is such that in merits detailed analysis Several situations must be considered:

4b.1 - A term is transfer. J to the next equation in absolute value. These errors almost exclusively concern terms containing the unknown: 39-39-3x=125-39 ; 3x=86

There is conceptu... interference at the begining of the learning process. it seems strange to work with the ... wn preceded by the "-" sign. In the post-test, these errors, are due to a failure to che... transfer of terms

4b.2 - Only the absolute value of the result of an addition is tranferred. This error appears when numerical terms and terms containing the unknown are treated:

-37-65=6v+65-65 ; 102=6v

Part of the algebraic work is carried out on the equation. transformations are written down. On the other hand, numerical calculations resulting from transformations are performed outside the equation; pupils sometimes transfer only the absolute value they fail to check whether the result in the following equation matches the written operation.

4b.3 - The absolute value of the result of a division is transferred: this omission of the "-" sign concerns the unknown and the numerical terms:

-3x/3=86/3 ; x=86/3

4c - A "-" sign is added to a term in the next equation: 109y-25y becomes -84y

General observation: the errors which we have just analysed are not conceptual in general. Most of them are due to the absence of a type of checking process which is proper to algebra Some authors call them "slips" (errors due to lack of attention).

Now. this classification seems too general to us since it hides a very important phenomenon: checking what has been written is a part of the algebraic methd of solving equations. Checking is manifested by a constant movement from the equation which has just been written down to the previous one and vice versa until the terms transferred have been thoroughly checked as well as the validity of the numerical results written down. Checking what has been written down as vell as the validity of the chosen transformation form **an operational invariant** of procedure: the algebraic checking process.

5 - Erros **made** in numerical calculation.

5a - Errors in numerical calculations with directed numbers:

- there are slips of the rules of addition of directed numbers: -A-B becomes -(A-B) and A-B becomes A+B: -56-20=-36 or 57-12=69

The error occurs in most cases in the treatment of operations of the type -A-B. In the post-test; lack of checking may be at cause.

- slip of the rule governing signs in the multiplication and in the division of directed numbers: 60=(-4) (-9m)+90 ; 60=-36m+90 or -18/-5=-3,6

These errors are probably conceptual at the begining of the learning process.

5b - Errors in mental calculation of the absolute value of numerical operations:

- results of divisions wich differ from the correct numerical result (mental calculation): 99/-9=-10 or 44/4=10

- result in additions of directed numbers which differ from the exact numerical resault (the error sometimes concerns the order of magnitude): 125-39=90

General observation: these errors correspond to a dysfunction of the "schèmes" which governs numerical operations. These "schèmes" belong to the "schème" which governs the solution of algebraic equations The large number of errors in the post-test shows a failure to check the result of numerical calculations

TOWARDS A NEW COGNITIVE MODEL:

Our cognitive model is a description of the functioning of the "schème" which governs the algebraic solution of equations

The exercice of algebra may begin with a transformation which leads to shifting a term to the other side of the equation The pupil must therefore. chose a relevant transformation among all permitted transformations. this implies the conceptualisation of a whole set of permitted transformations To choose a permitted transformation implies the respect of an operational invariant of our conceptual field. the **conservation of the equality.** This operational invariant. conceptual in nature, is a kind of principle which encompasses a good number of mathematical theorems. the violation of which leads to erroneous transformations

But once the transformation is applied into a new equation. it has to be checked against the previous equation. even if the correct rule is used To conduct the algebraic process correctly implies a particular type of checking· checking the **validity of the transformation** performed; ie that what has just been written corresponds to what was intended to be done

Once the algebraic transformation has been determined, the pupil must perform arithmetical operations on numbers. Now, the equation is sometimes perceived as a succession of terms between which competing arithmetical operation exist: the error concerning the priority of operations occurs. The same analysis holds for the development of a product of factors or the reduction of terms on one or other side of the equation. **Respect of the priority of arithmetical operations** is an operational invariant of a c nceptual nature in the solution of algebraic equations.

Once the arithmetical operations on numbers have been performed, the results obtained must be written into the next equation as well as all unchanged terms. Now, the large number of errors observed in the post-test which are due to incorrect rewriting shows in a negative way the importance of **checking what has been carried to the next equation**.

Checking what has been carried forward as well as checking the validity of the transformation performed implies a constant movement to and from the equation just written and the previous one. These two types of checking processes constitute an operational invariant of procedure: **algebraic checking**; the disregard of which is the origin of many errors.

The correct functioning of the "scheme" which governs the solution of algebraic equations is based primarily·

- on the operational invariants seen previously
- on other "schemes", in particular those which governs the arithmetical operations of addition, subtraction, multiplication and division: which also have their own checking system.
- on mathematical concepts (such as those of equation and unknown and many others) which are invariants of another nature

CONCLUSION: In our model the conceptualisation by pupils of mathematical contents is based in the mental construction of a rectricted number of ever more general operational invariants which encompass particular rules of action Consequently, the study of the cognitive development of pupils proves to be unavoidable. **These operational invariants** are proper to the conceptual field under consideration.

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DIDACTIC MODELS, COGNITION AND COMPETENCE IN THE SOLUTION OF ARITHMETIC & ALGEBRA WORD PROBLEMS

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Elements of a Local Theoretical Model based on experimental results regarding the skills necessary for the use of three methods for solving word problems are presented. The notion of Family of Problems is emphasized in order to analyze the underlying complexities which expose the need to be competent in increasingly abstract and general uses of the logical and mental representations required to achieve full competence in the algebraic method par excellence, referred to here as the Cartesian Method. These skills are contrasted with those required by the other two methods which are more rooted in arithmetic: MAES *and* MIAS.

INTRODUCTION

This paper is based on the results obtained from experimental classroom studies and clinical interviews with students between the ages of 15 and 16 years. The studies looked at the students' solution of arithmetic and algebra word problems. The work was carried out during the 1989-90, 1990-91 and 1991-92 academic years at the National Autonomous University of Mexico's high school in the south of Mexico City and the "Centro Escolar Hermanos Revueltas", a high school also in Mexico City. Some preliminary results can be found in Filloy E. and Rubio, G. ([4]). The studies are concerned with the application of three Didactic Models for the solution of arithmetic/algebraic problems: 1) MIAS.- The statement of the problem is conceived of as a description of "a real situation" or "a possible state of the world"; the text is transformed by means of analytic sentences, that is, using "facts" which are valid in "any possible world". Logical inferences are made which act as a description of the transformation of the "possible situation" until one which is recognized as the solution to the problem. We will call this method the **Method of Successive Analytical Inferences (MIAS)**. This is the classic analytical method for solving these problems using arithmetic only. 2) MAES.- A method of solution which uses numerical explorations in order to begin an analysis of the problem and thus reach a solution. We will refer to this method as the **Analytical Method of Successive Explorations (MAES)**. 3) MC.- Some of the unknown elements in the text are represented by algebraic expressions. Then, the text of the problem is translated into a series of relations expressed in algebraic language, leading to one or various equations whose solution, via a return to the translation, brings about the solution to the problem. This approach to the solution of problems is usual in current algebra textbooks and we will call it the **Cartesian Method (MC)** (see [8] for a description in terms of the Rules for Conducting the Spirit of Descartes).

The project aims to describe the types of difficulties, obstacles and facilities produced by the use of any of the three methods when solutions to word problems appearing in algebra textbooks

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are attempted. We are interested in what type of skills generates the use of **MAES** so that the user can come to be competent in the use of **MC**. And, which skills generated by **MIAS** are necessary for the competent use of **MC**.

The use of **MAES** which, as the previous paragraph indicates, is conceived of as a didactic artefact which the user will abandon as his competence in the use of **MC** progresses, provides elements for closing the gap between syntactic and semantic development, required for competent use of algebra. For this reason, our research project contemplates, among others, the relationship between the solution of problems with competent syntactic use of algebraic expressions, in such a way that we can (a) detect and describe the knowledge or skills needed to set the analysis and solution of certain types of problems in motion; (b) find out if the transference of algebraic operativity is propitiated and/or reinforced in the context of the problems via the application of each one of the three methods for problem solution; **MAES** in particular, since this method is consonant with certain pre-algebraic and arithmetic preferences which the student uses spontaneously to approach problems when he reaches high school (see [4]); (c) show if a proposal for teaching which uses **MIAS** and **MAES** among its didactic methods may allow the student to use language strata close to algebra, but more concrete than algebra itself, in the solution of problems.; (d) analyze how the use of the elements of the System of Signs of Arithmetic, via **MIAS**, can lead the user to give meaning to the symbols which are found in the algebraic expressions called equations; (e) make it possible that algebraic expressions are given meanings external to mathematics; and that with the development of these skills, the foundations be laid so the student can, (f) confer meanings from within algebraic language itself on the symbolic formations of algebra; and, (g) study the relationships between the ability to carry out the logical analysis of an arithmetic- algebraic word problem in **MIAS**, with the ability to analyze and solve a problem via the other two methods, **MAES** and **MC**.

THE RESEARCH PROJECT

This experimental project has the following phases:

PHASE 1. Exploratory study.

a) Theoretical analysis in order to formulate a Local Theoretical Model (see [2]) which would allow the empirical observations to be analyzed in terms of three components: 1) Formal Competence, 2) Teaching models. 3) Cognition. In [4] there is a brief description of each of these components; a more complete version can be found in [5] and [6].

b) Experimental process: the classroom activity of groups of students was monitored (1990-1992). The groups were classified according to three key areas of competence, essentially, the predominance of the following prerequisites in the solution of problems: I) Arithmetic, II) Pre-algebraic and syntactic and. III) Semantic.

In the teaching component of this phase, only **MIAS** was used in some groups and **MAES** in others.

This article describes the theses that led to this phase of the work and which constitute the frame of reference used for designing the following phase.

PHASE 2 (1991-92). In this phase, five exploratory questionnaires were used to classify the population which was monitored in class. The results were used to select cases for clinical study, in order to observe in detail the theses presented later on and those which arose as the observation progressed throughout the school year. These data are currently in analysis. Essentially, MIAS and MAES were used together in the teaching process so that, at the end, the student's skills with MC could be analyzed.

PHASE 3. (1992-93). An experimental format similar to that of Phase 2 will be set up, but the three methods will be pres 'nt in the teaching sequences. The results of the research regarding MC , carried out jointly with Fernando Cerdán and Luis Puig [1], will be added to the theoretical framework.

FIRST RESULTS

In this section we enumerate some of the theses which arose after the initial experimental work (PHASE 1, above) described in [4]. The reader should add the seven results reported there. The theses bring together ideas, both explicitly and implicitly, from different fields of knowledge such as history, epistemology, didactics, psychology, mathematics, etc. However, these assumptions were obtained both from the interaction with the students in class, solving problems, and as a result of the interpretation of the results of their endeavors via the theoretical framework briefly described in [4] and [6].

I) Brief description of the phases of MAES. On the basis of the experimental results obtained, we can describe four important phases in the implantation of MAES in terms of didactic teaching sequences:

PHASE 1. Reading and making the unknowns explicit. The initial representation of the problem through reading is the consequence of a first analysis of the situation that the student is trying to understand. The "quantities" that are to be determined (explicit unknown quantities) are separated out; the result of analysis, this "separating out", helps some users to understand the problem and thus establish the relationships between the data and other unknown quantities in the problem (secondary unknowns).

If one of the unknown quantities is not made explicit, the student cannot always carry the analysis through since, in Phase 2, the unknown, for which a numerical value will be proposed as a solution, may or not be known.

PHASE 2. A hypothetical situation is introduced, proposing numerical values for unknown quantities, thus assuming a possible solution to the problem and, from this, obtaining consequences.

In this phase we have a first situation of analysis, especially in the creation of new unknowns. However, there is still not necessarily a notion of the relationships between the elements of the problem.

If the problem situation is understood, that is, if the information is recognized, it compared with some previously formed scheme which is stored in the long term memory. Cognitive mechanisms which anticipate the solution come into play, sketching a solution, showing themselves via a

representation which uses language strata dominated by the user within a system of mathematical signs.

However, if the problem situation is not understood, the user needs to bring other cognitive mechanisms into play, among which are those which lead to a deeper analysis.

One possibility for a deeper analysis of the problem is to do so by means of the traditional method of translation, an adaptation of the Cartesian Method (MC). However, the difficulty with this approach is that it requires the student to carry out the analysis with a strata of language which uses the System of Mathematical Signs (SSM) of algebra, whose signs and rules have less semantic meaning for a problem solver of 15 or 16 years old, who only spontaneously uses the SSM of arithmetic. For this reason, the use of the strata of intermediate language provided by MAES allows the construction of a bridge between the two strata and a numerical approach can lead to the unfolding of the analysis of the problem situation and thus, its solution.

PHASE 3. In this phase a comparison must be established between two quantities which represent the same in the problem, but at least one of them is the consequence of interrelating the data with the unknown or unknowns of the problem (in the Cartesian Method this would correspond to formulating an equation representing the problem).

PHASE 4. On the basis of what is obtained in Phase 3 of the comparison, the operations that have been carried out are recovered and finally an equation is obtained. The construction of meanings for the representation of problems using equations has to pass through a stage of analysis of the situation where the relationships should come to have meaning to the student. Then, employing the intermediate language stratum used by MAES, it is hoped that the user can give meaning to the relationships in the problem and, finally -and perhaps more importantly- confer a meaning to the construction of the relation of equivalence represented by the equation, which points towards the solution of the problem.

II) When they make use of algebraic language, some experts and many beginners spontaneously *employ numerical values* (and arithmetic operations) to explore and thus resolve some word problems in algebra. This is because the use of numbers and arithmetic operations spontaneously confers meanings to the relationships found immersed in a problem and, in many cases, opens up more opportunities for the logical analysis of the problem to be set in motion. Since algebraic language is more abstract, it is more difficult to grasp the sense of the symbolic representations and, thus, to find strategies for solving the problem.

III) In order to solve more complex problems, further competence is needed to carry out the logical analyses of problemat situations. The analytic reasoning required to solve the problems is set in motion when 1) there are no obstructors such as 2) y 4) in [4], 2) there is no uncertainty about the tactics that should be put into practice to solve the problem. In order to advance with the above, it is also necessary to advance in 3) the use of intermediate tactics immersed in the uses of a) algebraic expressions, b) proportionality, c) percentage, d) the uses of multiplication within the schemes: $? \times A = B$, $A \times B = ?$, $A \times ? = B$, e) the handling of negative numbers, etc. That is to say, competent use prevents the user falling into certain *cognitive tendencies* (Filloy, [3]) which obstruct the possibility of making expert use of MC to solve word problems, such as a) the presence of appellative mechanisms which lead to the setting in motion of erroneous processes (for example, if a type of equation appears in the solution of a problem which the user does not know how to solve), b) the presence of obstructions deriving from semantics as to syntax

and viceversa, (for example, by solving problems and giving meaning to algebraic symbols, the subject is predisposed towards a good use of syntax), c) the presence of inhibitory mechanisms (for example, when one of the whole numbers in a problem which has been solved is replaced with a rational one), etc.

IV) The domination of such intermediate tactics should collaborate in the development of "positive" cognitive tendencies which appear in processes of learning of more abstract concepts, such as a) the return to more concrete situations when, in a situation of analysis, this is necessary in order to improve competence with MC; or b) the presence of a process of abbreviation of a concrete text in order to produce new syntactic rules, (for example, in the solution of problems, when the user is operating with numerical values which are assigned to the unknown in a problem in each exploration with MAES, and then, little by little, the user begins to introduce the abstract meaning of the unknown as he solves the equations with the rules of algebraic language, no longer making reference to the concrete situation), etc.

V) The competent use of MC to solve arithmetic-algebraic word problems implies an evolution of the use of symbolization in which, finally, the competent user can give meaning to a symbolic representation of the problems that arises from the particular concrete examples given in the process of teaching, thus creating Families of Problems whose members are problems which are identified by the same scheme for solution. Use of the Cartesian Method makes sense when the user is aware that by applying it, he can solve such Families of Problems. The Cartesian Method of problem solution is not conveyed by the unarticulated revision of examples (as is encouraged by conventional didactics, see [7]). The integrated conception of the method needs the confidence of the user that the general application of its steps will necessarily lead to the solution of these Families of Problems.

VI) One way of observing the complexity of a problem consists of an analysis which looks at the difficulties produced by inventing problems similar to a previously solved problem. Varying the data allows one to see if the user perceives that the problems are the same from a logical point of view (from the point of view of MIAS) and that the difficulty lies only in finding relations between data and unknowns. In the same way, the complexity of the relationships of a problem can be observed by inventing problems similar to one that has already been solved, starting with the solution, that is, knowing the value of the unknown or unknowns (to start by assigning a value to the unknown when a similar problem is invented is not a natural tendency among users, Filloy-Rubio, [4]). This process puts the relationships established previously to test and opens up a way forward to the recognition of a Family of Problems when there is a need to create the data of an analogous problem. The creation of problems similar to one solved previously tests the forms of mental representation or comprehension that were used to analyze the original problem.

VII) If a student is to become a competent user of the mathematical system of algebraic signs, which we will abbreviate as SMS_2 (see [3]), he has to be competent in other, less abstract systems of signs, such as a) the mathematical system of arithmetic signs (SMS_0), used in MIAS, and be able to handle systems of signs between the two, such as b) in the Analytical Method of Successive Explorations (which would use a SMS_1 system).

VIII) To give full meaning to the Cartesian Method (MC), to solve word problems in algebra, the (competent) user has to have the ability to return to systems of signs with a greater semantic content, e.g. SMS_1 above or SMS_0. The acceptation of MC in the solution of problems requires the users to recognize the algebraic expressions used in solving the problem as expressions involving

unknowns. We can say that there is competent use of expressions with unknowns when carrying out operations between the unknown and the data of the problem makes sense. In earlier stages, the pragmatics of these systems of signs leads to the use of the letters as variables, passing through a stage in which the letters are only used as names and representations of generalized numbers and a later stage in which they are used only to represent the unknown in the problem. Both of these are quite distinct and precede the use of letter as algebraic unknowns and the use of algebraic expressions as mathematical relations of quantities and magnitudes, in particular as functional relations.

IX) The didactic model based on the Analytic Method of Successive Explorations (MAES), when used to solve verbal algebraic problems, serves as a bridge joining syntactic development with semantic development by means of the construction of meanings for arithmetic-algebraic operations, on the way from the use of the notion of variable to that of unknown.

X) The meanings of arithmetic operations, their properties and results, as they are used in MIAS and MAES, serve as antecedents for elaborating the meanings of algebraic relations established between the use of expressions with unknowns and with data, furthermore, of the meanings of complex unknowns in a verbal problem (when using MC).

XI) MAES propitiates different algebraic interpretations of the word problem, and does not always follow the order of the statement as usually occurs in the teaching sequences used to illustrate MC.

XII) The dimensional analysis of the equations obtained from numerical relationships established between quantities involved in a verbal problem helps understand the notion of mathematical relations between quantities and magnitudes (arising from the verbal problem) and in general, to create meanings which lead to the notion of equivalence between algebraic expressions that involve the use of unknowns as the element common to two algebraic relations, expressed in an equation.

XIII) MAES in computing (for example, using spreadsheets) has didactic relevance in itself because it gives meaning to numerical methods for solving the equations which arise when problems are defined.

XIV) In the case of certain problems, a strategy for solution via the Cartesian Method or MAES is not always the best path. A direct arithmetic logical analysis (MIAS) may be a better option.

XV) The symbolic representations of problems in MC makes the use of the work memory more efficient. When the subject manages to relate data and unknowns, the information is integrated into more complex "chunks". When the subject succeeds in making these relationships, the use of syntax obviates the need to recharge the work memory with semantic descriptions linked to the statement of the problems.

XVI) Algebra (and MC) are required to simplify the more complex arithmetic and algebraic problems found in the arithmetic and algebra of the pre-university level. Part of the complexity of the arithmetic/algebraic word Families of Problems is derived from the difficulties their arithmetic logical analysis presents. To understand this progress in the competent use of MC one should,

a) Explore the tensions between the uses of the concepts of name, representation of a generalized

number, representation of the unknown, unknown, variable and relation. To this end it is necessary to understand what happens with the difficulty of a problem and of the solution of the equation representing it when the data of the problem are varied.

b) Analyze the relationship between the complexity of a Family of Problems and the development of algebraic syntax/semantics (from the operation of negatives, the use of rational numbers, the simplification of algebraic expressions, the solution of equations, etc.). This is connected with the explicitation of what we might call the diverse uses of algebraic expressions indicated in a).

XVII) It is necessary that in the First Phase of the Analytic Method of Successive Explorations (and probably in any method), which consists of the reading and comprehension of the text of the problem, the subject makes a logical sketch of the situation of the problem. This would involve, among others, a logical-mental representation of the problem where the fundamental information of the problem situation is integrated and where the central relationships necessary for developing a strategy for solution are identified; here lies the importance of competence with MIAS. However, it is not enough to have a comprehension or a logical mental integrated representation of the problem to be able to go on to use MAES and the Cartesian Method. It is also necessary, as part of logical analysis, to have developed skills competence for,

a) Separating out the principal question of a problem which is given generically. In MAES this becomes Phase 1 of this didactic model: making each one of the unknowns of the problem explicit.

b) Take the problem apart in such a way that if there is an implicit unknown, it is made explicit and, furthermore, transformed into the principal unknown (skill to change the unknown).

c) Create new unknowns starting from the problem situation, and use these to design strategies of solution.

d) Represent relationships between diverse unknowns.

e) Identify representations of relationships in order to find a common element to one or various of these mathematical relations.

f) Represent the above identification using an equation.

g) Use algebraic procedures to solve equations as a tactic in the search for the unknown in a problem situation.

All the above competences are important and necessary so that the user grasp the sense of the Cartesian Method.

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ALGORITHMIC MODELS AND THEIR MISUSE IN SOLVING ALGEBRAIC PROBLEMS

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Abstract

The paper deals with two main types of errors frequently made by high school students in solving algebraic problems: a) The use of the distributive property applied inadequately to exponents, and b) the confusion between terms and factors in the simplification of algebraic fractions. It has been found that many students, who know the corresponding correct formulae [for instance $(a + b)^2 = a^2 + 2ab + b^2$] forget them under the pressure of the more elementary model of the distributive property, [for instnace, since $2(a + b) = 2a + 2b$] they may consider, in their solution, attempts that $(a + b)^2 = a^2 + b^2$. The general idea of the paper is that, very often, various types of mistakes are the effect of conflicting models in the mind of the student, the more primitive, elementary ones being those which dictate the solving strategy.

Difficulties encountered by students in learning algebra have, since long, attracted the preoccupation of teachers and researchers. In the last years, the interest has been focused in systematic errors and their sources. (See, for reviews, Radatz, 1980, and Kieran, 1990.) Thus, it has been pointed out that students face difficulties in interpreting the meaning of literal symbols (Kucheman, 1981), in perceiving the structurality of an equation and consequently using algebraic rules inadequately (Larkin, 1989) etc. Laursen (1978) refers to the fact that students tend to confuse terms and factors. An important paper, with regard to the souices of errors in solving algebraic problems, is that of Matz (1980). According to Matz the errors are the result of reasonable, although unsuccessful, attempts to adapt previously acquired knowledge to a new situation. The student extracts a rule from a prototype (or directly from the textbook) and, sometimes uses it inadequately.

The A refers mainly to "linearity errors" (for instance $\sqrt{A+B} = \sqrt{A} + \sqrt{B}$) and "errors made by overgeneralization". For instance, since from $(x - 3)(x - 4) = 0$ one deduces $x = 3$ and $x = 4$, the student writes that from $(x - A)(x - B) = K$ follows $x - A = K$ and $x - B = K$.

Our present research has been inspired by this line of thought. The data reported here are a part of a larger project. There were two main hypotheses which inspired our investigation. The first is that many systematic errors in solving algebraic problems have their source in inadequate applications of rules serving as models. The term model implies the notion of

structure which is richer than that of rule. The student does not blindly, automatically, use a rule. He is inspired by a model, by a certain solving strategy rendered meaningful in his view by previous experience. For instance, when writing $(x + a)^2 = x^2 + a^2$ he does not simply apply a rule. He has never learned such a rule. What he has in mind, we assume, is the distributive property $m(a + b) = ma + mb$, older in his experiences, more acceptable intuitively and visually. This property becomes the prototype of a large category of situations (for instance, $\sin(a + b) = \sin a + \sin b$). The notion of "model" allows us to establish hierarchies with regard to their degree of intuitiveness, their familiarity, their direct meaningfulness, the apparent coherence. According to these criteria, their impact may be predicted.

A second hypothesis of the present research has been that solving models may conflict in the students' mind. As an effect, the "stronger" ones may impose themselves over the "weaker" ones and eliminate them. The fact that the student knows the formula $(a + b)^2 = a^2 + 2ab + b^2$ does not imply that he will use it necessarily, correctly in a problem solving situation. He may fall back on an incorrect formula with a higher degree of intuitiveness: $(a + b)^2 = a^2 + b^2$.

A third hypothesis of the project not checked by us so far, is that it may be beneficial to the student for overcoming his difficulties to become aware of the sources and mechanisms of his mistakes.

In the present paper we focus our attention on the inadequate application of two solving models: One is the abovementioned distributive property used inadequately in the case of exponentiation. For instance $(a + b)^2$ becomes $a^2 + b^2$ similarly to $2(a+b) = 2a + 2b$. The second inadequate use of an algorithmic model refers to the confusion between terms and factors (Laursen, 1978). In an expression like:

$$\frac{a \cdot b}{b}$$

the students tend to cancel -b and +b and write:

$$\frac{a \cdot \cancel{b}}{\cancel{b}} = a$$

as they would do in an expression like

$a + b - a = b.$

Methodology

The subjects were 71 students enrolled in two 9th grade classes in a high school in the Tel Aviv area. The students were supposed to know the basic formulae, rules and theorems on which the solving procedures were based. The questionnaire used contained two part sections. The first part contained 10 algebraic expressions. The students were asked to simplify them.

The second part contained six items asking the students to write the respective formulae (if they exist).

The questionnaires were administrated collectively in each class and one hour was allowed. One has to emphasize that the classes were considered ranging high with regard to the mathematical performances of the students.

Results

In what follows, we discuss the reactions of the subjects, separately for each item (problem). We will concentrate on the erroneous answers their frequency and their hypothetical causes The results of the first part of the questionnaire are presented in Table 1.

Section A of the Questionnaire

Items (1) and (2):

$\frac{x - 3}{3}$, $\frac{a - b}{b}$

The items have the same mathematical structure and it has been assumed that a part of the subjects will reduce mistakenly -3 and +3 and respectively -b and +b.
Seventy -five percent answered correctly to item 1. Thirteen percent of the subjects wrote:

$\frac{x - 3}{3} = x$ and, respectively, $\frac{a - b}{b} = a$ (as assumed).

But it was also a second type of mistake, namely:

$\frac{x - 3}{3} = x - 1$ and respectively $\frac{a - b}{b} = a - 1$.

Another 17% gave one of the answers:

$x^2 \pm 1$ or x^2

and 24% wrote: "impossible".

Section B of the Questionnaire

We considered it interesting to determine whether the subjects know the formulae which intervene in the questionnaire.

In the table below, we present in percentages, the results obtained (see Table 2):

The most important finding is that many of the students who know the formulae which intervene in the problems of the questionnaire, do not use them correctly when attempting to solve.

Table 2: Systematic errors when considering the various formulae which intervene in Section A (in percentages). N = 71.

		Systematic Errors					
	Correct	1		2		Non-systematic Errors	No Answer
$a^2 - b^2$	56	No formula	24	$a^2 - b^2 = (a - b)^2$	8	4	8
$a^2 + b^2$	65	$a^2+b^2 = (a+b)^2$	24	$a^2 + b^2 = c^2$	4	5	2
$(a + b)^2$	84	No formula	4	$(a + b)^2 = a^2+b^2$	6	6	-
$a^3 - b^3$	4	No formula	45	$a^3 - b^3 = (a - b)^3$	12	17	22
$(a - b)^2$	75	$(a-b)^2 = a^2 - b^2$	7	$a^2 - ab + b^2$ or $a^2 + 2ab + b^2$	7	11	-
$a^2+b^2+c^2$	80 (no formula)	$a^2 + b^2 + c^2 = (a + b + c)^2$	8			6	6

Let us consider some of these items:

$$\frac{4x^2 - 9}{2x - 3}$$

Table 1: Synoptic table presenting the various types of errors in solving algebraic problems by 9th graders (in percentages) N = 71.

Items	Correct Answer	Errors							No Answer
		Non-Systematic	Systematic						
			Category 1		Category 2		Category 3		
	(%)	(%)		(%)		(%)		(%)	(%)
$\frac{x-3}{3}$	75	3	$\frac{x-\cancel{3}}{\cancel{3}} = x$	13	$\frac{x-\cancel{3}^{1}}{{}_{1}\cancel{3}} = x-1$	8	-		1
$\frac{a-b}{b}$	72	6	$\frac{a-\cancel{b}}{\cancel{b}} = a$	13	$\frac{a-\cancel{b}^{1}}{{}_{1}\cancel{b}} = a-1$	8	-		1
$\frac{4x^2-9}{2x-3}$	32	24	$\frac{{}^{2}\cancel{4}x^{\cancel{2}}-\cancel{9}^{3}}{2\cancel{x}-\cancel{3}} = 2x-3$	25	$\frac{4x^2}{2x} - \frac{9}{3} = 2x-3$	8	2x-3 without expl.	8	3
$\frac{12a^2-3b^2}{6a-3b}$	27	30	$\frac{{}^{2}\cancel{12}a^{\cancel{2}}-\cancel{3}b^{\cancel{2}}}{\cancel{6}\cancel{a}-\cancel{3}\cancel{b}} = 2a-b$	21	$\frac{12a^2}{6a} - \frac{3b^2}{3b} = 2a-b$	8	2a-b or 2a±3b without expl.	10	4
$\frac{(x-2)^2}{x^2-4}$	31	29	$\frac{(x-2)^2}{x^2-4} = \frac{x^2-4}{x^2-4} = 1$	27	$\frac{\cancel{x^2}-4x+\cancel{4}}{\cancel{x^2}-\cancel{4}} = 4x-1$	13	-		0
$\frac{(a-b)^2}{a^2-b^2}$	27	18	$\frac{(a-b)^2}{a^2-b^2} = \frac{a^2-b^2}{a^2-b^2} = 1$	28	$\frac{\cancel{a^2}-2ab+\cancel{b^2}}{\cancel{a^2}-\cancel{b^2}} = -2ab$	10	$\frac{\cancel{a^2}+ab-\cancel{b^2}}{\cancel{a^2}-\cancel{b^2}} = ab$	13	4
$\frac{a^2+4}{a+2}$	27	18	$\frac{a^{\cancel{2}}+\cancel{4}^{2}}{\cancel{a}+\cancel{2}} = a+2$	28	$\frac{(a+2)^2}{a+2} = a+2$	10	a+2 without expl.	13	4
$\frac{a^2+b^2}{a+b}$	37	20	$\frac{a^{\cancel{2}}+b^{\cancel{2}}}{\cancel{a}+\cancel{b}} = a+b$	15	$\frac{(a+b)^2}{a+b} = a+b$	7	a+b without expl.	20	1
$\frac{a^2+b^2+c^2}{a+b+c}$	37	15	$\frac{a^{\cancel{2}}+b^{\cancel{2}}+c^{\cancel{2}}}{\cancel{a}+\cancel{b}+\cancel{c}} = a+b+c$	13	$\frac{(a+b+c)^2}{a+b+c} = a+b+c$	13	a+b+c without expl.	21	1
$\frac{x^3-1}{x-1}$	6	25	$\frac{x^{\cancel{3}^{2}}-\cancel{1}}{\cancel{x}-\cancel{1}} = x^2-1$	17	$x^2 \pm 1$ or x^2 without explanation	17	impossible	24	11

In this case, the students knew that the "reduction", being based on a division ($\frac{-3}{-3}$ and $\frac{-b}{-b}$ the result of the division is 1. Eight percent of the subjects gave this result. We have here a mixture of an error (the confusion between "term" and "factor") and the reminiscence of a correct rule.

With regard to item 2, ($\frac{a - b}{b}$), the results are similar, which indicates the consistency of the subjects in having this type of problem. There were 72% who answered correctly, 21% who made systematic errors and 6% who made other various types of errors. Exactly as with the previous case, 13% wrote:

$$\frac{a - \cancel{b}}{\cancel{b}} = a$$

and 8% wrote $\frac{a - b}{b} = a - 1$

identifying $-\frac{b}{b}$ with 1.

The item 3, $\frac{4x^2 - 9}{2x - 3}$

Thirty three percent gave the answer:

$$2x - 3$$

Eight percent did not justify at all the solution; twenty five percent reduced in the following way:

$$\frac{{}^{2}\cancel{4}x^{\cancel{2}} - {}^{3}\cancel{9}}{2\cancel{x} - \cancel{3}} = 2x - 3$$

Eight percent wrote :

$$\frac{4x^2 - 9}{2x - 3} = \frac{4x^2}{2x} - \frac{9}{3} = 2x - 3$$

Here, we have a new type of error as an effect of the intervention of a multiplicative model. The subjects decomposed the fraction, separating their terms as they would have been products of factors ($\frac{a \cdot b}{c \cdot d} = \frac{a}{c} \cdot \frac{b}{d}$ and, then, reduced as if two separate fractions were involved.

The next question asked the students to reduce the terms of the fraction:

$$\frac{12a^2 - 3b^2}{6a - 3b}$$

In order to solve the above problem, one has first to factorize the terms of the fraction:

$$\frac{3(4a^2 - b^2)}{3(2a - b)} = \frac{3\,(2a - b)\,(2a + b)}{3\,(2a - b)} = 2\,a + b$$

As in the case of the previous fractions a part of the students (21%), first, incorrectly operated some reductions on the terms of the numerator and denominator as if the numerator and the denominator consisted only on products of factors :

$$\frac{{}^{2}12a^{\cancel{2}} - \cancel{3}b^{\cancel{2}}}{\cancel{6}\cancel{a} - \cancel{3}\cancel{b}}$$

and thus obtained the incorrect answer : $2a - b$.

Other students, (8%), separated the terms at the numerator and the denominator and obtained:

$$\frac{12a^2}{6a} - \frac{3b^2}{3b} = 2a - b$$

Finally, a third category wrote $2a \pm 3b$, or $2a - b$ without any explanations.

In principle, items 3 and 4 reveal the same types of misconceptions supported by the same types of algorithmic models.

The main error consists in separating the terms of the numerator and the denominator and operating reductions on the corresponding terms thus obtained. The basic model applied, inadequately, is that of operating reductions as if the numerator and the denominator consisted on products of factors.

Questions 5 and 6:

$$\frac{(x - 2)^2}{x^2 - 4} \text{ and } \frac{(a - b)^2}{a^2 - b^2}$$

The two item are of the same type. It has been supposed, from previous observations, that the students will use for the two numerators, $(x - 2)^2$ and $(a - b)^2$, a wrong formula

$[(a - b)^2 = a^2 - b^2]$. It is what happened to 27% of subjects. They wrote:

$$\frac{(x - 2)^2}{x^2 - 4} = \frac{x^2 - 4}{x^2 - 4} = 1$$

There were also subjects (13%) who used correctly the formula but reduced incorrectly, confusing terms with factors:

$$\frac{x^2 - 4x + 4}{x^2 - 4} = 4x - 1$$

With regard to item 6, exactly the same strategies were used.

$$\frac{(a - b)^2}{a^2 - b^2} = \frac{a^2 - b^2}{a^2 - b^2} = 1 \quad (28\%)$$

or

$$\frac{(a - b)^2}{a^2 - b^2} = \frac{a^2 - 2ab + b^2}{a^2 - b^2} = -2ab$$

Surprisingly, in this case, the subjects were more consistent, simply "reducing", in fact eliminating, the term a^2 and b^2.

Questions 7 and 8:

$$\frac{a^2 + 4}{a + 2} \text{ and } \frac{a^2 + b^2}{a + b}$$

Our prediction was that the main error would be to start with a wrong formula:

$$a^2 + 4 = a^2 + 2^2 = (a + 2)^2$$

but this happened only to 10% of the subjects, who wrote:

$$\frac{(a + 2)^{\cancel{2}}}{\cancel{a} + \cancel{2}} = a + 2.$$

Twenty eight percent of the subjects were attracted by a less sophisticated strategy. They "reduced" the squares of the numerator and got again a + 2:

$$\frac{a^{\cancel{2}} + 2^{\cancel{2}}}{\cancel{a} + \cancel{2}} = a + 2.$$

Item 8:

$$\frac{a^2 + b^2}{a + b} = a + b$$

Fifteen percent reduced incorrectly:

$$\frac{a^{\cancel{2}} + b^{\cancel{2}}}{\cancel{a} + \cancel{b}} = a + b$$

Twenty seven percent simply gave the answer a + b which may be obtained either by reducing as above, or by identifying:

$$a^2 + b^2 = (a + b)^2$$

Item 9:

$$\frac{a^2 + b^2 + c^2}{a + b + c}$$

It has been assumed that the main error will consist - as for the previous items - in reducing incorrectly, the corresponding terms of the numerator and the denominator. Thirty seven percent answered correctly and 15% made various mistakes, but 47% used systematically incorrect strategies. Out of these, 13% used the incorrect reduction procedure:

$$\frac{a^{\cancel{2}} + b^{\cancel{2}} + c^{\cancel{2}}}{\cancel{a} + \cancel{b} + \cancel{c}} = a + b + c$$

Another 13% identified first:

$$a^2 + b^2 + c^2 = (a + b + c)^2 \text{ and then they got:}$$

$$\frac{(a + b + c)^2}{a + b + c} = a + b + c$$

Another 21% simply wrote the incorrect answer:

$$a + b + c.$$

Both basic types of mistakes, (the inadequate use of certain types of solving models) encountered so far, appeared then in this case: a) The incorrect reduction of corresponding terms as if they would represent factors and, b) The use of a wrong formula inspired by the distributive property: $(a + b + c)^2 = a^2 + b^2 + c^2$.

The last item was:

$$\frac{x^3 - 1}{x - 1}$$

Seventeen percent reduced incorrectly:

$$\frac{x^3 - 1}{x - 1} = x^2 - 1$$

From the above table we learn that 56% of the subjects answered correctly that $a^2 - b^2 = (a + b)(a - b)$. Nevertheless, only 32% used this formula correctly in solving the above problem. The rest of them made incorrect reductions under the pressure of the more elementary model of direct reduction!

Let us consider the items:

$$\frac{a^2 + b^2}{a + b} \quad \text{and} \quad \frac{a^2 + 4}{a + 2}$$

Inspecting the table, we may find out that 84% wrote correctly that $(a + b)^2 = a^2 + 2ab + b^2$. Nevertheless, only 37% solved correctly the first item and only 27% solve correctly the second one.

The rest of them identify:

$$\frac{a^2 + b^2}{a + b} = a + b \quad \text{and}$$

$$\frac{a^2 + 4}{a + 2} = a + 2$$

These students either reduce directly the corresponding terms or equalize $a^2 + b^2$ with
$(a + b)^2$ and reduce after the transformation.

The important finding in the above examples is that formulae like

$(a + b)^2 = a^2 + 2ab + b^2$, $(a - b)^2 = a^2 - 2ab + b^2$

or $(a + b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$

seem to be unsupported intuitively. Even after being memorized as such, they remain fragile, and are, consequently, in danger of being rendered non-effective under the pressure of other models which fit better a primitive, intuitive attitude (in the present case, the distributive property involving multiplication and addition).

The same principle applies also in the case of reductions to lowest terms. The model corresponding to additive structures, which consists in reducing terms by subtraction ($a - a = 0$ or $2a - a = a$) is applied inadequately, also, in the case of division, that is when dealing with the terms of a fraction.

For instance

$$\frac{a + b}{a} = b$$

The following general conclusion may be drawn from the above findings: Various algorithmic models (solving strategies) may conflict among them under the effect of superficial

similarities. Very often, it is not the adequate strategy which is used by the solver, but the older one, the more primitive, the more acceptable intuitively.

We suppose that it would be very profitable for the student if such predictable, systematic confusions are discussed in the classroom and their psychological causes clarified.

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A Clinical Interview on Children's Understanding and Misconceptions of Literal Symbols in School Mathematics

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Abstract

This paper aims to clarify children's understanding and misconceptions of literal symbols, and to investigate persistency of misconceptions through a clinical interview. In the interview, subjects were carefully paired so that the two children had inconsistent ideas on understanding literal symbols. To identify such children a written problems task was conducted beforehand. The methodology of this careful and purposeful identification of subjects for interview is one of the characteristic features of this study. The interview revealed that some students have the misconception that the same literal symbol does not necessarily stand for the same number. This phenomenon is considered as a lack of understanding of the "definite aspect" which is one of the key aspects of the concept of variables.

Introduction

Mathematics can be characterized as some sort of language. To understand mathematics as a language, it is crucial for students to grasp the grammar, vocabulary and rhetoric method besides understanding the contents. Moreover, comprehension of such conventions is fostered through long learning experiences. However, it should be noted here that conventions acquired by students may not necessarily be "correct" or "reasonable" from a teacher's point of view. Actually, researchers have been identifying many misconceptions so far as incorrect rules grasped by students such as "letter as objects". But understanding is surely a reflection of each student's own activity. Therefore, in any teaching process it is reasonable for students to have arbitrary understanding or misconceptions.

So far, among English speaking students, the misconception that different literal symbols necessarily stand for different numbers is well-documented (Kuchemann, 1981; Booth, 1984). That is the inverse of the correct proposition that the same letter stands for the same number. The " letter as object " can be characterized by this incorrect convention.

One of the most plausible sources for an incorrect convention such as "letter as object" is language itself ; that is English. At the beginning stage, a teacher may say that *a* and *b* are abbreviations for apples and bananas, which would create the "letter as objects " misconception. However, this is unlikely to happen in Japanese schools, because the Japanese writing system is quite different from English. From the Japanese students' point of view the literal symbols in algebra seem to be unconnected to the Japanese written language. The unfamiliarity of Japanese with literal symbols in algebra is a problem for Japanese especially at the beginning stages of learning Algebra.

Furthermore, this unfamiliarity would create another distinctive source for misconception for Japanese students. In this paper the author focuses on the misconception, which is not often-reported in English speaking nations, that the same letter does not necessarily stand for the same number, which is the negative proposition of the correct convention :the same letter stands for the same number. Focusing on that incorrect convention held by Japanese students, this paper aims to clarify Japanese students' understanding of literal symbols in

algebra through a clinical interview.

Preliminary Survey Identifying Interview Subjects

The Preliminary Survey Problems

Prior to the interview, subjects were carefully chosen, then paired and interviewed together. In this way paired subjects could be released from the pressure in comparison with a one to one interview situation. Additionally, the subjects were carefully chosen so that the two students had different ideas on their understanding of literal symbols. Specifically "different" in this context means that the two students held inconsistent conceptions. The interview context created a conflict which allowed students to express their ideas explicitly to each other. The methodology of this careful and purposeful identification of subjects for interview is one of the characteristic features of this study. To identify such students, a written problem task was conducted beforehand as shown below:

Problem 1

Find the number that is appropriate for x in this expression.

Akiko-san answered in the following manner. If you agree, put ○ in ( ) and in the case of disagree, put × in ( ). Also, write the reason for your answer.

( ) 2,5,5 reason :
( ) 10,1,1 reason :
( ) 4,4,4 reason :

Problem 2

x+y=16

Find the number that is appropriate for x and y in this expression.

Yoshiko-san answered in the following manner. If you agree, put ○ in ( ) and in the case of disagree, put × in ( ). Also, write the reason for your answer.

( ) 6,10 reason :
( ) 9,7 reason :
( ) 8,8 reason :

The preliminary survey revealed the same results as another study conducted by the author in which subjects were divided into two types : A and B as shown below. It was rare for a student to get *both* problems correct(Fujii, 1990).

Type A : Holding the misconception : the different letter stands for different number
Student got Problem 1 correct.
Student got Problem 2 incorrect by an ×in ( ) for 8,8.

Type B : Holding the misconception : the same letter does not necessarily stand for the same number.
Student got Problem incorrect by putting ○ in all ( ).
Student got Problem 2 correct.

The paired students for interview were chosen one each from the two distinct groups ; Type A and Type B. The classroom teacher was asked to arrange the A-B pairs for the interview. Eventually seven paired elementary school students (6th graders) and six paired junior high school students (8th graders) were selected and interviewed. Thus, the total number of students interviewed was 26.

The Interview Tasks and Procedures

While the preliminary problems 1 and 2 were used for the interview too, another task for

interview was prepared by modifying the task used in the study conducted by Takamatsu (1987). Takamatsu reported that some 6th grade student expressed the relation between side and perimeter of a square by using x, as $x+x+x+x=x$. At the first stage of the interview, subjects were introduced to this expression written on paper, with a square, and an explanation as follows:

> Hiroshi-kun expressed the relation between the sides and perimeter of a square by using x as $x+x+x+x=x$. Is this a correct or incorrect expression ? State the reason why and explain your own way of expressing of the relation.

At the second stage of the interview, subjects were asked about any inconsistencies between their responses in the interview and those in the preliminary survey task results: responses to the problem 1: $x+x+x=12$. For instance, if a student identified the expression $x+x+x+x=x$ as an incorrect one, then his responses on the expression $x+x+x=12$ interpreted as 2+5+5=12, 10+1+1=12 besides 4+4+4=12 were critically examined. On the other hand, if a student identified the expression $x+x+x+x=x$ as a correct one by saying that the letter x can be any number, then his responses on the expression that the expression $x+x+x=12$ should be interpreted 4+4+4=12 only was critically examined. At the third stage, students were asked about the relationship between responses in the interview and responses on the counter-example tasks which were prepared before the interview :

Counter-example tasks
(1) Simplify $5x+7x$, $x+x+x+y+y$.
(2) Solve the equation $x+x+x=12$ for x.

Results

Analysis of View Points
For deeper analysis, this study was trying to clarify students understanding through the following three steps.

Step 1 : Analysis focusing on students' conclusions of interview tasks.
Step 2 : Analysis focusing on students' thought lying behind their conclusions.
Step 3 : Analysis focusing on students' confidence in their thought.

Step 1 aimed at analyzing the responses in terms of the convention: the same letter stands for the same number. In the process of the interview, if two students agreed with each other on the type A, we could identify their opinion based on the proposition that the same letter stands for the same number. However, that is an interviewer's or a third person's judgment. Whether students arrived at the proposition by themselves, and whether they are able to describe it in their own words, are different types of problem to consider. Step 2 aimed at clarifying this aspect. In Step 3, the analysis hightlights discord between both students and also between tasks in order to clarify students' confidence toward their description which comes out of the process of the interview.

Step 1 Analysis: Overview Findings
Analyzing students' responses between two expressions $x+x+x+x=x$ and $x+x+x=12$, the four groups, G1, G2, G3 and G4 were identified as shown in Table 1. The G1 students answered quite consistently with their types : The type A student identified that $x+x+x+x=x$ was an incorrect expression, while the type B student reacted as it was a correct one. On the expression $x+x+x=12$ they answered as in the same items taken from the preliminary survey. In other words, through the whole interview process, their answers were distinct or

parallel to each other. There were 5 pairs in G1 : 4 pairs in elementary school and 1 pair in junior high school.

The G2 students agreed with the expression $x+x+x+x=x$ identifying it as an incorrect one but they disagreed with the interpretation of xs in the $x+x+x=12$: The type A students thought that xs stand for 4 only : 4+4+4=12, but type B students interpreted those as 2+5+5=12 and so on. There were only two junior high students' pairs in G2.

The G3 pairs both agreed with each other about the two tasks given in the interview by responding in a way of the type A. In other words the type B students in these pairs came to agree with the type A in the process of the interview. There were 6 pairs in G3 : 2 pairs in Elementary and 4 pairs in junior high.

The G4 students, contrasting with the G3, responded in a way of type B, that $x+x+x+x=x$ was correct. Incidently the G4 did not show a clear conclusion for the expression $x+x+x=12$ since they were totally involved in considering the correctness of the expression $x+x+x+x=x$.

Table 1: Summary of the interview results

		$x+x+x+x=x$	$x+x+x=12$	Elementary	Junior high
G1	Type A	incorrect	Only as 4+4+4=12	4 pair (8 students)	1 pair (2 students)
	Type B	correct	Such as 2+5+5=12, etc.,		
G2	Type A	incorrect	Only as 4+4+4=12	0 pair	2 pair (4 students)
	Type B	incorrect	Such as 2+5+5=12, etc.,		
G3	Type A	incorrect	Only as 4+4+4=12	2 pair (4 students)	4 pair (8 students)
	Type B	incorrect	Only as 2+5+5=12, etc.,		
G4	Type A	correct	not answered	1 pair (2 students)	0 pair
	Type B	correct	not answered		

Step 2 Analysis : Students' ideas lying behind their responses
Two situations were identified where students' ideas became more explicit in expressing their own terms. One was the situation of substituting a number in the place of x, because students needed to consider the *signifie* of x . The other was a place where they discuss the appropriateness of *signifiant*.

The convention, the same letter stands for the same number, was more explicitly revealed for $x+x+x=12$ rather than $x+x+x+x=x$. Concerning the expression $x+x+x+x=x$, some students, especially elementary school students did not consider it in terms of the convention. Rather they were concerned with the appropriateness of the left side of the expression : $x+x+x+x$. They stated that the expression was false because $x+x+x+x$ should be changed to $x\times4$. (This is a correct way to change from addition $x+x+x+x$ to multiplicative expression *in Japan.*) The tendency to focus on the left side of the expression and not consider the right side or the whole expression was revealed in the other pairs interviewed, and also in an other survey (Fujii, 1990). Furthermore, it should be noted here that the misconception accepting the expression $x+x+x+x=x$ does not relate to the misconception concerning the omission of the coefficient, particularly when it is 1.

On examining the expression $x+x+x=12$, students become more fluent in describing the

convention, particularly the type A students could describe the convention lying behind their answers or conclusions by their own words, such as :

Xs are the same letters, therefore they must stand for the same number.
The xs refers to the same number in one expression.
The x, three together, must be the same number.

On the other hand, the negative proposition of "the same letter stands for the same number" was rather implicitly described by type B students. Elementary school students' descriptions were more concrete reflecting the expression given in the task, such as ;

We can make 12 anyway.

Junior high students gave arguments such as:

The x is an unknown number, then we can use it to refer to any number.
The x can be any favorite number.
Because all the numbers are unknown, then we can write x for it.

Step 3 Analysis: Students' confidence in their thought

When the different ideas were revealed in the interview, students who were paired deliberately according to the preliminary survey were inevitably faced with at least two considerations. One was how to react to the different idea held by their counterpart. The other was a reconsideration of their original thought. The reflection on a person's own thought may be more serious when he/she discovers an inconsistency in his/her own thoughts. These situation could reveal their confidence in their idea behind their descriptions.

Confidence in their thought : facing a different opinion

First of all, students have to identify the counterparts's idea. This was not such an easy task especially for elementary students. Although the type A students seemed to recognize their counterpart's idea, the type B students seemed to be puzzled or not to recognize their counterpart's idea. The following extract illustrates the point :

(I : Interviewer ; At :Takahashi-kun, type A, Br : Re-kun, type B)
I: At-kun you chose only at 4,4,4, can you read the reason you wrote ?
At: X is same, then x goes in same number.
Br: Is it so ? (Looking at At's face. Looked puzzled)
At That what I thought, any way, I'm not sure though.

Br's expression "Is it so ?" with a suspicious look shows that he has never considered the proposition before that the same letter stands for the same number. After identifying different opinions between the two students paired in the interview, students must consider which was the correct or the reasonable one. The elementary students seemed to have no source to form a conclusion. In fact, they were never taught formally the convention that the same letter stands for the same number according to the National Course of the Study issued by Monbusho. The following extract illustrates the point.

I: I understand that you have a different opinion. Can you decide which is the correct one, or have you ever taught at school of a way decide which is true ?
At:Br: No, we have never been taught.
I: So, you don't know how to think about this ?
At;Br: Un('means "No").

I: Is this(Br's response) also likely to be correct, in your eyes, At-kun?
At: It is(with a nod implying "Yes").

Confidence in their thought : faced with inconsistency in own ideas
In the interview, students were asked two tasks in terms of the convention that the same letter stands for the same number : $x+x+x+x=x$ and $x+x+x=12$. For $x+x+x+x=x$, if the type A student identifies that it is a incorrect one and the type B students does it as a correct one, then these responses seem to be quite consistent with their preliminary survey task ; $x+x+x=12$. In fact, four elementary pairs and one junior high pair answered in this way (see Table 1). On the other hand, if the two paired students both agreed or disagreed against the expression $x+x+x+x=x$ and/or $x+x+x=12$, then the contradiction would be revealed between their responses, within each type. The interview revealed a variety of reactions such as follows :

(Ay, Bo : 6 graders in elementary school)
I: You identified that the different number could put in here($x+x+x=12$), but you just say that it does not apply here($x+x+x+x=x$).
Bo: Yes, I said so.
Ay;Bo: (Laughing a little, then looking at the expression seriously.)
I: You don't know which is true ?
Ay;Bo: Un(means "No").
I: What if you are forced to decide which is correct ? You don't know ?
Bo: I don't know.

On the other hand, some students, particularly type B students, tended to modify the answer in the preliminary survey, trying to be consistent with responses in the interview. Furthermore, the other students seemed rather persistently to reject considering the two expressions together. They tended to insist that the two expression $x+x+x+x=x$ and $x+x+x=12$ were unrelated, and that they cannot be considered at the same time. The following extract illustrates the point.

(Ba; 8th grader at junior high.)
Ba : I feel quite OK. When the different numbers are substituted for xs in $x+x+x=12$, but the other one $x+x+x+x=x$, x stands for the same, then $x+x+x+x=x$ should become $x+x+x+x=4x$.

Students' Reaction To the Counter-example.
According to the present curriculum in Japan, it is rather difficult to find appropriate counter-examples for elementary school students concerning the convention that the same letter stands for the same number. On the other hand, we can prepare a counter-example for junior high students such as simplifying the algebraic expression or solving linear equations as shown above. Nevertheless, in the interview, these counter-example tasks seemed to be not so relevant for students to consider the convention. Students tended not to relate the task in the interview with the counter-example. The following extract illustrates the point.

(Bs : 8th grader at junior high.) After confirming that x+x+x=12 became 3x=12.
I : Yet, Bs-san, your interpretation was 2+5+5=12 on the (preliminary survey) task. Then (at the counter-example) in this case, if x is 2,5,5, then what would happen ?
Bs : This one (equation x+x+x=12) is an ordinary expression, but that is a substitution problem x stands for 2, and, x would not be considered any more, then 2+5+5 became 12. On the other hand, this one (equation) is an ordinary expression, we do not substitute any number for x, merely x, then we get 3x. ---(an omission)-- There are many ways of thinking, that is, the equation is to be considered in terms of this kind of thought, and the substitution problem we use a different way of thinking. If someone decides to do so, that is a way to do.

Bs could be a typical example of someone who tends to think that manipulating literal symbols may differ between contexts: substitution and equation. Her responses were consistent with the convention that the same letter stands for the same number in the equation context but in the substitution context she supported the negative of that proposition. Asking her, then, about a foundation for simplifying the algebraic expression by manipulation the similar terms, she explained as follows :

I: Why does y+y become 2y ?
Bs : The y is 1y, 1 is omitted in writing, so it really is 1y+1y is 2y.
I: Well, I am asking why 1y+1y is 2y ?
Bs : We don't write as 1y, 1 is omitted. We only calculate between numbers, but not these letters, you don't care how many there are, no matter how many letters, you just let them be.

For Bs, the foundation for simplifying the similar terms is a very procedural oriented rule, it is not related to the convention that the same letter stands for the same number at all. A similar tendency was revealed in the interviews with other students. This tendency implies that they are good at solving procedural-oriented tasks, but they would not realize the reason why the procedural rule works.

Discussion

Relation between the Concept of Variable and Misconceptions.
The concept of variable has two aspects : "Definite" and "Unspecified" aspects. Although these two aspects seem to have a conflicting nature as Van Engen(1961) already pointed out, the concept of variable could be formed through dissolving this conflict or integrating these inconsistent concepts. The definite aspect of concept of variable is materialized in the convention, that the same letter stands for the same number. Students' misconceptions described as "X can be any number" emphasizes only the unspecified aspect of a variable. This misconception would not be revealed in the expression which contains only one literal symbol. But in the case of an expression containing more than one literal symbol, this misconception will be materialized. Students' reaction saying $x+x+x+x=x$ is correct, and interpreting $x+x+x=12$ as 2+5+5=12 are a result of considering only the unspecified aspect of the concept of variable. The misconception which lacks the definite aspect of the concept of variable is also revealed at the tertiary education level. Koseki (1988) reported that university students could not identify the incorrectness of the expression : $(2n+1)+(2n+1)=2(2n+1)$ as a mathematical expression of "An odd number plus an odd number equals an even number".

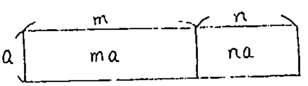
On the other hand, the misconception : that different letters stands for the different numbers, could be characterized as a lack of sufficient understanding of the unspecified aspect of variable, in a sense that students persistently reject substituting the same number for different literal symbols. Although the domain of variable does not depend on the literal symbol itself, the interview revealed that students tend to relate the surface character of literal symbols, such as differences in letter, with the domain of variable.

Source of Misconceptions
The *empty boxes* of pre-algebra conceptions play a significant role as a bridge between number and algebra. In Japan, the empty boxes are introduced in the 3rd grade, and literal symbols in the 5th grade in elementary school. Then in junior high school, algebraic expressions are studied at some length, although the algebra does not exist as an independent subject in school mathematics in Japan : it is systematically included in various

parts of the mathematics curriculum. This carefully planned curriculum must surely be a good foundation for Japanese students to understand literal symbols in algebra. Nevertheless, it also seems to cause the Japanese students' some misconceptions, such as, the same letter does not necessarily stand for the same number. Because the empty boxes' conventions are slightly different from those of literal symbols. In fact, students may interpret the empty boxes as places to fill in answers with emphasis on their *unspecified* nature. The discrepancy between the conventions of literal symbols in algebra and of the empty boxes may become a source of misconceptions.

Although it is difficult to find an association between the literal symbols of algebra with Japanese orthography : Hiragana and Katakana, it is interesting to note here that these literal symbols in algebra may be thought of as Japanese ideographs : Kanji characters. First, the way to introduce a new literal symbol in school mathematics, in 5th grade in elementary school, is exactly the same as the way to introduce a new Kanji. That is, students learn how to pronounce the symbol, how to write it, then its meaning : x means an unknown number or quantity. In that context, the *unspecified* aspect is likely to be emphasized, and may create a source of the misconception.

Beside identifying the source of the misconceptions, we also have to consider and identify the teaching materials which may play a role as counter-examples against the misconception, and place emphasis on the *definite* aspect of a variable. The *definite* aspect of a variable may be revealed in operating mathematical expressions, such as in solving equations or simultaneous equations. Unfortunately, these materials have not been considered in that context. Rather, they are taught in a procedural-oriented way, or presuppose the *definite* aspect implicitly. For instance, in a textbook, a foundation of simplifying similar terms is concealed in the geometrical figure itself without considering the *definite* aspect of the concept of variable (see, Fig.1).



(Fig 1)

Phenomenon of Compartmentalization

By analyzing students' way of resolving inconsistencies or contradictions, the compartmentalization phenomenon in organizing the two conflicting knowledge are identified. This may be associated with the nature of the variety of meanings of a variable depending on the contexts, such as specific unknown, generalized number, constant number, variable and so on. Even though students may not be efficient enough to deal with these different and also somehow conflicting meanings of literal symbols in each context, we surely need to study this aspect since there would be a gap between the teacher's expectations and the students' reality. In fact, the interview revealed the students' belief system of the nature of mathematics, that is, students seem to tolerate accepting contradictions or inconsistencies in their mathematical knowledge.

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SEEING A PATTERN AND WRITING A RULE

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One route to algebra, currently recommended in Australia and elsewhere, is via the investigation of geometric patterns, number sequences and function tables. This paper discusses difficulties experienced by 14- and 15-year-old students when interpreting function tables and formulating algebraic rules for generating values of the variables. Written tests were given to 143 students who were in their third year of learning algebra. A further 15 students were interviewed individually. The data show that more students could find and use a relationship for calculating than could describe it verbally or algebraically. Major causes of difficulty in formulating algebraic rules were (i) focussing on recurrence patterns in one variable rather than on relationships linking two variables; (ii) inability to articulate clearly the structure of a pattern or relationship using ordinary language.

The *National Statement on Mathematics for Australian Schools* (Australian Education Council, 1991) and the *Curriculum and Evaluation Standards* (National Council of Teachers of Mathematics, 1989) recommend that algebra learning should be based on the study of patterns. In this paper we report a study involving students who were tested on their success in recognizing number patterns from tables of ordered pairs and describing them algebraically. (Examples of tables are shown in Fig. 1). Herscovics (1989), commenting on the results of national testing in USA, points out that although most students could recognize a simple pattern in a table of ordered pairs connected by a simple relationship (e.g., add 7) the majority were unable to generate the corresponding algebraic rule (e.g., $y = x + 7$). Other studies (Arzarello, 1991; Pegg & Redden, 1990) have also shown that generating algebraic rules from patterns and tables is difficult. The ability to perceive a relationship and then formulate it algebraically is fundamental to being able to use algebra. Some of the reasons why students find it a difficult task are presented in this paper. We conclude with some suggestions for teaching in the light of new curriculum approaches which are being promoted in response to the *National Statement.*

Arzarello (1991) found that when students had generated a table of ordered pairs from a geometric pattern, they focused on the difference between successive values of each variable. They searched for a recurrence rule that would predict a number from the value of its predecessor rather than a functional relationship linking pairs of numbers. For example, in Item C (see Fig. 1) many students may notice that x goes up by 1 and y goes up by 3, but be unaware of the functional relationship that links the two variables (i.e., $y = 3x + 2$). A similar observation has been made by Clement, Narode and Rosnick (1981), MacGregor (1991) and Pegg et al. (1990).

In this paper we discuss students' responses to test items designed to find out what aspects of the task of recognizing patterns and describing them algebraically present most difficulty; in particular,

(i) what patterns and relationships students perceive in function tables;

(ii) the association between being able to work numerically with a relationship and being able to express it clearly in either natural language or algebraic notation.

The results of our study show that whereas most students can perceive patterns in tables easily, many do not perceive the functional relationship. Even amongst those who "see" the functional relationship sufficiently clearly to calculate with it, a substantial proportion cannot express it in natural language. Most

students who cannot express it in natural language also cannot write the relationship in the symbolic code of elementary algebra. These findings have important implications for the teaching of beginning algebra.

Procedure

Pencil-and-paper tests were given to all 143 students in year 9 (approximate age 14) at a school in a low socio-economic area of Melbourne, Australia. These students were taking the third year of an integrated mathematics course which includes some algebra in each year. The textbook they used introduced algebra as a way of expressing relationships between quantities described by tables and spatial designs. Fifteen students at a similar school were interviewed and audio-taped while they worked on test items. In this paper we discuss the students' responses to the three items shown in Figure 1. Item A involves addition, item B multiplication, and item C both multiplication and addition.

A. Look at the numbe· in this table and answer the questions.

x	y
1	5
2	6
3	7
4	8
5	9
6	..
7	11
8	..
..	..

(i) When x is 2, what is y? (ii) When x is 8, what is y? (iii) When x is 800, what is y?
(iv) Describe in words how you would find y if you were told what x is.
(v) Use algebra to write a rule connecting x and y.

B. The results of an experiment that measured two quantities L and Q were.

L	Q
3	9
5	15
9	27
21	63

(i) What would you expect Q to be when L is 30? (ii) What would L be when Q is 99?
(iii) Describe in words how you would find Q if you were told what L is.
(iv) Use algebra to write a rule connecting L and Q.

C. Look at the numbers in this table and answer the questions.

x	0	1	2	3	4	5	6
y	2	5	8	11	14	17	..

(i) When x is 6, what is y? (ii) When x is 10, what is y? (iii) When x is 100, what is y?
(iv) Describe in words how you would find y if you were told what x is.
(v) Use algebra to write a rule connecting x and y.

Fig. 1. Three test items

Results

Complete results for each part of the three items may be seen in Tables 1, 2 and 3. Almost all students could read tables correctly (evidenced by A(i)) and extend them (evidenced by A(ii) and C(i)). Interview data suggest that these answers were usually given from the recurrence relation, that is, as you move down (or along) the table, the value of y goes up by 1 (or 3). In the written test, many of the incorrect answers to C(ii) were 29, 31 and 33 which probably also result from incorrectly using the recurrence relation, for example by adding 3 the wrong number of times or making arithmetic errors. When use of a functional relationship was essential because the recurrence relation was impractical, as in A(iii), B(i) and C(iii)), success rates were very much lower. For all three items, there were students who had perceived the functional relationship sufficiently clearly to use it for calculating but who could not describe their rule in words (A(iv), B(iii), C(iv)) or write it algebraically (A(v), B(iv), C(v)).

As shown in the bottom row of Tables 1, 2 and 3, for the two easier items (A and B) approximately one-third of the students wrote a correct algebraic rule, one-third were wrong, and one-third made no attempt. Item C, which combines addition and multiplication, was more difficult.

Table 1
Responses for item A (n = 143)

Task	Correct	Wrong	Omit
(i) Read table	139 (97%)	1 (1%)	3 (2%)
(ii) Extend table	138 (97%)	2 (1%)	3 (2%)
(iii) Calculate (hard)	88 (61%)	47 (33%)	8 (6%)
(iv) Use words	70 (49%)	48 (34%)	25 (17%)
(v) Write formula	50 (35%)	46 (32%)	47 (33%)

Table 2
Responses for item B (n = 143)

Task	Correct	Wrong	Omit
(i) Calculate (direct)	90 (63%)	17 (12%)	36 (25%)
(ii) Calculate (reverse)	66 (46%)	36 (25%)	41 (29%)
(iii) Use words	78 (55%)	20 (14%)	45 (31%)
(iv) Write formula	53 (37%)	38 (27%)	52 (36%)

Table 3
Responses for item C (n = 143)

Task	Correct	Wrong	Omit
(i) Extend table	127 (89%)	5 (3%)	11 (8%)
(ii) Calculate (easy)	96 (67%)	35 (25%)	12 (8%)
(iii) Calculate (hard)	27 (19%)	77 (54%)	39 (27%)
(iv) Use words	24 (17%)	51 (36%)	68 (47%)
(v) Write formula	17 (12%)	50 (35%)	76 (53%)

Perceived patterns and relationships

The interview data show that students easily observed patterns in the tables, but only some of these patterns were useful for other parts of the questions. For example, during her oral response to Item A, Sarah (Year 10) said "x it starts at 1, 2, 3, 4 and builds up . . . it's going in order" and "the odd numbers are with odd numbers and the even are with even". Michael (Year 7) added the numbers in each row and noticed a recurrence pattern in the totals, saying "When you're adding both numbers they are always increasing, like 1 and 5 is 6, 2 and 6 is 8, 3 and 7 is 10, ... each numbers go up by 2". One of the most striking findings from both the written testing and the interviews was the variety of patterns perceived and the large proportion of generalisations expressed verbally that cannot be expressed in the elementary algebra that students are learning.

Both the interview data and the high success rates for extending tables (A and C) indicate that most students easily perceived the recurrence relations, either as a link between the two variables ("as x goes up by 1, and after 5 so does y") or as two separate sequences (e.g. "x starts at 1 and goes to 8, and y starts at 5 and goes to 11"). However we were surprised at the many immature ways in which simple addition was expressed, for example (item A) "In between x and y there is four", "There's three numbers missing", and "You plus x all the time".

Students who perceived only recurrence patterns in A and C would have difficulty with B, where there are no constant differences to be seen in the table. This prediction was confirmed by the relatively high omission rate for B(i) (25%, compared with 6% and 8% on similar parts of A and C) which indicates that many students had not seen the very simple relationship "3 times". During the interviews we noticed that students took a little longer to recognize this rule, but once found it was used easily, correctly and with confidence for calculating further values. However one of the 15 students interviewed seemed unable to see a link between columns in B, in spite of strong hints from the interviewer, and persisted in looking for a recurrence rule. One other student interviewed was severely handicapped by lack of number sense. Even with considerable prompting, she could only describe the relationship in item B as "the x is a bit less than half the y", when in fact x is exactly one-third of y.

The success rates for A(iii) and B(i), in which students needed to use the functional relationship rather than a simple counting procedure, are 61% and 63% respectively. We conclude that approximately 40% of the students in the sample were unable to recognize in table format the simple relationships "y is 4 more than x" and "Q equals 3 times L".

Methods for calculating further values

The written responses from the main sample and the interview sessions both show that some students had used one rule for simple calculations and another rule for larger values of the variables. For example a student described the rule for item A as "If there is one x there will be three numbers and then y". He used this rule successfully for simple calculations. However for A (iii) (find y when $x = 800$) he decided to multiply 800 by 5, and explained that he had gone back to the first row where x is 1 and y is 5. Another very common wrong answer for A(iii) was 1200, also based on direct proportion, obtained by multiplying the correct y value for $x = 8$ by 100. The incorrect use of a simple direct proportion when

large numbers were involved was common for items A and C, and has been reported in other contexts (Stacey, 1992). One student justified the answer 1200 for A(iii) by writing the explanation "You add 4 if it is under 100 and if it is over 100 you add 400". It is likely that many students do not clearly understand that they are not applying the same rule to larger numbers, and think that by using direct proportion they are merely taking a shortcut to the answer .

Writing an algebraic rule

We had expected that students would find it hard to write a rule for C, both in words and algebraically. However it is disturbing to see how many students could not use algebra to express the simple relationships in A and B. In many cases the students' verbal descriptions were not helpful as a basis for clear thinking about the function and its algebraic representation. For example, explanations of A included "Every y number is 4 times larger", "x equals three digits y", "Minus 4 from each", "You count four places", and "With the x, for the y, you put 4 on it". Attempts at algebraic rules for A included the following: $x + 4y$ $\quad x = y + 4$ $\quad x + y$ $\quad xy + 4$ $\quad x = 1y$ $\quad x + 5$ $\quad x = 1, y = 4$

The rule for item C contains two operations: multiply by 3 and add 2. Success in writing the rule should depend on recognizing each of these operations and writing them in the two earlier items A and B. Of the 62 students who attempted to write algebraic rules for A and C, 31 were wrong for A and with the exception of two individuals, were also wrong for C. Students who were correct for A had an even chance (16:15) of being correct for C. Of the 59 students who attempted to write algebraic rules B and C, 26 were wrong for B and again with the exception of two individuals were also wrong for C. Students who were correct for B had an even chance (18:15) of being correct on C.

Association between working numerically and expressing relationships

As would be expected, the interview data and statistical tests on the written responses confirm that, for the sample as a whole, there was a strong association between difficulties in correctly calculating with the pattern and incorrect or incoherent verbal expressions and incorrect algebraic formulations. Of special interest is the subset of students who demonstrated that they understood the functional relationships in the tables by successfully calculating new values as required by A(iii), B(i) or C(iii). By this measure these students "knew" what the relationships were. Were they also able to articulate them clearly and to express them algebraically? For many students, this step was the stumbling block. Two examples of students who could calculate A(iii) correctly but not express their procedures algebraically are shown below.

Sarah (Year 10) calculated correctly that if $x = 100$ then $y = 104$, and if $x = 1000$ then $y = 1004$. When the interviewer asked for an explanation of a general rule, Sarah offered the following five alternatives:

"The x value is lower than the y by 4".

"It goes up 4 for every x that they have".

"Whatever number you have for the y, you add 4 onto it".

"For the y, it is four numbers higher than the x".

[writes $x =$ [illegible]] "With the x, for the y, you put 4 on it". (For a discussion of the reasoning behind this form of reversed equation, see MacGregor, 1991; MacGregor & Stacey, in press a)

After many prompts from the interviewer, such as "You know how to get y from x. Could you write a rule that starts with y?", Sarah finally wrote $y = x4$.

Roberto (Year 10) used incorrect direct proportion rules for large number calculations but was able to use the "add 4" rule for small numbers without being able to say what he was doing. In the interview he was constantly reminded to look for a simple way of saying his rule, but his only explanations were:

"With the x and y numbers there is always *four* [later corrected to *three*] numbers in between".

"If there is x there will be three numbers and then y".

"Whatever the x is, the y must be three, four digits".

[writes $x = 3y$] "x equals three digits y".

The fact that many students doing the written test could calculate values but not write explanations or algebraic equations indicates that for them, as for Sarah and Roberto, recognizing and articulating the structure of the relationship was the stumbling block. Figure 2 shows that, of those who did understand a relationship well enough to do the difficult calculation, only three-quarters could explain it clearly and only one-half could write it algebraically.

The data in Fig.2 refer only to the students who were correct on the difficult calculations in items A, B and C. Note that since there were many incorrect or omitted answers to A(iii), B(i) and C(iii), the total of responses for the 143 students on three items is only 203, less than the possible total of 429. Verbal descriptions were marked correct if they included evidence that the student had seen the functional relationship. For example in item A, "You always add 4" and "The numbers are 4 differents" were marked correct, whereas "Just add 1" and "Find a pattern for y and time it by x" were marked wrong. As may be seen in Figure 2, one-quarter of the students who were able to calculate new values correctly were not able to describe verbally the relationship they had used. This was despite the fact that two of the three articulations required were extremely simple (i.e., "add 4 to x" and "multiply x by 3").

		Algebraic rule		
		Correct	Not correct	Total
Verbal description	Correct	92	60	152
	Not correct	13	38	51
	Total	105	98	203

Fig. 2. Association of success on verbal descriptions and algebraic rules for students who could calculate successfully

Figure 2 shows that students who gave a correct verbal description were more likely than other students to write a correct algebraic rule. Of those students who did not articulate their patterns clearly, one-quarter were able to write a correct algebraic rule. A strong and highly significant association between performance on the verbal and algebraic tasks is evident. $\chi^2 (1, N = 203) = 18.77, p<0.0001$.

Conclusions and Implications

The route from perceiving a pattern to writing an algebraic rule is complex. There are many critical steps along the way when students may lack necessary skills and knowledge or make a wrong decision. We have shown that some of these critical steps in moving from a function table to an algebraic rule are:

• looking beyond recurrence patterns and finding a relationship linking the two variables;

• being able to formally articulate the relationship used for calculating numerical values (e.g., being able to say "Add 4" rather than "You count four places");

• knowing what can and cannot be said in elementary algebra (e.g., "Every time x goes up by 1, y goes up by 4" cannot be easily translated to an equation);

• knowing the syntax of algebra (e.g., "$x = y4$" does not mean "Start with x, and add 4 to get y")

Students who could not perform the difficult calculations had seen a variety of patterns, particularly the recurrence relationships. Many of these patterns, although valid, are not helpful and do not lead to an idea that can easily be expressed with the algebra that students are learning. It seems to us to be important that teachers recognize that their students see many patterns. Students beginning algebra need to discuss why some of the patterns and relationships they see are more useful than others.

Some students in the study perceived a functional relationship in an immature way, which they could not express clearly (e.g. "x equals three digits y"). Our study shows that saying even a simple relationship such as "add 4" can be difficult. One essential prerequisite for using algebra is that students can put their informal arithmetic knowledge into a formal arithmetic structure - to know, for example, that doubling ("You plus it by the same number") is multiplication by 2.

Recommendations for teaching

A close look at the way in which current Australian textbooks implement the pattern-based approach to beginning algebra, as well as the suggestions in the *National Statement* and *Standards*, reveal different routes from seeing a pattern to deriving an algebraic rule. These differences may be of crucial importance for learners. In one common approach, students extend a geometric pattern, notice the recurrence patterns, make a table of values, and derive an algebraic rule that will produce the numbers in the table. In another approach, trialled by Pegg et al. (1990), students describe features of a geometric pattern orally and in written sentences, and then suitable verbal descriptions are expressed algebraically. Our findings suggest that the verbal description is an important and perhaps necessary part of the process of recognizing a function and expressing it algebraically. This suggestion is timely because new curriculum materials following the *National Statement* recommendations introduce algebra as a language for expressing relationships between two variables, often beginning with a geometric pattern or number sequence. As far as we know, no evaluation of the pattern-based approach has been carried out. Pegg et al. (1990) observed how the approach was used in four classes of 11-year olds, and concluded that it appeared to be successful and that it promoted a positive attitude to algebra. MacGregor and Stacey (in press b) tested 500 students in several schools who had learned algebra in different ways. Some had learned through a pattern-based approach whereas others had learned in a traditional way. The results

indicate that the pattern-based approach, as implemented in the schools taking part in the study, was no more effective than traditional approaches.

We suggest that spatial or numerical patterns where recurrence relationships are very evident, such as "you need three more matches each time you add a square" (Australian Education Council, 1991, p. 191) are probably not a good way to begin algebra. If the recurrence relation is the easiest to see, but does not lead to the algebra that teachers want, it is silly to pick examples which lend themselves to it (especially when this is done exclusively, as is the case in some textbooks). We recommend the frequent use of items such as B, set in a real context, where there is no recurrence pattern in the x values.

Finding and using relationships that fit data presented in tables offers an example of students' strengths in informal mathematics (e.g., seeing patterns and calculating with them) and their weaknesses in formal mathematics. Many students are able to calculate but cannot describe what they do. They can "work out answers" without being fully aware of the procedures they are using or the structures of the relationships involved. This conscious awareness of mathematical procedures and structures, and the ability to describe them verbally, is crucial for learning and using algebra.

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Towards an Algebraic Approach: The Role of Spreadsheets

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This paper presents the results of a Mexican/British collaborative pro; t which is investigating the ways in which students use a spreadsheet to solve algebra problems. Two groups of 14-15 year old students (one in Mexico and one in Britain) worked on a sequence of spreadsheet activities and were interviewed before and after the teaching sequence. The majority of the students in this study became successful at using a spreadsheet to solve algebra word problems. These students had all been relatively unsuccessful with mathematics and the majority could not solve the problems at the beginning of the study. The symbolic nature of the spreadsheet took on an important mediating role in this problem solving process. The students used their spreadsheet experiences to develop strategies for solving algebra word problems when working away from the computer. These strategies involved working from the unknown to the known. Working in this way is an important and difficult aspect of algebra and we suggest that the existence of these strategies could provide an important basis for developing algebraic methods

Introduction

This paper presents the results of a Mexican/British collaborative project which is investigating the ways in which students use a spreadsheet to solve algebra problems. In the first phase of the project we carried out a study with two groups (one in Mexico and one in Britain) of eight pre-algebra pupils (aged 10 11 years). Pupils engaged in a sequence of spreadsheet activities which focused on the ideas of function and inverse function, equivalent algebraic expressions and the solution of algebra story problems (Sutherland & Rojano, 1992). The results of this study showed that the majority of these 10 11 year old pupils did not spontaneously think in terms of a general algebraic object when first working in a spreadsheet environment. Their thinking was initially situated on the specific example with which they were working. In order to think algebraically pupils need to have access to thinking both with the specific situation and in general and the spreadsheet environment supported these pupils to move from focusing on a specific example to consideration of a general relationship. The spreadsheet also supported pupils to accept the algebraic idea of working with an unknown, an idea which most pupils find difficult (Filloy & Rojano, 1989). They used a spreadsheet cell to represent the unknown number and then, with the mouse or the arrow keys, expressed relationships in terms of this cell. Dealing with the unknown in this numeric way could be considered a beginning step in accepting the possibility of operating on a symbolic unknown in a traditional algebra situation.

This paper focuses on the second phase of our collaborative project in which we worked with two groups of eight 14-1: ear old pupils (one in Mexico and one in Britain). All these pupils had a history of being unsuccessful with school mathematics, the British pupils having very little previous experience of algebra and the Mexican pupils having negative previous experiences. Both groups of pupils engaged in a sequence of spreadsheet activities and were tested and interviewed individually before and after the teaching sequence. The spreadsheet activities and interviews were almost identical to those used in the study with pre-algebra pupils and in a later paper we shall compare the results of these two groups.

Theoretical Background

The theoretical rationale for the study was influenced by previous research on pupils' understanding of algebra (for example Booth, 1984; Filloy & Rojano, 1989; Kieran, 1981; Küchemann, 1981) and a consideration of the potential of computer-based environments for learning algebra. Results from work with computers often conflicts with established results on pupils' difficulties with traditional algebra, particularly with respect to pupils' understanding of literal symbols. For example in computer environments pupils accept the idea that a literal symbol represents a general number and readily use unclosed algebraic expressions (Sutherland, 1992).

There are a number of aspects of Vygotsky's work which influenced this research. Sign systems are considered to be mediators of action and these include "various systems for counting; algebraic symbol systems; works of art; writing schemes; diagrams, maps, and mechanical drawings, all sorts of conventional signs" (Vygotsky, 1962, pp 137). Vygotsky also stressed that mediational means are sociocultural in the sense that mediated action cannot be separated from the social milieu in which it is carried out. From the point of view of research on teaching and learning in the classroom this suggests that studies of pupils cannot be separated from influences such as the mathematics curriculum and culture.

Methodology and Teaching Sequence

The methodology used within the project was essentially longitudinal case study. Two groups of eight 14-15 year old pupils were studied simultaneously, one in Mexico and one in Britain. Both groups engaged in two blocks of spreadsheet activities over a period of 6 months working in pairs for approximately 12 hours of "hands on" computer time. The British and Mexican pupils worked as whole classes in a computer room during one of their mathematics lessons. The spreadsheet activities were very similar to those used in the pre-algebra pupil study and consisted of the following two blocks of activities (described in detail in Sutherland & Rojano 1993; Rojano & Sutherland, 1992).

Block 1-Introductory Problems Within this sequence pupils were introduced to the following spreadsheet and mathematical ideas; entering a rule; replicating a rule; function and inverse function; symbolising a general rule; decimal and negative numbers; equivalent algebraic expressions (for example 5n and 2n +3n). The problems were analysed so that mathematical and spreadsheet ideas were introduced simultaneously.

Block 2-Algebra Story Problems. Within this sequence pupils were introduced to solving algebra story problems in a spreadsheet by: representing the unknown with a spreadsheet cell; expressing the relationships within the problem in terms of this unknown; varying the unknown to find a solution.

Both groups of pupils engaged in pre and post interviews. The pre-interview items were similar to those used within the pre-algebra study (Sutherland & Rojano, 1992) and consisted of items on

expressing generality; symbolising a general relationship; expressing and manipulating the unknown; function and inverse function; solving algebra story problems[1].

Strategies for Solving Algebra Word Problems

In this section, we discuss the strategies used by the students when working on the algebra word problems. A fuller account of this study with 14-15 year old pupils is given in Rojano and Sutherland, 1992. We shall focus on two algebra word problems which were presented to the pupils in the pre-interview and the post-interview. Before discussing the results of the study we firstly discuss the ways pupils might solve these problems in both a paper and a spreadsheet setting by considering the following problem

Rectangular Field
The perimeter of a field measures 102 metres. The length of the field is twice as much as the width of the field. How much does the length of the field measure? How much does the width of the field measure?

An algebraic approach This problem could be solved on paper using the following algebraic approach which involves working from the unknown to the known.

$$
\begin{aligned}
\textit{Let the width of the field} &= X \textit{ metres.}\\
\textit{Let the length of the field} &= L \textit{ metres.}\\
\textit{Then } L &= 2X \text{ --- } (1)\\
\textit{and } 2L + 2X &= 102 \text{ --- } (2)\\
\textit{So by substituting } (1) \textit{ in } (2) &\\
4X + 2X &= 102\\
6X &= 102\\
X &= 17 \textit{ metres.}
\end{aligned}
$$

Many studies indicate that pupils are more likely to use a non-algebraic approach when solving this type of problem (Bernarz et al, 1992; Lins, 1992). Within this paper we shall discuss the two non-algebraic strategies which were used by students within our study

A whole/parts strategy illustrated by the way a pupil solved the rectangular field problem in the pre-interview *"I did 102 divided by 6 . . I just did two of the lengths to make it sensible .... I just thought there must be two of those in one length...... ."*. This solution involves working with a known whole (the perimeter) and dividing this into parts to find the unknown lengths of the side of the field. In some senses this method is in opposition to the algebraic approach which involves working with the unknown

[1] There were some differences between the way the test/interview was administered in Mexico and the Uk. The Uk students were first asked to solve the test independently as a whole group and away from the computer and were then interviewed on their responses to the test items. In the interview they were offered support if they were not able to answer a question alone If appropriate it was suggested that they use the spreadsheet to solve the problem. In Mexico the students were asked individually to solve the problems as they wished. After a while if a student could not make any attempt the interviewer asked him (or her) to use the spreadsheet If the student had solved the problem with paper he or she was then asked to solve it with a spreadsheet

lengths (called for example X and L) and expressing these unknowns in terms of the known perimeter (for further discussion of this strategy see Lins, 1992).

A systematic trial and error approach illustrated by the way a pupil solved the rectangular field problem in the pre-interview *"well I tried 40, it was 120.....so I knew it must be smaller than that...in the 30's...and when I tried 36 and it was 108...I knew it couldn't be 35 so it must be 34.......*"This strategy has also been identified by a number of researchers (for example Bernarz et al, 1992; Filloy & Rubio, 1991) and involves working from the unknown to the known.

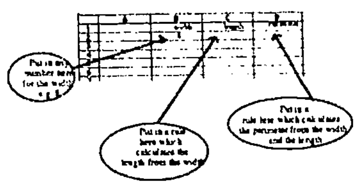
A *spreadsheet approach* Within this study the pupils were explicitly taught a method for solving the algebra word problems with a spreadsheet. This involved working from the unknown to the known. The unknown is represented by a spreadsheet cell. Other mathematical relationships are then expressed in terms of this unknown. When the problem has been expressed in the spreadsheet symbolic language pupils can then vary the unknown either by copying down the rules or by changing the number in the cell representing the unknown (see for example Fig. 1)

Rectangular Field

The perimeter of a rectangular field measures 102 metres.

The length of the field is twice as much as the width of the field

Use a spreadsheet to work out the width and the length of the field.



Then change the number for the width until you can answer the question.

Figure 1: Rectangular Field Problem

Table 1a presents an overview of the students' responses to the rectangular field problem in the pre and post interview and Table 1b presents an overview of the strategies used.

	Pre interview		Post Interview	
	Mex	Brit	Mex	Brit
Correct soln	1	0	6	8
Incorrect soln	[illegible]	5	0	0
No soln	3	3	1	0
Total	7	8	7	8

Table 1a Overview of pre and post interview results for the Rectangular Field Problem

	Pre interview		Post Interview	
	Mex	Brit	Mex	Brit
Paper-based				
Algebraic	0	0	0	0
Whole/parts	1	4	1	2
Trial & Error	3	1	2	2 (1)+
Spreadsheet	0	0	4	4
No soln	3	3	0	0
Total	7	8	7	8

+ bracketed no is no of students who used this approach before using spreadsheet

Table 1b Overview of pre and post interview strategies used by students when solving the Rectangular Field Problem

There was a marked improvement in the facility rate between the pre and post interview. The vast majority of pupils were able to obtain a correct solution in the post interview (14/15) compared to only 1 out of 15 pupils in the pre-interiew. This was the same for both Mexican and British students. In the pre-interview, only half of the students (in Britain and Mexico) attempted this problem and those who did used non-algebraic methods. Of these solutions the predominant approach in Mexico was a trial and error approach and the predominant approach in the UK was a whole/parts approach This could relate to the fact that the Mexican students had some previous experience of algebraic equation solving, whereas this was not the case for the British students. The trial and error method has some similarities with the algebraic method (Filloy & Rubio, 1991).

In the post-interview there was a marked difference in the approaches used by the students which can be summarised as:

- an increased level of awareness of all the relationships between unknowns and between givens and unknowns;
- the use of spreadsheet symbolism to express general relationships;
- the appearance of systematic trial and error strategies for the British students when working away from the computer;
- the integration of whole/part and trial and error non-algebraic methods with a spreadsheet method
- an evolution towards a "more general and algebraic method" consisting of proceeding from the unknown to the known.

These changes relate very much to the experiences the students had with the spreadsheet. For example Gisel, a Mexican student, who could not answer the rectangular field problem in the pre-interview, used a step by step approach in the post interview. That is, she worked the problem out with the semiperimeter (Width+Length) and expressed this in the spreadsheet as A1+B1 where B1=2*A1. She was able to complete the solution process with support from the researcher. In this case, it was crucial for Gisel to have tried specific values for the width in the semiperimeter formula which provoked her to modify this formula to the correct perimeter formula (2*(A1+B1)). Symbolising and varying one of the unknowns (i.e. proceeding from the unknown to the known) played a crucial role in the solution process.

Edgar used a trial and error approach, showing that he was able to deal with the relationships between unknowns and data as well as being very good at estimating. It was only at the end of this mental solving process that Edgar expressed the perimeter formula in the spreadsheet, just to verify his tentative answers. He only tried three values for the width when working at the computer (one greater and one less than the correct one). Carla, one of the British students used a trial and error in the pre-interview. She wrote down 50cm and 1cm and although she knew that this was incorrect she was not able to refine her solution. In the post interview she was able to successfully use a trial and error strategy (Fig. 2) when working on paper although she said *"it was difficult... it says the length is twice the width so you find the length first .. half that and give it width so I just kept trying..."*

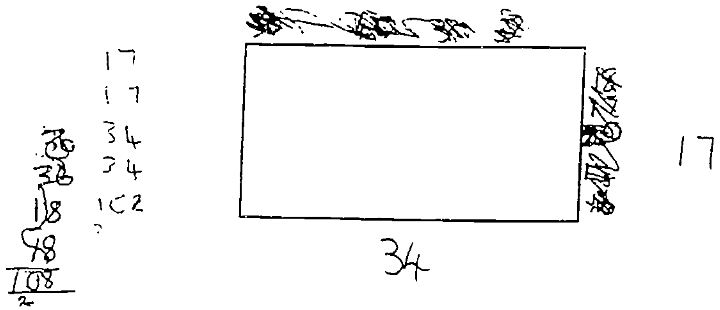


Fig. 2 Carla's solution to the Rectangular Field Problem in the post interview

The Chocolates Problem

Chocolates Problem
100 chocolates were distributed between three groups of children. The second group received 4 times the chocolates given to the first group. The third group received 10 chocolates more than the second group. How many chocolates did the first, the second and the third group receive?

Table 2a shows that approximately half of the students (in Britain and Mexico) tried to solve this problem in the pre interview. Trial & error and whole/parts were the strategies used in most of the cases. Not taking into account the relationships between the unknowns led some children to incorrect answers. Some of them applied an equal share whole/parts strategy and others used an unequal share strategy without the restrictions of the problem statement.

	Pre interview		Post Interview	
	Mex	Brit	Mex	Brit
Correct soln	2	1	7	8
Incorrect soln	1	3	0	0
No soln	4	4	0	0
Total	7	8	7	8

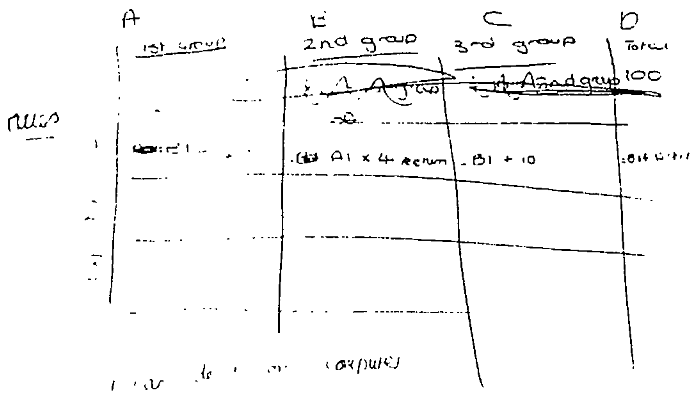
Table 2a Overview of pre and post interview results for Chocolates Problem

	Pre interview		Post Interview	
	Mex	Brit	Mex	Brit
Paper-based				
Algebraic	0	0	0	0
Whole/parts	0	2	0	0
Trial & Error	3	2	2	2 (2)+
Spreadsheet	0	0	5	6
No soln	4	4	0	0
Total	7	8	7	8

+ bracketed no is no of students who used this approach before using spreadsheet

Table 2a Overview of pre and post interview strategies used by students when solving the Chocolates Problem.

As with the Rectangular Field problem there was a marked improvement in the facility rate between the pre and post interview. In the post-interview almost all of the pupils could symbolise in the spreadsheet the relationships present in the problem and used such expressions to find out the solution to the problem proceeding from the unknown to the known (i.e. by means of varying one of the unknowns). Jo a British student was not able to solve this problem in the pre-interview but her solution in the post interview (with no computer present) illustrates the way in which the spreadsheet code was beginning to play a role in her thinking processes



She had drawn a spreadsheet on paper to support her solution and had correctly written down all the rules represented in the problem On paper she had not specified the unknown but when she worked on the spreadsheet the "circular reference" error message provided feedback on this error In the post interview Jo was asked "If *we call this cell X what could you write down for the number of chocolates in the other groups*" and she wrote down:

= X = X x 4 = X x 4 + 10

A number of the other students were able to use their spreadsheet experiences to represent the word problems with algebraic symbolism and we shall be focusing on this transition in our ongoing work.

Making a decision about which of the three unknowns to vary represented a difficulty for some students They needed support to realise that once they had already expressed the relations in the spreadsheet, the only cell to be varied was the one that did not have a formula in. This difficulty did not arise in the rectangular field problem; a plausible explanation for this could be that in the chocolates problem, the number of the unknowns was increased.

Some Concluding Remarks

The majority of the students in this study became successful at using a spreadsheet to solve algebra word problems similar to the ones described in this paper. These students had all been relatively unsuccessful with mathematics and the majority could no' solve the word problems at the beginning of the study. The symbolic nature of the spreadsheet took on an important mediating role in this problem solving process. The taught spreadsheet approach involved working from the unknown to the known and this is also a characteristic of the trial and error approach to solving word problems. A number of students developed this trial and error approach for solving algebra word problems when working away from the computer. Working from the unknown to the known is an important and difficult aspect of algebra and we suggest that the spreadsheet activities could provide a basis for developing algebraic methods. We recognise that we have only worked with a limited set of problems and that we still need to investigate the links between a spreadsheet approach and an algebraic approach. We shall be working in this direction in our ongoing work.

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TASK-BASED INTERVIEWS FOR A LONGITUDINAL STUDY OF CHILDREN'S MATHEMATICAL DEVELOPMENT*

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An initial group of 22 elementary school children from schools participating in an intensive, constructivist-oriented mathematics project are being studied for a three-year period. Sources of data include videotapes of students' individual problem solving and small-group mathematical activity, both in and outside of class. This paper discusses the development of individual task-based structured clinical interviews, two per year, designed to elicit a rich variety of problem-solving behavior. The goal is to be able to observe the growth of the children's internal representational capabilities in some detail. We describe the underlying principles and structure of the interviews, the design of the first two interview scripts, the preparation of clinicians, and some highlights of early outcomes for illustrative purposes.

Mathematics education has come to focus less on procedural and algorithmic learning, and much more directly on conceptual understanding, complex problem solving processes, and children's internal constructions of mathematical meaning (Davis, Maher, and Noddings, 1990). Reforms in school mathematics endeavor to encourage guided explorations by children, small-group problem solving, and the discovery of patterns and ways of reasoning about them, in place of teacher-centered, direct instruction. In this context it becomes increasingly important to be able to *describe* and *assess* the mathematical development of individual children over time; to find ways of observing that permit inferences about the deeper understandings we seek to achieve through these emphases (Lesh and Lamon, 1992). The present report describes one component of an exploratory, in-depth, descriptive study being conducted at Rutgers University toward this end.

An initial group of 22 students are being observed over a period of three years; it is expected that at least half will remain for the full term of the longitudinal study. Observation began during the 1991-92 school year, with children in grades 3 and 4 from a cross-section of New Jersey communities: two urban schools (5 third-graders and 4 fourth-graders); one school in a predominantly blue-collar, "working class" community (7 fourth-graders); and one in a suburban, "upper middle-class" district (6 third-graders). These schools are already participating in an intensive, con-

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structivist-oriented mathematics teacher development project (MaPS: Mathematics Projects in Schools), directed by Carolyn Maher and Robert Davis. One reason for the study is to be able to assess some of the project outcomes in relation to individual children's mathematical understandings as they grow over time. Sources of data are to include videotapes of the children's individual problem solving, as well as their small-group mathematical activity both in and outside of class.

This paper discusses the challenge of developing individual, structured, task-based clinical interviews for the study--two per year, to be conducted with each child.

Theoretical Basis and Goals of the Task-Based Interviews

Our broad goal is to be able to describe in as much detail as possible the growth of the children's internal representational capabilities in mathematical contexts. The tasks must therefore be able to elicit a sufficiently rich variety of problem-solving behavior to permit inferences about these capabilities. While some psychologists have proposed cognitive models based exclusively on propositional encoding, or combining propositional encoding and visual imagery (so-called "dual code" models), or including an emotional code as a third type of representation, we have in mind a descriptive assessment based on the model of Goldin for problem-solving competency structures (1987, 1988a,b). This model involves the following kinds of internal representational systems, all regarded as equally fundamental to the psychology of mathematics education [some related inferences we hope to make from the interviews are noted in brackets]: (1) *verbal/syntactic* systems [use of mathematical vocabulary, developing precision of language, and self-reflective (metacognitive) descriptions]; (2) *imagistic* systems [evidence for non-verbal, non-notational mental models embodying mathematical understandings (such as visual and kinesthetic representation in two and three dimensions), uses of external concrete representations, gestures in description]; (3) *formal notational* systems [uses of notation and underlying concepts, relation of notation to conceptual understanding, evidence of the creation of new notations]; (4) a system of *planning, monitoring, and executive control* [spontaneous and/or prompted use of trial and error, special cases, charts, or other heuristic plans and devices, and outcomes of strategy use]; and (5) a system of *affective* representation [not only general "global affect" toward mathematics, but changing "local affect" during problem solving, critical emotional events, interactions between affect and executive decisions, and related belief systems].

As this is a longitudinal study, we are especially interested in how systems of representation *develop* over time, and in changing student conceptions and misconceptions. As we study this systematically, looking for specific, anticipated processes such as those indicated, we are also conducting a continuing search for *unanticipated* events whose fine details we hope to describe.

There are inherent limitations to the process of inferring internal cognition and affect from behavior. Systems of internal representation are conjectured, context-dependent constructs; they are one possible (not necessarily unique) way of understanding observed mathematical behavior. Some essential ambiguity is present, even when the competencies described are fairly well-defined. Affect

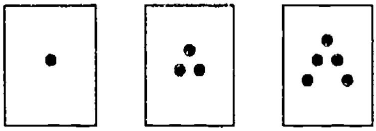
is inferred, with limited reliability, from external indicators such as facial expressions and tone of voice. Our wish to be open to new possibilities also means that we cannot define *a priori* in too definite a way how particular behaviors will be interpreted. In this sense we are conducting an entirely *exploratory* study. However an essential feature for scientific progress is that our structured interview scripts permit a degree of *comparability and reproducibility:* other researchers can carry out "the same" interviews with other children, and our tentative methods of inference can be made explicit, explored, refined and improved, or rejected.

Principles and Structure of the Interview Scripts

In this section we describe the underlying principles and structure of the interviews, the design of the first two interview scripts, and the preparation of clinicians. Though each interview has its own particular focus, certain broad characteristics are maintained in all of them. These constitute a further development of the structure of clinical interview scripts described by Goldin (1985), Goldin and Landis (1985, 1986), and Bodner and Goldin (1991). They include the following: (1) Each interview is based on particular mathematical ideas appropriate for the age group of the children being interviewed (grades 3-6). The goal is to choose topics having associated meaningful, semantic structures as well as formal, symbolic structures: e.g., additive or multiplicative structures, schemata underlying rational number concepts, etc. We want topics that can be studied in depth, but are flexible enough to allow for evidence of widely differing capabilities on the part of the students. (2) Each interview consists of a series of questions, posed in one or more task contexts. The questions become increasing difficult, culminating in some that can be attempted by all the children but will pose major challenges even to those who are most mathematically astute. (3) The children are encouraged to engage in *free problem solving* to the maximum extent possible, exploring the strategies that they themselves bring to bear, using whatever method or methods seem most appropriate to them as they work on the task. They are reminded occasionally to talk aloud about what they are doing, and to describe what they are thinking. Hints and prompts, or new questions, are offered *only* after the opportunity for free problem solving, and are then followed by a further period of observing how the child responds without directive intervention. (4) All student efforts are "accepted" during the interview, without imposing preconceived notions about appropriate ways to solve the problem, and treating "wrong" answers similarly to "correct" answers (with occasional, specified exceptions). Responses elicit follow-up questions without indication of their correctness. The rare exceptions (guiding the students toward particular understandings), decided in advance, are made only where the understandings are essential for subsequent interview questions to be meaningful. (5) Multiple external representations are available for student use, varying from task to task: paper and pencil, markers, cards, chips and/or other manipulatives, paper cut-outs, computer terminals, etc. (6) Each interview includes questions about affect, and retrospective questions.

Task-Based Interview #1. The first interview script (about 45 minutes) was completed during the academic year 1991-92, and administered during May and June 1992. The task (motivated by a

National Assessment of Educational Progress problem designed for the high school level) involves laying out for the child three cards, one at a time, as follows: "Here is the first card, here is the second card, and here is the third card."



The cards are seen to come from a stack in an envelope, so that the child may infer from the context that there is a deck of cards larger than what has been shown, and (possibly, tacitly) that there is a pattern in the cards. After a brief pause to allow for spontaneous response/detection of a pattern, the child is asked, "What do you think would be on the next card?"

The materials placed ahead of time on the table in front of the child, within arm's reach, are: blank index cards (the same size as the ones with dots on them), magic markers of different colors, round chips (checkers) in red and black, a pad of paper, and a pencil. The child can select any or all.

After a complete, coherent response has been elicited, the child is similarly asked (in slow succession) the following questions: * What card do you think would follow that one? * Do you think this pattern keeps going? * How would you figure out what the 10th card would look like? * Here's a card (showing one with 17 dots, in the "inverted V" pattern). Can you make the card that comes *before* it? * How many dots would be on the 50th card?

For each question, the following stages exist: * Posing of the question (free problem solving). * Heuristic suggestions (if not spontaneously evident): e.g., "Can you show me using some of these materials?" * Guided use of heuristic suggestions: e.g., "Do you see a pattern in the cards?" * Exploratory (metacognitive) questions: e.g., "Do you think you could explain how you thought about the problem?" The clinician seeks always to elicit a complete, coherent verbal reason *and* a coherent external representation before proceeding (except that for the question about the 50th card, an external representation was not required). A complete, coherent reason means one based on a described pattern; this could take place verbally, with a drawing, with a chip model, etc. However, it is not required that the "canonical" card (e.g., the fourth card having 7 dots in an inverted V pattern) be drawn, or that the canonical pattern be described, in order for a response to be considered a complete coherent reason and a coherent external representation.

This "non-routine" task embodies an underlying, additive arithmetic structure, with opportunities for the child to detect numerical and/or visual patterns, to use visual, manipulative, and symbolic representations, and to demonstrate reversibility of thinking.

Task-Based Interview #2. This interview design was completed in the Fall of 1992. The script is (as this is written) being used in individual interviews administered during January 1993, with the same children (now in the fourth and fifth grades). As in the first interview a pad, a pencil, markers, and checkers are placed ahead of time on the table in front of the child, available for use. First, some preliminary questions are asked (to explore the child's imaginative and visual processes).

The child describes whether s/he is right- or left-handed. Then the child is asked to imagine a pumpkin, to describe it, to manipulate the image in various ways (including cutting the pumpkin in half), to spell the word "pumpkin", and to spell it backward. In slow succession, the child is then asked a series of questions: * "When you think of one half, what comes to mind?" * "When you think of one third, what comes to mind?" * "Suppose you had 12 apples. How would you take (one half)(one third)?" * [Next cut-outs are presented in succession: a square, a circle, and a 6-petal flower.] "How would you take (one half)(one third)?" * [Depictions are presented of a circle cut-out with one-half, one-third, and one-sixth represented conventionally (as in a pie graph), and unconventionally (the portion representing the fraction translated to the center of the circle).] "Can this card be understood to represent (one half)(one third)?" * [A 3-by-4 array of circles and 6-petal flowers is presented.] "How would you take (one half)(one third)?" For each question, the child is also asked, where appropriate, "Can you help me understand that better?" and "Are there any other ways to take (one half)(one third)?" * The child is then asked to write and interpret the (usual notation for) the fractions "one-half" and "one-third". * The child is shown a solid wooden cube and asked some preliminary questions about its characteristics (number of faces, edges, and corners), guided if necessary toward an understanding. Next the child is asked to think about cutting the cube in various ways: * "Now think about cutting this cube in half. What would the two halves look like?" * "Suppose we painted the cube red and then cut it the same way. How many faces are painted red, for the smaller pieces you told me about?" * Similar questions are then asked about cutting a series of up to 5 additional cubes, marked with lines at designated vertical and/or horizontal positions (resulting in pieces that are respectively 1//3, 1/4, 1/8, 1/9, and 1/27 of the original cube).

The script contains numerous suggested exploratory questions, and retrospective questions at two points points in the interview. It is designed especially to detect and explore in greater depth imagistic systems (visual/spatial and tactile/kinesthentic) in problem solving, while attending to affect and to other kinds of internal representation (see also McLeod and Adams, 1989; Goldin, 1992).

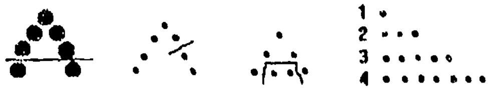
Plans for Successive Interviews. In the third interview we plan to focus more deeply on affect, in much the same way that the second one focuses on visualization and imagistic thinking. Similarly the fourth interview will focus on strategic thinking. In the fifth interview, we hope to include computer software designed for student use as a mathematical tool for problem solving. The sixth and final interview will return explicitly to selected mathematical ideas from the earlier interviews.

Preparation of Clinicians. The clinicians are graduate students working toward advanced degrees at Rutgers University, and one professor; all have teaching experience. The team members design and develop the interview scripts, practice by interviewing each other, and make preliminary script revisions. The scripts are next tried out with individual children (personal acquaintances of the clinicians), and revised a second time. Final revisions take place after a pilot study at different (urban) school, where each clinician conducts a taped interview. The clinicians discuss the pilot tapes, and evaluate critically each others' techniques.

Some Highlights of Early Outcomes

As this is written, the data analysis from task-based interview #1 is in progress, and interview #2 is being administered; thus it is too early to report overall findings (which extend well beyond the scope of a brief report). Some highlights of the early outcomes are included here for illustrative purposes.

We observed a rich variety of behaviors. Here are some examples of solution strategies employed by fourth-graders working on the problem from interview #1:



Otto **Marcia** **Diane** **Jacqueline**

(schematic representation of student productions)

"Otto" used checkers to describe adding two to the bottom of each pattern, as in the figure. (This was an anticipated solution strategy.) He later converted this visual idea to an arithmetic procedure of doubling the number on each side and then "adding one" to represent the one at the top. "Marcia" drew dot patterns, and also added two to each pattern; but the two she added were not at the bottom of the previous pattern as anticipated: instead she went up the left side, and then down the right side of each design. For example in drawing the fourth card (with 7 dots), she drew the bottom left dot, went up the left side and made the 4th dot at the apex of the drawing. Then she drew a 5th dot coming down the right side, and said, "then you add two more, like this." She then drew the 6th and 7th dots coming down the right side of the pattern , completing the expected design (but not in the expected fashion). Later, as she pondered cards with more dots, she changed her thinking more than once but retained the idea of a "left side" and a "right side", arriving at the idea that the number of the card was the number of dots on the left side, the number on the right was one less, and those two numbers were to be added. "Diane" revealed a perception of "triangularity"; when drawing the 4th card (with 7 dots), she first drew the five dots in the pattern of the 3rd card, and then squeezed two dots between the bottom dots of the pattern, forming an equilateral triangle. For later cards she avoided drawing triangles, but expressed an idea that two dots were to be added to each picture somehow forming a triangular design. (Other children "filled in" the inverted V with the additional dots.) "Jacqueline", in response to a heuristic suggestion that she make a chart, disregarded the presented pattern and created a chart with one dot in the first row, three dots in the second row, five in the third row, etc., continuing until she had completed it for the 10th card.

When asked to describe the 50th card, many children switched strategies from those they had used to that point, determined to find "an easier way." Several made charts of numbers; others began to multiply, referring to a lower card. For example "Myron" knew there were 19 dots on the 10th card, so he concluded that the 50th card would have 19 x 5 or 85. He later corrected his multiplication error, but clung to the strategy. We observed (among many other things) differences

in the mathematical lexicons used at different schools; one child in response to the heuristic suggestion to make a table, drew a picture of a table (the kind people sit at); in her school, the word "chart" had been used for this, but the synonym "table" had not. In general the students demonstrated a rich variety of problem solving behaviors. Some were more visually oriented, seeing geometric patterns in the cards. Others demonstrated more numerical approaches: adding two, focusing on odd numbers, doubling, or adding or subtracting one from a number that had been doubled. Most demonstrated reversibility in telling about the card that "comes before" the one shown.

Our success in observing a variety of behaviors and capabilities leads us to think that we will be able to describe in considerable detail the development of the children's representations.

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Count-on: The Parting of the Ways for Simple Arithmetic

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This paper considers the pivotal role that "count-on" plays in the development of qualitatively different forms of thinking in simple arithmetic. Viewed as a compression of the "count-all" procedure, "count-on" is seen as one example of a procedure that leads to a bifurcation in mathematical thinking between those who operate flexibly with procepts (in the sense of Gray & Tall, 1991) and those who use inflexible procedures. Qualitative and quantitative evidence is presented which exhibits the diverging strategies of children being tested in the early stages of the British National Curriculum in Mathematics (D.E.S., 1989). This shows that children can attain the same level and yet some may be operating in a different way which inhibits subsequent success.

Diverging approaches to basic arithmetic have been identified by a number of authors (e.g. Carpenter, Hiebert & Moser, 1981; Gray, 1991) and often these provide an indication as to whether the child operates flexibly with number or uses inflexible procedures. This paper considers a point in development when the parting of the ways may occur. Evidence is offered, both qualitative and quantitative, to suggest that a significant divergence occurs from the application of the "count-on" procedure for addition: it may simply be a procedure to solve any given addition problem, or a means through which new number knowledge is developed. The contention is that for the latter to happen the concept of number requires a flexible meaning. Gray & Tall (1991) used the word "procept" to describe a symbol such as 3 + 2 which could evoke both a process (of addition) and a concept (of sum). Number as a procept may be composed and decomposed in many flexible ways: 4 + 4 is 8, so 4 + 5 is "one more", which is 9. In the context of such flexible thinking, "count-on" may be viewed as a process to build up relationships between flexible number procepts. But if the thinking is only procedural, then "count-on" is simply a procedure to be carried out in time to solve a given addition problem with each problem seen in isolation. Subtraction problems are treated very differently. In proceptual terms, if 16 plus something is 18, then the "something" must be 2, so subtraction is another way of viewing addition facts. In procedural terms, the reverse of the "count-on" procedure is "count-back". Counting back 16 starting from 18 is a far more difficult operation to carry out. Therefore the difference between seeing "count-on" as a procedure and as a means of obtaining flexible number knowledge may be the parting of the ways between an increasingly inflexible method in which arithmetic becomes more and more difficult, and a flexible method which can lead to long-term success.

Retrieving the solution to number combinations

There appears to be general agreement that, over time, there is gradual change in the way many children handle basic number combinations. Although in essence this change is manifest by a decline in counting methods and an increase in fact retrieval methods, the issue of how responses are made to

basic number combinations is a contentious one which generates models that focus either on individual fact retrieval (e.g. Thorndike, 1922; Siegler & Shrager, 1984;), or on both individual fact retrieval and rule and procedure generated responses (e.g. Baroody & Ginsburg, 1986).

The place of counting in the development of number awareness is unquestionable, and it is common, certainly in England, for children to be exposed to a variety of number combinations in the belief that by solving them through counting they may 'understand' addition and subtraction. However, an implicit objective behind such exposure and practice is that children will learn the combinations they have practised. Ashcraft (1985) indicated that incidental learning would predict the eventual memory of most combinations but Steffe, Richard & von Glaserfeld (1981) point out that a problem with practice and exposure methods is that factual knowledge, viewed in the context of operations involving counting, would seem to involve reflective abstraction. Carpenter (1986) implicitly supports this notion by indicating that learned procedures may not ensure that related conceptual knowledge has been acquired.

The Compression of Counting Procedures.

This relationship between the ability to use a learned counting procedure such as "count-all", or a compressed procedure such as "count-on", and the ability to recall or "derive" the solution to a number combination, brings us face to face with the procedural/conceptual interface, an issue addressed by Hiebert & Lefevre (1986). For Gray & Tall (1991, 1992) such an interface is a cognitive manifestation of the characteristics inherent in mathematical symbolism. They analyse the various counting procedures in proceptual terms.

"Count-all" is seen as three counting procedures; count one set, count another, put the sets together and count that. Its inverse, "take away", carries out the reverse process of counting the whole set, counting a part to be removed and counting what is left. As "count-all" occurs in time as three distinct procedures, it is hypothesised that this will not easily lead to the development of flexible known facts.

"Count-on", although superficially a compression of "count-all", is a sophisticated double-counting in which 4 + 2 involves counting "five, six" at the same time as keeping track that two numbers have been counted. The inverse of "count-on" is "count-back", starting at the larger number and reciting the number sequence backwards to count off the number to be subtracted. Although "count-back" may be seen as a compression of the take away procedure, it also involves double counting but the counting goes in opposite directions: incrementing and decrementing in ones at each count. To do this children generally need some form of counting aid to maintain a check of the amount counted back. An alternative approach is "count-up" but this involves recognition of a proceptual link between addition and subtraction. The relationship between a number triple $5 - 3 = \square$ can be identified by incrementing using a procedure in the same way that $3 + \square = 5$ can be solved by incrementing with a procedure. Here, however, it is conjectured that a proceptual link has been recognised between the subtraction of three from five and the addition of something to the three to make five.

Count-on: The Parting of the Ways

It is conjectured that the use of "count-on", and its complementary procedures of "count-up" and "count-back", leads to a divergence between those who apply flexible thinking through the use of procepts and those who think procedurally. Such a divergence has been termed the proceptual divide (Gray & Tall, 1991). Figure 1 places "count-on" in a pivotal role prior to this parting of the ways.

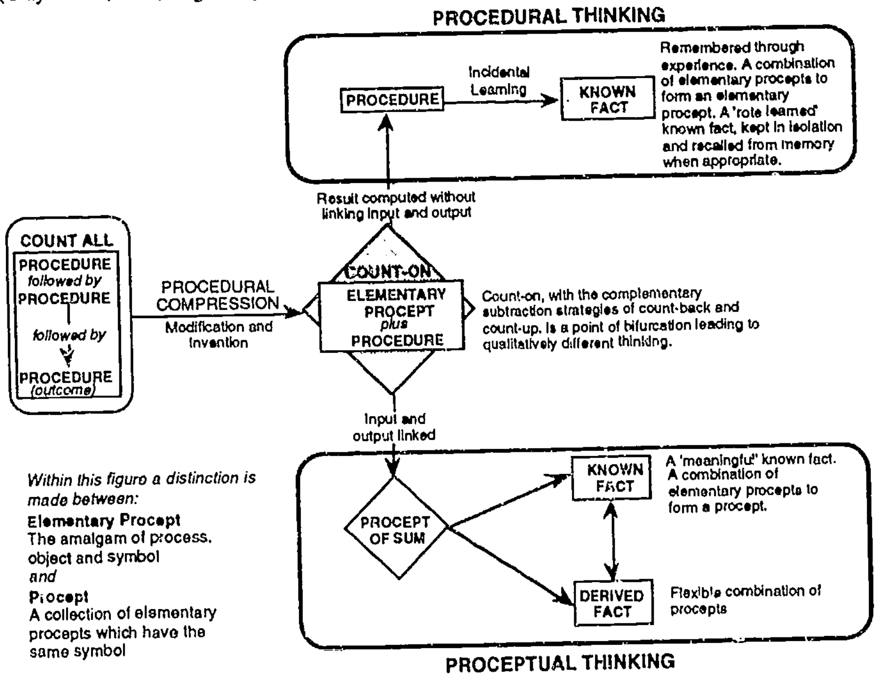

Figure 1: Count-on. The pivot for a proceptual divide in simple arithmetic.

The cognitive compression of the "count-all" strategy into "count-on" implies a compression in the number of procedures used. It does not imply compression in the time taken to implement the procedure. This may happen if the physical or mental supports used are suitable and then "count-on" may have the potential for reflective abstraction. If the inputs and the outputs which are the results of the incremental or decremental procedures are linked there is a possibility of reflection; $3 + 5 = 8$ may now be seen as the procept of sum. Of course this may be remembered as an isolated piece of factual knowledge and herein lies the problem of identifying whether of not a known fact is rote learned or whether it is a proceptually known fact -- the distinction made between an elementary procept and a procept (Gray & Tall, 1992) reflects this observed cognitive reality. In isolated instances the answer to

this question is not easily resolved. Indeed resolution may only come with evidence of one known fact being used to form another, but, this could be almost instantaneous! The distinction between proceptual thinking and procedural thinking may be identified through the integrated use of known facts, derived facts and procedures on the one hand and the extensive use of procedures and isolated known facts on the other. It is through the absence of the ambiguity that we may identify the proceptual divide. From a procedural point of view the essence of strategies such as "count-on", "count-back" or "count-up" is that they may be refined to such a degree that though they may become very efficient at one level they may not only mitigate against reflection but also against success at the next higher level.

Evidence for the bifurcation caused by count-on

Method

Evidence for the parting of the ways arises from the analysis of the responses made by a class of mixed ability children (N=29), aged between 6 years 8 months and 7 years 7 months, in the numerical components of a series of Standard Assessment Tasks (SAT), (SEAC 1992) allied to the National Curriculum of England and Wales. The tests were administered during the summer term of 1992. The numerical components were part of a broader spectrum of Mathematics Assessment Tasks (MAT) which included the option of Data Handling and Probability.

The numerical components included addition and subtraction number combinations. The maximum time allowed for each item was five seconds but, for a child to achieve a particular level of attainment, only one error in addition and one error in subtraction was allowed. As a result of their responses children were identified as having the following levels of achievement:

Level 1 (L1): Could add and subtract objects where the numbers involved were no greater than ten.

Level 2 (L2): Achieved the above and illustrated that they were able to recall the number combinations to ten without calculation.

Level 3 (L3): Achieved the above and illustrated that they were able to recall the number combinations to twenty without calculation.

The children were recorded on camera during the formal elements of the tests, where problems were presented orally, and then interviewed separately in a second interview, where problems were presented orally and through written symbols, within three weeks of the formal testing.

Results

The analysis of both test video and interview video indicates that for many of the children, even though the purpose of the time limit was to prevent calculation, counting was the dominant means through which solutions were obtained.

Children who were unsuccessful at L2, not only failed to recall the solutions to most of the number combinations but then attempted to use a procedure which was either inefficient or too lengthy to satisfy the timed criteria. For example, when attempting the formal component Simon tried to obtain

solutions through "count-all" using his fingers as a procedural anchor. His procedure was so inefficient and lengthy that he not only ran out of time to obtain a solution, but, his concentration on its application inevitably meant that he also failed to hear the first part of the subsequent combination. Joseph, on the other hand, tried to carry out all of the counting in his head with no external physical support. There is no evidence to indicate whether he was attempting count-all or count-on. Whichever, he found his strategy very hard and would sit for extended periods with no obvious sign of action but he, "*...liked trying to do things my head. I like them to be harder because when I grow up I will be able to do harder things.*" Joseph failed to obtain the solution to any of the L2 addition or subtraction combinations.

When they counted, most of the children successful at L2 used their fingers to support a "count-on" procedure. Frequently a subitised display of fingers equivalent to the second set was used to maintain a check on the amount counted. When subtracting, many children used an approach which involved the immediate display of the large amount through extending a number of fingers equivalent to it, immediately curling a number of fingers equivalent to the value of the small set and the subitising of the value of the remaining extended fingers. This **enactive subitising** involved no actual counting, the child seemed to need visual support of the numbers in a concrete form; to see that, three add five equals eight.

At the formal level, achieving L2 took no account of the means by which the level was attained; there was no differentiation between those who extensively used counting and those who solved every combination by recalling the solution to the addition and subtraction combinations. Only when the children began to attempt the L3 of the MAT that the real differences began to emerge.

Many children who failed to achieve the L3 level of attainment not only knew very few of the combinations to twenty but also attempted to use a procedure which they could not generalise within the time limit. The general pattern that emerged from those who achieved L2, but failed to achieve L3, was that they recalled solutions to combinations such as 17 + 0, 7 + 7 and 15 – 0 but for all of the others i.e. 9 + 6, 4 + 11, 15 + 2, 17 – 6, 11 – 9, and even 18 – 10 they attempted to use a counting procedure. Children successful at L3 not only knew many combinations but their solution of others demonstrated a considerable degree of flexibility through the use of alternative approaches: 9 + 6, "*You get nine, add one to make ten, and then add five.*"; 19 – 13, "*You have thirteen and count on to the nineteen–you add some of the nineteen onto the thirteen.*"; 11 – 9, "*I took one away from the eleven. That leaves ten. You take one away from that and that leaves nine.*" This flexibility also included the efficient use of a procedure. Jonathan, for example, recalled all of the solutions to the combinations apart from two. To obtain the solution to 17 – 6 he counted back six from seventeen. When attempting 20 – 5 he knew that "*fifteen add five is twenty so twenty take away five is fifteen*". However, though it was not unusual for children who achieved L3 to demonstrate a limited use of counting some recognised its limitations. They did not use it if they felt it was an unreasonable approach. Anthony didn't do 19 – 13 because "*...it was a bit too hard and I knew I couldn't count it in quickly*".

Figure 2 illustrates the overall percentage of number combinations attempted through the use of a counting procedure whilst figure 3 shows the percentage of occasions when these attempts counted towards the achievement of a particular level of attainment.

Of particular interest in figures 2 and 3 is the overall extent with which counting was used and the extent with which it led to success. The difference between those who did not achieve L2 and those who achieved L3 is particularly striking. *Not only did the latter use counting considerably less frequently but if they used it they did so with more success than children within the other groups.*

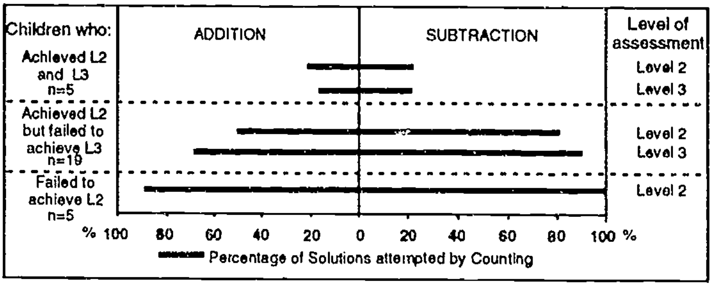

Figure 2: Percentage of solutions to basic number combinations attempted through counting

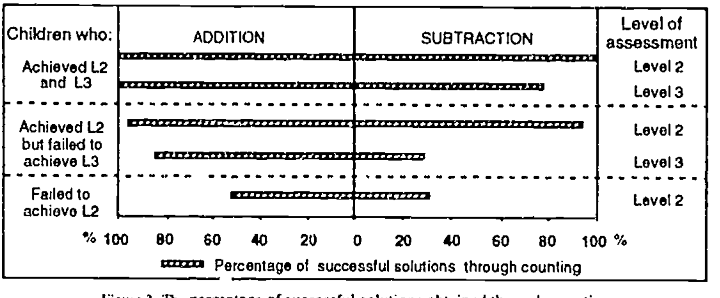

Figure 3. The percentage of successful solutions obtained through counting.

For the children who failed to achieve L2 and for those who failed to achieve L3 the focus appeared to be on the *doing* aspect of the arithmetic. An inability to recall the solution to a number combination appeared to turn their attention to the identification of an action which would enable them to arrive at a solution; their concentration was then on the implementation of this action. The more successful children demonstrated that they had alternatives available to them; through this flexibility they either established the solution to a combination through a derived fact or used a very efficient counting procedure.

Conclusion

In one sense the MAT component relating to the knowing and using number facts achieved its purpose of differentiating between children over three levels. However a longer term prognosis, which only takes note of a child's current level of attainment as a starting point without noting how the attainment was achieved, may lead to a very different outcome.

Clearly, asking children to attempt what is considered to be the same range of problems presents each child with a different level of difficulty. This is not only dependant upon what they know and the way in which they use what they know but it is also a function of their procedural approach and particularly:

- The frequency with which the procedure was used by an individual child at the lower level,
- The efficiency of the procedure,
- The ease with which the procedure can be generalised,
- The ability to recognise the limitations of the procedure.

In the context of simple arithmetic, it is believed that at some point all children use "count-all" but within a varying amount of time this is compressed into "count on". "Count-on" provides them all with not only a potentially more efficient procedure to handle problems but it also acts as a springboard to a different quality of thinking. In this sense it acts as a "junction box"; it can cause a bifurcation that leads to a parting of the ways between those who are successful and those who are not successful. The evidence from the sample points to what is happening at the time of bifurcation. In the longer term it is hypothesised that many of the L2 children will develop a proceptual view of simple arithmetic. Through their experience of "count on" they will move to an even faster track – that provided by proceptual thinking – which provides them with greater flexibility (see for example, Gray, 1991). Other children may oscillate between the relative speed obtained through fairly efficient "count-on" procedures, and, in isolated instances, attempts at proceptual thinking. Gray & Tall (1992) illustrate examples of below average ability children who use derived facts to obtain some solutions to basic number combinations. In some cases their use is almost procedural, for example, *"when I have to add 4 and 5 I always say two five's and then take away one"* (Thomas, age 11). In other cases the derivation is so cumbersome that eventually it is felt that the child will stop using the approach and resort to procedural methods.

The difficulties that children have in establishing a proceptual view of simple arithmetic should be a signal us to the difficulties they may have in more complex areas of arithmetic. How may we reasonably expect children to understand the multiplication and division if in simple addition and subtraction that are still procedural? Difficulties that children have in establishing a proceptual view of simple arithmetic should be a signal to us of the difficulties they will have in developing a proceptual view of fractions. However there is a big difference between the two. Whilst in simple arithmetic it is *advantageous* for the child to think proceptually, when operating with rational numbers it is *incumbent* upon them to do so. For the child to operate the addition and subtraction of fractions successfully they need to be able to see the same fraction in many different ways.

It is conjectured that the problem of the proceptual divide that occurs in simple arithmetic is a microcosm of the problems that occur as mathematics becomes more complex; at each higher level a proceptual divide occurs. Some children take to the fast route fairly easily to become successful, others take the slower, procedural route to achieve success at one level only to be faced with another parting of the ways through which they take an even slower route which eventually leads to failure.

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IDENTIFYING SOLO LEVELS IN THE FORMAL MODE

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The SOLO Taxonomy of Biggs and Collis has become a useful framework for categorising the quality of student responses to questions. Its value has been demonstrated in a number of subject and topic areas (e.g., Biggs and Collis, 1982; Pegg and Davey, 1989; Pegg, 1992). However, the major focus of this work has been directed at only one of the five modes of functioning described by the Taxonomy. This mode, the concrete symbolic, is that most closely associated with primary and secondary schooling and typical of adult functioning in every day life.

The purpose of this paper is to explore student responses in the mode of functioning referred to as formal. A sample of 60 students (17-20 years of age) were tested on a series of questions. Success in these questions required satisfying conditions attributed to formal thinking. The results highlighted that the ability to consider various cases which take into account a 'range of possibilities' and 'associated limitations' is a key indicator for performance in this mode.

Background

The SOLO Taxonomy evolved out of identified shortcomings of the views of Piaget. In particular, the notion of a characteristic way of thinking for a student across a variety of content areas, and even within the same topic area, was found not to be supported by empirical data from the classroom. To overcome this issue Biggs and Collis (1982) directed their attention towards the nature or quality of the response and away from development. There are two basic categorisations associated with the Taxonomy. One is concerned with modes of functioning and the other with a series of levels associated with each mode. The reader is directed to recent reports (Collis and Biggs, 1991; Pegg, 1992) for more detail.

However, relevant to this paper are: two modes, namely, concrete symbolic, and formal; and three levels, namely, unistructural, multistructural, and relational.

Brief descriptions of the two modes are:

Concrete symbolic (accessible from about 6 years) concerns the use of second order symbol systems, such as, written language and written music. Responses in this mode are linked closely with experiences or observations of the real world.

Formal (accessible from about 16 years) concerns the use of abstract constructs. Responses in this mode go beyond particular circumstances by consideration being given to various possibilities and alternatives.

A recent evolution of the Taxonomy related to modes of functioning is that the developing mode is no longer seen to replace an earlier mode of functioning. Two important implications of this are (i) an

earlier mode can be assessed by a student; and, (ii) an earlier mode can be used to provide support for a response.

The key characteristics of the three levels are:

Unistructural - the response uses only one relevant aspect.

Multistructural - the response uses several relevant, but apparently isolated aspects.

Relational - the response includes relevant aspects, but also takes into account relationships between the aspects.

Associated with cues or information used in a question are other features which help explain the level of response provided. Most important, in this present context, is the working memory capacity available. Not only does a student need the necessary memory space to carry out the procedures but also to monitor the procedures being used. As students gain more familiarity with certain concepts their procedures become more routineised and hence more memory space is cleared for additional mental processing.

The Study

Introduction

Despite a large and growing research effort directed at the SOLO Taxonomy there has been scant attention paid to the formal mode. In particular, there has been no research directed at verifying the existence of a unistructural-multistructural-relational cycle in the formal mode.

A useful direction for such an investigation flows from earlier endeavours by Collis (1975) and Küchemann (1981) regarding the notion of a variable. Küchemann stated that the "concept (of a variable) implies .... some understanding of how the values of an unknown change "(p. 110). However, he qualified this comment by adding "precisely what this might mean is hard to pin down... because many items that might be thought to involve variables can nonetheless be solved at a lower level of interpretation" (p. 110). While there are some differences between the stances taken regarding the hierarchical nature of the understanding of letters, both researchers attribute abilities associated with variables with higher order (i.e., formal) functioning.

The Research Questions

The study aimed to explore two research questions, namely,

Was it possible to identify empirical evidence, associated with students' understanding of a variable, for:

(1) the qualitative difference between the concrete symbolic and formal modes of response; and,

(2) a series of levels within the formal mode which satisfied the level conditions of the SOLO Taxonomy. That is, would there be examples of students' responses which focused on one aspect only, several independent aspects, and several related aspects?

Research

To investigate the formal responses, 60 students from Year 11 to Year 1 at University (17-20 year olds) were asked to complete a series of questions which involved the notion of a variable. Two questions are considered below in detail. These were chosen as they are representative of the questions asked and encapsulate the general thrust of the overall results.

Question 1

If p is a real number, discuss the following

$\frac{1}{p} > p.$

This question was developed to address a number of issues.

1. To minimise possible rote responses the word 'discuss' was used rather than 'solve' and 'p' was chosen as the unknown rather than 'x'.

2. The question allows for at least one apparently 'successful' but incorrect response. To achieve this a student simply manipulates the symbols to obtain $1 > p^2$ and then $1 > p$. This type of response would be seen as belonging to the concrete symbolic mode as the student has simply worked within the system of given symbols.

3. A consideration of different cases would be an indication of a formal response. In such answers, students would need to maximise their working memory space by not reaching an answer until a range of various possibilities and associated limitations for the variable were taken into account.

Results/Discussion

An analysis of the responses enabled a clear dichotomy to be identified. The key element that separated the students was the inability of one group to consider different cases. This group simply worked within the system given. Three different types of responses were identified. Typical examples

included:

Type 1. "$\frac{1}{p} > p$ | "$\frac{1}{p} > p$

$1 > p^2$ or $1 > p^2$

$\therefore$ p must be smaller than 1" | $p < \pm 1$"

Students who gave this type of response worked as they might have done when solving an equation. No special consideration was given to either the inequality sign or the multiplication of both sides by p.

Type 2. "not true e.g.: $\frac{1}{7} > 7$"

"no p value holds for this equation"

In both the examples above students have selected a positive integer greater than one, tested it and then drew a conclusion based solely on this information.

Type 3. "$p \neq 0$ $\frac{1}{p}$ can't be $> p$ $\therefore$ the statement is false"

Some students were aware that $p \neq 0$ but made no use of the knowledge. While this might be considered an indication of taking a constraint into account it seemed that, for most students, this was a drilled or rote response. Follow up interviews confirmed that "teachers expect you to write it down".

Responses that did take into account different cases represented a qualitative different answer from those described above. These responses are considered to belong to the formal mode as they satisfied the necessary requirement described earlier. An analysis of the responses enabled each to be considered where appropriate under unistructural-multistructural-relational headings although there were often different forms of responses at each level. Typical examples and a brief discussion of each are provided below.

Unistructural responses (formal mode)

There were two groupings identified here.

Group 1. "p must be a fraction e.g. if $p = \frac{1}{2}$ $\therefore \frac{1}{\frac{1}{2}} > \frac{1}{2}$

$2 > \frac{1}{2}$"

"$p < 1$"

Group 2. "If p is a positive number then the equation is false, however, if it is negative it is true"

"p must be a negative number"

In both groups the focus has been on the consideration of **one case**. In Group 1. the focus was on positive real numbers and consideration was given to values on either side of $p = 1$. The word "fraction" was commonly used and was meant to be a "number less than one". Zero was not included. Often students felt the need to provide concrete symbolic support, that is, a particular example with their answer to help clarify the meaning.

The answer of "$p < 1$" was relatively common. No manipulation of symbols was provided and students, in interview, described what they did as "working in my head", i.e., substituting numbers and generalising it to a result.

Group 2. responses are also unistructural. Here the focus was also on one case but the comparison was between positive and negative numbers. This has arisen when students had chosen (say) $p = 7$ and $p=-7$. They saw that $p = -7$ "works".

Multistructural responses (formal mode)

Again separate groupings of responses were identified. Typical examples were:

Group 1. "$p \neq 0$, p must be less than 1 and greater than zero i.e., $\frac{1}{2}$, for the statement to be true."

"p must be a fraction or decimal between 0 and 1 for this statement to be true. $p \neq 0$ and $p \neq 1$"

Group 2. "$p < 0$ and $p \neq -1$"
"$p < -1$"
"p has to be not = to 0
false when $p > 0$ $p \neq -1$
true when $p < 0$
$\therefore$ true for all $p < -1$"

Group 3. "p must equal a proper fraction, this will allow the statement to be correct. If p was a negative number the statement would also hold true. P will equal all values when $p < 1$ except when $p = 0$"

In each of the above groupings **two or more cases** were considered. However, in each case there were clear inconsistencies. Group 1. responses continued to focus only on positive reals and have generally taken into account the possibilities within this restricted domain. Group 2. responses have considered the negative reals. The third group, Group 3., represents those responses who managed to cover many of the cases but have fallen short of a complete overview of the question. Other features of

these answers include a tendency for long responses, often with ideas repeated. There were also instances of concrete symbolic support.

Relational responses (formal mode)

The students whose responses were relational had a complete overview of the problem. Typical answers were:

"$0 < p < 1$ and $p < -1$"

"$p \neq 0$, $\{p: p < -1\} \cup \{p: 0 < p < 1\}$"

The responses to the question analysed above were similar to other questions asked of the students. Questions which involved inequalities such as $p > p^2$ and $\frac{1}{p} > \sqrt{p}$ provided further support for the types of categorisation provided above.

Question 2

What can you say about x given the following expression

$$\frac{\sqrt{4-x^2}}{x+1}$$

Results/Discussion

This question was different from the earlier question in that there is clearly nothing to 'solve'. It was thought that this feature may have a cueing effect and assist students to focus their attention on the various cases. There was some evidence of this effect. However, a number of students were not able to respond in this manner and chose to work within the system provided. These responses could clearly be coded as concrete symbolic. There were two basic types of responses identified.

Type 1. Students attempted to manipulate the numerator $4-x^2$ either by factorising into $(2-x)(2+x)$

OR

by performing a 'simplification' such as $\sqrt{4-x^2} \rightarrow 2-x$

Type 2. Students attempted to work with the denominator usually by multiplying the expression by $x-1/x-1$

As in the previous question some students did include $x \neq -1$. However, once again, it was usually written as an initial statement and then ignored.

Formal responses, those that considered various cases, could be again coded using the unistructural-multistructural-relational cycle.

Typical examples included:

Unistructural responses (formal mode)

"$x = -1$ does not exist, ie does not make a real number"
"x cannot equal ± 2"
"$x \leq 2$ because we cannot have a $\sqrt{\text{negative}}$"

Here the students focused on only one condition, either in the numerator or the denominator but not both. There was also evidence of some confusion as to whether the numerator was "allowed to be zero".

Multistructural response (formal mode)

"$x \not> 2$, $x \geq -2$"

"$x \not> 2$, $x \leq 2$ to exist

$x \neq 1$ to exist"

"$x \leq 2$, $x \neq -1$"

Again more than one condition was considered but students have not achieved a complete overview of the question.

Relational responses (formal mode)

"$-2 \leq x \leq 2$ and $x \neq -1$"

Students' responses at this level were able to take into account all the necessary restrictions.

Conclusion

This paper has provided examples of student responses which: (i) highlight the qualitative difference between the concrete symbolic and formal modes of functioning; and, (ii) detail empirical support for levels within the formal mode. The focus of the investigation was on the understanding of a variable and in particular the spontaneous use of different cases to determine possible values of a pronumeral.

A feature of the answers was the degree of consistency of student responses across the items asked. For example, it was usual for a student who provided a concrete symbolic answer for one question to give the same type of response in another question - despite different wording or question structure. This is consistent with features of the Taxonomy. While it is possible for students to answer

questions at almost any level, at or below their ability, it would be expected that students who were giving their best would answer close to their potential.

Further support for these findings was evidenced in the robust nature of the responses when the students were interviewed. Even under extensive prompting few students who had provided a concrete symbolic answer were comfortable or capable of considering various cases. Those students who had given unistructural or multistructural answers in the formal mode, were able, in general, to provide under prompting, more detailed answers. However, very few of the students who gave unistructural responses were able to attain a relational answer. Once again these findings are consistent with features of the SOLO Taxonomy.

Clearly much more work is needed in this area if real advances are to be made in the way we identify and teach higher level skills. It is important that empirical evidence is gathered for a range of learning experiences across the formal mode. Only with such knowledge will the SOLO Taxonomy achieve its potential as an assessment tool.

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ON TEACHERS' CRITERIA TO ASSESS MATHEMATICAL ACTIVITIES

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Summary:
We will concentrate on some results concerning the use of a certain type of open-ended problems in the research project "Open tasks in mathematics". The project tried to clarify the effect of open-ended problems on pupils' motivation, the methods and how to use them. In the project, altogether 20 open-ended problems were used in classrooms during the years 1989–92 (in grades 7–9). Here we will deal with the interviews of four teachers from a suburban junior high school in Helsinki (Finland). Criteria given by the teachers to assess open-ended problems in six follow-up interviews are discussed. The criteria could be classified into three main categories: Convenience to use, Pupils' motivatedness, and Support in learning objectives. Into these classes, they were distributed almost uniformly.

The method of using open-ended problems in classroom for promoting mathematical discussion, the so called "open-approach", was developed in Japan in the 1970's (Shimada 1977, Nohda 1988). About at the same time in England, the use of investigations in mathematics teaching became popular (Mason 1991), and the idea was spread more by Cockcroft-report (1982). In the 1980's, the idea to use open problems (or open-ended problems) in classroom spread all over the world (see Pehkonen 1991), and research of the possibilities to use open problems is especially now very vivid in many countries (e.g. Pehkonen 1989, Silver & Mamona 1989, Nohda 1991, Zimmermann 1991, Clarke & Sullivan 1992).

Here, we will concentrate on some results concerning the use of a certain type of open-ended problems in the research project "Open tasks in mathematics" (Pehkonen & Zimmermann 1990). The purpose of this paper will be to answer the following question: Which criteria do the teachers use when assessing open-ended problems used in the experiment? But firstly, the research project is briefly described.

Research project

The aim of the project. The purpose of the project was to improve mathematics teaching in junior high schools, especially to develope and foster methods for teaching

problem solving. The project tried to clarify the effect of open-ended problems on pupils' motivation, the methods and how to use them. Thus, open-ended problems were used here as a method for change in mathematics teaching. In realizing the experiment, we tried to stay within the frame of the "normal" teaching, i.e. in the frame of the valid curriculum, and to take account the teaching style of the teachers when using open-ended problems.

The theoretical framework for the research project was the constructivist understanding of learning (e.g. Davis & al. 1990). In order to see, whether there are any changes in mathematics classes, one should try to see and understand mathematics lessons from "inside". This implies interpretative research methodology which means here interviews, classroom observations and careful interpretation of questionnaire results.

On the realization of research. The project started in the autumn 1987, and ended in the summer 1992. In the pilot study of the project during 1987–89, the research design was tested, measurement instruments were developed and the problem material was worked into its final form. The main experiment was planned to begin in the autumn 1989 both in Hamburg and in Helsinki with ten grade 7 classes and to be continued with those classes through the whole Finnish junior high school (up to grade 9), i.e. to the summer 1992. Unfortunately, the German counterpart could not get any finance for his part of the research project. So, the main experiment was realized only in the Greater Helsinki area (Finland).

Both in the beginning and at the end of the experimental phase, teachers' and pupils' conceptions of mathematics teaching were gathered using questionnaires and interviews. In the main experiment, experimental group 1 (nine teachers) and experimental group 2 (six teachers) differed in the point that from the mathematics lessons of experimental group 1 about 20 % (i.e. once a month about 2–3 lessons) was reserved for dealing with open-ended problems. There was a questionnaire for each open-ended problem in which the pupils' evaluations of using that open-ended problem were ascertained. The teachers' evaluations of using open-ended problems were obtained with short interviews after each term. The teachers in experimental group 2 were told that they were participating in an experiment, whose aim was to investigate the development of pupils' problem solving skills. They were told nothing about open-ended problems nor the

experimental group 1. Pupils in both experimental groups solved in their class work some open-ended problems which were the same for both groups.

The problems and the teachers

Open-ended problems used. In the project, altogether 20 open-ended problems were used in classrooms during the years 1989–92. On each grade level (grades 7–9), there were about seven open-ended problems. In the autumn when the school began, the teachers were given a booklet of open-ended problems for the school year. In the booklet, nine open-ended problems were described, and the teachers should select seven of them to use during the school year. The description of each open-ended problem was contained in average in four pages. Firstly, there was one page of background knowledge which was structured as follows: grade level, topic area in the curriculum, mathematical entries, objectives, preknowledge, materials, teaching hints; the other pages were for tasks.

Behind most of the open-ended problems, there is a well-known problem (e.g. Tangram) from which a sequence of problems has been developed. In classroom, an open-ended problem will take about 2–3 lessons to deal with. For example, the following tasks with figures on two pages were given for Tangram: (1) Make a tangram-puzzle according to the figure given! Which bigger pieces can you make with the smaller ones? (2) The area of the whole puzzle is 100 cm^2. Calculate the area of each piece! (3) Which "wellknown" polygons can you make, if you are using only a) 2 pieces, b) 3 pieces, ... , f) 7 pieces? (4) Calculate the area of each polygon you have found!

The test subjects. The research project was realized in junior high schools in Helsinki and in Järvenpää. The four mathematics teachers in question here were all from the same suburban school in Helsinki.

Anja (36; 10)[1] is a silent introvert person who gives the impression that teaching is for her only a job. She does not look at all to be an eager teacher, but she is very seldom showing her eagerness of any matter. In the first interview, she stressed beside "number mathematics" also mathematical applications and understanding of

[1] Numbers refer to the physical age and to the teaching experience of the teacher, both in years.

computational rules.

Jari (48; 23) is the natural leader of the mathematics teachers in the school. He is always active and full of life and is experimenting often new approaches in his class, and he is always ready to share his results with his collegues. In the first interview, he emphasized that mathematics is on the first place calculations. But in his teaching, he seems to lay very much emphases on promoting pupils' thinking skills.

Pentti (55; 15) is acting as a mathematics teacher in the junior high school since 15 years. He is very good company in the school, who is all the time joking, also about himself. Usually everything will suit for him. In the first interview, he told that mathematics reminds him of many matters: basic calculations, logical thinking and applications.

Toivo (32; 1), a fresh teacher, is a silent thinker who tries his own ideas in all silence. But when he has worked some useful material into an applicable form, he is ready to bring it to his colleagues. In the first interview, he transmitted the view that mathematics means computing and following of the rules.

Some preliminary findings

In the follow-up interviews about the use of open-ended problems, the teachers were asked to assess the problems used during the last term. As the last question of the interview, they were invited to order the open-ended problems according to their preference. The question used was as follows: *"Give the order of your preference for the open-ended problems used."* At the same time, they were asked to give their reasons for the order.

Firstly, the teachers were invited to order the last three or four open-ended problems used. And secondly to try to order these with all earlier problems. If the teachers wondered what the order of preference could mean, they were said that it means the order of selection which they as teachers will use. As the criteria – for and against – given by the teachers in six different interviews were gathered together, one could find in them a clear pattern.

On the criteria in general. The criteria given by the teachers could be classified into three main categories: Convenience to use, Pupils' motivatedness, and Support in learning objectives. Into these classes, they were distributed almost uniformly. The

responses reflecting the teachers' personal preference has been left out, e.g. *"I think this has been one of the best problems"*. This categorization will also be supported by the teachers' reasons for why some open-ended problem has not been considered to be as good as the other ones (a so called negative criterium). In categorization, one criterium entirety was used as a unit. Thus, about ten positive criteria and some negative ones came into each main caterogy (Table 1).

	Anja	Jari	Pentti	Toivo
Convenience to use	6	0	7	2
Pupils' motivatedness	4	3	1	4
Support in learning objectives	4	7	2	3

Table 1. The distribution of the criteria given by the teachers.

The following characteristics of open-ended problems pertain to the group Convenience to use: the minor amount of work required from the teacher, facility to realize, and material at hand. Some examples of reasons which represent these aspects are e.g. *"one is not compelled to do much work for it"*, *"it was the easest one to realize"*, and *"material is always at hand"*. Some reasons representing negative selection are e.g. *"you should work much for it"*, and *"it was not so easy to realize"*.

The group Pupils' motivatedness contains pupils' eagerness and success. This type of reasons are as follows: *"The pupils worked on it really eagerly"*, and *"the pupils reached many of these alternatives rather nicely"*. One can also see the same meaning of eagerness in negative selections, e.g. *"I somehow did not have an inspiring effect on the pupils"*.

The group Support in learning objectives is compound of two areas, on the one hand of promoting thinking and on the other hand of teaching managements. These groups of reasons are represented e.g. by following statements: *"Then something is happening also in pupils' brain"*, and *"since it is suitable in so many places"*. Negative reasons of this group are e.g. *"it is more entertainment"*, and *"it was not so promoting"*.

On criteria according to each teacher. The diversity of the teachers in question can

be seen in the favour of different criterium categories (Table 1): The responses of Pentti (55; 15) were mainly gathered into the category Convenience to use. Also the most of Anja's (36; 10) reasons are placed into the first category (Convenience to use), althrough she also gave many reasons which were categorized into the second (Pupils' motivatedness) and the third group (Support in learning objectives). The responses of Jari (48; 23) concentrated in the category Support in learning objectives, none of the responses were in the first category. The responses given by Toivo (32; 1) were distributed, perhaps, most uniformly into these three categories.

Beforehand, one could have expected that the category Pupils' motivatedness will be the largest one, since teachers when selecting a "different" task usually explain it with the need to motivate pupils. But behind the surprisingly large category Convenience to use, one may find a very human characteristics of teachers: desire for convenience.

If we compare the distribution of the reasons given by the teachers with their pictures given a couple of pages before, we can see some connections. Pentti is very good company, and for him everything seems to be a joke, also teaching. A tendency to minimize the work done for teaching suits very well to his picture. Anja has also taught so long that the job might be formed already a routine for her. She tries, according to the role of a "good teacher", to take the different components (as pupils' needs and mathematical demands) into consideration, but she will not forget to take care of her own amount of work. Jari is an idealistic teacher for whom the teaching has important meaning on his life. He is still able to begin the pondering of teaching situations from the point of mathematics and pupils. His own amount of work is not so important. Toivo represents a fresh teacher who has still in good memory the ideas of teacher education, and who likes to ponder and experiment himself new teaching situations.

Discussion

The criteria described for selection of open-ended problems are gathered from six different interviews, but each time the question was the same. The interviews were realized always in the beginning of the next school term (i.e. each half year). The four first interviews were done by the author, and the two last ones by a research assistent. The reasons given by the teachers were held during these three years very similar, thus one can think the information gathered to be reliable.

On the other hand when evaluating the findings, one should take account that the teachers in the experiment represent the best ones in the teacher population. They participated voluntary the experiment which had caused them some extra work, and they are interested in their job and respected among other teachers. Just their type of teachers will be used as a specialist of teaching practice by the school administration and by publishers of learning materials. Therefore, the results are not necassary representative.

The research findings gave more information to the "pictures" of the teachers described earlier. On the other hand, the pictures of the teachers explained in a certain amount the distribution of the reasons into the main categories. However, the diversity of the teachers may be grounded on their different conceptions about mathematics teaching. Therefore, the search of explanations might demand to outline and understand their "mathematical world view".

Questions based on the research findings. Based on the research findings, some questions arose. Firstly: On which reasons do the teacher actually form his assessment of the selection of an open-ended problem (or more generally of mathematical teaching material)? It seems that a part of teachers base their assessment on the convenience to use the material. This explanation can be heard by the publishers when they demand from the authors of learning materials that the material should be easy to use. And this demand seems to be justified.

Secondly: Which kind of objectives should we pose for those conducting the change in teaching? In the research project, one aimed with open-ended problems to cause change in mathematics teaching. In the research findings, we see that about one third of the reasons given by the teachers are connected with the convenience to use the material. Thus in order to reach change with the aid of teaching material, one can choose at least between two ways: (1) One emphasizes the pupil-centerness and the mathematical content of the tasks. This leads to the problem of teacher in-service education. (2) One is satisfied with the offering of easy-to-use materials to teachers. This leads to the problem of producing materials.

At the end, it will be given a short comment about the relevance of the findings to the research itself: If a researcher will ask teachers to assess teaching material according to its applicability, how reliable is such an information? On which criteria do

the teacher base such an assessment? It seems that the researcher should also be ready to clear out teachers' conceptions about mathematics teaching, i.e. teachers mathematical beliefs at least in some range.

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FACILITATING IN-SERVICE MATHEMATICS TEACHERS SELF-DEVELOPMENT

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This paper reports on a project involving elementary and secondary mathematics teachers engaged in a process of using autobiographical stories of their teaching as a means of professional development. The stories were used to determine "frames" that influenced the teachers actions in the classroom. The meaning and roots of these frames were determined by reflecting on professional and social experiences. The outcome indicated that the process helped the teachers to reflect more deeply about their teaching thus developing a personally meaningful understanding of it and how they could enhance it.

The focus of this paper is the use of mathematics teachers' autobiographical stories of their teaching to help these teachers to understand their teaching and thus bring about their own development.

BACKGROUND

The changing climate in mathematics education seems to be reflected also in how educators and researchers are beginning to view mathematics teachers. Traditionally, the trend in research on the learning and teaching of mathematics has generally been to "by pass" the teachers who were treated more as inert transmitters of facts. But growing dissatisfaction with this systematic control of the teacher and teacher related variables (for eg. Silver, Lester. Thompson and Grauws, all in Silver 1985) seems to be resulting in increased interest in the teachers' role in the teaching and learning of mathematics.

Thompson (1992) reported a significant increase in studies on mathematics teachers with particular focus on teachers' belief about mathematics and mathematics teaching and learning. C. Hoyles, in her plenary address at PME (1992), also provided an overview of recent studies that focus on mathematics teachers' belief. The general pattern of the findings of these studies is that teachers' belief influences classroom behaviours. In resonance to this, and perhaps the changes teachers are being required to adopt, the impact of teachers' beliefs on teachers' change has also gained increase attention. Hoyles (1992) reported on a set of studies that focused on in-service teacher training where it was argued that it made sense to explore the belief systems of teachers <u>before</u> attempting to introduce changes. My work with in-service teachers reflects this perspective, but it does not seek to explore teachers' belief systems as if they were isolated entities. Instead, it engages teachers in a process of understanding their own belief systems to understand their teaching and thus facilitate

change. This process uses autobiographies -- teachers' stories of their teaching and personal lives. Since this process defines the methodology of the study being reported, it seems appropriate to briefly describe the body of literature that validates it.

Over the past decade, teacher narratives have emerged as a significant means of research on teaching and teacher development (Solas 1992 contains a comprehensive review/reference of this). Biography (Butt & Raymond 1987), autobiography (Grumet 1980, Pinar 1981), biographical narrative (Berk 1980), narratives (Connelly & Clandinin 1990, 1991), and story (Elbaz 1991) have all been promoted, with varying definitions, as means of understanding teacher thinking and classroom behaviour. These approaches recognize the personal dimension of teaching and the teacher. They focus on the personal experiences --- professional and social, past and present --- of teachers. They capture events in the teachers' past personal and professional lives that were and are influential in shaping how they think and act in current classroom situations. They recognize that "each teaching action and the thinking associated with it is nested within uniquely personal, situational and contextual determinants and influences" (Butt & Raymond 1987). They promote that to understand our actions, we do not simply observe and theorize about them, but we look at our history or biography and our intentions to recover the meaning to them, for, as Bruner (1990) points out, the narrative we construct or the stories we tell, reflect who we are and thus contain the meaning of our actions.

These processes, however, are not limited to researching teaching, but have been used to facilitate development and growth of in-service teachers. In the book, "Understanding teacher development" (Hargreaves & Fullan 1992), several situations are described involving these processes. My use of stories falls in this category of teacher development with a similar goal of raising awareness of the relationship between thought, action and experience. However, unlike most of the situations cited, in my work the telling of the stories or analyzing each story is not an end in itself. The stories are used to identify "frames" that influence the teachers' thinking and behaviour.

"Frames" refer to the underlying assumptions or unconscious dispositions that influence teachers' actions in the classroom. As Barnes (1992) notes, they are not discrete entities, but patterns that help us to organize and understand the complex events in which we take part. In discussing the effects of frames and framing on teachers behaviour, Barnes (1992) further points out, "When teachers theorize ... the theories are not always closely related to their actual behaviour in lessons. This is not because they wish to deceive but because they are often acting upon a set of priorities (frames) of which they are not fully aware." This could explain the inconsistency between teachers' professed beliefs and their practice in some studies on mathematics teachers noted by

Holyes (1992). Simply interviewing teachers about their beliefs will not by itself necessarily recover underlying frames since the teachers are unlikely to be aware of them. This is why the story telling process has to move beyond the telling of the story to help bring into focus basic dimensions of teaching that are taken for granted. The remainder of this paper will describe the portion of my ongoing work with in-service teachers that deal with recovering frames and their meanings.

THE PROCESS

The participants were nine in-service mathematics teachers (4 elementary, 5 secondary) with teaching experiences of 5 to 25 years. I functioned as facilitator and met with the teachers as a group and/or individually to discuss the process being investigated. This process, described below, provided the data for the study, i.e. teachers' stories, teachers' reflective journals and written feedback, my responses and my notes on group and individual discussions. The process consisted of the following four stages:

I. The participants wrote 7 stories over a period of 7 weeks. They were told that each story should be of a personal experience in their teaching. The stories should include as much details as possible (when, where, with whom, feelings, actions, etc.) and should be telling of their teaching. They were not to analyze, theorize, philosophize or do any other "ize" with the stories as they reported them. They were only to describe the stories as they happened, i.e. a detailed literal description of the actual experiences. Each story was submitted to me for feedback before the next was written. The only feedback I provided at this time was whether the stories satisfied the above criteria.

II. Each participant reviewed his/her seven stories for themes or patterns underlying his/her behaviour in the stories. The teachers were instructed not to analyze the individual stories, but to try to get some sense of the holistic picture with respect to their behaviour. This required standing back from the stories and considering, for example, the conflicts, tensions and/or harmony embodied in the actions of the actors in the stories in relation to the teacher. They were then to reflect on the patterns or frames they recovered in terms of what these frames meant from their perspective. This was done with no intervention by me. Their reflections were recorded in journals.

III. I also reviewed the stories for "frames" before seeing what the teachers recovered in stage II. (In most cases mine and theirs bore some resemblance to each other.) They were given the "frames" I recovered at the same time I received theirs and their journals. They were required to reflect on my suggested frames from their perspective. This reflection was not to take place only within the boundaries of the stories, but within those of the teacher's life as a person and a mathematics teacher. Thus they were to go beyond the literal meanings reflected in the stories to get to a deeper

understanding of the meaning and roots of the frames. These reflections were recorded in journals.

IV. Finally, the teachers wrote journals reflecting on the consequences of their findings on their ongoing teaching.

The process was not as simple as it may appear on the surface. The teachers usually found it frustrating deciding on stories to tell, difficult writing the stories without theorizing and challenging finding underlying patterns. This made the process time consuming for them. Thus the process was spread out over 3 months. This period also seemed to be necessary to allow their reflection to evolve with depth.

SAMPLE DATA AND RESULTS

Because of the restriction on space, it isn't possible to include a complete set of stories to illustrate the process. Since the stories were dealt with holistically, fragmented pieces from each could be misleading. So only one story will be presented in its entirety (with no editing) as an example of the nature of the stories.

> This story takes place in a mathematics 30 [grade 12] class. We had discussed properties of the parabola and I thought that it was time for the class to be exposed to a more formal and rigorous approach to the topic. As a result, I sat and planned a lesson that would reveal the derivation of the general formula of a parabola at the origin.
>
> I had previously exposed my class to the idea of a directrix, vertex and focus and had given the locus definition of the parabola. We had spent a lesson looking at parabolas and identifying these features and a second lesson looking at equations of parabolas and seeing how we could identify the coordinates of the focus and vertex and the equation of the directrix. I thought that the students, once seeing the specific cases, would appreciate the derivation of the general formula.
>
> I began my planning by taking out a sheet of paper, drawing a parabola at the origin (pointing up) and labelling the vertex (0,0), the focus (0,f) and the directrix y = c. I saw that my placement of these features were not accurate so I erased my parabola and continued to plot points by measuring a constant distance between focus and directrix. I plotted about six such points and then joined them to make a smooth curve. I was much more pleased with the result.
>
> Once plotted, I turned to the locus definition and substituted the words with the general coordinates that I had chosen. As I waded through the algebra I was struck by the elegant and intimate relationship between the definition, the graph and the algebra. After measuring with a ruler one or two more times to satisfy myself that I had drawn an accurate parabola and double checking my manipulations of the algebra I took this sheet and photocopied it for the class. The next day I entered class, handed out my photocopied sheet, went to the blackboard and "re-did" the derivation for them. When I was done I turned to the class and asked if there were any questions. The students were silent. I took this as an indication that they followed along quite nicely. I left the derivation of parabolas pointing down, left and right as exercises. The students diligently put pencils to paper. As they progressed I circulated to see how they were doing. I found that most of them used the photocopied sheet

as a guide and simply re-oriented the coordinates of the focus, vertex and directrix to make them consistent with the parabola that opened in the direction they wished. None of them tackled the problem in the manner that I did; by drawing their own parabolas and working from the definition.

I questioned a student to check for her understanding of the exercise. She told me that things were going "O.K.". I asked her to give me the definition of the parabola. She did. I asked her to tell me what the coordinates of the focus and vertex were. She did. I asked her what the derived equation meant to her. She couldn't answer that one. I probed and asked her what each literal symbol meant. Her answer indicated that she couldn't distinguish a variable from a constant. To her all the letters were "variables". She was unable to see the difference between "some value" and "any value". I started checking with other class members and found that this was a common misunderstanding. It was only then that I saw that perhaps the students weren't ready for the lesson that I had just delivered.

My observation of the set of stories of this teacher was that he tended to treat content and teacher's interest as the most significant part of the mathematics curriculum. This was consistent with what he had observed on his own. He noted in his initial journal:

> ...there seems to be a conflict at the heart of most of my stories, that is, my struggle with mandated curriculum. I found it very surprising in reviewing my stories that the bulk of them dealt with the way that I delivered subject matter. I always thought of myself as a teacher who cared more about the students rather than the content....

Excerpts from his follow-up reflections (after becoming aware of my observations):

> ...It became obvious to me as I re-read my stories that I am struggling with the demands of content in my teaching. Most of my stories centered around concerns of planning and delivering concepts to students. I found it enlightening that in practically all of my stories there was a conflict between how I thought students would react and how they actually did. In my stories, I shared experiences of frustration when the content that I delivered was either over the heads of my students or not challenging enough for them. Yet, I continued to put the content first.
> ...I realize that my responsibility is to plan new ways of delivering the content so that my students will be able to enact with it in a more meaningful way. However, I do not see that I must "throw away" content concerns in order to be true to needs of my students.
> ...I realize that I must teach content to students; my concerns over their growth as human beings is motivated by my duty to teach content.
> ...I must be sure that the content does not overshadow the needs of the students.
> ...I plan lessons that stimulate my interest. I present math to my students in a manner that makes obvious the wonderment that I feel when I first engage with the mathematical ideas. ...I realize that my interest must include a vision of the world that students live in. I wrote a story that dealt with planning a lesson that revealed the connections between algebra, geometry and logical reasoning in proving the locus definition of the parabola. I was stimulated by the exercise in planning for the lesson. The connections were obvious and fascinating TO ME. However, I failed to take into account the view of my students. To them this lesson came off as dry and separate knowledge. I failed at looking at the world through their eyes. I should have included

> some work that dealt with their mathematical world; perhaps a hands on approach with manipulatives, a computer application, or an artistic approach. In retrospect all of these would have stimulated my interest and met the students at a place where they were comfortable....

Further reflection by him included a storying of historical personal experiences that could have shaped his belief about the math curriculum in terms of content, of changing personal social experiences (eg. becoming a parent) that have been reshaping his view of his students and of how he now understands reconciling the conflict between teacher-content, student-content and teacher-student relationships.

This teacher's situation reflects those participants whose stories revealed something significant of which they were unaware. Another example is the teacher who became aware that her frame of caring was dominating her desire of giving more responsibility to her students for their learning and behaviour. This conflict resulted in negative consequences in her teaching and interaction with her students. With her new understanding of the situation, she is now working on developing a more compatible balance in her behaviour.

Not everyone, however, had surprising revelations. For a few, the outcome was enlightening as a reinforcer and a way of understanding or confronting the underlying meanings of what they were aware of on the surface. For example, one way in which another participant now understands the constant changes he has made over the years of his teaching, is in terms of the tension between the traditional approaches the mathematics curriculum and text books dictated and his seemingly uncontrollable desire to teach mathematics from a personal problem solving perspective. He referred to this problem solving perspective as "learning by experience" which he traced to his childhood experiences outside of school and described how it has continued to influence his behaviour as an adult. He noted: "My initial way of dealing with anything new or different is to try and figure it out without anyone telling me, showing me or by some written explanation of how or what to do." However, although problem solving has now become an integral part of the curriculum, the tension has continued because of what he now sees as a conflict between his personal perspective of it and those prescribed in text books. He has begun to work on establishing a compatible balance between the two.

The effectiveness of the process depended on the nature of the stories. Stories that described the teachers' actions and thoughts as they occurred were more revealing of underlying assumptions. Only for a few of the participants were such stories written consistently. The others would at times focus on a student or a colleague and include themselves as a minor or passive actor. In such cases,

the stories were more telling of what the teacher consciously valued or dislike in his/her teaching. Thus they were reinforcing but not necessarily enlightening. However, of the seven stories written, a minimum of three that focused on the teacher was sufficient to get a meaningful pattern.

The teachers were surprised at how much they were able to learn about themselves from the process. One noted: "I am learning that one cannot hide from reality when the storying process is used." However, the process didn't seem to make sense to them until they started looking for "frames". This reinforced for me that simply telling the stories is not necessarily meaningful particularly for mathematics teachers who are likely to find this activity in conflict with what they would associate with a mathematics teacher. In fact most of participants found looking for the frames the "most fun part" because they perceived and approached it as problem solving.

In general, the process increased the teachers' awareness of their teaching of mathematics, some of the underlying assumptions consciously and unconsciously influencing their teaching and the possible roots of these assumptions. It provided them with a personal understanding of how to alter their behaviour to enhance their teaching. This was evident not only from the suggestions they made but from follow-up journals on what they had actually started to do or tried in their classrooms with encouraging results.

CONCLUSION

The autobiographical process seems to be a viable way of facilitating mathematics teachers' development, of helping them to understand the meaning of their actions and, hopefully, to become more liberated and effective teachers. However, more work is needed to determine the long term effect of the process on teachers' behaviour.

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THE STUDY OF UNDERSTANDING IN ARITHMETIC EDUCATION
UNDERSTANDING PHASE AND LEARNING PROCESS MODEL
- IN CASE OF AREA-LEARNING FOURTH GRADERS -

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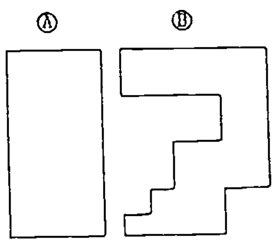
This study focuses on analyzing the pupils' understanding process of arithmetic. To analyze this is far-reaching and beyond our ability. Then we made the first step in viewing the example we get through practice. One is to present the fourth graders' understanding process of area-learning. The other is to present instruction learning process to improve a deeper understanding. The former is modality model concerning area learning and the latter is process model. The purpose of this study is to make a logical framework of the model which grew out from the classroom. In order to do this, arrangements will be made of the several grounds concerning understanding process. Then through comparing those grounds with models, general logical hypothesis will be given as to model structuring and the results will be considered.

1. Introduction:

We are interested in a fact when pupils have solved F-1 problem after finishing area-learning. Most pupils tried to solve the problem using (length × width) formula carefully. Then they can solve area problem Ⓐ but in case of problem Ⓑ in spite of the time spent dividing into pieces, we got a lot of incorrect answers because of using the formula many times. However, there were some pupils who solve the problem by formulating adequately voluntary unit (1/8 of the figure of problem Ⓐ) after watching the figure sometime without trying to solve at once. To those pupils, this problem was like a puzzle and fun, and the given answer was correct. This was quite a shock to us and made us think what it is to understand deeply the concepts of area.

F-1 Which is larger Ⓐ or Ⓑ ?



It so happened that a phrase has changed in the course of study and guidelines for arithmetic. "Have pupils to understand" has changed into "pupils understand". What seems to be a drastic change concerning understanding process has not been seriously considered. Most people don't pay a serious attention to the change. We have focused our study upon "understanding" because we were doubtful of the conventional teaching and we wanted to know the understanding process of arithmetic.

What is it for pupils to "understand" arithmetic. This is exactly what we try to find out in our study. But to solve this is far reaching and beyond our ability. Then we present our first step through one example we got in the classroom. It may be difficult to investigate pupils understanding process of arithmetic but it may not be so dificult to investigate what it is like for pupils to understand the area-learning (fourth grade). We will elaborate upon it in the following chapter.

2. Preceding Study:

(1) The interpretation of R.R.Skemp

The core of the famous work of Skemp is the distinction between relational understanding and instrumental understanding. He stated in the report, "He distinguishes these two understanding: relational understanding and instrumental understanding. The former means that both how and why are understood. instrumental understanding was not considered as understanding until recently. Formerly it was called regulation without reasoning"*(2)

Without knowing the reason one can obtain the circle-area only with the knowledge that radius × radius × 3.14. This is instrumental understanding. On the other hand, relational understanding is knowing the reasoning of the circle-area.

(2) The interpretation of S.I.Brown.

The core part of Brown is the distinction between internal understanding and external understanding . Brown defined the two understandings as follows: "To understand internally what X means is to know the relationship inside the X itself and to understand X externally is to know how it is related with others while considering X as a whole."*(3) As a matter of fact, to obtain the circle area by adopting "radius × radius × 3.14 formula, and also to know the formula is in the category of internal understanding. On the other hand, "What is the use of inducing area formula in measuring area?" and "Why do people try to measure the area?" seem in the category of external understanding.

(3) The interpretaion of Yutaka Saeki

Yutaka Saeki mentions in his book "Understanding" as follows: "To understand deeply things or phenomenon will coincide with the following accumulated factors. That is: ① solving the practical problems (problem solving), ② showing the reasoning of things (grounds for it), ③ correlations between actual society and culture (practicing socially), ④ widening of correlated society (developmental enlargemnt)"*(4) That is, understanding occurs when the practical problems are solved, the reasoning is shown, and accordingly combination of actual society and culture, then widening of correlated society.

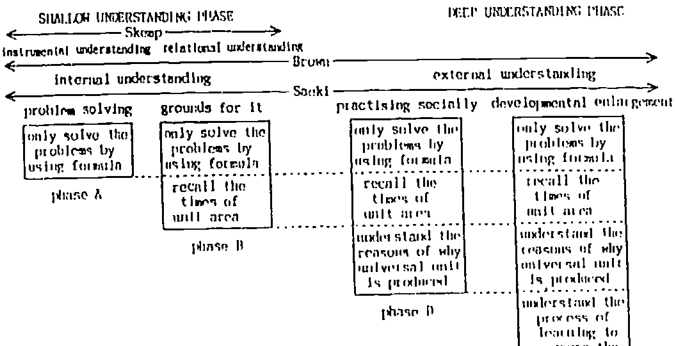
3. Modality model and Process model in area-learning:

We made the study of understanding, focusing upon "understanding the area" in the children's behaviour. As a result, based upon the F-1 research problem, the other problems, which you will find later in the paper, suggested that there are phases in the understanding of area learning. This is obviously endorsed by the result from the problem 1 solving the area of 3 cm × 5 cm rectangle. When we had our pupils explain the meaning of 15 cm^2, pupils came into two categories: One could only explain "as a result of the formula" and the other could explain how many times of area unit perspective. We thought that the framework of the two groups corresponds with that of Skemp's instrumental understanding and relational understanding.

Moreover, according to "Adding up and Consideration of Achievement Analysis" at Tokyo Arithmetic Education Study Group, 1990 *(5), there is as low as 47 percent of correct answer to the problem asking the area of classroom among 34143 fourth graders. The problem is a multiple choice: to choose 60 cm^2, 600 cm^2, 6000 cm^2, 60 m^2, 6 km^2. The pupils came into two groups: one group have the ability of calculating in the situational setting and the other group do not have such ability. This shows that there are more phases of understadning besides Skemp's instrumental understanding and relational understanding which we discussed before. And there need to be another set phases to explain our result.

Then we call the Skemp's set of phases of instrumental understanding and relational understanding, "shallowing understanding phase". We thought that the set of phases, with which we cannot explain our result, corresponds with Brown's external understadning or Saeki's practicing socially and developmental enlargement. And we set up a hypothesis and call it, "deep understadning phase".

Then the fact suggests us that the following four phases (modality model).



MODEL OF UNDERSTANDING PHASE (MODALITY MODEL)"

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Phase A: Pupils cannot solve the problem by using the formula. Phase B: Pupils can use the formula and also can explain. Phase C: Pupils can understand the birth of universal unit in addition to phases A and B. Phase D: Pupils can understand the necessity of using larger units or smaller units in addition of all the phases above mentioned.

These models are bases upon the classroom teaching at elementary school and the phase D pupils can be considered to have reached the "deep understanding" which we believe to be ideal. The consideration of phase D pupils can develop the creative activities themselves through which to find "unit". With the unit they can adjust themselves after entering a wider society, such as play-ground and parks, out from the classroom situation. Phase D pupils can also distinguish themselves at changing perspectives. Also, they are the children who can name their own unit because of the attachment of the unit based upon thoir own classroom activities. Using the unit, they further can create the new units one after another according to the new situations. As a result, they will have grown into the children who can understand the background of the brith of universal unit.

We considered the phase D pupils to have a good understanding of the area and we made a format of learning process model and practised at some elementary schools.

"LEARNING PROCESS MODEL OF UNDERSTANDING"

FIRST: Understanding why universal unit is produced
- (1) Creating the area of my own. (territory gaining game)
- (2) This is the unit of my own. (naming the voluntary unit of one's own finding) "1 Othello", "1 Sakata", "1 Mario", "1 Domino".
- (3) This is the unit of our own. (producing the common unit usable in classroom society) "1F" F is the initial of homeroom teacher.
- (4) My size is "14F" and my size is "18F"

SECOND: Understanding how people learn to mearure the larger size and smaller size.
- (1) The unit of our own usable anywhere: "1F"
- (2) Let's create the larger unit...1F
  Let's create the smaller unit...1f
- (3) The size of our classroom is 100F.
  The sizo of a telephone card is 8f.

THIRD: Understanding the grounds of area formula being porduced.
Let's produce the formula. It's troublesome to lay units all the while.
(AREA)=(WIDTH)x(LENGTH)

FOURTH: Understanding the reasons of how the universal unit is produced.
We want to know the convenient unit commonly used in the world.
cm^2, m^2, km^2.

4. Comparison Between Modality Model and Understanding Theory:

In the process of our study we were amazed at the fact that Skemp took up as an example of area solving when he discusses relational understanding and instrumental understanding. In this case he defines the instrumental understanding the solution getting by the use of the formula width × length. He calls the knowledge of why to

multiple, relational understanding.

This is to say that our notion of phases arising from solving the area of 3 cm × 5 cm rectangle corresponds with distinguishing instrumental understanding and relational understanding.

Let us compare phase-concept we presented with "relational understanding" and "instrumental understanding".

Our phase A of being able to solve the problem by using the formula is in the category of instrumental understanding and our phase B of being able to use the formula and also to explain it, is in the category of relational understanding. But the difficulty lies in the discussion of phase C and phase D pupils. Those pupils who understand the birth of universal unit are the pupils who have the knowledge of understanding the background of cm^2 and m^2 in the society, because they have experienced the birth of valid unit. We find it rather difficult to put those pupils in the category of relational understanding. Another category would be needed.

The same is also true of phase D pupils, who are able to change perspectives on a case-by case basis. Those pupils are also difficult to discuss within the framework of relational understanding. To approach changing perspective mathematically, we think that in addition to relational understanding, we would also need intuitional understanding and logical understanding proposed by two Canadians, Byers and Herscovics.

Here we must distinguish between internal understanding and external understanding before comparing internal understanding and external understanding, as Brown put it, and our understanding model.

As mentioned in the first section, Brown defines the distinction between the two understandings. Hirabayashi mentions that we hold an "understanding rope" and that new object is to be hung upon the rope to identify its location. This is what he thinks external understanding and pupils begin examining the object "hung un the rope". This is like a spider approaches the hunt on the web and this process can be considered to be internal understanding. By *(3), this idea of external and internal understandings can be thought to be set in a definite framework. Accordingly, the problem is what kind of framework is to be set. Hirabayashi says, "Math education in Japan can be characterized as perhaps something underpinning internal understanding of materials and lacking external understanding. This is exactly like "Being unable to see the wood for the trees" which is often quoted to imply today's educational system. Therefore, the difference between the framework of internal understanding and external understanding is that the concept of internal understanding is analyzing and external understanding is how to locate the concept in the whole area. In connection of our model, phase A in which pupils can analyze its formula can be thought to be internal understanding. And to be able to understand the birth of

universal unit is to be able to understand its background process, so this could be framed into external understanding. Phase D in which pupils can understand in addition to the phases above mentioned, the necessity of using larger units or smaller units could be in the framework of external understanding.

This brings us to believe that "understanding phases" we presented, that is, modality model, is the model which includes both internal and external understandings. The development of class through "learning and studying process model" which created phase D pupils can the thought to be the learning-process model bases upon external understanding, the type of model never developed in Japan before.

Saeki mentions in his book *(4) that there are four conditions in the process of understanding ① problem-solving, ② grounds for it, ③ practising socially, ④ developmental enlargement of it. This is to say, the phase in which concrete problems can be solved and the reasoning can be presented corresponds with model A and B. If practicing it socially is to be something including universal unit of social background, it corresponds with phase C. And if developmental enlargement of it is to be considered to create unit from the different perspective, it fits exactly phase D and the model we presented.

Accordingly, what Skemp, Brown, and Saeki say about understanding corresponds wonderfully with our model.

5 Results and Consideration:

In order to verify the effectiveness of "understanding learning models" we propose which helps pupils reach deep understanding (phase D), we made evaluating problems 1 to 5 and considered the results. Here we evaluted phase A as problem 1, phase B as problem 2, phase C as problem 3, and phase D as problems 4 and 5.

"RESULTS OF UNDERSTANDING MEASUREMENT"

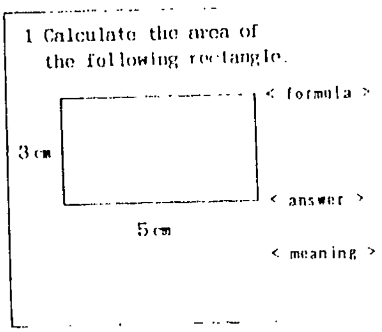
PUPILS IN EACH PHASE	GENERAL GROUP	HYPOTHITICAL GROUP
staying in phase A	41 %	0 %
staying in phase B	34	11
staying in phase C	16	15
having reached phase D	9	74

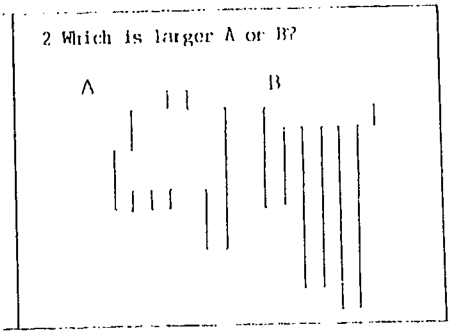
Note: Phase A pupils can only solve the problems by using formula.
Phase B pupils can understand the meaning of formula.
Phase C pupils can understand the birth of universal unit.
Phase D pupils can understand the larger and smaller units.

The results indicated that among pupils in general group those who are not able to understand the meaning of formula and try to obtain the answer formally consist 41 %.

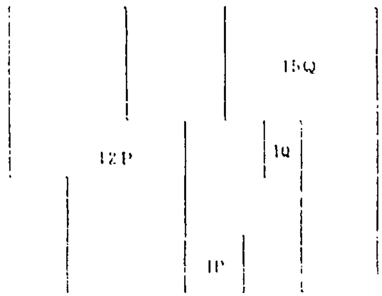
On the other hand, those who stayed in phase A consist 0%. This means that pupils in general group stayed in Skemp's instrumental understanding , whereas most pupils who had experienced this learning model attained relational understanding. The excessively low ratios of pupils among general group in phase C and phase D and high ratio (74%) among experimental group indicate that pupils who experienced this learning model have reached not only inner understanding buc also external understanding, as Brown put it. We should like to omit the other researches and problems for lack fo space. But in discussing the pupils in the process of experimenting this learning model, the phase in which pupils solve the practical area-problem and are able to explain the grounds for it corresponds with Saoki's solving the practical problems and showing the reasoning of things. And those who reached phase C have grown into pupils who find voluntary unit and create new units one after the other because of a deep attachment to units. They go on to understand the background of the birth of universal unit now being used. In other words, they recognize the necessity of creating common unit when a society meets another society. This can be regarded as practising socially. And phase D, the deepest form of understanding we proposed involves, added to the three other phases, the ability to conceive units corresponding to each scene. Pupils in phase D we just discussed are those who develop activities which correspond to wider society out of classroom situation, such as play-grounds or parks. And in these activities they can create their own units and see things in a different perspective. This can be looked upon as widening of correlated soiety.

For reasons mentioned above, "understanding phase model" we proposed is a new concept on understanding (framework of understanding) which is not only based on Skemp's concept but also combined with the understanding concepts of Brown and Sacki. Our "understadning model" surely helps pupils attain what we call "deep understanding". We firmly believe the validity of our learning model.

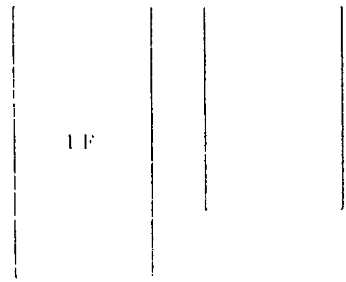




3 Which country is larger? Solve the foll wing based upon "1P" and "1Q".



4 Suppose the size of the left is "1F". Describe the size of the left by using "1F".



5 Suppose the size of the left is "1P" Describe the size of the right by using "1P"

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「考えることの教育」 佐伯 胖
「Young Children Reivent Arithmetic (IMPLICATIONS OF PIAGET'S THEORY)」
Constance Kazuko Kamii 平林 一栄監修

CAUSAL RELATIONSHIPS BETWEEN MATHEMATICS ACHIEVEMENT, ATTITUDES TOWARD MATHEMATICS AND PERCEPTION OF MATHEMATICS TEACHER ON SECONDARY SCHOOL STUDENTS

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(Abstract)
The purpose of this study is to find causal relationships between student mathematics achievement, attitudes toward mathematics, and student perception of mathematics teacher on two models. Model 1 reveals influence from perception of mathematics teacher and attitudes toward mathematics to mathematics achievement. Model 2 reveals influence from mathematics achievement and perception of mathematics teacher to attitudes toward mathematics. Path diagrams were constructed with significant standard partial regression coefficient based on two models.
There were many significant relationships between perception of mathematics teacher and attitudes toward mathematics. There were few significant direct relationships between perception of mathematics teacher and mathematics achievement.

1. Introduction

Recently evaluation of attitude toward mathematics has been a topic of concern, and the topic related to the affective variables have been attended at length in Japan . There have been several extensive reviews of research on attitudes toward mathematics and affective variables as related to mathematics learning. (Aiken 1970, 1976, Kulm 1980, Lorentz,1982, Reyes 1984)

Lorentz's review revealed that it was not only important to find a relationship between mathematics achievement and attitudes toward mathematics, but it was also important to find relationships among affective variables relating to attitudes on mathematics.

There are internal variables and external variables relating to attitudes toward mathematics. The former are mathematics achievement and intelligent level, and the latter are teacher variables, learning environoment and one's home environoment.

Haladyna etc.(1983) found a causal relationship among student motivation , teacher quality, social psychological class climate, management - organization class climate, and attitudes toward mathematics.

Etington and Wolfle (1984) examined the reason men and women differ in mathematics achievement by means of a covariance structure in a causal model of mathematics achievement.

In Japan Minato and Kamata (1988) tried to find the causal relationships between attitudes toward mathematics and mathematics achievement with CLPC analysis. They found certain relationships about them, but didn't find any clear relationships with teacher variables.

Imai(1985) found relationships between attitudes toward mathematics and mathematics

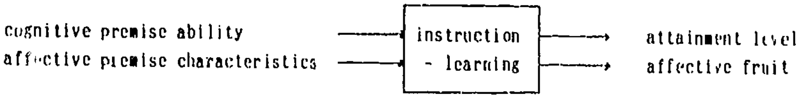
achievement, between attitudes toward mathematics and perception of mathematics teacher.

Sasaki and Imai(1984,1985) found relationships between attitude toward mathematics and student feeling of parental attitude toward mathematics.

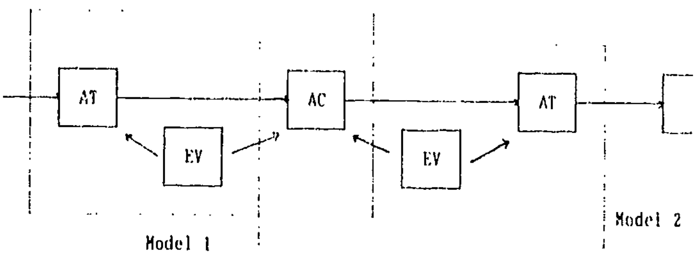
In this study I will focus to find the causal relationships between mathematics achievement, attitudes toward mathematics, and perception of mathematics teacher

.2. A model for some variables as related to mathematics achievement, attitude toward mathematics and perception of mathematics teacher

Bloom,B.S. set up the instruction - learning process model about cognitive premise ability, affective premise characteristic, attainment level, affective fruit.



These variables are shown with time passing in this model. In addition to this model, I think teacher variables like external variables affect mathematics achievement and attitudes toward mathematics. I will try to find the relationships between attitudes toward mathematics (AT) as affective variables, mathematics achievement (AC) as cognitive variable, and perception of mathematics teacher (teacher variable) as external variable (EV), using these models. We can find influence from attitudes toward mathematics and perception of mathematics teacher to mathematics achievement on Model 1; from mathematics achievement and perception of mathematics teacher to attitudes toward mathematics on Model 2



3. Method

(1) Sample

159 junior high school students(ST1) and 63 senior high school students (ST2) participated in this study. All students were taught by the same teacher.

(2) Instrument

Mathematics Achievements (AC1, AC2) were results of periodic examination in their school. AC1 was a result of first examination (junior high school: number and caluculation, simple equation, senior high school: triangle function, experimental function) and AC2 was a result of final examination (junior high school: number and

caluculation, simple equation, direct and inverse proportion, geometry, senior high school: elementary function, number line, calculus). I measured attitudes toward mathematics by the Mathematics Attitude Inventory (Sandman,1973, MAI; Motivation (MO), Enjoyment (EN), Self-concept (SC), Value (VA), Anxiety(AN)). There are two reasons why this particular instrument was used: One reason is that MAI is constructed by multidimentional scales. The other reason is that Aiken estimated this instrument as being useful in his review.

Brassel investigated junior high school student attitudes toward mathematics with MAI. (Brassell,1980)

Imai(1986) analyzed the varidity and reliability at MAI on juior high school students of Japan. I used the instrument measuring perception of mathematics teacher (PMT), which I(1985) developed in order to measure teacher variables relating to student mathematics learning in a previous study . This scale consisted of 14 subcategories, 7 categories and 29 items.

Difficulty toward mathematics(DI) and willingness about mathematics learning (WI) were measured on seven point scales with "Math is difficult " - "Math is easy", and "I want to study math willingly" - "I don't want to study math willingly".

< Variables >

AC1, AC2		Mathematics achievement
AT		Attitude toward mathematics
	MO	Motivation in mathematics
	EN	Enjoyment of mathematics
	SC	Self-concept in mathematics
	VA	Value of mathematics in society
	AN	Anxiety toward mathematics
	DI	Difficulty toward mathematics
	WI	Willingness about mathematics learning
FV		External variables
	T1	Teacher character
	T2	Favor toward math teacher
	T3	Confidence toward math teacher
	T4	Teacher's attitude toward math
	T5	Teacher's instruction
	T6	Device of motivation
	T7	Teaching how to think and solve problem

(3) Procedure

I measured attitudes toward mathematics in the final term, and I measured students' perception of the mathematics teacher in first term and final term. The administrator read all items to the junior high school students. High school students read all items themselves and responded to them.

I caluculated 4,3,2,1 points to responses for " strong agree "," agree ", " disagree ", " strong disagree " on MAI and I caluculated 5,4,3,2,1 points to responses for " strong agree "," agree ", " disjudgement ", " disagree "," strong disagree " on PTE .

Data were eliminated for students who completed only part of inventory. All responses were anonymous to encourage honest responses and to ensure confidentiality.

(4) Analysis of data

Internal correlation coefficients among 14 scales, AC1 or AC2, MO, EN, SC,VA, AN,DI or WI, T1, T2, T3, T4, T5, T6, T7, were caluculated. To investigate the validity of these models and to explain causal detemination of relationships among them, the technique of path analysis was used (Ethington & Woefle, 1984). This procedure is useful for examining direct and indirect effects of the constructs. Statistically ordinary least-squares regression analysis was used to estimate path coefficients (standard partial regression). I analyzed the significance of standard partial regression coefficients from independent variables to dependent variables. Significance of standard partial regression coefficients were estimated with F statistics which were caluculated from the difference of R_f^2(full model) and R_n^2(restricted model).

4. Results and discussion

(1) Model 1

On path diagram of junior high school students(Fig.1) there were significant path relations from EN, SC and AN to AC2, but not from PTE (T1 ~ T7) to AC2. These results reveal that the number of significant relations from attitudes toward mathematics to AC2 is more than the number of significant relations from perception of mathematics teacher to AC2. There were significant path relations from DI to some attitude variables (for example : MO, EN, SC, AN). There were significant path relations from T4(Teacher attitude toward mathematics) and T5(Teacher's instruction)to DI. There were some significant path coefficients from perception of mathematics teacher to attitudes toward mathematics. These were significant relations about T2 → EN, T2 → SC, T2 → VA. These results suggest a humanistic relationship between teacher and students influence the students' att toward mathematics. There were significant path relations from T6 to EN and VA. These results stress that for enjoyment of mathematics to increase and for favorable feelings about the value of mathematics to be achieved the teacher's influence must be positive. On the path diagram of senior high school students there were significant path relations from MO, SC and T2 to AC2. Relations from EN and AN to AC2, which were significant on junior high school students, were not significant on senior high school students. But relations from T2 to AC2, which were not significant on junior high school students, were significant on senior high school students. This result reveals that students' favorable relationship toward their mathematics teacher relates to mathematics achievement as fruits of mathematics learning. Relations from T1 to attitudes toward mathematics were not significant on junior high school students, but relations from T1 to MO,EN and SC were significant on senior high school students. These results suggest that teacher's

characteristic(T1) relate to more attitude variables on senior high school students than that on junior high school students. We would like to pay attention to the results of the relation from T5 to EN, which were not significant on junior high school students, but were significant on senior high school students. This result insists that teacher's instruction influences students' enjoyment on mathematics in senior high school. There were significant path relations from DI to SC and AN in senior high school. And there were significant path relations about T4, T5 → DI → MO, EN, SC, AN in junior high school. The latter result reveals that DI is a medium variable between perception of mathematics teacher and attitudes toward mathematics on junior high school students.

(2) Model 2

On the path diagram in junior high school (Fig.3) WI related to MO and T5 significantly. There were significant relations from AC1 to attitudes toward mathematics (MO,EN,SC,AN), from perception of mathematics teacher to attitudes toward mathematics (T2 → EN, VA, T6 → EN, VA, T5 → AN).

I think that T2,T5 and T6 relate to attitudes toward mathematics greatly. Some relations, from perception of mathematics teacher and mathematics achievement to attitudes toward mathematics about AC1 → MO, T1 → MO, EN, SC, T4 → EN, T5 → EN, T5 → SC, T5 → AN, T6 → VA, T6 → AN were significant on senior high school students. Results about senior high school students(Fig.4)that had no significant relationship from perception of mathematics teacher to mathematics achievement, stress that perception of mathematics teacher are not relative variables to mathematics achievement. There were significant relations from T6 to VA on both junior and senior high school students. These results stress that the teacher's influence on motivation(T6) affect students' feeling on the value of mathematics and mathematics learning. Some relations about T2 → EN, T2 → AC1 → EN , T2 → VA which were significant on junior high school students, were not significant on senior high school students. Relation from T5 to AN was significant path relation on junior high school students and on senior high school students. These results reveal that methodlogical instruction aspects, for example the degree of the ease to understand, relate greatly to students' feeling of anxiety toward mathematics and mathematics learning. A significant relation between T6 and VA stresses that perception of mathematics teaching about instructional influence on students' mathematics learning translate to the students' feeling about the value of mathematics.

5. Conclusions and implication

This research is a pilot study concerning the causal relationships between mathematics achievement as cognitive variables, attitudes toward mathematics as affective variable, and perception of mathematics teacher as external variables on Japanese secondary school students.

I found many causal relationships on two models. The following were the major points :

・There were many significant causal relationships between attitudes toward mathematics and perceptions of mathematics teacher on junior high school students and senior high

school students.

· There were few significant direct relations between perception of mathematics teacher and mathematics achievement.

· Difficulty toward mathematics is mediate variable between perception of mathematics teacher and mathematics achievements.

Perception of mathematics teacher was the only external variable used with the two models in this study. I would like to analyze the causal relationships on various models containing many external variables in the future.

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Table.1 Standard partial regression coefficient (Model 1)(ST1)

	AC2	DI	MO	EN	SC	VA	AN
T1	-0.096	0.034	0.039	-0.069	-0.013	0.103	-0.145
T2	0.151	-0.049	0.061	0.296**	0.238**	0.249*	0.042
T3	0.107	-0.107	0.125	0.018	-0.043	0.043	-0.019
T4	0.085	0.245**	0.061	0.109	0.155*	0.071	-0.021
T5	-0.023	-0.293**	-0.026	-0.025	-0.034	-0.100	-0.167
T6	-0.145	-0.009	0.157	0.196*	0.032	0.189*	-0.092
T7	-0.016	0.041	0.117	-0.104	-0.062	0.050	0.121
DI	-0.043		-0.230**	-0.480**	-0.532**	-0.122	0.474**
MO	0.191						
EN	-0.381**						
SC	0.449**						
VA	0.007						
AN	-0.294*						

Significant level : ** 1% * 5%

Table.2 Standard partial regression coefficient (Model 1)(ST2)

	AC2	DI	MO	EN	SC	VA	AN
T1	0.089	0.191	-0.315*	-0.449*	-0.418*	-0.112	0.122
T2	-0.327*	-0.170	0.091	0.078	0.020	0.276	0.189
T3	0.149	-0.248	0.091	0.059	0.126	-0.019	0.050
T4	0.062	0.065	0.067	0.306*	0.185	0.157	-0.064
T5	-0.162	-0.097	0.263	0.390*	0.336*	0.020	-0.610**
T6	-0.078	-0.175	0.104	0.176	0.132	0.342*	-0.257
T7	-0.128	-0.023	-0.012	-0.075	-0.170	-0.043	0.237
DI	-0.130		-0.220	-0.139	-0.338*	-0.006	0.329**
MO	0.205*						
EN	-0.052						
SC	0.254*						
VA	0.084						
AN	-0.094						

Significant level : ** 1% * 5%

Table.3 Standard partial regression coefficient (Model 2)(ST1)

	WI	AC1	MO	EN	SC	VA	AN
T1	0.026	-0.094	0.050	-0.068	0.002	0.053	-0.160
T2	-0.121	0.212*	0.029	0.260*	0.189	0.242*	0.087
T3	0.047	0.132	0.123	0.045	-0.032	0.048	-0.027
T4	0.143	0.012	0.002	-0.010	0.020	0.040	0.099
T5	0.202*	0.079	0.024	0.100	0.093	-0.069	-0.281*
T6	-0.091	-0.098	0.179	0.218*	0.071	0.197*	-0.128
T7	0.019	-0.024	-0.113	-0.119	-0.076	-0.047	0.133
AC1	-0.080		-0.204**	0.186*	0.353**	0.061	-0.321**
MO	0.191*						
EN	0.213						
SC	0.056						
VA	0.081						
AN	-0.151						

Significant level : ** 1% * 5%

Table.4 Standard partial regression coefficient (Model 2)(ST2)

	WI	AC1	MO	EN	SC	VA	AN
T1	0.055	-0.065	-0.339*	-0.471**	-0.478*	-0.101	0.183
T2	0.172	-0.005	0.133	0.102	0.078	0.278	0.133
T3	-0.104	0.201	-0.019	0.008	0.030	-0.056	0.137
T4	0.044	0.158	0.036	0.302*	0.197	0.129	-0.081
T5	-0.163	0.164	0.238	0.391*	0.359*	-0.007	-0.638**
T6	-0.041	-0.209	0.201	0.217	0.204	0.354*	-0.321*
T7	0.046	-0.019	-0.001	-0.069	-0.162	-0.039	0.228
AC1	-0.149		-0.277*	0.078	0.059	0.173	-0.027
MO	0.192						
EN	0.276						
SC	0.089						
VA	0.078						
AN	-0.115						

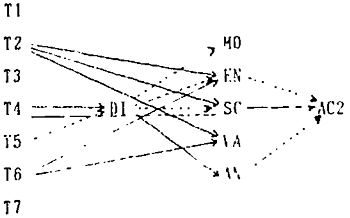

Fig.1 Significant relationships on Model 1 (ST1)

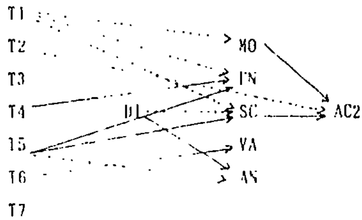

Fig.2 Significant relationships on Model 1 (ST2)

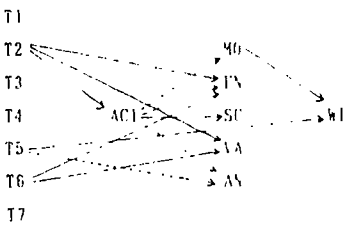

Fig.3 Significant relationships on Model 2 (ST1)

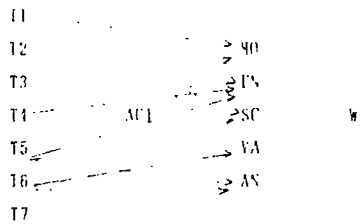

Fig.4 Significant relationships on Model 2 (ST2)

Table.5 Sample items of the scale measuring perception toward math teacher

category	sample item
T1: Teacher character	My mathematics teacher teaches me kindly when I ask him a question.
T2: Favor toward math techer	As my mathematics teacher is unsociable, I do'nt like him.
T3: Confidence toward math teacher	As my mathematics teacher is earnest, I respect my mathematics teacher.
T4: Teacher's attitude toward math	My mathematics teacher is interested in mathematics.
T5: Teacher's instruction	My mathematics teacher teaches mathematics intelligibly.
T6: Device of motivation	My mathematics teacher gives us various practice activities in our mathematics class.
T7: Teaching how to think and solve problem	My mathematics teacher makes us always think why a problem can be solved in a particular way.

A SURVEY OF CURRENT CONCEPTS OF PROOF HELD BY FIRST YEAR MATHEMATICS STUDENTS

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This paper presents results of a questionnaire concerning concepts of proof held by first year undergraduates. Students were asked to comment on arguments which were presented to them as "proofs". Their reasons for judging the validity of arguments, rather than being based on logic or rigor, mirrored Hanna's (1989) description of the practising mathematicians' criteria for accepting theorems. Three modes of judgement have been identified: empirical, logical and aesthetic. Problem areas for students included following the chain of reasoning, using mathematical concepts and knowledge, and making sense of mathematical language.

Introduction

Although formal proof plays a prominent role in higher (degree level) mathematics (Hanna, 1989, 1983), attempts to teach the notion of proof are somewhat haphazard at all stages of British mathematics education, particularly at school level. Furinghetti & Paola (1991) describe a similar situation in Italian schools. Students are expected by their university tutors to have built up a concept of 'formal proof' from examples given in the classroom, with very little or no guidance. The fact that the term 'proof' has a substantially different meaning in mathematics to that in everyday life, and the current philosophical confusion about what proof really is (Tymoczko, 1979) are further hindrances. The student therefore faces the unenviable task of building a construction of the concept of proof to match that held by the established mathematicians who set the examinations, if they are to be able to provide the required answers. As Senk (1985) has shown, a high proportion of students fail to achieve this.

In this paper we present an analysis of a questionnaire to investigate the current concepts of proof held by first year undergraduate students at South Bank University, in the first semester of the Mathematical Contexts and Strategies course. This analysis is the first stage of an ongoing project designed to develop and evaluate materials for teaching the notions of proof to sixth form and first year undergraduate students.

The questionnaire attempts, initially, to determine what experience of proof students had at school. Subsequently the theorem that the angles of any (Euclidean) triangle sum to 180° is presented, and the students are asked to comment on the validity of a selection of "proofs" given to establish this fact.

Proof at school

The first question asked was: 'Whilst at school, did you ever "prove" anything in mathematics? If your answer is yes could you also give a short example of something you proved.' The word "prove" was placed in quotation marks to draw attention to it, and yet leave its exact meaning purposely vague. Typical examples of theorems proved at school included a variety of trigonometric identities, such as $\tan x = \dfrac{\sin x}{\cos x}$, Pythagoras' Theorem, various theorems connected with circles, and a few analysis theorems such as the fact that the sum of two even numbers is even.

One of the answers given stood out.

'No, but I was asked to write down a proof of something, e.g. $\frac{d(\tan x)}{dx} = \sec^2 x$ '

It seems that the student has read the question 'did you "prove" anything at school' to mean 'did you personally "prove" anything new', the emphasis being placed on 'you' and not on '"proved"'. (If this is a correct interpretation of the answer given would need to be established in a later interview.)

The fact that over two thirds of the students answered 'No' or left this question blank indeed suggests that there is some confusion concerning the notion of proof in British school mathematics. Proofs of various theorems will have been presented at school, but possibly without the fact that they are proofs being made explicit. This may have resulted in some students being unsure what was meant by the term "prove". Even those aware of the term may have been expressing the view indicated in the quote above, namely that their writing out a proof does not constitute "proving" a theorem.

A theorem about the sum of the interior angles of a triangle

The next section of the questionnaire offered six arguments purporting to prove the theorem that the angles of a Euclidean triangle sum to 180°. The students were asked to consider each argument in turn, and discuss, with reasons, whether it was a 'good proof' or not. The following table categorises the comments written by the students on each proof:

Proof number	Good	Not Good	Lack of understanding	Undecided/ Blank
1	11	17	0	5
2	11	11	2	9
3	21	5	1	6
4	21	2	2	8
5	13	9	1	10
6	7	6	5	15

fig. 1 - A Table Classifying Student Comments on a Selection of Proofs

We have used **Lack of understanding** to indicate that the student expressed an opinion which clearly indicated a lack of comprehension of the proof given - such as writing 'I don't understand this proof'. Where the decision to include a comment in the **Lack of understanding** column instead of in another column was difficult, the strategy of giving other columns precedence was adopted. For example when one student stated 'This is not a good proof because the angles on a straight line might sum to 200°' his comment was tallied under **Not Good**, and not in the **Lack of understanding** column, because despite being unaware that two right angles equal 180° (an obvious case of lack of understanding) the student made a clear statement that he was against the proof.

Proof 1 The 'proof' given for the theorem involved measuring and summing the angles of a thousand triangles, and showing that the average of these sums was 180° by using a protractor. At most, this could be considered to be an empirical verification of the theorem for a number of test cases. It is not a mathematical proof.

Comments made by students indicating that this was not a good proof included:

'There are an infinite number of triangles in existence. One of these could possibly disprove this formula.'
'This proof is inaccurate For a topic such as mathematics accuracy is a key feature'

'An approximation and therefore not a definite conclusion'

As can be seen above, the two reasons given for disliking proof 1 were that it was inaccurate, and that there might be an unexamined counterexample. None of the criticisms of the proof mentioned that it did not use any form of logical reasoning.

Some of the comments made by students suggesting this was a good proof were:

> 'It is a good proof because if you are a mathematician proving something the different results that you get their difference mustn't be more than 5 or 4 points.'
> 'Yes it's right that the sum of the angles of a triangle is equal to two right angles because we can proof the angles of a triangle add up to 180°.'
> 'A good proof as he found the conect measurement.'

The reasons given for liking the proof were usually unclear. In a few cases, such as the first quote, an empirical justification was given - namely that if we try enough cases, and get a low error then it is highly probable that the conjecture is correct. The statement '5 or 4 points' is possibly a reference to the convention in school science of considering an experiment successful if the estimated error is less than 5% of the result obtained.

Proof 2

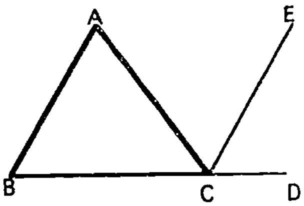

fig. 2 - The Diagram for Euclid's Proof.

This proof of the theorem is that given in book 1 of Euclid's *Elements*. It relies on previous propositions of the equality of alternate and corresponding angles, and the construction of a line parallel to one of the triangle's sides, and passing through the remaining vertex (See fig. 2). Comments made indicating that this proof was good included:

> 'I approve of this method as it shows logic.'
> 'I agree with the proof as I have used its assumptions of alternate and corresponding angles when I was at school.'

Most comments also included labelling the unlabelled angles of the diagram given, and writing out the algebraic sums of the angles involved with the proof. It is noteworthy that although this is a classic example of a textbook proof, only one third of the students considered it to prove the theorem.

Some of the comments made against this proof included:

> 'Doesn't prove it Statements don't follow and are not relevant to proof'
> 'This proof seems vague to me as it does not mention anything about the extended BC, nor the parallel line CE'
> 'It doesn't follow from what has previously been said in the proof. If the angles in a triangle added up to 200°, everything else about the proof would still be correct except for the last line.'

The most common reason for disliking the proof was due to a misunderstanding of one of the steps in the proof, by not understanding how one step led to the next, or by disagreeing with one of the initial premises.

Problems arose about the extension of BC, and the construction of CE, with most of the students wanting to know why this was done.

Other problems arose with the hidden premise that the angles on a straight line sum to 180°. The convention of labelling the angle of a straight line with the symbols '180°' is usually stated as plain fact at school, without its implications on later propositions being considered. Some of the students wrote that they could see that the three angles of the triangle corresponded to three angles on the line BCD, centred at C, but for some reason failed to see that therefore the sum of these three angles must equal 180°. The results on this part of the questionnaire verify studies by Schoenfeld who says:

> '... much of the mathematical knowledge that the students had at their disposal, and that they should have been able to use, went unused in problem solving.' (1985, p. 13)

This statement is true for students analysing proofs as well as for those involved in problem solving.

In a couple of cases it was clear the students had failed to grasp some of the mathematical concepts required to understand the proof, for example:

> 'No values for the angles are given, therefore it is not possible to find the sum.'

It is possible that the student cannot cope with the concept of a general triangle, or the fact that it is sometimes not necessary to know the values of variables in an equation to find their sum (e.g. $4x - 5x + x = 0$ for all x). Another student did not understand the meaning of 'Extended BC to D', and showed this by making the following comment:

> 'BCD may not be a straight line'

We interpret this example as a lack of understanding of mathematical language, rather than a lack of understanding of the mathematical concepts involved.

Proof 3

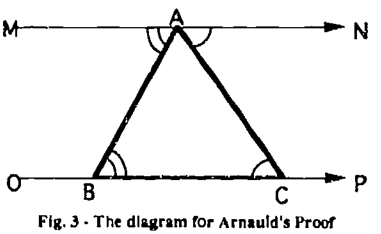

Fig. 3 - The diagram for Arnauld's Proof

This proof of the theorem is that given in Book VIII of Arnauld's *Nouveaux éléments de géométrie*. The proof removes the assumption of corresponding angles, and relies purely on Euclid's parallel postulate and the equality of alternate angles, by constructing a triangle with two vertices on one line, and the third vertex on a parallel line (see fig. 3). Proof 3 can be viewed as a clearer version of proof 2 (Barbin, 1989), although the perception that a proof is "clear" is obviously subjective and requires further investigation. Some of the positive comments made about this proof were:

> 'I feel that the proof is self-explanatory.'
> 'This tells us about angles on a straight line, therefore it is possible to prove.'
> 'Yes, proof works. Easy to understand Clear.'
> 'This proof is like the last one.' (*The student wrote positively about proof 2.*)

The main reason for liking the proof in each case was due to the fact that it was clear, and seemed 'reasonable'. It is interesting to note that only two students remarked on the fact that proof 2 and proof 3 are almost identical. However, the extra clarity of proof 3 resulted in the number of positive statements made about it being double the number made about proof 2.

Some of the reasons given as to why proof 3 might be a bad proof were as follows:

> 'This would not convince me to believe the theorem, because nothing particular points to the actual figure of 180°.'
> 'To prove something we have to use pure mathematic way not use shape or drawing something.'
> 'Too abstract'

As can be seen from the comments above, the reasons for disliking proof 3 were varied. Some students disliked the proof because they could not follow its reasoning. Other students indicated that the proof was too abstract, or not abstract enough. Once again some students failed to see why the sum of the angles of a triangle should be 180°, even though the previous line of the proof convinced them that the three angles of that triangle were equal to three angles on a straight line.

Proof 4

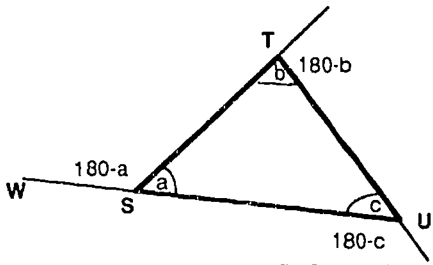

Fig. 4 - The Diagram for a LOGO-Style Proof

This proof is based on a LOGO style of presentation, and asks the students to consider the angle a 'robot' would turn through as it moves along the edges of a triangle. Although not found in any textbook, the proof is valid, and has the advantage of being a general proof for the sum of the interior angles of any convex polygon (see fig. 4). Comments made in favour of the proof included:

> 'This gives a much clearer proof. Giving a specific formula is easier to prove.'
> 'The logic behind [this proof] is very straightforward.'
> 'This is very clear. I would be convinced by this proof, and it is very simple to follow.'
> 'I think it's best way to proof because each step of proof has been explained and a person which doesn't know anything about triangle can understand the reason of $a + b + c = 180$°'

The reasons for liking proof 4 were similar to those given for proofs 2 and 3, namely that it was straightforward and logical. The fact that the workings of the proof could be reconstructed visually, by actually imagining a 'robot' moving around the triangle may have helped. Finally, the proof relies on fewer, more intuitive premises, unlike the Euclidean proofs in 2 and 3 (the concepts of corresponding and alternate angles, and the construction of parallel lines are not present), and it is likely that this made the proof clearer and therefore more credible to the students (Tall, 1992, 1979).

Only two of the students disliked this proof. Their comments were:

> 'We do not know the angles a, b and c.'
> 'Poor.'

This proof had the smallest negative response of all the arguments given.

That almost a quarter of the students made no response or gave no decisive opinion is most likely due to the fact that it is the longest proof presented in the questionnaire - it contains at least twice as much text as any other, and some students may have been put off by this.

Proof 5 The theorem is demonstrated for a particular triangle, by cutting it in three, and placing the angles on a line. Although visually appealing, this does not constitute a proof, as the theorem is only tested on one triangle. Responses in favour of this proof included:

> 'Does not involve complex mathematical equations.'
> 'It is easier to understand things pictorially'

As the comments show, the reasons for liking this 'proof' were mainly because it was a visual, 'hands-on' argument, and was very easy to understand.

Only a few of the comments indicated dissatisfaction due to the lack of generality of the demonstration:

> 'Good for one triangle only Poor to assume all triangles'

Most comments against the 'proof' centred on the fact that using paper and scissors would not be accurate, or that it would be easy to force the corners of the triangle to fit onto a straight line, for example:

> 'We could say that some of this has been fiddles, i.e. inaccurate cuttings, adjusting parts of the cut triangles'

Proof 5 is written in 'art and crafts language', referring to scissors, card, and ripping up paper triangles. This may have caused many of the students to switch out of a mathematical mode of thinking and into an every-day mode As a result there are very few criticisms of the lack of formality or rigor of the argument given, with most comments focusing on the limitations of paper and scissors.

Proof 6

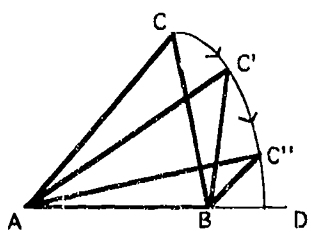

Fig. 5 - The Diagram for Proof 6

This argument relies on the continuous variation of a triangle. Mathematical terms, a complicated diagram (see fig. 5), and authoritative language such as 'intuitively we feel' are used to trick the reader into believing the argument constitutes a proof. The method of continuous variation is not valid in this case, there is no evidence that all possible triangles have been considered, and the result of 180° is obtained by calling a line segment a triangle.

A high proportion of the students gave no opinion on this argument. This is probably due to the fact that they did not feel they understood it sufficiently to write something valid, combined with the fact that it was

one of the later proofs, and some of the students may have felt less enthusiastic about writing comments by this time. There is however also some evidence against this conclusion, as it was clear from notes written on the questionnaire that some of the students answered the questionnaire backwards, working from proof 6 to proof 1!

Five students actually wrote 'I do not understand' or words to that effect, which was by far the highest result in the **Lack of understanding** column. In previous proofs many students seemed unwilling to admit that they failed to understand the arguments given, but it was seen as acceptable to confess to confusion in the case of this proof.

Seven of the students wrote comments indicating a positive attitude towards the argument. Responses included:

> 'Confusing but logical.'
> 'Also a proof, but rather confusing to start off with and not as straight forward as some of the others.'
> 'The proof is correct because it is easy to follow and is straight forward and can be done practical quickly and demonstrate easily.'

The last comment was unusual for this 'proof' - most comments indicated that the 'proof' was very difficult to follow. It seems likely that the students had been coerced by the authoritative mathematical style used (De Villiers, 1991), in particular, the claim that the proof is 'logical' has no other basis

Six students disliked the proof. The reasons they gave included

> 'Very hard to follow. Diagram is confusing as is proof.'
> 'Poor, using "intuitively we feel"'
> 'This is too complicated and difficult to understand.'

Only one of the comments picked up on the lack of justification for some of the steps taken in the 'proof'. The rest of the comments indicated that the 'proof' was poor because it was hard to understand.

Conclusion

In *More than Formal Proof* (1989) Hanna states the following:

> 'The development of mathematics and the comments of practising mathematicians suggest that most mathematicians accept a new theorem where some combination of the following factors is present
> 1. They understand the theorem, the concepts embodied in it, . and there is nothing to suggest it is not true.
> 2. The theorem is significant enough to have implications in one or more branches of mathematics (and is thus important and useful enough to warrant detailed study and analysis).
> 3. The theorem is consistent with the body of accepted mathematical results.
> 4. The author has an unimpeachable reputation as an expert in the subject matter of the theorem.
> 5. There is a convincing mathematical argument for it (rigorous or otherwise), of a type they have encountered before.'

Hanna is writing about the acceptance of theorems, but she could equally have been writing about the acceptance of proofs. She goes on to say

> 'If there is a rank order of criteria for admissibility then these five criteria all rank higher than rigorous proof.'

When we look at the comments made by the students on the proofs of the triangle theorem, we find a surprising number of points in common with the way Hanna suggests the mathematical community accepts theorems. Students consistently ranked the proofs on their clarity, usefulness, consistency, how convincing they were, and how easy they were to understand before considering if they were logical and rigorous. Hanna's fourth point on reputation holds as well - the arguments were all called proofs, and as a result there were only a few cases where students wrote that the argument presented was 'not a proof'.

By focusing on the responses of each student individually we have classified three modes of thought used to judge the 'proofs' given, namely empirical, logical, and aesthetic. The students tended to think predominantly in one mode. Students thinking empirically favoured those arguments providing *evidence* supporting the triangle theorem, namely proofs 1 and 5. Students thinking logically favoured proofs providing a *rational argument* demonstrating the theorem from acceptable premises, namely proofs 2, 3 and 4. Students thinking aesthetically favoured proofs which they found *visually* or *intuitively appealing*, such as proofs 4 and 5. It is the logical mode of thought that is normally considered to be the most mathematical.

A closer look at the comments given on proofs 2, 3 and 4 allowed us to identify three problem areas in understanding formal proofs. failing to follow the chain of reasoning, misunderstanding a mathematical concept, and misunderstanding the language used The term 'misunderstanding' is used as a shorthand for the phrase 'failing to interpret ... in the same manner as the teacher'. Failure in following the chain of reasoning frequently occurs when one of the steps in the argument is considered to be 'obvious' and is therefore left out. Mathematical concepts are misunderstood in several ways - examples include using axioms from an alternate mathematical frame, or failing to use previous definitions and premises (e.g. the case when one of the students wrote that the angles on a straight line might sum to 200°). Failure to understand the language used occurs when a word or phrase has a specific meaning in mathematics, but is interpreted according to its everyday usage, or when the student has never encountered the word or phrase before.

The results obtained from this paper will be used to evaluate further stages of the research project on developing teaching materials on notions of proof through the use of computer software, currently running at South Bank University.

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On Shifting Conviction in Conceptual Evolution

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Abstract. *How does a learner shift his conviction through the event in the process of conceptual evolution by overcoming the epistemological obstacle? In conclusion, we can see the learner's justification of an event as social adaptation. For this, through the ninth grade classroom observation, conceptual evolution is seen as the decision-making situation, and the notion of utility is useful to the study.*

1. Introduction

We can see the phenomenon which the learner's previous knowing turns out to a defiance or a difficulty when he/she is going to acquire a new concept. Some previous studies call such difficulties the epistemological obstacles (Brousseau,1983, Sierpinska,1985, Cornu,1991). Learners should overcome these obstacles for sound and firm conceptual evolution. Therefore, research on the epistemological obstacles and processes for overcoming them suggests that a teacher should decide how to teach his/her students and conjectures how students could learn.

This report rather focuses on the overcoming process for epistemological obstacle. However, the present study is part of wider one which aims to reconstruct mathematical teaching and learning process for introducing the notion of epistemological obstacle to make clear the above mentioned phenomena, and it's just beginning. So we consider the conviction as a basis of learner's way of knowing especially in this report. The research problem is "*How does a learner shift his conviction through the event in the process of conceptual evolution?*"

2. Conceptual Evolution and Decision-Making

In this report, we use a model for a learner's overcoming epistemological obstacle as a

framework to understand his/her conceptual evolution (Mizoguchi,1992). This model consists of the relation with four kinds of nodes, that is, notion, conviction, event, and epistemological obstacle, so conceptual evolution is characterized as overcoming obstacle.

Notion is learner's ambiguous idea, knowledge, or mental model. *Conviction* is connected with the notion for a background. Conviction means learner's attitude towards mathematical knowledge, and it could be a criteria which the learner evaluate the notion. A set of notion and conviction forms learner's way of knowing. The notion also corresponds with some *event*.. Event means learner's experienced things for acquiring his/her previous notion, that is, solving problems, examples, or metaphor, so it includes the thing which the learner his/herself generated as well as the thing presented by other students or a teacher . The learner doesn't evaluate the notion itself directly, but through the event as indeed. Hence conviction is also a criteria to evaluate the event.

At first the learner has a naive notion(N_1) and a conviction(C_1) for a background of N_1. The learner's way of knowing at this phase(K_1) consists of N_1 and C_1. K_1 acts on a event(E_1) like some problems which the learner has solved successfully until then, or something understandable. A new situation in a face to a new event(E_2) may occur, where the learner needs to understand the new notion(N_2) to accept E_2. If the learner sticks to the previous conviction(C_1), his/her way of knowing becomes the epistemological obstacle(EO) for accepting E_2. It is needed to shift towards a new conviction(C_2) corresponding to N_2 for overcoming EO. By shifting towards C_2, the learner could accept N_2, and, at last, arrive at the new way of knowing(K_2) which consists of N_2 and C_2. Thus, the learner could overcome the EO. Figure 1 illustrates this discussion.

We can also see such learner's conceptual evolution as a situation of his/her decision-making. In the report, the act of decision-making is represented with choosing some *alternatives*. Each alternative has a related *outcome* which is assigned *utility*, which gives an extent of subjective preference or rationality (Baron,1988, Fishburn,1988).

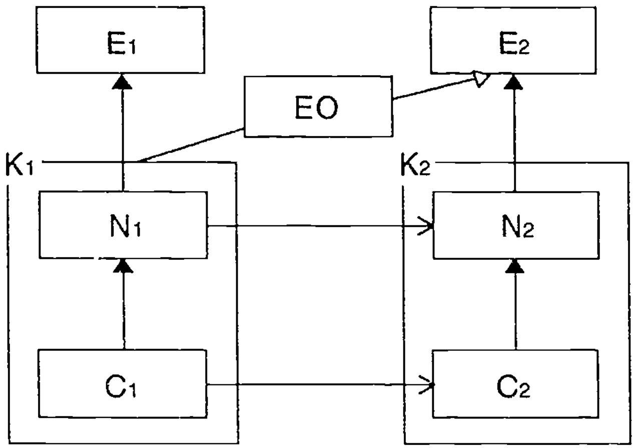

Figure 1

Now, the previous notion(N_1) and the new notion(N_2) corresponding to the new event(E_2) can be seen alternatives in decision-making. It is decided which the learner chooses any one, not by comparing alternatives directly, but by referring to outcomes(i.e. events) connected them. At this time, the learner's conviction evaluates the event. If the previous conviction(C_1) evaluates the event E_2, the new notion(N_2) and the event E_2 connected to N_2 are rejected, so the previous notion(N_1) is chosen, or maintained. However, by shifting the learner's conviction(C_1) towards C_2, he/she can evaluate the event E_2 on C_2, so the new notion is chosen, or the notion evolves.

Utility is a criteria or rationality for a learner's decision-making. So utility means a preparatory condition with the learner's special making. Thus utility is the learner's attitude in this sense. Though notion and alternative, or event and outcome are in corresponding relations, utility is attitude towards more general making than conviction or attitude towards mathematical knowledge, and it is prescribed as a including relation. It is conjectured that learners make decisions based on attitudes which are not always convictions. By introducing the notion of

utility, we could grasp learners' actual decision-making activities when we analyze.

In brief, conceptual evolution is seen as a decision-making situation, and we could consider the learners' attitude, which evaluates events, as utility, which assigns subjective rationality to outcomes, by introducing this notion.

3. Classroom Observation

3.1 Method —subjects, material, and procedure—

The ninth grade class at a private junior high school in Tokyo was observed. The class had 10 students. The observed lesson was aimed to introduce square root. The lesson was done by a ordinary teacher, and observed by VTR. A position set VTR is at the left in back of the classroom, and all the students, the teacher, and the blackboard were put in one frame of VTR.

At some other day, a student RIK was extracted and interviewed . Here, we discuss RIK's performance especially.

3.2 RIK's Performance and Discussion

The implemented lesson was aimed to inform that numbers which cannot be represented by finite decimal fractions exist, and to introduce a new representation(symbol $\sqrt{\ }$) with such numbers. A posing problem was to find a length of a side of a square which area was 50. In the lesson, beginning to use that the length of a side of a square which area was 100 was 10, they looked for the length of a side of a square which area was half of the first(see figure 2).

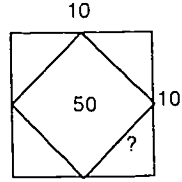

Figure 2

The lesson made progress as follows;

[1] Students already found that the length was about 7 in the previous lesson;

[2] In this lesson, beginning to confirm that, they looked for a more accurate value by using calculators;

[3] After investigating to a certain extent, the teacher suggested that it is impossible to represent it with a finite decimal fraction;

[4] The teacher informed that a new representation was needed to represent such a number;

[5] Introducing square root($\sqrt{\ }$).

At the progress [2], the following communication with the teacher and the students was observed. (T:teacher, R:RIK, S:other students)

T So, if the area is 50, the length of the side is (longer) than 7.06... ("longer" was not said because the adjective is the last in a sentence in Japanese)

S Longer.

T: Yes, it needs to be longer.

S. 49 9849

T. What?

S 7 07

T It becomes closer, but a little insufficient, so it is (longer) than 7.07

S Longer

S: Hi, 7 071.

T: If 7.071

S 49.999041.

T It becomes much closer, but not enough If 7 072

R All Right! 7 0711

T What?

R It becomes 50 000

At the end of above, RIK answered "50.000" as 7.0711^2, and it was the length of a side of a square which area was 50 for him. This means that RIK deals with every number as finite. And at the interview, it was shown how RIK dealt with it. He said as following at the beginning of interview.

I: What is the number which is 10 by square?

R $\sqrt{10}$ (description)

I: Can you represent it with decimal fraction?

R: (using calculator) 3.1622776···(description)

I: What "···" means?

R: Forever.

I Forever. (showing protocols of the lesson · calculating $\sqrt{50}$ using calculator) Why did you say "50 000" without last 455 at that time?

R: I counted fractions of .---5 and over as a unit and cut away the rest ("Shishagonyu" in Japanese)

As shown above, RIK dealt with the infinite number as finite by rounding in the lesson. But he describe $\sqrt{10}$ as 3.1622776···. He said the followings for this reason.

I. So, how do you distinguish rounding and describing "···"?

R. On problem.

And he said the followings about his previous notion.

R I had rounded any number

I Why did you change your thinking?

R Because I knew irrational numbers

I Have you ever seen the number using "···"?

R A few. But in fact, such a number exists, and it can be represented using $\sqrt{\ }$

As seen the above protocols, in alternatives with the treatments of infinite decimal fractions, that is, "rounding every number" or "rounding dependent on problem", RIK chose the latter at last. In compliance with the former, it is interpreted that RIK could get an outcome $\sqrt{50}=7.0711$ on the utility which means empirical, actual necessary, so it is also conviction. However, for the outcome of introducing the symbol $\sqrt{\ }$, that is, the number 7.0710678··· may represent $\sqrt{50}$, it was prompted to shift towards the utility or the conviction of formalism of mathematical representation, and he makes decision to choose the latter.

Why was RIK able to evolve his notion, or why was his treatment with infinite numbers able to change?

At first, RIK was not able to represent an accurate length of a side which area is 50 by his

usual representation, thus he performed the operation of rounding in the process of approximating one by one. Then the symbol $\sqrt{\ }$ was introduced by the teacher. RIK could be interpreted as experienced the following activities to accept a symbol presented by his teacher. First, this new representation was recognized "true" for RIK. Because not only RIK indeed approximated one by one, but also it was presented by his teacher. If RIK chooses his previous notion or "rounding every number", he turns out to get risk in decision-making. Because to choose it is not necessary to introduce a new representation, and it is considered that he judged that doing so produced bad results. RIK learns mathematics instructed by his teacher from now on. If he didn't accept his teacher's instruction, he might perform another mathematics. So RIK proceeded to remove this risk. If we call the risk removing *justification*, RIK shifted the conviction of empirical, actual necessary towards the conviction of formalism of mathematical representation to justify the event of introducing a new symbol $\sqrt{\ }$ by his teacher. This means that RIK accepts his teacher's presentations and statements, and it turns out to produce good results for his learning from now on. Thus we call that *justification of an event as social adaptation* by RIK.

4. Concluding Remarks

As seen above, RIK shifted his previous conviction to new one. And it was in the state evolved mathematically. However, the process seen there is never desirable. Clearly, it is a product which RIK concluded an interest between he and his teacher We, educators, rather expect for RIK to conclude mathematically. And for this, we have to inquire how events are needed. They may require teacher's instruction, or to prepare the situation where learners may discover or create something.

We also have to deepen the inquiry on conviction for the background of notion. Although some previous studies (e.g. Sierpinska, 1987, English, et al., 1992) investigated students' attitudes, it is learners' attitudes towards mathematical knowledge that we need exceedingly, because our

research focus is on epistemological obstacle. It was the epistemological obstacle related to limits that RIK showed in his performance. As taken notice of conviction, the author now thinks necessary to investigate succesively from the elementary level relative to notion concerned rather than investigation at one point in time (e.g. RIK was at the ninth grade). In the case of research on epistemological obstacle related to limits, it is corresponding to search for the area of a circle at the fifth grade in Japanese curriculum. On a gap between to search for the areas of polygons and of curved figures which are essentially required the concept of limit, what is the learner's previous conviction, and what is the learner's new conviction? These are our next issues.

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CATALYSING TEACHER ATTITUDE CHANGE WITH COMPUTERS

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Teachers' attitudes towards mathematics and mathematics teaching are often seen as constraints in attempts to reshape the mathematics curriculum. Viewed from this perspective, teachers are viewed as obstacles in the way of new pedagogical ideas, and in need of 'treatments' which will enable them to grasp, and subsequently 'implement' them. Some authors (e.g. Sikes, 1992; Hoyles and Noss, 1992), however, have criticised this kind of view, preferring instead to recognise that schools are settings with their own rules of discourse, and that teachers are not so much constrained by their setting as reciprocally acting on it and influenced by it. In this scenario, teachers do not need 'treatments' to cure their lack of 'enthusiasm, thoroughness, and persistence', they need to be offered ways of thinking about their practice which fit with their own ideas and beliefs, and for which they can play the major part in developing alternatives. Unfashionable though this approach is, it seems possible that it may provide a way forward. Noss, Sutherland and Hoyles (1991) designed a year-long computationally-based course which highlighted the complexities of teachers' development with the computer, and pointed to some general descriptive and analytical conclusions concerning the ways it might become a central element of the existing mathematical culture.

Both in Clark and Peterson's (1986) perspective, and in Noss *et al's* (1991) standpoint the centrality of the teacher appears established. However, there are many questions that remain unanswered. What are teachers' purposes and motivations to engage with innovations? How do teachers relate to the innovations in the course of their implementation? What factors are at stake in explaining teachers' interactions with the innovations? How do innovations change as teachers work through their implications in practice?

This paper addresses a set of questions related to these issues. In particular, it documents the nature of teacher change by looking at the cases of two teachers who attended a course aimed at introducing primary teachers to a Logo-mathematical culture, and attempting to identify what was at stake for them in interacting with such a culture.

Course design and aims

A mathematical Logo-based in-service course which aimed at providing a setting for personal and professional exploration and change was designed and implemented. The title of the course, "Teaching and Learning Mathematics with Logo in the Primary School", emphasised that the ultimate goal of the course was on mathematics. Moreover, it placed the onus not only on pupils, but also on the teachers' own development with the subject. The course took place during the Summer term of the 1988/89 academic year, and was run by the first of the authors.

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The course was designed to last for an academic term and to comprise twelve sessions, each of the sessions lasting two-and-a-half to three hours. The first five sessions were concentrated into a fortnight. This was intended: (a) to provide the participants with the basics of the Logo language, the definition of procedures, as well as the technicalities of managing the computer in a relatively short period of time; and (b) to prevent them forgetting what they would have learnt and developed from one session to another. After the fifth session the participants were introduced to more complex features of Logo (e.g. recursion, multiple turtles, words and lists). They were also expected to use Logo with their pupils in the classroom. From session nine onwards, the development of a Logo microworld was the main focus of the participants' activity. The frequency of the sixth through twelfth sessions was mostly once a week with the scheduling left flexible allowing for changes to suit the convenience of the participants.

The objectives of the study were specifically:

- To explore and describe the attitudes towards mathematics and mathematics teaching that the teachers who volunteered for the course held initially, as well as their motives for joining such a course.
- To examine the extent to which and the ways in which the attitudes towards mathematics and mathematics teaching of the participating teachers that were recognisable at the beginning of the LogoMaths course interacted with other factors throughout their participation in the course, and to discover these factors.
- To investigate what kind of influences, if any, might the participating teachers' involvement with the LogoMaths course had on their attitudes towards mathematics and mathematics teaching.

A framework for understanding teacher change

The key construct used in this study in connection with teachers change with regard to mathematics and mathematics teaching was *attitude*. Our interpretation of this construct is in line with those social psychologists who use the term in a broad sense and avoid distinguishing between attitudes and other constructs such as values, thoughts, beliefs, and opinions (e.g. Petty and Cacioppo, 1986). Making sense of attitudes in this way allowed us to explore a range of issues which are commonly found in the literature dealing with mathematics and mathematics teaching. These include: (a) teachers' views about mathematics and school mathematics; (b) their personal feelings towards mathematics in terms of both enjoyment and confidence; (c) their opinions about the aims of teaching mathematics in school; (d) their pedagogy and teaching strategies; and (e) their opinions about the role of computers in children's mathematical learning. Concomitantly, based on what is known about perspectives on people's attitudes and attitude change (not just teachers), we identified three most basic considerations to be used as action guidelines in the study:

Principle 1: Attitudes can be understood only as the history of attitudes. The point here is that attitudes are developed and are continually evolving as a result of a wide range of experiences. Hence, it is relevant not only to recognise what teachers' attitudes are, but also to know why they hold such attitudes. It is not suggested that *all* the experiences contributing to teachers' attitudes should or could be brought to light, but that efforts should be made to uncover some of the factors that might explain those attitudes. In particular, it seems important to attempt to trace teachers' past experiences with mathematics as pupils.

Principle 2: Change can be understood only as the history of change. In attempting to document eventual changes in teachers' attitudes towards mathematics and mathematics teaching

associated with a course, it is essential to try to understand the trajectories followed by the participating teachers. In particular, special attention should be given to the teachers' motives for joining the course, and whether or not they had already felt a need for change.

Principle 3: The main agent for educational change is reflection. This third principle captures the spirit of Olson's (1985) reflexive principle of change, namely in that bringing teachers to come to know new ideas can stimulate them to try new ideas in their jobs, and act as a catalyst for promoting them to reflect upon their old attitudes and practices. But it takes further Olson's reflexive principle by pointing out that those interested in promoting change should also reflect upon their attitudes and practices to bring about change. We were interested in what the participants did with the course, not just what the course did to the participants.

Data collection and analysis

The participants in the study were ten primary teachers (all female) from six schools of the Oporto area (Portugal) who volunteered to attend a Logo-based mathematical course. The data collection plan for this study was modelled on that of Hoyles, Noss and Sutherland's (1991) investigation. Four main different sources of data were used to assess the course participants' attitudes throughout the study: (1) semi-structured interviews, (2) attitude questionnaire, (3) participant observation, and (4) dribble files (permanent computer records of the participants' work with the computer saved in disks). Participating teachers were interviewed twice, once before the course and then again after the course. Before and after the course, and prior to being interviewed, all the course participants were administered the attitude questionnaire which had been used in a previous study of teachers' attitudes towards mathematics and mathematics teaching (Moreira, 1991). Details may be found in Moreira (1992).

The procedures used to analyse the data were designed to map the participants' interactions with the course activities in a systematic manner, one which would allow for both illuminating the history of each participant throughout the course and making comparisons across participants. To this end, an *interaction profile* was created for each participant. As a first step, each participant's dribble files were run and analysed, and the Logo work records were transformed into descriptions of the character of this work and the strategies used. Once all the dribble files for a single participant were analysed, a second-level analysis took place. This consisted of putting together the data provided by the dribble files analysis and field notes for each participant for each of the twelve course sessions. The basic unit of analysis was still the participant's work during a particular session, but this work was analysed in terms of common categories and components across the sessions. These components and characteristics were qualitative in nature. They included, for example, the kinds of activities a participant was engaged in one particular session, with whom he/she was working, his/her level of involvement with the work, the Logo programming features he/she was using, and the nature of mathematical activity associated with the Logo work.

The participants' answers to the pre-interviews together with their answers to the attitude questionnaire before the course started were used to make sense of their interactions with the course, and at the same time these interactions were used to understand their attitudes. The participants' interactions were then used to explain possible changes in their attitudes. From the start we were interested in a set of factors which included participants' attitudes towards mathematics, and towards mathematics teaching, as well as their motives for joining the course; but we were prepared to refine our category system by including additional categories and discarding some of those initially considered. The categories which came finally to be considered

were as follows: (1) motives for joining the course, (2) readiness to use computers, (3) sensitivity to mathematics, (4) views of (school) mathematics, (5) pedagogical mathematical expertise, (6) pedagogy/teaching strategies, (7) learning orientation, and (8) course climate and cultural norms.

The analytic process was simultaneously inductive and deductive. Data from the individual participants were used to generate cross-case findings, and these were in turn used to write the story of each participant. As a final analysis step, we focused upon the cases of those participants who (a) appeared, *ab initium*, to be motivated to change their attitudes, and (b) had already been using computers and Logo.

Two mini-stories

In the remainder of this paper, we first outline two miniature stories of two participants designed to locate the issues which played a part in the participants' reactions to the course, and the ways in which they interacted with it (for case studies see Moreira, 1992). We then draw some brief conclusions from these two cases, and outline more general implications of the studies.

Alice: a case of attitude change

In her mid 40s, Alice has been a primary teacher for more than 25 years, eight of which were in her current school. At the time of the Logo course, she was teaching a class of fourth graders. (As is common practice in Portugal, Alice has taught these pupils through their four years of primary schooling). Although committed and dedicated to her profession, Alice was probably not fully satisfied with it. She has not had other professional experience apart from teaching primary school children, but teaching did not seem to be her vocation. Looking back to her decision to become a primary teacher, she remarked that she would have rather preferred to go into something related to the arts and that she had yet relinquished this aspiration.

The advent of computers in education in Portugal might have tapped into Alice's relative frustration with her job and her aspiration to do something different. She belonged to the first wave of teachers who volunteered to implement computers in primary schools in Portugal as part of the MINERVA project (the national project to introduce computers in Portuguese schools). She described herself as "fascinated and attracted" by computers.

On the other hand, Alice's feelings of enjoyment of and confidence with mathematics seemed to be relatively lower than those of the majority of the Portuguese participants. She spoke of hating the subject and finding it difficult when she was in school, and even of wanting to give up studying because of maths, but she was able to get the equivalent of O-level in the subject.

Her view of the subject was connected to her views of children's enthusiasm for it: *"I don't know whether it's because they have mental laziness... Maths demands the learning by heart of tables, doing calculations... All of that is probably too dull for them..."*

Enrolling in the Logo course provided Alice with the avenue she needed to challenge her attitudes towards mathematics and mathematics teaching with which she appeared to be already uncomfortable. Although on entering the course she was not primarily concerned with mathematics (as most of the participants, Alice was most interested in trying and learning about the use of computers and Logo in school), she was prepared to justify to herself the necessity of making a mathematical input during her involvement with the Logo course.

Her trajectory in the course was most influenced by her strong personal enthusiasm and confidence in using computers. Logo captured and sustained her interest during the course, and enabled her to reshape her attitudes to mathematics. In particular, such an interest and enthusiasm for the computer seemed to have spilled over into a desire to engage in Logo programming activity. Moreover, entering the course with more experience than most of the other participants seems to have helped her to feel comfortable and resourceful, thus increasing her self-confidence.

In addition, the relative freedom and flexibility offered by the course approach matched Alice's orientation to learning, and she came to develop a deeper confidence with and

appreciation of mathematical activity. A sense of pride and satisfaction were reflected, for example, in her post-interview remarks about a mathematical investigation (an activity totally unknown to Portuguese teachers): *"I didn't have time to do it during the course sessions, but I did it here in school... I found it very stimulating... I had to use my brains a lot, I had to do a lot of calculations, but it was quite satisfying and fun..."*. The course also provided her with the grounding of a new kind of cognitive and metacognitive mathematical literacy. As the course progressed, she had to evolve her perspective of Logo and mathematics in order to accommodate to the kinds of activities that were suggested. By the end, Alice appeared to have shifted away from a predominantly rule-oriented perspective of mathematics to a more problem-solving one.

Alice's involvement with the computer and Logo during the course had, however, two major unintended effects. The first is that with a relatively weak background in mathematics, Alice seemed to have found some of the features of programming with Logo difficult and demanding a lot of effort. As a result, she came to consider that mathematics is a subject suited for people 'with a special mind'. The second effect involves what might be called excessive personal involvement with the computer. At the beginning of the course, Alice indicated that she was mainly interested in the applications of Logo in the classroom. As the course progressed, however, she appeared to have developed more interest in increasing her own competence with the language. Her comments in retrospect in the post-interview confirm this tendency: *"..my expectations at the level of what I have to teach in school were completely fulfilled... but what I want to know goes beyond what I have to do with pupils... It's out of my own personal interest... Because I'm eager to know things..."*.

A disadvantage of this development is that it appears to have curbed Alice's ability to participate fully in the course activities focussing on the use of Logo by children and hindered her reflection upon pedagogical issues. Thus, while it would be over-optimistic to argue that Alice's views and beliefs underwent radical revision, there was at least some evidence that the course produced some perturbation in those attitudes. The possibility remains that shifts occur as Alice's experience in working with the computer and Logo in the classroom accumulates. As she put it: *"things take time to percolate..."*

A different evolution: a sketch of Diana

Diana, in her late 30s, has been a primary teacher for 16 years, 14 in her present school, a private catholic school in Oporto city. Diana had already has been using computers and Logo with her pupils (a class of second graders) for about one year. Taking advantage of parents' social background, participation, and involvement with children's school life, she managed to buy two small microcomputers for her class. She considered that computers were inevitable and therefore the sooner they were introduced to kids the better. She was the first and only teacher in her school to do so, and found in the father of one of her pupils (an engineer) the technical support that she needed. The influence of the computer in Diana's teaching appeared to be negligible. In fact, she evidenced little realisation of the potential of computer for creating qualitatively different kinds of learning environments. The computer was more like a new subject to teach than a tool to teach old subjects.

Like Alice, in joining the course, Diana seemed to be primarily interested in broadening her perspectives of how to use Logo in school. To a lesser extent, Diana, too, appeared to have seen the course as a means of redefining her teaching of mathematics within the context of a computer environment. Indeed, she seemed to have reached a stage in which she was no longer satisfied with her approach to teaching the subject:*"To tell you the truth I teach mathematics in a way that I don't like. Maybe I'm compelled to do so owing to constraints such as time, the curriculum..., but I think that I teach in a very structured way. I try to be less structured as possible, but I reckon that it is still quite structured..."*

Diana saw mathematics as something to be presented in small, straightforward steps -- above all in a way that was 'fun'. Nevertheless, she admitted that of her 32 pupils, only two liked mathematics. Much of this may be explained in terms of her narrow conception of mathematics — that of the primary school curriculum. But Diana's business-like conception of teaching may also be greatly influenced by the school ethos. Discussion of her attitudes to teaching was frequently accompanied by suggestions that she felt isolated in the school and that she had to frame her work with her pupils in terms of parents' expectations and

pressures. What we saw when we visited Diana's classroom was that her pupils inherited the business-like air that seemed to characterise Diana's management of her teaching.

What is fascinating about Diana is her struggle between a desire to teach differently and her difficulty in realising her wish. This apparent impotence did not appear to derive from her lack of awareness of alternatives, even if these were only perceived in general terms: she expressed an eloquent dissatisfaction with her initial-training, and a desire to *'go into a teacher training course again, so that I could learn more..'* Diana was looking for a 'less structured' approach, but the tension she felt was largely based on the institutional context in which she worked. But we must also consider Diana's personal characteristics.

Despite Diana's initial motivation for changing her attitudes towards mathematics and mathematics teaching, the course seemed contributed in only a limited way to catalysing such a shift. In fact, certain factors made Diana's development with the Logo course less than easy. As a teacher, Diana might have felt the course activities and suggestions were too remote from the reality of her classroom and of her school. These, in general, did not suit her teaching style based upon transmitting information within a structured environment. Interestingly, however, she accepted well some of the innovatory Logo ideas suggested in the course, such as the Floor Turtle activities. The tendency to favour these activities was perhaps related to the fact that -- precisely because they were new — they did not collide with her established ways of doing things.

As a learner, Diana appeared to feel uncomfortable with the course approach which emphasised interaction among peers. Coming from a school where there was little co-operation among the staff, Diana had some troubles in relating to her peers as they *"had a different way of approaching things"*. Further disappointment appeared to have emerged from the clash between the course emphasis on self-direction and Diana's orientation to learning as mainly a matter of being told what to do and follow the instructions. Equally problematic was her relative low background in mathematics coupled with her narrow views of the subject. Thus, the course environment did not constitute for Diana an opportunity to use mathematics in an integrated and meaningful way and to generate new perspectives about the subject.

At the end of the course most of her expressed views remained largely similar to those at the start. In particular, Diana continued to regard the computer in education as a form of preparation for pupils' future lives, and she continued to hold an undifferentiated and fragmented view of mathematics dominated by the mathematics of the school curriculum. In addition, she did not appear to have moved far away from her entering perspectives about teaching mathematics. Furthermore, at the end like at the beginning of the course, Diana appeared to be willing to extend her attitudes and practices associated with mathematics teaching, but was sanguine as to the extent to which she could realise a change in her school practice.

Some conclusions

We outline a small number of conclusions around the theme of course/participant interaction. These are drawn from the two cases above, although we would want to refer the interested reader to the more detailed conclusions derived from a much larger empirical base in Moreira (1992).

First, we note that the main differences between Alice and Diana does not lie in the length of their teaching career, their formal education, or their previous experience with computers. Both Alice and Diana, were experienced teachers, and had been using computers for little more than one year. They also had similar mathematical backgrounds and had pursued identical programmes of teacher training. Their contrasting reactions to the course have then to be described in terms of variables other than these *professional/cognitive* factors. What seems to be at the heart of the differences in Alice and Diana's involvement with the Logo course were factors that were more fully integrated into their *personal/emotional* lives, as well as the contextual conditions of their schools.

The intensity of Alice's involvement with the course might have emanated, first of all, from her feelings towards computers, coupled with her view of the computer as a tool to support

the curriculum. These two factors together seem to have set her at his ease, freed her to think about questions of learning (and to a lesser extent teaching) mathematics with Logo rather than focussing on the surface features of the course. Diana had also incorporated computers based on her own initiative, but she saw the computer very much as an end in itself — a technical object which involved little intellectual or emotional commitment.

It also seems that Alice and Diana's different reactions to the course were seriously influenced by the match (or mismatch) between their preferred personal ways of learning, their style, and the course orientation. The course represented for Alice an opportunity to be an active participant in her learning process: she clearly enjoyed its freedom. Diana, in contrast, felt uneasy throughout the proceedings, although this was not evident at the beginning of the course. Diana's business-like orientation to teaching extended to a similar orientation to learning (or *vice versa*) — indeed she resented the cooperative stance of the course, as well as being expected to be in control of her own learning. In fact, the potential mismatch between Diana's style and the course's approach *might* have been less serious in a longer course; and this points to a limitation of the study in general. Noss *et al* (1991) have shown how even considerably longer courses only scrape the surface of teachers' beliefs and attitudes, and — not surprisingly — how the evidence suggests that time is a key element.

A further point that emerges from these two case studies is that related to the culture of the teachers' school. In other words, it is clear that institutional contexts generally are at least as important as individual considerations. Alice's school was a school in movement. Some of the teachers in this school had joined the MINERVA project and were already using computers in their teaching. Two other teachers were enrolled in the Logo course. One may well say that at the time of the study, computers were seen as a school priority. In such a setting, it is not surprising that Alice was able to discuss and reflect on course events with her colleagues. In Diana's school, she was alone in her attempts to work with computers. Her headteacher and colleagues were not supportive, and given that her school was private (unlike Alice's), it was to be expected — in the Portuguese system — that she would be extremely constrained in the extent to which she was free to innovate, and her autonomy severely compromised by the authority of the headteacher.

Diana enrolled on the course looking for alternative approaches and willing to consider changes in her pedagogical approach towards mathematics. But the alternative simply did not mesh with what she saw as possible within her context. It is possible that she saw new possibilities on the course which she was prepared to consider, but she simply did *not* see a possibility of integrating these possibilities with her own beliefs and — more importantly — her practices. Not surprisingly, she had no alternative other than sticking with her tried and tested views rather than plunging into the unkown territory represented by LogoMaths.

In this respect, we ought to view this kind of phenomenon as a limitation of the course, rather than of Diana. Or perhaps, that Diana's belief-system did not fit with those of the course. But it would be more useful to consider Diana's beliefs *in practice* rather than in the abstract; it is not that she 'believes' this or that about mathematics education or computers, it is that these beliefs are situated in particular practices — Diana as teacher, Diana as staff-member in a private school. This creates a methodological difficulty regarding interviews. If it is unhelpful to regard beliefs as abstractly held and then applied, what were we finding when we interviewed Diana at the outset of the course? Interviews are not 'decontextualised', they constitute a context which is different from the beliefs *in practice* with which Diana operated in her professional practice. To

be sure, an interview with a researcher is a context with its own rules of discourse, but hardly the ones we were hoping to investigate.

We might also note how the two sketches illustrate the coarseness of the distinction in the literature between personal and professional motivation. Alice and Diana *both* came to the course 'intrinsically' motivated — Alice's primarily personal, and Diana's largely professional. But their expectations differed. Alice used the course to develop her personal interest in computers, and used that as a bridge to reevaluate aspects of her professional practice. For Diana, the personal element was never kindled by the course.

A final point. It is often assumed that teachers (and incidentally, pupils) develop attitudes towards innovatory practices in an incremental way, and that radical visions of new possibilities might serve to alienate them from the innovations being proposed. Whatever truth there may be in this assertion in general, it is worth noting that in Diana's case, the facet of the course with which she engaged most closely was Turtle geometry, an activity which was radically *different* from her traditional practice and which therefore did not conflict with her traditional practice.

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ATTITUDES TO TEACHING MATHEMATICS
THE DEVELOPMENT OF AN ATTITUDES QUESTIONNAIRE

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The attitude of teachers, particularly primary teachers, to the teaching of mathematics have long been considered as important precursors to the formation of positive attitudes to mathematics among pupils (e.g., Aiken, 1976). While the research evidence that such a relationship does exist is somewhat tenuous, there is a strong belief among pre-service teacher trainers that positive attitudes need to be fostered in teacher education courses. Unfortunately, the research evidence suggests that high proportions of pre-service teachers hold negative attitudes towards mathematics (e.g., Kelly & Tomhave, 1985). Although many instruments measuring affect in areas like self-concept, anxiety etc. have appeared in the literature over the years, no comprehensive instrument on attitudes is available to help teacher trainers monitor attitudinal changes among pre-service teachers to the teaching of mathematics over the period of training. This research re-examines attempts to develop such an instrument in Australia (Nisbit, 1991) and posits an alternative and refined version.

Definitions of attitude generally include the idea that attitudes manifest themselves in one's response to the object or situation concerned. One such definition states... "Attitude is a mental and neural state of readiness, organised through experience, exerting a directive or dynamic influence upon the individual's response to all objects and situations with which it is related" (Allport in Kuhn, 1980, p 356). When exploring the attitudes of pre-service teachers toward mathematics it is necessary to not only consider their attitude towards the subject itself, but also their attitude regarding the teaching of mathematics.

The attitudes of pre-service teachers are of particular importance because of their potential influence on pupils. Although the research evidence is certainly not conclusive, it has been sufficient to suggest that positive teacher attitudes contribute to the formation of positive pupil attitudes (Aiken, 1976; Sullivan, 1987). Some studies have indicated that teacher attitudes towards a subject and the teaching of that subject influence the instructional techniques they employ and that these, in turn, may have an effect on pupil attitudes (Carpenter & Lubinski, 1990; Williams, 1988).

Although there is little hard evidence that holding positive attitudes towards mathematics is actually beneficial, it is difficult to argue against their desirability. The widespread belief in the relevance of positive pupil attitudes towards mathematics is reflected in the following extract from the National Statement on Mathematics for Australian Schools "An important aim of mathematics education is to develop in students positive attitudes towards mathematics and their involvement in it..." (Australian Educational Council, 1991, p 31).

One argument presented in support of the need of positive attitudes is that such attitudes can enhance achievement in mathematics at primary, secondary and tertiary level (Dungan & Thurlow, 1989). Most studies on the relationship between attitude and achievement have revealed a low but significant correlation (Aiken, 1976; Kulm, 1980). However, the nature and direction of this relationship is yet to be unravelled (Daane & Post, 1989, Kulm, 1980; Suydam, 1984).

Research into attitudes has explored various components of attitude such as anxiety, enjoyment, self-concept and belief in the usefulness or value of mathematics. One component that has received much attention is that of 'mathematics anxiety'. There is no doubt as to the existence of 'mathematics anxiety' and a number of instruments for measuring levels of anxiety have been developed and implemented (Richardson & Suinn, 1972). However, Sovchik, Meconi and Steiner (19[illegible]) suggest that the construct 'mathematics anxiety' is not as well defined and measurable as assumed by some mathematics researchers. There is some doubt as to whether anxiety is in fact a separate construct. It may just be a reflection of some deeper attitude (Wood, 1988). Anxiety's relationships to other factors such as enjoyment, general attitude towards mathematics and performance are unclear. There is growing evidence that self-concept is a better measure of how people feel about themselves as teachers of mathematics, and that self-concept has an influence on the formation of attitudes. Studies have also found a consistently high positive relationship between self-concept and mathematics achievement (Hackett & Betz, 1989; Marsh, Cairns, Relich, Barnes & Debus, 1984; Reyes, 1984).

If it is accepted that it is highly desirable for teachers of mathematics to exhibit positive attitudes then the high proportions of pre-service teachers found to hold negative attitudes towards mathematics is somewhat alarming (Becker, 1986; Kelly & Tomhave, 1985; Sullivan, 1987). Recent research with pre-service teachers has begun to reveal a series of links between mathematics attitudes, the choice or avoidance of mathematical studies, self-concept and attitudes towards the teaching of mathematics. It appears that students (both male and female) with low self-concepts in mathematics are less likely to pursue mathematical studies. Not surprisingly then, studies have revealed that most pre-service teachers who exhibit negative attitudes towards mathematics have not chosen to study mathematics in their final years of high school (Aiken, 1976; Sullivan, 1987).

The potential of teacher training courses to change the negative attitudes of pre-service teachers towards mathematics needs to be considered. Sullivan (1987) found that almost half of the students entering a teacher training course possessed negative attitudes towards mathematics. He states...."The course improved their attitudes overall, but those who started with negative attitudes still had the most negative attitudes at the end" (Sullivan, 1987, p1). He concluded that if these initial attitudes are so significant, teacher education courses may need to establish entry criteria based on the mathematics background of the applicants.

If the attitudes of pre-service teachers are to be improved, there first needs to be a reliable instrument with which to measure levels of attitudes and perhaps to identify groups of students with special needs (Aiken, 1976; Nisbet, 1991; Watson, 1987). Nisbet (1991) attempted to develop an instrument which consisted of the Fennema-Sherman (1976) 'Mathematics Attitude Scales' plus some parallel items constructed by Nisbet to cover the 'Attitudes to Teaching Mathematics' aspect.

The subsequent factor analysis resulted in a 22 item questionnaire with 4 factors (Anxiety, Confidence and Enjoyment, Desire for Recognition and Pressure to Conform). An analysis of these results suggests that the factor analysis conducted by Nisbet arguably may not have isolated the most accurate or useful factor solution. The data was therefore re-examined through similar factorial analysis. Because no self-concept items were included in Nisbit's questionnaire our initial questionnaire was supplemented with items on mathematics self-concept developed by Marsh (1988).

METHOD

The information reported by Nisbet did not include complete correlation tables on factor loadings therefore it was not possible to reanalyse the data. Instead we opted to re-administer the battery of sub-scales used in the initial developmental stages as well as some additional self-concept items and to factor analyse the new data set to see whether a similar pattern of results would emerge.

Subjects

The battery of tests was administered to 345 pre-service students in the Diploma of Teaching/Bachelor of Education program at the University of Western Sydney, Nepean. All students in the three years of undergraduate study were invited to participate in the study in their normal class time for lectures in mathematics education. Approximately 20 students declined to participate. The subjects ranged in age from 17 years to 43 years with 39.6% classified as mature age students, over 21 years of age. The majority of the sample was female (79.1%), was represented in similar proportions within the mature age group (73.8%), had a large proportion of recent school leavers (81.6%) and had studied maths to Year 12 (88.5%).

Materials

Nisbet argued against the inclusion of mother's and father's *attitudes towards one as a teacher of mathematics* on the grounds that a large proportion of the sample (up to 50%) indicated that they were undecided or did not respond to these items. He does not include these items in his final scale. In the

interest of limiting the number of item. included in our questionnaire and in order to include the Marsh self-concept items without lengthening the questionnaire unduly, we decided to eliminate this set of items from our questionnaire. Therefore the range of items described by Nisbet and which were derived from the Fennema and Sherman (1976) attitude scales and the Marsh (1988) Self-Description Questionnaire were included in our 65 item questionnaire designed to measure general attitudes to the teaching of mathematics.

The sub-scales include the *Confidence in teaching mathematics* (items 1, 9, 11, 17, 26, 41, 49, 53, 60); *Mathematics teaching anxiety* (items 2, 18, 27, 32, 37, 38, 42, 51, 54, 61); *Attitude toward success in teaching mathematics* (items 3, 12, 19, 43, 55, 62); *Mathematics teaching as a male domain* (items 4, 13, 21, 28, 22, 44, 56, 63); *Usefulness of teaching mathematics* (items 6, 14, 22, 29, 34, 46, 57, 65); *Effectance motivation in teaching mathematics* (items 7, 16, 23, 31, 36, 39, 47, 52, 58, 64); *Perception of teacher's/lecturer's attitudes towards one as a teacher of mathemaics* (items 8, 24, 48, 59); and, *self-concept in mathematics* (items 5, 10, 15, 20, 25, 30, 35, 40, 45, 50).

RESULTS

In order to determine how the sample matched Nisbet's with reference to training in high school mathematics, we cross tabulated age group with study in high school mathematics, here defined by the NSW syllabus as Maths in Society, 2 unit, 3 unit and 4 unit at Year 12 level. For this group 10.8% studied Maths in Society and 51.7% and 12.9% two unit and three unit respectively. No students entered the program having studied four unit maths.

These results clearly indicate that mature age students in general have studied significantly less (Chi-Square = 70.06, $p<.001$) mathematics than their younger counterparts and conforms with Nisbet's findings. This further emphasises the serious implications for this group as entrants into primary teacher education programs without the requisite background in mathematics as recommended in the Discipline Review of Teacher Education in Mathematics and Science (1989).

A six factor solution (see Table 1) using Principal Axis Analysis and varimax rotation resulted the clearest distribution of items into identifiable sub-scales. Two major factors emerged which while similar, measure two different aspects of self perceptions related to mathematics, that is, individual attitudes of pre-service teachers as *teachers of mathematics* (ATM) and their self-concept *as mathematicians* (MSC). In addition, we found evidence for a distinct *mathematics teaching as a male domain* (MTMD) scale, *usefulness of teaching mathematics* (UTM), *excellence as a teacher of mathematics* (ETM) scale and an *other's perceptions of me as teacher of mathematics* (OTM) scale.

The first scale which consists of an amalgamation of 30 items from a variety of the original subscales tends to reflect a general attitude towards the teaching of mathematics. In fact all bar two items

Table 1 Factor loading of 65 item questionnaire on six factors

Factor Item No	ATM(1)	MSC(2)	MTMD(3)	UTM(4)	ETM(5)	OPTM(6)
7	84					
32	81					
2	75					
16	74					
37	73					
27	72					
60	71					
36	70					
1	66					
61	65					
42	65					
53	65					
58	64					
17	64					
39	63					
18	61					
51	59					
49	59					
8	59					
64	56					
52	56					
5	55					
38	55					
41	54					
50	53					
26	51					
11	49					
9	46					
47	46					
54	40					
35		73				
45		68				
25	44	65				
20		60				
40		60				
10		55				
15	40	54				
30		53				
28			68			
33			67			
4			58			
31			57			
44			57			
13			53			
21			49			
56			47			
63			43			
23			.			
22				79		
34				77		
14				72		
6				61		
57				54		
29				46		
65				45		
43					66	
55					59	
12					47	
62					.	
3						57
19						42
46						-42
48						.
59						
24						

Note: Decimal points are omitted. Values less than .40 are not included

(1) ATM: Attitude to teaching mathematics
(2) MSC: Mathematics self-concept
(3) MTMD: Mathematics Teaching as a male domain
(4) UTM: Usefulness of teaching mathematics
(5) ETM: Excellence as a teacher of maths
(6) OTM: Other's perceptions as a teacher of maths

specifically mention the act of teaching as part of the content. The coefficient alpha reliability was .96. The second factor (coefficient alpha = .89) on the other hand, reflects eight of the ten Self Description Questionnaire items and is specifically relevant to personal performance on mathematical tasks. All of these items load to some extent on the first factor but form a distinct separate factor. The third factor (coefficient alpha = .79) includes all of the items from the *mathematics teaching as a male domain scale* and two items from the *effectance motivation in teaching mathematics* scale but the loading of one of these items is very low. The fourth factor (Coefficient alpha = .78) contains seven of the eight *usefulness of teaching mathematics* scale. The last two factors which emerge are not as clearly delineated, nor as reliable as the first four, but may be identified as *excellence as a teacher of mathematics* (coefficient alpha = .57) scale and *other's perceptions of me as a mathematics teacher* (coefficient alpha = .29) scale.

One of the other major differences in the findings of this research in contrast to Nisbit's is the inclusion of *mathematics teaching as a male domain* scale in the item pool to be analysed. Not only did this set of items clearly stand out as a separate construct but further analyses of the scale showed that there are significant differences ($t=4.17$, $p<.001$) in response with females scoring higher than males (Female: $M=74.4$, $SD=6.2$ vs Male: $M=70.7$, $SD=6.6$). It is debatable whether such a statistical difference is important as both groups score very high overall (maximum score possible 80). The only other significant difference ($t = 3.9$, $p<.001$) in response between males and females was found for the *mathematics self-concept* scale with males ($M= 45.1$, $SD= 10.4$) scoring higher than females ($M=39.5$, $SD=11.9$) (maximum score possible = 64). This is consistent with all the current literature which does indicate a consistent trend for males to rate themselves more highly than females on this construct.

DISCUSSION AND CONCLUSION

This factor structure does not conform to the Nisbet factor pattern which defines both a confidence and an anxiety factor. He states "Anxiety and confidence in teaching mathematics are independent factors. They are not opposite extremes of the one continuum. The most confident students are not necessarily the least anxious." (p 45).

The factor structure we propose does not differentiate between anxiety and confidence rather it combines these two scales along with the restructured Sherman and Fennema (1976) *Effectance Motivation* scale, into a conglomerate scale which seems to reflect general attitude to the teaching of mathematics and personal feelings towards this activity. Interestingly, the self-concept items derived from the Marsh (1988) Self-Description Questionnaire constitute a distinct factor but one which shares considerable variance (55%) with the initial factor, suggesting that attitudes and confidence (here derived from the self-concept variable) are inextricably associated. In contrast to Nisbet's results, these

results suggest that personal perceptions of one's adequacy as a mathematician should impinge on attitudes to teaching mathematics and are part of a continuum.

The evidence for mathematics anxiety as a separate construct, particularly as a construct different from confidence, is not convincing as has been argued in the literature review. The factor structure which emerged from the analyses of this set of data provides additional evidence for this point of view.

Further analysis of our results has led to the development of a comprehensive 20 item attitude questionnaire. It includes only two factorial sub-scales, attitude and self-concept. The 30 item ATM scale was reduced to a 12 item questionnaire by eliminating items with very high intercorrelations. Despite the reduction from 30 to 12 items the reliability of this scale was reduced only marginally from .96 to .90. Surprisingly this scale is an amalgam of three of the Fennema-Sherman subscales, Confidence in Teaching Mathematics, Mathematics Teaching Anxiety, and Effectance Motivation in Teaching Mathematics. The Marsh self-concept items retained their integrity as a scale with only one item drifting to the attitude scale. The remainder of the subscales were eliminated on the grounds of their poor reliability or, as in the case Mathematics Teaching as a Male Domain, because of their lack of variability and extreme response profiles.

It is our intention to test the reliability and validity of this new instrument as part of an on-going development program. We believe that attitudinal changes among pre-service teachers is an important affective consequence of a viable and successful training program. To measure such change through the use of a reliable and valid instrument will allow tertiary educators to more accurately evaluate the success of their programs in this vital area.

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THE PSYCHOSOCIAL LEARNING ENVIRONMENT IN THE HONG KONG MATHEMATICS CLASSROOM

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Introduction

Learning problem is of major concern in many countries where compulsory education is implemented and the creation of a learning environment conducive to learning has become one of the major tasks of school teachers. The importance of classroom environment to learning can be reflected the conducting the IEA (International Studies in Educational Achievement) classroom environment study in the past few years (Anderson, Ryan & Shapiro, 1989).

Since the late 60s, educational researchers have shown increasing interest in the study of classroom learning environment. Though most of these investigations focused on actual environment and academic achievement, students' approaches to learning were found to have a close relationship with the classroom environment that they preferred (Hattie & Watkins, 1988). Those students who prefer a certain type of classroom environment are more likely to take up a particular approach to learning.

Another research focus has been student achievement as a function of person-environment fit. It is shown (Rentoul & Fraser, 1980; Fraser & Fisher, 1983a, 1983b) that cognitive outcomes can be predicted by actual-preferred congruence, that is a student tend to achieve better in a learning environment close to his or her preference.

Thus, both the preferred and actual classroom environment

would influence the approaches to learning, which would, in turn, generate effect on the learning outcome.

Despite the great amount of effort devoted to the analysis of classroom environment in Western countries. there are not many researches of this kind on Asian students. Actually, the Hong Kong classroom has its particular features. It has a large size and is crowded (Visiting Panel, 1982). In addition, the curriculum is examination driven and modern Chinese parents place great emphasis on the achievement of their children (Ho, 1986). Great emphasis is laid on lecturing, rote-learning and preparation for in-school and public examinations in the Hong Kong classroom (Morris, 1985, 1988).

In fact, satisfactory reliabilities were not obtained in a number of researches using Western instruments (such as Classroom Environment Scale) in the local situation (Chan, 1991; Cheng, 1985; Cheung, 1982). It is therefore necessary to analyse the Hong Kong situation, and there is a need to develop instruments to assess its classroom environment in order to further investigate how it affects students' approaches to learning and learning outcomes.

Methodology

Qualitative research by using open-ended questions (appendix 1) and semi-structured interviews (appendix 2) was performed among thirty five Grade Nine students from seven different secondary schools in Hong Kong. The interviews were transcribed and content analysed. Descriptors were extracted and the dimensions of the learning environment of the Hong Kong mathematics classroom were constructed. The instruments will thus be developed.

Results

By content analysis, the following results were obtained.

1. The most important factor leading to good mathematics results as perceived by students is making effort and doing of more exercises.
2. Students also think that the teacher is crucial in the mathematics classroom. Whether the classroom is "good" or not depend solely on the teacher.
3. A good mathematics teacher as perceived by students is one who
   - explains clearly,
   - shows concerns about students and treat them as friends,
   - makes sure that the students understand,
   - teaches in a lively way,
   - is conscientious and well-prepared,
   - answers students' queries

   (in that order).
4. A good mathematics teacher should also provide more exercises and should generate a lively atmosphere but keep good order.
5. A good learning environment is one
   - which is not boring,
   - in which the classmates are quiet,
   - with classmates engaged in learning,
   - with order observed,
   - in which discussion with classmates possible after lesson

   (in that order).
6. Factors leading to making effort are
   - previous experience of success,
   - perceived ability,
   - understanding the lesson.
7. Having someone (family members, classmates, tutors) to ask after school is also helpful to learning.

By extracting the descriptors, the following dimensions of the mathematics classroom were identified, which are *enjoyable, order, involvement, achievement orientation, teacher led, teacher support, teacher involvement* and *collaborativeness*.

The resulting instrument consists of 8 items in each dimension. The reliabilites obtained in the pilot study range from .55 to .90.

Direction of further research

In the forthcoming study, about 360 students from 9 Grade Nine classes will be invited to participate and the following research model would be investigated (Fig. 1). The Learning Process Questionnaire (Biggs, 1986) will be used to identify the approach to learning, cognitive variables will be measured by the Attainment Test and School Mathematics Grades, the affective variables will be measured by Attitude Towards Mathematics Scale (Minato, 1983) and part of the Self-Description Questionnaire (Marsh, 1988), whereas the preferred and actual classroom environment will be measured by the newly developed scale.

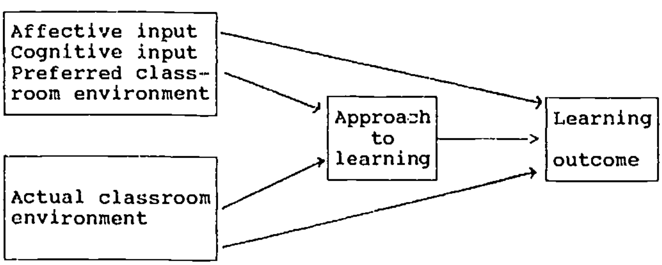

Figure 1. The research model

We will also take into consideration the approach to mathematical problems. For this purpose, open-ended questions were being asked among some 240 Secondary Three students. From the preliminary analysis, three dimensions were identified, which are *understanding, revising & working hard* and *asking others for help.* Another instrument to measure the approaches to tackling mathematical problems will be developed according to the above dimensions. However, further qualitative research will be needed to clarify the concept of understanding in the context of mathematics

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Appendix 1 Written form for the qualitative research.

Students are requested to respond to some open-ended questions at the beginning of the interview. This will last for 10-15 minutes.

Please use any adjective you think appropriate to describe the followings.

1. Mathematics is ______________________________.
2. My mathematics teacher is ______________________.
3. My mathematics classroom is ____________________.
4. My classmates in the mathematics class are ____________
5. My mathematics learning is _____________________.

Appendix 2 Questions asked in the interview will include the following areas.

A. Do you enjoy your mathematics class ?
   1. What elements of the class do you like most ?
   2. What elements of the class do you like least ?
   3. What are the differences between the mathematics class this year and that of last year ?
   4. What elements of the class do you like to retain when you are promoted.

B. What elements of your class do you think are conducive to learning ?
   1. What elements of your class are conducive to learning ?
   2. What are the elements of your class that obscure learning ?
   3. How can your class be improved so as to make it more conducive to learning ?

C. How do you revise your mathematics lesson ?

1. Would you revise differently in another class ?
2. Would you revise differently if you are taught be another teacher ?
3. Would you revise differently in another subject ?
4. Would you revise differently if you are with another group of classmates ?

D. What would you do when you have difficulty in mathematics ?

1. Would you ask your classmates ?
2. Would you ask your teachers or private tutors ?
3. Would you look up text books/reference books/notes ?

E. What kind of activities are often carried out in class ?

1. What are often carried out in class ?
2. What do you prefer ?
3. Do you frequently have group activities ?

F. How would you prepare for your mathematics tests ?

1. How would your prepare for your tests ?
2. How satisfied are you with your results in the recent mathematics test ?
3. What do you think leads to this result ?
4. How does the class affect you in your tests ?

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PME XVII

JULY 18-23, 1993

Proceedings of the Seventeenth International Conference

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PME 17 PROCEEDINGS

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PREFACE

The first meeting of PME took place in Karlsruhe, Germany in 1976. Thereafter different countries (Netherlands, Germany, U.K., U.S.A., Belgium, Israel, Australia, Canada, Hungary, Mexico, Italy) hosted the conference. In 1993, the PME conference will be held in Japan for the first time. The conference will take place at the University of Tsukuba, in Tsukuba city. The university is now twenty years old. It is organized into three Clusters and two Institutes. There are about 11,000 students and 1,500 faculty members. The Institute of Education at the University of Tsukuba has a strong commitment to mathematics education.

The academic program of PME 17 includes:

- 88 research reports (1 from an honorary member)
- 4 plenary addresses
- 1 plenary panel
- 11 working groups
- 4 discussion groups
- 25 short oral presentations
- 19 poster presentations

The review process

The Program Committee received a total of 102 research proposals that encompassed a wide variety of themes and approaches. After the proposers' research category sheets had been matched with those provided by potential reviewers, each research report was submitted to three outside reviewers who were knowledgeable in the specific research area. Papers which received acceptances from at least two external reviewers were automatically accepted. Those which failed to do so were then reviewed by two members of the International Program Committee. In the event of a tie (which sometimes occurred, for example, when only two external reviewers returned their evaluations), a third member of the Program Committee read the paper. Papers which received at least two decisions "against" acceptance, that is a greater number of decisions "against" acceptance than "for", were rejected. If a reviewer submitted written comments they were forwarded to the author(s) along with the Program Committee's decision. All oral communications and poster proposals were reviewed by the International Program Committee.

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We also wish to express our heartful thanks to the following local committee and local supporters who contributed to the success of this conference:

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HISTORY AND AIMS OF THE P.M.E. GROUP

At the Third International Congress on Mathematical Education (ICME 3, Karlsruhe, 1976) Professor E. Fischbein of the Tel Aviv University, Israel, instituted a study group bringing together people working in the area of the psychology of mathematics education. PME is affiliated with the International Commission for Mathematical Instruction (ICMI). Its past presidents have been Prof. Efraim Fischbein, Prof. Richard R. Skemp of the University of Warwick, Dr. Gerard Vergnaud of the Centre National de la Recherche Scientifique (C.N.R.S.) in Paris, Prof. Kevin F. Collis of the University of Tasmania, Prof. Pearla Nesher of the University of Haifa, Dr. Nicolas Balacheff, C.N.R.S. - Lyon.

The major goals of the Group are:

- To promote international contacts and the exchange of scientific information in the psychology of mathematics education;
- To promote and stimulate interdisciplinary research in the aforesaid area with the cooperation of psychologists, mathematicians and mathematics teachers;
- To further a deeper and better understanding of the psychological aspects of teaching and learning mathematics and the implications thereof.

Membership

Membership is open to people involved in active research consistent with the Group's aims, or professionally interested in the results of such research.

Membership is open on an annual basis and depends on payment of the subscription for the current year (January to December).

The subscription can be paid together with the conference fee.

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THE USE OF COMPUTERS IN LEARNING TO CORRELATE ALGEBRAIC AND GRAPHIC REPRESENTATIONS OF FUNCTIONS

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ABSTRACT : We present a research whose aim is on the one hand, to study the adaptation processes developed by students faced with a computer game based on the interplay between graphic and algebraic representations of functions and the way these processes can be related to the construction of mathematical knowledge and on the other hand, to study the possibility of integrate some elements of this process analysis into the software in order to provide the teacher or the user with a detailed and pertinent account of a game session.

I - INTRODUCTION

A mathematical concept is not a monolithic object. A single concept may be understood from several points of view and may have several different representations; in mathematics one needs to be able to move freely between these points of view and representations, adapting them to the setting in which a concept is used [Douady, 1984]. Acquiring this mobility, however, requires a substantial investment of effort and training. For the concept of function, a growing body of research over the past decade has attempted to identify the problems involved, to make precise the cognitive procedures that underlie the corresponding learning, and to find ways to optimize the relationship between teaching and learning (cf. the survey [Leinhardt & al., 1990]). Our research takes place in this global framework but it is limited to investigating how students learn to correlate algebraic and graphic representations of a function, using specially designed software. Therefore it is most similar to research like that of [Schoenfeld & al., 1990].

II - DEFINITION OF THE PROBLEM AND RESEARCH METHODOLOGY

1 - The problem

The goal of our research is to study how students learn to play a computer game based on the interplay between graphic and algebraic representations of functions. We are trying to answer the following questions: what leads to success in the game? Does success necessarily imply the acquisition of mathematical knowledge that can be used outside the game? Can it be achieved in a relatively short interaction with the software and if so, under what conditions? Do game strategies exist that favor success or make it less likely? Are different strategies equally effective at teaching mathematics?

These questions are essentially cognitive, but they lead to more theoretical and methodological questions. The record of the student/machine interaction, which is the basis for our analysis, provides us with extremely detailed observations of behaviour. How can these data be used, both efficiently and economically? How can we discern significant patterns from them? How can we proceed from this

microscopic level to interpretations that are more global in terms of schemes, knowledge, concepts? Is it possible to incorporate feedback into the game, so that the software responds to the student's behaviour, changing the parameters so as to optimize her learning of mathematics?

Of course, our research does not claim to answer all these questions, but we hope to contribute to progress in answering them.

2 - Methodology

After a standard search of the literature, we designed software that was intended both to develop students' ability to infer the algebraic representation of a function from its graph, and to enable us to study the effect of certain variables on students' success at the game (variables that are thought to be didactic, that is, likely to modify the way students play the game and/or the sense of observed behaviour).

The Software

The principle of the game is straightforward: a curve appears on the screen and the student must produce an equation in a specified algebraic form. The version used in the research dealt with straight lines and parabolas; for the latter, three algebraic forms were used: F_1 (Ax^2+Bx+C), F_2 $(A(x-P)^2+Q)$ and F_3 $(A(x-R)(x-S))$. The current version also includes trigonometric functions, homographic functions of the form $(Ax+B)/(Cx+D)$, logarithms and exponentials. To enable us to study the effect of the presumed didactic variables, the game depends on parameters that can be freely modified by the researcher (starting from default values) using a "dialogue" module. The principal parameters are: type(s) of function and choice of the type (imposed or fixed by the student for each game), algebraic form for a given type (also fixed by the researcher or chosen by the student), the number of points assigned the player at the beginning, the availability and cost in points of assistance such as additional sketches, displaying of coordinates, algebraic checking of answers coefficient by coefficient, graphic checking by drawings...

The experiments

The two experiments completed and analyzed so far were organized the same way: pre-test, computer session and post-test. Pre-test and post-test consisted of paper and pencil exercises testing the ability to correlate algebraic and graphic representations. The two experiments differed in the functions treated and in whether or not these functions were still being taught as part of the regular classroom teaching at the time of the experiment.

--The first experiment involved 33 students 16 to 18 years old from two classes in the last two years of high school (première and terminale); they worked with second degree polynomial functions in the forms F_1, F_2 and F_3.

--The second experiment involved 21 students 14 to 15 years old (troisième, or 9th grade); they worked with functions of degree one in the form x --> ax+b.

In the first case, the students had studied the functions, but the correlation of graphic and algebraic representations, even if it hadn't been mastered, was no longer the official object of classroom teaching. The second experiment took place immediately after the students were taught equations for straight lines. The following discussion concerns only the first experiment.

The notion of atoms of knowledge

To study the students' ability to infer algebraic representations of functions from their graphs, we have introduced the notion of *atoms of knowledge*.

-- An atom of knowledge expresses the relationship between a property of a coefficient of an algebraic representation of a function and its graphic representation.
-- This relationship should in theory be accessible by simply "reading" the graph.

For example, for a parabola of form F_1, F_2 or F_3:
-- the equivalence K_1 (the parabola is cup-shaped if and only if A>0) is an atom of knowledge;
--the equivalence K_2 (the parabola has a positive slope where it crosses the y-axis if and only if B>0) is an atom of knowledge;
-- the equivalence K_3 (the vertices of the parabolas P_1 and P_2 have the same x-coordinate if and only if $B_1/A_1= B_2/A_2$) is not an atom of knowledge because it simultaneously deals with two coefficients.

This notion of atom of knowledge meets several needs:
-- It takes into account the complexity of the cognition underlying the ability to switch between forms of representation (a complexity shown, for example, by [Duval, 1988] and [Schoenfeld & al, 1990].
-- It makes it possible to recognize partial success at correlating graphic and algebraic representations.
--It facilitates the automatic treatment of data, in particular in entering successive coefficients in a file for each student.

The pre-tests and post-tests were designed to test the most elementary atoms of knowledge. In the case of parabolas we tested, for the coefficient A, students' ability to determine its sign (signA), to determine whether the two coefficients A_1 and A_2 were equal or opposite (A_{eg} and A_{op}), and to rank parabolas according to the magnitude of A (when both coefficients A_1 and A_2 are positive (OrdA+), both are negative (OrdA-) or the two are of different signs (OrdA)).

For the coefficient B of form F_1, we tested the ability to interpret B=0 (B_{nul}) and equal or opposite values of B for parabolas with identical openings (B_{eg}) and (B_{op}); we did not test the atom of knowledge K_2 (cf. above), since it is too complex. For the coefficient C of form F_1, we tested students' ability to interpret its sign (SignC), its magnitude (C_{ord}), the case where C=0 (C_{nul}), and

whether two coefficents were equal or opposite (C_{eg}) and (C_{op}); the same atoms of knowledge were tested for the coefficients P and Q of parabolas of the form F_2. For the coefficients R and S of form F_3, we tested the ability to interpret the equality of R and S (R_{seg}) and their signs (RS++, RS--, Rs+-).

The notion of atoms of knowledge was also used to analyze the way students played the game. Since the coefficients in the software were necessarily whole numbers, we looked at only the following atoms of knowledge: sign, approximate size (to within one in absolute value), the magnitude of coefficient A, and the sign and exact value for the other coefficients.

Analysis:

In the first stage, we established a file for each student that summarizes:

-- results of pre-test, post-test and computer session, in terms of atoms of knowledge,

-- game strategies, based on the choice of forms and options and on possible changes observed during the session;

-- global results: number and type of exercises treated; success, both overall and by type; number of parabolas encountered...

These results were subjected to qualitative statistical treatment: factorial analysis, hierarchical analysis of similarities and implicative analysis [Larher,1991]. We then applied them to our original questions. On some points the analyses converged and gave clear answers but on others they remained difficult to interpret or even incoherent.

In the next stage we chose a certain number of student files which from the point of view of statistical analysis seemed to be typical or particularly problematic. The detailed study of these files led us to refine our original analysis of game strategies, which had guided us when we first set up the files. This enabled us to formulate hypotheses on the possible sources of the ambiguities and incoherences we had seen; it also enabled us to see cases where a student's ability to move from a graphic to an algebraic representation suddenly "crystallized," and to identify features of the game that were likely to promote this crystallization: what we call "catalysts."

In the third stage we tested, on all the files, all the hypotheses we had made in the second stage, concerning strategies, the phenomena of crystallization and catalysts.

III - RESULTS

1 - Pre-test

Pre-test results confirm the findings of the research cited above on the failure of standard instruction to teach students how to correlate algebraic and graphic representations. The percentages of success for the three forms F_1, F_2 and F_3 and the related atoms of knowledge are as follows:

Forme F1 :

SignA	Aeg	Aop	Aord	Bnul	Beg	Bop	SignC	Cnul	Ceg	Copp	Cord
56	11	11	6	17	0	0	6	6	6	0	0

Forme F2 :

SigA	Aeg	Aop	Aord	Pnul	SigP	Peg	Pop	Pord	SigQ	Qnul	Qeg	Qop	Qord
53	27	13	7	27	13	7	7	7	27	20	20	13	13

Forme F3 :

SignA	Aeg	Aop	Aord	RS++	RS+-	RS--	RSeg
50	18	9	6	9	9	3	3

These results highlight the students' inability to correlate algebraic and graphic representations: the only task at which more than 50 percent were successful was interpreting the sign of A (regardless of the form, F_1, F_2 or F_3). They also show a consistent ranking; for example, for A, we see:

Sign A >> A_{eg} > A_{op} > A_{ord}, for all three forms.

In addition, the pre-test shows that students consistently have more trouble with relationships of magnitude than they do with equalities or oppositions, and that they have a great deal of difficulty reading graphs point by point (for example looking at intersections with the axes), even though that approach is often considered to be the easiest.

2 - Post-test

Post-test results show a definite improvement which it is reasonable to attribute to the computer session, the only time spent officially on this subject (students were not told what they did right or wrong on the pre-test). The results are:

Forme F1 :

SignA	Aeg	Aop	Aord	Bnul	Beg	Bop	SignC	Cnul	Ceg	Cop	Cord
89	78	78	22	50	17	17	67	50	56	60	33

Forme F2 :

SigA	Aeg	Aop	Aord	Pnul	SigP	Peg	Pop	Pord	SigQ	Qnul	Qeg	Qop	Qord
87	73	67	33	87	83	80	73	60	80	73	60	53	67

Forme F3 :

SignA	Aeg	Aop	Aord	RS++	RS+-	RS--	RSeg
85	79	70	24	73	67	61	52

The overall success was 59 percent, compared to 13 percent on the pre-test. Some of the initial rankings also changed: for example, while students still did best overall on interpreting the sign of A, they did almost as well on recognizing equal or opposite values of A. Similarly, students generally did well on the different atoms of knowledge associated with the coefficients P and Q, and R and S. However, they still had trouble interpreting magnitude, especially for A. In this regard the earlier ranking remains valid. In addition, the ability to interpret B, the most complex task, was little affected by the session at the computer.

We see then that a brief, one hour long interaction with a simple computer game produced a qualitative change in the cognitive organization of many of the students. How was this change brought about?

3 - Game strategies and learning

In our initial analysis of the game, we identified three basic strategies: the use of coordinates; reading and estimating; and trial and error. Each type included several distinct strategies. For example, a coordinate-based strategy could be crude (and thus useless for forms F_2 and F_3), it could use particular points, or even choose particular points depending on the algebraic form under consideration. Similarly, strategies of reading could differ depending on the knowledge invested at the level of reading, and estimation could be done globally or coefficient by coefficient...

In fact, the student files do not tell us what strategies students used, but rather tactics characterized by certain patterns or lack of patterns in the choice of options ("coordinates", "sketch," "answer") and of their parameters. It is from these data that we must infer a possible game strategy or conclude that a student had no organized strategy. The incoherences uncovered by our first statistical analyses pointed out the danger of a too rapid automatic system of inferring game strategies. The detailed analysis of certain files enabled us to refine the criteria that were first used in this process. Consider, for example, the tactic of answering without asking for assistance, a tactic used by the end of the session by 75 percent of the students. Closer analysis shows that it corresponds to three fundamentally different situations, in terms of strategies:

a) The student plays in an unorganized way, without using feedback. This is associated with very rapid play (little time for thinking between answers), frequent changes of algebraic form and a lack of success. It is rarely found at the beginning of a session; it is rather a defeatist strategy adopted after repeated failures. It is the sign that the computer session is a failure.

b) The student knows enough to be able to adopt an elaborate strategy of reading and estimation. For example, for F_2 and F_3, she is able to read P,Q,R and S and to interpret the sign of A, and she is also sufficiently familiar with the openings to permit her to succeed, after two or three attempts, with the proposed coefficients (most often whole numbers between [-5,5]). Since she is allowed three tries,

she can succeed without using additional sketches to check answers (although such sketches are cheap); and the suspense involved in answering directly keeps her interested in the game.

Very few students were capable of adopting this strategy from the beginning; usually it is a strategy found at the end of the session, but is easily distinguished from the defeatist strategy discussed in a). In the case of form F_1, we sometimes see an efficient variant using the option "coordinates"; the student reads C, estimates A, views the coordinates of a point, calculates B from the estimated value of A, and, in case of failure, redoes the calculation based on a new estimate of A. This was the tactic generally used by those students using the reading/estimating strategy successfully; no student managed to interpret B directly.

c) The student develops an organized game (different from b) from the beginning of the session, trying to take into account the feed-back.

How well does success at the game carry over to success in the usual environment? Here our research clearly shows that all strategies are not equally effective. In particular, the predominant use of coordinate-based strategies, even when quite elaborate and successful, led to little improvement from pre-test to post-test. In contrast, strategies b) and c) discussed above led to an almost complete transfer of knowledge that apparently was acquired during the session. The only gap that was observed concerned the magnitude of A. Some students were able to estimate A rapidly in several tries, indicating a knowledge of OrdA+ and OrdA-, but failed to transfer this ability to the task of ranking parabolas P_i by the magnitude of the corresponding coefficients A_i, even for coefficients of the same sign. It is as if (referring to the theory of conceptual fields and schemes [Vergnaud, 1991]), the invariant constructed during the game is not sufficiently well anchored, conceptually, to permit the necessary transfer to a different context.

4 - Phenomena of crystallization and catalysts

In the first experiment, our analysis showed that a student's grasp of the meaning of coefficients often occurs abruptly: suddenly, during a game, a coefficient makes sense and the student's behavior reflects this during all the subsequent exercises. This occurred much more seldom in the second experiment with younger students: this is no doubt explained in part by their lack of familiarity with the objects, leading to a more fragile cognitive network.

We have called these sudden realizations "crystallizations" and have tried to get a precise idea of the context in which they occur. We have identified a certain number of catalysts for this crystallization, of which the principal are:

-- meeting a particular parabola,

-- changing to a different form,

-- identifying particular points.

Of course, these factors do not always catalyze crystallizations. In addition, they do not produce the same effect independent of time and form (F_1, F_2 or F_3). When a student meets a particular situation early in the game, it often has no effect. Similarly, too simple a situation often results in only local, restricted success and does not spark a true crystallization. But the above factors are seen in almost all of the observed crystallizations.

IV - CONCLUSION

These first experiments are limited but without any doubt, they prove the effectiveness of the software designed for this research. Moreover, they do help us to understand the process by which such a computer game can impart mathematical knowledge, to see how data from such experiments should be analyzed, and to set our sights on software that could take this analysis into account in real time, using a student's behaviour as feedback to guide him in a series of games. As a first step the present version incorporates a real-time analysis module that provides a record of the play, various statistical data and a graphic image of the student's cognitive evolution from the beginning to the session, expressed in terms of strategies and atoms of knowledge.

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THE EFFECT OF CALCULATOR USE ON THIRD GRADERS' SOLUTIONS OF REAL WORLD DIVISION AND MULTIPLICATION PROBLEMS

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As part of a study of the long-term effects of calculator use, a random sample of 48 grade 3 children from five schools were observed while tackling real world problems based on quotition division and repeated addition multiplication in a realistic context. Calculators and concrete materials were provided, as well as pencil and paper. Children with long-term experience of calculator use performed significantly better on all multiplication items and the most difficult division item. While they did not make significantly more use of calculators than the children without such calculator experience, they were better able to attach meaning and interpret their answers, especially where knowledge of decimal notation or large numbers was required.

Introduction

The *Calculators in Primary Mathematics* project is a long-term investigation into the effects of the introduction of calculators on the learning and teaching of primary mathematics. It is based on the premise that calculators have the potential to significantly change mathematics curriculum and teaching (Corbitt, 1985, p.14; Cockcroft, 1982, p.109). Apart from the *Calculator-Aware Number* (CAN) project (Shuard, Walsh, Goodwin & Worcester, 1991), there is little evidence that such changes are commonly occurring (Curriculum Development Centre, 1986, p.18; Hembree & Dessart, 1986, p.83; Reys, 1989, p.173). The project commenced in 1990 at kindergarten and grade 1 level and will continue to grade 4 in 1993. In 1992, there are 45 kindergarten to grade 3 classes in six schools. All children are "given" their own calculator to use whenever they wish. Teachers are provided with systematic professional support to assist them in using calculators to create a rich mathematical environment for children to explore.

The effect on children's long-term learning is one of the research foci of the project. All children at grades 3 and 4 levels (approximately 450) are being given a written test and a test of calculator use in each of the years 1991, 1992 and 1993, with a random 10% sample of these children also taking part in a 25-minute interview. The 1991 children and the 1992 grade 4 children (none of whom have been part of the calculator project) form the control group for the study. At the time of testing, the 1992 grade 3 children and all of the 1993 children will have been part of the project for $1\frac{1}{2}$ and $2\frac{1}{2}$ years respectively. Among the hypotheses for the long-term study is an expectation that children involved in the calculator project will perform better on "real world" problems, by selecting appropriate processes more frequently and by making better use of calculators.

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Groves (1992) reported on the processes and strategies adopted by the grade 3 and 4 children in 1991 when attempting a "real world" problem amenable to quotition division - a simplified version of a question from a problem solving test (Stacey, Groves, Bourke & Doig, in press). Children predominantly used informal addition and subtraction based strategies rather than division, confirming earlier findings (Kouba, 1989; Hart, 1981, p.47; Bergeron & Herscovics, 1990, p.32; Neuman, 1991, p.76; Stacey, 1987, p.21). Although children were provided with calculators and concrete materials, they overwhelmingly chose mental computation as their "calculating device". Difficulties encountered by those children who used calculators confirmed the mathematical sophistication required to interpret the answers obtained. While reliable mental methods and an ability to use calculators (together with an understanding of the meaning of the operations and the real world problems which they model) may be sufficient for all practical purposes (Hart, 1981, p.47; Bell, Fischbein & Greer, 1984, p.130; Bergeron & Herscovics, 1990, p. 34), these results confirmed the importance of attaching meaning when using calculators and the necessity to develop skills such as estimation and approximation, together with a strong intuitive understanding of aspects of the number system such as decimals.

This paper compares the results for the grade 3 children in 1992, who had been part of the calculator project for $1^1/_2$ years, with those of the 1991 grade 3 children for the quotition division question referred to above, and a question based on "repeated addition" multiplication - also referred to as "multiple groups" (Bell, Greer, Grimison & Mangan, 1989), "equal groups" (Greer, 1992) and "isomorphism of measures" (Vergnaud, 1983).

Method

In 1991 and 1992, a random sample of about 30 grade 3 children were given a 25-minute interview. The interview was designed to test children's understanding of the number system; their choice of calculating device, for a wide range of numerical questions; and their ability to solve "real world" problems amenable to multiplication and division, with or without calculators. Throughout the interview, children were free to use whatever calculating devices they chose. Unifix cubes and multi-base arithmetic (MAB) blocks were provided as well as pencil and paper and calculators. Many of the questions were expected to be answered mentally. This paper focuses on interview results from five schools (the sixth school having had a different pattern of participation in the project) on the "real world" problem amenable to division and the "real world" multiplication problem. In the first two parts of the division question, children were presented with clear bottles containing the appropriate number of white, medicine-like tablets (actually sweets). The bottles were attractively labelled with the contents and the amount to be taken each day - for example, in M1 the label clearly displayed "15 tablets take 3 each day", as well as the distracter "$7.43". For the remaining three parts, accurate volumes of coloured liquid were used with information such as "120 ml take 20 ml each day" and a price. For this example (the first using liquid "medicine"), 20 ml was poured from the bottle into a clear medicine measure. In each case, children were asked how many days the medicine would last. In the multiplication problem, children were shown a paper clip chain, consisting of 17 multi-coloured clips, which they were told was made by

a grade 3 child at another school. They were then asked how many paper clips would be needed if 10 children each made a similar chain; if a whole class of 27 children each made such a chain; if a school of 295 children made such chains; and finally if 1 million children made such chains.

As well as their answers, children's choice of calculating device were recorded. For the purpose of this analysis, these have been classified as calculator, mental (which may include the use of fingers), and other (which also includes no answer given). For the two tablet medicine items, an additional category of Unifix or drawing has been included. For the division question, the mathematical processes used have been classified here as counting on/multiplication, division and other, with the additional category of quotition using concrete materials or drawing for the tablet items. For the multiplication question, the processes have been classified as multiplication, counting on and other.

Results

Tables 1 and 2 compare the 1992 and 1991 frequencies of correct and incorrect answers, use of calculating devices and solution processes for the tablet and liquid medicine questions respectively. In each case, the left side shows choice of calculating device against correctness of answer, while the right side shows solution processes. In those parts of the question where remainders occur, an extra category of answer is included to indicate answers which, while incorrect, give the correct number of whole days.

Table 1: Comparison of frequencies of correct and incorrect answers, use of calculating devices and solution processes, on tablet medicine items for grade 3 children in 1992 and 1991

Question	Device[1]	C	M/F	U/D	O/–	Total	CO/M	DI	Q	O/–	Process[2]
M1 15 tablets, 3 per day How many days?	√ [3]	2* (1)	15 (14)	5 (4)	0 (0)	22 (19)	11 (13)	4 (1)	5 (2)	2 (3)	√ [3]
	X [3]	0 (0)	2 (3)	1 (0)	0 (1)	3 (4)	2 (1)	0 (0)	0 (0)	1 (3)	X [3]
	Total	2 (1)	17 (17)	6 (4)	0 (1)	25 (23)	13 (14)	4 (1)	5 (2)	3 (6)	Total
M2 21 tablets, 4 per day How many days?	√ [3]	4 (0)	4 (5)	4 (2)	1 (1)	13 (8)	5 (7)	3 (0)	3 (0)	2 (1)	√ [3]
	I [4]	1 (3)	4 (2)	1 (1)	0 (0)	6 (6)	1 (2)	1 (3)	1 (1)	3 (0)	I [4]
	X [3]	1 (1)	4 (5)	0 (0)	1 (3)	6 (9)	3 (2)	0 (1)	0 (0)	3 (6)	X [3]
	Total	6 (4)	12 (12)	5 (3)	2 (4)	25 (23)	9 (11)	4 (4)	4 (1)	8 (7)	Total

* Figures in table represents 1992 frequencies of responses, with 1991 frequencies in parentheses

1 C - calculator; M/F - mental (inc. use of fingers); U/D - Unifix/drawings; O/– - other (e.g. written)/no answer

2 CO/M - counting on/back (repeated addition/subtraction)/ multiplication ; DI - division; Q - quotition using concrete materials or drawings; O/– - other / no answer given

3 √ - correct answer; X - incorrect answer or no answer given

4 I - incorrect answer with integer part correct (e.g. 5, 5+, 5 remainder 3, 5 remainder 25)

Table 1 indicates that, while the 1992 children performed better on both tablet medicine questions, the overall pattern of use of the various categories of calculating device was similar, as were the processes used. Mental calculation, using counting on or multiplication, was the most popular in both years. In 1992, there was an increased correct use of calculators for M2, which was due to the increase in children's ability to interpret the answer 5·25, as well as an increased correct use of quotition division by those children who used Unifix or drawing. None of the differences are statistically significant.

Table 2: Comparison of frequencies of correct and incorrect answers, use of calculating devices and solution processes, on liquid medicine items, for grade 3 children in 1992 and 1991

Question	Device[1]	C	M/F	O/–	Total	CO/M	DI	O/–	Process[2]
M3 120 ml, 20 ml /day How many days?	√[3]	5* (4)	12 (10)	2 (1)	19 (15)	11 (6)	6 (6)	2 (3)	√[3]
	X[3]	0 (0)	2 (1)	4 (7)	6 (8)	4 (0)	0 (0)	2 (8)	X[3]
	Total	5 (4)	14 (11)	6 (8)	25 (23)	15 (6)	6 (6)	4 (11)	Total
M4 300 ml, 40 ml /day How many days?	√[3]	4 (1)	2 (1)	2 (1)	8 (3)	4 (2)	4 (1)	0 (0)	√[3]
	I[4]	2 (6)	2 (2)	2 (0)	6 (8)	3 (3)	2 (5)	1 (0)	I[4]
	X[3]	1 (1)	5 (3)	5 (8)	11 (12)	5 (2)	1 (1)	5 (9)	X[3]
	Total	7 (8)	9 (6)	9 (9)	25 (23)	12 (7)	7 (7)	6 (9)	Total
M5 375 ml, 24 ml /day How many days?	√[3]	9 (0)	0 (0)	0 (0)	9 (0)	1 (0)	8 (0)	0 (0)	√[3]
	I[5]	4 (5)	0 (0)	1 (0)	5 (5)	1 (0)	4 (5)	0 (0)	I[5]
	X[3]	4 (9)	1 (1)	6 (8)	11 (18)	2 (3)	3 (3)	6 (12)	X[3]
	Total	17 (14)	1 (1)	7 (8)	25 (23)	4 (3)	15 (8)	6 (12)	Total

* Figures in table represents 1992 frequencies of responses, with 1991 frequencies in parentheses

1 C - calculator; M/F - mental (including use of fingers); O/- - other (e.g. written) / no answer

2 CO/M - counting on/back (repeated addition/subtraction)/ multiplication; DI - division; O/- - other/no answer

3 √ - correct answer; X - incorrect answer or no answer given

4 I - incorrect answer with integer part correct (e.g. 7, 7+, 7 remainder 80, 7 remainder 2) or 8

5 I - incorrect answer with integer part correct (e.g. 15, 15+, 15 remainder ?, 15 $^3/_4$)

Table 2 shows 1992 children also performing better on the liquid medicine questions. For all of the liquid medicine items, the use of calculating device was similar in both years. For M3 and M4, the counting on/multiplication process was used by even more children than in 1992 than in 1991. This use was not always straightforward: several children used their calculators for repeated addition, keeping a tally on their fingers or paper, while one child consistently counted backwards on the calculator until a negative value was reached. In 1992, three instances were recorded of children doing mental calculations such as the following for M4: "40 x 5 = 200, 40 x 2 = 80, 7 days with 20 mls left". No similar calculations were recorded in 1991.

The most difficult item, M5, produced a significant difference in the number of children who obtained a correct answer - over a third in 1992, all using a calculator, compared to none in 1991 (χ^2=10·625, df=2, $p \leq 0{\cdot}01$). In 1992, over half of the 17 children who attempted this item using a calculator were successful, compared to none of the 14 in 1991. The 1992 children were able to use their calculators more effectively largely because they were able to correctly interpret their answer of 15·625. This is not surprising when one considers the results from two earlier items on the interview. The first item asked children to read 5·42 and then select from 542, 2, 5, 54·2 and 6, the number closest to 5·42. For the other item, children were shown 278 + 39 and "the answer found by someone using a calculator" - i.e. 7·1282051. They were asked firstly to read the number and then to say "about how big" it is or give a "number close to it".

Table 3: Comparison of frequencies of responses on decimal recognition items for grade 3 children in 1992 and 1991

Response	Reads 5·42		Selects 5 or 6		Reads 7·1282051		Integer close to 7	
Yes	13#	(0)	6[1]	(1)	14	(2)	5[2]	(0)
No	12	(23)	19	(22)	11	(21)	20	(23)

Figures in table represents 1992 frequencies of responses, with 1991 frequencies in parentheses

1 Although 5 is the correct answer, both 5 and 6 have been regarded as correct here

2 Integer answers in the range 5 to 9 were regarded as correct here

Table 3 shows a significant improvement in 1992 children's ability to read 5·42 and 7·1282051 (χ^2=16·402, df=1, $p \leq 0{\cdot}001$ and χ^2=12·063, df=1, $p \leq 0{\cdot}001$, respectively). Over 20% of the 1992 children knew the approximate size of these decimals, compared to only one correct response for the 1991 children (not significant at the 0·05 level).

Table 4 shows that approximately twice as many 1992 children as those in 1991 obtained correct answers for each of the multiplication items (all significant at least at the $p \leq 0{\cdot}05$ level). Although there was no significant difference in patterns of use of calculating devices, children in 1992 made more use of calculators. This change was partly due to less use of other devices (such as counting the paper clips or drawing) and also to fewer children making no attempt to answer. For P1, the item requiring multiplication by 10, while more children in 1992 used a calculator than in 1991, more children also used mental computation. For each of the items, more children in 1992 used multiplication than in 1991 (significant at the $p \leq 0{\cdot}002$ level for P2 and P4, but not significant at the $p \leq 0{\cdot}05$ level for P1 and P3).

Table 4: Comparison of frequencies of correct and incorrect answers, use of calculating devices and solution processes, on paper clip items for grade 3 children in 1992 and 1991

Question	Device[1]	C	M/F	O/–	Total	M	CO	O/–	Process[2]
P1 10 paper clip chains,	√[3]	11# (4)	6 (5)	1 (0)	18 (9)	13 (6)	3 (2)	2 (1)	√[3]
17 clips each.	X[3]	0 (3)	6 (3)	1 (8)	7 (14)	0 (0)	2 (1)	5 (13)	X[3]
How many paper clips?	Total	11 (7)	12 (8)	2 (8)	25 (23)	13 (6)	5 (3)	7 (14)	Total
P2 27 paper clip chains,	√[3]	16 (8)	0 (1)	1 (0)	17 (9)	16 (7)	1 (1)	0 (1)	√[3]
17 clips each.	X[3]	1 (3)	2 (2)	5 (9)	8 (14)	2 (0)	0 (1)	6 (13)	X[3]
How many paper clips?	Total	17 (11)	2 (3)	6 (9)	25 (23)	18 (7)	1 (2)	6 (14)	Total
P3 295 paper clip chains,	√[3]	18 (6)	0 (0)	0 (0)	18 (6)	18 (6)	0 (0)	0 (0)	√[3]
17 clips each.	X[3]	0 (6)	0 (2)	7 (9)	7 (17)	1 (4)	0 (1)	6 (12)	X[3]
How many paper clips?	Total	18 (12)	0 (2)	7 (9)	25 (23)	19 (10)	0 (1)	6 (12)	Total
P4 1 million paper clip	√[3]	11 (2)	4 (3)	0 (0)	15 (5)	15 (4)	0 (1)	0 (0)	√[3]
chains, 17 clips each.	A[4]	4 (3)	0 (0)	0 (0)	4 (3)	4 (3)	0 (0)	0 (0)	A[4]
How many paper clips?	X[3]	1 (2)	0 (2)	5 (11)	6 (15)	1 (1)	0 (0)	5 (14)	X[3]
	Total	16 (7)	4 (5)	5 (11)	25 (23)	20 (8)	0 (1)	5 (14)	Total

\# Figures in table represents 1992 frequencies of responses, with 1991 frequencies in parentheses
1 C - calculator; M/F - mental (including use of fingers); O/– - other (e.g. written) / no answer
2 M - multiplication: CO - counting on (repeated addition); O/– - other / no answer given
3 √ - correct answer; X - incorrect answer or no answer given
4 A - correct answer obtained with incorrect calculation (e.g. 1000 x 17 = 17 000)

Conclusion

The *National Statement on Mathematics for Australian Schools* (Australian Education Council, 1990) supports an increased emphasis on developing number sense through mental computation in recognition of the role of the calculator. Classroom observation and teacher reports from the *Calculators in Primary Mathematics* project schools suggest that many young children are dealing with much larger numbers than would normally be expected, as well as negative numbers and decimals. Preliminary

results from extensive teacher questionnaires indicate that, as a result of their experiences, teachers are changing their expectations of children's performance in these areas (Groves & Cheeseman, 1992).

Grade 3 children who had been part of the calculator project for $1^1/_2$ years performed better than those who had not been in the project on all the "real world" division and multiplication items on the interview, although there was no significant difference for the easier division items and no significant difference in their choice of calculating device for any of the items. The significantly better performance of project children on the division item which gave 15·625 as the answer, together with project children's ability to better interpret decimal notation, suggests that long-term experience in using calculators has provided at least some children with the intuitive understanding of decimals which is required to make meaningful use of calculators in such situations.

Wearne and Hiebert (1988) recognise the growing importance of decimals in a technological age. As part of their theory for the development of symbol competence, they stress the importance of referents with rich associations for the student - usually non written material. either in everyday use, such as money, or specially designed, such as MAB blocks (p. 224). We believe that it is precisely by acting as such a referent that calculators can play a critical role in children's development of concepts, such as place value and decimal notation, by enabling children to experiment with and manipulate symbols in a way which is impossible to achieve with concrete materials.

The 1992 project children used multiplication more often than the 1991 non-project children for all of the paper clip items, significantly so for two of the four items. Not surprisingly, children appeared more likely to choose an operation such as multiplication (as opposed to the less sophisticated counting on) when they appeared able to find the answer with their calculators. At least some of the improved performance was due to their ability to interpret large numbers on the calculator display. There has been much research on children's choice of operation when not required to actually calculate an answer. Fischbein, Deri, Nello and Marino (1985, p. 5) suggest that some "intervening intuitive model" (such as repeated addition in the case of multiplication) may inhibit children from choosing the correct operation for problems which require different models (such as those involving multiplication by a decimal). Anghileri (1989, p.384) seeks links between the structure of a multiplication task and the solution strategies. However, a different possible further question for research may be the link between children's selection of operation for a problem and their ability to calculate the answer if required.

Preliminary results discussed here indicate that children with long-term experience of calculators are better able to tackle "real world" problems which would normally be beyond their paper and pencil skills. While they did not make more use of calculators than the children without this experience, they were better able to attach meaning and interpret their answers.

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Coordinating Sets and Properties when Representing Data: The Group Separation Problem

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This paper reports on a clinical study of students' productive understanding of database record/field structures. Using a data analysis tool with which they were familiar, students were asked to create a database structure that would allow them to produce a desired graph. A recurring pattern was observed in which subjects produced a set-based structure instead of the required property-based structure.

This paper describes and discusses some of our observations in an exploratory study relating to students' understanding of data structure. We have been working for some years on implementing data-based inquiry activities with students aged 10-15, in school classrooms and in small groups (Hancock, Kaput & Goldsmith, 1992). In these pilot activities students are often responsible for collecting data, entering it on computers and analyzing it, using a prototype database/data analysis software tool called the Tabletop. The present study arose from our observing in these contexts that students sometimes had intriguing difficulties in constructing useful databases.

Although the Tabletop is different from traditional data analysis programs, it is similar in one important respect: it provides a standard database format for entering and modifying data, and a separate, "read-only" facility for graphing and analyzing the data. While one often uses the tool in order to produce graphs, data cannot be entered in the form of a graph. This property sets the stage for a recurring kind of mathematical problem: Given that one would like to produce a particular kind of graph, how must data be entered in the database in order to make such a graph possible? (Kaput & Hancock, 1991). While students often manage this without difficulty, we have seen it become problematic in some interesting cases. For example, one group of 11-year-olds conducted a survey in the school cafeteria over three days in order to find out how lunching behaviors varied depending on the meal being served. The students wanted to make a graph that separated the questionnaires according to the day on which they were administered. They were unable to produce the graph, but did not realize that this was because their database was missing the crucial "day" information.

Two plausible causes might be postulated for difficulties like these: Students may not realize that the computer doesn't know what they know; and/or, they may not know how to encode this information in a way that the computer can use. Closer study in a clinical setting has shown us that both issues are important, and that they are bound up with each other. This paper, however, focuses on a particular phenomenon that clearly relates to the second issue: students' tendency to gravitate to a set-based representation, rather than the property-based representation required by the tool.

Tabletop Essentials

First, a brief description of the Tabletop's data formats. The Tabletop offers two coordinated representations of a database: the row/column window and the "Tabletop" window. The row/column window is used for database construction and data entry. Information in the row/column window is

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arranged in a rectangular array. If we think of a database as describing a collection of objects, each row contains all the information about one of those objects. In a database of cats, each row describes one cat; in a questionnaire database, each row describes one respondent. Each column of the array represents a particular property of the objects (the cats' ages, say, or the answers to question 3). It includes a column heading at the top (the "field name") and contains one piece of data for each object in the database. Rows correspond to database records, columns to database fields.

The Tabletop window presents the database as a set of small animated icons. Each icon represents an object in the database; by double-clicking on it, the user can view all available information about that object. These icons can be arranged automatically into Venn diagrams, scatter plots and other arrangements useful for data analysis. The Venn diagram, with which we are particularly concerned here, is a spatial alternative to the traditional database query. It can have between one and three overlapping loops. In each loop the user may specify a set constraint in three parts: a field, specified by choosing from a menu of all the fields in the database; a comparison operator (=, >, < etc), again chosen from a menu; and a value, which can be typed in. Objects for which the mathematical statement so specified is true will slide automatically into the loop and all others will slide out. The Tabletop window is a "read-only" view, which allows flexible exploration and analysis of an existing database; to add or modify information the user must return to the row/column view.

The Group Separation Problem

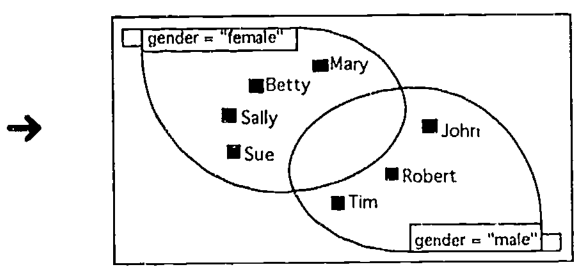
We devised the following task, which we call the "group separation problem," to see whether students would realize they needed to make explicit to the computer certain implicit information that they "already knew."

The subject is given a database consisting of a list of names. (The database is created in the subject's presence.) Some of the names are male names and some are female names. The task consists of getting the computer to place the icons representing the male names in one Venn ring and those representing the female names in another. Thus the task involves the separation of a collection of data into two disjoint sets or groups. Since the subject knows, simply by looking at the names, which name is a girl's name and which a boy's, the task requires the student to objectify implicit knowledge. To accomplish this task, the student needs to create a new field with the heading "sex" or "gender" or "boy/girl" or some other such designator and then specify for each name whether the person is "male" or "female," "boy" or "girl." Once this is done, two loops can be made in the Tabletop window with constraints referring to the new field.

Name
John
Mary
Sally
Tim
Betty
Robert
Sue

For example:

Name	Gender
John	male
Mary	female
Sally	female
Tim	male
Betty	female
Robert	male
Sue	female



Pilot Results

Thirteen students (four fifth-graders, five sixth-graders, and four seventh-graders) participated in a pilot study. These students had between one and two academic years' worth of experience and familiarity with the Tabletop program. They had used the Tabletop to analyze many databases, and had created databases of their own during classroom and small group activities. In some of these databases they had even used a gender field. However, of the thirteen subjects, only two (seventh graders) could solve the problem outright. The majority could only solve it with varying degrees of hints and guidance. A few could not solve it at all. For these students, production of record/field structures clearly lags far behind comprehension in the context of tool use.

Many subjects began confidently. They would proceed to create a constraint for the first loop, beginning with the menu of field names. When the menu came up with "Name" available as they only choice, some of them visibly hesitated. We have the impression that, had a "Gender" field been available at that point, subjects could have used it successfully, as they often did in class activities. Instead they lost momentum. Their behavior ranged from not knowing what to do at all, to poking around at the various menus hoping that something would happen if they could only find the right button, to constructing endless permutations of the constraint rules in the Venn rings. Most of the students did not recognize a need to supply the computer with additional information. When prompted with questions such as "Does the computer know who is a boy and who is a girl?" and "How can you tell the computer who is a boy and who is a girl?" they would realize that they had to change something in the database. Their problem then became how to enter this information in a form usable to the Tabletop program.

Here a striking and unanticipated pattern emerged. Independently of each other, most of the subjects (9 out of 13) created similar data structures of a kind which, although intelligible to humans, is quite inappropriate in the Tabletop program. They added not one but two columns to the database— one for girls and one for boys. In each column they simply listed all the names of that gender, without regard for the alignment of information in rows. A typical example is shown here.

Name	Boys	Girls
John	John	Mary
Mary	Tim	Sally
Sally	Robert	Betty
Tim	♦	Sue
Betty	♦	♦
Robert	♦	♦
Sue	♦	♦

This led of course to bizarre data records. Double clicking on icons in the Tabletop, for example, would yield object descriptions such as:

Name: John	Name: Mary	Name: Tim
Boys: John	Boys: Tim	Boys: ♦
Girls: Mary	Girls: Sally	Girls: Sue

Some vocabulary would be helpful here. We will call the representation consisting of a field of individual names and a parallel field of corresponding properties a *property-based* representation. This is the representation that the Tabletop program requires to solve the problem. A *set based* representation, on the other hand, consists of two set names, each with its own list of members

Property-based		Set-based	
Name	Gender	Boys	Girls
John	male	John	Mary
Mary	female	Tim	Sally
Sally	female	Robert	Betty
Tim	male	♦	Sue
Betty	female	♦	♦
Robert	male	♦	♦
Sue	female	♦	♦

We can also say that the property-based representation has an *individual-centered* structure (a list of individuals and a corresponding list of their properties), whereas the set-based representation has a *group centered* structure (sets or groups are defined as lists under group designators—here "boys" and "girls").

Variations of the task

After our initial pilot study we created new variants of the group separation problem. After beginning with the standard boys/girls problem, the interviewer could choose from among the following tasks to assess the robustness of the subjects' knowledge and to pinpoint sources of difficulty.

1. Given a set of seven pictures of dogs and cats with their names, the subject is asked to make a database and then separate the dogs from the cats. Whether an animal is a dog or a cat is clear from the picture, but not written anywhere.
2. Given a set of seven pictures of people with names and numerical ages, the subject is asked to separate the children from the adults. The numbers must be converted into adult/child categories, either when the data is entered or when the loop constraint is written.
3. Given a prefabricated, *set-based* database, the subject is asked to separate the married from the single people. The problem can only be solved by restructuring the data in proper record/field form. This "perverse" database does seem highly effective for revealing fragility of students' structural knowledge. However we have some concern about whether such a question is ethical. We have used it with only a few subjects.

Married	Single
Mr. Jones	Miss Ross
Mrs. Rodriguez	Timmy
Ms Smith	Cindy
Mr. Davis	Ms. Steinem

Armed with these variations, we interviewed six additional subjects. Two sessions are summarized here.

Tanya
Age: 13; Finished Grade 7

Given the boys/girls problem, Tanya tries a variety of Venn diagrams. A Loop labelled "Name = 'girl'" fails to attract any icons. She can explain why ("no one is named 'girl'"), but because she doesn't have any better ideas, she's trying everything she can think of. She tries "name='Sally'" and one icon moves in. She tries "Name = 'Sally, Laura, Elizabeth'", but that doesn't work. "Too many names," she explains. She tries Name=girl, Name >= girl, Name < girl. Then she tries changing 'girl' to 'girls.'She tries "Name='Sally'" again just to make sure that something works, even if it doesn't solve the problem. Then she tries "Name='boy'."

We hint that Tanya might need to give the computer more information about who is a boy and who is a girl. She goes to the row/column window, creates a new field called "Girl," and enters the three girls' names under the heading. She tests in the Tabletop window: "Name=girl" still doesn't work; the other operators still don't work either. She returns to the row/column window and creates a new field called "sally" and enters a single item, Sally. This doesn't help at all when she returns to the Tabletop. Now she is stuck

To generate a possibly helpful example, we create another field in the database called "favorite color," with some made-up data. How could one separate the people who like red? Tanya does this with no difficulty. Having seen how the favorite color field was automatically added to the menu of options for the first part of a loop constraint, Tanya quickly gets the idea to solve the boy/girl problem by adding a field named "sex."

Next we pose the dogs/cats problem. Tanya says she'll solve it the same way as boys/girls, but in fact she creates a set-based representation. She creates a field called "dog" and adds the four dogs. Then she makes another field called "cat," and adds the three cats.

Dog	Cat
Spot	Samantha
Ralph	Fluffy
Maxine	Scruffy
Strudel	

When she is about to return to the Tabletop window, we ask her how many icons she expects. She answers seven, but there turn out to be only four. She cannot explain why. Again she is stuck. We remind her about the previous database, where all the people were in one column. With this hint, Tanya returns to the row/column (database) window. She deletes the two existing fields and adds a new one called "animals," in which she enters the seven animals' names. She then makes a second field called "kind of animal," and enters "dog" and "cat" next to each animal's name. With this database she is able to make the required separation in a Venn diagram. She goes on to solve another variant, creating a field called "age" in order to separate adults from children.

Animals	Kind of animal
Samantha	cat
Fluffy	cat
Scruffy	cat
Spot	dog
Ralph	dog
Maxine	dog
Strudel	dog

Tanya can adjust to the need for property-based structures when given hints, but she seems to forget easily on the next task. Two very different structures are "the same way" to her.

Robbie
Age 12; Finished Grade 5

We begin by posing the girls/boys problem. Like Tanya, Robbie at first tries various permutations of constraint rules, and hunts for a "magic button" in the program's menus. Seeing that this isn't getting anywhere, he suddenly says, "How about putting in another database?" - by which he means adding a new field to the existing database. Robbie adds a new field and calls it "Sex." He enters "M" or "F" appropriately for each record. Asked to explain these changes, he says "Because 'Name' just says the names, and 'sex' means, like, boy or girl." He feels confident that this will work, and it does. Asked if he has done something like this during the school year, Robbie says no, he "just invented it off the top of my head." (Two recent projects in his class did in fact use sex or gender fields.)

We next give Robbie the dog and cat cards, and ask him to make a database that will allow him to put the dogs in one ring and the cats in another. Confident that he can solve the problem, Robbie injects remarks such as "you're not going to trick me this time" as he produces the following:

Animal	Sex
Ralph	D
Fluffy	C
Spot	D
Maxine	D
Samantha	C
Scruffy	C
Strudel	D

With this database and the Venn loop constraints "Sex = 'D'" and "Sex = 'C'," Robbie solves the problem. Asked whether another person would understand the resulting graph, Robbie says "no." He decides to change "Animal" to "Name" and "Sex" to "Animal," and to replace "D" and "C" with "dog" and "cat." Asked if he can also separate the male animals from the female animals, Robbie correctly creates an additional field called "sex."

Next we present Robbie with the "perverse" database (described above). Robbie notices that not all the names are present in the Tabletop window. There are only four icons, and these can be labelled with names of the single people or the married people, but not both. Double-clicking on an icon shows two names together — Mr. Jones and Miss Ross, for example. Robbie goes back to the row/column window and drags the "Single" field farther from the "Married" field, explaining "If they're too close, they'll be in the same box" (referring, apparently, to the pop-up window appearing when one clicks on an icon). But this proves to have no effect. Then Robbie suspects that we are trying to trick him. "I know how your mind works," he adds at one point. Thinking that the "Ms." abbreviation is confusing the computer, he changes all the "Ms." titles to either "Mrs." or "Miss." But this, too, has no effect.

Sensing frustration on Robbie's part we propose that he build a new database from scratch. We give him a set of cards showing pictures of people, their names, and whether they are "single" or "married." Robbie recreates a set-based structure, with columns for "Married," "Single" and "Kids" (some of the single people are children). Then he removes the Kids column and puts the children into the Single column. When he goes to the iconic window he sets up two Venn loops with constraints "married=married" and "single=single." When these fail he experiments with various operators in place of the =.

We draw Robbie's attention to the difference between the number of icons (5) and the number of people (7) and the fact that some icons reveal two names in their pop-up windows. Robbie says, "*Oh*, I know what to do... If it's the same row it's, like, the same subject." To our surprise he then constructs a database consisting of two rows as follows.

m-s	Name	Name	Name	Name	Name
single	Yung-chi	Mr. Jones	Miss Ross	Franklin	Cindy
married	Mrs. Rodriguez	Mr. Smith	◊	◊	◊

Robbie is astonished to discover only two icons present in the iconic view, corresponding to the two rows in his database. After nearly an hour and half of work, we judge that Robbie has had enough, and show him the "correct" solution. Robbie himself recognizes this as similar to the solutions he had achieved earlier in the session.

In Robbie's last construction, the rows do indeed all contain the same "subject," but the subjects are groups, not individuals. This is actually another set-based representation, with the horizontal and vertical dimensions transposed!

Discussion

This study offers just a first glimpse of the difficulties students seem to have with conventional database representation. Much more work is needed to explore this phenomenon more fully. Meanwhile, we would like to venture some ideas and questions about students' apparent tendency to gravitate to set-based rather than property-based representations.

To begin with, we should note that the problem contains a paradox. The goal is to separate two sets that were mixed into one, but the nature of the computer tool requires the creation of a unified structure in which the male and female names remain mixed in the "name" field and a parallel field encompassing the two possible categories, male and female. The need for unification in the database in order to achieve separation of the icons in the Venn rings might not occur to students without a good understanding of the mechanics of the tool; creation of two separated columns would seem like a more direct route.

Visual salience of rows and columns may condition subjects' perceptions of the data. There are few visual cues in the row/column window to demarcate the horizontal data records, whereas the vertical columns underneath each field name stand out as quite evident. Beginning with a single column of data may also be a weakness in our method, since it accentuates the vertical even more. Perhaps children also

have more experience in their daily lives with vertical lists than with horizontal ones. This cannot be a complete explanation, however, since, as Robbie's final construction shows, set based representations can dominate even when horizontal and vertical are switched.

Logical structure

To solve the problem one must not simply see that there are two dimensions; one must also devise a logical structure that coordinates them appropriately — a property-based structure. A property-based structure is logically more complex than a set-based one. In a set-based representation there are two kinds of objects: the sets (girls, boys) and their members (the individuals). In a property-based representation there are three: the individuals, the property heading (gender) and the property assignments or categories (girl, boy). These are not three levels in the sense of objects, classes and superclasses. A true superclass of "boys" and "girls" would be "People." "Gender" is not a superclass, but a *metaclass*: its members are the two classes (or class-defining properties) "male" and "female."

The set-based/property-based distinction is very close to the distinction between two fundamental mathematical objects, the set and the relation. This in turn suggests a connection to a phenomenon that has been reported concerning students' perception of mathematical functions represented as tables, as in the following diagram. When asked to figure out the pattern, many students apparently treat each column as a self-contained sequence of numbers, rather than looking for a common relationship between corresponding numbers on the left and the right (J.Kaput, personal communication cited by R. Noss). Again students appear to be using a simpler two level model (sequence and element) rather than a more complex three level model (function, domain and range).

x	f(x)
1	0
2	3
3	8
4	15
5	?

Linguistic context

Language often provides us with a bias towards describing certain relationships in terms of set membership and others in terms of property assignments, even though the two kinds of descriptions are logically isomorphic. In English we ordinarily say "Sally is a girl." We are less likely to say "Sally's gender is female," and "gender" is certainly a less common and more abstract word than "girl." On the other hand we say "Sally's favorite color is red" rather than "Sally is a XX [red-liker?]" — we do not even have a word for the class of people whose favorite color is red. It is thus a plausible and testable hypothesis that students' gravitation toward set-based versus property-based representations would vary according to whether they are accustomed to speaking about the topic in terms of set membership or property assignment.

Moreover, constructing a property-based data representation is presumably more difficult when the word or phrase to be used as a property heading (such as "gender") is not readily available to the student. One must invent a property heading such as "boy/girl" (or, as one student proposed, "What are you?") Alternatively, one can import a known word from a different context. For example, Robbie, familiar with the word "gender" but not with "species," initially listed the dog/cat assignments under the heading "Gender." In a similar situation, another student asked if "dog and cat could be a kind of gender."

Directions for further research

In light of the preceding discussion, it would clearly be worthwhile to repeat these interviews, varying spatial cues as well as the kind, size and number of categories into which the objects are to be separated. In addition, since we seem to be catching students at a transitional point in their understanding of record/field structure, we would like to adopt a longitudinal format, so that we can try to track the learning process. Finally, we would also like to move beyond the group separation problem

to the challenges of encoding other kinds of data within a record-field database, as well as using software that supports other kinds of data structures (hierarchical, dimensional and relational) more directly.

Implications for tool design

Graphical improvements to the software may indeed help students become more aware of the correspondence of rows to icons, so that they are better alerted to the need to enter data in property-based form. More interestingly, we can envision a tool that spares users this need by accepting data in set-based form. If such a tool also accepted data in property based-form, and could translate readily and reversibly between the two (not unidirectionally, as current data tools do), then students would be able to explore and work in a transformational space of data structures. We suspect that this would put them in a much better position to reflect on the differences and relationships among different data structures.

Implications for pedagogy

In addition to being important in its own right, the ability to organize information into regular structures is fundamental to many mathematical activities. For example, we believe that data structure issues underlie many difficulties which students experience in learning and doing data analysis and statistics, even though research and curriculum tend to focus on more well-known concepts of traditional statistics. As the foregoing observations show, the ability to organize data appropriately should not be taken for granted. We look forward to an increase in knowledge of, and interest in, the challenges of learning about data structure.

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USING A DISCRETE COMPUTER GRAPHING ENVIRONMENT IN ALGEBRA PROBLEM SOLVING: NOTIONS OF INFINITY / CONTINUITY

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This research focused on a functional approach to the teaching of early algebra and made extensive use of computer-assisted graphical representations as tools for solving a range of problems. The aim was to uncover areas of ease/difficulty experienced by seventh graders in learning how to produce, interpret, and modify graphs. They worked in pairs at a computer during approximately twenty-five problem-solving sessions. The paper describes the ways in which one pair of pupils coped with the two types of infinity they encountered in a dynamic graphing environment that plotted intervals of discrete points rather than continuous curves. In addition to helping pupils become aware of the use of graphical representations as problem-solving tools, the environment provided a rich context for learning about density of points, infinity, continuity, and other issues that tend to be ignored until the calculus.

Introduction

There are several ways to introduce algebra to beginning students, for example, as a notation for expressing generalization of numerical relations and numerical/geometric patterns (Bell, in press; Mason et al., 1985), as a method for finding the value of an unknown number that involves equation-solving and expression-simplification (Herscovics & Kieran, 1980), and as a means to represent and operate upon functions (Fey, 1989; Heid, 1988; Kieran, Boileau, & Garançon, 1989; Schwartz & Yerushalmy, 1992). This last category includes both process-oriented approaches that involve interpreting functional representations from a computational perspective and object-oriented approaches that involve interpreting algebraic representations and their corresponding graphical representations from a more structural perspective (Sfard, 1991). Schwartz and Yerushalmy (1992) have claimed not only that function is the "primitive algebraic object" (p. 264), but also that it should be the focus of algebraic instruction right from the outset of a first algebra course. They also emphasize that graphs of functions ought not to be delayed until after students have spent considerable time working with algebraic representations. The aim of this research paper is to describe a study that is situated in the third category above, that is, one which comprises a functional approach to the early teaching of algebra and which incorporates extensive use of computer-assisted graphical representations as tools for solving both non-quantitative and quantitative problems. The research questions of the study reported herein focused on uncovering some of the areas of ease and of difficulty experienced by seventh graders in learning how to produce, interpret, and manipulate Cartesian graphs.

This study is part of a long-term research project on the use of a particular computer environment, CARAPACE, as a tool for introducing certain conceptual aspects of algebra, such as generalization and uses of variables, and in the bridging power of the computer to link numerical approaches to problem solving with more traditional algebraic ones. In our past presentations at PME and PME-NA meetings, we have, in turn, reported on: a) the guess-and-test numerical solving methods that are used when data are displayed and operated upon in tabular computer representations (Kieran, Garançon, Boileau, & Pelletier, 1988); b) the processes used by beginning algebra students to represent and solve problem situations by means of multi-line functional procedures (Kieran, Boileau, & Garançon, 1989); and c) a synthesis of three years of research with the CARAPACE environment (Garançon, Kieran, & Boileau, 1990). In 1990, we incorporated a graphing component into CARAPACE, and piloted it in 1990-91 (Kieran, Boileau, & Garançon, 1992). In 1991-92, we carried out the study reported herein, one which not only included features of the teaching approaches documented above, but also focused especially on the use of graphing representations.

Students have traditionally encountered difficulty in interpreting graphs. Kerslake (1981), in presenting the graphing data from the large-scale longitudinal CSMS study of 13- to 15-year-olds, pointed out, for example, that pupils having plotted points and joined the line on which they lay were unable in general to recognize that other points also lay on the line and that there were indeed infinitely many points both on a line and between any two points on the line. Similar findings have been reported by Herscovics (1979) and Leinhardt, Zaslavsky, and Stein

(1990). Brown (1981) found that the results were the same for the CSMS questions related to the infinite nature of the set of numbers expressible as decimals (e.g., "How many different numbers could you write down which lie between 0.41 and 0.42?"). In a study on intuitions of infinity involving 500 11- to 15-year-olds, Fischbein, Tirosh, and Hess (1979) found that, for a question on whether the successive division of a segment into smaller and smaller parts would eventually come to an end, the majority at each age level believed that the process could not go on forever. Only 5.5% believed that the process is infinite because there is an infinite number of points. On another question of the Fischbein et al. study involving successive division of a segment in two and whether any one of the points of division would coincide precisely with a given point of the segment, the percentage of wrong answers was equally high (e.g., 91% for the 12-year-olds).

The question of student understanding of infinity / continuity is, thus, one that we decided to include in this study. It seemed reasonable to us that the dynamic features of computers might have a positive effect on enhancing student understanding of this conceptual domain. However, very few studies have addressed this issue (Romberg, Fennema, & Carpenter, in press). One notable exception is the work of Goldenberg and his colleagues (Goldenberg, 1988; Goldenberg, Lewis, & O'Keefe, 1992). Goldenberg et al. have reported the positive effects of using a graphing environment that "instead of presenting the graph as a gestalt--either a fully formed static picture or the canonical smooth left-to-right sweep of graphing software--presented one point on the curve at a time" and which was based on the observation "that students *can*, in fact, lose track of the points when they face a continuous curve, and the consequence includes failure at tasks that require consciousness of the points" (p. 238). Goldenberg (1988) has also remarked that:

> In a pedagogy that makes regular use of translation between graphic and symbolic representations of functions, we cannot avoid dealing with issues such as these [the finite and the infinite, ...], and yet notions of infinity and the continuous nature of the real line are totally foreign to beginning algebra students. Research needs to be done to find appropriate ways of dealing with these issues. (p. 164)

It is hoped that our study is a step in this direction.

Overview of the Graphical Environment of CARAPACE

With CARAPACE, it is possible to use several different representations in one's problem solving. A procedural, algebraic-algorithmic representation permits the translation of a problem into one or several functions (see previous reports for details of the non-graphical representations). This procedural representation is executable once the pupil enters values for the input variable(s). The results of the execution are conserved in a table of input-output values and can be displayed in graphical form in the Cartesian plane. Three different types of graphing tools are available in CARAPACE: construction, manipulation, and analysis.

Constructing graphs. During students' initial contacts with the graphing environment, it is important that they learn what a Cartesian graph is and what its links are with the procedural representation of the function that they have already generated. Thus, they must do most of the work of graphing with very few hidden aspects carried out by the computer. CARAPACE also permits more advanced levels of functioning; but the initial level requires plotting points one by one: the pupil enters an abscissa, CARAPACE shows the step-by-step evaluation of the function for this abscissa, and then returns to the graphing screen to remind the student of the values of the abscissa and its result. The student then, with the aid of the mouse, locates the abscissa and ordinate on the horizontal and vertical axes and, when these have been correctly located, the point is plotted. Later levels do not show the evaluation of the function, nor require the location of the abscissa and ordinate on the axes. A more advanced level, for example, permits the graphing of a series of points by specifying an interval and step-size: Naming the interval [1, 10] with a step-size of 0.7 (see Figure 1) would result in the plotting of points whose abscissas are 1, 1.7, 2.4, ... 9.4. At the most advanced level, the pupil can specify an interval by means of a mouse-drag along part of the horizontal axis, which creates the illusion of a continuous curve since the function is calculated for the abscissas corresponding to each pixel (see Figure 2).

In sum, these different levels cover the range from plotting point by point to the plotting of a large number of points geometrically without referring to specific numbers. The crucial aspect here is that the graph is presented as it is: a finite graph. It is the result of evaluating a function by means of a finite number of abscissas, which have all been supplied by the pupil and which are conserved by CARAPACE in a table of values.

Manipulating graphs. Choosing the size of a graphing region is known to be a problem for students (Goldenberg, 1988). CARAPACE is equipped with a default region; but since our aim is to get the students directly involved in the construction and manipulation of their graphs, they are soon taught how to use the

different tools that allow them to change the graphing region. First, there is a double set of axes: the traditional axes containing the origin at their point of intersection, which are displayed within the graphing window, but which may not be visible at all if the portion of the plane that is being graphed is distant from these axes; an auxiliary set of axes that sit outside the graphing window do not intersect and correspond to the abscissas and ordinates of the points in the represented region (see again Figure 1). These auxiliary axes are one of the means available for determining the region to be represented. Each of these axes holds two movable labels. If, for example, one places the label 10 at 1/4 of the horizontal axis and the label 50 in the middle of the axis, then the implicit unit is such that 1/4 of the axis corresponds to 40 units and the entire horizontal axis covers the interval [-30, 130] (see Figure 3).

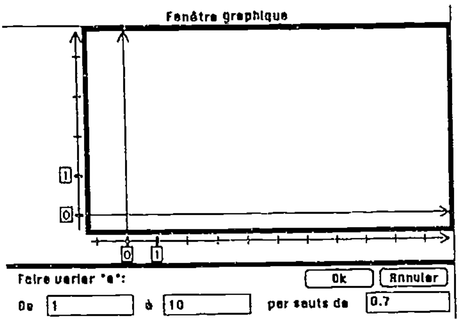

Fig. 1 The graphing window of CARAPACE, showing the default region. Note also the double system of axes: one inside the window and the other outside (bearing the labels and scale graduations). Below the graphing window is the dialogue window for specifying the plotting of several points by means of an interval and step-size

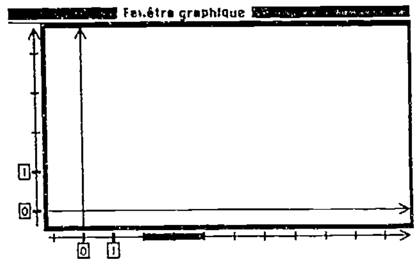

Fig 2 The selection of the interval [2, 4] by means of a mouse-drag

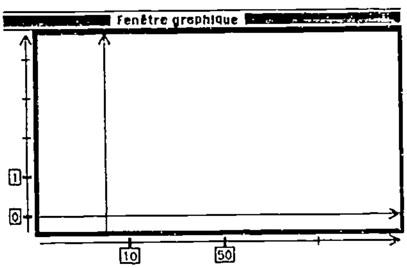

Fig 3 A modified horizontal axis

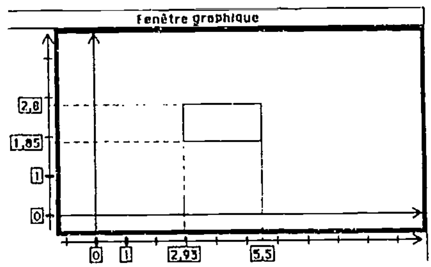

Fig 4 The selection of a ZOOM-IN region Note the accompanying axis-labels

It is possible at any time to modify the size of the represented region by changing the position and value of the labels. The points that have already been plotted move accordingly. This is a dynamic process: While the label is being moved by a mouse-drag, the points move simultaneously--keeping their correct position vis-à-vis the changing set of axes. These dynamic tools for manipulating the represented region not only help maintain the contact between the geometrical object--the point--and its numerical components but also provide a concrete way of experiencing the link between modifying the units of an axis and modifying the appearance of the graph (slope, curvature, continuity, etc.).

Other manipulation tools of CARAPACE include the ZOOM-IN. With the aid of the mouse, one specifies a rectangle within the graphed region; the labels on the axes are tied to this rectangle and represent the abscissas

of the vertical sides and the ordinates of the horizontal sides (see Figure 4). When a zooming-in is requested, this rectangular region enlarges dynamically to occupy the graphing screen. The points of the graph, as well as the labels, are involved in the animation and move at the same time that the rectangular region enlarges.

Analyzing graphs. In CARAPACE, there are two ways to find the coordinates of a point of the graph. First, one can add graduations to the axes containing the labels (see, for example, Figure 1). Second, one can move a graphics cursor, which is a movable point in the plane, tied to the mouse, and which carries with it two auxiliary labels that move along each axis, specifying the numerical value of the coordinates (see Figures 5 and 6). It should be noted here that there is a problem of precision in that an interval contains an infinity of real numbers, but is represented on the screen by a finite number of pixels, each one corresponding to an infinite number of coordinates. In CARAPACE we have chosen to label with the coordinates having the least number of decimals. Thus, for example, if all of the possible abscissas of a pixel are within the interval [0.173, 0.256], we would label with the number 0.2

Why the Graphing of Discrete Points Rather Than a Continuous Curve?

Correspondence with word problems. In an environment such as CARAPACE where we use word problems that the pupils represent in a functional way by means of a program, the domain of definition of the functions can be quite varied. For example, a program that associates the profit of a travel agent with the number of clients is a function whose domain (the number of possible clients) is the set of non-negative integers. To plot a graph of the function for the points whose abscissas are not integers would be nonsense in the context of this problem.

Coherence between representations. In addition to permitting functional algorithmic representations of problems (by programs), CARAPACE allows for the use of other modes of representation, for example, the table of input-output values containing the abscissas for which the function was evaluated and the corresponding values of the function. Since we wanted to maintain the equivalence between the tabular and graphical representations, it was necessary that CARAPACE be constructed in such a way that both these modes represent the same thing at all times. Since a table of values is a discrete representation of the function, it followed that the graphical representation also be discrete.

Communication of current state of graph. When one represents a function by an algebraic expression or by a program, the domain of definition of the function has the potential of being infinite or even continuous. In designing the pedagogical environment of CARAPACE, we decided that it would be important for the pupils to take on as much responsibility as possible in the construction of different representations. For the case of graphical representations, the pupil has to decide not only which region of the Cartesian plane to represent on the computer screen but also which scales to use and for which abscissas the function will be evaluated and represented. As we have already pointed out, several different graphing levels are available, from a point-by-point selection to a much larger--but always finite--number of points. One can, however, give the illusion of continuity by plotting all the representable points (all the pixels) between two abscissas; but, even in this case, only a finite number of points is plotted and this can become evident with a change of scale. Thus, in CARAPACE, what one sees on the graphing screen is not an "ideal" representation, but the current state of the representation as it is being constructed by the pupil.

Disclosure of pupils' strategies. When pupils work with CARAPACE, they represent a situation in a functional way by means of a program. But in solving the problem, they generally use some of the other representations that are available to them. For this reason, when pupils construct, for example, a graphical representation, their aim is not to obtain the most complete representation possible for the function but rather something that is adequate to arrive at a problem solution. In order to generate this representation, they must make several decisions regarding the size of the region, appropriate scales, and the points to plot. The observation of these decisions allows us to obtain an idea of the strategies they are using in their problem solving and of their conceptions of what constitutes an adequate representation.

The Pupils

The two pupils whose work we describe in this report are fairly typical of the other pupils who participated in the study. They are seventh grade students, 12 to 13 years of age, who are considered to be of average mathematical ability by their teachers. They worked together during each session, sharing one computer. During

each of the 25 videotaped sessions with these two pupils, generally two one-hour-long sessions per week, the researcher-intervenor presented them with two or three problems to be solved in the CARAPACE environment. In addition to the transcribed protocols of the videotapes, the data-source included computer dribble-files of each session. Details of the types of problems used and the non-graphical approaches for introducing pupils to CARAPACE can be found in our previous reports mentioned above.

The mathematical background of these pupils did not include any algebra. Their prior experience with Cartesian graphs was limited to some plotting of points. In the 16 CARAPACE sessions preceding the ones we describe in the next section of this report, they had worked with representing word problems by functional-algorithmic programs, solved them numerically by using the CARAPACE tables of values, learned how to use the graphics part of the environment to represent functions graphically, and solved problems with the aid of graphical representations. Note that even though they had had contact with the graphing module during the six preceding sessions, they had not been taught particular strategies for using these tools. They were only made aware of the existence and functioning of these tools. The different strategies that were observed, whether new ones or already used in previous sessions, were their own. The observations that we now describe are those related to their handling of the emerging concepts of infinity / continuity as they manifested themselves during Sessions 17-22 of the study.

Behavior of the Students in the Face of Infinity

When a function is defined on the real number line, its domain is infinite in two ways: first, because it extends indefinitely; second, because an interval contains an infinite number of points. In presenting only finite graphs--thus, incomplete from two points of view--CARAPACE induces a confrontation with these two types of infinity. Note that the pupils of our study had already learned decimal numbers, which they had studied in class, and knew that between two given numbers there exists an infinity of decimal numbers. Moreover, they had already solved certain problems numerically in the CARAPACE environment, using the table of input-output values; thus, they had developed strategies involving successive approximation and had not shown any significant difficulties in using decimals.

In the graphing environment, in order to confront pupils with the two types of infinity, we had them work with "hidden problems"--that is to say, programs representing functions that have previously been entered into a CARAPACE file, but to which the pupils did not have access. For example, suppose the function that had already been entered by us into the computer was equivalent to $y = (x - 13.1)(x - 24.4) + 63$, we asked them to "Find all of the inputs that would give 63 as output" (*input* corresponds to x and *output* to y). They had to not only come up with approximations to the solutions, but also evaluate the global behavior of the function in order to have an idea of the number of solutions.

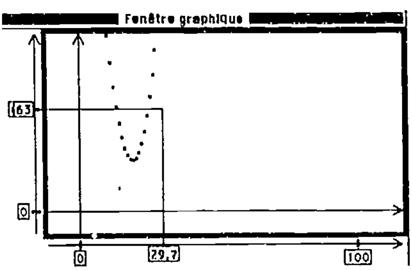

Fig 5 · A discrete plotting and the graphics cursor

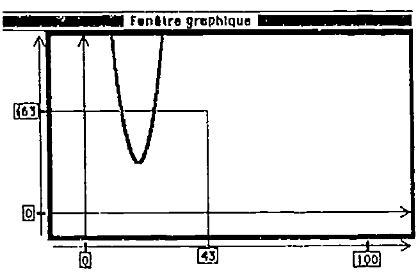

Fig 6 · A "continuous" plotting and the graphics cursor

When confronted with this problem, these pupils began by plotting the interval of points whose abscissas were between 0 and 100, with a step size of 1 (see Figure 5). Then, upon seeing that among the points plotted there was not one whose ordinate equalled 63, they declared with confidence that there was no solution. There were, however, points whose ordinate was greater than 63 and others whose ordinate was less that 63 (e.g., the points with abscissas 13 and 14). In a purely numeric mode, such as a table of values, one might think--based on what

we had observed in previous sessions--that they would probably have continued by trying abscissas between 13 and 14. Here, in the graphing mode, one in which the numerical-graphical link had been made explicit over the course of several sessions, they seemed to lose partial awareness of this link. We say *partial* because even if they seemed to think that the graph did not contain any points other than the ones they had plotted and did not think of the possibility of carrying out successive approximations, it was still clear that for them a solution is the abscissa of a point whose ordinate is 63. Nor did they seem to think that there might be solutions in a domain other than the one they had used to plot their points.

In the session that followed, the researcher-intervenor asked them if there might not be decimal solutions for the above problem. Their first reaction was a categorical refusal to use decimals--as if these numbers had nothing to do with graphing. Then, with the further suggestion from the intervenor that they try, they decided to plot the interval from 0 to 100 with a step size of 0.1. This time, the graph gave the illusion of continuity (see Figure 6). In the face of the apparently-continuous graph, they could "see" points whose ordinate was 63 and, thus, believed that there were some solutions. Since these solutions had only one decimal, they were found without any difficulty. When the intervenor asked them if they had found all the solutions, they replied, "yes," without hesitation. Their justification was that since the curve was increasing (they were speaking of the visible part of the graph for increasing values of x), it would keep increasing. On the other hand, when the intervenor reformulated the question to ask, "What would it be like if there were other solutions?" one of the pupils replied that, "it would go like this," and with his hand made a wave of a continuous graph that increased for a bit and then began to decrease for increasing values of x.

During the following sessions, we posed similar problems, but ones for which the solutions had several decimals. We observed the same phenomenon: After having plotted the interval from 0 to 100 with a step of 0.1 (we note that they did alter the step size), and in not seeing a point with the sought-for ordinate, they stated that there was no solution. When the intervenor suggested that there might be a solution with more than one decimal, they expressed some discouragement and remarked that it would require plotting too many points. But they did try, using a step size of 0.01; however, they ran into an unexpected problem--the computer they were using did not have enough memory to register the 10000 points that they wanted calculated. It was this constraint that pushed them to use a process of successive approximation in the graphing environment. Thus, they began to zero-in on their solutions by, first, choosing an interval with a step-size of 1 to localize the solution area; then, chose a smaller interval with a step-size of 0.1. They iterated this process of choosing smaller intervals and step-sizes until they obtained the precision they needed. They were later shown how to use the ZOOM-IN feature and the method of plotting all the pixels without having to specify a step-size. They continued with this method of successive approximation, developing more and more expertise with it.

As far as extrapolating from the visualizable section of the graph is concerned, the observation noted above continued throughout the remaining sessions. The pupils did not experience difficulty with imagining that the graph extended over all of the number line. However, we did find that the pupils tended to imagine the extension according to the form displayed in front of them. They seemed to believe that it was useless to extend the graph in order to check their hypotheses. However, after a few more sessions during which they found that their predictions were often incorrect, they began to take more notice of the curvature of the graph. For example, if the graph was decreasing but its slope appeared to be increasing, they would no longer risk predicting that it would always decrease; they became aware of the possibility that the graph could reverse its tendency and begin to increase (see Figure 7).

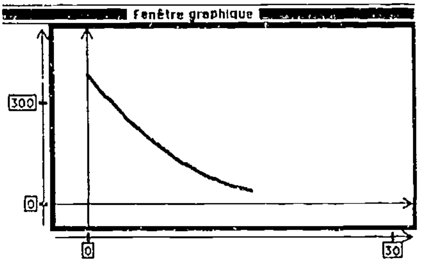

Fig 7 A decreasing graph with an increasing slope

Pupils' Use of the Graphing Tools

We have already discussed the pupils' use of the construction tools; we now look at their handling of the manipulation and analysis tools. Pupils had three different objectives when using the axis-labels: to determine a graphing region, to change the appearance of a graph, and to specify a particular point that was worthy of their attention. They learned very quickly how to use these labels to achieve their objectives. For example, with the "hidden" problem presented above, they began by setting up a region that was [0, 100] by [0, 100]. They did this by placing the labels of the horizontal axis at both extremities and placing within them the values of 0 and 100. But, for the vertical axis, they left 0 in the lower label and placed a label containing 63 at about 2/3 of the height of the axis (see again Figure 5). Since their aim was to obtain 63 as the value of the function, they chose to have 63 clearly displayed as one of their axis-labels. Even before plotting a single point, they seemed to have the clear idea of comparing the height of the points of the graph with 63; so, they opted for an approach that would make the position of 63 quite evident on the vertical axis. This strategy was used quite often.

Other strategies that we observed included the following:
a) To isolate a particular point, the pupils would often use as values for the labels numbers which were very close to and framed the desired coordinates of the point.
b) To obtain a global view of the graph and/or to give it an appearance of continuity, they would bring the labels closer together and/or give them values which were rather far apart.
c) To bring the apparent slope of the graph close to 0, they would stretch the horizontal labels and/or give them values which were very close to each other--this would help them to differentiate more easily the abscissas of the points of the graph and improve their reading of coordinates by means of the graphics cursor. It is to be noted that the graphics cursor was their preferred tool for reading coordinates; they rarely used the axis-graduations.

Conclusions

Discrete plotting of points, along with the availability of different options for actually placing the points on the screen, not only permits pupils to maintain for a longer period of time a link between graphical and numerical aspects of functions but also provides for a very gradual transition from algebraic expression representations to Cartesian graph representations. In spite of this, we observed that, in the beginning, students tended to easily lose partial awareness of the link between the two representations and that the strategies of successive approximation which were used in the numerical context were not immediately called upon in a graphing context. We also observed an initial tendency to rely on the appearance of graphs and to extrapolate from what was visible.

It is impossible here to make comparisons with software that plots "continuous" graphs that are defined for "all the real numbers," since, if the problems that we are raising do not occur in continuous graphing environments, we do not know if it is because they do not exist or because the conditions necessary to provoke them are not present. In any case, we believe that there is a fundamental difference between reading the abscissa of a point from a ready-made graph and generating an abscissa in order to construct a point of a graph.

Nevertheless, we saw a rapid improvement during the six sessions with the two pupils who were described in this report: the development of successive approximation strategies in the graphing context, and the replacement of an approach favoring the search for continuity and a multitude of points by an approach favoring the search for local density of points--sufficient to solve a problem.

We also observed that the dynamic tools for modifying scales, geometric (position of the labels) as well as numeric (value of the labels), could be efficiently manipulated by young pupils with either geometric objectives (e.g., isolating a point, global view of graphs) or numeric ones (e.g., increasing the precision of the graphics cursor). This type of tool provided pupils with an almost physical "contact" with graphs and permitted them not only to experiment in a dynamic way but also to actually feel the relation between changing scale and transforming a graph. We are reminded of a very appropriate comment by Goldenberg (1988):

> Our students, even at early stages, cannot escape having to deal with notions of continuous functions and discrete points, infinity and infinitesimals, the invisibility of points, and other issues we tend to ignore until the calculus. We must consider appropriate ways of introducing these ideas much earlier than we typically do. The behaviors of some young students who become comfortable changing scale on graphs suggests an "intuitive calculus" long before the algebraic manipulations for formal calculus are present. (pp. 171-172)

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GRAPHIC CALCULATORS IN THE CLASSROOM: STUDENTS' VIEWPOINTS[1]

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This paper presents the results of a study about the views and attitudes of students of a low achieving 11th grade class who were involved in an innovative experience with graphic calculators for all academic year. Contrasting the results obtained from a questionnaire and from interviews, it concludes that students tended to point some improvements in the mathematics class, but attributed their origin more to their teacher' style and personality than to the use of this technology.

Graphic calculators are a quite powerful new technology for mathematics teaching (Demana & Waits, 1990). They have obvious curricular implications, especially at high school and college level. They point towards significant content changes, emphasizing graphical representations, stressing issues such as units and scale; and they may even favor a shift in learning styles, with more exploration and student activity. This paper presents a study that investigated students' views and attitudes towards graphic calculators as well as mathematics classes and its effects in their personal relation with this subject.

Theoretical background

The general conceptions, views and attitudes of the students regarding mathematics and mathematics classes are increasingly seen as crucial factors affecting their performance (Schoenfeld, 1989). It is of great interest to know how much can these be addressed by curriculum innovations. In a recent experience of a new national curriculum, 7th grade students' views and attitudes were found to improve significantly, in close relation with the introduced methodological changes. However, in the same experience, the views' and attitudes of college-bound 10th graders showed no positive change, but rather an increase in anxiety and distrust regarding the system (Ponte et al., 1992).

Students' response to innovations in mathematics teaching is not always what the innovators seek. With powerful technologies, the students' approach to mathematics tasks may differ significantly, in an impoverished or even counterproductive way, from the original intentions (Hillel, 1992). And also, students' agendas and personal expectations regarding mathematics classes may resist to what they perceive as departures from the usual, and, on their view productive learning activities (Ponte e Carreira, 1992).

[1] This paper reports some results from the project "Dynamics of Curriculum Innovations", supported by JNICT (Portugal) under the contract PCTS/C/FCT/12-90. Also in this project are H. Guimarães, L. Leal and A. Silva

Ruthven (1990) studyed the effects on the performance of advanced upper secondary school students of extended use of graphing calculators. He reports that this technology had a strong influence both on the mathematical attainment and on students' approaches to specific tasks, especially what he called symbolization items. However, one should also consider the effects of such technologies in students' general views regarding mathematics learning. These views are closely related to cultural factors, as suggested by Vygotsky's (1978) approach to the study of social cognition. Their role should be considered to understand, and possibly influence, students' behavior in class (Schoenfeld, 1992).

The experience with graphic calculators

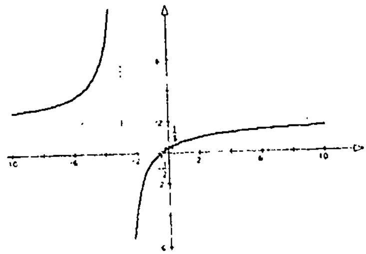
This experience took place in a suburban school not very far from Lisbon. The mathematics teacher has some tradition in using calculators. She participated in 1989/90 in a year long inservice program focusing in technology. In the following year she used graphic calculators with a 11th grade class. In 1991/92, she also used this instrument with a new 11th grade class, adding some new activities to those previously developed. She stated the following intentions for this experience:

> *Regarding the students:* develop the intuition, establish connections between analytical and intuitive reasoning, show the students the importance of discovery, stimulate self-confidence, promote investigations and formulation of conjectures, develop the ability to formulate problems, develop communication and argumentation skills.
>
> *Regarding the class:* change the climate, centering the class in the student and not in the teacher, adjust the technology used in class to the technology used out of school.
>
> *Regarding the curriculum:* establish connections among different topics, usually viewed as compartmented, deepening the study of mathematical functions.

The class begun with 18 students but ended the school year with just 15. They were regarded as the weakest 11th grade class of this school, with a quite low achievement in mathematics (and in other subjects) and frequent behavior problems. In the beginning of the year the students were loaned a graphic calculator by the mathematics teacher, that they took home to use in their homework assignments and were supposed to always bring to class. They were also given an hand-out with the instructions to produce graphs of functions in the calculators. These were used in class in two ways: (a) to work on specifically designed activities, proposed in worksheets, generally to be carried out in groups of four; (b) occasionally, during the remaining mathematics classes, in an individual basis The calculators were also used freely in tests. Some of the activities were intended to promote investigation, observation and discovery, exploring the graphical representations (Questions 1 and 2). Others were standard exercises that could be made both with the calculator or using the classical analytical approach (Question 3).

1. The figure represents the graph of the function
$y = (ax+b)/(x+c)$ (a, b and c are real numbers)
Find the values of a, b e c. Explain your thinking.

2. a) Draw the graphs of the functions:
$f(x) = x^2$, $g(x) = x^2 + 2$ e $h(x) = x^2 - 1$
b) Compare the functions in what concerns their domain, range, zeros and monotony.
c) How should be the graph of $y = x^2 + k$ (k in IR)?



3. Draw the graph of $g(x) = |x| + 2$.
Study its domain, range, zeros, sign and monotony.

The classroom climate was informal and relaxed. But most of the students were frequently not much attentive in class and had difficulty in getting really involved in working in the proposed activities. By the middle of the year several of the students were no longer bringing their calculator to the class, as initially agreed.

Method

This study is part of a larger investigation dealing with two innovative activities, initiated by the teachers of this secondary school. Information has been gathered by a variety of ways, including classroom observations, interviews with teachers and students, and documental analysis. The data reported in this paper came from two main sources. One is a questionnaire made by the teacher to ascertain the opinion of the students about the work with the calculator that was responded in class by the end of the school year. It was a free-response instrument containing seven questions dealing with the use of the calculator and its implications in mathematics learning (see Figure 1). Another data source is the interviews made with three students. We wanted them to provide a variety of cases: one student interested in mathematics but with a weak reaction to the calculator, another not much involved with mathematics but adept of the calculator, and a third one interested in mathematics and frequent user of the calculator. Among these there should be boys and girls. These interviews lasted for about 45 minutes each, were audio-taped and transcribed. Analysis of the questionnaire responses and the interviews were made by the technique of "content analysis", following the main categories: (a) views of calculator, (b) view of mathematics classes, (c) personal relation with mathematics. We choose to contrast the results provided by these two instruments because we felt they would be most informative about the issues proposed for this study.

Results from the questionnaire

The responses that 12 students gave to the questionnaire are summarized in Figure 2.

Figure 1—The questionnaire

1. In the beginning of the school year you were loaned a graphic calculator. Did you use it frequently or not? If not, explain why; if you used it a lot explain in what situations and disciplines.

2. Did the use of the calculator helped you to better understand any particular concept? If yes, which one?

3. Do you prefer using the calculator just by yourself or with other colleagues?

4. What seem to you the advantages and disadvantages of using the calculator in mathematics classes?

5. Do you think that the use of the calculator contributed to changes in the role of the teacher in the classroom, as well as in her relationship with the students? In what way?

6. Did the use of the calculator contributed in any way to alter your view of mathematics?

7. Give your opinion about the positive and negative aspects of this year's mathematics classes.

Figure 2—Results of the questionnaire

Questions	Responses		
1.	Frequently 10	Average 1	A little 1
2.	Yes 10	May be 1	No 1
3.	In group 4		Individually 8
4.	Just advantages 6	More advantages than disadvantages 1	As much advantages as disadvantages 5
5.	Perceived some changes 8	No changes perceived 3	No answer 1
6.	Yes 8	Partially 1	No 3
7.	Just positive aspects 8	Mostly positive aspects 2	No answer 2

Views about the calculator. Most of the students reported to use frequently the calculator in mathematics classes, especially to make graphs of functions. Four of them indicated that the calculator favoured a more global understanding of functions and another four spoke of the possibility of confirming the results obtained by analytical methods. They reported having used the calculator both to solve exercises in class and to verify responses in tests. They also used the calculator in physics to perform calculations and to store formulae in memory (a facility they discovered themselves).
All the students identified advantages in the use of calculators in mathematics classes. Ten of them stressed the help in understanding mathematics concepts (such as monotony, zeros, asymptotes), six referred they could make graphs and computations faster, and two indicated the possibility of obtaining complex graphs. A half of the students referred as a disadvantage a single aspect: the eventual dependence from the calculator with a consequential "difficulty in thinking by our own mind" or "making graphs by hand".

Views about mathematics classes. For students, the calculator had no significant influence in the mathematics class, although eight of them pointed some differences. Among these, three referred that the student-teacher relation was strengthened because the teacher had to support them using the machine; two thought that classes became more interesting and less monotonous than usual; and another girl pointed to different teaching style, with a more active participation from the students. The undertaken of Group work, that was made only with calculator worksheets, was viewed as positive just by four students who valued the possibility of interchanging ideas with their colleagues. It should be noted that, from the ten students that indicated positive aspects regarding the mathematics class, only two referred the use of the graphic calculator. The remaining ones referred to more general aspects such as the good relationship with the teacher, her availability to respond to their questions, the light and fun climate of the class, and their increased motivation. As negative aspects, one student just referred the behavior of their colleagues and the fact that not all the syllabus had been covered.

Relationship with mathematics. Three students indicated that the use of the calculator did not affect in any way their view of this subject, whereas nine indicated some changes. From these, five indicated that the calculator made the topic "less complex", giving them some advantage over the students that do not use the machines, and three referred that they were enjoying mathematics better. These changes mostly concern the facility with mathematics and the personal relation with this subject. They do not point towards a different view of mathematics.

Results from the interviews

Susana. She was regarded by the teacher as a good student but not much interested by the calculator.

Views about the calculator. In fact, Susana was almost untouched by the activities with the graphic calculator. She indicated to have some curiosity in the beginning but never come to be a frequent user. On the one hand she had technical difficulties that she never overcame. On the other hand, her private mathematics tutor indicated that she should do the mathematics questions by "paper and pencil", "analytically"—and she felt more confident with this approach. She just used the calculator in the tests to make sure that her graphs were correct. Making graphs and computations faster is the single advantage that Susana identified in using the calculator in the study of mathematics. In contrast, she stresses two disadvantages: one is the imprecision and lack of rigor of the machine that "sometimes fails to provide the coordinates of the points that one is looking for"; another has to do with the habits of "dependence" that it promotes.

View about mathematics classes. Her view of the mathematics classes was not affected by the graphic calculator. This student is particularly sensitive to the attitude that she feels from her teachers, and this was the most important aspect that she mentioned in this year—she felt that the teacher was quite interested about their students and made efforts to help them learning mathematics. The single influence that she pointed concerning the use of the calculator come across from the group work—however, she mostly valued the interaction with her colleagues and not the use of the machine.

Relation with mathematics. Susana recovered this year the enjoyment with mathematics that she had in junior high school. In 10th grade, she did not well in this discipline, attributing the responsibility to her teacher who "did not explain well and was not much concerned with the students". But this year Susana understood most of the subject matter, both in class and in her out-of-school tutoring. She valued this aspect as very important since for her mathematics is a discipline "that must be understood", which "cannot just be studied for the tests", requiring a continuous effort. Another idea that she hold concerning mathematics is related to rigor, as she speaks about "the need of the analytical determination of required points". This idea is certainly associated to the fact that Susana found the graphic calculator insufficient in the study of functions.

Leopoldo. This student was indicated by the teacher as generally not interested in mathematics but with a good relationship with the calculator.

Views about the calculator. In the beginning of the year he learned to make the graphs in the machine, feeing confident in using it. He thought that the calculator was of good help in the study of functions. Having used it both in class and in the tests, he pointed as its main advantage the possibility of quickly making the graphs of functions and confirming the results obtained by analytical means. As a disadvantage, he pointed to an eventual dependence from the machine that he admitted to have developed.

View about mathematics classes. Leopoldo thought that the use of the calculator "has made classes more interesting", since it was a different thing that break the usual

"monotony". But the great differences that he noted in the classes referred to the "way of teaching" the subject by the teacher, to the enjoyable climate, and to the group work. It is interesting to note that in class Leopoldo seemed quite involved with the machine, but during the interview never referred spontaneously to this instrument.

Relation with mathematics. Mathematics never was Leopoldo favorite topic and with time his interest for it has been decreasing. He indicated that this is closely related to his difficulty in understanding the subject: "If one can understand, it is a nice discipline, it is interesting. But when one understands nothing, it is quite boring..."

Ana. This student was indicated by her teacher as very interested in the calculator.

Views about the calculator. During the interview Ana never mentioned the calculator, but when questioned about it she was quite positive: "It is very cute!" During the year she used frequently the machine in class and in tests, both to make graphs and computations. She thought that the graphic calculator facilitates the understanding of graphs, as well as the results obtained by analytical means. As a disadvantage, she also indicated the eventual "dependence" from the machine. She said that, without the machine she is just able to make "not too complex graphs", what she regarded with some apprehension.

View about mathematics classes. She confessed a great dislike from classes in general, saying that studying "was quite boring". Nevertheless, she said that this year she enjoyed mathematics classes, which she attributed to the teacher. On one hand she found her "marvelous", given the relation established with the class. On another hand, she liked the teaching style, explaining, making exercises, responding to questions. She also enjoyed group activities. She did not point to any influence of the calculator in class.

Relation with mathematics. Mathematics is not a topic of high interest to Ana. She never got much involved with this discipline that "requires much out-of-class work"— which "is not among her priorities". She identified a "theoretical" and a "practical" side in mathematics. For her, success in mathematics implied the regular practice of exercises: "In mathematics it is not enough to memorize for a couple of hours, you need to practice." Her lack of study shows up in her results, that are sometimes acceptable and other times not. For this student, the calculator did not arose a different relation with mathematics.

Conclusion

It is quite salient the lack of interest of most of these students for mathematics. This is related to their general lack of interest for school and to the difficulties that they find in this subject. They tend to regard mathematics classes as quite monotonous. The rather homogeneous views concerning the disadvantages of the calculator may be explained by pressures coming from different sides—from other mathematics teachers, from their

private tutors, even from some parents. The calculator was seen as an useful add-on, that quickly draws graphs and enables confirmation of results obtained "analytically" by hand, but not as something that radically changed the nature of the work.

The calculator did not improve dramatically the global achievement of this class. It was not regarded by students as a major influence in their way of learning mathematics. But one should note the great importance that they ascribed to the relation with their teacher and the global appreciation of her efforts.

There is much in common between what the students said in the questionnaire and in the interviews about the advantages and disadvantages of the calculator and how they regard mathematics classes. However, if one looks carefully, there is a distinct flavor in the results coming from there two instruments. From the questionnaire, one gets the picture that the calculator was valued as a quite important addition to their mathematical work. From the interviews, one in stuck by their lack of reference to the calculator, and to the stress that they put into the relation with the teacher. This subtle difference may be due to the different ways the questions were posed to them: the questionnaire focused mostly on the calculator whereas the interviews sought to provide students' perspective about the whole mathematics class. (More data, concerning these and other students, will be provided at the meeting.)

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Constructing Different Concept Images of Sequences & Limits by Programming

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As a transition between an informal paradigm in which a limit is seen as a never-ending process and the formal ε–N paradigm we introduce a programming environment in which a sequence can be defined as a function. The computer paradigm allows the symbol for the term of a sequence to behave either as a process or a mental object (with the computer invisibly carrying out the internal process) allowing it to be viewed as a flexible procept (in the sense of Gray & Tall, 1991). The limit concept may be investigated by computing s(n) for large n to see if it stabilises to a fixed object. Experimental evidence shows that a sequence is conceived as a certain kind of procept, but the notion of limit remains more at the process level. Deep epistemological obstacles persist, but a platform is laid for a better discussion of formal topics such as cauchy limits and completeness.

The difficulties faced by students in coming to terms with the limit concept are well-documented (e.g. Cornu, 1991), from the coercive effects of colloquial language (where words like "tends to", "approaches" suggest a temporal quality in which the limit can never be attained) to difficulties coping with the formal definition and the quantifiers involved. Here we investigate the effects of introducing a computer environment allowing the student to construct some of the concepts through programming.

Following Dubinsky (1991), Sfard (1991), we formulate our observations using a theory whereby mathematical processes are subsequently conceived as objects of thought. We use the term *procept* (Gray & Tall, 1991) for the amalgam of process and concept where the same symbol is used both for the process and the output of the process. We hypothesise that different mental structures for the limit concept $\lim_{n\to\infty} s_n$ are produced by different environments. both for the term s_n as a computational process and a mental object, and also the limit itself, as process and object. These contain potential conflicts which require cognitive reconstruction to pass successfully from one paradigm to another.

We consider three separate paradigms:

(I) a (formula-bound) *dynamic* limit paradigm,

(II) a *functional/numeric computer* paradigm,

(III) the *formal ε–N* paradigm.

Paradigm (I) occurs in UK schools for students aged 16 to 18. In this curriculum the limit of a sequence (s_n) is studied only briefly in a dynamic sense as n increases, the main focus being on arithmetic and geometric progressions and convergence of the latter. Many students use the words "sequence" and "series" interchangeably in a colloquial manner. This approach emphasises the *potential* infinity of a process which cannot be completed in a finite time. The terms are seen as being given by *formulae* and the specific geometric (and arithmetic) progressions studied also have partial sums which can be

expressed as formulae However, a general partial sum $s_n = a_1+...+a_n$ will rarely be given by a closed formula, so is again more likely to be seen as a process of addition rather than a limit object. The accent on process creates a "generic limit" concept in many students (Monaghan, 1986) in which the varying process is encapsulated as an indefinite "variable" object such as $0\cdot\dot{9}$ (which is "just less than 1") or $\frac{1}{\infty}$ which "just exceeds 0" (Cornu, 1991). This contains the seeds of conflict with the formal paradigm (III).

Paradigm (II) is the current focus of the paper. The individual may specify a procedural function taking a natural number as an input and outputting a real number. The computer language chosen allows such functions to be specified in a wide variety of ways – as a formula, or as a procedure involving logical decisions, loops, iteration, recursion. The symbol $s(n)$ may be considered either as representing the programmed procedure, or the output of the function and therefore behaves as a procept. The numerical computation is performed internally by the computer: we call such a procept a *cybernetic* procept. The computer language used has no built-in limiting process and the limit cannot be programmed in a proceptual way. Instead it may be investigated by computing the values of $s(n)$ for large values of n, say $s(1000)$ or $s(10000)$, to see if the value of the term stabilises. This gives a numerical value of the limit, allowing the limit to be studied as a (numerical) object. However, this produces the notion of a *cauchy limit* in which the terms become indistinguishable to a given level of accuracy rather than computing the exact numerical value of the limit. Paradigm (II) therefore allows the notion of sequence as a cybernetic procept with the limit being both process and object, yet differing subtly from the full proceptual structure of the formal paradigm.

Paradigm (III) is the eventual target paradigm, which will be studied later in the degree course. The notion of sequence will be defined formally as an *arbitrary function* $a:\mathbb{N}\to\mathbb{R}$ from the natural numbers to the real numbers, with the notion of limit given in terms of the ε–N definition. There are cognitive difficulties both with the notion of sequence as a function and with the limit. For instance, the definition of a sequence as a function from $\mathbb{N}$ to $\mathbb{R}$ includes the requirement that the function be specified simultaneously for *all* values on the infinite set $\mathbb{N}$, involving *actual infinity* rather than potential infinity. The definition of "limit" is formulated in terms of an unencapsulated *process* (given ε, an N can be found such that ...) rather than being described explicitly as an object. It involves several layers of quantifiers which exceed the short-term memory processing capacity of many students. There is a severe problem of the *status* of the limit notion – can one *define* an object linguistically, or does it need to have an independent existence? For example, if a decimal such as $0\cdot\dot{9}$ (nought point nine recurring) is believed to exist as a number *less* than one, can it be *defined* to be something *equal* to one?

The plan of action is to use paradigm (II) as a transitional environment to provide students with experiences which will lay the cognitive foundations for the formal definitions and to study what cognitive changes occur and what obstacles prove resistant to change.

The experiment

The study took place at Warwick University in a 20 week (60 contact hours) course on programming and numerical methods using BBC structured BASIC. The students were first year trainee mathematics teachers with nominal minimum UK A-level grade C in mathematics and one other grade D. This places them in the top ten to fifteen percent of the total population but few have the required grades (A in mathematics, plus two other Bs) to study mathematics in the university mathematics department.

Each week one of the three contact hours was available for introduction of new topics and discussion of difficulties with the lecturer. Students were encouraged to work together from printed notes. Assessment was by four assignments, so there was no need for rote-learning for an examination.

The first term introduced fundamental programming constructs such as variables, FOR:NEXT loops, REPEAT:UNTIL loops, graphical commands, procedures, functions, and structured programming, including the development of a structured graph plotter and a project to write a computer game.

The second term concentrated on programming numerical methods and investigating their properties. Topics studied included solution of equations in the form $x=f(x)$ by iteration, $f(x)=0$ by bisection, decimal search, Newton-Raphson, calculating numerical gradients, areas, solutions of differential equations, the order of accuracy of various algorithms, sequences, series (and their possible limiting behaviour), calculating functions by procedural methods, including Taylor series.

To broaden the concept image of sequence beyond a formula, functions were defined such as:

```
1000 DEF FNiterate(n) : LOCAL x,k : x=1 : FOR k=1 TO n : x=COS(x) : NEXT k : =x
```

(The LOCAL command localises the variables so that their values are not affected elsewhere.)

`PRINT FNiterate(30), FNiterate(50), FNiterate(100), FNiterate(1000)` gives

```
0.739087043      0.739085135      0.739085135      0.739085135
```

so that the sequence stabilises to $x=0.739085135$ for which $x = \mathrm{COS}(x)$ within computer accuracy.

Other sequences defined in the course, to give more flexible concept imagery, included

```
1100 DEF FNo(n) : IF n MOD 2 = 0 THEN = 0 ELSE = 1/n

2000 DEF FNprime(n)
2010 LOCAL k,ok : ok=1
2020 IF n=1 THEN = 0 ELSE IF n=2 THEN = 1
2030 FOR k=2 TO SQR(n)
2040   IF n MOD k = 0 THEN ok=0 : k=n
2050 NEXT k
2060 =ok
```

`FNo(n)` returns 0 if n is odd and $1/n$ if n is even, `FNprime(n)` returns 1 if n is prime and 0 if composite.

A sequence of terms `FNa(n)` may be summed iteratively or recursively, as follows:

```
3000 DEF FNs(n) : LOCAL k,s : FOR k=1 TO n : s=s+FNa(k) : NEXT k : =s
4000 DEF FNS(n) : IF n=1 THEN = FNa(1) ELSE = FNS(n-1)+FNa(n)
```

The formal definition of sequence was given in the notes as a function from the natural numbers to the real numbers and the concept was further discussed in seminars. The students were invited to program various sequences and series and to investigate the behaviour for large values of n. When stabilisation occurred, it was emphasised that the greater the accuracy required, the larger the value of n necessary for the terms to be indistinguishable to this accuracy. This was used in a seminar to lead to the formal ε–N definition of a limit of a sequence.

The convergence of the sequence 0.9, 0.99, 0.999, ... was discussed, with nth term $s_n = 1 - \frac{1}{10^n}$ and its limit was demonstrated to be 1 using the definition. The meaning of an infinite decimal expansion such as π=3.14159... was re-defined to be the limit of the sequence (s_n) where s_1=3.1, s_2=3.14, ... and , in general, s_n is the decimal including the first n places of the decimal expansion. In particular, the sequence with $s_n = 0\cdot\underbrace{99\ldots9}_{n\text{ places}}$ has limit 1 and is written as $0\cdot999\ldots = 0\cdot\dot{9}$. Its value is therefore 1.

The effects of the experiment

Data was collected from the experiment in four ways: a pre-test with questions on limits of sequences and series, a post-test with essentially the same items (Li, 1992), interviews with selected students, and written work submitted for assessment.

In response to the question:

> If you can, explain in your own words what is a sequence,

the pre-test revealed the overwhelming sense that a sequence needed a formula or pattern (Table 1).

What is a sequence?	pre-test (N=25)	post-test (N=23)
terms given by a formula or pattern	17	9
mentioning "series"	4	0
function from natural numbers to reals	0	3
function from reals to natural numbers	0	4
mention of *no formula* for terms	1	3
other	2	4
no response	1	0

Table 1: What is a sequence?

Responses revealed the colloquial interchangeability of "sequence" and "series", for instance,

> *A sequence is a series of numbers connected to the numbers before and after by a formula.*

The change from pre-test to post-test showed a reduction in students mentioning a formula and an increase in those mentioning the function definition or denying the need for a formula. However, the definition was poorly remembered (it did not need to be rote-learnt for an exam) and four students reversed the order (from reals to natural numbers) which was subsequently explained verbally by one of them reading the term from left to right as "s_n is a real number which is related to the number n."

This confirms the often noted phenomenon that students in such a course do not rely on the concept definition to do mathematics, instead they evoke a concept image from their experience.

The notion of *series* changed substantially from pre-test to post-test. Table 2 shows that there was a strong move from terms being "given by a formula" to adding terms of a sequence. Specific mention of geometric and arithmetic sequences also diminish in the light of a wider variety of examples.

What is a series?	pre-test (N=25)	post-test (N=23)
adding terms of a *sequence*	0	15
adding numbers given by a *formula*	2	2
numbers related by a *formula*	13	4
geometric or arithmetic progression	7	0
other	7	2
no response	1	0

Table 2. What is a series?

This shows that, although students may have difficulty in expressing their knowledge and rarely evoke the definition, there is a general shift in understanding that a series is *technically* a sum of terms, even though the word "series" and "sequence" may continue to be used informally on occasion.

Two successive questions revealed interesting contrasts which changed little from pre-test to post-test:

(A) Can you add $0\cdot1+0\cdot01+0\cdot001+\ldots$ and go on forever and get an exact answer? (Y/?/N)

(B) $1/9=0\cdot\dot{1}$. Is 1/9 equal to $0\cdot1+0\cdot01+0\cdot001+\ldots$? (Y/?/N)

The favoured response on both pre-test and post test is *No* to (A) and *Yes* to (B) (Table 3).

Responses to (A)/(B)	Y / Y	Y / N	N / N	N / Y	N / ?	? / N	nr / Y
pre-test (N=25)	4	0	1	18	0	1	1
post-test (N=23)	2	2	2	14	1	0	0

Table 3: a recurring sum

How can the equation $0\cdot1+0\cdot01+0\cdot001+\ldots=1/9$ be *false* but $1/9=0\cdot1+0\cdot01+0\cdot001+\ldots$ be *true*? One may hypothesise that each is read left to right and that the first represents a potentially infinite process which can never be completed but the second shows how 1/9 can be divided out to get as many terms as are required. Interviews suggested shades of meaning consistent with this but sometimes with a different emphasis. For instance, several students said that the initial statement "$1/9=0\cdot\dot{1}$" in (B) coloured their view and that they used this to equate $0\cdot\dot{1}$ and "$0\cdot1+0\cdot01+0\cdot001+\ldots$" One claimed not to know how to convert a fraction to a decimal, other than divide it out on a calculator; in this case 1/9 gives 0.11111111 and this would be sufficient in his estimation to show that the digit 1 was repeating.

Despite the experience of a sequence of terms becoming indistinguishable, and the seminar explanation of the definitions of limit and infinite decimals, there was little change in the response to the question "is $0\cdot\dot{9}=1$? (Y/?/N)." (Table 4).

Is $0\cdot\dot{9}=1$ *?*	Y	N	?	no response
pre-test (N=25)	2	21	1	1
post-test (N=23)	2	21	0	0

Table 4: Nought point nine recurring

Interviews revealed that students continued to conceive $0\cdot\dot{9}$ as "a sequence of numbers ... getting closer and closer to 1", or not a fixed value "because you haven't specified how many places there are" or "it is the nearest possible decimal below 1". The programming experiences did not change this view, and it is important to note that one cannot compute the *exact* limit by programming in this environment, so the limit concept cannot be *constructed* through programming.

Another generic limit did, however, prove to change (Table 5), in response to:

Complete the following sentences: 1, 1/2, 1/4, 1/8, ... tends to ________
The limit of 1. 1/2, 1/4, 1/8, is ________

"tends to" / "limit"	0 / 0	0 / $\frac{1}{\infty}$	$\frac{1}{\infty}$ / $\frac{1}{\infty}$	0 / ?	2 / 2	0 / 2	0 / 1
pre-test (N=25)	0	11	1	5	0	2	2
post-test (N=23)	8	3	3	0	4	0	2

Table 5: The evoked meanings of "tends to" and "limit"

The response "2" may indicate the sum of the *series* $1+\frac{1}{2}+\frac{1}{4}+\ldots$. An interview revealed the response "1" for the limit related to an interpretation of the "limit" of the sequence as the largest term. The most commonly occurring response changed from "tends to 0, limit $\frac{1}{\infty}$" to "tends to 0, limit 0" suggesting that the idea of $\frac{1}{\infty}$ as an indefinite number, arbitrarily small, is being replaced by the numeric limit 0.

There were considerable successes in programming. It is quite apparent that the students were able to program functions as procedures yet use the name of the procedure as an object in another piece of programming. For example, the *programming* in the following problem was successfully completed by 23 students out of 25:

Define a function FNb(n) which returns the value b_n where

$$b_n = \begin{cases} 1/n^2 \text{ if } n \text{ is prime} \\ 1/n^3 \text{ if } n \text{ is not prime and even} \\ 1/n! \text{ otherwise} \end{cases}$$

(Hint: it may help to define a function `FNprime(n)` which returns 1 if n is a prime and 0 if not, and another function `FNcalc(n)` which returns $1/n!$)

Calculate the sum $b_1 + b_2 + \ldots + b_n$ for n = 1000. Does the series Σb_n converge?

The function `FNprime(n)` mentioned earlier was often used in a function of the following kind:

```
10000 DEF FNb(n)
10010 IF FNprime(n)=1 THEN =1/n^2 ELSE IF n MOD 2 = 0 THEN = 1/n^3
      ELSE = FNcalc(n)
```

Twenty three students used such a function to calculate $b_1 + b_2 + \dots + b_n$ and seventeen of these programmed the partial sum as a function which added up the terms FNb(k) for $k=1$ to n. Thus there is considerable evidence to show success in using the function notion as procedure or object.

However, the calculation of a sum took longer for a larger number of terms. For instance it might take five seconds to compute a sum of 1000 terms, fifty seconds for 10000 terms and five hundred seconds for 100000 terms. This temporal aspect is illustrated by the following response to $\Sigma\ 1/n^2$:

> *$\Sigma 1/n^2$ converges – with count 1000 it appeared to converge towards 1.6440 (4 dps). However, on another occasion with count 7996, the sum was 1.6448 (4 dps). I think that the series does converge, as successive values get smaller and the difference between successive sums becomes smaller, but it takes a long time to converge – longer than I spent!*

It was also notable in class that, when students had programmed the partial sum as a formula, some became obsessed with the numerical values and no longer focused on the internal process of adding terms. For instance, it was possible for a student to program FNs(n) adding together $1/k^2$ for $k=1$ to n, and note that when $n=1000$ the sum is still changing in the 6th decimal place without explicitly noting that the thousandth term added on is $1/1000^2$. For many students, the computer laboratory work seemed to focus on the syntax of the programming and the use of the program to investigate numerical values rather than any reflection on the symbolic processes occurring.

The environment was able to provide a spectrum of phenomena from series such as $\Sigma 1/n!$ which stabilise in a few terms, to those such as Σn which clearly diverged, and in between, series like $\Sigma 1/n$, $\Sigma 1/n^2$ which grew by smaller and smaller amounts and were open to question. In the seminar proofs were given for convergence of $\Sigma 1/n^2$, and divergence of $\Sigma 1/n$. (In *theory* the latter diverges, but *in practice*, it grows so slowly that on today's computers it will not exceed 100 in a human life-span!)

The sum Σb_n proved to be very interesting. The procedures took considerably longer, so only a small number of terms could be computed (certainly not more than 10000). But the nth term at various times could be $1/n^2$ or $1/n^3$ or $1/n!$, depending on the value of n, so the amount by which the sum increased as an extra term was added could vary considerably:

> *In places it looks as if it is not converging, but other parts of the series looks as if it is.*

Other students believed the series not to be converging:

> *The series does not converge, although it is increasing very slowly.*

This example of Σb_n proved to be very fruitful. Each term was less than or equal to $1/n^2$, so the nth partial sum was less than the nth partial sum of $\Sigma 1/n^2$ and the latter partial sums could be proved to be increasing and bounded above. There was a spirited dialectic argument in one seminar about whether an increasing sequence bounded above necessarily converged to a limit, or whether it could continue creeping up, never reaching the upper bound, and never actually converging. The *completeness axiom* therefore arose as part of a natural student conversation.

Reflections

The course provided an environment in which certain sequences were seen to stabilise after a certain number of terms, and the more accurate the required stabilised value, the further along the sequence one may have to go. Experience was provided for the definition of the limit, in the *cauchy* sense as well as the sense of tending to a specific value. In this context discussion of the completeness axiom occurred naturally.

But deeper epistemological questions remained. For many students, the meaning of an infinite decimal as a limit of a sequence was not established. It already had a different stable meaning and in the programming paradigm (II) such a limit cannot be *constructed*, only approached within reasonable practical accuracy, which fails to disturb the earlier meaning.

Some gains were made – the proceptual programming of a function as procedure and object, a clearer distinction between sequence and series, and some progress towards the perception of the limit object as a specific number rather than an indefinite generic limit. However, it is essential to examine the nature of computer-constructed objects with greater care. The programmed function is a *cybernetic* procept which auto-calculates the value and has subtle differences from the formal notion of function. The focus is taken away from the relationship between process and product which would be given by experiencing the calculation itself. The latter construction may therefore not be performed and necessary relationships may not be constructed. Deeper epistemological obstacles are likely to remain. Further cognitive reconstruction is necessary for transition to the formal ε–N paradigm, but at least experiences have been gained which may be fruitful as a basis of discussion.

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A LEARNING MODEL FOR FUNCTIONS IN A COMPUTER GRAPHICAL ENVIRONMENT

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In this study a learning model with three stages is used to describe the learning of trigonometric functions in a graphical computer environment. The stages are free Exploration (1), Analysis and Comparison (2), Experimentation and Practice (3). Worksheets with learning activities to be done with a function plotter were designed, according to the model, for a group of highschool students. A second group of students worked the same trigonometric concepts with a paper and pencil version of the worksheets. A significant difference favoring the computer group was observed on both the posttest and the delayed retention test. One possible interpretation of the results is that the visual approach emphasized in stages 1 and 3 in the computer group favored the learning of the concepts.

Introduction

In this study a learning model for functions was used to describe the learning of trigonometry functions with a function plotter on a microcomputer. According to the model, the learning process has three stages:

Stage 1: Free exploration (FE)

In this first stage the student explores a previously unknown concept with the function plotter. By means of sequenced questions the students is being introduced to the concept.

Stage 2: Analysis and Comparison (AC)

The student compares members of new families of functions with already known functions and analyses the properties of those functions until he develops a more formal mathematical concept.

Stage 3: Experimentation and Practice (EP)

The student goes through guided experimentation activities and works with more examples of the acquired concept in order to assure learning and retention.

The model is a dynamic and recursive process model and was used as the background for a didactical strategy which was developed in a graphical computer environment. The learning activities suggested by the strategy, lead students to the construction of concepts related to the basic characteristics of trigonometric functions $y = a \sin bx + d$ and $y = a \cos bx + d$: amplitude, period and position of the graph in the coordinate

system. The concepts are introduced in an increased order of complexity. For each new concept, the three stages Free Exploration, Analysis and Comparison and Experimentation and Practice are revisited.

Background the study

The learning model was developed on the assumption that learning is a constructive activity (Glaserfeld, 1987, Goldin, 1992) and that different representations are the basis of reasoning. Emphasis was put on visual reasoning (Dreyfus, 1991) and the transition from graphical vs. algebraic representations. Some authors speak in this context about multiple linked representations (Goldenberg, 1986).

Related studies have shown that students can find analogies between graphical and algebraic representations if they are trained to do so (Schwarz, Bruckheimer, 1988. Schwarz and Dreyfus, 1989).

A basic ability involved in learning in a graphical computer environment can be identified as visual processing (Bishop, 1983) or visual reasoning.

In order to be able to interprete function graphs correctly, perceptual experience and mathematical knowledge are necessary, because graphs have there own ambiguitos (Goldenberg, 1987).

It is clear that with the availability of graphics software decreases the need to graph functions (with pencil and paper), but increases the importance of interpreting graphs.

Eisenberg, Dreyfus (1989) suggest that many mathematical concepts and processes can be related to visual interpretations or visual models. It is important to discuss the pedagogical potential and the problems which arise in the context of such visualization. Students have to learn to coordinate different representations, for example, symbolic with graphical ones. A variety in representations of the same concepts enriches learning but also increases complexity.

In previous studies (Wenzelburger, 1989, 1990a, 1990b, 1991, 1992) results were found which favored the learning of function concepts of the experimental groups in a graphical computer environment which had the characteristics of a "generic organizational system" (Tall, 1985).

Method

The learning model FE- AC - EP is based on a constructivist approach which suggests that students are the builders of their concepts in a graphical computer environment which facilitates the continuous transition between algebraic and graphical representations and the interpretations of both.

A didactical strategy was developed which reflects the three stages of the learning model and was tried out with a group of highschool students (twelfth grade) in a Calculus class. The teaching method is based on worksheets used by students on an individual basis. In order to control for the effect of the worksheets, the class of 25 students was randomley split into two subgroups - the experimental group (14 students) which learned with the worksheets and a computer graphics program about the trigonometric functions and the control group (11 students) wherein students used only worksheets. The main hypothesis of the study was that the graphics tool for the microcomputer plays a decisive role in the learning and retention of the function concepts and not the worksheets which required each individual student to be actively involved in the learning.

In the pretest, the posttest and the retention test (4 months later) administered to both groups, certain abilities were measured. The mathematical content were trigonometric functions $y = a \sin bx + d$, $y = a \cos bx + d$. Students had:

-To interprete the parameters of an equation or a graph of a function.

-To associate a graph with an equation or an equation with a graph

-To distinguish between different equations and graphs which were given and recognize them as members of a family of functions.

-To draw graphs given an equation.

-To write the equation, given the graph.

Both groups worked during five weeks (15 50-minute sessions) with the worksheets, and the computer in the case of the experimental group. In the computer center every student had his own computer and the investigator was present during the sessions. The control group was monitored by the classroom teacher.

The teaching strategy was based in the three-stage learning model: Free Exploration, Analysis and Comparison, Experimentation and Practice. The content was organized in a sequence with increasing order of difficulty. The parameters: a, b and d were real numbers.

Topic 1, 2: Amplitud of functions $y = a \sin x$, $y = a \cos x$.

Topic 3: Period of functions $y = a \sin x$, $y = a \cos x$.

Topic 4, 5: Amplitud and period of functions $y = \sin bx$, $y = a \sin bx$.

Topic 6, 7: Amplitud and period of functions $y = \cos bx$, $y = a \cos bx$.

Topic 8, 9, 10: Characteristics of the functions $y = \sin x + d$, $y = a \sin x + d$, $y = a \sin bx + d$

Topic 11, 12, 13: Characteristics of the functions $y = \cos x + d$, $y = a \cos x + d$, $y = a \cos bx + d$

Topic 14: Amplitud, period and shape of the function graphs $y = a \sin bx + d$, $y = a \cos bx + d$

As an example we describe the activities students in the computer group had to do in the case of Topic 8. They were supposed to learn about the characteristics of the function graphs of $y = a \sin x + d$

1. FE: Free exploration with the function plotter - students were asked to vary d and observe the effect on the graph.

2. AC: Comparison between $y = \sin x$ and $y = \sin x + d$. Student were asked to draw a conclusion about the parameter d. They were asked to focus on the line $y = d$ and its relation to $y = \sin x + d$. They had to analyze numerous examples according to symmetries, y-intercept, maxima and minima.

3. EP: Experimentation and practice with the function plotter, but students also had to draw graphs by hand, determine the parameters from graphs or equations, write equations for given graphs and draw a final conclusion about the characteristics of $y = \sin x + d$.

A similar approach was proposed for all topics. Students in the control group skipped the free exploration stage and their experimentation and practice stage was restricted to paper and pencil work, since without a graphics tool which quickly generates families of functions, these stages become very cumbersome and time-consuming.

The learning objectives which were measured with the exams were the following.

Objective 1: Determine the amplitude and period of the functions $y = a \sin bx$ and $y = a \cos bx$, given the graph.

Objective 2: Determine the amplitude and period of the functions $y = a \sin bx$, $y = a \cos bx$, given the equation.

Objective 3: Given the graph of $y = a \sin bx + d$, $y = a \cos bx + d$, write or identify the equation.

Objective 4: Given the equation of $y = a \sin bx + d$, $y = a \cos bx + d$ draw or identify the graph

Objective 5: Determine the amplitude, period and the horizontal axis of symmetry of y = a sinbx + d, y = a cosbx + d given the equation.

Objective 6: Determine the amplitude, period and the horizontal axis of symmetry of y = a sinbx + d, y = a cosbx + d given the graph.

Objective 7: Differentiate the parameters a, b and d in functions y = a sinbx + d, y = a cosbx + d.

The diagnostic test, the posttest and the retention test consisted of a closed part (20 items) and an open part (8 items).

The basic assumption of the study was that the graphical computer environment would play a decisive role in the application of the three stage learning model by means of worksheets. Therefore two hypothesis were tested:

1. Students in the computer group would do better on the posttest than students in the control group.

2. Students in the computer group would do better on the delayed retention test than students in the control group.

Results:

The pretest showed that students participating in the study had some previous knowledge of functions of type y = a sinbx and y = a cosbx. Therefore gainscores were computed and a student's t was calculated. Table 1 shows the average percentages of gainscores for the retention test.

TABLE 1
AVERAGE GAINSCORES IN %

	NUMBER OF STUDENTS	AVERAGE	STANDARD DEVIATION
Computer Group	14	78.57*	16.48
Control Group	11	70.54*	13.25

*Significant at the 0.10 level.

The retention test was given to all students four months later without prior notice. The average scores in % are shown in Table 2.

TABLE 2

Average Scores in %

	Number of Students	Average	Standard Deviation
Computer Group	12	78.83**	13.41
Control Group	9	65.88**	17.76

**Significant at the 0.05 level.

The reliability of all tests was computed for both groups with the splithalf-method and turned out to be good (values between 0.83 ad 0.92). Also the indices of item ease were determined for all items and found to vary between 0.21 and 1.00 (percentage of correct responses of total responses given).

Discussion and Conclusions

The results reported show that in the posttest, students in the computer groups did somewhat better than the control group, while in the delayed retention test the difference between both groups was more marked and favored the computer group. This is in agreement with results found in previous studies by the same researcher.

The main difference between both treatments was in the first and in the third stage of the learning model since free exploration is difficult and guided experimentation is limited without an electronic graphics tool. Students in the computer group had the opportunity to observe and analyze a great variety of function graphs and could manipulate freely families of functions while students in the control group had a more restricted exposure the function graphs since they were limited to paper-and-pencil work. Students in the computer group performed better on tasks related to Objectives 3 and 6 which consisted in the interpretation of a graph in two different ways: determine parameters of a function given a graph and write an equation of a function given a graph. But also on items which measured the achievement of Objectives 2 and 7, the computer group did better: students had to interprete equations of functions.

For Objectives 1, 4 and 5 basically no difference between both groups was found. Tasks associated to these objectives required students to draw or identify function graphs, given the equation or to interprete equations in order to determine parameters of a function.

We conclude that the teaching strategy based on the three stage learning model FE - AC - EP in a graphical computer environment fosters visual reasoning which requires the interpretation of graphs. While the teaching strategy based only on worksheets produced satisfactory results for tasks which focused on the drawing of graphs and the interpretation of equations.

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WHAT DO CHILDREN BELIEVE ABOUT NUMBERS? SOME PRELIMINARY ANSWERS

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The research described here is part of a larger investigation of children's beliefs in aspects of mathematics. A particular focus is the contrast in beliefs between children who have had unrestricted access to electronic calculators and those who have not. The beliefs investigated in this paper relate to the use of numbers, the magnitude of numbers, the symmetric nature of the number system, and the possibility of there being other types of mathematics. The methodology used is comparatively new in mathematics. Some two hundred year three children (fourth year of school in Australia) of whom one hundred were calculator users and one hundred were non-users were asked about their beliefs on these questions. Their responses were analysed and the beginnings of a continuum describing children's number beliefs established. Advantages and disadvantages of the research methodology and suggestions for future research questions are provided.

Introduction

In a recent article commenting on the children and teachers involved in the CAN (Calculator-Aware Number) project Janet Duffin echoes an oft heard claim, viz 'Children use[d] the calculator to extend the traditional mathematics curriculum ... it introduce[s] them to both decimals and negative numbers at a stage much earlier than would have previously been the case.' (Duffin, 1992: 24). While not rejecting her claim outright, it does appear to be based on an assumption about the knowledge of non-CAN project children. Although the traditional curriculum may not introduce, for example, negative numbers until some later stage, do we know that children have *no* prior knowledge before formal teaching commences? Research into so-called 'misconceptions' would indicate exactly the opposite (see for example, Confrey (1990) for a review of this research). Similarly those holding a constructivist view of learning would find it difficult to accept an 'empty vessel' status for the non-CAN children. In today's economic climate large numbers appear regularly in news reports, negative numbers figure in (winter) temperature readings, percentages can be found in many shops offering discounts, and decimals are found in money dealings. Surely many, if not all, children are exposed to a considerable variety of number situations before any formal programme is attempted? Such exposure will be the foundation of children's beliefs about what

numbers are and what they are for. Whether their beliefs are the same as ours, or even antagonistic to later teaching, needs to be examined before claims like that above (which may appear reasonable and may well be true) can be treated with anything but caution.

The present study is a preliminary attempt to collect such data and perform such analyses as will help in determining the nature of children's beliefs about number. As Ausubel said '[T]he most important single factor influencing learning is what the learner already knows' (Ausubel, 1968: vi).

The evidence of research into learner beliefs (prior to teaching in science) shows that they are indeed critical to the outcomes of instruction and have been well documented (Adams, Doig and Rosier, 1991). In mathematics however the role of affective variables has not received the same attention. In his review of research into affective variables McLeod (1992) categorizes this research into the following categories: beliefs about mathematics; beliefs about self; beliefs about mathematics teaching and beliefs about the social context. However none of the reported studies focus on learners' beliefs about specific concepts of mathematics, such as learners beliefs about numbers. Research into learners' mis-beliefs (usually denoted misconceptions) is similarly reticent about fundamental ideas such as the use of numbers.

Research questions

The investigation reported here forms a small part of a larger study involving number and calculators which is an adjunct to CAN-like projects being conducted by Deakin University (Groves, Cheeseman, Allan and Williams, 1992). While these projects have sought to answer many questions, the specific foci selected for this report are a description of children's beliefs about some aspects of number and a comparison of the beliefs of calculator-using children and non-calculator users. In order to achieve this, the following quetions were posed:

1 What do we use numbers for? The purpose of this question was to explore the children's beliefs about the use to which we put numbers. Apart from counting and calculating, adults use numbers for labelling (for example house numbers) and position (for example first place in a race). (A fuller discussion of these aspects can be found in Doig, 1990).

2 How big can numbers be? The purpose of this question was to explore the range and extent of the numbers that children considered to be the greatest possible. In Victorian schools children at year three are expected, by the end of the year to understand, read, and write whole numbers up to 1100 (MOE, 1988). Curriculum

planners do not necessarily say that this is the largest number that children may either use or know, but leave the issue open. To date there is no evidence to tell us what the range of 'largest' numbers may be.

3 How small can numbers be? Like its 'largest' counterpart, there is no evidence to tell us just what children believe to be the smallest possible number. The Victorian curriculum suggests that children at year three can deal with decimal fractions as small as tenths (MOE, 1988). Whether this was the real limit was to be investigated.

4 Is there a symmetry to 'large' and 'small' numbers? The idea that the real numbers are in some way symmetric about zero is usually raised in the junior secondary school (year eight in Victoria) when negative numbers are introduced (MOE, 1988). However, for those younger children who are aware of negative numbers there is the possibility that this idea has already emerged. A comparison was to be made of children's responses to the two previous questions to see if any 'symmetry' was exhibited.

5 Is there maths other than ours? At first glance this question seems to be unconnected to the previous four. However if one considers that at year three the dominant aspect of number is positive integers, usually operationalized through counting, it seems sensible to investigate the beliefs of those about to be confronted in the next few years with negative integers, rational and irrational numbers. How nonsensical are these 'new' numbers to those raised in the belief that numbers are whole? Does a belief that there is only 'our' (whole number) maths impede, or make more difficult, the learning of fractions or negative numbers? Is this a case of Hawkins' 'critical barriers'? (Hawkins, 1978).

Methodology and instrumentation

To gather information on children's beliefs at year three (approximately nine years of age) would usually involve one-to-one interviews. However to gather sufficient data to be able to make justifiable inferences makes interviewing not feasible. Fortunately there has been developed recently techniques for gathering and analyzing such data using interview-like written formats and modern statistical tools. To date these have been used in science and social science only but there was no reason to doubt that the technique would apply equally well to mathematics. For a full description of these formats and their application to science see Adams, Doig and Rosier (1991) and for partially similar methods in mathematics Streefland and van den Heuvel-Panhuizen (1992) and Tirosh and Stavy (1992). The particular format selected for this investigation was that of a short story entitled 'What happened last night'. In this story an alien visitor

asks questions of the child (reader) who responds by completing 'gaps' left in the text. In all administrations the entire story was read to the children by the author then the children read and completed the story in their own time, approximately thirty minutes.

Subjects

The subjects were from four Melbourne (Victoria) schools. Two schools were where the children had had complete access to calculators and two non-calculator schools matched on socio-economic variables . The number of subjects in each school is presented in Table 1.

Table 1: Number of year three subjects in calculator and non-calculator schools.

Calculator schools		Non-calculator schools	
E1	E2	C1	C2
Subjects = 54	Subjects = 51	Subjects = 55	Subjects = 39
Total subjects = 105		Total subjects = 94	
Total subjects = 199			

Analysis

As outlined earlier, the methodology used was that of written responses to leading questions. This meant that two hundred scripts each of five responses had to be prepared for analysis. The procedure used was that pioneered by Adams and Doig (Adams et al, 1992) in their study of science beliefs. First all responses to a particular question are read to give an overall 'feel' for the range of responses. Theoretically each response is unique, but in practice responses tend to 'group' themselves in a qualitative sense. Thus after the initial reading, it is possible to describe tentative qualitative categories. All responses are then placed into one of these mutually exclusive categories. If necessary this process is repeated until all responses can be accommodated. Each category is now given an integer 'level' label, which describes its ranking from being the most to the least sophisticated response. The level labels cannot be equated across questions, and in some cases two qualitative responses have been assigned to the same level. The analysis of the labelled data was via the Quest® interactive analysis program (Adams and Khoo, 1992). The analysis was of three forms, a simple frequencies analysis, chi-square and a Rasch partial credit analysis (Wright and Masters, 1982). The application of this model enables the construction of a developmental continuum for the set of questions as a group. This is the 'Beliefs about numbers' continuum.

Results and discussion

Below are the response percentages by category for each of group of children, in calculator and non-calculator schools. For each question the highest value label indicates the response most concordant with school (curriculum) beliefs.

Table 2: Question 1 What do we use numbers for?

Purpose: To explore the children's beliefs about the uses of numbers.

LABEL	Calculator schools	Non-calculator schools	DESCRIPTION
3	67 6	45.7	four operations
2	16 2	30 9	counting
1	3 8	4.3	learning
1	5.7	17 0	maths
0	6.7	2 1	uninterpretable

By year three most of the numerical experiences of Victorian children have been to do with counting, with some work on addition and subtraction, mainly non-algorithmic. It is no surprise then that the overwhelming majority of children believe that either counting or calculating (addition was the operation most frequently mentioned) are the major uses for numbers. Very few children mentioned uses such as house numbers, telephone numbers and so on. The most striking aspect of the responses to this question is the greater percentage of 'calculator' children looking beyond counting to operations.

TABLE 3: Question 2 How big can numbers be?

Purpose: To explore the range of numbers that children know.

LABEL	Calculator schools	Non-calculator schools	DESCRIPTION
4	18 1	14 9	$+\infty$
3	20 0	24 5	finite – > display
2	25 7	21 3	finite – < display
1	11 4	13 8	physical size
0	24 8	25.5	uninterpretable

At the age of eight the idea of infinity has not been raised in school, but some notion of it exists in a considerable number of children. Whether this belief is in an 'infinity' the same as ours is probably doubtful for many of these children. Those children who used a

same as ours is probably doubtful for many of these children. Those children who used a finite 'big' number for their response were divided between those whose 'big' number was shorter than a calculator's display (eight digits) and those whose 'big' number was longer than the display. The noteworthy point is that the 'calculator' children appear to be constrained by their familiarity with such displays.

TABLE 4: Question 3 How small can numbers be?

Purpose: To explore the range of numbers that children know.

LABEL	Calculator schools	Non-calculator schools	DESCRIPTION
5	5.7	2.1	-∞
4	9.5	3.2	-finite - > display
3	10.5	1.1	-finite - < display
2	34.3	51.1	0 or 1
1	4.8	3.2	+finite
1	13.3	17.0	physical size
0	21.9	22.3	uninterpretable

As in the previous question, the fact that there exists a group of children of this age who have notions of a negative 'infinity' is quite surprising. The fact that none of the children, 'calculator' or 'non-calculator', used fractions was unexpected, especially in the light of comments such as Duffin's. The large number of children who nominated 0 or 1 as their choice of 'small' number may either be swayed by the school curriculum's emphasis on counting, or be unaware of the existence of negative numbers.

By matching children's responses to the previous two questions it was possible to gauge to what extent the children could be said to have notions of a symmetry within the number system. While it would appear that slightly more 'calculator' children had some idea of a symmetry, this was mainly true for those who gave infinity for their responses to the preceding two questions. For those whose responses were a finite number, more 'calculator' children certainly favoured a symmetry, but the vast majority of children gave responses indicating no symmetry at all.

TABLE 5: **Question 4 Is there a symmetry to large and small numbers?**

Purpose: To explore whether children have a symmetric notion of the number system.

LABEL	Calculator schools	Non-calculator schools	DESCRIPTION
3	3.8	2.1	$+\infty/-\infty$
2	12.4	2.1	+finite/-finite
1	44.8	53.2	non-symmetric (numerical instances)
0	4.8	1.1	non-symmetric (mixed instances)
0	11.4	13.8	physical
0	22.9	27.7	uninterpretable

TABLE 6: **Question 5 Is alien maths the same as ours?**

Purpose: To explore whether children believe that there can be 'other' maths.

LABEL	Calculator schools	Non-calculator schools	DESCRIPTION
2	9.5	11.7	the same
1	31.4	29.8	different
1	32.4	25.5	examples
0	26.7	33.0	uninterpretable

Those children who gave examples of alien mathematics generally chose to show the symbols of this alien system, without indicating whether this was the only difference between us and them mathematically. However, both groups of children were firmly of the opinion that 'other' mathematics was possible. A large percentage in each group were unable to answer the question, perhaps because there is little discussion of the nature of mathematics at this level of schooling.

The chi-square analyses showed that the two groups of children differed on only two questions. There was a considerable difference in the children's responses to question three, on the 'smallness' of numbers, (chi-square = 44.67) where nearly a quarter of the 'calculator' children gave a negative number in their response compared to about six percent of the other group. In responses to question four there was also a significant

difference (chi-square = 11.63), with four times as many 'calculator' children believing in a symmetry in the number system.

The questions and their responses reported here, while only a subset of a larger study, do indicate that there are reasons to believe that there is a systematic set of beliefs being developed by children, and that those of the 'calculator' children do differ from the 'non-calculator children. Overall it would seem that to some extent the claims made about the effects of using calculators with children is validated, especially with regard to negatives. But more work needs to be done. The high number of responses that were uninterpretable indicates that better questions need to be formulated and follow-up interviews conducted. (The third form of analysis, providing a 'Beliefs about numbers' continuum, is not included here due to space restrictions, but will be presented at PME XVII).

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RESEARCH INTO RELATIONSHIP BETWEEN THE COMPUTATIONAL ESTIMATION ABILITY AND STRATEGY AND THE MENTAL COMPUTATION ABILITY: ANALYSIS OF A SURVEY OF THE FOURTH, FIFTH, AND SIXTH GRADERS IN JAPAN

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ABSTRACT *The purpose of this research is to obtain performance data on the computational estimation and mental computation skills of Japanese students in grades 4, 5, and 6, to clear effects of the teaching and learning of rounded numbers and rounding strategies on computational estimation, and to inquire the relationship between computational estimation ability (CEA), strategy (CES) and mental computation ability (MCA). This paper shows that: (1) the teaching and learning has strong effects on computational estimation containing only numerical data positively and on computational estimation in problematic situations negatively; and (2) there are significant relationships both between CEA and MCA, and CEA and CES, but there is no significant relationship between CES and MCA. These results suggest that the reasonable and efficient computational estimation requires flexible rounding of numbers based on sound number sense as well as mental computation ability.*

BACKGROUND FOR RESEARCH

Recently estimation has been emphasized in school mathematics curricula in Japan (Japanese Ministry of Education, 1989) as well as in the United States (National Council of Teachers of Mathematics, 1989). The major emphasis on estimation in elementary and intermediate mathematics curricula is based on such recognition that in the age of calculator/computer students should learn estimation to make sense of number or quantity and make a reasonable and adequate decision.

In general estimation is divided into three categories; i.e. computational estimation, measurement estimation, and numerosity estimation (Sowder, 1992). Among them computational estimation has been a main topic addressed by mathematics researchers (Reys, Bestgen, Rybolt, and Wyatt, 1982; Rubenstein, 1985; Reys, 1986; Reys, Reys, Nohda, Ishida, Yoshikawa, and Shimizu, 1991; Dowker, 1992). As a result of these research, the three general cognitive processes and a number of specific strategies in computational estimation were identified (Reys et al., 1982). Moreover it was suggested that computational estimation would be closely related with mental computation and number sense (Rubenstein, 1985; Sowder, 1992; Dowker, 1992).

In this research computational estimation is defined as a process consists of rounding numbers and mental computation with these rounded numbers without paper and pencil or other external aids. Therefore, in a problematic situation which requires computational estimation, it is very important to round numbers in consideration of a purpose of the situation (reasonable rounding) and in accordance with estimator's ability of mental computation (efficient rounding).

PURPOSE

In Japan, according to the previous curriculum of elementary school mathematics (Japanese Ministry of Education, 1977), it was necessary for students to practice mental computation in the third grade and to know the rounded numbers and how to round numbers by using strategies of "kiriage" (rounding up), "kirisute" (rounding down), and "shishagonyu" (if the following digit is more than 5 round up, and if less than 5 round down) in the fourth grade. Estimation was not highly emphasized. Moreover the relationship between mental computation and estimation was not explicitly stated. On the other hand, in the current curriculum (Japanese Ministry of Education, 1989) which has been implemented in 1992 school year, estimation is emphasized to a greater degree while the emphasis on mental computation is diminished.

In this ironical situation, a total of 351 upper graders in elementary school were administered a computational estimation and mental computation test in June 1990. The purpose of this research was to obtain performance data on the computational estimation and mental computation skills of Japanese students in grades 4, 5, and 6, to clear the effects of the teaching and learning of rounded numbers and rounding strategies at the fourth grade in the previous curriculum on computational estimation, and to inquire the relationship between computational estimation ability, strategy, and mental computation.

METHOD

Subjects A total of 124 fourth graders, 143 fifth graders, and 84 sixth graders in four public elementary schools participated in this research. The schools were selected to represent a range of social and economic backgrounds. One school in a small town (Hyogo), one school in a city (Nara), and two schools in a large city (Hiroshima) were involved. In all classes students were heterogeniously grouped, as is the custom in elementary schools in Japan.

Material The computational estimation and mental computation test (MCE test) consisted of three parts was developed for this research based on a pilot test and previous research in Japan (Itoh et al., 1987, 1988). The first part of the MCE test contained 3 problems (one problem was in open-ended and the others were in multiple-choice format) of computational estimation in some problematic situations. The second part included 6 items of computational estimation containing only numerical data, with their explanations of how to estimate. The third part included 12 mental computation items. All items were related to four operations with whole numbers relevant to Japanese students.

All items of the first and second part were included on two sheets of paper and then these papers were distributed to students with uniform instructions in each school by a classroom teacher. Students were given 30 minutes to complete these items.

The all items of third part were presented either in a written test or in an oral test. In the written test, all items were included on a single sheet of paper and then this paper was distributed to students with the instruction that they mentally compute exact answers for all problems and that if they cannot compute mentally, they mark X in a given frame. They were given 10 minutes to complete the written test. For the oral test, all items were prerecorded on a recorder and presented to students at a specified period of time (5-10 seconds per item) with the instruction that they compute mentally

exact answers (without copying the problem) and write their answer on an sheet of paper provided.

RESULTS AND ANALYSIS

The first part of MCE test

Table 1 summarizes the percent answers to the problem 1; "There are 48 boys and 54 girls in a park. You will give two candies to everyone. About how many candies do you need enough for the purpose?". This table shows that 69.4% of the fourth graders, 69.9% of the fifth graders, and 76.2% of the sixth graders gave answers ranging from 200 to 220 candies and no significant difference was found among them. However focusing on answers (204, 210 or 220) which are reasonable in the purpose of this problem, whereas 54.9% of the fourth graders gave these reasonable answers, only 18.2% of the fifth graders and 22.6% of the sixth graders gave them. About half of students at 5 and 6 grades made their estimation using "shishagonyu" strategy and answered 200 candies. A cross test showed a significant difference between fourth graders and fifth graders (χ^2=53.47, df=4, p<.0001). Considering the fact that the fourth graders had not learned any rounded numbers and rounding strategies in schools, this analysis suggests that the teaching and learning of rounding strategy "shishagonyu" has an effect on a stereotyped estimation.

Table 1 *Percent Answers to The Problem 1*

	Percent Answers				
Grade	200	204	210	220	other
4 (n=124)	14.5	19.4	26.6	8.9	30.6
5 (n=143)	51.7	9.1	7.0	2.1	30.1
6 (n=84)	53.6	10.7	8.3	3.6	23.8

Table 2 summarizes the percent answers to the problem 2; "When you buy 46 apples at 83 yen, is 4500 yen enough for you to buy them? Choose one way among the followings to check by mental

Table 2 *Percent Answers to The Problem 2*

	Percent Answers								
Grade	80×40	80×50	90×40	90×50	100×40	85×45	other	enough	not enough
4 (n=124)	12.1	21.0	1.6	33.1	2.4	10.5	17.7	88.7	0.0
5 (n=143)	7.0	75.5	0.0	7.0	1.4	4.9	3.5	83.2	4.9
6 (n=84)	0.0	73.8	2.4	14.3	1.2	4.8	3.6	90.5	2.4

computation whether 4500 yen is enough or not, and then mark one of two alternatives of enough or not enough". In this problem, it is a reasonable and efficient estimation strategy to round up both numbers. Whereas 33.1% of the fourth graders chose this strategy (90×50), only 7.0% of the fifth graders and 14.3% of the sixth graders chose it. About 75% of students at 5 and 6 grades chose the "shishagonyu" strategy (80×50). Although the majority of students at each grade level chose a correct alternative "enough", some students seemed to do it based on unreasonable estimation. A cross test showed a significant difference between the fourth graders and the fifth graders in regard of estimation strategies (χ^2=62.47, df=2, p<.0001). This analysis also suggests a negative effect of the teaching and learning of rounding strategy "shishagonyu" on computational estimation in a problematic situation.

Table 3 summarizes the percent answers to the problem 3; "When 456 candies are shared with 87 persons, about how many candies does one person get? Choose one way among the followings to solve this problem by mental computation, and then write your answer". This table shows that the fourth graders tented to choose a way (456÷90) similar to an exact way (456÷87) in this problem. It is very interesting that only 8.1% of the fourth graders, 11.9% of the fifth graders, and 21.4% of the sixth graders chose a most reasonable and efficient strategy to estimate the number of candies. This fact tells us that it is difficult for the majority of students at each grade level to round up one number and round down the other at the same time in a flexible manner.

Moreover we should note that only 34.7% of the fourth graders and 54.5% of the fifth graders could answer a reasonable number "about 5" candies and that almost of unreasonable answers were "about 50" candies. This fact suggests that it is difficult for them to mentally compute a division of 3-digit number by 2-digit number which could be reduced to a division of 2-digit number by 1-digit number and that they seems to be lack of number sense.

Table 3 *Percent Answers to The Problem 3*

	Percent Answers								
Grade	456÷90	450÷87	450÷80	450÷90	500÷90	500÷100	other	about 5	other
4 (n=124)	38.0	6.5	8.1	8.1	17.0	4.8	17.7	34.7	65.4
5 (n=143)	11.9	5.6	4.9	11.9	52.4	5.6	7.7	54.5	45.5
6 (n=84)	14.3	4.8	7.1	21.4	25.0	6.0	21.4	78.6	21.4

The second part of MCE test

The second part of MCE test included 6 items containing only numerical data. In order to obtain both performance and strategy data, students were asked to roughly compute each of items without using paper and pencil and then to write a method of rough computation. The strategies used by students could be classified into three types. The R type was that student firstly rounded number(s)

and then computed them mentally (e.g. 347-248→350-250=100). The C type was that student computed an exact answer mentally (e.g. 347-248=99). The M type was that student firstly rounded number(s), then computed them mentally, and finally rounded an answer again (e.g. 357+245→350+250=600→610).

Performance levels on 6 items and the range for acceptable estimates for each of them are shown in Table 4. The percentages of the three types of strategies used by students who could estimate reasonably are also shown in this table.

The fact that the percent correct on each of items increased gradually with grade level showed that performance of computational estimation improved with grade. This table also showed that the percentage of using the R type strategy dramatically increased from the fourth grade to the fifth grade. A cross test showed significant differences at all items (for the third item $p<.001$, for the others $p<.0001$) between students using the R strategy and those using the C strategy. This analysis suggests a positive effect of the teaching and learning of rounded numbers and rounding strategy "shishagonyu" on computational estimation containing only numerical data.

Table 4 *Percent Correct on 6 Items and Types of Strategies Used for Computational Estimation*

Item	Acceptable Interval	Grade 4 (n=124)				Grade 5 (n=143)				Grade 6 (n=84)			
		Total	R	C	M	Total	R	C	M	Total	R	C	M
375+245	[590,610]	80.6	34.7	40.3	5.6	83.9	66.4	14.0	3.5	92.9	75.0	15.5	2.4
307+699	[1000,1010]	75.2	38.0	31.5	5.6	81.8	63.6	15.4	2.8	94.0	76.2	15.5	2.4
347-248	[99,100]	61.3	40.3	18.5	2.4	83.2	70.0	12.6	0.7	90.5	75.0	15.5	0.0
21×48	[900,1100]	21.8	8.9	11.3	1.6	55.2	46.9	7.7	0.7	73.8	61.9	9.5	2.4
98÷9	[10,11]	58.9	32.3	20.2	6.5	70.6	57.3	6.3	7.0	89.3	64.3	11.9	13.1
16+58+83+41	[180,210]	57.3	30.0	24.2	3.2	84.6	66.4	14.0	4.2	96.4	78.6	13.1	4.8

The third part of MCE test

Performance levels on 12 items of mental computation in the written test and the oral test are shown in Table 5. As expected, this table showed that performance on both the written and oral tests generally improved with grade. In grades 4 and 5, performance on the written test varied by operation, with addition being the easiest and division the most difficult. There was a significant difference between the performance on the written and oral tests. This difference likely resulted from a variety of factors including the format of presentation and timing of each test. Other factors which may have contributed to better performance on the written test include a decreased need to "memorize" numbers. This conjecture seemed to be supported by the performance on the item 209+542. If students had used a mental version of the paper/pencil algorithm, it would be difficult to memorize the numbers and carry out this strategy in only 5-10 seconds. On the other hand, if they had computed the problem

mentally by using the "toukaho" strategy (e.g. 209+542=209+500+42=709+42=751) the difference in performance might nct have been so great.

This analysis suggests that although performance on mental computation of four operations with whole numbers improved, many students seem to employ a mental version of the paper/pencil algorithm instead of making use of the advantage of Japanese numerical system.

Table 5 *Percent Correct on 12 Items of Mental Computation*

	Written Test			Oral Test	
Item	Grade 4 (n=92)	Grade 5 (n=106)	Grade 6 (n=84)	Grade 4 (n=32)	Grade 5 (n=37)
43+58	96	87	96	47	81
375+32	92	89	95	72	65
209+542	90	85	94	25	27
53-8	95	87	93	78	87
42-13	83	84	93	31	73
145-46	82	80	87	56	57
15×4	79	81	90	53	89
324×2	90	87	93	44	76
54÷9	84	84	98	72	87
96÷8	62	73	92	38	76
603÷3	60	72	93	50	81
284÷4	65	70	91	28	70

Relationship between the computational estimation ability and strategy and the mental computation ability

Using data on the second and third parts of MCE test, the relationship between computational estimation ability (CEA), computational estimation strategy (CES), and mental computation ability (MCA) was analyzed. In this research CEA, CES, and MCA were defined as the followings.

CEA: It was measured by the frequency of reasonable estimation on 6 items in the second part of MCE test. Students who made more than 5 reasonable estimates were grouped in the high-ability class (EH) and the others were in the low-ability class (EL).

CES: It was determined by the type of strategy used to estimate 6 items in the second part of MCE test. The R type strategy and the C type strategy were differentiated.

MCA: It was measured by the frequency of correct mental computation on 12 items in the third part of MCE test. Students who computed more than 10 correct answers were grouped in the high-ability class (MH) and the others were in the low-ability class (ML).

As a result of the cross tests, firstly it was shown that there was a significant relationship between CEA and MCA in the written and oral tests (χ^2=28.24, df=1, p<.001; and χ^2=15.01, df=1, p <.001, respectively). Secondarily there were significant relationships at each of items except for the fifth item "98 ÷ 9" ($7.05 \leq \chi^2 \leq 23.58$, df=1, p<.01) between CEA and CES. Thirdly there was no significant relationship between CES and MCA. These results are illustrated in Figure 1. This suggests that the mental computation ability contributes to the computational estimation ability. For flexible computational estimation strategies, however, the reasonable and efficient rounding numbers is more important than the mental computation ability. In other words, it might be said that the reasonable and efficient computational estimation requires flexible rounding of numbers based on a sound number sense as well as mental computation ability.

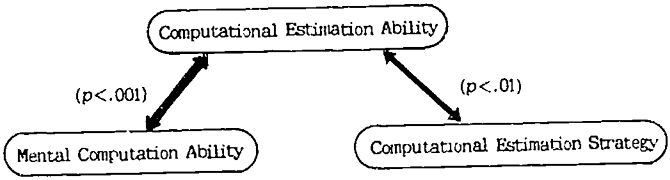

Figure 1

CONCLUSIONS AND IMPLICATIONS

Through this small research the follows were found out mainly.

(1) The teaching and learning of rounded numbers and rounding strategies in the fourth grade had strong effects on performance on computational estimation at the fifth and sixth grades in Japan. Whereas in the computational estimation containing only numerical data the teaching and learning of rounded numbers and rounding strategy "shishagonyu" had a positive effect, it had a negative effect on the computational estimation in some problematic situations and been led to stereotyped estimation.

(2) There were significant relationships both between computational estimation ability and mental computation ability, and computational estimation ability and computational estimation strategy. There was, however, no significant relationship between computational estimation strategy and mental computation ability. These results suggest that the reasonable and efficient computational estimation requires flexible rounding of numbers based on a sound number sense as well as mental computation ability.

These are similar to the following indication and observation by Reys et al. (1991) which focused on the computational estimation performance and strategies used by fifth- and eighth-grade Japanese students. "Indeed it is possible that the overlearning of written algorithms may inhibit a number of important factors that contribute to success at estimation, namely, flexible use of numbers, tolerance for error, use of multiple strategies, and adjustment techniques. Students at both grade levels tended to apply learned algorithmic computational procedures mentally" (p.55). The second finding in this research, however, is rather new and could support that computational estimation would be closely

related with mental computation and number sense.

Although all results of this research may be of limited generality and validity, they suggest that the formal teaching of rounded numbers and rounding strategies could lead students to a rigid and stereotyped estimation. Therefore the teaching computational estimation should help students round numbers flexibly, estimate value reasonably, and make an adequate decision in a problematic situation. To do it we must put more emphasis on mental computation and number sense than the complicated paper/pencil computation in elementary school mathematics curricula.

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VOLUNTARY INTERACTION GROUPS FOR PROBLEM-CENTERED LEARNING

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In a problem-centered learning approach compatible with a constructivist view of knowledge and learning, social interaction among students and attempts by students to make sense of their own and each other's constructions lead to the development of shared meanings and to individual students' constructions of increasingly sophisticated concepts and procedures. Even within the same basic teaching approach, different patterns of social interaction and the role that the teacher assumes in the learning process heavily influence learning outcomes. Our present evidence points towards granting young students the latitude to construct both mathematical meanings and the necessary social support systems, and questions the validity of some present assumptions about suitable support systems.

Introduction

Contrary to an empiricist view of teaching as the transmission of knowledge and learning as the absorption of knowledge, research indicates that students construct their own mathematical knowledge irrespective of how they are taught. Cobb, Yackel and Wood (1992) state: "...we contend that students must necessarily construct their mathematical ways of knowing in any instructional setting whatsoever, including that of traditional direct instruction," and "The central issue is not whether students are constructing, but the nature or quality of those constructions" (p. 28).

A problem-centered learning approach to mathematics teaching (e.g. Cobb, Wood, Yackel, Nicholls, Wheatley, Trigatti & Perlwitz, 1991; Olivier, Murray & Human, 1990) encourages students to construct their own knowledge and also attempts to establish individual and social procedures to monitor and improve the nature and quality of those constructions. This approach is based on the view that the construction of mathematical knowledge is firstly an *individual* and secondly a *social* activity, described as follows by Ernest (1991):

"(i) The basis of mathematical knowledge is linguistic knowledge, conventions and rules, and language is a social construction.

(ii) Interpersonal social processes are required to turn an *individual's subjective* mathematical knowledge, after publication, into *accepted objective* mathematical knowledge.

(iii) Objectivity itself will be understood to be social" (p. 42, our italics).

Whereas a traditional, transmission-type teaching approach necessarily leads to subjective knowledge which is largely re-constructed objective knowledge, a problem-centered learning approach reflects the belief that subjective knowledge (even if only in young children) should be personal *constructions* and not re-constructed objective knowledge. (When

we aim at children constructing their own knowledge, as opposed to re-constructing existing objective knowledge, we do not imply that children are actually creating knowledge that does not already exist as objective knowledge; we do state that the children in this approac construct *their* knowledge as new.)

Optimal learning environments for these two modes of learning (i.e. the personal construction of an individual as opposed to the re-construction of objective knowledge) probably differ significantly. The descriptions and tentative evaluations of different social interaction patterns in this paper are given from the perspective of the *construction*, and not the *re-construction*, of knowledge.

Classroom culture in problem-centered learning. Some salient characteristics of our problem-centered learning classrooms include:

1. Students are presented with problems that are meaningful and interesting to them, but which they *cannot* solve with ease using routinized procedures or drilled responses.
2. The teacher does not demonstrate a solution method, nor does she steer any activity (e.g. questions or discussion) in a direction that she had previously conceived as desirable, yet she expects *every* student to become involved with the problem and to attempt to solve it. Students' own invented methods are expected and encouraged.
3. It is expected of students to discuss, critique, explain, and when necessary, justify their interpretations and solutions.

The combination of characteristics 1 and 2 show why a problem-centered learning approach is incompatible with traditional transmission mathematics teaching. Characteristic 1 requires that the problems posed lie in the student's zone of proximal development (Vygotsky, 1978), which implies that this type of problem can only be solved by the individual if help is available. Characteristic 2, however, means that the teacher does not supply that help, even in the indirect form of steering conversations towards a *particular* goal. To resolve the apparent conflict between characteristics 1 and 2, characteristic 3 implies that opportunities for discussion and justification among *peers*, with the teacher as facilitator (but not mediator), will provide the necessary support for students to solve problems which lie in their zones of proximal development. This leads to the negotiation of a classroom culture where students perceive themselves as autonomous problem-solvers, accept the responsibility for their own learning and also learn to respect and evaluate other points of view.

Vygotsky describes the zone of proximal development as the distance between a learner's "actual developmental level as determined by independent problem solving" and the higher level of "potential development as determined through problem solving under *adult guidance* or in collaboration with *more capable peers*" (p. 86, our italics). We, however, circumscribe the teacher's role to that of facilitator, chairman, or if necessary, devil's advocate, and regard *students'* mathematical discourse as the main vehicle for learning. We regard this type of discourse as compatible with what Richards (1991) calls *inquiry math:* "asking mathematical questions; solving mathematical problems that are new to you; proposing conjectures; listening to mathematical arguments" (p. 15).

Social interaction serves at least the following purposes in problem-centered classrooms:

- Social interaction creates opportunities for students *to talk about* their thinking, and this talk encourages reflection. "From the constructivist point of view, there can be no doubt that reflective ability is a major source of knowledge on all levels of mathematics... To verbalize what one is doing ensures that one is examining it. And it is precisely during such examination of mental operating that insufficiencies, contradictions, or irrelevancies are likely to be spotted." Also, "... leading students to discuss *their* view of a problem and their own tentative approaches, raises their self-confidence and provides opportunities for them to reflect and to devise new and perhaps more viable conceptual strategies" (Von Glasersfeld, 1991, p. xviii, xix).
- Students learn, and learn from each other, by *listening* to and trying to make sense of other procedures and concepts being explained.
- Through classroom social interaction, the teacher and students construct a consensual domain (Richards, 1991; see below) of taken-to-be-shared mathematical knowledge that both makes possible communication about mathematics and serves to constrain individual students' mathematical activity. In the course of their individual construction of knowledge, students actively participate in the classroom community's negotiation and institutionalization of mathematical knowledge (Cobb *et al*, 1991).

The problem-centered learning approach is not prescriptive about how teachers should organize their classrooms to facilitate social interaction between students. In the sections that follow we briefly describe and discuss three teachers' attempts to establish an inquiry mathematics culture in their classrooms. We will present anecdotal evidence to show that students sometimes need to work individually, that when they work cooperatively they function better when interacting with peers of equal ability (contradicting Vygotsky's notion above), and that students understand and cope with learning *through* problem solving and social interaction.

Two classrooms

It was possible to observe and videotape two third-grade classrooms in the same school on a regular basis during the whole of the 1991 school year, as well as to analyze all written work done by the students in these classrooms. According to local educational practice, both teachers had divided their classes loosely into three ability groups, but the range of ability (or "quickness of understanding") within each group was still significant.

The one teacher, Roxanne, established the following procedure. She would pose a challenging problem to a group of students which they would then tackle *individually*. When most of the students had solved the problem, she would initiate a group discussion about the different methods used, requiring explanation and justification from each student. Any errors and misconceptions were identified and resolved *by the group* under her guidance. All strategies were accepted and she tried not to show preference for a particularly short or elegant strategy for fear of pressurizing students into imitating a procedure they did not really understand. Roxanne handled the whole-group discussions very competently; she was

non-evaluative and awarded each student's contribution equal time, encouraging students to discuss not only their strategies but also their reasons for making certain choices. There was no pressure, even covertly, from her for students to adopt certain strategies.

The other teacher. Helen, typically posed a problem to a group of students and required each student to tackle the problem individually. After some time she would suggest that they start co-operating informally in small sub-groups of two or three, sometimes suggesting partners but always leaving the final choice to the students themselves. The small sub-groups were not stable, but broke up and re-formed according to the needs and interests of the different students. The most important social norm which governed this type of interaction was that a student could only be helped with a solution strategy that he had already initiated. Helen spent little time on whole-group discussion and frequently neglected to elicit reasons for certain mathematical decisions taken by students. She did, however, insist that no problem-solving session be terminated unless each student had solved the problem successfully by means of a strategy that *he himself* had initiated, and that it was the *group's* responsibility to effect this, not hers. Methods which had produced erroneous results were analyzed in detail by students until the error was identified, and this was then reported back to the teacher or the group.

Some observations about social interaction. We observed that students in Helen's class seemed to choose peers relatively equal in cognitive ability rather than more able students to cooperate with. We pursued this matter with some lower ability students in a series of informal interviews. Martin commented on his reluctance to cooperate with more able students as follows: "This means that they start telling me what *they* think, and then I don't have the time to think for myself."

The perceptions of the lower ability students can be summarized as follows:

> "We don't like working with the quicker students. We need time to think things over and to talk things over with people who don't think faster than we do. We want to think for ourselves."

Many students' responses indicated that they were explicitly aware of at least one function of social interaction: They did not actually need direct suggestions from their group about possible solution strategies, but felt that working *with* them would definitely enable *all of them* to solve the problems.

This seems a good intuitive description of a consensual domain of taken-to-be-shared meanings and practices that leads to successful mathematical discourse. A consensual domain is established when two or more organisms interact in response to the others, and ".. are acting *as if* they have to come to an agreement regarding their underlying assumptions" (Richards, 1991, p. 18). Richards also states that "Through reflexivity, participants establish that they are in the same ball game. *Only then can we begin to speak about communication*" (p. 18, our italics), and "For communication to occur, *both* participants must have the potential for change" (p. 19). The main difference between Helen's and Roxanne's classroom interaction patterns is that Helen's students engaged in intimate, reflexive discussions with *chosen* peers of near equal ability, whereas Roxanne's discussions involved larger groups with

a wider ability range. It was therefore possible for Helen's students to establish consensual domains where these *could* be established successfully, i.e. where a reflexive relationship *could* be built and where both (or all the) participants *could* (were willing to) change. It seems that students' preference to work with equal-ability peers and not with higher-ability peers is determined by the intuitive realization that they *can* establish a consensual domain with equal-ability peers, but not with higher-ability peers.

It seems from the episodes reported that these students obtain more satisfaction from *constructing a solution to a problem* than from simply producing the right answer (note their expressed need to be allowed to think for themselves), and furthermore, that *they themselves* believe that this process is aided by collaboration with peers of equal ability and not by collaboration with more able students. This intuitive perception of optimal mathematics learning is of course heavily influenced by the classroom culture, which is in turn influenced by the teacher's perspective on the nature of mathematics learning as inquiry; had the teachers at any time intimated that desired learning constitutes the successful (faultless) implementation of prescribed procedures to solve set problems, students' perspectives on optimal learning processes would probably have been quite different, and might have predisposed them towards learning *from* more able peers (or, of course, from the teacher herself), in the school mathematics tradition.

Observed differences in learning outcomes. Although the upper ability students in both classes progressed in much the same way and evolved similar computational strategies, there was a marked difference between the rates of progress of the other students in the two classes (Mathieson, 1992). We briefly illustrate this difference with reference to the development of students' division strategies.

In Helen's class, the middle and lower ability students' computational strategies developed steadily and surely over varying periods of time depending on each student's abilities. There were no signs of students imitating strategies they did not fully understand. The *most primitive* division strategy in use by the end of the school year consisted of a judicious mixture of doubling and the addition of multiples of the divisor (Murray, Olivier & Human, 1992). For example, Freda (lower ability group) solved 988 ÷ 38 as follows:

$20 \times 38 \to 760 \quad 5 \times 38 \to 190 \quad 950 + 38 \to 988$
$988 \div 38 = 26$

In Roxanne's class, progress was much slower. For example, by the end of the school year Nanette, a middle ability student, solved 324 ÷ 18 as follows:

18, 36, 54, 72, 90, 108 ... (continuing to add one 18 at a time until she reached 324, then counting the number of 18s).

The weakest students in Helen's class were therefore performing at a higher level than the middle ability students in Roxanne's class, implying a substantial difference in understanding and skill between the weaker students of the two classes. In spite of the obvious weakness of Helen's approach (little higher-level mathematical discussion), the very great difference in performance between the two classes suggests that a social interaction pattern

where students are initially required to tackle a problem individually but are then encouraged to form loose groups *of their own choice* for further support, results in faster progress and the construction of more stable concepts and procedures than an interaction pattern where students are required to explain, justify, listen and evaluate in larger group context, no matter how supportive and non-evaluative the teacher is. This is not necessarily true for higher ability students — our present (incomplete) data indicate major differences for middle- and lower-ability children only.

A third classroom

During the 1992 school year we monitored a culturally diverse third grade classroom to establish whether other variables than ability would determine students' choice of partners to cooperate with. Although the medium of instruction in that school is English, only 13 of the 32 students in that classroom had English as mother tongue; other mother tongues included Sotho, Tswana, Xhosa (African languages), Cantonese, Korean, Chinese and Afrikaans.

The teacher, Olga, reports that she experienced severe difficulties in establishing a classroom culture in which students were willing to accept responsibility for their own and their peers' learning. After four months the following model had evolved. Olga had divided the class into four ability groups of approximately eight students each. Three groups would be given written assignments or practical tasks to complete, while Olga would take one group and pose a problem verbally to the group as a whole. She would then leave them, because she had come to realize that she tended to interfere if she remained with the group. The students themselves would then decide how to proceed—usually they would discuss the problem first so as to understand the problem, then solve it individually or cooperatively according to choice. Olga did not, like Helen, require that each student initiate a solution method on his own; this rule proved unnecessary in her classroom (see for example Keba's comment below). Cooperation was strictly by choice and usually a number of smaller subgroups of two or three students each would be formed. Sometimes students preferred to work individually, and different students were several times observed warding off someone who wanted to join him, preferring to work on his own. In the final phase, however, the group as a *whole* had to reach consensus about the answer and resolve conflicts. Only then was Olga invited back and a general report-back to her followed, including not only explanations of different *methods* used, but especially also descriptions of initial misunderstandings and incorrect strategies, and how these were resolved.

The academic progress made by the class was extremely satisfactory and on par with other classes in the problem-centered learning approach.

At the end of the 1992 academic year the students were asked to write individual free comments on the problem-solving sessions "on the mat" (as they call their problem-solving sessions on the floor). These comments clearly emphasized five main issues:

- A preference for individual work in some circumstances.

  Andrea: "I like to work alon if the sum is easy."

  Thato: "Sometimes if they don't agree with me I work alone."

- A preference for working with *particular* students. No conflicting choices were made, identification was mostly mutual.

  Jiyun: "I like working with Ziona because she understands me."

  Ziona: "I like to work with Jiyun."

- A clear understanding of *why* discussion and cooperation are beneficial to the learning process.

  Margaret: "...discust the sum and we found the sum becam easy."

  Kanyiso: "When I am on the mat I learn lots of different ways."

  Kenneth: "we discussed our answer than if we got differnt answer than we see were did we did our mistack."

  Rishaad: "It is nise to work in groups becase you will lern good."

- A clear understanding of what constitutes efficient group functioning.

  Keba: "If you working in groups you must not just copy and you must think of diferent ways."

  Dinea: "We explain the sum to the others if they got their answer wrong we dont say it is wrong because we dont nou if ours is right or wrong."

  Dintiragetse: "...if we dont agree then we check our mistaick."

  Oscar: "We discuss the answer when all is rit [when everybody has obtained the right answer] we tell our teacher then we go to our seats."

- A tolerance towards the personal needs of other students.

  This is illustrated in their comments on how mistakes and disagreements are handled; in the way that they respect a student's right to work by himself if he so wishes, but also in statements like the following, which recognizes the need to give everyone a turn:

  Darryl: "I like explaneing to my friends and they like explaneing to me."

There is no indication that any of these five categories of perceptions are peculiar to any of the variables of ability, gender or language. Students' indicated choice of peers for cooperation was found to be dependent on ability, but not on language or gender. The informal small groups identified by the students varied from the homogeneous Jonathan, Alvin and Jason (all English boys), to Arend (English boy), Jo-Anne (Afrikaans girl) and Tshegofatso (Tswana girl). In fact, all the sub-groups except the first mentioned included different home languages.

While Helen suggested social interaction patterns, Olga made no suggestions at all, leaving in entirely up to the students. In as much as each student in Helen's class was expected to assume responsibility for his mathematical growth and to evolve solution strategies which suited him personally, each student in Olga's class was also expected to explore and evolve social interaction patterns according to his own needs.

Conclusions

The conclusion that we are tempted to draw from the above observations is that the construction of mathematical concepts and processes is both more uniquely personal and more dependent on social interaction than we ever realized before. The students reported here demand *both* complete freedom from interference of any kind to construct their own knowledge, *and* a social interaction system that makes possible the establishment of consensual domains to support true mathematical discourse. The successful establishment of these so cial structures is not more difficult, but simply more *sensitive* to outside interference than we have suspected. Students, even very young students, should not only be trusted to construct their own mathematical knowledge, but should also be trusted to evolve the social structures that they need for optimal learning.

Current perspectives on mathematics teaching recognizes the value of *the learning process itself* as part of *doing* mathematics. Our students seem to be intuitively aware of their need to be *allowed to learn* while they solve their problem, a need that is not fulfilled when inappropriate help is provided by a teacher or by a peer. The perspective of learning *through* problem solving as well as learning *to* solve problems is not new; it is, however, interesting to note that even young children are possibly aware of this mode of learning and can, when given the opportunity, create environments favourable to this mode of learning.

Our observations of the nature of the collaborative interactions among students seem to contradict Vygotsky's notion of learning through collaboration with *more capable* peers. It seems that, in a problem-centered environment for mathematics learning, communication and the construction of mathematical knowledge are facilitated when students interact with peers of near equal ability rather than with peers of higher ability.

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TEACHERS' AND STUDENTS' VIEWS ON STUDENT REASONING

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Teachers' ability to help students learn mathematics requires an understanding of student ways of thinking. But this is not sufficient in itself: no less important is what teachers do with this knowledge; i.e., the nature of decisions and actions that teachers make based on this knowledge so that meaningful learning is emphasized and powerful constructions of mathematical concepts are fostered. This paper explores these issues in the light of teachers' and students' views on student reasoning. Two interviews were conducted: One with two junior-high school teachers, and the other with two eleventh grade students. The interviews consisted of two parts: (1) responding to students' questions and ideas, and (2) reacting to teachers' responses. The paper concentrates on teachers' and students' points of view on the issue of helping students to reason about problem situations. Ritual and general aspects of responses are investigated, and the understanding of student ways of thinking is explored.

INTRODUCTION

The last two decades of extensive cognitive research on student learning, has yielded much useful data on student conceptions and thinking in mathematics. Many studies have shown that students often make sense of the subject matter in their own way, which is not always isomorphic or parallel to the structure of the subject matter or the instruction (e.g., Even, in press; Hershkowitz et al., 1987; Kieran, 1992; Markovits et al., 1986; Schoenfeld et al., in press).

Teachers' ability to help students learn mathematics certainly requires an understanding of student ways of thinking. The importance of this aspect of teacher knowledge is a natural consequence of adapting a constructivist point of view of learning. However, only in recent years was the teacher role in knowledge construction recognized (Confrey, 1987; Maher & Davis, 1990; von Glasersfeld, 1984). A teacher who pays attention to where the students are conceptually, can challenge and extend student thinking and appropriately modify or develop activities for students. Starting from students' limited conceptions the teacher can help build more sophisticated ones. However, teacher understanding of student ways of thinking is only the first stage. No less important is what teachers do with this knowledge; i.e., the nature of decisions and actions that teachers make based on this knowledge so that meaningful learning is emphasized and powerful constructions of mathematical concepts are fostered.

Our research project aims at investigating two interrelated aspects of pedagogical content knowledge: A passive one being teacher knowledge and understanding of student conceptions; an active one, teacher responses to student questions and hypotheses concerning subject matter. The goals of the research are: (1) to study teacher pedagogical knowledge according to those two aspects, and (2) to investigate the potential use of activities based on the research in teacher education.

The contextual content in which we conduct our research is mathematical functions. We selected this topic as the focus of the study both because of its importance in the discipline of mathematics and in precollege and college mathematics curricula, and also because researchers have provided interesting findings about students' knowledge and understanding about mathematical functions.

This research project is multi-phased. The first phase (Even & Markovits, 1991, in press; Markovits & Even, 1990) investigated teacher awareness of student conceptions, and kinds of teacher responses to student questions and hypotheses. This paper concentrates on the second phase which is built on the initial findings from the first phase. After analyzing how teachers react to other teachers' responses in the first phase, we examine students' reactions to teachers' responses as well. We also look closely at the interaction between teachers and the interaction between students when they react to teachers' responses. In addition, after analyzing more than one hundred questionnaires in the first phase in order to get a general impression of teachers' ideas, we pursue a deeper investigation of two issues that emerged from the analysis of the first phase which are critical for meaningful learning: the help given to students (a) to reason about problem situations, and (b) to construct knowledge. This paper reports findings related to the first of these two issues.

METHODOLOGY

The subjects in this phase of the study were two teachers and two students. The students were good eleventh grade students. Both studied advanced mathematics courses and were very good friends. The teachers also knew each other very well. Sharon, one of the teachers, has taught junior high mathematics for eleven years. Dan, the other teacher, has taught mathematics for five years in junior- and senior-high school. Both teachers were finishing a two year in-service certification program for junior-high school teachers, in which they were considered as two of the best participants with respect to mathematics knowledge and mathematics teaching.

Two interviews were conducted; one with the teachers and one with the students. We chose to interview them in pairs in order to allow conversation and exchange of ideas. For this reason we chose subjects who knew each other very well. The interviews consisted of two parts: (1) responding to students' questions and ideas, and (2) reacting to teachers' responses.

In the first part the following three situations (see Figure 1) were presented in written form, one at a time. The subjects were asked to read the situation and then to explain what they think the student had in mind, and how they would respond. In the second part of the interview, the subjects were presented with responses that were given by other teachers (from the first phase of the study) to the same situations (see Figure 2), and were asked to react to these responses. Each response highlights certain characteristics of the help given to students (a) to reason about problem situations, and (b) to construct knowledge. Three teacher responses were presented for each situation. In response A.1 the "rule" is just repeated in different words; response A.3 does not suggest anything new to what the student has probably already done. Only response A.2 aims at helping the student to reason. In B.1 it seems that the teacher, suggesting that the student did not understand the whole topic, did not understand the student difficulty. Therefore, s/he neither could help the student to reason nor help him/her to construct his/her own knowledge. In C.1 the teacher is not specific about the student misunderstanding. Responses B.2, B.3, C.1, C.2 and C.3 illustrate the difference between "telling" the student the answer,

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Situation A

A student tells you that he noticed that the graph of a quadratic function $f(x) = ax^2+bx+c$ looks like $\cap$ when $a<0$, and like $\cup$ when $a>0$. The student asks you why this happens. How would you respond?

Situation B

A student was asked to find the equation of a straight line through A and the origin O (Fig.1).

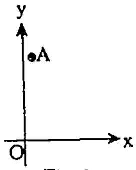

Fig. 1

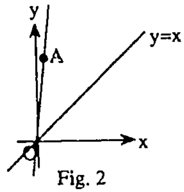

Fig. 2

The student said: "Well, I can use the line y=x as a reference line. The slope of line AO should be about twice the slope of the line y=x, which is 1 (see Figure 2). So the slope of line AO is about 2, and the equation is about y=2x, let's say y=1.9x."

a) What do you think the student had in mind? b) How would you respond?

Situation C

A student is asked to give an example of a function that satisfies $f(2)=3$, $f(3)=4$ and $f(9)=15$. The student says that there is no such function.

a) Why do you think the student answered this way? b) How would you respond?

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Figure 1. The Situations

and helping the student to construct a solution by and for him/herself. Response C.3 reflects "superficial constructivism": the student is involved, but only in a relatively technical activity. He is asked to plot the points, while the teacher is the one who does the actual solution by drawing the function.

HELPING STUDENTS TO REASON

Teachers

Ritual and general aspects of responses. The teachers differed in the way they responded to the situations presented to them. At first Dan seemed to care about responding in a way that would be meaningful for the student. He said that his reaction to Situation A would depend on the stage where the student was. Also, when presented with the teachers' responses, Dan stated that one of the responses is too complicated for a student. Dan seemed to realize that meaning is subjective and not objective; that

the student's background and knowledge have to be taken into consideration when deciding on the kind of reaction.

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<u>Situation A</u>

A.1 When $a<0$ the function has a maximum for some point, and smaller values for the others. But when $a>0$ the function has a minimum and higher values for the other points.

A.2 $x^2>0$, if $a<0$ then the term will be negative and will grow in absolute value. That's why the function increases in the domain where $x<0$. $a>0$ the term is always positive and the function increases for $x>0$.

A.3 Substitute some points in particular examples and you will see why.

<u>Situation B</u>

B.1 The student has some idea about slope, but the idea is not well established. The topic of slope and the way slopes are calculated should be reinforced.

B.2 The student's claim was based on the idea that if the angle is twice as big, so is the value of 'm'. I would ask the student to draw the graph of the function $y=2x$ on the given drawing, so s/he can see the angle when $m=2$.

B.3 The student meant the angle and not the slope. In order to clarify for him the difference between angle and slope, I would draw several examples: $y = x$, $y = 2x$, $y = 3x$. Thus the student will understand the difference between slope and angle.

<u>Situation C</u>

C.1 The student does not understand the concept of function. I would show him an example of a function:

$$f(x) = \begin{matrix} x+1, & x < 3 \\ x+6, & x > 3 \end{matrix}$$

C.2 He gave that answer because he could not think of a rule of correspondence, and that is why he said that there is not such function. I think that I would plot the points A(2,3), B(3,4), C(9,15) and show the student that one can draw an infinite number of functions through these points.

C.3 The student meant that there is no such linear function. I would ask the student to plot the points and then I would draw a function through the points and ask him if the graph describes a function.

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Figure 2. Teacher Responses

But then Dan gave two optional responses to Situation A--both lacked an orientation of helping students to reason about the situation. The first suggestion had a strong ritual component. It was aimed at students who had not yet formally learnt how to analyze a quadratic function. Dan suggested telling the student (the emphases in the following quotations were added): "We <u>know that</u> the coefficient 'a',

that means in the meantime, at this stage it is given that the coefficient 'a', the coefficient of x^2 determines the direction of the branches upwards or downwards." Interestingly, Dan remarked that he was asked the very same question in his 10th grade class, which had not learned yet how to analyze quadratic functions. As he recalled, his answer then was the same: "Look, at this stage, we haven't yet learned it, just accept it: the [coefficient] 'a' determines the direction of the function." The same attitude also came into play when Dan chose response A.1 and stated: "It is a good answer. Just accept it. When a>0 then maximum."

Dan's second suggestion to Situation A was aimed at students who had already learned how to analyze quadratic functions. He suggested that an appropriate response would be: "We have already seen, during the course of our study, that the coefficient of x^2 determines the location of the vertex upwards or downwards." This kind of general answer would not seem to have the potential to help the puzzled student reason more than he could before.

In Situation B Dan responded in a similarly general way. He "sent" the student back to his notes: "I would send him back to the material that we have learned, to see how we calculate a slope. I would tell him that there is a lot of stuff in his notebook about slope because we spent a lot of time on slope." Again, besides reminding the student that he has already learned this material, this kind of answer does not do much for developing the student's understanding. Only when presented with the teachers' responses, in which two suggested to sketch graphs, Dan decided that this is a good idea. He remarked that by sketching the graphs the student would see the angle and realize what the connections are.

Situation C was the first in which Dan tried to respond in a way that aims at developing student reasoning from the beginning. He suggested to ask the student to draw a coordinate system and represent the points on it. Then he said that he would ask the student if there was a line that passed through those three points. Does it have to be a straight line? Is it the only one? Is it a function at all? By asking this kind of questions, Dan assumed that the student would realize that there are functions which are not straight lines as well as that there are infinitely many functions that pass through the three given points: "At the end of this process we will reach the conclusion, I assume that we will reach the conclusion that... ." This approach of asking questions whose aim is to help the student reason about the situation and develop a better understanding of the concept of function is very different from the approach Dan used when he responded to Situations A and B.

In contrast with Dan's responses, Sharon seemed to care about responding in ways that would be meaningful for the student and encourage him/her to reason mathematically throughout the three situations. Her responses were always specific and not general or ritual oriented as Dan's. She, for example, suggested to the student to substitute numbers and sketch graphs for Situation A; use square paper in Situation B and check whether for every horizontal step there are two vertical steps; tell the student that there are infinitely many functions that can pass through the three given points in Situation C. She also took into consideration the student's background and knowledge when deciding on the kind of response to Situation C: She said that she would not choose response C.1 because piece-wise functions are too complicated for students at this point.

Understanding student thinking. A closer look at Sharon's responses revealed that even though it may seem at first glance that her responses were meaning oriented, aiming at fostering student reasoning, this was not always the result. The cause for this mismatch seemed to be her misunderstanding of the student's ways of thinking. She, for example, suggested to the student to substitute numbers and sketch graphs for Situation A in order to see that "when 'a' is negative it is upside down and when 'a' is positive it is straight". But this suggestion does not relate to the student's question of why this happens. What Sharon actually did was to repeat what the student has already known without relating to his specific "why?" question at all.

After both teachers finished to discuss their own and the other teachers' responses to Situation A, it appeared that they still did not relate to the student's question: "why?" The interviewer then remarked that actually the child came to the teacher already saying that he noticed the situation and asking why this happens. Interestingly, this remark changed neither Sharon's nor Dan's response. Dan did not respond at all to the interviewer's remark. Sharon claimed that the student could use specific cases to draw a general conclusion. She stuck to her response even after the interviewer remarked that "he [the student] has already drawn the conclusion. He has noticed that the graph of the function is... ."

While the mismatch between the student's question and the teachers' responses to Situation A was not immediately apparent, Dan explicitly admitted that he did not understand the student's way of thinking in Situation B. At first Dan suggested to ask the student how he determined that the slope is 2. Sharon then raised the issue of the angle ratio of 1:2. She said: "I think that he said that approximately this angle equals that one, so if this is one, then that is two." Only then Dan realized that he did not understand the student. He said: "Actually, now that you are saying. I did not think that this is one and that is two. Yes, I did not think in this direction... ."

Situation C was the only case where both teachers seemed to understand and concentrate on the student's way of thinking. Illustrations for that were given in the previous section.

Students

Ritual and general aspects of responses. The students, unlike the teachers, were more similar in their consideration of meaning. They cared very much about responding in a way that would be meaningful for the student and develop his/her mathematical reasoning. They themselves did not always know how to do that but were very critical about previous experiences with their own teachers as well as the teacher responses presented to them. For example, when presented with Situation A, Eleanor said that when she studied this material she had a lousy teacher who, whenever was asked such questions used to say: "That's the way it is"--an answer similar to the one that Dan gave. Both Eleanor and Dafna strongly criticized this kind of answer.

The students were also critical of the teacher responses presented to them. For example, they claimed that A.1 is not an explanation. "All it does is giving him [the student] another rule without an explanation", said Dafna and Eleanor agreed with her.

The students considered A.2 (which emphasizes reasoning) to be the best explanation, something that neither Dan nor Sharon did. However, after debating the issue among themselves, they felt it is incomplete and suggested a way for improving it. The students, unlike the teachers, searched for a response that has the potential to foster student mathematical reasoning. More than that, they criticized other responses presented to them with respect to reasoning developing--something that the teachers did not do.

The students were also critical about the teacher responses to Situation B. They did not like B.1, (the one that Dan chose) because it was too general. Dafna said that the teacher has to be more specific and focused. Eleanor was even more extreme and categorized this response as lousy.

Understanding student thinking. As we saw earlier, both students understood the presumed student's way of thinking in Situation A. They, themselves, were not sure how to respond, but recognized that the teachers were not concentrating on the student's specific question of "why does this happen?" The students opposed teacher response A.3 that does not explain "why": "It does not explain to him why it is. It just shows him that it is really like this." Interestingly, Sharon, the teacher who seemed to care about responding in a meaningful way, did choose this response.

As with the teachers, one participating student understood the student's way of thinking in Situation B while the other did not at first. Eleanor read the three teacher responses and commented: "What is the connection to an angle? He did not say anything about an angle." Only after Dafna explained to her that one angle was twice the other, Eleanor realized the connection. The student's presumed way of thinking in situation C was understood by both students.

CONCLUSION

Helping students to reason is extremely important in learning to do mathematics meaningfully (Lampert, 1988; Lappan & Even, 1989; NCTM, 1989, 1991; Schoenfeld, 1992). The teacher's role in developing students' mathematical reasoning is crucial because "mathematical reasoning cannot develop in isolation. ...The ability to reason is a process that grows out of many experiences that convince children that mathematics makes sense" (NCTM, 1989, p. 31).

In order to help students to reason, teachers need to understand student thinking and conceptions, and respond in ways that challenge and extend student thinking, so that meaningful learning is emphasized and powerful constructions of mathematical concepts are fostered. This phase of the study, similarly to the first-phase findings (Even & Markovits, 1991, in press; Markovits & Even, 1990), illustrates several cases where teachers ignore student ways of thinking and their sources. Even when teachers understand the student's difficulty, they sometimes respond in ways that are too general (e.g., they decide to explain the whole topic again) or emphasize rituals instead of pertain to meanings.

Interestingly, the students in this study concentrated on helping students to reason mathematically more than the teachers did. The students criticized teacher responses with respect to the help given to students to reason. They tried to work cooperatively towards a common goal: finding responses that

emphasize reasoning. It seems that the method used in this study for research purposes, has the potential to be useful in the classroom for instructional purposes. The use of open-ended questions based on vignettes describing hypothetical classroom situations involving mathematics, which has proven useful in teacher education (Even & Markovits, 1991, in press), has the potential to elicit students' knowledge, and to create a community of learners in the classroom with considerable responsibility for judging, validating and helping others.

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Teaching Functions

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Abstract
We report about a study how to teach the concept of (linear and quadratic) functions by the use of computers. With a simple functionplotter software and special worksheets we emphasize guess-and-test procedures ("One-Way-Principle") to concentrate on the equation↔graph relation without the use of tables.

1. The One-Way-Principle

Basically there are three main aspects for the concept of function and therefore six operational modes of transfer:

$$\begin{tikzcd} & \text{Table (Set of pairs)} \arrow[dl, leftrightarrow] \arrow[dr, leftrightarrow] & \\ \text{Equation} \arrow[rr, leftrightarrow] & & \text{Graph} \end{tikzcd}$$

But in mathematics education traditionally tables are the only link between equation and graph. This may explain the difficulties of many students to identify a given graph with the appropriate equation and vice versa. To strengthen the equation↔graph relation we use the computer.

Our approach in using the computer is based upon a psychological fundamentum. We distinguish two modes of working with the computer, the syntactical use and the semantical use (Meissner 1983). Working syntactically means pressing keys and waiting for the output to take it as the result. There is a lot of syntactical working in mathematics education, e. g. paper-and-pencil algorithms or pressing calculator keys. You work on a sequence of symbols, step by step. You just concentrate on the actual step, forgetting quickly all the symbol-manipulations which already are done. Numbers and operations (like addition or multiplication) degenerate to a sequence of meaningless symbols and thus also the result often is taken as a (meaningless) sequence of symbols. The syntactical working very often leads to an *instrumental understanding* only (Skemp 1976).

In the semantical use the computer is a computational aid. There is a guess or a numerical assumption at the beginning and the computer result will tell you how good (or bad) your assumption was. Example: You bought paper clips for 10 $. You got 76 pieces and 12 cents back. How much is a paper clip? First you guess 15 cents. You press $\boxed{1}\,\boxed{5}\,\boxed{x}\,\boxed{7}\,\boxed{6}\,\boxed{=}$ to get 11.40 $. Not very good, but your next guess with 13 cents will be excellent. Or: You get the graph of a function and need the equation. You make a guess, the computer shows you the graph of your guess, and you get hints for a better guess.

For one formula there are different problems, i. e. a "family of problems", according to which of the variables are given and which variable is unknown. Mathematicians use algebraic transformations to transfer the "usual" formula into that formula which allows a syntactical calculation to get the value of the unknown variable. Examples: Percentages, volume or area of geometrical solids, ..., and also the four basic operations or functions. But we can avoid the algebraic transformations. We just make a guess and work syntactically with the "usual" formula. The output then gives a hint for a better guess.

Our observations show, that many young students (but also many adults) often work like that (Meissner 1982). They have only an intuitive, global approach to a given problem. They are not conscious of a formula or a function, but they know a procedure how to compute (syntactically) to get the result: They unconsciously apply a formula. This is the reason why one problem is quite easy and why the other types of problems of the same "family" are much more difficult. For those problems you first need the formula consciously and second you must run through algebraic transformations. Also in daily life we often observe guess-and-test procedures. Our problem solvers do not do an algebraic transformation or select the inverse operation, they just make a guess and use the already known procedure to find out how good the guess was (see example paper clips above).

This is the idea of the One-Way-Principle (OWP). We use the calculator or the computer to train the easy type of the "family" (syntactical use) and then all the other types are solved by guess and test (semantical use). Thus the "family" of these problems becomes a unity, all described by the one same (simple) sequence of calculations or key strokes (Meissner 1979).

We think this is true also for the unity "equation-graph" of a function. We start syntactically with the one simple sequence of key strokes to get the graph for the given equation. Then we continue semantically by guess and test. Here the graph is given. Make a guess for the equation and let the computer show you the graph of your guess. Compare this graph with the given graph. Will you try again?

The OWP is a teaching method. It starts with a syntactical use of the calculator or computer. This "one way" will be trained to become an easy skill for the learner. Related problems then will be solved with the same skill as a tool, if necessary by guess and test. There will be no reverse operations or algebraic transformations or inverse functions. We remain in the one-way-direction, therefore One-Way-Principle.

Each student works on the OWP-level and thus individually has the chance to take off to a more abstract understanding. Starting with instrumental skills the OWP leads via guess and test (semantical use) to a *relational understanding* (Skemp 1976). "Functions as a process" develop to "functions as an object" (e. g. see Dreyfus 1990).

3. Design of the Study

We will use the computer and the OWP in a sequence of lessons with the topic "Linear and Quadratic Functions". In the heart of these lessons there is a set of self guiding worksheets.

Together with a computer and a function plot program this material allows the students to study the relations between graphs and equations of functions without the help of a teacher.

3.1. Hardware and Software

From the very beginning we planned our lessons for the kind of hardware one usually finds in German schools (e. g. elder IBM compatible computers or Atari, black/white monitors, no harddisk, no plotters) and for standard function-plotter software which one easily can buy or can get as a public domain software for those computers. The development of own software is too expensive and if at all then there is no guaranty of access to everybody or of a general acceptance. In this case it also is not necessary because of so much existing good and cheap function-plotter software. But the standard software does not guide the learning experiences of the students. Therefore we needed supplementary working materials (see 3.3).

3.2. The Teaching - Learning - Concept

In our lessons we want to give the students many examples and counterexamples for the concepts to be taught before any systematization. Concepts are constructed and used context-bounded, as the theory of "subjective domains of experiences" shows (Bauersfeld 1983). Also the distinction between "concept images" and "concept definition" (see e. g. Vinner & Dreyfus 1989) proves that the performances of students are more influenced by practical experiences with examples of a notion than by a mathematical definition they have been taught. Therefore it does not make sense to define, analyse or systematize before students possess a rich repertoire of examples and counterexamples and a variety of experiences with them. According to the One-Way-Principle this means that first a lot of (syntactical) tasks "equation→graph" are solved by using the function plotter. Then the tasks "graph→equation" are solved by guess and test. In this case to guess an equation means to find the right type of function and to estimate the parameters, e. g. the slope of a straight line. Thus estimating is a highly semantical activity.

Most of the time in our lessons the students work by their own, with the help of a set of worksheets and a computer plot program. Usually two students use one computer. Both of them have their own set of written materials. The teacher more is a "manager" of the students' activities than a "direct instructor". The teacher helps when technical problems occur, distributes the worksheets, and is at the students' disposal when they have questions. But there are extremely few hints for solutions or suggestions by the teacher, and no advice at all for formulations etc.

3.3. The Worksheets

Following the mentioned principles of instruction we developed a series of worksheets for linear and quadratic functions. As an example we will summarize worksheet QF 1, the first worksheet about quadratic functions:

1. Let the computer draw the graphs of the quadratic functions with the following equations: $y = x^2 + 2$, $y = x^2 - 3$, $y = x^2 + 4.5$, $y = x^2 - 1.5$ and $y = x^2$. - Write down what all the graphs have in common. - What distinguishes the five graphs? - What distinguishes the five equations? - What do all equations have in common? Write a general equation.

2. Which relations do you see between the shape of the graph and the equation? Write down your conjectures.

3. How does the graph of the quadratic function with the equation $y = x^2 + 1$ look like? Draw it with a pencil into the (given) coordinate system. - Let now the computer draw the graph of $y = x^2 + 1$. Copy it with a green pencil into your coordinate system. Copy first some characteristic points of the computer graph. Then label this graph with $y = x^2 + 1$.

4. How does the graph of $y = x^2 - 2$ look like? Draw it. - Let the computer draw the graph of $y = x^2 - 2$. Copy it in green (chracteristic points). Label the graph with $y = x^2 - 2$.

5, 6. You see (picture given) the graphs of two quadratic functions. Which equation belongs to graph 1? Control whether it was the right equation by typing it into the computer. Does the computer show the same graph? - Which equation belongs to graph 2? - Type this equation into the computer. Does it show graph 2?

7. Which quadratic function is represented (graph given)? Write down the equation. - Control with the help of the computer.

8. Do you want to re-formulate your conjectures from task 2? If yes, do it now.

In contrast to the traditional curriculum we directly start with the equation↔graph relation. Two types of tasks have to be solved, type 1 (equation→graph) and type 2 (graph→equation). Type 2 (tasks no. 5, 6 and 7) demands guessing an equation and testing it with the function plotter. For the students, who do not have any experiences with quadratic functions, this type of task in the beginning only can be solved by blind guessing. But every guess enriches their knowledge and guessing becomes a more and more accurate estimation. To shorten the phase of blind guesses we invite the students to let the computer draw the graphs of several quadratic functions (task no. 1). But also in the simpler tasks of type 1 (no. 3 and 4) the students are challenged to guess and test. The computer allows the further control of their predictions.

We know from calculator games (Meissner 1987) that, while working on tasks like these, the students build up their own "theories" (consciously or unconsiously) about the hidden relationships. These theories they then use in the next task. They check their thesis, improve it, use the improved thesis again and so on. From time to time we ask the students to formulate their experiences (no. 1, 2). In the beginning we allow them to use their own language, their own notations and names they invent. Lateron we offer the opportunity to re-formulate these summaries when they got more experiences (no. 8).

We developed several worksheets about linear and quadratic functions which are of the same structure like the given example. They treat the following topics:

Linear functions:	Quadratic functions:
$y = mx$, vary m	$y = x^2 + c$, vary c
$y = mx$ and $y = -mx$	$y = (x - b)^2$, vary b
$y = mx + b$, vary b	$y = (x - b)^2 + c$, vary b and c
$y = mx + b$, vary m and b	$y = ax^2$, vary a
	$y = ax^2$ and $y = -ax^2$
	$y = a(x - b)^2 + c$, vary a, b and c

In both types of tasks the feed-back the computer gives is a graph on a screen which has to be compared with a graph on a worksheet. To allow a real comparison, which also reveals the degree of differences, both graphs have to appear on the same medium, here the worksheet. Therefore the students have to copy the graph from the screen into their worksheet with the help of some "characteristic points". It is their decision which points they choose. These copies from the monitor are painted always in green (green = computer graph). For their own painting the students have to use another colour than green. The use of the two colours allows easily to see what has been done by the students or by the computer. (This is also important when analysing these worksheets lateron as learning protocols).

There are additional worksheets besides the systematic ones we just introduced here. We developed several sheets with exercises which the teacher can give to those students who work faster than the others or who have a need of additional exercises concerning special aspects. The tasks on these sheets are sometimes more complex, but also more interesting. They involve "nice pictures", there are tasks with infinite many solutions or tasks where a rough approximation is sufficient. (We use similar ideas as in Dugdale 1982.) And there is also a self-guiding worksheet about how to handle the computer and the software. It allows the students with absolutely no computer experiences to handle the new technology appropriately.

4. The Investigation

We tested our material with several groups of students in 5 voluntary courses for weak students. In these courses the students had the chance to repeat what already had been taught in grades 8 and 9. The courses took place in the afternoon, after normal school time, normally 2 blocks weekly of two lessons each in two or three successive weeks, and they all were taught by Susanne Mueller-Philipp. In four gymnasiums of Muenster one course each was established for students of grade 10, and in one of these schools we had one additional course with a group of 12 weak students of grade 11, who ought to repeat linear and quadratic functions before starting calculus.

From 36 students we now have a complete set of data, that is pretest and posttest (see below) and all the worksheets they worked upon. The average number of lessons includes one lesson for an introduction into the course and the handling of the computer, but excludes the time for testing. The posttest was given immediately after the course. It is important to mention that none of our experimental students had any experiences with function plotters, almost all had absolutely no computer experiences.

group	girls boys	age (Ø)	lessons (45 min)
A	7 + 1	15.1	7
B	2 + 0	15.0	6
C	3 + 5	15.6	7
D	5 + 1	15.3	8
E	5 + 7	16.6	6
all	36	15.7	

The students who participated in the courses were selected according to their performances in a pretest which was carried out in a normal school lesson shortly before the courses started. Altogether 14 classes were tested (295 students) and the students with bad results in this pretest were recommended to take part in our course. Because participation was voluntary we did not always reach the students with the worst test results.

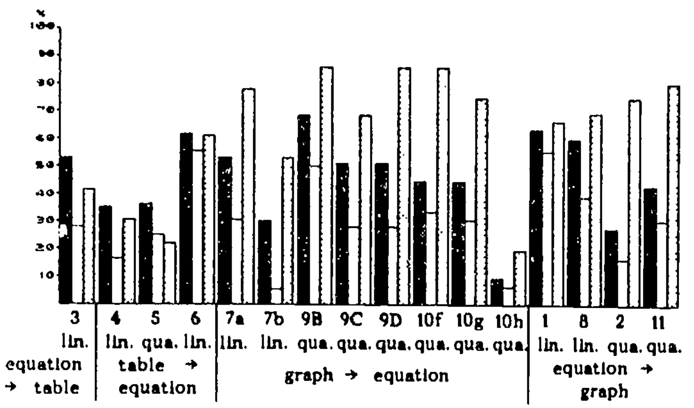
The pretest was a paper-and-pencil test closely following what is usually taught in grades 8 and 9 in a German gymnasium about linear and quadratic functions. The emphasis of the test is on the ability to translate graphs into equations and vice versa. (The translations equation↔table were tested by three items, the translations table↔graph not at all). All calculations and drawings had to be done on the test sheets so that we got as much informations as possible about the solution paths. There were two parallel test versions. The second version was given as posttest to our experimental students. Of course the test was not based on any computer experiences.

5. Results

Our experimental students after the course worked much faster and on more problems:

	pretest all	pretest exp.gr.	posttest exp.gr.
test time	29.9 min	33.6 min	21.4 min
worked on	12.7 probl.	10.9 probl.	14.0 probl.

There was an increase of correct solutions for the 16 test items:

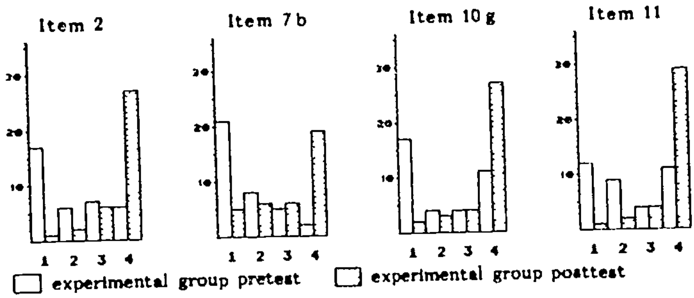


Percentage of correct solutions: ■ all students pretest (n = 295)
□ experimental students pretest (n = 36)
⬚ experimental students posttest (n = 36)

Still rather bad are the performances on item 7b (the equations $y = 1{,}5\ x - 2$ respectively $y = -1{,}5\ x - 2$ had to be found) and on item 10 h (we looked for the equation of a stretched and horizontally and vertically transformed parabola).

We also observed a change of solution strategies. While in the pretest half of our experimental students solved or tried to solve the task to draw a graph of a linear function (item 1) and of a quadratic function (item 2) by calculating long tables of ordered pairs, in the posttest only two students did so, each of them in one task only. And in the posttest more students used a graph to find an adequate equation to the given table (items 4 and 5).

The percentages of correct solutions give only limited informations about the understanding for the relationship graph $\leftrightarrow$ equation. E. g. they do not distinguish between "complete nonsense" and "almost right". Therefore we classified the students' "solutions" on a scale from 1 to 4. We present some examples. The improvement of understanding in the posttest is obvious. Classifications 1 and 2, which dominated in the pretest, diminished, while the classifications 3 and 4 clearly increased:



1. obviously no understanding (e. g. drawing a straight line as the graph of a quadratic function)
2. some small correct relationships (e. g. writing a linear equation belonging to a given straight line, but totally wrong parameters)
3. correct idea, but small mistake(s) (e. g. forgetting the stretching factor of a parabola or having the correct slope but with the wrong sign)
4. correct solution

We summarize. The results of our study are more solution attempts, less test time, better solution strategies, better performances, and an improved understanding. The study shows that an instruction like the one described here seems to help students to build up a relational understanding for the concept of function. An analysis of the students' worksheets which is not yet finished will give us more informations about the learning processes we initiated.

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GEOMETRICAL PROOFS AND MATHEMATICAL MACHINES
AN EXPLORATORY STUDY

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We are reporting the early results of an exploratory study on geometrical proving at high school level (11th grade). The teaching experiment which the data are from is a part of a research project for high school, where geometrical reasoning and historical perspective are introduced to frame students' own activity on a special kind of cultural artifacts, i.e. the so-called mathematical machines. We shall describe and interpret a short segment of a teaching experiment concerning the shift from the level of expérience mentale *to the level of* démonstration *as defined by Balacheff in his study of the process of proof. Our research is not framed by the extension of the concept of reflective abstraction by Piaget (as in Dubinsky) but has been developing by means of appropriation of concepts elaborated by activity theory (e.g. Vygotsky, Davydov, Leont'ev), where the teacher's role as well as the function of object-oriented action are meant differently than in piagetian frameworks.*

1. Introduction: short presentation of the research project.

The teaching experiment where the data are from aimed at introducing 11th graders to proof in geometrical setting. It was a part of a broader research project for high school developed in cooperation between high school teachers and university researchers. The leading motives of the project can be described under the following keywords: *history - geometry - machines* (Bartolini Bussi & Pergola to appear). We shall briefly discuss only the third, to frame the description of a crucial part of a teaching experiment and the interpretation of the teacher's role and the function of object-oriented action.

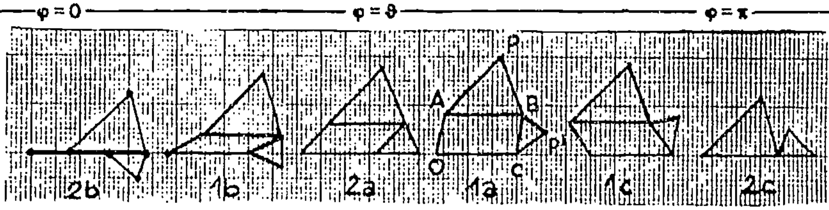
2. The semantic field of mathematical machines.

Mathematical machines lie in the intersection between the field of mechanical experience and the field of geometrical experience. Machines are based on physical as well as on geometrical principles: the former concern materials, energetic sources, the distribution and regulation of acting forces and so on, while the latter concern the relative motion of each part, the trajectory of each moving point and so on. The separation of the above principles dates back to the start of modern science (e.g. the discussion of *material hindrances* in *Two New Sciences* by Galileo), even if only at the beginning of the 19th century Ampère (1830) explicitly proposed the creation of *kinematics* as *the science of motion apart from the causes of motion themselves*. In the following decades the so-called *kinematic geometry* was developed to exclude time process (e.g. speed, acceleration) from consideration. Most of the machines built in our project can be studied within kinematic geometry: their action is *to force either a point or a line or a whatever geometric figure, supported by a suitable material structure which makes it visible, to either move in the space or be transformed according to an abstract mathematical law* (NRSDM 1992).

An example of such a machine is given by the *pantograph of Sylvester*, a mechanism *(linkage)* consisting of eight articulated bars, fixed to a tablet by means of a pivot in the point O. Whatever the configuration of the linkage, the points P' corresponds to P under a rotation around O of the same angle θ (the *invariant* of the linkage). Sylvester himself (1875) described this *pantigraph* (sic!) as a generalization of

the pantograph of Scheiner (1631), already used by Renaissance painters as a tool for perspective drawing. Later pantographs were considered as elements of the *theory of linkages* where different individual linkages were studied as members of a theoretical system. In this theory more general proofs were given, which could apply to the pantograph of Sylvester as well as to other linkages (Lebesgue 1950).

$\varphi = 0$ $\varphi = \vartheta$ $\varphi = \pi$



the strip illustrates configurations for increasing values of the angle φ = AOC from o to π (ϑ = PAB)

fig 1-2

The history of pantographs represents the genesis of contemporary *scientific-theoretical knowledge* (contrasted to *empirical knowledge*, that refers to more ancient phases as well as to contemporary common sense knowledge) as based on the *dialectical relationship between specific and general, between concrete nd abstract*. If the starting point for the construction of a *concept* is the *substantial abstraction from the concrete* (realized by means of the *analysis of function of a certain relation of things within a structured system)*, its result is a *developed theory*, where the specific manifestations are deduced and explained from their general foundations (the so- called *ascent from abstract to concrete, from general to specific* - Davydov 1972-79 ch. 7). In this process, there are *experimental phases* too where individual examples are observed according to questions that are *theoretically formulated.*

Every machine embodies both empirical and theoretical knowledge and can be read in many different ways. Whether the analysis of a machine is empirical or theoretical depends on *which questions are posed*; morevover, we have different theoretical analyses of the same machine, if the reference system is changed (e.g. elementary geometry, theory of linkages). According to the above discussion, it is not possible for the researcher to split this aspect of the human experience without falling into a reductionist perspective (the empirical use of a pantograph; the study of its configurations at the abstract level only): in this sense, the context of mathematical machines is an example of *semantic field* (Boero 1989).

The above interpretation is not within student capability: they have separate either geometrical or mechanical experiences, at the perceptual as well as at the rational level. A possible dialectic between them is known by the teacher who is aware of the historical development of geometry as strictly interlaced with mechanics. Yet for students machines are opaque for geometry (and viceversa) and can become *transparent*, only by means of a suitable *activity* (Meira 1991). Students do not necessarily recapitulate the historical process of creating or discovering an artifact and its properties, as it had been realized by a collective complex process that lasted centuries and that depended on a lot of individual and cultural

factors from inside as well as from outside mathematics: because of its complexity and of the dependence from so many factors, it is not a good candidate to modelize the classroom process. Student activity could be better described as the *appropriation* of an existing cultural artifact (e.g. the pantograph) as well as of the modes of its study (appropriation is meant in the sense of Leont'ev (1964-76) as *the process that has as its end result the individual's reproduction of historically formed human properties, capacities and modes of behaviour*) . Learning is not based on individual immediate relationship to reality but rather on the *mediation* (Vygotsky 1978) between individuals and objects, realized by means of the artifacts - either tools or symbolic systems - created by the human beings in their century development. *No personal individual experience, however rich it may be, could lead to the development of either logical or abstract or mathematical thinking and to the autonomous formation of the corresponding conceptual systems* (Leont'ev 1964-76 p.338).

3. Some implications for didactics.

The above discussion, that alludes to some basic ideas of activity theory, gives rise to two theses:

1. Dialectic between specific and general is a basic motive for teaching-learning activity in school setting. Object-oriented action, as concrete manifestation of a research task, is the driving force for the teaching-learning process.

2. The teacher is the responsible for mediation in the classroom, to introduce students in the systems of cultural artifacts, that have been produced by generations of human beings.

In the following, we shall argue how the above theses were realized in the planning of teaching experiments as well as in the functioning of classroom processes.

4. A teaching experiment.

This teaching experiment was carried out in a 11th grade classroom and concerned a detailed study of different linkages by different groups of students. Four-month observation was fulfilled during the school year 1991-92. Both teacher's lectures and the group work of five students (engaged in the study of a specimen of the pantograph of Sylvester) were recorded. We do not detail the overall structure of the teaching experiment (Bartolini Bussi & Pergola to appear), which consisted in an *introductory lecture* concerning some historical notes about linkages and their study, two two-hour sessions of *small group work* concerning the study of five different linkages by five different groups of students and a *whole class study* of each individual linkage, lead by the teacher.

We shall briefly describe (this section) and interpret (§5) a part of the small group work. It had been structured in advance by means of a written list of eight questions (app.1) to be answered in writing. An excerpt of the final text edited by the observed group is in app.2. We shall refer to the drawing (fig. 3) that was produced by the students together with the written text, even if the process had been carried out pointing at the linkage without referring to any coded point.

Answering the *first question* (one full hour) resulted in the sequence of the following partial answers:(1) some bars are equal (direct measuring); (2) there is a deformable parallelogram and two

undeformable triangles; (3) a point (O) is fixed; (4) the triangles are isosceles (observing the configuration of fig.2a); (5) the triangles are similar (idem); (6) it is equivalent to fix the angle PAB or the ratio of PB to PB'; (7) the bars representing the bases of the triangles have to be cut and pierced in order to have the same proportion as the bars representing the sides (details in Cavani 1992).

The process of answering the *fourth question* resulted in three distinct phases: (1) conjecturing the invariant; (2) looking for a proof; (3) writing down the proof. The first phase was solved by means of a joint activity with the teacher. The conjecture was proposed by a student, accepted by the teacher, but rejected by the others and accepted after checking the linkage.

The second phase (one full hour) was carried out *without the teacher*. The collective building consisted in a sequence of moves: each move is coded as either E (*experimental*) if it is based on either visual or tactile experiments or L (*logical*) if it is deduced from already accepted statements (in turn either experimentally - E - or logically - L - stated). (cfr. fig. 3 for the codage of points).

0 (E) *conjecture to be proved:* OP is equal to OP' and the angles POP' are equal for all the configurations of the linkage.

1 (L) *proof:* if the triangles OAP and OCP' are congruent (already proved) then OP is equal to OP'

2 (E) *problem:* is there a relation between the motion of P and the motion of B? the search is interrupted.

3(E,L) *observation:* the length of PP' is not always the same . The statement is confirmed by theory, observing that it is not sufficient to know two sides to determine the whole triangle.

4 (E) *observation:* even if OP is equal to OP' then the length of OP is not constant.

5 (L (E,L)) *proof:* if POP' is constant (conjectured in move 0) and POP' is isosceles (proved in move 1), then all the triangles POP' of the infinitely many configurations are similar.

6 (L) *problem:* there is a constant ratio (move 5) between PP' and OP (= OP'); what is the ratio?

7 (L) *proof:* for the configuration of the fig. 2a it is proved that the angles POP', BCP' and BAP are equal.

8 (E) *conjecture:* the triangles PBP', OAP and OCP' are similar.

9 (L) *proof:* as the triangles BCP' and BAP are similar (hypothesis) then the proportions follow: CP' : AB = BP' : BP and, as OC=AB, CP' : OC = BP' : BP.

10 (L (E)) *proof:* if PBP' and OCP' are similar (move 8), the ratio of OP and PP' (equal to the ratio of OP' and PP') is constant.

11 (L) *proof:* the triangles OPP' of the infinitely many configurations are similar.

12 (L) *proof:* if all the triangles POP' are similar the angle POP' is constant (inverse of the move 5).

13 (L) *proof:* the angles P'BP, OCP' and OAP are equal (based on the angular propertles of poligons)

14 (L) *proof:* the triangles P'BP, OCP' and OAP are similar (cfr. move 8). (Cavani 1992 for details).

5. Discussion.

The discussion is divided into two parts, that refer to (a) *the organization aspects* (designed in advance); (b) *the functioning aspects* (observed in the classroom process).

5.1. Organization. The *first choice* concerned the definition of the task with reference to *a physical*

object (the linkage, actually given to the students and not only evoked). It did not define only an early context for activity to be detached from (as in the shift from specific to general, that is emphasized in the piagetian framework) but it was just a pole of the pair *physical object - ideal object* (i.e. specific - general) to be put in a dialectical relationship, according to the main features of theoretical thinking (§2).

The *second choice* concerned the structure of small group work by means of the *eight questions*. Their aim was to force the transformation of the *specific* linkage into a determined *ideal object* and, in the same time, to either recall or structure the *theoretical setting* (elementary geometry, theory of linkages) where to describe its *general* role and function. They were designed as *mediating tools* (Vygotsky 1978), which *inhibited the direct impulse to react* to the external stimulus given by the action on the pantograph; they formed an (incomplete) *orienting basis* (Gabay 1991). The eight questions have not the same status. For instance, the first one is, in a sense, transitional, as it could have been answered also within an *empirical* setting (even if it did not happen, as we shall show in the following): the students could have written the construction rules, referring to the conjuction of eight bars of given length by means of hinges and pivot (actually some ancient descriptions of machines were of this kind and probably we could have similar descriptions if the same task were posed out of school setting to a not literate person). The *second and third questions* forced the students to turn to some basic tools of the *theory of linkage:* they have sense only after Lagrange's (1787) introduction of the concept of *generalized coordinates of a mechanical system*. The fourth question introduced the *ideal object* into the theory of elementary geometry. The last four questions - and mainly the seventh - had to realize *the ascent from the general to the specific*, as a rotation of a whatever given angle ϑ was linked with the geometric properties of an individual pantograph. In a few words, the whole of questions were designed *to realize the dialectic between the specific linkage and a general theory of its configurations*.

The *third choice* concerned the acceptance of phases of *joint activity* between teacher and students in the small group work. Joint activity between adults and young people in problem solving is consistent with the concept of *zone of proximal development* by Vygotsky (1978). It does not arise from the professional need of the teacher to keep the experiment within suitable time limits, but on the *cultural* need of direct students' efforts toward the appropriation of the century products of human activity. The phases of joint activity cannot always be designed, as they depend on the actual process, even if some crucial points can be tentatively indicated in advance.

5.2. Functioning. In this section we shall address the same issues in a different order. The *eight questions* structured classroom process *from outside*. Yet they are good candidate for the *internalization* process (Vygotsky 1978) to be transformed into crucial elements of an individual student' methodology to study some classes of linkages (but this hypothesis can be verified only in a long term study, that is now in progress).

Some phases of *joint activity* were observed. For instance, in the conjecturing phase, the teacher focused on angular properties of the linkage and suggested to consider angles that were not visible

(actually the lines OP and OP' do not refer to the physical but to the ideal object). Its intervention was requested by the students who had suggested a lot of trivial *geometrical properties* (such as *the perimeter of the exagon is always the same*) without really understanding what was the meaning of this task. Then the conjecture was verbally formulated by a student and accepted by the teacher; yet the others did not trust the teacher and turned at the linkage to check it; they put the linkage in several configurations and moved the finger along the lines PO and OP' to concretize different manifestations of the ideal angle. Their utterances changed from *impossible* to *true yet astonishing*.

The last observation leads us to the third aspect (i.e. *the function of object-oriented action*). At the very beginning of the group work (question 1), the students measured the bars to be sure that some of them were *exactly equal*, but they immediately claimed that measure was not relevant *(we write how this is built, but he* - the fictitious addressee - *can make it as he likes)*. Actually, they were not all sure that the empirical procedure of measuring was suitable for the ideal geometrical setting where the properties of figures are studied. Yet the results of measuring were immediately interpreted within a theoretical setting, transforming the linkage (the *physical object*) into a schematic figure (the *ideal object*). The following process concerned still the physical object, as the students handled the linkage for a whole hour. They observed that the linkage assumed infinitely many transient configurations (the "generic" configurations (fig.1) and some "special" ones (fig. 2)) and that it was possible to pass from one to another by means of "small" movements. Sometimes the link between physical and ideal object were not easy to manage. For instance, after realizing that the triangles PAB and P'CB were similar, the students recited well some theorems concerning sides and angles of similar triangles, but did not realized that the proportionality of the sides PB and BP' could have solved the concrete problem of describing the (relative) lengths of the bars. A (short) joint activity with the teacher lead the students to link geometrical properties to physical objects. However, some embryos of the *ascent* from the *general* to the *specific*, necessary to complete the dialectic movement *between them* were present. For instance, during the early observation of the linkage, a student told that it was not necessary to keep the same angle PAB in building a new linkage: *If the angle is right, that one* (fig.2a) *will be more flattened*. Yet the movement to and fro specific and general was not always interiorized by the students: sometimes the teacher intervened just to suggest the shift (*try and check it on the linkage* for a verbal statement obtained by means of logical deduction; *try and prove it* for a verbal statement obtained by means of experiments). The special configuration of fig.2a played a twofold role: in the first question it suggested the similarity of the two triangles PAB and P'BC. Later, in the proving process it did obstruct the solution: it happened that the linkage assumed a configuration close (not equal) to the special one of fig.2a; immediately a student cried: *Please, move it! I feel mixed up: I cannot see the sides* (OP and OP') *any more* and she put the linkage in a more "generic" configuration. To sum up, *unverbalised* (either *visual-tactile* or *visual-image*) *activity* was observed in the whole process: its volume is consistent with the results on adult problem solving by Tikhomirov (1988 p.90), who stated that *[thought activity] consists not only of processes subordinate to*

a consciously realised goal but also of processes subordinate to an unverbalised anticipation of future results and that *in activity, the second type of processes may have a larger share than the purposeful actions proper*.

The process of proof building led to a *démonstration*, in the sense of Balacheff (1988). Actually the written text was not complete, as the crucial move 10 was lacking: it was considered verbally several times while the students were editing the text, but not written down, as something obvious. Moreover, the proof was not complete, as it referred only to the "generic" case of fig. 1a (the other cases require small adaptions). The problem of varying the configurations had been considered but rejected:*Have we to check out other cases? We haven't. We have made a proof* (with emphasis). They could have appealed to something similar to the *continuity principle* (Poncelet 1822) but they were sure that the text was not only *convincing* but *true*: the responsibility for truth was given to the theoretical feature of the setting where the process had been carried out, i.e. the system of elementary geometry. Even if in some moves (e.g. moves 5 and 10) the students were supposed to give the same status to experimental informations and to logically derived statements, *the overall process was theoretical*, because of the students' tension towards a complete justification within the system of elementary geometry of all the moves.Yet *its genesis* (and some tracks appeared still in the ordering of written statements - cfr. the column of comment in app.2) *was realized by means of a shift to and fro the experimental and the logical level*.

6. Concluding remarks.

Our results address two questions for didactical research. The relevance of unverbalised activity is stressed also by research in computer environments (Dreyfus 1991), yet the use of computer allows only the analysis of the interaction between verbal-logical and visual-image activity. *What is, if any, the role of visual-tactile activity in advanced geometrical thinking?*

Most of didactical research emphasizes the learner's responsibility for learning to contrast traditional teaching. We have argued (§2) that the appropriation of cultural artifacts is determined by the adult mediation. *How is realized the teacher's role concerning cultural mediation in the classroom?*

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Appendix 1: the eight questions

1. Represent the linkage with a schematic figure and describe it to somebody who has to build a similar one on the base of only your description
2. How many degrees of freedom has the point P? Draw the set R of the positions of P during the motion. Choose the parameters to determine the position of P.
3. How many degrees of freedom has the point P'? Draw the set R'of the positions of P' during the movement. Choose the parameters to determine the position of P'. Can you use the same as for P?
4. Are there some geometric properties which are related to all the configurations of the linkage? Try to prove your statements.
5. Can we say that the linkage realizes a biunivocal correspondence between R and R', i.e. every point of R has a correspondent in R' and viceversa?
6. If P and P' are two writing points, draw two corresponding figures (considering the above corrdresponence between R and R') . Are both the figures traced in the same time? Are the features of motion - speed, acceleration and so on - relevant?
7.Which are the common properties of the figures traced in point 6? Are they superimposable? Does it exist a simple motion which superimposes them? describe it.
8. Does the same motion superimpose also R and R'?

Appendix 2: collective answer to question 4 (fig.3)

Thesis. POP' is constant.
The angle POP' is constant as the triangles POP' obtained by means of the deformations of the mechanism are always similar, whatever the position of P and P'. **cfr. move 12**
In fact OP=OP', because the triangles OCP' and OAP are congruent, as **cfr. move 2**
CP'=OA,CO=AP and OCP'=OAP (BCO=OAB and P'CB=BAP)
The above triangles are also similar to a third triangle PBP', because, **cfr. move 9**
as the triangles BCP' and BAP are similar, it follows: BP' : BP = CP' : CO **cfr. move 10**
and the angle P'BP = OCP' as **cfr. move 13**
(posing that CP'B = CBP' = α and CBA = β) we have
PBP' = 360 - (2α+β)
OCP' = 360 - (2α+β).
This is true because prolonging the line BC from the side of C the angle supplementary to BCP' is equal to 2α and the angle supplementary to BCD is equal to β as two contiguous angles of a parallelogram are always supplementary.

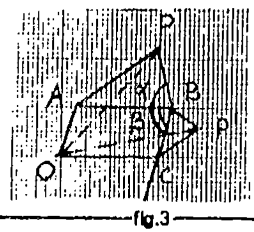

fig.3

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THE CONCEPT FORMATION OF TRIANGLE AND QUADRILATERAL IN THE SECOND GRADE

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Understanding of the concept of n-gon can be characterized by a commutative diagram which is composed of four ways to judge a given figure as n-gon. And it is examined through three cases on the introductory instruction to the concept of triangle and quadrilateral. In each of these cases, it was observed that some children gave out the persuading arguments containing a critical idea on the concept formation represented as a structure-preserved transformation in the diagram. Following the diagram, four stages on the concept formation and its development are introduced. According to two operations in the diagram, essentially two kinds of teaching tools can be distinguished. And the meaning of teaching tools are discussed.

1. Introduction : Structure on the concept of n-gon

It is well known that there are some difficulties on the identification and construction tasks of elementary figures. One of the elements that cause such difficulties is referred to the fact that children (or even adults) cannot use physical or mental rotation. On that point, Vinner and Hershkowitz distinguish three elements in identifying concept examples: (1) the concept image, (2) the concept definition, (3) a group of operations (mental or physical) (Vinner and Hershkowitz 1983). Hasegawa argues that there are four ways to judge a n-gon, presupposing that its boundary is made up of straight lines:

(1) Counting the number of its sides.

(2) Counting the number of its vertexes.

(3) Using a motion (a congruent transformation).

(4) Using a general transformation.

The ways (1) and (2) are taken generally in a mathematics classroom. Meanwhile (3) or (4) means to identify by transforming a given figure to a regular figure. For (3), Vinner and Hershkowitz point out as a group of operations. For (4), Miyazaki and Ueno attribute the understanding of something to the continuous movement of viewpoint (Miyazaki and Ueno 1985). These ways (1)(2) and (3)(4) are complementary and understanding of the concept of n-gon is characterized that these ways constitute a commutative diagram as in Figure

1 (Hasegawa 1985).

$$\begin{tikzcd} F \arrow[r, "\phi"] \arrow[d, leftrightarrow, "\varphi"'] & \mathbb{N} \arrow[d, equal] \\ F' \arrow[r, mapsto, "\phi"] & \mathbb{N} \end{tikzcd}$$

Figure 1

In the diagram, F and F' are n-gons, $\mathbb{N}$ is a set of natural numbers, $\phi: F \longmapsto \mathbb{N}$ is a function which takes a n-gon F into the number of its sides or vertexes and $\varphi: F \longleftrightarrow F'$ represents a transformation based on the ways (3) and (4). Notice that if F and F' are quadrilaterals and φ a continuous deformation of F (or F'), φ does not make sense until its domain is restricted to $\phi^{-1}(4)$. Generally speaking, the concept formation of n-gon means to construct the equivalence classes based on an equivalence relation $\sim_n$ which imposes partition into a set of polygons, where the relation $F \sim_n F'$ means that F and F' are both n-gons and it is constructed under the mapping φ : $\varphi(F') = F$, where φ is restricted to $\phi^{-1}(n)$. And the definition of n-gon is given using the inverse of ϕ, as $\phi^{-1}(n)$ or "a figure that is enclosed with four straight lines is called a quadrilateral," for instance, based on ϕ which is constructed through the classification activity by children.

The ways of judgement of n-gon were represented with the diagram, where two operations ϕ and φ were distinguished. We can distinguish these two in another concept formation processes based on the construction of equivalence relation such as the area of polygon or the angle sum in a triangle and so on (Hasegawa 1990). One of the purposes of teaching the concept of n-gon or another concepts mentioned above in a mathematics classroom is to construct the function ϕ which quantifies an object. On the other hand, ϕ becomes firmer when it has φ as a foundation. The transformation φ is structure-preserved and qualitative, and is a realization of an equivalence relation, but φ itself is meaningless. Its meaning depends on ϕ. These two are complementary and in a classroom, it is necessary to treat both of them.

In this paper, three cases on the introductory instruction to the concept of triangle and quadrilateral in the second grade that were previously reported (Hasegawa 1985. Hasegawa and Takahashi 1990) will be reexamined collectively. In these cases, it was observed that children asserted critical ideas in order to persuade other children. "Critical idea" is an idea which relates to the

essential part of concept formation, physical or mental rotation of a figure, for instance. It emerges as a persuading argument in a classroom discussion. According to these cases, the relation between the critical idea and the operations and the meaning of teaching tools will be discussed.

2. Three cases on the introductory instruction to the concept of triangle and quadrilateral in the second grade

Case 1. Under the theme of "Let's build our zoo", children constructed several polygons, each of which was a cage of an animal drawn by children linking some points given around each picture of the animal. Then children cut out them from a ground paper and classified according to the figures such as (a rectangle), (an isosceles triangle) and "the others" that a teacher showed to the children (fig. 2).

Figure 2

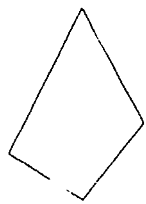

Figure 3

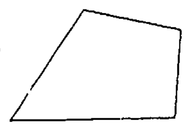

Figure 4

The teacher took up one child's classification. He classified a quadrilateral (fig. 3) into a set of triangle and a parallelogram into a set of "the others" and so on, and many children gave out their ideas against the classification. As some ideas noticed the number of its corners or sides were given out, the teacher asked to the children to count the number of corners of the polygons, and through that activity children began to understand that the figure (fig. 3) was also a quadrilateral. Then the definition of triangle and quadrilateral was introduced to the classroom. At that time a child asserted, "This figure (a triangle in an irregular position) doesn't look alike a triangle, but (rotating it) moving so, it looks like a triangle, and the figure (fig. 3) also looks like a quadrilateral if I move it (rotating it (fig. 4))." ---

In this case, physical rotations were used for the judgement and it seems that the assertion with action was given out for the purpose of persuading

other children. Especially the figure (fig. 3) that many children had not deemed as a quadrilateral and thought as an exception of quadrilateral turned to a general example of a quadrilateral through the classroom discussion (cf. Lakatos 1976).

Case 2. This case was also carried out under the theme of "Let's make cages of animals." The number of figures used to classify was less and each shape was nearer to a regular figure than that of the case 1. Prototypes for classification were not presented. After the classification activity, two children's classifications were presented on a chalkboard. One of the two classified "the cages" into two classes according to the number of corners, but the other classified them into three classes (fig. 5: A picture of an animal was drawn in each figure.) and explained that the figures in a trapezoid class "have four corners but become narrow." One child asserted in order to justify the latter classification, "If reducing this figure (the rectangle), it becomes the square (with reducing action by hands), and these two (the trapezoid and the isosceles trapezoid) also are same if this point is moved (with rotating the trapezoid and expanding action by hands)." ----

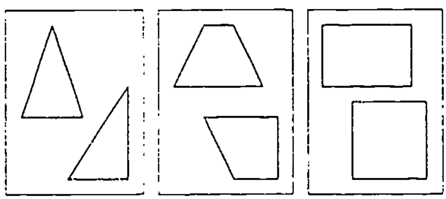

Figure 5

In the case, a transformation which is not a motion is used to justify the classification. If a figure F is a prototype (a rectangle in the case 1, which was deemed as a quadrangle by many children), ϕ means the generalization or the extension of prototype F into F' or the resolution of F' into the prototype. In the case 2, where the mapping ϕ functioned as the justification of equivalence of trapezoid and isosceles trapezoid, an equivalence class of quadrilateral will be accomplished to make operate ϕ on two subclasses, the rectangular class and the trapezoid class, of quadrilateral class in the classification that the child presented.

Case 3. School children in the primary grades often judge an asteroid-shaped plane figure (fig. 6) as a quadrilateral since it has four corners. The next case was carried out using the asteroid-shaped as an introductory subject matter to the concept of quadrilateral. The following illustrates part of the lesson.

T(Teacher): What is this figure? (A teacher showed the figure (fig. 6). Some children answered that it was a quadrilateral and one third of the classroom agreed them.) Why is it a quadrilateral?

C(Child): There are four corners. (He pointed the sharp points of the figure. Hearing his assertion, half of the children agreed him.)

C: Is it also a quadrilateral if edges curve more? (The teacher drew a figure (fig. 7) on a chalkboard with his assertion.)

C: It is also a quadrilateral, since it has four corners.

C: Please curve more its sides. (The teacher drew a figure (fig. 8).)

C: It is not a quadrilateral. (He asserted the figure as in Figure 7 a quadrilateral.)

C: It is a quadrilateral, since the corners are sharp.

C: If I curve more, it becomes cross-shaped. Is it also a quadrilateral? (The teacher drew a figure (fig. 9).)

C: No!

C: If so, we cannot judge to what extent it is a quadrilateral when we curve the sides and to what extent it is not. ---

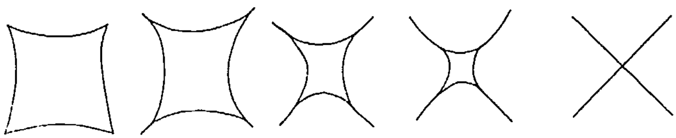

Figure 6 Figure 7 Figure 8 Figure 9

The transformation $\phi: F \longleftrightarrow F'$ in the cases 1 and 2 was restricted to a set of figures that are enclosed with straight lines. In the case 3, children who asserted that the asteroid-shaped was not a quadrilateral made ϕ a deformation which preserved the asteroid-shaped, and in consequence the assertion that it was a quadrilateral drove to a conflicting situation. The asteroid-shaped and the cross-shaped are not topologically equivalent. However, it must be given attention to the argument that extended other children's assertion to the

"limit" in order to drive it into conflict.

3. Four stages on the concept of n-gon and its development

Mey distinguishes four levels in the development of information processing: monadic, structural, contextual and cognitive (Mey 1982). In the same way, we can introduce four stages on the concept formation of n-gon and its development.

(1) A monadic stage, where each figure is isolated and treated as a single object. Judging whether a given figure is quadrilateral (or triangle, pentagon, and so on) or not is carried out according to comparing with some stored canonical forms in mind and evoking a congruent or similar figure with its name. At this time, ϕ acts as an enlargement or a reduction of size of the figure without activation of a rotation of a figure.

(2) A structural stage, where a n-gon is decomposed into its components such as corners and sides, and according to the number of them, it is judged as n-gon. Where ϕ takes the n-gon into $n \in \mathbb{N}$. On the other hand, ϕ as a rotation is not necessarily activated.

(3) A contextual stage, where the diagram (fig. 1) is accomplished. For the sake of the commutability of diagram or the continuous deformation of figure, n-gons are settled in one sequence in which children can see the invariants in the changes of state or shape (fig. 10, 11).

(4) A cognitive stage, where children can create a context which is useful for their aim. Through a classroom discussion, as we have observed in the case 3, they can aware the restriction of ϕ to the domain $\{F ; F \in \phi^{-1}(n)$ and the boundary of F is made up of straight lines.$\}$ and the definition of n-gon more clearly.

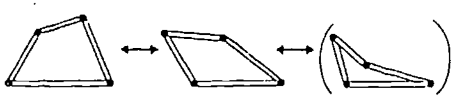

Figure 10. "Pantograph."

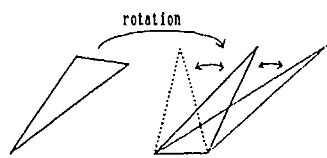

Figure 11. The use of rubber band.

These stages are characterized by an awareness of ϕ. However, they, especially (3) and (4), don't necessarily occur successively in the order. According to the figure(s) presented to children, either (3) or (4) may be evoked, and in some cases, these two may progress to overlap each other.

Peer interaction tends to induce persistent comprehension activity, and it creates and amplifies surprise and perplexity, produces discoordination: the awareness of a lack of coordination (Hatano and Inagaki 1987). In the cases mentioned above, children's active discussion was observed and it is supposed that motivation for comprehension was enhanced, and the critical idea on the concept formation was given out with the intention of persuading other children.

The critical idea in the persuading argument, which is regarded as a verbalization of ϕ, arose from classroom discussion depending on the given figures. The construction of equivalence classes with equivalence relation $\sim_n$ which forms the relation between isolated objects is regarded as a reflective abstraction. And it is the classroom discussion that enables the abstraction. We can expect such a discussion in a mathematics classroom turns to a mathematical proof or motivation for it ("from interpersonal into intrapersonal", see Vygotsky 1978).

It is necessary to expand the awareness of ϕ and the critical idea that characterize these stages to a common thinking in a classroom. And there, teaching tools (teaching aids) manipulatable by each child are substantial.

4. Teaching tools

Following two kinds of operations in the diagram, essentially two kinds of teaching tools can be distinguished: one is a task-setting teaching tool (TTT), "the cages of animals" in the cases 1 and 2, for instance, from which ϕ is constructed, and the other is an explanatory teaching tool (ETT), "a pantograph (fig. 10)", for instance, which is a materialization of ϕ. TTT is characterized by means of the terms discreteness, finiteness, logicality and analysis, and ETT by continuity, infiniteness, intuition and synthesis. TTT sets a situation or a task to be explored by children. On the other hand, ETT throws light on the relation between objects through its continuous movement and forces children to alter their criteria of judgement (Hasegawa 1990).

In the cases we have examined, except paper cutouts in the cases 1 and 2, continuously movable ETT had not used for explaining the concept. But we can materialize children's critical idea as ETT, by which the critical idea is to hold common in the classroom. Through the activity of manipulating ETT and classroom discussion, understanding of the concept with its movable mental

image can be fostered. The effect of ETT has not examined experimentally in a classroom situation, but the importance of manipulatives in learning geometry seems to be accepted commonly.

Afterwards the mathematics learning progresses to the direction where the transformation ϕ is restricted to an Euclidean transformation such as congruence or similarity, and ϕ itself becomes the object of learning. In such a stage, some teaching tools manipulatable by each child are still necessary for the concept formation with its useful mental image. Teaching tool is a source from where children's mathematical activities start and becomes a mathematical object beyond a mere physical entity in consequence of being given a mathematical meaning through the activities.

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STUDENTS' UNDERSTANDING OF GEOMETRICAL FIGURES IN TRANSITION FROM VAN HIELE LEVEL 1 TO 2

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ABSTRACT

This study investigated states of students' understanding of geometrical figures in transition from van Hiele level 1 to 2. The level of the eighth grade students were determined by the van Hiele Geometry Test, then a clinical interview was administered for level 1 students. Consequently, the four states of understanding in transition from level 1 to 2 were specified by analyzing what kind of geometrical figure a level 1 student considered the drawing as her visual model of the geometrical figure. These states enabled to classify whether a student understood inclusions and whether the drawing was consistent to a student's concept definition of a geometrical figure.

In this study, I investigated several states of students' understanding of geometrical figures in transition from van Hiele level 1 to 2.

Dina van Hiele-Geldof and Pierre Marie van Hiele presented the theory of geometric thought. They demonstrated that learners moved sequentially from the basic level to the highest level, assisted by appropriate instructional experiences. Van Hiele(1984) described the level 1 and 2 of geometric thought as follows. "At the First level of geometry, ...so that a square is not necessarily identified as being a rectangle." "At the Second Level properties are ordered. ... The square is recognized as being a rectangle because at this level definitions of figure come into play." In short, for level 1 students, a square is not identified as being a rectangle, so he does not understand the relation between a squaro and a rectangle.

However, a student do not jump from the state at which a student does not understand inclusions(*) of geometrical figures to the state at which a student understands, that is, from level 1 to 2. To move from level 1 to 2, students develop through several different states between level 1 and 2. Burger et al.(1986), Fuys et al.(1988), Usiskin(1982) have reported that students, especially those in transition, are difficult to classify reliably, for level 2 and 3, so the levels are not discrete. Thus, there are students who are not classified by any levels, whose state of understanding is different from levels. According to their idea, the continuity of levels have been clarified, but the kind of state of understanding between two continuous levels have not yet been clarified.

Many previous research have shown that students do not understand inclusions of geometrical figures(The Ministry of Education,1984; Koseki et al.,1977). It hasn't been made clear enough the reasons why students do not understand inclusions. By analyzing students'

(*)Inclusion means the interrelation between two sets that is obtained when all the members of the first set are members of the second.

understanding in detail, states of students' understanding can be clarified.

This study shows that there are several states of students' understanding of inclusions between van Hiele level 1 and 2.

THEORETICAL BACKGROUND

To clarify the states of understanding of inclusions, I present concept image and concept definition. Using them, some phenomena in the process of the learning of the concept were analyzed(Vinner,1983). The set of properties together with the mental picture is called 'concept image,' by 'concept definition,' we mean a verbal definition that explains the concept. The process of students' concept formation of geometrical figures is based on their concept image and concept definition of the geometrical figures. The transition between level 1 and 2 is a part of the process of concept formation, concept image and concept definition expand closer to the concept. In this paper, in particular, concept image of a geometrical figure means student's mental picture of it, concept definition means student's verbal representation to define a geometrical figure.

To analyze students' understanding of inclusions, I present the visual model. It means the drawing which is acceptable for a person as his concept image and consistent to his concept definition if the drawing is idealized as a mathematical object. When a person represents a geometrical figure by drawing it, it is considered that he represents his concept image, so the drawing is acceptable as his concept image of the geometrical figure. When a person explains a geometrical figure by the drawing which represents the geometrical figure, it is regarded that the drawing is acceptable as his concept image. When a person describes a geometrical figure in words, verbal description is equal to his concept definition of the figure. Also, when a person agrees to the verbal description, the description becomes his concept definition of the figure. Then, when a student interprets the drawing which represents a geometrical figure or represents a geometrical figure by drawing, the state of student's understanding of inclusions is clarified by what kind of geometrical figure the student consider the drawing as her visual model of the geometrical figure.

METHOD

1.Subjects

One hundred nineteen students at a private girl's junior high school participated in this study. This school ranks high in Tokyo. The eighth grade students were tested.

At the time of the study, the students had studied constructing a proof, but they still were not been able to write a proof completely. They had no instruction concerning how to prove inclusions of quadrilaterals.

Seven students from these participants were interviewed individually at a later time.

2.Problems

In this study, the students were tested by items determining levels 1 and 2 in van Hiele Geometry Test used in the Chicago project(Usiskin,1982). The problems consisted of five items for each level. From quotes of the van Hieles themselves regarding student behaviors to be expected at each level, questions were written for each level that would test whether a student was at that level. After piloting, the test items were modified and the test constructed, so the test has a high reliability to determine the van Hiele level of the student.

In the interview, the following question(Item 13) which determines the level 2 of students was used.

Item 13: Which of these can be called rectangles?

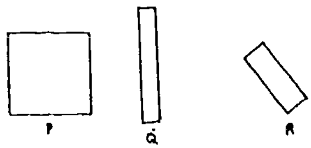
(A)All can

(B)Q only

(C)R only

(D)P and Q only

(E)Q and R only



3.Procedure

The items which will determine whether the students belong to van Hiele levels 1 or 2 were administered for about ten and eight minutes respectively. The criterion used in the Chicago project was utilized in this study. The student who answered 3 to 5 correctly, the items determined for level 2, was given level 2 regardless of her answer for level 1. The clinical interview had been done for seven level 1 students selected three months later after the paper test was administered. The interviewer had taught them geometry two classes a week for eight months. In this paper, four students were considered. Two of the students answered (E), two did (A) in the Item 13. They were interviewed in a room separated from the classroom after school and an audio tape was made so that transcripts could be analyzed. They were given their answer sheet of written van Hiele test, asked to explain what they had answered and why. Their inner states of understanding of geometrical figures was shown through the clinical interview. First, they were explained only in words, then described using the given drawing or their own drawing. The interviewer further asked the students about their answers and confirmed them, but did not instruct. She asked each student several items for about twenty minutes.

By analyzing the transcripts, it was shown what kind of geometrical figure a student consider the given drawing in Item 13 as visual model of the geometrical figure. When a student explained the reason why she answered using the given drawing of a geometrical figure, she said, "I have such image," and so on. The drawing is considered as being a reflection of her concept image of it. When a student said, "I have such image," for her own drawing of a geometrical figure, the drawing is regarded as being a reflection of her concept image of .. When a student described a geometrical figure in words, her verbal de-

scription becomes her concept definition of it. When a drawing is acceptable as student's concept image of a geometric figure and is consistent to her concept definition of it, it is considered as her visual model of the geometrical figure.

RESULTS AND CONSIDERATION

1.Written test

The table 1 shows the van Hiele levels and the corresponding number of students.

[Table 1] Van Hiele levels and the corresponding number of students.

level	n
3	60
2	53
1	6

Forty five percent of the eighth grade students were determined level 1 and fifty percent of them were determined level 2. According to the results of analyzing textbooks in Japan(Yamazaki,1991), the eighth grade students are required to have achieved level 2. But half of the eight grade students had not achieved level 2.

2.Interview

By considering level 1 students' answer in Item 13 and examining their understanding of the relation of a rectangle and a square, states of their understanding were clarified. By an analysis of what kind of geometrical figure a student consider the given drawing in Item 13 as her visual model of the geometrical figure, from her concept image and concept definition of the geometrical figure, her understanding of geometrical figures was clearly shown. The transcripts of four level 1 students were analyzed. First, Student YN answered that figures Q and R in Item 13 were rectangles.

(Interviewer asked Student YN why she answered (A) in item 13, required of her to explain it in words.)
YN:①In a rectangle, its opposite sides are equal, equal and parallel.
②It seemed that its opposite sides were equal in P.
...
YN:But, well ... (felt silent).
Int:Explain. Oh! Explain using the drawing, go ahead.
YN:(took out a ruler, checked sides of the figure with it.)
Well, I used a ruler and verified the sides. The opposite sides were parallel, P was a rectangle. Ah, I thought that this side and this side were parallel and the same length. ③It seemed that there was a gap between this side and the ruler.
Int:Is this side distorted?
YN:④If it is distorted, it is not parallel to the opposite side, so P is not a rectangle. ⑤The sides of Q were fit to the ruler parallel, so I thought that Q was a rectangle.
...
Int:Did you think P is a square? Yes or no?
YN:Ah! When I only see without drawings, I think yes.
Int:You said yes when you saw it, but what do you answer if you use a ruler?
YN:Off course, if the upper side of P is distorted, off course, no.

In this example, according to ⑤, Student YN explained her answer pointing Q and R,

so they were acceptable as her concept image of a rectangle. In accordance with ①, P was consistent to her concept definition of a rectangle, i.e. the opposite sides are same length and parallel. Therefore, Q and R became her visual model of a rectangle.

However, P did not become Student YN's visual model of a rectangle. According to ②, P was acceptable as her concept image of a rectangle, but based on ③, P was inconsistent to her concept definition of it, i.e. the opposite sides are parallel, because she found a small gap between the upper side and a ruler(④). Thus, P didn't become her visual model of a rectangle. Also P did not become her visual model of a square because P was consistent to her concept definition of a square, i.e. the opposite sides are parallel.

Secondly, Student YY answered that Q and R were rectangles.

> Int:In item 13, why did you choose(E)? Explain it in words.
> YY:Yes, well, ①the length of vertical sides was not the same length of horizontal. ②P seemed to be a square, so I did not choose P. ③I chose Q and R as a rectangle because they do not have four equal sides. ④And I seemed that two diagonals were equal in Q and R.
> Int:Yes.
> YY:⑤I did not choose P because I thought P was not a rectangle. I thought ⑥P was not a rectangle because P seemed to have four equal sides, so I didn't choose it.

Student YY explained her answer by pointing the sides and the diagonals in Q and R, so Q and R were acceptable for Student YY as her concept image of rectangles. According to ①,③ and ④, Q and R were consistent to Student YY's concept definition of a rectangle, i.e. the length of the horizontal sides are not the same as the length of the vertical sides, four sides are not equal, and two diagonals are equal. Therefore, Q and R became her visual models of a rectangle.

In accordance with ②, P was acceptable for Student YY as her concept image of a square, and it was consistent to her concept definition of a square, i.e. a square has four equal sides based on ⑥, thus P became her visual model of a square. But P didnot become her visual model of a rectangle because P was not acceptable as her concept image of a rectangle and inconsistent to her concept definition of a rectangle based on ①, ⑤ and ⑥.

Thirdly, student KN answered that P, Q and R were rectangles.

> Int:Explain why you chose (A) without drawings, but only in words.
> ...
> KN:Yes, ①P seemed to be a square, so Q and R might be chosen. ②But it was not written that P was a square. So, we were able to understand that P has right angles. I, I, at that time ....
> Int:Explain your answer not now, but at that time.
> KN:At that time, ③I saw that Q and R were rectangles, so I chose only Q and R, but it was not written that P is a square .... I answered that all three quadrilaterals were rectangles ④because opposite sides seemed to be parallel.
> Int:I see. Explain your answer of your choice (A) with drawing.
> KN:Yes, oh. ⑤P has four right angles and ⑥Q has also four right angles and R has four right angles, and ⑦for P, Q and R, opposite sides are all parallel. Ah, so, P is a square, but it has not been proved that P

is a square. ⑧It was written that four sides were equal in P, so ⑨In a rectangle, opposite sides are parallel. Ah, P, Q and R were consistent as a rectangle, so I chose all three.

According to ③, for Student KN, Q and R seemed to be rectangles, so Q and R were acceptable as her concept image of a rectangle. Based on ⑥ and ⑨, her concept definition of a rectangle, i.e. four angles are right angles and opposite sides are parallel, so Q and R were consistent to her concept definition of a rectangle. Thus, Q and R became her visual models of it.

P was acceptable for Student KN as her concept image of a square based on ①, but, in accordance with ② and ⑧, she thought that the sides of P might not necessarily be equal, so P was inconsistent to her concept definition of a square, i.e. a square has fourequal sides. Thus, P did not become her visual model of it. However, P was consistent to her concept definition of a rectangle, i.e. four angles are right and opposite sides are parallel based on ⑤ and ⑦, and P was acceptable as her concept image of a rectangle(④). Therefore, P became her visual model of a rectangle.

Fourthly, Student MT answered that P,Q and R were rectangles.

(Interviewer asked Student MT why she chose (A).)
MT:①If I verified P, Q and R with my ruler, they had about four right angles, so I chose three.
Int:Well, why did you think that P was a rectangle? You said that P was a rectangle. Why was P a rectangle?
MT:②Because P has four right angles.
Int:All right angles? Yes, I see. Then, is P a square?
MT:③At first, I thought that P was a square, and I was going to choose Q and R, but ④as a square is a particular rectangle, I thought P was a rectangle.

In accordance with ①, student MT verified the angles of Q and R with her ruler, so they were acceptable as her concept image of a rectangle. They were consistent to her concept definition of a rectangle, i.e. a rectangle has four right angles based on ① and ②. Thus, Q and R became her visual model of a rectangle.

Based on ③, P was acceptable for student MT as her concept image of a square, and P was consistent to her concept definition of rectangle(②). However, she said that a square was a particular rectangle(④), and four angles were all right angles(②), so P also was consistent to her concept definition of a square. Thus, P became her visual models of a square. Moreover, P was acceptable as her concept image of a rectangle, P became her model of it.

It has been definitely shown by the transcripts of four level students that each of them had different visual models for figure P in Item 13. Table 2 shows whether figure P is acceptable for each student as his concept image of a rectangle or a square, whether P is consistent to her concept definition, and whether P becomes the visual model of a rectangle or a square.

[Table 2] For figure P in Item 13, consistence of four students' concept images, concept definitons, and being visual models

Name	rectangle C.I.	rectangle C.D.	square C.I.	square C.D.	rectangle visual model	square visual model
YN	X	X	O	X	/	/
YY	X	X	O	O	/	●
KN	O	O	O	X	●	/
MT	O	O	O	O	●	●

C.I.: concept image

C.D.: concept definition

(O: consistent, X: inconsistent, ●: visual model, /: not visual model)

When figure P did not become a student's visual model of a rectangle, it is considered that she did not understand the relation between a rectangle and a square. In contrast, when the figure P became a student's visual model of a rectangle, it is considered that she understood the relation. The former is the state A, and the latter is the state B.

Then the state A and the state B are divided into two detail states respectively. When figure P did not become a student's visual model of a square, the state A_1 and the state B_1 were shown. Otherwise, the state A_2 and the state B_2 were shown. Student YN and YY were at the state A. For Student YN, P was not acceptable as her concept image of a square, but a side of P seemed to be distorted, so P was inconsistent to her concept definition of a square. Thus P did not become her visual model of it, so Student YN was at the state A_1. However, for Student YY, P was acceptable as her concept image of a square and was consistent to her concept definition of it, thus P became her visual model of it, so Student YY was at the state A_2.

On the other hand, Student KN and MT were at the state B. P was not the Student KN's visual model of a rectangle, but P was the student MT's visual model of a square. For Student KN, P was not acceptable as her concept image of a square, but it was consistent to her concept definition of a rectangle. Thus, she thought that her concept definition of a rectangle which includes wider meaning than her concept definition of a square, and P was consistent to her concept definition of a rectangle, so it became her visual model of a rectangle, not a square. Therefore, Student YN was at the state B_1. However, for Student MT, P was not only acceptable as her concept image of a rectangle and was not only consistent to her concept definition of it, but also was acceptable as her concept image of a square and consistent to her concept definition of it. Thus, P can become her visual model of a rectangle and a square, so Student YY was at the state B_1.

CONCLUSION

When a person interprets the drawing which represents a geometrical figure, several states of students' understanding in transition from level 1 to 2 can be clarified by

analyzing for what kind of geometrical figure level 1 students consider the drawing as her visual model of the geometrical figure. That is, the difference of visual models shows different states of understanding of geometrical figures in transition of levels.

This study suggests that by diagnosing the state of a student's understanding of geometrical figures with the visual models, teacher can know the student's state in detail, thus we can improve learning and teaching for concept formation of geometrical figures. We have to examine whether the states of understanding of another geometrical figures are specified by the visual model or whether the states of understanding in another transition of levels are clarified by the visual model. Consequently, we can make clear the validity that the visual model shows several states of students' understanding in transition of van Hiele levels.

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BIG AND SMALL INFINITIES: PSYCHO-COGNITIVE ASPECTS

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Abstract

The concept of infinity has played an important role in almost every branch of human knowledge. Nevertheless, such an interesting subject for cognitive psychology hasn't been studied deeply enough. We believe that different concepts of infinity, associated with different qualities and features, activate different cognitive processes, and because of this they should be studied systematically. Using a taxonomy based on the idea of iteration, in which different features are considered (i.e. convergence, divergence, cardinality, spatial measure), some intuitive aspects related to infinity in plane geometry were studied. Subjects aged 8, 10, 12 and 14 years participated in the study (n=172). Results show that for some cases of the taxonomy older subjects were better equipped to deal with these situations, whereas for others cases (related to convergence) they had much more problems than the younger subjects. Some differences concerning intellectual performance were found, mainly for the groups of 10 and 12 years.

Resumen

El concepto de infinito ha jugado un importante rol en casi todas las disciplinas del saber. Sin embargo, este sí interesante objeto de estudio para la psicología cognitiva no ha sido lo suficientemente estudiado. Creemos que conceptos distintos de infinito asociados con distintas calidades y características, activan distintos procesos cognitivos, y por esta razón deben ser estudiados sistemáticamente. Usando una taxonomía basada en la idea de iteración, que considera distintas características (p.e. convergencia, divergencia, cardinalidad, medida espacial), algunos aspectos intuitivos relacionados con el infinito en geometría plana fueron estudiados. Participaron en el estudio 172 sujetos cuyas edades eran de 8, 10, 12 y 14 años. Los resultados muestran que en algunos casos de la taxonomía los sujetos mayores estaban mejor equipados para afrontar estas situaciones, mientras que en otros casos (relacionados con la convergencia) ellos tuvieron más problemas que los menores. Se observaron algunas diferencias en cuanto al rendimiento intelectual, principalmente en los grupos de 10 y 12 años.

In an article published in the late seventies E. Fischbein and collaborators stated that, "it is surprising that psychology has done so little in exploring the fascinating concept of infinity, whose importance for science, mathematics and philosophy is undeniable (Fischbein, Tirosh and Hess, 1979, p. 3). Today, the situation remains the same.

If we analyze the history of the concept of infinity, we find that since the dawn of civilizations this concept has played an important role in almost every branch of human knowledge, fascinating and thrilling philosophers, theologians, scientists and mathematicians. In mathematics, already since the early time of Zeno, its study has presented many difficulties and disputes. Full of counter-intuitive features, infinity, has always been a controversial and elusive concept. In mathematics education, it is widely known that students face many difficulties when they study notions related to infinity,

such as calculus, infinite series or set theory. Such an important and particular concept of human mental activity should be an interesting subject for cognitive psychology for two reasons. First, because of the important role that this concept has played in the different disciplines of human knowledge, and second, because it is a rich and representative concept of a dimension of mental activity not based on direct experience. Paradoxically, none of the different theoretical approaches in cognitive psychology has studied this concept deeply enough (Fischbein et al. 1979; Núñez, 1990); not even approaches close to mathematics, logic and formal disciplines, such as the information-processing approach, or Piaget's genetic approach. Both being close to formal disciplines because of the objects they intend to study and because of the conceptual tools they use to model findings.
In cognitive psychology, literature on infinity is in general over-represented by educational oriented research, being the fundamental or basic research very scarce (Núñez, 1990). In addition, efforts seem to be isolated, discontinuous, and poor in theoretical ties. Besides, there is a lack of precision: often the term "infinity" confines different features and properties not well identified (i.e. potential and actual infinities; huge and small infinities; referred to quantity and to spaces; etc). In order to overcome these difficulties and to build a theoretical corpus that permits the creation of new concepts for a better discrimination and explanation of the phenomena that now aren't well identified, we suggested to consider the cognitive activity that infinity in mathematics requires as an independent scientific object for cognitive psychology (Núñez, 1990). We believe that different qualities and features associated with endless iterations and with other concepts related to infinity, activate different cognitive processes, and because of this they should be studied systematically. According to this idea, we have conceived a taxonomy of "different" infinities, which has been described elsewhere (Núñez, 1991) and which lies at the base of the present study.

The taxonomy

The taxonomy supposes that at the very base of the notion of infinity lies the operation of iteration and it was conceived to study intuitive psycho-cognitive aspects of the concept of infinity as *pure* as possible (i.e., trying to isolate the mathematical training). By intuition we understand "direct, global, self-evident forms of knowledge" (Fischbein et al, 1979, p. 5). For this reason we have focused our study on the domain of plane geometry, in which it is possible to ask children about number and sizes of figures without recalling specific school-learned knowledge or technical notions. The cases of the taxonomy are conceived as different types of transformations applied to plane figures (i.e. polygons). In order to simplify and to operationalize our approach, we have assumed that a plane figure can be transformed (its attributes being operated iteratively) in terms of number (cardinality (#)), and/or in space ((S); vertically and/or horizontally, being

enlarged (D), or diminished (C)). Thus, this taxonomy intends to differentiate the following aspects related to infinity:

1) ***Type*** of infinity: whether it is conceived after a **divergent** ($T_{(D)}$) or **convergent** ($T_{(C)}$) operation of iteration.
2) ***Nature of the content*** of the operation of iteration: whether it is **cardinal** ($N_{(\#)}$; infinite quantities) or **spatial** ($N_{(S)}$; infinite big or small spaces).
3) ***Coordinations*** of type & nature ($T_{(D)}N_{(\#)}$; $T_{(D)}N_{(S)}$; $T_{(C)}N_{(S)}$): when more than one attribute being transformed is present.

Related to the first quality, ***type*** of infinity, and this from a developmental point of view, important evidence shows that the idea of convergence is mastered and understood much later than that of divergence (Piaget & Inhelder, 1948; Langford, 1974; Taback, 1975). The study of the history of the concept of infinity in mathematics reveals a similar tendency (Núñez, in press). Concerning the second quality, ***nature of the content***, we believe that a distinction between a cardinal content and a spatial content should be made, especially when talking about infinity. We share the opinion of D. Tall who says "cardinal infinity is ... only one of a choice of possible extensions of the number concept case. It is therefore inappropriate to judge the 'correctness' of intuitions of infinity within a cardinal framework alone, especially those intuitions which relate to measurement rather than one-one correspondence" (Tall, 1980, p. 271). Now, if we consider that there are two attributes iterated simultaneously, we have another interesting aspect to be studied: the ***coordinations*** of the qualities (***type*** and ***nature***). In our opinion, what is interesting is not only to observe how many attributes should be coordinated in a given situation, but to observe what are the attributes of these elements.

These different qualities of the operations related to infinity could be represented in a 3-dimensional space (Núñez, 1991), a space of transformations (Figure 1), where axes are each of the most simple qualities that an attribute infinitely iterated may have ($T_{(D)}N_{(\#)}$; $T_{(D)}N_{(S)}$; $T_{(C)}N_{(S)}$). Points in the space are each of the combinations of coordinations which hypothetically activate different processes from a cognitive point of view. We believe that such a model facilitates the conception and the operationalization of our hypotheses. For instance, among other hypotheses we could mention that, in general the closer one gets to the origin (no transformation), the easier to deal with the transformation; that situations on axis $T_{(C)}N_{(S)}$ are more difficult than those on axis $T_{(D)}N_{(S)}$ (due to the differences of difficulty levels between divergence and convergence mentioned above); that situations related to infinity represented by points on plane $\mathfrak{C}$ are much more difficult to deal with, than those on $\mathfrak{A}$ or $\mathfrak{B}$ (because there is a coordination of qualities different in type and nature, which demands an activity of a superior level than that demanded in plane $\mathfrak{A}$ or $\mathfrak{B}$, where at least qualities of the same order are

present, type (divergence) and nature (space) respectively). Besides, these aspects may vary according to age, intellectual or academic performance and/or gender.

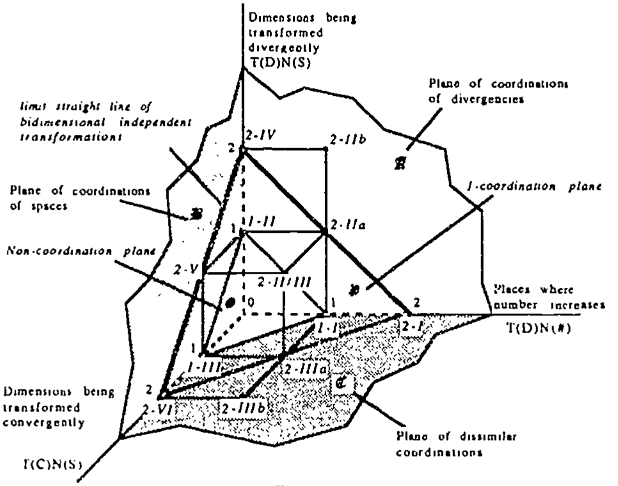


Figure 1

Method

Procedure

The study considered two parts. The goal of the first part was to explore with a quantitative approach the different cases of the taxonomy, by means of questionnaires. In the second part we studied in depth, by means of individual interviews (approx. 50 minutes), four of the cases of our taxonomy (cases 2-V, 2-II/III, 2-IIIa and 2-IIIb; see Figure 1), judged to be most interesting ones after the analyses of the results obtained in the first part. In this article, only the results of the first part will be analyzed.

Two variables related to performance were studied: Intellectual performance measured by the Raven test and academic performance (studied only in the second part) measured by the marks obtained both in mathematics and in french (official language at school). In the first part the Raven test and the questionnaires were administrated during school time in the classroom. The individual interviews took place six months later in a room located in the same building. The interviews were videotaped.

Subjects

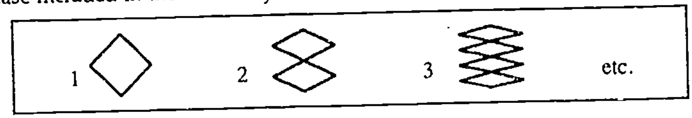
In the first part participated 172 students of two schools of the city of Fribourg, aged 8, 10, 12 and 14 years. Boys and girls were equally represented. Nearly 20% of this population

participated in the second part (N=32; 8 subjects by age group, half high and half low intellectual-academic performers in each group).

Material

Concerning the cases of our taxonomy, they were operationalized by figures which were transformed according to a certain rule (i.e. growing or diminishing in size, or number). The subjects were asked to explain what would happen with the "path around the figure" (perimeter) and the "amount of painting needed to color" the figure, as the transformation continues further and further "without stopping". The following is an example of a case included in the taxonomy (case 2-IIIa; see Figure 1):

Case 2-IIIa:



This case considers an increasing number of figures (*type* divergent and cardinal *nature*: $(T_{(D)}N_{(\#)})$) and a diminution of the height of each figure (*type* convergent and spatial *nature*: $(T_{(C)}N_{(S)})$). The other cases were constructed following the same principle.

Results

The answers to what happen with the perimeter and the area were grouped in 4 categories in order to analyze the general tendencies of their distributions by age (*Plus* or more, remains *Equal*, *Less* or diminishes, and *Missing*, for answers difficult to classify or no answers). We were interested in the distributions of these categories rather than in the correctness of the answer itself. The reasons that led the subjects to give the answers were to be studied in the second part of the research. According to age, results of the first part of the research reveal differences between the distributions of the answer categories for 5 cases of the taxonomy (cases 1-II, 2-V, 2-II/III, 2-IIIa and 2-IIIb). For certain cases the "correct" answer is more frequent as the age increases. The following is an example related to the perimeter:

Case 2-IIIb:

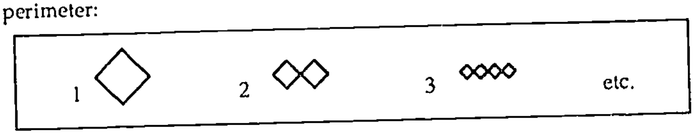


Figure 2 shows the distributions (%) of the four categories by age. The bold line represents the "correct" answer. A χ^2 test between the categories *Less*, *Equal* and *Plus* show that the differences are highly significant $\chi^2(6, N=159) = 34.92$, $p<.0001$.

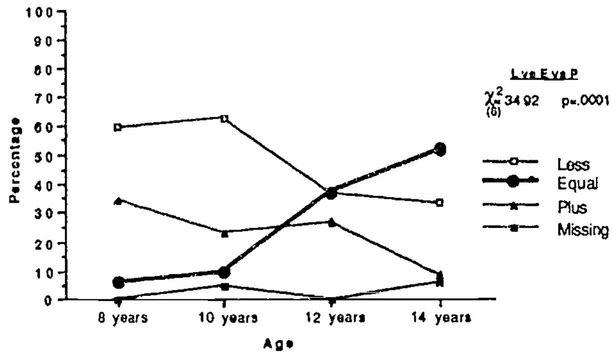


Figure 2: Percentage of subjects by answer and by age group (case 2-IIIb; perimeter). N=172.

For other cases, especially those asking about the surface and presenting a convergent iteration, the frequency of "correct" answers decreased with age, as it can be seen in the following example:

Case 2-II/III:

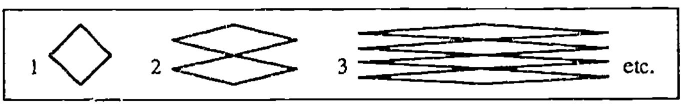


Figure 3 shows the distributions (%) of the four categories by age. The term "Conv. (F)" (for False Convergence) at the right of the graphic, indicates that 18% of the 14 years-old subjects who responded *Less* said that there will be no area, or that the figure becomes a line. A χ^2 test between the categories *Less*, *Equal* and *Plus* show that the differences are significant $\chi^2(6, N=155) = 14.76, p<.02$.

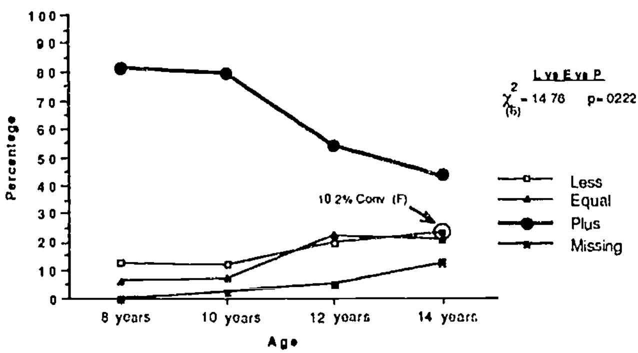
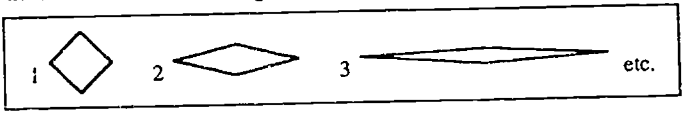


Figure 3: Percentage of subjects by answer and by age group (case 2-II/III; surface). N=172.

For other cases, although the "correct" answer was relatively stable according to age, differences were found concerning the type of "wrong" category. For example in the following case in which the area of the figure remains the same,

Case 2-V:



the "wrong" answers *Less* and *Plus* are almost equally distributed at 8 years (ratio *Less/Plus* nearly 1/1) but the latter decreases with age and the former increases with age (ratio *Less/Plus* nearly 16/1, see Figure 4). Moreover, for those 14 year-old subjects who responded *Less*, 35.5% of them said that the figure will become a line without surface (Conv. (F)). A χ^2 test between the categories *Less, Equal* and *Plus* show that the differences are highly significant $\chi^2(6, N=157) = 25.47$, $p<.0003$.

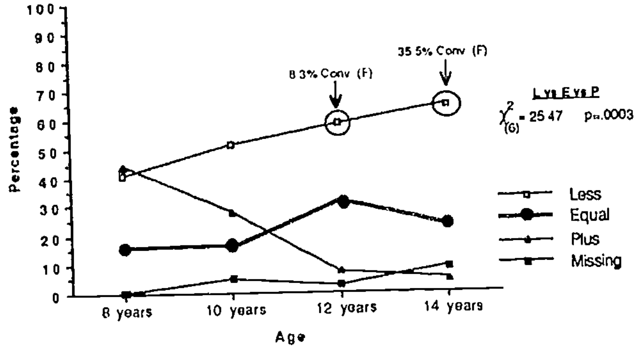


Figure 4: Percentage of subjects by answer and by age group (case 2-V; surface). N=172.

As far as intellectual performance[1] is concerned, differences between low and high performers were found only at age 10 and 12. At age 10 the "wrong" answer *Less* to the question about the surface of the case 2-V (Figure 4), was associated with high performance whereas the "wrong' answer *Plus* was associated with low intellectual performance ($\chi^2(1, N=33) = 3.88$, $p<.049$). For the same question at age 12, high performers answered more frequently the "correct" answer *Equal* than the low performers ($\chi^2(1, N=40) = 7.93$, $p<.0049$). A similar tendency concerning the "correct" answer was found at age 12 for the question about the perimeter of the case 2-IIIb (Figure 2) ($\chi^2(2, N=41) = 10.53$, $p<.0052$).

Finally, no gender differences were found.

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[1] For these analyses the Raven scores were dichotomized by age group into high and low performers.

Discussion

The taxonomy and the material we have used revealed that for certain combinations of features related to infinity, interesting differences between age groups could be found. In those cases in which the complexity increases because of the *coordination* of different *type* and *nature* of the transformations (such as the case 2-IIIb on the plane $\mathfrak{C}$ of Figure 1), age seems to help in dealing with these situations. Nevertheless, older subjects seem to loose their developmental advantage as soon as the effect of convergence appears in situations concerning surfaces in which the level of operability decreases. Thus, in a case like 2-II/III (Figure 3) in which measure and estimation become difficult, as age increases, the fact that the height of each figure decreases progressively is much more important than the increasing number of figures and the increasing width of the figures together. The growing effect of convergence as age increases is comparable to a "black hole" which attracts any other effect that might alter the situation. Even in more simple cases like 2-V (on plane $\mathfrak{B}$ of Figure 1), where a compensation of height and width is possible, convergence seems to play a much more important role as age increases. For certain of these cases, at age 10 and 12 , maybe when a certain intuition of convergence begins to develop, intellectual performance show similar tendencies as age does.
These results suggest that depending on the features of the attributes which are iterated to conceive infinity, different cognitive processes take place. The relation between cognitive development and the role of convergent iterations in the situations shown could be interpreted in the perspective of the concept of "epistemological difficulty" coined by the philosopher G. Bachelard, which says that it is in the act of knowing itself that appear, like a functional necessity, troubles and backwardness, and that these difficulties could be studied throughout history and educational practice (Bachelard, 1938).

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THE ATTAINED GEOMETRY CURRICULUM IN JAPAN AND HAWAII RELATIVE TO THE VAN HIELE LEVEL THEORY

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This paper compares how students in Hawaii and Japan are distributed relative to the van Hiele levels. To measure the attainment of van Hiele levels three tests were developed. Students in grades 3,6,8,10 were tested in Hawaii and in grades 4,7,9,11 in Japan. The results show that the Japanese students are about two years ahead of the Hawaii students. Part of this difference can be accounted for by the curriculum and by instruction in verbal communication. Also, language and context can influence how a student responds to a test item.

Purpose: This study is part of our larger study entitled " Comparison of Japan and Hawaii Geometry Curriculum, Instruction and Students Using the van Hiele Theory and the National Council of Teachers of Mathematics (NCTM) Standard" The purpose of the larger study is to compare the curriculum and teaching and learning of geometry relative to the van Hiele level theory and the NCTM Standards in Japan and Hawaii in order to gain insights into the problems of learning and teaching geometry. The objectives are (1) to analyze and compare current geometry curriculum as evidenced by American and Japanese text series in light of the van Hiele model and NCTM Standards, (2) to determine how students in Hawaii compared to Japan at the start of grades 4,7,9, and 11 are distributed with respect to the levels of the van Hiele scheme, and (3) to compare and analyze how classroom teachers effectively instruct to assist students in moving from one level to the next. This paper will focus on the second objective,that is, on the attained geometry curriculum both in Hawaii and Japan.

The Van Hiele Theory: There are three aspects to the van Hiele Model: Insight, obstacles (levels) and instruction (phase). This study pertains to the obstacles or levels aspect of the model. According,

1. There exists five levels of understanding (thinking) with reference to specific topics areas in geometry.

2. The properties of these levels as suggested by Hoffer (1983) are:

a. Level O (Visualization)- Figures are judged by their appearance as a whole without regard to properties of their components.

b. Level 1 (Analysis)- The student begins to discern the properties of figures; figures are recognized as having parts and are recognized by their parts.

c. Level 2 (Order)- The student logically orders the properties of concepts and figures.

d. Level 3 (Deduction)- The student can construct proofs, understand the role of axioms and definitions, and supply the reasons for steps in a proof.

e. Level 4 (Rigor)- The student can understand the formal aspects of deduction and can interrelate different axiomatic systems of proof.

3. These levels are sequential. The student must master one level of understanding (thinking) before proceeding to the next level. Knowledge of the language structure is pertinent to the advancement in level of thinking.

Selected Previous Findings: Since some of our previous findings are related to this paper, a brief summary of them is presented. On the average, in grades 1-8, American textbooks devote about 8 percent of their pages to geometry, whereas the Japanese texts allocate about 27 percent. More geometry content is covered in American texts than in Japanese texts, however the Japanese texts cover more concepts. American texts repeat concepts taught in pevious grades (Whitman and Komenaka, 1990) The U.S. and Japanese textbook series both satisfied the majority of the NCTM Standards on Geometry for grades K-4 (Kimura, 1991). Of the 40 NCTM Curriculum Standards, 4 are devoted to the subject of geometry. The Standards implicitly recognizes the importance of the van Hiele model when it states "Evidence suggests that the development of geometric ideas progresses through a hierarchy of levels. Students first learn to recognize whole shapes and then to analyze the relevant properties of a shape. Later they can see

relationships between shapes and make simple deductions. Curriculum development and instruction must consider this hierarchy because although learning can occur at several levels simutaneously, the learning of more complex concepts and strategies require a firm foundation of basic skills." (p. 48 NCTM, 1989) Geometry does not exist as an independent subject in Japan. In the elementary school it is integrated with the study of arithmetic and pre-algebra. In junior high school it is integrated with algebra and probability/statistics. (Nohda, 1991) In America, commonly a formal course in geometry exist in high school. The geometry content found in the Japanese curriculum in grades 7-9 is that commonly found in the United States high school geometry course (Nohda, 1991). This means the Japanese curriculum is about two years ahead of the American geometry curriculum. Both the American textbooks and the Japanese series provide students with tasks at van Hiele levels 0,1,2. However, the American texts provide significantly less tasks at level 1 and 2 than the Japanese series. Level 2 tasks appear only in grade 6 and is very negligible. The exercises in the American series tend to fluctuate from one van Hiele level to another (level 0 and 1), within a grade level and from one grade to another whereas in the Japanese text the students are generally taken through the levels in sequence within each section. (Whitman and Komenaka, 1991)

Procedure: (1) Test Instruments. Based on the translated writings of the Van Hieles' (Fuys et al., 1984) and studies by Fuys and D. Geddes (1984) and Z. Usiskin (1982), three tests were developed and pilot tested in Hawaii. These tests are Geometry 1a (22 items) for third/fourth graders, Geometry 2a (24 items) for sixth/seventh graders, and Geometry 3a (33 items) for eighth/ninth graders and for tenth/eleventh graders. Geometry 1a tested for levels 0,1,2, attainment, Geometry 2a tested for levels 0,1,2 and Geometry 3a tested for levels 0,1,2,3. Some of the items were common to all three tests. After the tests were piloted and revised they were sent to Japan for review and translation. A mathematics teacher in Hawaii back translated the tests the Japanese had translated and modified. Adjustments were made to the English version of the test based on the Japanese suggestions. In some instances changes had to be made due to language and

cultural differences. How the test should be administered and scored were communicated to the Japanese. In general, the details were agreed to by both parties.

(2) Sample. Both the Hawaii and Japan students were tested in April or May. Since the Japan school year begins in April and the Hawaii school year ends in early June, the Hawaii students were tested in grades 3,6,8, and 10 whereas the Japanese students were tested in grades 4,7,9,and 11.

(a) Japan Sample. The Japanese sample consisted of 131 third graders, 113 seventh graders, 109 ninth graders and 91 eleventh graders. Students in the neighborhood of Osaka made up the grade 4 sample. Students from Sapporo made up the grade 7 and grade 9 samples. And students from Nagoya made up the grade 11 sample. The students for all grade levels were average students.

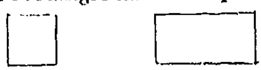
(b) Hawaii Sample. The Hawaiian sample consisted of 99 third graders, 232 sixth graders, 159 eight graders and 159 in grades 9-12 with the great majority being in grade 10. The third graders were selected from among students from 7 elementary schools. These schools represent below-average, average, and above average students in Hawaii according to standardized test scores. The sixth graders were from three schools. These schools are average to above average according to standardized test scores. The eighth graders were chosen from two schools. About a third of the eight graders were considered by their teachers to be average and the remainder represented a range of ability from average low to high-high. The tenth graders were students from three average high schools.

Selected Results: Table 1 shows the percent of correct responses on the geometry tests in Japan and Hawaii. In both places for each part of the geometry test, the percent of correct responses increased as the grade levels increased. Also for both Hawaii and Japan within each grade level the percent of correct responses decreased from Part A to Part D. In Japan, in grade 11, it decreased from 96 percent to 58 percent. In Hawaii, in grade 10, it decreased from 83 percent to 22 percent. There is about a two years gap between the Hawaii and Japan results. This difference may be accounted for partially by the geometry curriculum found

in these locations. In general, the Japanese euclidean geometry curriculum is completed by grade 9 whereas in the United States (including Hawaii) it is completed by grade 10. Also, the geometry found in the curriculum of grades 7 and 8 in Hawaii is essentially a review of elementary school geometry. In Japan, in grades 7,8, and 9 the study of euclidean geometry takes place. Using the criterion of 70 percent of level items correct to decide that a student has attained a particular level, we found 4 percent of the grade 4 students attained beyond level 0. Of the grade 7 students 10 percent attained level 1 and 32 percent level 2. Of the grade 9 students 31 percent attained level 1, 41 percent level 2, and 3 percent level 3. Of the grade 11 students 21 percent attained level 1, 34 percent level 2, and 40 percent level 3.

Table 2 shows the percent of correct responses of Hawaii and Japan students to the following item (item #8):

Explain how the rectangle and the square are different.

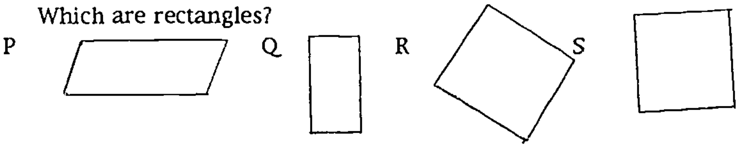


Explanation: ______________________________

Both Hawaii and Japan students' correct responses increased with higher grade levels. In general, the Hawaii students except for those currently studying formal euclidean geometry gave explanations in terms of visusal knowledge. For example, "the rectangle has two long sides, and the square has all short sides". Of the Japanese students, 72 percent of the seventh graders and 63 percent of the ninth graders used definitions in their explanations. In responding to this item, the Japanese used mainly geometric knowledge whereas the Hawaii students used visual knowledge. In terms of the van Hiele levels, the Hawaii grades 6 and 8 students functioned more at level 0 and the Japan grades 7 and 9 students more at level 1. The difference may be because the Japanese geometry curriculum is about two years ahead of the Hawaii curriculum and that the Japanese elementary curriculum is more intense than that of Hawaii (Whitman and Komenaka, 1990). Another reason for the difference is that the Japanese students are well instructed on how to give verbal explanations (Stigler, 1988)

Table 3 shows the percent of Japan and Hawaii students who selected the correct response to the following item (item # 13):

Which are rectangles?



(a) Q only (b) P and Q only (c) P and R only (d) Q and S only
(e) Q, R, and S only

In Hawaii, as the grade level increased, the percent of correct responses to this item increased, whereas in Japan no improvement was seen after grade 7. It appears that the translation of this problem, due to the fine nuances of the language, implied a single response. `Also the nature of the question implied a response based on what the student sees visually - and no more. This interpretation is strenthened when we analyze the responses by Japanese seventh graders to both items 8 and 13. Of the 50 percent of seventh graders who erroneously chose Q as the correct response, 90 percent of these also explained the difference between a rectangle and a square by the use of defintions and characteristics of figures. The students appear to have shifted levels of thinking in moving from one problem to another.

Conclusions: The percent of correct responses on the geometry test shows the Japanese students to be ahead of the Hawaii students by about two years. Part of this difference can be accounted for by the geometry curriculum and instruction in verbal communication in both places. Language and context can influence how a student responds to a test item. This is especially true in cross-cultural studies.

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Table 1 Percent of Correct Responses on the van Hiele Geometry Test in Japan and Hawaii

	Japan				Hawaii			
	Grades							
	4	7	9	11	3	6	8	10*
Part A (Level 0)	73	87	87	96	55	61	66	83
Part B (Level 1)	36	66	78	86	29	47	55	74
Part C. (Level 2)	31	51	62	77	22	34	40	63
Part D (Level 3)	--	--	24	58	--	--	04	22
Number of students	131	113	109	91	99	232	159	159

* Grade 10 contains students in grades 10-12 who are enrolled in a conventional tenth grade geometry class.

Table 2 Percent of Correct Responses of Hawaii and Japan Students To Item Number 8

	Grades			
	3(4)	6(7)	8(9)	10(11)
Japan	---	93	95	100
Hawaii	---	49	58	81

Table 3 Percent of Correct Responses of Hawaii and Japan Students To Item Number 13

	3(4)	6(7)	8(9)	10(11)
Japan	09	22	17	25
Hawaii	06	12	36	50

TYPES OF IMAGERY USED BY ELEMENTARY AND SECONDARY SCHOOL STUDENTS IN MATHEMATICAL REASONING

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In this investigation the use of imagery in two groups of students, one in fifth grade and the other in eleventh grade was studied. Seven fifth grade students and 6 eleventh grade students were given a series of clinical interviews in which their understanding of mathematics was probed. The results showed that all the students interviewed used some type of imagery. The younger students used imagery in many of the same ways as the high school students. Great individual differences were found between students in both groups in the types and facility of imagery used. Students with a greater relational understanding of mathematics tended to use more abstract forms of imagery such as dynamic and pattern imagery, while students with less relational understanding tended to rely on concrete and memory images.

There is a long history of interest in the relationship between spatial ability and mathematical knowledge (Bishop, 1980, 1989). The results of this research, however, were frequently conflicting and unclear. The research area became much more fruitful when researchers started to think of spatial ability and spatial sense in terms of imagery, rather than in the sense it was used by factor analytic psychologists. Recent work (Brown & Wheatley, 1989, 1990; Reynolds & Wheatley, 1992; Presmeg 1985, 1986a; 1986b) in which individual student's thinking was probed in clinical interviews shows that students do use imagery in the construction of mathematical meaning. Since much of school mathematics is highly instrumental (Skemp, 1987) and visual reasoning has traditionally been held with low regard by the mathematical community (Dreyfus, 1991), it may not be expected that students use imagery in mathematics classes. In the construction of meaningful, relational mathematics, however, it might be expected that students use imagery much more frequently.

In this paper we have made several assumptions about the nature of mathematics learning and the imagery involved. First, what happens in mathematics classes in school cannot be considered meaningful learning of mathematics. Some students do construct meaningful mathematical relationships from the material presented, however. When we speak of "weak" or "strong" mathematics we are referring to the students relational understanding, not indicators of classroom performance. Second, learning mathematics must be considered as mathematical relationships that are constructed meaningfully by the individual. This learning frequently involves the use of imagery. Finally, the definition of imagery cannot be restricted to the traditional view of the picture in the mind. It also must include very abstract and vague forms of imagery.

The purpose of this current investigation is to confirm and extend our understanding of how students use imagery in school mathematics and in their relational understanding of mathematics. To do this the thinking of younger students was studied in addition to that of high school students and a variety of tasks to measure relational understanding was included in the clinical interviews.

Methods

Selection of students. The 13 students used in this investigation all attended the same local public school. Six of the students were in the eleventh grade and enrolled in an Algebra II course. The other seven students were in the two fifth grade classes that are not ability grouped. The eleventh grade students were selected because they scored above the median in mathematical visuality on a test of mathematical processing (Presmeg, 1985). The fifth grade students were selected because they scored in the moderate range on a test of mental rotations (Wheatley, 1978). None of these students was considered as doing extremely well in their school mathematics by their teachers and some were considered to be performing poorly.

Procedures. After selection, students were given a series of individual clinical interviews designed to probe their imagery and understanding of mathematics. All interviews were videotaped or audiotaped and subsequently transcribed for analysis. During the experiment data about the individual students was collected from a variety of other sources such as student records, interviews with teachers, and classroom observations, but the findings reported here are based on the interview data alone.

Interview tasks. Because the level of mathematical sophistication between the fifth and eleventh grade students varied greatly different tasks were used in the interviews for these two groups. Only one imagery task, a pattern replication task, was given to all students. All interview tasks, were designed to probe similar aspects of students' thinking.

In the first interview of the series eleventh grade students were given toothpicks and asked a series of questions concerning the construction of squares from the toothpicks. In the first series of tasks the student was given 24 toothpicks and asked to construct 1, 2, 3, 6, 7, 8, and 9 squares using them. In the next series the interviewer first used four toothpicks to build a single square (Figure 1a) and asked the student how many squares there were in the pattern. After they agreed there was one she then extended the pattern so that there were two small squares in each dimension (Figure 1b). The student was then asked how many squares were in the pattern. After they agreed there were five she then extended the pattern to three-by-three (Figure 1c) and four-by-four (Figure 1d), each time asking the student how many squares were in the pattern. The investigator then asked the student to imagine a five-by-five pattern and to predict how many squares would be in it. Finally, the student was asked how many squares would be in the pattern for the general case, n-by-n.

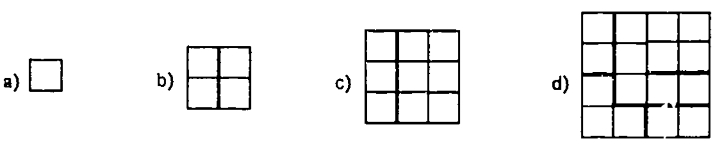

Figure 1. Patterns of toothpicks in which students were to find the number of squares present.

In another interview, students were asked to solve to Algebra problems from their textbook. These problems were taken from material that had recently been covered in their Algebra II class. One of these concerned the trajectory of an object shot upward from the earth and acted upon by gravity. The other concerned a rectangular sheet of plastic with squares cut from the corners and folded to make an open pan of known volume. Students were to find the original dimensions of the sheet.

In another interview, students were asked to solve two nonroutine problems that required mathematical reasoning, but no algebra skills. The two problems were presented in written form as follows:

The Faces of the Cube

There are 216 small cubes are arranged in a 6 by 6 by 6 large cube. One layer of small cubes is removed from each face of the large cube. How many cubes remain?

Tigers in Cages

There are 15 tigers and 4. How many ways can the tigers be put into cages so that no 2 cages have the same number of tigers?

In two interviews fifth grade students were asked to solve problems from a pencil and paper test that was designed to measure relational, rather than instrumental understanding of mathematics. Items were designed to test understanding of inverse operations, numerical patterns, nonroutine and multistep problems, geometric areas, and fraction concepts. During this interview, students were asked to read each problem and think aloud as they worked.

In another interview, students were given a choice of four nonroutine problems to solve. Of these the two discussed here are:

Tables and Chairs

We have 12 square tables, each of which seats 1 person on a side. If we push them together to form one long table, how many people can be seated?

Horses and Ducks

In a field there were horses and ducks. When Tom looked through the fence he could see seven heads and 20 legs. How many horses were there? How many ducks?

In a final interview these students were asked about their school mathematics. In particular they were asked what topics they enjoyed doing the most and how they remembered the algorithms involved.

Results

Analysis of the results of these interviews showed that all the students used imagery at some point during the interviews to solve the mathematical tasks. This is a particularly important finding since these groups of students were not selected for high imagery ability. Many were weak mathematics students and tasks designed specifically to probe their visual imagery indicated it was not strong. Other

students whose imagery was judged to be stronger frequently used it in their mathematical reasoning to great advantage and with great efficiency. Perhaps the best way to examine how students used imagery is to show specific examples of how this was done.

Types of Imagery and Examples

Presmeg (1985; 1986a) identified five types of imagery used by students in mathematical reasoning in school mathematics. Examples of these five types were found in the current investigation. In this section these five types are explained, related to other terms in the literature and several examples of each included.

Concrete Imagery. This type of imagery may be thought of as a "picture in the mind." In classical conceptualizations it is thought of as the only type of mental imagery, but currently this view is thought not to be adequate. Piaget and Inhelder (1966) have used the term "static" images and Johnson (1987) "rich" images to describe this type of image. The salient characteristic of concrete images is that they consist of a single, nonmoving but often highly detailed picture.

It has been suggested (Brown and Wheatley, 1989) that this type of imagery has little to do with relational understanding of mathematics. It has also been found (Presmeg 1985; 1986a) that the use of a concrete image in mathematical reasoning may lead to many difficulties. In spite of these difficulties, however, it was found that many students do use concrete imagery in mathematical reasoning.

The results of this current investigation confirm these previous findings. Many instances were found where both fifth grade and eleventh grade students used concrete imagery in their mathematical reasoning. During the counting squares portion of the match stick task several students used the concrete image of the five-by five square to reason the answer to this problem. One student in particular, Jerry, used a method for counting squares in the five-by-five and even the six-by six that indicated his image was extremely detailed and accurate. Jerry also used a unique method involving a concrete image for counting the number of cubes removed from the large cube in that problem. Again, Jerry's solution involved the construction of a very detailed and accurate image.

Likewise, students in the fifth grade group frequently could construct concrete images which lead to viable solutions to problems. Two girls, Janet and Jean, whose imagery was otherwise not strong were able to construct concrete images of fractional parts and use these images to sort fractions into three categories: "about 1," "about 1/2," and "about 0."

It was also possible, however, to find many examples in which concrete imagery led students to solutions that were not viable. Most commonly in these cases, students constructed an image that was inconsistent with or failed to coordinate all the information given in a problem. In the counting squares problem one student constructed an image of the five-by-five square in which it was made by adding a row of small squares to the four-by four on all sides. Another student, Tim, began to solve the cube problem by drawing a very accurate diagram of the six-by-six-by-six cube. He then reasoned, however, that this cube with one layer of small cubes removed from each side was a five-by-five-by-five. In both these cases once the student had constructed these images they became confident in the

viability of their solution methods and it was difficult for them to realize other possibilities. In Tim's case, however, he finally did realize the inconsistency of his original image with the information in the problem. He then generated a much more dynamic image of the cube with all its faces being removed and was then able to see another solution that was quite efficient and consistent with the information in the problem.

Among the fifth grade students several students attempted to solve the horses and ducks problem by assigning the number of legs to one species and the number of heads to the other. These students all had a clear image of how this could happen and were confident of the correctness of their answer.

Memory Images. In her original study Presmeg (1985, 1986a) identified many visualizers who recalled a formula by visualizing it written on a blackboard or in a note book. She called this type of imagery as memory images for formulae. This classification, however, seems narrow. Younger students and those in noncollege intending curricula do make frequent use of formulae in their school mathematics, but it still may be possible that they use visual methods to recall information. It seems more generally viable to talk about memory images that students use to recall information from mathematics classes. Piaget and Inhelder (1966) referred to these images as "reproductive." Such images, when they occur, are concrete and though they may be accurate and detailed, may not contribute to a student's understanding of mathematics. The two examples given below illustrate this point.

In an example very similar to that cited by Presmeg (1986a), Carl used a memory image to recall the quadratic formula. After he visualized and wrote the formula he was then able to explain how it was used to solve quadratic equations. In similar fashion another student, Judy, had a good memory image for the diagram her teacher had drawn on the blackboard to help them solve the rectangular box problem. She could reproduce this diagram accurately and did so without prompting when she read the problem. As she attempted to use it in solving the problem, however, it was clear she had little understanding of how it related to the solution process. She was eventually reached a solution to the problem, but with a much difficulty. When prompted, several other students could reproduce the diagram of the parabola drawn by the teacher to illustrate the projectile problem, but they were unable to relate this diagram to the solution.

It is also possible that younger students may use memory images in a similar fashion to help them remember algorithms or procedures. In this investigation none of the students reported doing this, but it is a question for further investigation. It is also possible that students may use dynamic memory images, but in this investigation we found no examples where students reported doing this.

Kinaesthetic imagery. This type of imagery involves muscular activity of some type. Both in Presmeg's original investigation and with the students involved in this investigation the muscular activity was limited to the use of hands and fingers. Many examples were found in both groups of students.

One very interesting example of kinaesthetic imagery occurred when an eleventh grade student, Judy, attempted to solve the cube problem. On first reading of the problem she lacked an idea of how to begin. At the urging of the interviewer she drew an acceptable three dimensional diagram of the cube.

As she started to think about the problem, however, she abandoned this diagram and started to use her hands to demonstrate what would happen to the large cube as a layer of cubes was removed first from each side, then the top and bottom and finally the front and back of the cube. In this case the visual diagram, though she had drawn it herself, seemed of no assistance in finding a solution to the problem. The movement of her hands in space, however, allowed her to find a solution.

Kinaesthetic imagery was even more common in the younger students. One student in particular, Rob, was frequently observed to look out into space and point to solve problems that required counting. One student, Mandy, whose imagery was quite weak used kinaesthetic imagery to illustrate her solution to the tables and chairs problem. Her image was inconsistent with the information given in the problem, but her gestures showed her image was clear and the solution viable for her.

Dynamic imagery. Dynamic imagery involves the ability to move or transform a concrete visual image. In her original investigation Presmeg (1985) found only two instances of students using dynamic imagery and therefore was unable to study it in any detail. Other researchers (Brown and Wheatley, 1989), however, believe that the ability to transform images is essential to mathematical understanding. By accepting Kosslyn's (1983) position on conceptual acts in the imaging process they have excluded static imagery from consideration as a process involved in the construction of meaningful mathematics. In further work, Brown (1993; Brown & Wheatley, 1993) has identified several individual components of dynamic imagery that are particularly important to mathematical understanding.

In the current investigation the use of dynamic imagery was found to be much greater than in Presmeg's original work. Both students in the fifth and eleventh grades successfully used dynamic imagery to help them solve problems. In solving the rectangular box problem several eleventh grade students reported "seeing" the flat sheet of plastic being transformed into a three dimensional figure. Among the fifth grade students one series of area problems was particularly useful in evoking dynamic imagery. In this task, students isolated portions of a figure that could be moved and combined with other portions of the figure to make a whole unit.

Pattern imagery. Pattern imagery is a highly abstract form of imagery. It consists of pure relationships depicted in a visual-spatial scheme. This is the type of imagery Einstein seems to have used in his scientific thought when he talks about "certain signs and more or less clear images which can be 'voluntarily' reproduced and combined" and "this combinatory play" (Hadamard, 1945, p.142). This type of imagery, then, may be considered of major importance in mathematical understanding. Johnson (1987) has used the term "image schemata" to describe this process in a theory of meaning and Dörfler (1991) has shown how the concept may be applied in mathematical thinking. He goes on to delineate different types of image schemata and show how these can be produced by protocols.

In the current investigation pattern imagery was the least frequent type of imagery to occur. Examples were found in both an eleventh grade and a fifth grade student, however. Both students were

considered two of the stronger mathematics students in the investigation.

In the counting squares portion of the toothpick task one student, Tim, found a general solution for the n-by-n pattern of squares. He did this by constructing an image of several larger patterns and switching back and forth between visual and numeric patterns. Tim also used a pattern image in the solution of the tigers in cages problem. He began by drawing a diagram. As his work progressed, however, he was could see a pattern so that he proceeded simply by writing answers. It appeared his imagery became quite abstract and allowed him a very efficient solution. One fifth grade student, Brad, started to solve a multiplicative problem, involving the pairing of shirts and shorts to make outfits, by drawing a diagram. After he started drawing lines on the diagram to find the answer he recognized a pattern and could then count multiplicatively to find the answer without completing the diagram.

In this investigation we were not able to find any examples of imagery that could not be classified as one of the preceding types. This finding does not preclude the existence of her types, however. The use of different types of interview tasks may evoke many different types of imagery in students.

Discussion and Conclusions

The results of the current investigation confirm and extend previous findings. Both students in the fifth and eleventh grade groups use imagery of similar types and in similar ways.

There were, however, large individual differences in the imagery of students in both groups. Students who were judged to have a better relational understanding of mathematics in both groups frequently used imagery that was more dynamic, abstract and efficient than students whose relational understanding was judged to be weak. The weaker students tended to rely on concrete or memory images and these were, at times, inconsistent with information in the problem or unrelated to the solution.

This is not to suggest, however, that concrete and memory images do not contribute to relational understanding of mathematics. Examples were found where both were used to advantage. Teachers and mathematics educators need to be aware that each has associated problems. It is also possible that in curricula of the future where less emphasis is placed on symbol manipulation and learning of algorithms, and technology is readily available these more concrete types of imagery will become less useful. For the present these are open questions.

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FLUENCY IN A DISCOURSE OR MANIPULATION OF MENTAL OBJECTS?[1]

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Abstract. *In the representational view of the mind successful thinking in mathematics is based on the mental construction and availability of mental objects for mathematical notions. A critical discussion is carried out in regard to the characteristics ascribed to mathematical objects. This is complemented by a subjective report about my own mathematical experiences which exhibit the role of non-cognitive aspects like attitude, agreement, acceptance, willingness to enter a discourse.*

Mind as a Space for Mental Objects

In many papers, talks and discussions about mathematical thinking, terms are used like mental objects, mental entities, cognitive constructions and others, see for instance Gentner and Stevens, 1983, Greeno, 1988, Harel and Kaput, 1991. What is the use of such notions? They are used to model and explain thinking and problem solving in mathematics (but also elsewhere). Therefore, they are to be taken as theoretical terms within a psychological theory about how the human mind works and proceeds to produce a behavior which by an expert observer might be judged as mathematical (and correct, i.e. in accordance with established mathematical norms).

According to my interpretation of those papers the basic tenets and assumptions of that theory are the following points. First of all, the mind (or cognition) is viewed metaphorically as a kind of space which can contain something and which can be structured. As the product of cognitive or mental constructions in that mental space mental objects originate or are produced. These mental objects then can be manipulated, transformed, combined etc. with a kind of mental operations. And, what is even more important, the mental objects are representatives or replicas of mathematical objects. This means, they have properties and behave as the mathematical objects do. It is postulated that for an adequate understanding of a part of mathematics the learner has to construct mentally the mental objects corresponding to the entities of the mathematics in such a way that there holds a kind of isomorphism between those two realms. This is viewed to guarantee that the (mental) manipulation of the mental objects produces correct results. Other ways of expressing this model are in the discourse about mental representations of abstract objects (and the mathematical objects are of course abstract objects).

[1] Parts of this paper were the content of a plenary presentation in Working Group 4 at ICME 7

Mental Objects Support Understanding

The necessity to construct mental objects/units/entities (also termed: cognitive or conceptual ones) is considered being a prerequisite for adequately understanding certain mathematical constructions; for instance, addition of functions or quotient structures in algebra and topology. Since those constructions can only be applied to object-like entities, to accomplish them mentally presupposes to have constructed the respective mental objects. One has therefore to have available a mental object corresponding to a function or to a vector-space or to a group. The failure to understand those mathematical constructions is interpreted as being caused by a lack in the corresponding mental objects. All that is in full congruence with the representational view of the mind. What we know is, according to this position, a sort of a mapping, a picture or a representation of something else, of something outside of the mind, of the memory or the cognition. The working of the mind then consists in manipulating those (mental) representations. For being able to speak meaningfully and sensibly about mathematical objects of any kind one has to have available mental objects as mental representations of those mathematical objects. Those mental objects are, as so-called internal representations, well discerned from the usual representations (as graphs, symbols etc.) which are termed as being external.

An Example: Natural Numbers

Let me be a bit more concrete by interpreting this general description by some examples. The first example is already simple arithmetic of natural numbers. The theory of mental objects would stipulate the existence of mental entities which represent mentally numbers like 2,3,4,5, etc. At least, this is considered to be necessary for a deeper and adequate understanding of the operations with natural numbers, like 2+3, 4*5, etc. In arithmetic, those operations are no longer considered as being general descriptions of manipulations with discrete sets or manifolds; like 2+3 as being a symbolic description of combining two units of a certain kind with three others of the same kind. In arithmetic the referents for the numerals are no longer specific discrete sets but (natural) numbers qualified as mathematical objects. And those have to be represented mentally by respective mental objects (constructed by the learner to understand arithmetic). Those mental objects must be able to reflect (isomorphically) the quality of the represented numbers; especially they have to lend themselves to be added or multiplied, to be even or odd or prime or perfect etc.

Some Subjective Doubts

I feel tempted to formulate some doubts in the ecological validity of the sketched approach to explain mathematical thinking. First, my subjective introspection never permitted me to find or trace something like a mental object for, say, the number 5. What invariantly comes to my mind are: certain patterns of dots or other units, a pentagon, the symbol 5 or V, relations like 5+5=10, 5*5=25, sentences like five is prime, five is odd, 5/30, etc., etc. But nowhere in my thinking I ever

could find something object-like that behaved like the number 5 as a mathematical object does. But nevertheless, I deem myself being able to talk about this number, its properties, the relations and operations it takes part in; and I do add 5 and I multiply by 5 quite correctly. In other words, I have a huge amount of knowledge about the number "five" without having distinctly available for my thinking a mental object which I could designate as the mental object "5". Every (almost perhaps) property or relation regarding 5 I can express to myself and others in one way or another, e.g. by using conventional diagrams or formulas. But according to the common discourse these are just not to be equated with the number 5 itself but only are representations of it. In conclusion, I am constantly treating "five" as an object without having available either an abstract object outside of my mind or a mental object inside of it which could serve as that object. My knowledge about a (non-existing?) object is enough to give rise to my feeling of being entitled to talk as if there were such an object. Yet, as my analysis shows the object itself nowhere turns up, it is not needed in fact. I can deduce its further properties from those which I already know by deductive reasoning. I can construct material objects ("representations") to exhibit by way of an adequate interpretation at least some of those properties and relationships which are characteristic of the postulated (abstract or mental) object.

More Objective Doubts

Beyond my introspective experience with the number "5" there are other arguments which might question the adequacy of the theoretical discourse about mental objects or entities. For which natural numbers should I have a mental object available? Assuming that the mind has a only finite capacity for whatsoever it will possibly only contain finitely many mental objects. But give me any number as big as you want and I will treat it as an object: I can operate on it (half it, double it, add it to another one etc.), I can ascribe properties to it, put it into relations with other numbers, etc.

This I can do even for a number which I have never thought of before. Or, take very large numbers, like 10^{100}. What could be a mental object representing that number beyond essentially a symbolic or verbal description (10 to the power 100)? But those symbols do not have any mathematical properties; e.g. 10^{100} is neither even nor odd in contrast to the number it is taken to stand for. A similar remark applies to all kinds of (external) representations. Who is not yet convinced should take $(10^{100})^{100}$ or the 1000th power of that. Still, by using the common representations we can talk and argue about all these numbers without having access to any kind of object which could be viewed as being the respective number (neither in our mind nor in the abstract realm whatever that might be).

Some More Examples

Even worse is the situation with "objects" like $\mathbb{N}, \mathbb{Q}, \mathbb{Z}, \mathbb{R}$ or $\mathbb{C}$. They are infinite and thus cannot have a mental representation, a mental object sharing their constituting properties. But again, I do not have problems treating those sets as objects, e.g. by applying certain operations to them or by speaking of the properties and relations to other "objects" (e.g. $\mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q}$,$\mathbb{Q}$ is dense in $\mathbb{R}$,$\mathbb{C} = \mathbb{R}^2$ with a specific multiplication etc.). Of course, for those objects we even do not have a material representation. Any drawn number line by necessity is finite! And only in language and in discourse we can extend it infinitely. To be honest, I cannot **imagine** (create an image of) an infinite line! Thus, I do not have a mental object corresponding to an infinite line (or plane), and yet, I find it very sensible to talk about those things as objects.

What we do have is material models of a (very) local character which nevertheless are generic in the sense that they convey the general pattern or the general structure. Any finite part of the number line serves this purpose very well, at least if it is interpreted, viewed and accepted as a local model (for $\mathbb{N}, \mathbb{Z}$ or $\mathbb{R}$). Similarly, a small "window" into $\mathbb{C}$ instantiated as the Gaussian plane can be used as a generic model for all of $\mathbb{C}$.

Another Example: Rational Numbers

Let us consider some more examples. Notorious are the problems with fractions and rational numbers. As with "5", I have great difficulties with conceiving of a mental object for, say, 2/3. All kinds of things come to my mind when I see that symbol: part whole relations, measurements, a point on the number line, many numerical relations (like 2/3=0.666 ...), that I can add it to other numbers or symbols or use it as an operator (for taking two thirds) etc., etc. I have various so called representations for the number 2/3, many different instantiations, situations where it gets materialized but I can't get hold of neither an abstract object nor a mental object which could serve as the genuine number 2/3. Thus 2/3 to me appears rather as a fuzzy system of mutually connected and related situations, and actions (which commonly are termed applications of 2/3) and of operations with 2/3. That can't possibly be the mental object for 2/3! This mental object would have to be of a much more refined and restricted quality, stripped of all these material images and intuitions and just reflecting what counts as the mathematical object 2/3 (essentially viewed today as being a specific element of $\mathbb{Q}$ with its algebraic and topological structure). Of course here as well the arguments regarding infinity of $\mathbb{Q}$ etc. apply as I have made them for the natural numbers.

Who Manipulates Mental Objects?

It appears ridiculous to me to pursue a theory of the mind where the mind has or contains mental objects corresponding to the rational numbers. Then the mental objects for 2/3 and 3/5 would have to be combinable in ways corresponding to addition and multiplication in $\mathbb{Q}$. How can this be conceived of? Do the mental objects have a kind of extension (length, volume) to be added and,

simultaneously, the quality of operators to be multiplied? Who operates on the mental objects? Doesn't that lead to the notorious problem of the homunculus in our head? Thus, even when not regarding the problems with the infinity of $\mathbf{Q}$, the metaphor of mental object. mentally simulating rational numbers and their operations leads to unsurmountable contradictions and inconsistencies. Because of the exhibited numerous problems with that theory I also cannot believe that construction of mental objects - whatever that means - is a precondition for successful mathematical activity.

A More Advanced Example

Let me consider another example. It is argued that for a functional understanding of operations on (real) functions (like derivative, intergral) and of spaces of functions (like C(I),L_2) it is necessary to have constructed functions as mental objects (or entities and the like), or to have encapsulated the mapping process, to have reified the process, etc. Again, this is the argument that for treating something as an object or for carrying out operations on it which in a common interpretation need objects as operands one has to have at least mentally available adequate and appropriate objects. Those mental objects are (in theory) viewed to have the pertinent properties which then permit to manipulate and transform them accordingly. In a way in our mind, if trained adequately and extensively, there is to be a kind of imitation or isomorphic simulation of the mathematical processes and objects. Only then we can - it is assumed - genuinely understand the mathematics. What would this mean in our case of functions? Let us take the rather extreme example of $L_2(I)$, the Hilbert-space of (classes of) quadratic Lebesgue-integrable functions on the interval I. Not regarding the infinity of L_2 one has not even available a structural description of a generic member of L_2. There are various ways to define a limit process which when applied to functions in a certain space (like step functions or continuous functions) will by definition generate all members of L_2. But the latter can be very fuzzy and extremely irregular what prevents any generic description. How ever then could we have mental objects for the members of L_2? I do not believe that any mathematician has something available which would deserve that name. Yet, with relative ease one talks about the set of all those functions, of adding them, of their making up a vector space, of the integral as a linear mapping on that space, of the length or norm of a member of L_2, of the distance of two functions in L_2, etc., etc. And I repeat, all that is possible without having any kind of mental representation of L_2 and its members.

Mental Objects as Replicas?

Of course, the same argument applies to many (all?) other mathematical constructions which apparently have to be understandable without the support by any mental object or representation mirroring the mathematical structures and processes. To the very least, my arguments and deliberations should make it doubtful to pursue a strict theory of mental objects. Such a theory stipulates a kind of duplication of the mathematics in the sense that in our mind we have available and manipulate mental objects as replicas of the mathematical objects under study. Yet, it appears

we can get hold cognitively only of specific properties of those objects but the objects (they being abstract, mathematical or mental) themselves elude our awareness or consciousness. But do we need the objects? All the mathematical reasoning and arguments are exclusively concerned with mathematical properties and relations which, as it is, are mostly formulated as being attached to objects. Proving that differentiability entails continuity we only use those two properties, the functions themselves do not occur in any essential way. I could even think of formulations which avoid completely the function concept in a reified form. Might be, it is rather a matter of convenient expression and communication that one uses a language with objects as carriers of the properties and relations. It for sure is cumbersome to address properties directly.

An Alternative Approach

To exhibit an alternative way of viewing and describing mathematical thinking or mathematical cognition I decided to give a personal and subjective account on my own experiences with a specific mathematical topic and my accommodating to it. I chose complex function theory because I think a topic which is less familiar lends itself better to conveying the central features of my experiences and subjective understandings. Basic to complex functions is the field $\mathbb{C}$ of complex numbers. I have accepted that notion and all of its implications on the following grounds.

I carried out many additions, multiplications and divisions of specific complex numbers in different algebraic or geometric realizations. This permitted me to agree that those operations can be assumed to be applicable to any pair of complex numbers. It appeared sensible to me to speak of addition and multiplication of arbitrary complex numbers though I know that for most of them these operations cannot be carried out by me just by lack of time. The algebraic formulas, the geometric figures and the matrices worked as prototypic and generic exemplars justifying the generalization. I never could find anything like a mental object or representation of $\mathbb{C}$ as a whole in my thinking and I did not need anything like that. I cannot imagine the infinite Gaussian plane, I only can say something like "it stretches further and further", "it never comes to an end"; and yet, I find it useful to talk about the whole plane including the infinite point added to it. The tiny windows into the plane which I can draw serve me as a kind of generic schema, an image schema (Lakoff, 1987) for the whole plane.

Essentially, I interpret my "knowledge about $\mathbb{C}$" as the ability to handle the various models of complex numbers and their operations in an appropriate way. For that I do not need access to abstract objects which lurk behind the models (as their representations) though our language is very suggestive in this direction. But, I do not mind using this discourse since for me it is just a discourse which helps us communicating. For me the different "models" of $\mathbb{C}$ are simply symbolic constructions which from a certain point of view behave in the same way (i.e. are isomorphic). And I only and exclusively do know those "models" and I do not know anything beyond them and their mutual relations. And that is all what I really need. I should hasten to add that I

do not consider my understanding of $\mathbb{C}$ as being purely formal or symbolic. I have available for my thinking image schemata of a prototypic character for notions like complex number, addition, multiplication, norm, plane etc. which lend subjective meaning to my thinking and talking about $\mathbb{C}$. I am further aware that my accepting of those schemata (like the vector addition diagram) as being generic and universally applicable is a kind of agreement, of belief, of consent based on many prior experiences of the same kind where such an attitude, such a use of language (subjected to many other rules of discourse) proved sensible, plausible and even productive. Those experiences refer to my understanding and knowledge of natural, rational and real numbers, for instance, which shows the same features.

Let me now turn to the notion of a function $f : \mathbb{C} \to \mathbb{C}$. How do I associate a meaning with a sentence like: under f to every point (or number) in $\mathbb{C}$ corresponds a unique value, i.e. again an element of $\mathbb{C}$? Of course, any understanding of that presupposes already my having accepted the talk about $\mathbb{C}$ as admissible and sensible. Then I am reminded of various experiences: how we have talked about functions $f : \mathbb{R} \to \mathbb{R}$ and, foremost, of specific examples like $f(z) = z^2, f(z) = \sin z$, etc. It was for those examples that I had calculated $f(z)$ for some values of z, had drawn a diagram showing the relation between z and $f(z)$, etc. In each specific case I trusted it to be meaningful to talk about the function f having a value for every $z \in \mathbb{C}$ (or where f is definable): the formula or rule suggested itself to be applicable to every $z \in \mathbb{C}$. It was and still is for me again a sort of agreement with a style of talking, with a specific discourse which received its legitimacy from the schematized experiences with various examples. From accepting $f(z) = z^2, f(z) = \sin z, f(z) = \exp(z)$, etc. my confidence is widened to accepting talk about arbitrary functions $f : \mathbb{C} \to \mathbb{C}$ where I do not have available a rule or formula. What is specificly remarkable is that I am not aware of any "mental object" corresponding to, say, $f(z) = z^2$. I know what this tells me to do: take z and square it; I know various "properties" of f, e.g. how to map (mathematical jargon again!) circles around the origin, lines through the origin. Yet, I do not have available a representation of f in its totality as such which could serve me as the external basis for a mental object. Nervertheless, when I want to view f as an object I can do so and I can talk appropriately: I can talk about adding f_1 and f_2, about the correspondence $f \to f'$ being a linear mapping, etc. This very likely involves my being prepared to think of all values of f being evaluated; in a way, of f being completely accomplished. This amounts to conceive of the potentiality given by an expression like z^2 as being realized. The meaningfulness of this conception for me is based on the image schemata which I developed when handling specific cases (of z, of f) and examples. But I emphasize, those image schemata for me do not constitute a global view on f but are of a local character (e.g. how $1 + i$ behaves under $f(z) = z^2$). Still, they lend meaning to the generalized talk which is further justified through the ensuing development of the theory of complex functions: if we accept that discourse a wonderful and fascinating theory develops. Thereby, again and again similar agreements are necessary, accepting something as admissible, trusting in a way of talking, taking a specific case as

generic and generating general meaning.

Take as another example the vector space of all entire functions (or equivalently all everywhere convergent power series). I admit having no global conception of that beyond knowing a generic way of describing its elements, their addition and scalar multiplication. This, and many other pertinent pieces of knowledge and mathematical experiences, persuade me to use this talk and to associate meaning with it. For instance, I find it sensible to ask for the dimension, for a basis, for subspaces etc. I am fluent in the discourse which treats this notion as a complicated object without having any external representation of it or even a mental object reflecting its structure and properties. This discourse for me is a sensible and meaningful way of thinking and talking about a collection of already beforehand accepted things (= entire functions). But of course, one could dispense with this discourse, it is not unavoidable, and, one could reject it for various reasons. There is no cogent reason which forces me to accept the collection of all entire functions as something to talk about in a meaningful way. I could equally well stay with some specific entire functions and reject thinking of the totality of all such functions as a well-defined and surveyable entity. In a way I have to want and to like this way of thinking, and of exploring its consequences. For short, that is not just a cognitive question but one of attitude, of agreement, of socialization and experience.

To sum up my personal experience with complex function theory: For me my knowledge about it is a complicated web and fabric of image schemata based on specific exemplar experiences, generic examples lending meaning to generalized statements and definitions, agreements to and acceptance of ways of talking, etc. In all, for me it is fluency in a (highly specific) discourse which encompasses syntactic and semantic but also attitudinal and emotional aspects. And, for all that I feel no need for mental or abstract objects!

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AN ANALYSIS OF THE STUDENTS' USE OF MENTAL IMAGES WHEN MAKING OR IMAGINING MOVEMENTS OF POLYHEDRA

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Abstract

We report a part of a research project focused on the analysis of the mental images, processes and abilities of visualization used by primary school students when they manipulate polyhedra. Students worked in a hands-on/paper/computer environment integrated by real solids, plane representations, and 3D software. The tasks presented here asked the students to rotate some polyhedra and to imagine their position after some rotations. The main part of the paper is devoted to describe and analyze the characteristics of environment and tasks.

The students exhibited ways of solving those tasks that reflect different levels of proficiency in the use of abilities and processes of visualization and mental representation of spatial relationships.

Introduction

During the last years we have carried out a research project* aimed to develop units for teaching space geometry in Primary Schools. As visualization and plane representations are two basic components of the learning of space geometry, a part of the research focused on the ways students build, manipulate, and draw 3-dimensional objects. We created a microworld, integrating real solids, plane representations, and 3D software, in which different types of problems with polyhedra were proposed to the students: Movement, comparison, plane representation, and building up of solids from plane representations. Because of the limited space, we don't report here the whole research project, but a group of tasks in which students were supposed to build mental images of polyhedra and to make mental rotations of them.

The aims of this report are a) to identify different kinds of students' behaviours arisen when they worked on tasks dealing with the movement of polyhedra, and b) to analyze how these behaviours correlate with the use, or lack of use, of different mental images, processes or abilities of spatial visualization. In the following pages we describe the part of the microworld relevant for the tasks described, present the procedures used by our pupils for solving the tasks, and analyze these procedures in terms of the processes or abilities of visualization used by the students.

* This research project has been funded by the Institución Valenciana de Estudios e Investigación "Alfonso el Magnánimo" de Valencia (Spain).

A theoretical framework

Spatial visualization has been studied from several approaches. The one most directly related to Mathematics Education and our research distinguishes three components of people's spatial visual activity: Images, processes, and abilities.

Presmeg (1986) defined a visual image as "a mental scheme depicting visual or spatial information". She identified several kinds of visual images; the most relevant ones for our research are:

- *Concrete images* are pictorial figurative images of physical objects. Our pupils dealt ever with this kind of images of the polyhedra they manipulated.

- *Kinaesthetic images* are images involving the physical movement of hands, head, etc. Our pupils very often moved their hands to represent a rotation before choosing the appropriate button on the computer screen, or while explaining to a researcher or to another child the movements they had just made.

- *Dynamic images* are images involving the mental movement of an object or some of its parts. These images were necessary for the students' work, since in most of the activities they had to rotate polyhedra or compare two different positions of the same solid.

Visual images are the units manipulated by people when they make a mental activity of spatial visualization. So when a person manipulates visual images, there is a flow of information between external objects and mental images or between different images. The main visualization processes controlling this flow were defined in Bishop (1983; 1989)*. Some of their components were relevant to our experiment:

- The process of *interpreting figural information* (IFI), that is, of reading, analyzing and understanding spatial representations (such as plane representations or mental images of polyhedra) in order to obtain some data from them.

- The *visual processing of images* (VP), that is, "the manipulation and transformation of visual representations and visual imagery".

We have also to consider different visualization abilities which may be used by students when solving 3-dimensional geometrical tasks. Del Grande (1987; 1990) made a summary of such abilities. The following ones were relevant for our study:

- *Figure-ground perception* is the ability to identify a specific figure by isolating it from a complex background. Our pupils used this ability when they had to identify faces of a transparent solid, since usually the edges created distracting shapes.

- *Perceptual constancy* is the ability to recognize that an object has invariant properties such as shape in spite of the variability of its position. We shall show below that one of our

* Bishop (1983) defined abilities instead of processes, so he referred to the abilities to perform the corresponding processes.

students had problems with respect to this ability.

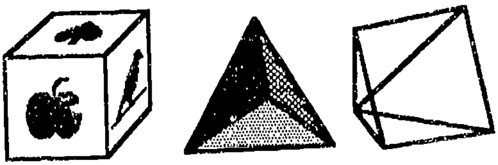
- *Visual discrimination* is the ability to compare several objects, pictures and/or mental images, and to identify their similarities or differences. As the students were asked to move or compare polyhedra, the had to use this ability.

The environment

We devised a microworld based on different solids: Cubes, tetrahedra, octahedra, square pyramids, and rectangular prisms. These polyhedra were presented in several ways:

• The appearances of the polyhedra were (figure 1): a) Figurative opaque cubes, b) shaded opaque polyhedra and c) transparent (straw) polyhedra.

• There were 3 physical embodiments for the solids: Real solids (made of cardboard or straws), perspective 3-D representations on a computer, and perspective 3-D representations on paper (hard copies from the computer screen). The figures or shades on the faces were the same in the 3 cases (hard copies from the computer).



- a - - b - - c -

Figure 1.

We used computers Macintosh SE with 2 programs: 1) We created a Hypercard stack allowing to rotate a figurative cube 90° around the 3 axes crossing its faces (figure 2): When one of the 6 rotation arrows is marked, the computer automatically rotates the cube 90° in that direction.

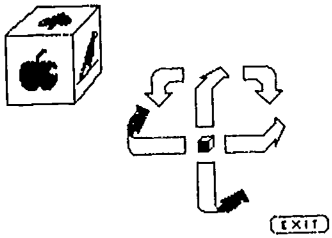


Figure 2.

2) The program Phoenix 3D allows to rotate freely solids around the axes X, Y, and Z (figure 3). After selecting one of the 6 direction arrows, the user has to determine the angle of rotation; for doing this, the software uses the metaphor of a hand turning in the chosen direction (in real time) a little pyramid that appears on the screen; when the user has decided the angle of rotation and releases the mouse button, the computer shows the new position of the solid.

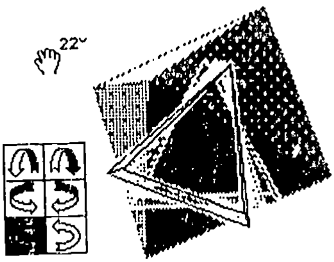


Figure 3.

The tasks

Several types of tasks, and several instances of each task, were proposed to the students, combining the different polyhedra, their appearances and embodiments, and the computer programs: Some times the students had to move a real solid and other times one on the computer; some times they had to compare a real solid to one on paper, other times to one on the computer, and other times solids on paper and the computer; etc. In this report we shall focus on two of those types of tasks: The first task asked the students to move a solid on the computer screen, from its current position into another position shown on paper. An example is this activity:

"Open the [Phoenix 3D] *file "Solid Tetrahedron". Move the tetrahedron on the screen to the first position you see below* [figure 4]. *Then, move it to the second one, and so on."*

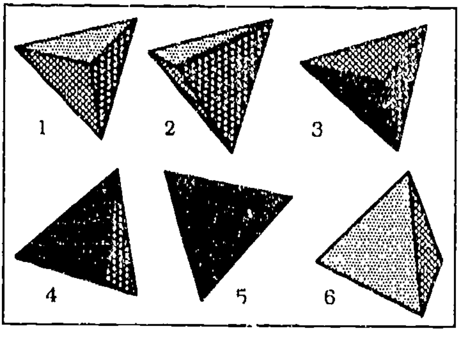


Figure 4.

For the second task, the students used a sheet of paper containing a picture of a figurative cube and several other views of the same cube with some blank faces. The students were asked to draw, if possible, the missing figures on the blank faces. An example is this activity:

"On the right there is a picture of a cube, and below there are some more pictures of the same cube [figure 8]. *Try to draw the figures on the blank faces of those cubes."*

Both tasks are related since the students should manipulate mental images of solids in both of them. The main difference between the tasks is due to the fact that the solid used can be rotated in the first task but no in the second one.

The subjects

Three 6th grade students (11-12 year olds) participated in the research. They had different ability levels: C was a high ability girl, E was an average ability boy, and M was a low-average ability girl. They had not received any previous instruction on techniques of visualization (mental rotations, plane representations, etc.). Their knowledge about 3-dimensional geometry previous to this experiment consisted on the learning of the main families of solids (prisms, pyramids, ...), their elements (faces, edges, ...) and basic properties (equalities, perpendicularity, ...). This knowledge was useful for our experiment, since the

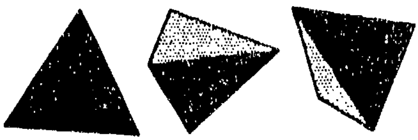
students only manipulated known polyhedra, so they did not suffered any difficulty for a lack of knowledge of some polyhedra's property.

In the tasks not requiring computers, each student worked alone, but interactions among them were allowed. When the tasks required the use of computers, as only 2 computers were available, C and E worked together, and M worked alone. This grouping was decided because when M and another child worked together, as M was the less able, it resulted in M's inhibition, being the other student who solved most of the activities.

Analysis of answers

To analyze the diversity of the students' answers we have taken in consideration the different elements of visual thinking involved; such analysis should help us to understand how the students' capabilities of visualization develop. In this section we show and analyze some students' answers to the tasks described above. The students carried out the first type of tasks (movement of solids on the computer) by using two different strategies, reflected in the excerpts included below:

• M had to move the shaded tetrahedron from its current position on the screen (figure 5-1) to the position shown by the model, a real tetrahedron (figure 5-2). The following dialogue took place between M and a researcher (R):



1 2 3
Figure 5.

[First M moved the solid to position 5-3 by and]

- M: *Ok.*

- R: *Is it in the same position?*

- M: *It is not the same face.*

- R: *This vertex here* [pointing to the model], *which one is here?* [pointing to the screen]

- M: *Ah! It is upside down.*

- [M made several rotations , but at the end the tetrahedron was nearly in the same position as before these turns]

- M: *Ok.*

- R: *Are they in the same position? It is the same as before. You have left it in the same position.*

[M started turning the solid by] *Tell me. What are you looking for?*

- M: *I don't know.* [she continued rotating the same direction] *Look. It is wrong.*

- R: *Is that movement useful to you?*

- M: *No, because ...*

- [R had to guide M, but she continued making complete rounds in a direction]

• In a similar activity, C and E had to move the shaded pyramid from its current position on the screen (figure 6-1) to the model's position (figure 6-2). E made the movements although, almost every time, C decided the movement to be made. This was the dialogue between C and E:

1 2

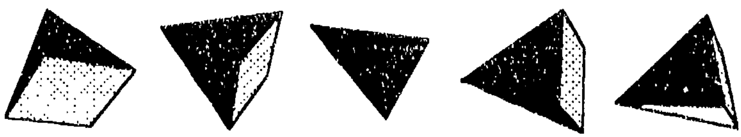
Figure 6.

- [C and E had moved the pyramid to the position in figure 7-1]

- C: *Now it has to be moved so that this* [pointing to the vertex in the screen] *goes there* [pointing to the vertex in the model].

- [E made the rotation , and the pyramid moved to position 7-2]

- C: *I see that. Now it has to be moved down* [E moves the pyramid by rotation , to position 7-3].



1 2 3 4 5

Figure 7.

- C: Ok. *Now, put it horizontal, horizontal. Do you know?* [C "drew" an horizontal line with her finger in front of the screen, and E moved the pyramid by].

- C: *And now we have to see this face* [the base].

- [C directed E to rotate the pyramid by and to reach position 7-4. Then C realized that they had mistaken the model's front face for another with a similar shade. Then she continued directing E in the same way as for the previous excerpt until they put the pyramid in position 7-5]

While M usually made a search by guess, just rotating the solid until it happened to stop in a position resembling the model's one, C and E used efficiently the IFI and VP processes, pre-planning the search and deciding which movement (or sequence of 2 or 3 movements) should be done by comparing the positions of the movable solid and the model.

• When the same task had to be done with a figurative cube, all the 3 students used the same strategy: They moved the cube until the figure on the front face of the model appeared

on the screen; then, they moved the cube to put that figure on the front face; finally, they turned the cube to put this figure in the correct position.

But having a correct strategy did not imply an accuracy in the movements: While C never needed to move the cubes more than a few steps, M some times made very long sequences of rotations and other times short ones. There is never a need of more than 5 rotations to move the cube to the position of the model, but often M made up to 10 rotations (in an occasion she needed 21 turns!). Such different behaviours mean different levels of proficiency in the use of visual abilities and processes: While C was able to transform correctly her mental visual movements into real rotations on the computer (i.e., to use properly the IFI process), many times M was unable to perform adequately such process.

• Finally, we refer to an E's answer to the second task described, in which it was evident that E had not completely acquired the ability of perceptual constancy. The students were provided with a sheet of paper containing a picture of a cube (figure 8 M), and other views of the same cube with some blank faces (figures 8 1 to 8-4). They were asked to draw, when possible, the missing figures on the blank faces.

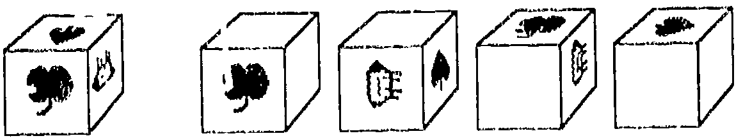


Figure 8.

The students could not move the model, so they had to imagine the shapes and positions. In the cube 8-4, E drew an apple on the front face and a sheep on the right one. This is a part of the dialogue between E and a researcher:

- R: [watching E's drawings] *It is not clear to me. Where does the sheep look at?*
- E: *Downwards.*
- R: [pointing to cube 9-M] *Doesn't it look at the apple?*
- E: *Yes.*
- R: *But here* [cube 9-4] *no longer*
- E: *Of course, because it moves.*
- R: *But do all faces move at one time, or only one face does?*
- E: *This* [face] *moves in this way, then that* [face] *moves in this way*
- R: *And, does the sheep continue looking at the apple or no longer?*
- E: *No longer.*
- R: *Then, by turning the cube, can you make the sheep look at different figures?*

- E: *Of course.*

Conclusions

• The use of 3 different types of solids (figure 1) was relevant in this experiment, since it was evident that the figurative cubes were confidently manipulated by our pupils, while the transparent solids were the most difficult for them. So students of different ages or ability levels shall react in different ways to the various kinds of polyhedra.

• When selecting software for moving solids, it is important to know how each program asks for data. Some programs allow only rotations of a specific angle; others represent graphically the rotation's amplitude (no input for angles is needed); and others ask to key in the value of the angle to be rotated. The first type of software is the best for young children, but it may inhibit the development of analytical reasoning. The third type is the best for good visualizers, who are able to imagine complex movements and to calculate accurately the angles of rotation. The second type is good for students in an intermediate stage of development of their capabilities of visualization.

• Different types of activities have to be proposed to students to promote a complete development of their visual reasoning: Movements, comparisons, drawing of, and building from, plane representations are types of activities in which different kinds of mental images, abilities and processes have to be used by the students.

• Some questions have arisen from this exploratory research about the students' behaviours and mental activities: The use of analytical or visual ways of reasoning. The kinds of mental images (for instance, static or dynamic, global or local) generated by students of different ages or abilities. These questions should be investigated in future phases of the research.

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PUPILS' DEVELOPMENT OF GRAPHICAL REPRESENTATIONS OF 3-DIMENSIONAL FIGURES
-- On Technical Difficulties, Conflicts or Dilemmas, and Controls in Drawing Process --

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ABSTRACT. In earlier papers we noted some developmental aspects of graphical representations of 3-dimensional figures in elementary school pupils. In this paper, on the base of attempts which pupils performed communication tasks on 3D-figures and the individual interview with some of them, we will observe what technical difficulties they have in drawing, what conflicts or dilemmas they are confronted with, and how they control their difficulties by themselves. In pupils' perspective drawings supplemented with any spatial information via words, symbols and in their attempts of redrawing, we will observe their difficulties or dilemmas, and their strategies to control them.

Purpose and Background of the Research

In earlier papers reported based on cooperative research, we identified developmental levels of perspective drawings of 3D-figures, where pupils were asked to represent cuboids, cylinders,pyramids and junglegym on a sheet of paper, seeing the models of them in one case and imaging them mentally without models < (1), (2)>. Our individual interview with 3rd and 4th graders showed their technical difficulties, conflicts or inevitable dilemmas between their awareness, seeing or imagery, and drawing, especially at C-level which we will show in Table 1. Though many of pupils at C-level, and some of them at lower levels also, recognized them, they could not control them.

This research has also connections with those which reported based on French-Japanease co-operative research <(3),(4)>.

In this paper we will observe what technical difficulties pupils have in drawing, what conflicts or dilemmas they are confronted with and how they control them, and we will further explore developmental aspects of graphical representations of 3D-figures.

Methods of the Research

Our experiment was conducted as follows.

Subjects: Elementary school pupils in Osaka, two classes for each grade.

Grades	1st	2nd	3rd	4th	5th	6th	Total
Numb. of pupils	45	57	58	62	40	56	318

Date: February, 1992

Tasks: Pupils were asked to represent each of 4 types of models of 3D-figures and to convey

spatial information of each of them to his/her friends staying far from here.

Instruction to pupils: "Here is a set of 4 types of models for each of you. You will receive them one after another. You would like to represent what you have on a sheet of paper and to convey the suitable representations so as to identify them to one of your friends staying far from here. You may give your explanations using drawings, words or symbols, as you wish."

Material: (1) wooden cylinder-models with a diameter of 5 cm and hight of 8 cm,
(2) wooden cuboid-models with edges of 5 cm, 5 cm and 8 cm,
(3) wooden regular-pyramid-models with edges of the base of 5 cm and hight of 8 cm,
(4) for 1st and 2nd grades: framed unit cube made of steel sticks and rubber holders,
for 3rd and 4th grades: framed cubes which 2×2×2 unit cubes will fit into,
for 5th and 6th grades: framed cubes which 3×3×3 unit cubes will fit into.
(Latter two models were called "jungle-gym" by pupils.)

Procedure: First each pupil received a cylinder-model. After he/she performed the task, he/she received a cuboid model, and so on.
Whole time for performing all tasks: about 50 minutes.
We interviewed with some pupils individually after the tasks.

Classification and Levels of Graphical Representations

We introduce our classification of pupils' graphical representations of 3D-figures and developmental levels in perspective drawings.

1. Classification of pupils' graphical representations of 3D-figures

(1) Graphical representations of any "state" or "form" of 3D-figures (we call them "static")
(P) Perspectives (or pictorial drawings, including one plan),
(E) Projections (two or more plans or orthogonal projections),

(2) Graphical representations of any construction of them after breaking mentally them up in any components (we call them "constructive").
(F) Nets (of solid figures)
(G) Constructive drawings (which contain "static" types P or E of representations of components and any information to direct how to build up or construct 3D-figures, such as poly-cubes or framed-cubes). Pupils' attempts to represent jungle-gym by using constructive drawing are shown in Fig.1.

2. Developmental levels in perspective drawings

In pupils' perspective drawings we classified 5 levels: N, A, B, C, and D level, which are identified similar typical representations, as shown in Table 1.

Results and Discussion

We observed pupils' developmental inclinations in graphical representations from following three aspects:

types of representations,

levels in perspective drawings, and

technical difficulties, conflicts or dilemmas and controls them in drawings.

1. Developmental inclinations in graphical representation types

In representation of cylinders, cuboids and pyramids, perspectives were used by most of pupils at 1st-5th grades, and two or more plans (or orthogonal projections) were used by a small number of them at 3rd and more at upper grades, gradually increasing. At 8th grade about half of pupils used two plans or orthogonal projections or nets.

(In Japan at 8th grade pupils learn orthogonal projections via two plans and nets of some solid figures.) Some of 5th and 6th graders represented a solid figure by using various types , as shown in Fig.2.

In the case of jungle-gym 4th-6th graders often used constructive drawings. We showed the examples used by 3rd and 4th graders, as shown in Fig.1.

2. Developmental levels in perspective drawings

(1) With a raise of grades levels proceeds to D-level, gradually and respectively for each figure.

1st and 2nd graders distribute mainly at A and B-level in the drawings of cuboids and pyramids, and at A, B and C-level in the drawings of cylinders. 3rd and 4th graders distribute at B,C and D-level, though rates of distribution of each level are different for each of 3 solid figures, for each of 3rd and 4th grades also. Most of 5th and 6th graders are at C or D-level and more 70% of them are at D-level in each drawings of 3 solids.

(2) In the drawings of cylinders, in comparison with other solids, more pupils are at D-level. In the drawings of cuboids the developmental inclinations described in (1) are showed remarkablly. On the other hand in the drawings of pyramids 4th-6th graders at D-level are less than in cylinders and cuboids. Pyramids are not familiar to pupils and besides it is often difficult for us to identify which level it is.

3. Developmdntal inclinations in technical difficulties, conflicts or dilemmas, and controls

(1) On supplementary information added to the perspective drawings

In this experiment, to perform the communication tasks many pupils used perspectives supplemented with any spatial information via words, numbers, symbols, etc., in other words, they used "mixed types" of perspectives. Supplementary information or legends are classified as follows.

Fig.1

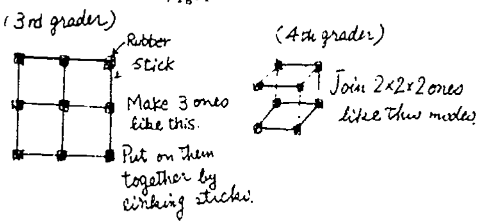


Fig.2.

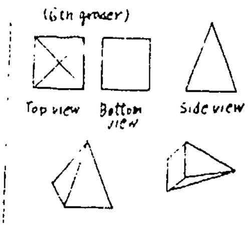


Table 1

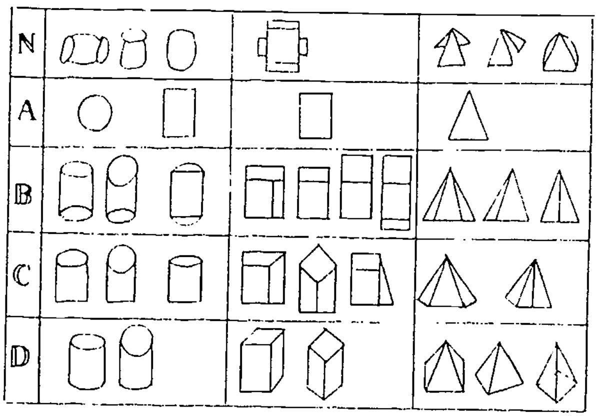
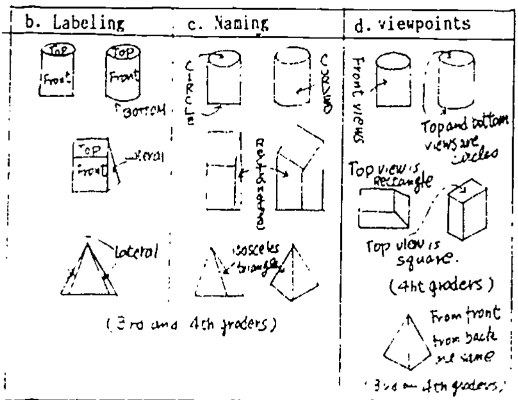


Table 2



1) Verbal description of a form or whole of a solid figure:

for cylinders: "Logs","Cups","Cans", etc,

for cuboids: "Wooden long square", "Boxes", "It is the most fine form of objects and it seems to overlook all",etc,

for pyramids: "Tents","Pyramids","The form is a isosceles triangle",etc.

2) Attributes of components, relations and structure (curved face, shapes and number of faces,etc.)

a. "Making up" faces or lines by shadowing and accentuating line,

b. Labeling the faces, such as "top","bottom" or "lateral" of a model,

c. Naming curved face and shapes of faces and numbering of faces,

d. Directing ones viewpoints, such as "where I saw it from",

(Pupils' attempts for each legend-b, c, and d are shown in Table 2.)

e. Drawing dotted lines.

3) Measurement

Pupils' "mixed types" of perspectives are summed up, as shown in Table 3.

1st and 2nd graders accentuate "where striked me" by using "making up" techniques.

Legends added to perspectives via words, symbols etc. are used by many pupils at 3rd and upper graders. 3rd and 4th graders supplement with "verbal description of a form of the solid" and "attributes of components", such as curved faces, shapes and the number of faces, on the other hand 5th and 8th graders supplement with "drawings dotted lines".

There are clear distinctions between legend-a,b,c and d. Labeling faces shows corresponding the figure drawn with the components of the model: "This is the top of the model". Naming curved faces or shapes of faces shows corresponding the figure drawn with any attribute of the components: "This is curved" or "This is a rectangle, though it looks like not so". Directing ones views shows corresponging the figure drawn with ones viewpoints: "This is a square, because I saw it from upper".

We observed pupils' difficulties, conflicts or dilemmas, especially in perspectives with legend-b and c. For example, pupils at C-level, who added legend-b, said: "I know the base is a circle, but I did not draw it. Because it must be flat and straight." or "Though this lateral must be a rectangle, this (figure) is not a rectangle, so I wrote 'rectangle'".

Legends-c directs their viewpoints: "Where I saw it from." or "I know the back view is same this front view". Legends-c are used more at 4th grade than at 3th grade. We may say that 4thgraders' viewpoints are freer than 3rd graders' ones.

5th and 8th graders drawing dotted liens are at D-level and they see hidden lines. Some of them drew some perspectives of a solid figure from free viewpoints. 5th graders use often legends of measurement and 5th and 8th graders' attempts give a set of spatial information of a solid figure.

(2) Redrawing and moving to other level or staying at the same level

The Table 4 shows changes of levels, that is, moving to other level or staying at the same

Table 3

Legends \ Grades / Solid Figures		3rd grade Cylinder	3rd grade Cuboid	3rd grade Pyramid	4th grade Cylinder	4th grade Cuboid	4th grade Pyramid	5th grade Cylinder	5th grade Cuboid	5th grade Pyramid	6th grade Cylinder	6th grade Cuboid	6th grade Pyramid
Verb. Descr. of Whole Form		6	5	5	12	9	4	2	0	1	2	1	1
Attributes of Components, Relations, Structures	Labeling Faces	2	0	1	4	5	4	0	0	0	0	0	0
	Naming Shapes	3	3	5	6	7	11	1	2	1	1	3	4
	View Points	0	1	2	4	2	2	1	3	1	4	1	2
	Dotted lines	0	0	0	1	1	1	7	5	5	4	11	10
Measurements		4	6	5	2	2	1	14	14	13	2	4	6

Table 4

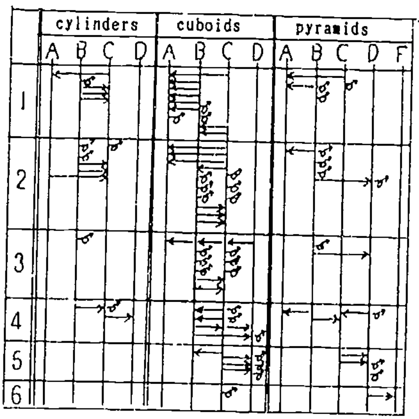


Table 5

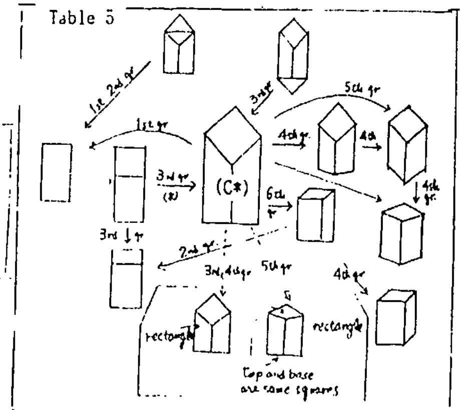
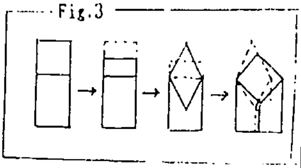


Fig.3



level after pupils' attempts of redrawing. A straight arrow in Table 4 means a moving, and the starting point and the end point of the arrow is corresponding to the level of drawings of pre-redrawing and the level of post-redrawing respectively. A curved arrow means staying at the same level.

In the case of cylinders pupils' levels of post-redrawing moved to upper levels respectively except for one attempts.

In the case of cuboids pupils' attempts are very complicated and some of them are shown in Table 5. In the case of pyramids some of pupils at lower grades changed their viewpoints.

At C-level, many pupils at 3th and 4th and a few at all other grades used one of the typical representations of cuboids, such as the drawing C* shown at center in Table 5.

Table 5 shows changes of levels in pupils' attempts around the typical drawings C* by redrawing or adding any legends.

1st and 2nd graders, who moved to A-level from the drawing C*, said "It(C*) looked like strange for me, and it's easy to draw it from front." 3rd graders, who moved to the C*, said "I know the base is a square, though I see it like a upside-down triangle. Drawing the triangle, it looks like another object, because it must be stand holizontally".

The 3rd grader's attempts shown in Fig.3 showed that he moved, through many steps, from B to C-level.

In Table 5 we observe inclinations as follows. The first, 1st and 2nd graders' attempts incline to move from the drawing C* to A or B-level in their redrawings and it seems that the drawing C* is not stable but strange or even doubtful for them as the representation of the cuboids.

The second, 3rd graders' attempts incline to stay at C-level by adding to the drawing (C* legend-a, b or c, though the drawing C* is strage or doubtful for them.

The third, 4th and upper graders' attempts incline to move to upper level.

SUMMARY

Graphical representations of 3D-figures are important means to communicate spatial information on them between a teacher and pupils, pupils and other pupils, and a pupil and him/herself.

In graphical representations of cylinder, cuboids and pyramids on the communication tasks pupils use mainly perpective drawings, and with raises of grades 4th and upper graders use gradually various graphical representations : perspectives from different viewpoints, two plan or ortogonal views, and nets of the solids. In jungle-gym 4th and upper graders use often "constructive" drawings, in which they seem to recognize somewhat of 3-dimensional figures.

With a raise of grades the level in perpective drawings proceeds to D-level, gradually and respectively for each solid figure.

Many of pupils , especially at C-level, and for pyramids some of them at B-level also,

have technical difficulties in drawing and recognize conflicts or inevitable dilemmas between their awareness, seeing and drawing. As we showed some examples, pupils feel against representing the base of cuboids using oblique lines, and instead of using them they add to their perspectives any legend or they change their viewpoints in order to represent more stably and more suitably one.

Adding legends to perspective drawings is their strategy to control technical difficulties ,conflicts or inevitable dlilemmas which they are confronted with in drawing. From this point of view Table 4 shows some aspects of developmental inclinations of graphical representations of 3D-figures as follows:

Verbal description of whole solid,

Attributes of components, relations and structure,

a. "Making up" faces or lines

b. Labeling faces

c. Naming shapes of faces

d. Directing ones viewpoints

e. Drawing dotted lines

Measuremsnts.

Many of 5th and 6th graders' attempts in drawing give a set of spatial information of a solid figure also from these aspects descrived.

When pupils feel strikingly technical difficulties, conflicts or dilemmas, they redrew. Some aspects of developmental inclinations were oberned in pupils' redrawing attempts: moving to other level or staying at the same level.

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CHILDREN'S USE OF DIAGRAMS AS A PROBLEM-SOLVING STRATEGY

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Abstract : Drawing diagrams is an often recommended strategy for problem-solving in the literature. This paper examines to what extent children actually use this strategy and discusses some specific examples of their drawings along with their use of diagrams presented by the authors.

Introduction

For many years now the ability to solve problems in mathematics has been recognised by mathematics educators as a key element of the mathematics curriculum (NCTM,1984; Cockroft,1982). School children however, have been noted to experience considerable difficulty in solving mathematical word problems despite the increased emphasis given to it in school mathematics. A step that is considered critical in solving word problems is understanding the problem and representing it in a meaningful way (Schoen & Oehmke,1980; Davis & Silver, 1982). There are at least 3 ways of representing a problem, (1) informal and linguistic, (2) physical or visual and (3) algebraic (Resnick,et al., 1981). Related to (2) is the use of drawing of diagrams as a strategy in solving word problems.

The drawing of diagrams as a problem solving strategy has long been advocated by mathematics educators (Polya,1957; Schoenfeld, 1985; Hyde.A.A. et al. 1991) and has of late received research attention (Gonzales, 1986; Bodner,1991) Although this has been the case, the extent to which children use this strategy in problem solving, or how spontaneously this strategy has been resorted to has not been fully investigated. In a recent study by the authors (Lopez-Real, Veloo & Maawiah,1992) analysing the performance of primary school children in a mathematics examination, it was noted that in only about 1% of the word-problems attempted by pupils was any kind of diagram evident. Furthermore, even if the word problem involved references to geometric figures such as rectangles and so on, the use of diagrams to solve the problem was rare indeed. The main focus of this paper will be on children's success in solving word problems after the hint of drawing diagrams was given.

Methodology

Ninety-six Standard 5 and Standard 6 primary school children were given a series of word-problems to solve. After marking the problems a selection of the unsuccessful instances were studied further using a "Newman" interview procedure (Newman,1977). That is, classifying the point at which error occurred under the categories Reading, Comprehension, Transformation, Process, and Encoding. A number of the problems used in this test had been chosen because, in the authors' view, the use of a diagram could be helpful, if not essential, in their solution. It is these particular problems which the rest of this paper is concerned with.

None of the children in the study failed to solve a problem due to a Reading error. The Comprehension and Transformation stages are often difficult to untangle even during interview, however they are both clearly distinguishable from a Process or Encoding error. Thus the errors that were identified in those former stages were now followed up with the following procedure. First, the pupil was asked to draw a diagram of the problem and to include any important information from the problem. They were then asked to try to solve the problem again, using their diagram. No help or assistance was given by the interviewers. If the pupil was unable to draw a diagram or was still unsuccessful after having drawn one, they were then presented with a diagram drawn by the authors and asked to try again using this diagram.

Overall Results

The main purpose of this paper is to discuss some of the idiosyncratic approaches of pupils using diagrams in problem-solving and to try to identify any significant elements. However, it is useful to look briefly at the overall pattern that emerges from the study. In the initial testing, the number of "diagram-suitable" problems that were marked was 693. Out of this number just 35 (i.e. 5%) involved the use of a spontaneous diagram. (Out of these 35 cases, 20 were solved correctly.) This is a striking indication of the fact that although the use of diagrams has been widely advocated in the literature, as mentioned in the introduction, in reality it is a very rarely used strategy by children. The follow-up results for those problems which had been unsuccessful at the Comprehension or Transformation stages are as follows :

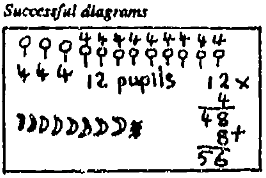
Able to draw Diagram		If Diagram Drawn		If Authors' Diagram Used	
Yes	No	Successful	Unsuccessful	Successful	Unsuccessful
107	19	41	66	52	26

(*Note: In a few cases, after an unsuccessful attempt using their own diagram, the authors' diagram was not subsequently used because it was almost identical to the diagram drawn by the pupil.*) The striking feature to notice here is that nearly one third of the attempts which had previously failed were now successful with no further assistance or help other than the suggestion to represent the information and situation of the problem pictorially. The responses to the specific problems used in the study are now discussed in more detail.

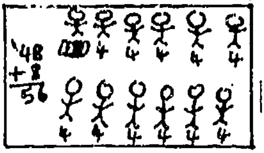
Discussion of selected diagrams

Problem1: A teacher bought a bunch of bananas to be shared equally among 12 boys. 8 of the bananas were bad and were thrown away. After this each boy received 4 bananas. How many bananas were in the bunch at the start?

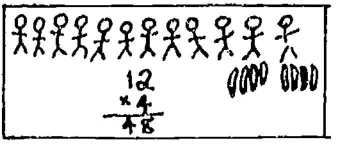
Successful diagrams



Izzudin

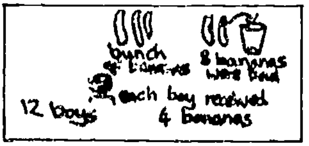


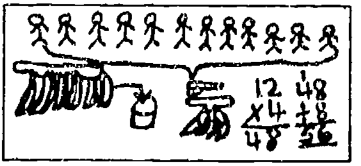
Termizi



Hasnal

Rebecca



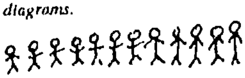


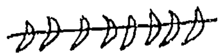
Pamela

The diagrams show that once a matching has been established between the students and bananas, pupils were able to obtain with ease the critical step 12 x 4. Izzudin's is an example of the more common form of helpful diagram. Termizi shows a slight variation in that be began by drawing 4 bananas (concrete) but then decided to write instead the number 4 (symbolic) for the rest of the diagram. Similarly Hasnal began by drawing 4 bananas per pupil but did not proceed beyond the 2nd pupil. The process of beginning to draw the picture itself was sufficient to enable him to establish the matching between pupils and bananas. Pamela and Rebecca are examples of a more symbolic form of representation. Pamela showed the relationship between the students and bananas by means of a long bracket. This diagram was meaningful enough for her

personally to work out the answer. On the other hand the function of Rebecca's diagrams appears to be to help her to isolate the important aspects of the problem.

Unsuccessful diagrams.



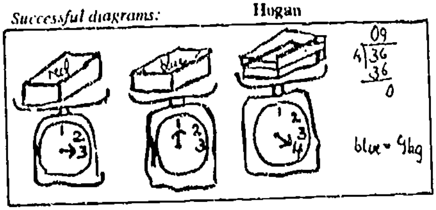


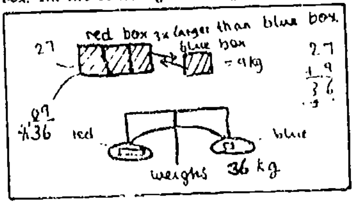
Two examples of unhelpful diagrams are presented above. In both these diagrams the important information has not been incorporated in the diagrams, and hence was of no help to the pupils concerned in solving the problem.

Problem 2. A red box weighs 3 times as much as a blue box. The two boxes together weigh 36 kg. What is the weight of the blue box.

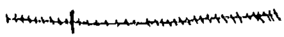
Successful diagrams:

Hogan





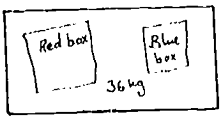
Rozie

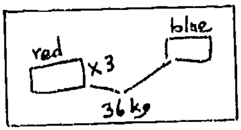


Izyan

Three different types of diagram were observed for this question, the most common being ones similar to that drawn by Rozie. The diagram expresses clearly the relationship between the red box and the blue box. An interesting representation of the problem was observed in the diagram drawn by Hogan. He drew three balances, the first with the pointer at 3, the second with the pointer at 1 and the third with the pointer at 4. Hogan's comment at this point was, " I am only marking 4, because 36 would be too big." This strongly suggests that Hogan has a clear view of the ratios involved and keeps the value 36kg(the total weight) in his head. He then successfully wrote down the step 36 ÷ 4. (Notice that his scales do not begin with zero though.) Izyan drew a line and marked out 36 parts to represent the total weight 36kg. He counted along the line and marked a longer division on 9 and said that the weight of the blue box is 9kg. He then proceeded to calculate the weight of the red box as 36-9. This is an example of an idiosyncratic representation, which was clear to Izyan personally but may not be to others.

Unsuccessful diagrams



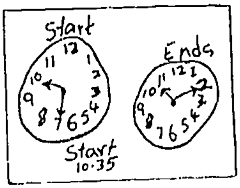


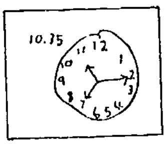
These diagrams lacked a clear visual relationship between the boxes. This appears to be a crucial factor as the pupils were found to solve the problem with ease when presented with the authors' diagram which was similar to Rozie's.

Problem 3. A lesson lasts for 35 minutes. It ends at 11.10 a.m. At what time does the lesson begin?

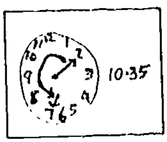
Successful Diagrams:

Omar





Faikis

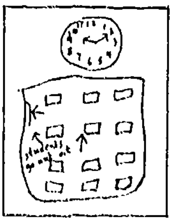


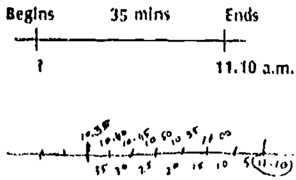
Emma

Although Omar has drawn two clocks his solution to the problem was carried out entirely on the "Ends" clock by counting back. Thus the drawing of the "Start" situation seems to be simply a way of recording the answer. However, it was noted that having drawn the solution Omar then counted forward, thus checking his answer. Another point to notice is that the hour hand is rather inaccurately placed and another person interpreting the diagram would be likely to read the times as 10.10 and 9.35. It appears that Omar keeps the "hours relationship" in his head while using the diagram to help him with the problem of the minutes. The same strategy is apparent with Faikis who draws the final situation, then counts back, and then inserts only the minute hand. But his final answer is nevertheless correctly written. This strategy is even more clearly illustrated by Emma who does not bother to include the hour hands in her diagram at all.

Unsuccessful Diagrams:

Adli





In Adli's case it is possible that he has spent so much time on irrelevant features of the final situation that he loses sight of the relational aspect of the problem i.e. the time difference between start and end. On being presented with the authors' diagram Adli's response was fairly typical. *Adli: But it doesn't show the minutes.*

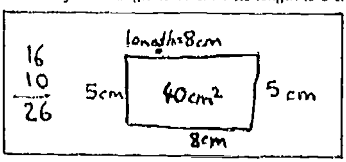
Interviewer: No, but if you like you can copy it and add anything you want.

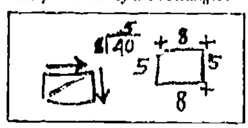
Adli then proceeded to draw his own version and successfully solved the problem. The markings at 5-minute intervals were spontaneous on his part.

Problem 6: The area of a rectangle is 40 cm^2. Its length is 8 cm. What is the perimeter of the rectangle?

Successful diagrams

Deirdre





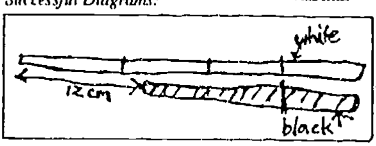
Izyan

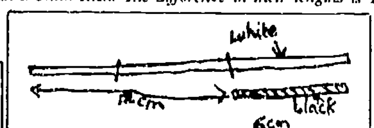
All pupils were able to draw a rectangle. Deirdre's solution was similar to the authors' diagram and was fairly typical of diagrams drawn by successful pupils. Izyan drew two rectangles, the first with arrows along the sides to indicate perimeter and the other with actual dimensions. He worked out the problem as shown.

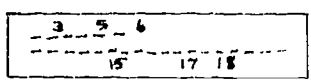
Problem 7. A white stick is 3 times longer than a black stick. The difference in their lengths is 12cm. What is the length of the black stick?

Successful Diagrams:

Kartika



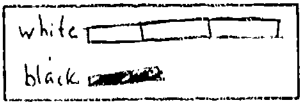




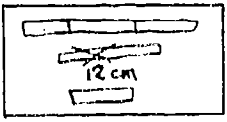
Marcus

Kartika made two attempts at a drawing without any prompting from the interviewer. Having completed her first drawing she immediately shook her head and said "No, that's not right". She proceeded to draw a second diagram in and then wrote down the answer 6cm without hesitation. Her second diagram is in fact almost identical to the authors' diagram for this problem. Marcus used a totally different strategy. His representation was in terms of "matchsticks" and he used a trial-and-error approach. Marcus was also unusual in that he focussed strongly on the difference between the two sticks being 12cm. He kept this parameter constant while varying the lengths and checking on the "3-times" condition. The figures in the diagram show his consecutive trials, i.e. (3,15), (5,17), and finally (6,18).

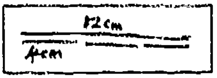
Unsuccessful Diagrams:



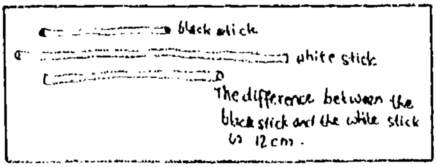
Hairol



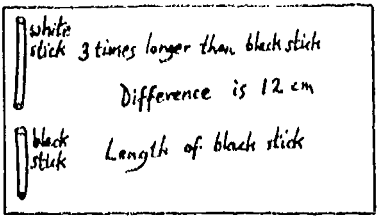
Ramziah



Farhana



Rafiuddin

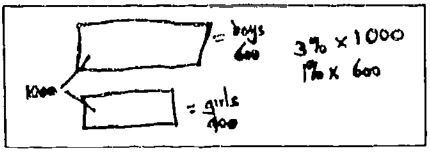


Azme

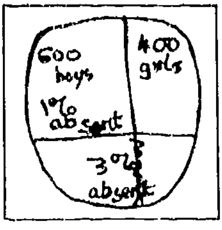
A number of the children were able to represent the "3-times" relationship effectively but were then unable to interpret the "difference" appropriately. Hairol has aligned the sticks in the most helpful way in terms of the "difference" aspect but does not put this information on his diagram. Similarly Ramziah and Farhana illustrate a lack of understanding of the difference concept. Both Rafiuddin and Azme were unsuccessful with their own diagrams but quickly solved the problem using the authors' version. In both cases they have drawn concrete examples of the sticks and the information from the problem is simply written alongside the diagram. Thus, the data and relationships stated in the problem are not effectively incorporated into the diagram. This was a common feature of many unsuccessful attempts for all the problems.

> *Problem 8. There are 600 boys and 400 girls in a school. One day 3% of all the children were absent. If 1% of the boys were absent, how many girls were absent?*

Successful Diagrams:



Pauline

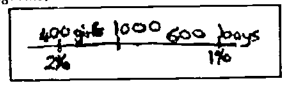


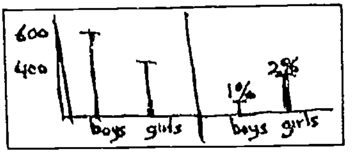
Sara

The key feature of both these cases is the fact that the 3% is clearly related to the whole population. However, Pauline indicates the number 1000 on her diagram which does not appear as data in the problem (i.e. she has already performed a calculation). On the other hand, Sara has only marked data given in the question. Her understanding that the 3% must be applied to the whole school is shown by her crossing out of the dividing line between boys and girls at the bottom of the diagram.

Unsuccessful diagrams;

Safrina





Malai

In the same way as Pauline above, both these cases have incorporated numbers which they have calculated. Unfortunately they have both jumped to the conclusion that the percentage of girls absent is 2%. This was by far the most common error for this problem. Without this error it is possible that Safrina's diagram would have been helpful but the separation of the data into two halves by Malai prevents the crucial relationships being seen.

Discussion of General Features

i) Initial attempts.

Before we consider features relating to the use of diagrams it is of interest to look at the kinds of strategies and behaviour evidenced by the children *prior* to being asked to draw a diagram. One of the most common was performing an operation initiated by a "cue" word in the problem. For example, the word "times" in Problems 2 and 7 often resulted in the operations 36x3 and 12x3, and the word perimeter in Problem 6 sometimes led to 2(40+8). Stuart's comment while trying Problem 1 highlights this type of strategy: "It says 'thrown away' so it's *got* to be a subtract somewhere!" The trouble is that such words act as mis-cues almost as frequently as they cue the appropriate operation. Unfortunately, we suspect that the strategy of looking for cue words is actively encouraged by many textbooks and teachers.

Even where it was not a case of being misled by a cue word, the speed with which children launched into operating on the numbers themselves (without apparent reference to their meaning) was very striking. Typically, 12x8 and 12-8 were frequent attempts at Problem 1, while 36+3 and 12+3 were even more common for Problems 2 and 7. Subtracting 1% from 3% in Problem 8 was an immediate reaction in almost all the pupils. Problem 3 was interesting in that many children chose the correct operation but subtracted the numbers as if they were ordinary decimals (i.e. 11.10 − 0.35 = 10.75). Moreover, a number of children spontaneously indicated that they knew their answer was not right − "There's only 60 minutes in an hour" − but were unable to see how to correct it.

ii) Function of Diagrams.

Given that nearly 1/3 of the pupils' attempts were successful after simply drawing a diagram, can we identify what function the diagram is playing in those cases? The Problems where a diagram was suggested by the interviewer were those where children had failed at the Comprehension or Transformation stages, so can a diagram help in some way to overcome those difficulties? Let us first consider Comprehension (or Understanding). We suspect that there are at least two elements involved in understanding to be considered here. The first arises from the Newman instruction " Tell me what the question is asking you to do." This is effectively asking whether the child understands the *structure* of the problem and does not probe any understanding of the second element which is the *concepts* involved. Thus it was clear that some children

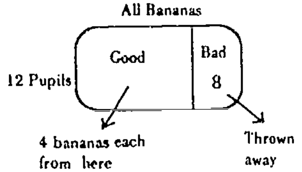
understood the "sticks" problem in terms of knowing they had to find the length of the shorter stick but did not have an understanding of the "difference" concept. Does this mean that understanding (in both senses above) is a *pre-requisite* for drawing an appropriate diagram? From our observations, it appears to us that the very act of drawing a diagram focusses the child's attention on to what the numbers *represent* rather than the numbers as abstract entities. In this sense the function of a diagram may be to act as an alternative form of "expressing the problem in one's own words" (which is another problem-solving strategy often advocated in the literature). It is as though it helps oneself become aware of one's own understanding. Having focussed on the meaning of the numbers and their relationships within the problem, it then appears that the diagram can act as a key aid in the Transformation stage. For example, as we saw before in the "bananas" problem, Hasnal breaks off the drawing process because his initial matching of pupils with bananas brings the realisation (*which he did not have before*) that he needs to multiply.

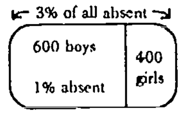
iii) Helpful Features of Diagrams

Despite the optimistic note of the last section, we must remember that for most of the children their own diagrams did not lead to success. Can we therefore identify any significant differences between successful and unsuccessful diagrams? Two specific elements featured regularly in the successful cases. First, the incorporation of numerical information from the problem into the diagram itself. Second, the representation of relationships in a clear, visual form. However, these observations need to be qualified with some further comments. It was clear from this study that the nature of the problem itself was a highly significant factor in the kind of diagram that was drawn, and this would either facilitate or hinder the ease of representing relationships visually. For example, if a realistic diagram of boxes is drawn for Problem 2 then it is unlikely that the "3 times" relationship will be shown visually (since size usually varies in three dimensions) whereas the "sticks" problem is essentially one-dimensional and hence *either* a realistic *or* symbolic diagram can easily show the relationship. In fact many of the diagrams could be classified on a continuum from Concrete/Realistic to Abstract/Symbolic but because of the above comment it was unclear whether or not they correlate with success or failure. Nevertheless, in this context, one thing is clear. When a problem moves from small to large numbers then a more symbolic representation becomes a necessity. This is well demonstrated when comparing Problems 1 and 8 and leads us to the next issue discussed in part (iv) below.

iv) Are there Stages in Diagram Development?

Sometimes children expressed puzzlement on first being asked to draw a diagram. One such reaction was "Do you mean like we used to do in Standard 1 and 2?" The workbooks used by these children in the first years of the primary curriculum contain a large number of diagrams, mostly of the concrete/realistic variety. However, during informal discussions with the children it was apparent that they were no longer encouraged to use diagrams in their mathematics and indeed may have been actively discouraged. The attitude of some of the children suggested that they viewed the use of diagrams as something they should have "grown out of". And yet more than a half of those who failed with their own diagrams (or were unable to draw one) were successful when presented with the authors' diagrams. One boy commented "I understood the problem before, but I understood it even better with the diagram". However, a sizeable proportion of the pupils were still unable to succeed with the authors' diagrams. Now, although it had not been our conscious intention, it is clear that the authors' diagrams were all at the Symbolic end of the continuum described earlier. (The diagrams for Problems 1 and 8 are shown on the next page.) It could well be the case that some children were not "ready" for such representations and yet might have been able to interpret a more realistic version. All these points lead us to conjecture that children may need to





progress through stages of development in the drawing and interpretation of diagrams for problem-solving but are not being given the experiences or encouragement to enable them to do so. (Compare this with the readily accepted stages in graphwork moving from pictograms and one-one identification up to histograms and line graphs etc.)

v) Final Comments

Most of the children were interviewed on 3 or 4 problems and one of the characteristics that emerged was their receptiveness (after initial uncertainty in some cases) to take on board the strategy of drawing diagrams. On many occasions a pupil would ask whether they could "draw a diagram this time" without being requested to. "I think a diagram would be useful here" was another comment from one boy. There are strong indications in this study that drawing diagrams can be an important strategy in problem-solving but is one that is not being exploited "at the chalk-face". The desire to push children too quickly into abstract manipulation and algorithms may be one reason for this. And possibly teachers have the perception (which is passed on to the children) that drawing diagrams is somehow less mathematically respectable than algebraic manipulation. However, if this strategy is to be taken seriously it must be realised that it needs to be *taught*. (But *not* in a rigid, algorithmic manner; the rich diversity of examples given in this paper should be encouraged.) Children need to be given a repertoire of techniques and strategies from which to select. If one accepts the broad visualiser/non-visualiser styles of thinking, it may be that non-visualisers would make little use of this strategy. But our study indicates that children are not even being provided with the *possibility* of choice which is surely a matter of concern. We feel that there is a rich field here for further research both in identifying the salient features of successful/helpful diagrams and in developing suitable curriculum materials.

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The influence of standard images in geometrical reasoning

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Abstract. In the frame of a research project on geometrical reasoning a particular phenomenon was highlighted: the influence of standard images on mental processes. The report presents the interpretation of this phenomenon according to the theory of Figural Concepts.[1]

Introduction. This report provides some results of a research project still in progress. The general subject of investigation is geometrical reasoning and the study of the dynamics between two critical aspects in such reasoning: the figural and the conceptual. The general reference frame is the theory of figural concepts [1][2][7].

The main thesis is that geometry (in elementary, Euclidian terms) deals with particular mental objects, the figural concepts, which possess, at the same time both conceptual and figural aspects. These aspects are usually in tension: the dialectic between figural and conceptual aspect is characteristic of geometrical reasoning.

The aims of the experimental research project are to observe and describe mobilities of this process. The collection of data is organized around simple problems about solids (for a description of the experimental design and other details see [5][6]). The following tasks, among others, were considered: counting the number of faces, vertices and edges of a specific solid, and drawing its unfolding.

The study was carried out at three different age levels and with pupils attending different schools in different social contexts. Particularly, three different age levels were studied, following the same experimental design: the end of the Primary School (pupils aged 10-11), the end of the Middle School (pupils aged 13-14) and the end of the High School (pupils aged 17- 18).

The method of observation was a standardized interview; that is to say, the schedule of the interview was fixed and the same list of questions was given to all the subjects. In this way each one

1 The reserch project is supported by CNR and MURST

was asked the same questions, but at the same time the subject was left completely free to use his own strategy for solution and to comment on it. Because the interviewer was present, it was possible to observe most of the process underlying the solution.

The analysis of the collected data highlighted a number of interesting aspects and a previous report gave a first account of them.

In this presentation a particular aspect is chosen and deeply discussed, in order to illustrate the idea of Figural concepts and their dynamics; this choice is based on the conviction that the following discussion is meaningful both within the framework of the theory of figural concepts and because of the fact that it goes on in the same direction as the results of other studies [3][4][7][8].

A particular phenomenon: the role of prototype.

Let us consider the counting task: a solid is shown and then it is hidden; the subject is then asked to count the number of its faces, vertices and edges. In other words, the main characteristic of the counting task is that the subject is asked to count in his/her mind.

Among different counting processes used by pupils a very common performance can be observed; frequently, and again at different age levels, the mental image of the object seems to be assimilated to a *standard image*, which doesn't correspond to the real object.

Consider the regular Tetrahedron. The solid is shown, the pupil look at it; then the object is hidden and the pupil is asked to count. One of the most frequent answers sounds like: "5 faces, 5 vertices and 8 edges". The mental image on which the pupil is doing the counting is not that of a regular Tetrahedron, but that of a square pyramid, and the answer is consistent with this substitute image.

This kind of phenomenon is very common at different age levels, but overall very persistent. The substitute image of a certain solid is always the same - i.e. the square pyramid for the regular Tetrahedron - and in this sense we speak about prototype effect. Furthermore, the substitute image seems to insinuate itself without any possibility of control (see the protocol enclosed).

Reasoning is submitted to the influence of such standard images: when the object is hidden, the percetual stimulus is interpreted

according to a prototypical image: few critical attributes present in the object draw the pupils' attention and this causes the image to be assimilated to a prototype. Afterwards, the counting is performed consistently with this image.

Thus, in achieving the counting task, the mental image is assimilated to a standard one, and this phenomenon agrees with other results such as those found in classification tasks [3][7]. What is interesting is that a similar phenomenon appears in a different domain of problems. In this case no classification is expressly required, no definition is formulated or must be checked. Pupils are merely asked to count certain items of a particular object. But, in the mental process of counting a mental image is involved and, as in the case of classification, a prototype effect seems to appear.

Standard images and paradoxes.

Moreover, a second aspect of the influence of standard images arises. Rather often, pupils' answers reveal a very strange phenomenon: the assimilation of the image of the object in question to the standard one is not complete. The resulting image appears like a *paradox*. In other words, pupils seem to reason on the mental image of an inconsistent object. The image derives from both the memory of the object - the perceptual stimulus - and the assimilation to the standard image.

The following situation is very common. The pupil is asked to count the edges of a regular Tetrahedron, the answer is: "7 edges ... three on the side, four on the base". His counting is clearly based on the inconceivable image of a solid formed by a square base and a lateral surface of three triangles. The effect of the standard figure leads him to consider the base in the form of a square, while the perceptual memory, coming from what was really seen, leads to three triangles on the lateral surface.

How is it possible to manage this kind of mental image, and not realize the cognitive conflict of such an impossible image? Actually the reality of the object seems to be completely absent, its figural appearance is not taken into account. This means that, in order to count, pupils consider an image which has no direct connection with the real object, but which is derived from the object by the combined means of a perceptual memory and the assimilation to a prototype.

The origin is in the visual code, but the image is affected by the conceptual organization: in the counting task, the structure of the algorithm used by the pupil - counting on the side and then on the base - leads him to separate the mental image into two autonomous parts, one originating from the perceptual memory of the object, the other from the influence of the prototype effect. The achievement of the counting does not require him to come back to the whole image, so that the counting is performed on the two parts separately, and cognitive conflict is not experienced. In fact, the subject realizes his/her error only in the presence of the physical object.

Persistence of the phenomenon.

It is interesting to observe the same phenomenon in the solution process of another task, the unfolding task. During the interviews, after the first step about the counting of faces, vertices and edges, the pupil was asked to draw the unfolding of a particular solid. Let us consider the case of a regular Tetrahedron: the solid is shown and then hidden. It is possible to have answers like the drawing of fig. 1. Once again the influence of the prototype is clear.

It is interesting to note the strength of this influence, which is evidenced by the fact that the error comes again in the unfolding task, even if it has already been corrected, discussed and checked on the object in the previous phase of the interview.

The protocol of Serena provides a good example.

Once again the pupil is referring to the hybrid image, coming from the two sources, the memory of the object and the prototype assimilation (see fig. 2), but this time the necessity of coming back to the real object in order to verify the unfolding transformation leads Serena to a conflict.

Discussion

The previous example can be interpreted according to the results of other studies where similar performances in terms of prototypes are shown. But it is possible to go further: what seems to happen is that the prototype effect works at a very general level, and shows a great persistence.

Not only are there 'figures' which are more popular than others, but their influence is so strong that it can overcome any control. Prototypes are not alone in leading to inconsistencies between

definitions and examples, but the effect of the prototype leads to inconsistencies between the mental image and the real object to which it is referred.

From the point of view of the theory of figural concepts, the mental object, which we deal with, originates in the figural domain, and is conceptually organized. The ideal harmony between the two aspects may not be complete, leading in certain circumstances to different interpretations. In this sense, the conflicts and errors, previously described, due to the prototype effect, may be interpreted in terms of the interaction between the figural and the conceptual aspect, and highlight a break between the two components.

The influence of the visual processes affects the figural component of geometrical reasoning. Visual processes, their own phenomena, and among them the effect of standard images, have a great influence and not always a positive one; but, over all that influence can avoid any rational control, and originates mental images which are unreliable, but very effective.

A correct and productive reasoning is possible only by a good harmony between images and concepts, in order to menage with this kind of phenomena.2

Protocols

Serena (11 years old) mentally counted the number of faces and vertices of the regular Tetrahedron correctly, then she is asked to go on in counting the number of edges.

- Fine! Now the edges, the spines.
- Well,... 7.
- How did you count them?
- Well, first those on the top, three on the top and the 4 at the bottom.
- Can you show me? [handing her the object] Then let us start with the faces.
- Then, 1,2,3,4. ...

[The counting goes on correctly and she realizes that she has made a mistake].

- What was your mistake, what do you think?
- I thought that there were 4 ..., here ... That the base was ... Yes, I was wrong.
- That the base was a square.[She nods. She is asked to draw the unfolding and Serena draws rapidly, the object having been hidden]
- Then, this comes vertically, this one too, this one too and then this one, we can say that it is the base [she points to the square in the middle (fig. 2)] and then they go together ... and it would come out...

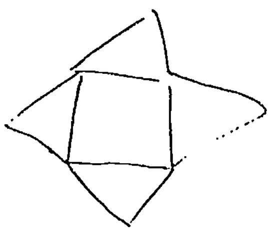
2 I want to thank P.Boero and M. Bartolini Bussi for their friendly and professional help to my work.

- Are you sure that it is right?
- But, There is a hole left.

There is a hole left. Can I put another face there?

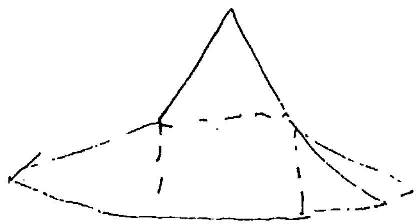
- I don't think so. However, let me try.
- Why don't you think so?
- I mean, because it has three faces ...
- On the top?
- Yes, on the top. And then a fourth, otherwise they would be 5.

[The discussion goes on, but she is not able to resolve the conflict. I give her the Tetrahedron. Serena checks the 3 triangles, she tries to draw again, then, finally, she raises the object and looks at the base ...]



b) CHIARA R. (anni 10:11)

fig. 1



DANIELE (anni 10 6)

fig.2

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A MODEL FOR ESTIMATING THE ZONE OF PROXIMAL DEVELOPMENT THROUGH STUDENTS' WRITING ABOUT MATHEMATICAL ACTIVITY

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In this article we concentrate on the use of a particular comparative indicator of structural difference in bodies of writing about mathematics. We apply this indicator to obtain a numerical comparison between the mathematics writing of 17 year 10 girls and a corpus of one million words of everyday English The distance measure is then combined with a measure of the elaboration of a student's writing to produce a measure of "engagement" of a student with the mathematics they are writing about. This measure of engagement correlates significantly with independent mathematics test scores.

ENGAGEMENT AND VYGOTSKY'S ZONE OF PROXIMAL DEVELOPMENT

The zone of proximal development

Vygotsky introduces the zone of proximal development in the following way.

> "This difference between the child's mental ages, this difference between the child's actual level of development, and the level of performance he achieves in collaboration with the adult, defines the zone of proximal development " (Rieber and Carton, 1987, pp. 209-214)

The definition and explication of Vygotsky's idea of the zone of proximal development occurs in section 4 of "The Development of Scientific Concepts in Childhood". This section deals with Vygotsky's basic notion that development and instruction are processes with complex interrelationships, and in it he explores these relationships through four studies, two of which are childrens' writing and the zone of proximal development.

Vygotsky's analysis (Rieber and Carton, 1987) of children's writing on the one hand and grasp of scientific explanation on the other indicates that, initially, these two activities are quite different from everyday action and speech, in that they require greater conscious involvement and a definite act of volition to become and stay involved. Consequently one might at first think that the more nearly a child's writing about scientific concepts had approached everyday writing the less a conscious act of engagement was occurring: the double burden of writing <u>and</u> writing about scientific matters would seem to indicate writing that bore little resemblance to writing about more everyday matters. However, as Vygotsky (Rieber and Carton, 1987) points out:

> "Thus, while scientific and everyday concepts move in opposite directions in development, <u>these processes are internally and profoundly connected with each other</u> (p. 219)

> "The development of scientific concepts begins in the domain of conscious awareness and volition. It grows downward into the domain of the concrete, into the domain of personal experience. In contrast, the development of spontaneous concepts begins in the domain of the concrete and empirical. It moves towards the higher characteristics of concepts, towards conscious awareness and volition. The link between these two lines of development reflects their true nature. This is <u>the link of the zone of proximal and actual development</u> " (p. 220)

Here and elsewhere in this chapter (p. 216, for example) Vygotsky states his belief that performance in language and thought related to everyday concepts merges with that for scientific concepts. For us, it difficult to separate Vygotsky's thinking on the development of scientific concepts, the conscious volitional nature of writing as compared with speech, and the rôle of the zone of proximal development. To us these matters are aspects of a unified vision: that, initially, everyday activity and speech communication is different from scientific activity and writing, the latter requiring a greater degree of conscious intent and volition on the part of a student, on the one hand, and involvement with a teacher-guide on the other.

A model of production of technical writing

Vygotsky argues convincingly that the connection between thought and language arises through a process of socialisation - a gradual mapping of language onto the inner processes of an individual. A significant part of these processes is mediated by gaining a vocabulary, however within the vocabulary of an individual at any one time not all words are functionally equivalent. The model we envisage for production of technical writing divides the vocabulary accessible to an individual into 3 functional levels. The first of these levels consists of those which, at any given time, form the semantic basis for the individuals expression of their understanding of their world. At this level vocabulary seems to be relatively stable and resistant to change., and the words at this level are highly tied to the emotions, expressing the experience of the individual. The second level consists of words that are accessible through a re-membering of experience - typically the words that an individual can read. This level is frequently supplemented by words novel to an individual. Functionally, this level is divided into a comprehension domain and a production domain, and typically the comprehension domain is quite a bit larger than the production domain. Words in the production domain are capable of being used with facility over a range of morphological forms with consistently acceptable syntax and in socially appropriate senses. This second level forms part of the interface between an experiencing individual and the society in which they are contextualized. The movement of words across the se two domains, and into the first level, are determined by, and we argue also assist in determining, an individual's sense of their world.

The third level consists of the vocabulary used in the production of particular texts. In other words, it is, from the point of view of an observer, an instance of the use of a specialised , connected, coherent part of the production vocabulary of level 2. We believe that the connectedness and coherence properties of the vocabulary are also part of the individual's cognition of their production process: in other words, what they produce is "about" something, and makes sense to them. this sense may, of course, differ from what an observer regards as the sense of the production.

The model is integrated from one level to another by a factor we call engagement. As a word is used in production it acquires a number of properties for the user: it may be close to or far from what the person wants to express, and it may be rich or poor in connections and associations for the person.

MEASURING ENGAGEMENT WITH MATHEMATICS THROUGH WRITING

Everyday writing and elaboration

There are several factors that might contribute to a measure of "engagement" of a mathematics student with mathematics, as evidenced by their writing. Following Vygotsky's observations on the growth of scientific and everyday concepts toward each other we might expect one such factor to be the degree to which a student's writing matches everyday writing. The quantitative tool we use to analyse these students' writing is based on a distance measure between linguistic corpora, devised for the purpose of providing us with a measure of how "far apart" two bodies of writing might be. The usual terms used to distinguish the form of a word, such as "the" from the actual occurrences of that word in a body of text are type and token: type refers to the form, or template, of the word and token refers to a specific instance of that type in a corpus.

Other factors implicated in "engagement" are the number of types used by a student, the extent to which any sampling of the types most frequently used contribute to the overall number of types, the volume of writing as measured by the number of tokens used, and the context-dependent nature of the writing as evidenced by the number of types used that are frequently used but not common in everyday writing.

The number of types is an important factor in any analysis of engagement: generally the use of a larger number of types can be regarded as evidence of the use of a larger active vocabulary that is brought to bear on a topic. There are two ways in which one might attempt to use the quantity of text to measure "engagement" by any person writing on a particular topic. The first is by the volume of work that they produce: the number of tokens. This variable is related to the motivational aspect of engagement. The second is by the number of types and their ranking by frequencies. This variable is related to the structural ways in which a person's mind actively engages with a subject.

The volume of writing - the number of tokens used - is, in part, a measure of the degree of engagement as measured by elaboration in a student's writing. The question of the correlation of the volume of writing with grades, for essay type examinations, has been discussed by Norton and Hartley (1986) and Lonka and Mikkonen (1989). Lonka and Mikkonen conclude that a significant factor in relating the volume of writing to grade, for an essay type answer, is the "elaboration" that occurs in the writing of able students. Their definition of elaboration, and its antithesis "reproduction", is as follows:

> "Reproduction meant that a fact was stated as it was presented in the source. Elaboration occurred when the writer presented reasons for or drew conclusions from the facts mentioned." (p. 223)

A distance measure between corpora

In its simplest form, imagine we have a corpus, or part of a corpus, A with types ranked by frequency of use: the most frequently used type has rank 1. Imagine we also have another such corpus, or part of a corpus, B with precisely the same collection of types W, but possibly different type ranks. We then define a vector $[A,B]$ with $|W|$ components: the k^{th} component of $[A,B]$ is the real number $\log_2(\mathrm{rank}_A(w)) - \log_2(\mathrm{rank}_B(w))$ where $\mathrm{rank}_X(w)$ denotes the frequency rank of the type w in the corpus, or part corpus, X. We then define the distance $d(A;B)$ between A and B to be the Euclidean length of this vector in Euclidean space $\mathbf{R}^{|W|}$. In other words,

$$d(A,B) = \sqrt{\sum_{w \in W} (\log_2(rank_A(w)) - \log_2(rank_B(w)))^2}.$$

The reason for applying logarithms to the ranks of types in a corpus is to reflect the empirical fact that actual corpora are distributed in an approximately log-normal way. Consequently a difference in type order between corpora, for types ranked at the head of the corpora, is more significant in measuring how far apart the corpora are than is the same rank difference toward the end of the corpora.

Typically, in order to make computation manageable, we are not so much interested in matching an entire corpus with another entire corpus, as we are in matching significant parts of corpora. Most of our analysis will center around the 100 most frequent types in any given corpus. So in the definition of d we take A and B to be the first 100 types in a given corpus here we are momentarily assuming that A and B consist of the types in a set W of size 100. It is straightforward to check that d is a metric on the set of all rank orderings of the types in the set W.

If, as happens in practice, two corpora do not have the same set of 100 most frequently used types in other words, if some types are on one list but not on the other - we assign any type on one list A but not on list B, a rank of 101 in B, and vice versa. This means we extend the definition of the metric d to include those terms $\log_2(101) - \log_2(w)$ where w is a type that occurs in one or the other lists of 100 most frequent types, but not in both lists.

Generally, for the students we have studied, the top 100 types account for 60% - 75% of all types used by a student. For students with a lower volume of writing in the course of the study, the top 100 types generally accounts for a percentage of the total number of types nearer 75%. That is, in general, we found that students who write less in a given period tend to overwork their top 100 word types, as compared with students who write a lot.

A measure of engagement

For students as old as those we studied - students in the 10th year of schooling we could expect a merging between the everyday and scientific modes of discourse, such as that described by Vygotsky, to have begun for many of the students. What we could expect therefore is for students with a high degree of conscious involvement, or engagement, in the scientific conceptual nature of their writing in mathematics, to produce writing that was <u>closer</u> to everyday writing than the writing of their classmates whose thought mode was still more dominated by unconscious everyday concepts.

The measure we use to discriminate individuals on the basis of their writing is defined as follows: for a corpus A we define the measure $E[A] = \frac{d[Brown;\ A]}{\log_2(vol[A])}$, where vol[A] is the total number of word tokens in the corpus A, and "Brown" refers to the corpus studied by Kucera and Francis (1967) at Brown university.

Why do we use the logarithm of the volume? Clearly we should scale the distance d[Brown, A] somehow by the volume in order to get a measure that we can use to compare writing across students. But why $\log_2(vol[A])$ and why not vol[A]? One reason has to do with the way we want to scale the variable vol[A]. If we imagine the students' volume of writing to be log-normally distributed, as it appears to be, then a scaling that will give us an approximately normal distribution is $\log_2(vol[A])$. Of course, this does not answer the question of why the students' volume of writing is, or could be expected to be, approximately log-normal.

The measure E[A] depends on a standard such as the Brown corpus: it will plainly alter somewhat when other corpora are used in its place. The essential feature of the Brown corpus for our purpose is that it is a moderately large corpus taken from everyday writing. The measure E[A] is smaller when the corpus A is close to the Brown corpus in the sense of the metric on corpora, and when the volume of writing that constitutes the corpus A is large.

We claim that this measure should give us a numerical indication of the extent to which a student at a certain level of mathematical development can go beyond that development to learn with the assistance of a teacher. In other words, we claim we have a measure of the degree of proximal development of a student in mathematics, in the sense of Vygotsky's zone of proximal development, based on students' writing about mathematical activity.

METHOD & RESULTS

The writing in mathematics class of 17 year 10 girls was collected over a period of one school term (approximately 13 weeks). The girls were students of the second author at Vaucluse College, Richmond, a secondary girls' school in inner-suburban Melbourne. Their writing was called "informal writing to learn", and it consisted of journal entries the girls made in reflecting on mathematics they had recently studied in class.

Token and type correlation

For the experimental group of students there was a very high correlation between the logarithm of the numbers of types and tee corresponding logarithm of the number of tokens:

Count	Covariance	Correlation	R-squared
17	.474	.963	.928

A simple regression gave $\#types = 7.1(\#tokens)^{0.54}$ with r^2 (adjusted) = 0.923. Why should there be such a strong correlation between $\log_2$(types) and $\log_2$(volume)? One possible answer has to do with the general character of power laws described by Newell and Rosenbloom (1981). The hypothesis favoured by those authors is a modification and elaboration of Miller's (1956) chunking hypothesis. However, in our study we do not have a single person carrying out repeated trails in time, rather we have many students carrying out simultaneous activity. If Newell and Rosenbloom's general argument on power-like laws were to apply in this situation it appears that we would have to assume a uniform model of production across the sample of 17 students. This, in turn, suggests to us a uniform cognitive model of writing to learn mathematics. This is entirely possible, but it was, for us, a most unexpected finding from the study. We feel that this issue needs to be explored thoroughly in larger-scale studies of a similar nature.

Volume and grade correlation

The volume of writing, that is the number of tokens used, by each student correlated well with grades:

Count	Covariance	Correlation	R-squared
17	1502.944	.651	.423

What is much more striking is the correlation of the logarithm of the volume with the same grades:

Count	Covariance	Correlation	R-squared
17	.764	.791	.626

Volume and distance correlation

There was almost zero correlation between the collection of distances from the girls' corpora to the Brown corpora, on the one hand, and the logarithm of the number of types for each girl on the other:

Count	Covariance	Correlation	R-squared
17	-1.449	-.012	1.383E-4

The measure E[A] for the corpus A for each student is as follows:

EMR	YL	ER	AVL	JJ	JT	LF	KB	GZ	VM	MHO	DO	AL	KOS	KK	DMB	MH
0.59	0.60	0.66	0.69	0.70	0.74	0.78	0.75	0.76	0.78	0.80	0.80	0.82	0.86	0.95	1.00	1.07

These measures are significantly negatively correlated with performance on tests:

Count	Covariance	Correlation	R-squared
17	-.102	-.838	.701

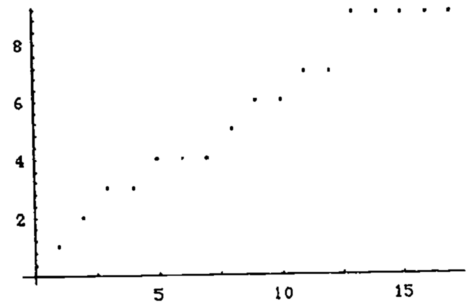
The measure E[A] is, as we suspected, giving us something other than that measured simply by volume. The fact that it significantly correlates with test performance suggests that it does indeed relate more closely to that part of "engagement" of a student in their writing that has to do with the structural aspects of mathematics. That volume, or the logarithm of volume, should have something to do with test performance, is as we have mentioned, suggested by the findings of Lonka and Mikkonen (1989): volume is greater for students who elaborate due to their writing style being more actively engaged in re-structuring and explanation, rather than mere description or recall. The fact, as it seems, that some students have more to elaborate on suggests that these students have reflected on their mathematical experience and are structuring, re-ordering, and communicating, their experience. In other words, through the socialisation process of writing to learn mathematics, elaboration is promoted by an active process of internalization of vocabulary: words that could be comprehended and then used, are now becoming words to describe the personal experience of an individual.

Significant words: We

One of the useful aspects of the distance measure between corpora is that it allows us to pick out word types that have a significant "shift". These are types that are used substantially differently in a students writing than in a standard collection of everyday writing: either significantly more or less often. One such word type is "we".

Pimm (1987, pp. 64 -70) remarks on the peculiar use of the word "we" in mathematics - generally he is concerned with speech in mathematics classes, but on p. 68 he discusses power, dominance and acquiescence in the context of textbook writers use of "we". How did the students in this study use "we" and how did this differ from the use of "we" as evidenced in the Brown corpus?

A rank order plot of the rank of the type "we" in each of the students' writing appears below:



The word type "we" occurs significantly relatively far more often in each of the students' corpora than it does in the Brown corpus, where its rank is 41. The elevation of "we" to rank 1 in the writing of one of the students, MH, is particularly significant, because a detailed examination of the ways in which MH uses "we" provides many indications of cognitive acquiescence throughout the corpus:

- We were told... • We had to ... • ...we had to divide 7 into quarters
- ...we had to try to figure out... • ...we had to divide each number ...
- ...we were given a sheet... • We got another sheet.

There were instances of reflective activity in MH's use of "we", such as that described below, however such instances of reflection were rare:

- That's where we got stuck. We thought it would be sixths but we couldn't prove it. Then we were told it was eighths but we couldn't prove that either.

On the other hand for another student, YL, whose use of "we" gave this word type a rank of 9, there was far less evidence of acquiescence to a teacher's real or imagined requests. Rather YL's use of "we" indicated a mind that was actively questioning and building:

- What can we do with it?
- We can: add numbers (+) subtract (-) multiply (X) divided by (D) square (X^2) square root
- ... as we approached the third steps of the work we found that these are some tough calculations we need to do.
- ...we started to feel unsure of what our aim was... • We solved our problem..
- When we have overcome our problem ... • We were puzzle about what 7/4 got to do with our problem ?
- We tried to do by ourselves...
- We never work on real numbers before, so this is one area we should know later on this year.

DISCUSSION

The results from the mathematics corpora of the students in this study are consistent with the view that they randomly use words from an everyday corpus, subject to the frequencies of word use in such a corpus. In other words, we hypothesise, it is as if they had psychic Tyches (the goddess of chance) dipping into a common reservoir of word patterns and use in everyday writing, and using words in mathematics this way. This hypothesis, which is not inconsistent with the experimental data and the analysis of distances of corpora, has a corollary. It is that for these students there was little extra structure in their use of words that would distinguish their mathematics writing as a genre. Principally however the hypothesis says that our measure could not distinguish the students' mathematics writing from an appropriately-random use of everyday words on the one hand, and was not close enough to an everyday corpus to be able to assert that they were writing in a significantly everyday sense on the other.

We hypothesise that the measure of engagement presented here will provide a more scientific way to study the coherence of given text and hence a way of quantifying the integration of the lexicon at the second of our proposed levels.

Presently we are extending this work in three ways. The first is by computer simulations of stochastic models of text production in which we aim to estimate the mean value of the distance function d[Brown; A] for appropriately stochastically determined corpora models A. The purpose of these simulations is to get a feel for whether simple stochastic models might provide a reasonable explanation of observed token frequencies, and also to get a feel for the magnitude of the distances involved. The second way we are extending the work is by a larger-scale study, using first and second year tertiary students, taking into account suspected factors that might influence the outcomes, such as gender, whether their writing language (in our case, English) is a second language, and whether students studied science subjects other than mathematics to an advanced level. The third extension is to update the corpus of everyday writing used as the base for, and so refine, our metric d[-,A].

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FORMATS AND SITUATIONS FOR SOLVING MATHEMATICAL STORY PROBLEMS

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The study was conducted for clearing the existence of the effect of format and situation of story problems on performance of low and middle grade students of an elementary school in Japan. Two kinds of format, drawn and verbal, and two kinds of situation, concrete and abstract, were dealt in the study.

The results revealed that there were significant differences both between two kinds of format and situation in the second and third grades, and a significant difference between two kinds of format in the forth grade. The study showed that in the second grade students the concrete situation affects more positively performance of verbal story problems and also drawn story problems than the abstract situation.

It is said that format influences on performance of elementary school students' solving problem. In the United States there are researches dealing with elementary school student's understanding of story problems with different formats (Neil, 1969; O'Flaherty, 1971; Threadgill-Sowder and Sowder, 1982; Threadgill-Sowder, Sowder, Moyer, and Moyer, 1985). With some exceptions, researches in the United States report that the drawn format is more effective than the verbal one for elementary school students' understanding of story problems.

Although in Japan there are few researches dealing with such a theme, except for two researches of Honma and Takahashi (1990), and Takahashi (1991), and although verbal explanations are excessivelly used in Japanese classrooms (Stigler, 1988), many pictures have been described in textbooks of elementary school matematics, and teachers of elementary schools often use pictures in their mathematics classrooms, saying that pictorial presentation may promote students' understanding of mathematical ideas. Now researches of effects on performance of story problems with different formats are needed especially in Japan

There are two kinds of picture used to learn story problems with numbers and operations in textbooks of elementary schools. One is the picture of concrete situation with concrete real items The other is the picture of abstract situation where the symbolic representation of tiles, dots, and tapes, are substituted for concrete items Except for the research of Takahashi (1991), there

is no research dealing with elementary student's understanding of story problems with different situations experimentaly or statistically in Japan.

In the United States some researches dealt with student's understanding of problems with different situations. Houtz (1974) found that among four problem forms, the models, slides, picture-book forms and abstract forms, the abstract form problems were more difficult than the others for second and third grade students. Caldwell and Goldin (1979) researched fourth, fifth and sixth grade students' understanding of word problems with different situations and contexts which are abstract factul(AF), abstract hypothetical(AH), concrete factul(CF) and concrete hypothetical(CH) problems, and found that the order of difficulty of the problems, from the easiest to the most difficult, was CF, CH, AH, AF, and regardless of problem's contexts any grade students solved more concrete problems than abstract problems. Heddens (1986) described that the semiconcrete level with pictures of real items and semiabstract level with pictures of tallies helped students to understand mathematics.

Then the problems of the study were as follows:

Problem 1. Are there any different effects between drawn and verbal formats on performance of solving story problems?

Problem 2. Are there any different effects between concrete and abstract situations on performance of solving story problems?

Problem 3. Are there any different effects between drawn and verbal formats on performance of solving story problems with each of the situations?

Problem 4. Are there any different effects between concrete and abstract situations on performance of solving story problems with each of the formats?

Problem 5. Are there any different effects among four versions of two formats and two situations on performance of solving story problems?

METHOD

Format and situation In the study, we defined format and situation as follows. Format is the written form of story problems, and we used drawn and verbal formats. Drawn format is a pictorial representation without numerals, and minimal words are used for presenting the structure and requesting the answer. Verbal format is the presentation of story problems with words and numerals without any pictures.

Situation is defined as the degree of the relation to everyday life in presenting story problems. Story problems of concrete situation are described with candies, crayons and so on according to

the content of problems, and story problems of abstract situation are described with dots (and tallies and tapes, but these are not used in the study) substituted for candies, crayons and so on in concrete situation.

Test instrument After preliminary test for students in other schools, we selected six types of operation significant and suitable for the subjects for each of the grades, which are as follows:

For the second grade, A: $a-b$, B: $a+b+c$, C: $a+b-c$, E: $a+b+c-d$, F: $a-b-c$, G: $a+b-c-d$.

For the third grade, F: $a-b-c$, G: $a+b-c-d$, H: $a\times b$, I: $a\div b-c\div d$ ($\div$: equal division), J: $a\div b-c\div d$ ($\div$: divisible), L: $(a\div b)\times c$ ($\div$: divisible), and

For the forth grade, F: $a-b-c$, G: $a+b-c-d$, K: $a\times b+c\times d$, L: $(a\div b)\times c$ ($\div$: divisible), M: $(a\times b)\div c$ ($\div$: divisible), N: $(a\times b)\div c$ ($\div$: with non-zero residual).

For each of the above types, we substituted each letter for a numeral in two ways, for example, the type G: $a+b-c-d$ was changed into two numeral phrases $2+8-4-5$, and $5+5-2-4$, which seemed not so difficult but not so easy for the subjects. Then we modified each numeral phrase into (Format)$\times$(Situation), and obtained four story problems. Figure 1 shows story problems with the same numeral phrase $2+8-4-5$ of type G of (Drawn format)$\times$(Concrete situation), and (Drawn format)$\times$(Abstract situation).

On the other hand, we developed Calculation Ability Test with some problems of calculations corresponding to the test instrument. The test were used for screening students.

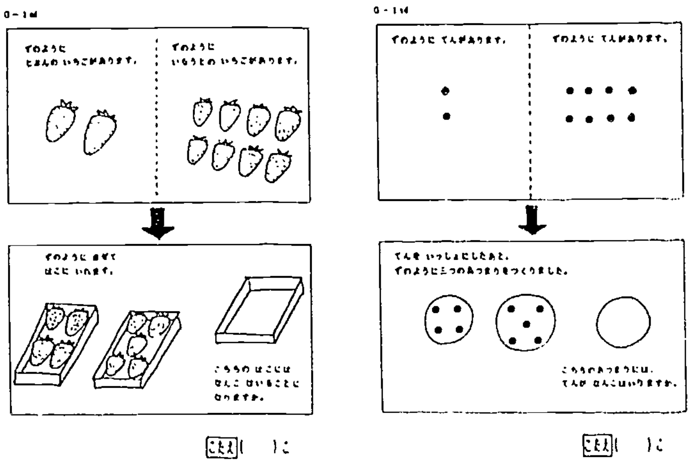

Figure 1. Examples of story problem with concrete and abstract situations.

Subjects The subjects of the study were the second, third and forth grade students of the Shinzan elementary school in Honjyo City, Akita, Japan. They were classified into two groups of good and poor calculators following performance of the Calculation Ability Test stated above. The students who were identified as good calculators and not absent from at least one of the tests were the subjects of the study.

The rates of excluding from subjects for poor calculators in the second, third and forth grades were 2%, 8%, and 11%, and the rates of excluding for their absence were 8%, 9%, and 5%, respectively. The students of 91%, 85%, and 87% of the second, third and forth grade students were the subjects of the study, and the numbers of the subjects were 164, 191, and 161, respectively.

Administration of instrument The test instrument and Calculation Ability Test were divided into three parts and were administered in three school hours, one of which is 45 minutes, from 6 to 24, July, 1992, at the near end of the first term which begins in April of the year.

The front cover of the test, there were blanks and a written sentence. The blanks were should be filled with students' classroom names and students' numbers, and their names. The sentence was written as, "The result of the test would not relate to your marks, but we expect you may pay your full attention and power: GANBAROH." The sentence was read aloud by their classroom teacher as the administrator of the test.

Data analysis In the study, all subjects were alloted to all the (Format)×(Situation). ANOVA suitable for such design as (Subject)×(Format)×(Situation) was undertaken for solving Problems 1, 2, 3 and 4, and Problem 5 was solved using the test of the difference between mean values. The result of ANOVA of a grade for Problems 1 and 2 is summarized in a table. The results of ANOVA for Problems 3 and 4 are indicated by fully abbreviated tables. In the following tables, ** and * denote significant at 1% and 5% levels respectively.

RESULTS AND DISCUSSION

Problems 1 and 2 The results of ANOVA for Problems 1 and 2 are summarized in Table 1 for the second grade, Table 2 for the third grade, and Table 3 for the forth grade.

Problem 1 was answered affirmatively in all the three grades. Problem 2 was answered affirmatively in the second and third grades. The mean values of performance of drawn and verbal formats of the second, third and forth grade were 19.70 and 18.51, 14.46 and 12.45, and 17.93 and 17.11, respectively for Problem 1. The mean values of performance of concrete and abstract situations of the second, third and forth grade were 19.69 and 18.55, 13.69 and 13.23, and 17.65 and 17.39, respectively for Problem 2. Throughout Tables 1, 2 and 3, with an exception of performance of situation of the forth grade (numerically not against the other cases), there consistently exists the

relation such that the mean value of performance of drawn format is significantly higher than that of verbal format, and the mean value of performance of concrete situation is significantly higher than that of abstract situation.

Table 1. ANOVA of the second grade.

Source	SS	df	MS	F	
Subject(S)	3596.93	163	22.07		
Format(F)	54.45	1	54.45	14.03	**
S×F	632.80	163	3.88		
Situation(T)	53.31	1	53.31	44.57	**
S×T	194.94	163	1.20		
F×T	2.82	1	2.82	2.29	
S×F×T	200.43	163	1.23		
TOTAL	4735.68	655			

Table 2. ANOVA of the third grade.

Source	SS	df	MS	F	
Subject(S)	6327.37	190	33.30		
Format(F)	193.01	1	193.01	58.39	**
S×F	627.99	190	3.31		
Situation(T)	10.14	1	10.14	7.74	**
S×T	248.86	190	1.31		
F×T	4.71	1	4.71	3.40	
S×F×T	263.29	190	1.39		
TOTAL	7675.37	763			

Table 3. ANOVA of the forth grade.

Source	SS	df	MS	F	
Subject(S)	4503.44	160	28.15		
Format(F)	27.47	1	27.47	8.10	**
S×F	542.28	160	3.39		
Situation(T)	2.87	1	2.87	2.55	
S×T	179.88	160	1.12		
F×T	2.13	1	2.13	2.21	
S×F×T	153.62	160	0.96		
TOTAL	5411.69	643			

Problem 3. The results of ANOVA for Problem 3 are fully summarized in Table 4.

Table 4. Results of ANOVA for Problem 3.

Situation	Grade		
	2	3	4
Concrete	*	**	–
Abstract	**	**	*

Table 4 indicates that on concrete situation Problem 3 is answered affirmatively in the second and third grades, and on abstract situation Problem 3 is answered affirmatively in the second, third and forth grades. On performance of story problems of concerte situation, the mean values of the version of drawn and verbal formats of the second, third and forth grade were 10.07 and 9.62, 7.27 and 6.42, and 8.98 and 8.68, respectively. On performance of story problems of abstract situation, the mean values of the version of drawn and verbal formats of the second, third and forth grade were 9.63 and 8.92, 7.19 and 6.03, and 8.96 and 8.43, respectively. On performance of concrete situation, problems of drawn format are significantly easier than those of verbal problems in the second and third grades, and on performance of abstract situation, problems of drawn format are also significantly easier than those of verbal problems in the second, third and forth grades.

Problem 4. The results of ANOVA for Problem 4 are fully summarized in Table 5.

Table 5. Results of ANOVA for Problem 4.

Format	Grade		
	2	3	4
Drawn	**	–	..
Verbal	**	**	*

Table 5 indicates that on drawn format Problem 4 is answered affirmatively only in the second grade, and on verbal format Problem 4 is answered affirmatively throughout the three grades. On performance of story problems of drawn format, the mean values of the version of concrete and abstract situations of the second, third and forth grade were 10.07 and 9.63, 7.27 and 7.19, and 8.98 and 8.96, respectively. On performance of story problems of verbal format, the mean values of the version of concrete and abstract situations of the second, third and forth grade were 9.62 and 8.92, 6.42 and 6.03, and 8.68 and 8.43, respectively. In the second grade, problems of concrete situation are significantly easier than those of abstract situation on performance of drawn format. On performance of verbal format, problems of concrete situation are significantly easier than those of abstract situation in all the three grades.

Problem 5. The mean values of performance of four versions of story problems are indicated in Table 6.

Table 6. The mean values of performance of four versions.

Format	Situation	Grade 2	Grade 3	Grade 4
Drawn	Concreat	10.07	7.27	8.98
			**	
Drawn	Abstract	9.63	7.19	8.96
				*
Verbal	Concreat	9.62	6.42	8.68
			**	** *
Verbal	Abstract	8.92	6.03	8.43

Table 6 indicates that the orders of larger or smaller mean values are at least numerically the same through the three grades. Table 6 also indicates that the difference between format are larger than that between situation on performance.

CONCLUSION

The study reveals that in the second grade there are significantly different effects between two formats of drawn and verbal, and also two situations of concrete and abstract, on performance of solving story problems. Despite that the effects are gradually decreasing as student grade increasing, the difference between drawn and verbal formats of story problems with abstract situation, and the difference between concrete and abstract situation of story problems with verbal format are obtained in the study.

Although the study also reveals that the different effect of formats is larger than that of situations, the findings of the study tells us that different effects between concrete and abstract situations of story problems on performance are not negligible especially for low grade students in elementary school mathematics classrooms in Japan.

Owing researches by Piaget, Bruner and others, elementary school teachers of Japan have been interested in the difference between two formats of drawn and verbal, and excessively used drawn formats in school mathematics classrooms, but they are not conscious of the value of students' such concrete and everyday life mode of thinking as revealed in the study, and they are not interested in the difference between concrete and abstract situations. The findings of the study implies the importance of the difference of situations for their presentation or translation of contents of school mathematics corresponding to the developmental level of students they teach.

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Children's Talk in Mathematics Class as a Function of Context

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In this paper we explore qualitative differences in the nature of student "talk" in various situations mathematics classes. Our analysis of episodes from inquiry mathematics classes indicates that students' interpretation of a situation and their perceptions of their expectations and obligations are more important in determining the nature of their "talk" than whether the setting is one of whole class discussion or small group problem solving.

Introduction and Background. In our work we collaborate with teachers to develop instructional approaches to mathematics which engage children in developing personally meaningful solutions to problems in small group collaborative settings and then discussing their solutions in a subsequent whole class setting. Classroom observations clearly show that the way in which children talk about their activity differs depending on their interpretation of the situation as a social event. This paper presents the results of an analysis which investigates these differences. Aspects of the children's "talk" that are of interest are the nature of the language used and the type of explanation given.

Distinctions between styles of speech and types of talk in the classroom have been made by researchers who study classroom language and disrourse, including Cazden (1988), Barnes (1976, 1977), and Pimm (1987). Barnes differentiates between what he calls "exploratory talk" and "final-draft (or explanatory) talk". Exploratory talk is speech that is thought out in the course of expression while final-draft talk is speech that is characteristic of planned expression. According to Barnes and Cazden, exploratory talk is more likely to occur in discussions between peers and final draft talk during a "report-back" discussion with the whole class. Pimm (1987) makes a similar distinction, namely, between student's "talking for themselves" and "talking for others".[3] The main purpose (and effect) of talking for oneself is to organize one's thoughts and reflect on one's own thinking, whereas the main purpose of talking for others is communicative.

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At the outset, these distinctions seemed useful to us to account for differences between children's activity and talk during small group work and in whole class discussion. Our purposes in organizing classroom instruction to include both small group work, where students develop solutions to problems, and whole class discussion, where they later explain their solutions to the whole class, were to facilitate opportunities for children to develop ways to talk about their thinking, to reflect on their activity, and to reconceptualize and reorganize their prior activity in order to explain it to others (Pimm, in press; Yackel, Cobb, & Wood, 1991). Therefore we expected that students' talk during whole class discussion would be indicative of such reflection, reconceptualization and reorganization and would be more mathematically sophisticated than talk in small group work. However, the data required a finer analysis to account for the observed differences. This finer analysis included taking the children's interpretation of the situation into account. It is the *context* that is relevant rather than the *setting*. Here we are using *context* in the sense of Cobb (1986). In this sense context is individual and includes the individual's interpretation of the situation including their expectations and obligations. Thus, the same setting, such as small group work or whole class discussion, may be completely different contexts for different individuals.

Discussion of Data and Results of Analysis. Videotape data and available transcripts were perused to identify cases in which children had been videorecorded as they attempted to solve a problem in the small group setting and then the same children described their solutions during the subsequent whole class discussion. Data analyzed include all available data from three months in the middle of one year-long teaching experiment and selected episodes across the school year from another year-long teaching experiment. Both were second grade classes but in different cultural settings.

The analysis of the targeted episodes indicates that for children the critical feature is not whether they are engaged in small group problem solving or whole class discussion but is the nature of the local situation realized as a social event. The presence of the teacher or of a researcher in small group work results in different interaction patterns than when the

children are alone. Children alter their explanations due to differing expectations, their perceptions of their obligations, and their own personal agendas. For example in the presence of the teacher a child's goal may be to establish that he had the correct answer rather than to elaborate on difficulties he had with a particular interpretation of the problem. On the other hand, the goal might be to elicit the teacher's assistance to resolve a conflict between two differing interpretations. When children are interacting alone in small group work they may be attempting to convince each other of the viability of their own personal interpretation, or they may be attempting to take each other's perspective. When children talk during whole class discussion it may be to report (and explain) a previously developed solution method. Or it may be to enter into the ongoing discussion using their prior small group activity as a basis for contributing to the discussion. In some cases, they abandon their prior small group work and develop a solution that addresses the questions and challenges of the immediate discussion. The child's interpretation of his obligations and his expectations of other participants in the immediate situation are critical to the nature of the activity (and talk) the child engages in. In making these judgements the child himself contributes to the development of the situation.

To illustrate children's differing activity and talk in various situations that they interpret as different social events we present the following example.

Example. John and Andy are working together on a problem in small group work, first alone, then in the presence of a researcher, and later in the presence of the teacher. Finally, they participate in the subsequent whole class discussion of the problem. The problem under discussion is the third problem in the sequence shown below. In each case the task is to figure out what number to put in the blank to make the left and right sides balance.

46, 46 / ____ 48, 48 / ____ 38, 38 / ____

Having solved the first task by adding 40 and 40 to get 80, adding 6 to 80 to get 86 and then adding on 6 more to get 92, and the second task by relating it to the first, "Just 4 more than that, what's that? That makes 96". Andy relates the third task to the second, "20, 20, take

away 20." John agrees to Andy's solutions of the first two tasks although he first proposed solving the second task by adding 8 and 8 and 80. John also relates the third task to the second but thinks you should "take away 10, take away 10." Each child insists on his interpretation of the problem. Since there is a dispute, an occasion for clarification has arisen. From the observer's perspective, it is appropriate for each child to provide a rationale for his suggestion. The first evidence of an attempt at doing so is given by John when he says, "Take away 10, take away 10 (pointing at 48 and then 38). Look, 40, 30. Not 20." Andy replies with, "I know, oh ... yeah." John "explanation", however, is not clearly articulated. It relies on pointing and is not a well developed verbal explanation of his thinking. Andy, on the other hand, gives no rationale. Further, his, "I know, oh ... yeah", indicates that he is unaware of the discrepancy between his and John's interpretations. He appears to agree with John acknowledging that this problem involves 30 whereas the previous problem involved 40. The subsequent verbal interchange that occurs does not address their differing interpretations of the problem task. Andy repeatedly restates the need to "take away 20", and reiterates the result as being "76 or in the 70's while John reiterates his interpretation of the task as "take away 10". John attempts to provide a rationale on three occasions, first saying, "Look, 40, 30, not 20." Later he says "No, take away 10. Look. It's 40, 30 (pointing to the two tasks)" and still later, "Because there is 48 and 43, that's just taking away 10. So take 10 away from that (pointing to the answer of the second task)." (We interpret the "43" as misspeaking.) John's last utterance is the only one in the entire dialogue that has the form of an explanation. His language "because ... so ..." is evidence that his intention is to give a rationale for the solution method he is advocating. However, his explanation fails to explicate the differences between his and Andy's interpretations.

There is a distinct change in the nature of Andy's utterance in the next portion of the episode when a researcher enters the scene. Now both boys, in response to the researcher's request, "How did you do it?" give a rationale for their solution activity.

Andy: Well, take away 10, 10 from each of these (points to the previous task). Take away 10 from each of these make 20, take away 20. He put 80. I think that's ... I think its seventy-six."

John: See, that's just 10 more ..., 10 less than that. So the answer has to be 10 less.

Andy's reply to the researcher elaborates his interpretation of the problem and explains the origin of the "20" that is to be taken away. Andy's previous comments to John did not provide this clarification. Andy may have assumed that John shared his interpretation of the task. There is some support for this hypothesis from the previous history of this pair (Yackel, Cobb, & Wood 1990). Another possibility is that Andy did not feel obliged to explain to John but did feel obliged to explain to the researcher. The researcher had previously established that he was interested in how the children thought about the tasks. In any event, Andy's reply to the researcher indicated that he understood which aspects of his thinking were critical to providing a complete explanation. For his part, John's reply to the researcher differed from his prior comments to Andy in that he now goes on to verbally relate his observation "just 10 more ..., 10 less than that" to the answer, "So the answer has to be 10 less", whereas previously he had only pointed to the answer. If either child had said to the other what he said to the researcher they may have resolved their disagreement. The difference in the explanations given by each child and in the nature of their "talk" indicates that they interpreted the presence of the researcher as a different social situation.

Later when the teacher enters the group the boys immediately tell her about their conflict over the problem even though they were already working on another activity page. Based on their previous experience in this class, they might expect the teacher to help them resolve their disagreement by discussing their solution procedures with them. However, the two solution methods discussed with the teacher are both different from the methods the boys were attempting to use.

Teacher: You know that 48 and 48 make 96, so 38, 38.

The teacher's initial comment signals a relationship to the previous problem. John's acceptance of the teacher's implicit reference to solve the problem by relating it to the previous problem is indicated by his next remark

John: Ten less than that.
Teacher: Not quite 10.

Simultaneously, Andy solves the problem by adding 30 and 30 and 8 and 8, using the same method he used to solve the first problem on the page.

Andy: thirty, sixty ... 68, 69, 70, ..., 76 (puts up fingers as he counts the last 8), 76.

Here Andy has used a counting-based method, one that would be undisputed by either child. In this class counting was always viewed as a method you could fall back on when you had no other way or to provide "absolute proof" of your answer. When the method of solution is counting-based no "explanation" was deemed necessary. Andy's giving this solution shifted the focus from relating the two tasks to the result of the computation. The teacher chose to followe up on the counting approach by suggesting an alternative way to figure out the sum of 30, 30, 8 and 8, namely adding the two 30's, the two 8's and then adding the partial sums.

The solutions discussed in the presence of the teacher are not the same as those the children attempted when working as a pair or in the brief interchange with the researcher. One interpretation is that, in the teacher's presence, Andy's purpose was to clarify which answer is correct. Another possible interpretation is that, in an effort to resolve the disagreement, he chose to use a method that had been established in the class as unquestionable. The verbal interchange with the teacher addressed the question of which answer is correct, but did not address the disagreement about the boys' differing interpretations of the problem. The teacher's comment, "Not quite 10", signals John that his interpretation is not appropriate but there is no explanation of why and the discussion does not return to this issue. A potential situation for explanation (Cobb, Wood, Yackel, & McNeal, 1992) exists but it does not develop since the teacher and Andy interactively constituted the topic of discussion to be developing a viable solution.

In the subsequent whole class discussion Andy first explained that he and John disagreed on the answer and then gave the variation of his solution method that the teacher suggested when she intervened in their small group work. It served the function of clarifying

that the correct answer is 76. The topic of discussion then moved to the difference the partners experienced in small group work.

Andy: Well, um ... John was a little mixed up cause he thought it was only 48 take away - and no 48. He thought just 48 and no other 48 and 38 and no other 38. And just take away 10 would be 86. But we had to take away 20.

Here Andy's explanation indicates that he might have an understanding of the difference between his and John's interpretation of the task, but there is no clear explanation like the one he gave to the researcher when he said "Well, take away 10, 10 from each of these. ... Take away 10 from each of these to make 20. Take away 20." These children understood that in interactions with the researcher, they would be asked to explain how they thought about the tasks, whereas, in whole class discussion the teacher would typically assist with explanation. The purpose of explaining one's solution was different in the two cases. When explaining to the researcher it was sufficient to describe and clarify your thinking. When explaining to the class, it was necessary to try to explain it so other students in the class, who presumably, represented a wide range of mathematical understanding, could understand. The teacher supports these expectations by asking, "Aha! Why did you pick 20?' Andy's response, "Cause 48 and 48, just take away - tak- 48 take away 10 makes 38" still provides no differentiation between his thinking and J hn's, who repeatedly said "take away 10" during their small group attempt to solve the problem. It is the teacher who goes on to give the clarifying explanation when she says, "Here is 38 (pointing to 48 and 38) and I know that's 10 less. And 48 and 38 is 10 less (pointing to the other 48 and 38). So how's that going to affect your answer?"

In summary, in this example, the language of explanation was limited to one instance of John's during small group work, and one instance of Andy's in the interchange with the researcher. Neither child used the language of explanation in the interchange with the teacher or during the total class discussion. Andy's use of explanation when talking to the researcher is consistent with what we would expect and with the notion of final-draft talk. However during class discussion teachers frequently intervene to fulfill the role of rephrasing

and elaborating on children's solution descriptions for the benefit of other students. By doing so the teacher serves both to assist the children to understand what constitutes an explanation and to interfere with their giving one. It serves to assist the children understand in as much as the teacher repeats and elaborates based on her understanding of the other children's potential for making sense of what is being said. It serves to interfere in that it relieves the student of the obligation to figure out for him/herself which repetitions and elaborations might be useful. In effect, the teacher is taking the responsibility for judging the fit between the explanation and what is taken-as-shared by the class.

Conclusion. This paper has documented the qualitative differences in student's talk and the nature of their explanations in various situations. The critical feature is the student's interpretation of the situation as a social event as it is interactively constituted by the participants.

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LEARNING OPPORTUNITIES IN AN IN-SERVICE TEACHER DEVELOPMENT PROGRAM

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This paper is focused on the analysis of teachers' mathematics pedagogy learning in the context of an in-service development program. Teachers' learning is considered as both an individual constructive activity and as an interactive activity of acculturation into the beliefs and practices of a specific pedagogical community. The analysis will concentrate on two teachers' learning processes. This analysis aims at a better understanding of the complexities involved in the teachers' learning process.

In recent years the studies related to teachers' education have replaced their perspective of a passive teacher with one that considers teacher as an active constructor of her knowledge. This change reflects a parallel shift in our conceptions about mathematics, and mathematics teaching and learning. Constructivism is the epistemology that underlies these changes.

Under this new perspective, several aspects of teachers' developmental process have been illuminated by many researchers (Cooney, 1985; Cobb, Wood, & Yackel, 1990; Lerman & Scott-Hodgetts, 1991; Simon, 1991). These studies are mostly referred to the teachers' individual activity in constructing new knowledge and they do not provide explicit data concerning the process of teachers' learning within the context of interactive communication with their colleagues. However, as Steffe (1991) emphasizes interactive communication that transpires between teachers and students, among students, or among teachers is one of the requirements of constructivism for mathematics education. This requirement springs from a view that respects students' as well as teachers' knowledge. Therefore, we should note that communication among teachers and between mathematics educators and teachers is necessary for teachers' development.

In the context of a teachers' development program we assumed that teachers - like the students - construct their own meaningful knowledge by reflecting on their experiences of their classrooms' incidents as well as by negotiating their own ascribed meanings to

these incidents within the community of their professional colleagues.

In a way analogous to the analysis of students' mathematical learning in a classroom (Wood, 1991), the purpose of this paper is to provide an initial analysis of teachers' learning as a constructive and interactive activity (Bauersfeld, 1988). This analysis is a part of a research program which aims to develop a model of the teachers' learning process.

Method

The examples used in this study for the investigation of teachers' learning come from a teachers' development program organized in order to help teachers construct a context of the class-room's events interpretation compatible with constuctivism. Ten third-grade teachers participated in this program three hours per week for three months. Prior to their participation these teachers were studied with respect to their beliefs and practices by being observed in their classrooms for four months. In the course of the program, these teachers were involved in several activities. Among these activities they were supposed to keep diaries with detailed comments on specific episodes from their own teaching that captured their interest. This activity was expected to develop their reflective practice. Some teachers' lessons were video-taped. These tapes gave opportunities for discussions in the seminar. These discussions were a means for the constitution of taken-as-shared meanings of the classroom events. Two representative teachers' teach-ing actions and reflections about these actions constitute the data for the analysis of their individual development. A transcript of the discourse developed among the program participants provided the data for the analysis of these two teachers' collective development. These analyses will be carried out by using constructs from the sociocognitive and socioanthropological context (Cobb, 1990). Analytical descriptive narrative was the method used for our analysis (Erickson, 1986).

Results

At first an analysis of two teachers' comments on specific incidents of their teaching will be given. In this analysis there will be shown the teachers' learning opportunities that were created by their activity of reflecting on their practice. It should be noted that both teachers used the same activity in their classrooms. This activity was suggested to the teachers by their educator for the purpose of helping them to become aware of their students' potential to construct non-standard algorithms for division.

Second an analysis of the discussion that took place between the teachers and their educator will contribute to an understanding of the related learning opportunities. Although this discussion happened at the end of the first month of the seminar, it should be mentioned that teachers and their educator were still in the process of establishing appropriate norms for their interactive communication.

A. Analysis of teachers' comments

The activity that both teachers gave to their students was the computation of the following sequence of divisions:

72 : 4 = ___ , 36 : 4 = ___ , 40 : 4 = ___ , ___ : 4 = 15 ,
80 : 4 = ___ , 84 : 4 = ___ , 168 : 4 = ___ , 160 : 4 = ___ ,
160 : ___ = 4 , 160 : 20 = ___ , 20 X ___ = 160.

It should be mentioned that both teachers use the same textbook. So two months ago their students worked on divisions with double-digit dividends. As both digits of the dividend were multiples of the divisor students had already been trained on the application of the distributive algorithm by analyzing the dividend into tens and ones. A set of activities on division, including the one above, was given to our teachers. These activities were supposed to be used in place of the textbook's second sequence of lessons concerning a further development of the standard division algorithm. However, the second teacher did not follow our suggestions and instead she used these activities as a supplement to the textbook's activities.

In the classroom of our first teacher - Maria - there was the following whole-class discussion after students had worked in small groups:

Teacher: Let's think again from the beginning, what's 72 divided by 4. What have you found? John, what did your group decide?
John: We found 18. We said 10 times 4 is 40. It doesn't do. We then took 20 times 4 and gave 80. Then we tried 18 and it gave 72.
Teacher: This is nice. Did another group find a different way for this?
Vasso: We said ... 10 times 4 is 40. 5 times 4, which is half of 40 is 20. 40 plus 20 is 60. Up to now we took it 15 times but we reached 60. 60 plus 12 is 72. 4 plus 4 plus 4 is 12. 15 plus 3 is 18.
Teacher: Is there a different way? Kiki and Marina what did you do?
Marina: We found 36 plus 36 is 72. 9 plus 9 is 18.
Teacher: Where did you find this nine?
Marina: In the second problem, Miss, where we found nine.
Teacher: How did you find that nine?
Kiki: As it is ... it makes nine.
Teacher: And why doesn't it make eight?
Kiki: If it was 26 divided by 4, it would be eight.
Teacher: And if I told you six?
Kiki: I will say 6 plus 3 equals 9 ... But shouldn't I think about this four at all?
Teacher: Do you agree to that? Tassia.
Tassia: If it was correct, then 72 divided by 4 would make 9 and that would be wrong.
[Students kept describing their strategies for the computation of the rest of the divisions].

The comments of Maria on the above excerpt of her lesson are the following:

" ... My students' methods in these divisions were wonderful. I was impressed by the way they were connecting the several results. So far I have not let all of them to describe their solutions. One correct method of solution was usually enough. However, I do not know where to find such problems. The textbook does not help. I was very glad with my students' happiness which they felt by participating in the classroom's discussion. I believe that these discussions help my students to develop a different picture of mathematics. By inventing their own strategies and not following my own, they stop viewing mathematics as mysterious and strange to them ... ".

The excerpt that follows is from the discussion that took place in our second teacher's - Eleni - classroom, after her students had worked individually for a few minutes:

Teacher: How much did you find in the first division?
Costas: I found 18 (He goes to the chalkboard and he applies the standard division algorithm).
Teacher: (Several students raise their hands whispering different answers) O.K., what did you find Rena?
Rena: I found 32. (She is asked to come up to the chalkboard but she can not complete her effort to apply the algorithm).
Teacher: You have to practice enough, otherwise you will have problems. Nikos, why are you raising your hand? What have you done?
Nikos: I also found 18 but I used a different way. I said ... how many fours equal 72 and then ... I added 4 plus 4 plus 4 ... until I found 72.
Teacher: This is alright, but it takes a lot of time; that's why it's better to use the way I showed you.
[Students continued with the next problems and solved each problem separately by using the standard division algorithm, the relationship between division and multiplication, or the basic facts]

Heleni's comments referring to the above episode of her lesson are the following:

... As I remember Rena's errors were due to her poor understanding of place value as applied to division. Besides, the technique of division is always a difficult topic for my students. I believe that these difficulties are related to the textbook. The textbook presents several ways of doing division and students get always confused. I usually insist on the standard technique because in this manner my students' problems progressively diminish ... ".

The episodes in Maria's classroom indicate that her students and herself have constructed an environment rich in learning opportunities for her. The variety of her students' strategies, their inventiveness and their attitudes trigger off a change in her context of interpreting her classroom's life. Maria's expectations for the activity of her students cease to be confined to "one correct solution method". These students' attitudes and practices might motivate her towards new expectations about their mathematical activity. Her future expectations might involve a more active participation and responsibility from her students in the construction of new mathematical knowledge. On the other hand, her reservations about her ability to design appropriate activities show that she begins to accept new obligations for her practice. Thus Maria's beliefs about the roles of her students and herself as well as about mathematical activity are in the process of reorganization.

In contrast, Eleni's lesson and her reflections on it do not seem to offer her any significant learning opportunities. The events in her classroom are not interpreted by her as subversive to her practice. Thus she does not have any reason to begin questioning her current practice. Moreover, by being attached to a tell-show-do method she overlooks whatever creative efforts of her students and she expects them to adopt her own methods. In other words, the relationship between her practice and beliefs is not dynamic.

B. Analysis of the discussion

Two episodes selected from a discussion that took place in our seminar will be used for the analysis of teachers' learning in the context of interactive communication.

First episode

The educator presented the excerpts of the two above lessons and initiated the following discussion:

Educator: What are your comments on the students' strategies in these two lessons?

Maria: I did not expect that they would do so well. I was surprised by the strategies they worked out.

Eleni: My students always find such strategies but I believe that we must not encourage them. After all these strategies are not effective when numbers are large. For this reason our primary goal should be to make students understand the standard techniques.

Maria: I can not see how your students develop such strategies. I disagree with you because in the textbooks there are no such activities.

Vassilis: I do not agree with you Maria. Some of my students use such strategies in the textbook's exercises.

Educator: Why does this happen?

Sophia: This happens of course after many exercises and usually with the higher ability students.

Eleni: This is also my opinion.

Giorgos: However, I found out that all my students, without my intervention, found many strategies by themselves in this activity.

Maria: I agree with you Giorgos. My problem is if all of my students understood the various solutions they heard.

Educator: I would like to dwell on what Sophia said. If students had not known the algorithm and had not been trained, could they have developed such strategies?

Eleni: No. Of course students should have been taught the algorithm. These strategies are dependent on the algorithm. What else could have happened?

The above discussion shows the incompatibility of teachers' interpretations concerning students' ability to generate their own strategies. For Eleni a students' ability to generate strategies depends on his prior knowledge of the related algorithm whereas for Maria is a matter of appropriate activities and good communication. This incompatibility stems from differences in their beliefs and practices. Eleni believes that the teacher is the sole source of knowledge. In contrast Maria seems to become aware of the teacher's difficulties in helping students to create their own strategies.

Teachers did not achieve taken-as-shared meanings of the their students' activity in constructing non-standard algorithms. The arguments heard from both teachers do not provide a rationale for their approaching. However, teachers had the opportunity to clarify and articulate their disagreement.

Second episode

Towards the end of the discussion the educator proposed a

comparison of the students' strategies to the teachers. As teachers compare these strategies Maria notes that their textbooks direct teachers to present the division algorithm based on the analysis of the dividend into tens and ones and other strategies are not mentioned. This comment leads Eleni to say "Now, I am convinced that all students' strategies are not related to the taught division algorithm".

These two teachers clarified the domain of students' experiences and achieved a taken-as-shared meaning of the students' activity. Thus they constructed a consensual domain for a further negotiation of the consequences of the students' activity on their practice.

Conclusions

The investigation of teachers' learning is necessary if we want to upgrade teachers' education. However, the analysis of teachers' learning as an individual and collective activity is a complex process. Teachers learn continuously from their experiences in their classrooms. But if we attempt to study their learning and develop appropriate models, it is essential to focus our attention on the design of suitable activities. These activities should reflect teachers' interests and at the same time be directed to topics that the research community of mathematics educators has institutionalized. In this way research on the analysis of teachers' mathematics pedagogy learning would be possible to keep pace with research on the analysis of teachers' mathematics learning.

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SOME THOUGHTS ABOUT CULTURAL TOOLS AND MATHEMATICAL LEARNING

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The study reported in this paper investigates the claim of socio-cultural theorists working in the Vygotskian tradition that cultural tools can serve as objective mediators that carry mathematical meaning from one generation to the next. To this end a longitudinal analysis was conducted of four second-graders' use of a particular cultural tool, the hundreds board. In general, the findings were inconsistent with the socio-cultural view that social and cultural processes drive individual thought. It was, however, possible to propose two phases in the constructive process by which children come to use the hundreds board as an efficient problem solving tool. The paper concludes with a discussion of assumptions implicit in the tool metaphor.

The analysis reported in this paper deals with the role that cultural tools play in mathematical learning. The issue is significant given the claim by socio-cultural theorists working in the Vygotskian tradition that cultural tools can serve as carriers of meaning from one generation to the next (e.g., Davydov, 1988; Leont'ev, 1981; Rogoff, 1990). From this theoretical perspective, children acquire their intellectual inheritance, the objective mathematical knowledge of the culture, as they learn to use the cultural tools of mathematics appropriately.

The particular cultural tool that was the focus of the investigation, the hundreds board, has printed on it a ten-by-ten grid with the squares labeled from "1" through "100" starting with the top left-hand square. The question addressed was whether the hundreds board serves as a so-called objective mediator that supports children's construction of increasingly sophisticated concepts of ten. To this end, a longitudinal analysis of four second-graders' (seven year olds') use the hundreds board was conducted. These children worked together on a daily basis in the classroom and the data consisted of video-recordings of the two pairs' of children's small group activity over a ten-week period. Additional data consisted of video-recorded interviews conducted with the children prior to the first small group session. It should be noted that the classroom was unusual (at least in the United States) in that the teacher had successfully guided the development of what, following Richards (1991), we call an inquiry mathematics tradition (Cobb, Wood, Yackel, & McNeal, 1992). Mathematical activity generally involved explanation and justification, and, further, the teacher and children together constituted a community of validators. Such a classroom provides a best-case scenario in which to investigate the socio-cultural hypothesis given that we are interested in the development of conceptual understanding rather than the shaping of behavior in a narrow range of situations.

Tens and Ones

A distinction that proved useful when accounting for the four children's mathematical activity was that between *ten as a numerical composite,* and *ten as an abstract composite unit* (Steffe, Cobb, & von Glasersfeld, 1988). This distinction can be illustrated by comparing two solutions to a task in which three strips of ten squares and four individual squares are visible, the child is told that thirty squares are hidden, and asked to find how many squares there are in all. One way in which children solve this task is to move their hands with all ten fingers extended as they count "34–44, 54, 64". Steffe et al. interpreted solutions of this sort by arguing that each of the three counting acts is a curtailment of counting ten units of one and thus signifies a numerical composite of ten ones. However, the way in which the children move all ten fingers as they count indicates that these composites are not discrete, conceptually-bounded entities or units for them. Other children solve the same task by sequentially putting up three fingers as they count "34–44, 54, 64". For Steffe et al., the children's acts of putting up individual fingers indicates that the composites of ten ones they count are single entities or units for them. As a consequence, the children are inferred to have created abstract composite units of ten, each of which is itself composed of ten units of one.

A second distinction that proved useful was that between the *image-supported* and *image-independent* creation of numerical composites and abstract composite units. In general, children's establishment of numerical meanings was inferred to be image-supported if they could rely on situation-specific imagery of items grouped in collections of ten when interpreting a task or another's mathematical activity. This would be the case in the two sample solutions given that the children could re-present hidden strips of ten squares. However, if there is nothing particular to a task that might lead children to visualize collections of ten items and, further, there is no evidence that they did so, we would infer that their establishment of numerical meanings was image-independent.

Findings

This section presents a summary of the case studies of the four children's mathematical activity. A more extensive account can be found in Cobb (1993). At the outset, all four children were limited to counting by ones when they used the hundreds board to solve arithmetic tasks. During the ten-week observation period, two of the children, one from each group, curtailed these solutions and counted by tens and ones. For one child, Janet, this subsequently became a routine way in which she solved a wide range of tasks. In contrast, although the other child, Brenda, solved tasks in this way in four different small group sessions, these solutions all seemed to involve situation-specific

advances. On the first occasion on which she counted by tens and ones, she was in fact working with Janet because Janet's partner, Chuck, was absent from school. Here, Brenda first attempted to solve the sentence 37 + 25 = _ by drawing three groups of ten tally marks and seven individual marks. This made it possible for her to create abstract composite units of ten in an image-supported manner when she interpreted Janet's counting solution, "(Points on the hundreds board) 25 plus ten makes 35, plus ten makes 45, plus ten makes 55. 1, 2, 3, 4, 5, 6, 7...62." Brenda then counted efficiently on the hundreds board during the remainder of this session.

The analysis of the video-recordings gives no indication that Janet's and Brenda's use of the hundreds board directly contributed to the advances that they made. Both children appeared to first establish abstract composite units of ten when interpreting a task and then to express those units by counting by tens and ones on the hundreds board. Significantly, there was every indication that Janet could establish abstract composite units of ten in an image-independent manner whereas Brenda could only do so by relying on situation-specific imagery. Further, it appeared that Janet's but not Brenda's use of the hundreds board supported her reflection on and monitoring of her activity of creating and counting units of ten.

Janet's and Brenda's attempts to explain their relatively sophisticated counting solutions to their small group partners, Chuck and John, were generally ineffective. Chuck, who could establish numerical composites of ten, seemed to understand how to count by tens and ones on the hundreds board but did not appear to understand why it made sense to do so to solve tasks. John, who could create neither numerical composites nor abstract composite units of ten, realized that there were more efficient alternatives to his counting-by-ones solutions. However, his occasional attempts to produce such solutions were singularly unsuccessful and it appeared that the only column on the hundreds board that signified the curtailment of counting by one for him was "[illegible]0", "30"...

Conceptually Restructuring the Hundreds Board

The findings briefly summarized above are inconsistent with the view that a cultural tool such as the hundreds board can serve as an objective carrier of meaning for the children. In general, it appears that learning to use the hundreds board efficiently as a problem solving tool involves the construction of increasingly sophisticated numerical conceptions. A comparison of the four children's mathematical activity suggests that two steps can be identified in this developmental process. The first results in knowing how to count by tens and ones on the hundreds board, and the second leads to the ability to use the hundreds board in this way to solve a wide range of tasks. The contrast between John's

and Chuck's problem solving efforts clarifies what is involved in making the first of these two developmental steps.

As was noted, Chuck seemed to understand how Janet counted on the hundreds board, but did not know why she did so to solve various tasks. This understanding appeared to be based on his realization that moving down a column from, say, "37" to "47" curtailed counting the ten intervening squares. Thus, it would seem that he gave numeral significance to regularities in the numerals (e.g., "7", "17", "27", "37"...) and, by so doing, structured the hundreds board in terms of composites of ten units of one. This inference is consistent with the way in which he immediately responded during his interview that he would perform ten counting acts to go from 43 to 53.

It seems reasonable to assume that John could also abstract these regularities from the numerals on the hundreds board. During his individual interview, for example, he produced the number word sequence "4, 14, 24...94" when the interviewer first put down four squares and then repeatedly put down a strip of ten squares. Nonetheless, the hundreds board did not seem to be structured into numerical composites of ten for him and only the column "10", "20", "30"... seemed to signify the curtailment of counting by ones. This inference is consistent with the observation that he had to count by ones during his interview to find out how many times he would count when going from 43 to 53. John's relatively unstructured interpretation of the hundreds board can be accounted for by first noting that a break or pause in the rhythm of counting by ones occurs at the end of each row. These breaks in sensory-motor counting activity might facilitate the isolation of the count of an intact row (e.g., "51, 52...60") as a composite of ten. Counting down the column "10", "20", "30"... would then signify the curtailment of counting along each row by ones. The possibility that John had made this construction is supported by his reference to the hundreds board during his interview to explain why he would perform ten counting acts when going from 40 to 50. In contrast, there are no similar breaks or pauses that might facilitate the isolation of counting, say, "38, 39, 40...47" as a numerical composite of ten. Consequently, the process of giving numerical significance to a sequence such as "7", "17", "27"... would seem to transcend sensory-motor counting activity. John appeared unable to create composites of ten units of one in this image-independent manner whereas Chuck was able to do so.

The second developmental step in learning to use the hundreds board as a problem solving tool can be clarified by considering why Brenda learned from Janet whereas Chuck did not. We saw that Brenda was able to make sense of Janet's efficient counting solution to the sentence 37 + 25 = _ by creating composite units of ten in an image-dependent manner. As a consequence, she knew why Janet had counted-on three tens and seven ones from 25. Chuck, however, seemed to misinterpret Janet's counts by tens and ones. On one occasion, for example, he assumed she had misread the sentence 39 + 19 = _ when she

said "39 plus ten". From his point of view, adding ten to 39 had nothing to do with solving the task. Although he structured the hundreds board by creating numerical composites of ten, there was no indication that these were single entities or units for him. Consequently, the abstract composite units that Janet expressed by counting on the hundreds board did not exist for him.

In contrast to Brenda, Janet could create abstract composite units in an image-independent manner. Further, there was strong evidence that her use of the hundreds board enabled her to reflect on and monitor her activity of counting by tens and ones. It seems reasonable to assume that, for her, the hundreds board was structured in terms of units of ten rather than composites of ten ones. The units of ten and one that she expressed by counting were then simply there in the hundreds board as objects of reflection for her. Thus, her ability to create abstract composite units of ten in an image-independent manner made it possible for the hundreds board to serve as a so-called cultural amplifier for her.

Discussion

Although the findings challenge the notion that cultural tools are carriers of mathematical meaning, the tool metaphor has been accepted uncritically thus far. In many situations, this is a perfectly reasonable way of talking. For example, the second-grade teacher often used this metaphor when she discussed materials such as unifix cubes and the hundreds board with her students. However, it is worth considering how assumptions implicit in the use of the metaphor influence the way that mathematical cognition and learning are characterized. First, it can be observed that talk of, say, the hundreds board as a tool tends to objectify it and to separate it from individual and collective mathematical activity. Newman, Griffin, and Cole (1989) note that a similar process of objectification occurs when researcher talk about "the task". Their analysis of the interactions that occur as a researcher interviews a child leads them to question the assumption that the researcher simply specifies the task for the child to solve. Instead, they propose viewing "the laboratory task as a kind of very tightly supervised instructional interaction" (p. 32). As they put it, "an enormous amount of psychological work goes into maintaining the psychologist's task as a focus of attention" (p. 18). Consequently, "tasks are *strategic fictions* that people use as a way of negotiating an interpretation of a situation. They are used by psychologists and teachers as well as children to help organize working together" (p. 135). This same argument can be made with regard to cultural tools. The analysis of the four children's mathematical activity indicates that there were significant differences in the ways that they structured the hundreds board. Consequently, if we want to preserve the metaphor, we might say that they used different cognitive tools when they

solved tasks. An enormous amount of social and cognitive work therefore went into maintaining taken-as-shared interpretations of the hundreds board. In the course of their small group interactions, both Janet and Chuck, and Brenda and John, were negotiating their differing interpretations of the hundreds board. In this sense, the notion of a single, unambiguous hundreds board was a strategic fiction that the teacher and children used to negotiated taken-as-shared interpretations of situations and to help organize working together. The uncritical acceptance of the tool metaphor obscures the differences in individual interpretations. As a consequence, the interactional and cognitive work necessary to communicate effectively and maintain the hundreds board as an objective entity tends to be overlooked.

To argue that learning to use a cultural tool involves active individual conceptual construction is, of course, not to deny that children make these constructions with the teacher's guidance as they participate in current classroom mathematical practices. For example, the advances that Janet made were not isolated, solo achievements but instead occurred against the background of whole-class discussions in which the teacher both legitimized counting by tens and ones on the hundreds board and indicated that she valued these solutions. Further, Janet contributed to the establishment of this use of the hundreds board as a classroom practice once she could create abstract composite units of ten in an image-independent manner. Thus, in this account, Janet's mathematical activity contributed to the establishment of the mathematical practices that both enabled and constrained her individual mathematical activity. This proposed reflexive relationship between individual activity and the taken-as-shared mathematical practices of the classroom community can be contrasted with socio-cultural theorists' tendency to argue that social and cultural processes drive individual thought. Saxe and Bermudez (1992) capture this reflexive relationship succinctly when they say that an understanding of children's mathematical environments requires the coordination of two perspectives.

> The first is a constructivist treatment of children's mathematics: Children's mathematical environments cannot be understood apart from children's own cognizing activities... The second perspective derives from sociocultural treatments of cognition... Children's construction of mathematical goals and subgoals is interwoven with the socially organized activities in which they are participants. (Saxe & Bermudez, 1992, pp. 2-3)

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DEVELOPING METACOGNITION DURING PROBLEM SOLVING

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Abstract: *The paper presents a case study that chronicles one path of the development of metacognition during problem solving and highlights some mechanisms that spurred that development and some of its effects on the subject. Data included journal entries, group and written problem solutions, four videotape sessions of talking aloud while solving a problem, a pre- and post-Attitude Inventory, and prompted retrospection. The development showed evidence of certain aspects of metacognition becoming automatic. Mechanisms that spurred development all had to do with externalizing thought processes in some way. The development had positive effects on the student's success in problem solving.*

Metacognition has been widely discussed and accepted as an important factor for success in problem solving. Yet, the path of its development, and the mechanisms that spur someone along that path, as well as the effect of that development have received much less attention (Lester, 1992). Earlier work established that self-report data can be used to study the development of metacognition (DeGuire, 1987, 1991). The purpose of this paper is to present a case study that chronicles one path of such development and highlights some mechanisms that spurred that development and some of its effects on the subject. Though there is no universally accepted meaning of the word "metacognition," all usual definitions include the monitoring and regulation of one's cognitive processes; this is the aspect that will be emphasized in this paper. Also, the word "problem" is used here to mean a task for which the potential solver does not know a set procedure, even though one might exist.

Context of the Case Study

The subject of the case study (Jackie) was a student in a course on Problem Solving: Topics in Mathematics for the Elementary Classroom, taught by the researcher. There were 12 students in the course, all preparing to teach students in the elementary grades (ages 5 or 6 through 12 or 13) and all with limited backgrounds in college-level mathematics.

The data for the case study were gathered throughout the semester-long course (two 75-minute sessions per week for fifteen weeks) on Problem Solving. The course began with an introductory phase, that is, four sessions devoted to an introduction to problem solving, several problem-solving strategies (e.g., make a chart, look for a pattern, work

backwards), and metacognition. After the introduction, the course progressed from fairly easy problem-solving experiences to quite complex and rich problem-solving experiences, gradually introducing discussions of and experiences with the teaching of and through problem solving and the integration of problem solving into one's approach to teaching. Throughout the course, students discussed and engaged in reflection and metacognition. Often, especially early in the course, students solved problems in pairs, with one student serving as the "thinker" and the other as the "doer" (Schultz & Hart, 1989). Most other in-class problem solving took place in groups of 3 to 5 students.

During the course, the students were given 8 problem sets to be solved and written up outside of class in order to be evaluated. The written report was to be an in-progress record of all work on the problem, including blind alleys, and was to include a separate column for "metacognitions." All solutions were evaluated so as to give more weight to the cognitive and metacognitive processes in the solution than to the final solution of the problem.

Students in the course also completed an Attitude Inventory (Charles, Lester, & O'Daffer, 1987, p.27) twice, once early in the course and once at the end of the course, and wrote 10 journal entries. The topics of the journal entries were chosen to encourage reflection upon their own problem-solving processes and their own development of confidence, strategies, and metacognition during problem solving.

Jackie was one of two students who volunteered to be videotaped while thinking aloud during problem solving. There were four videotape sessions at various times during the course. The problems used in the videotaping sessions are representative of the problems used throughout the course. In abbreviated form, they were the following:

<u>Videotape Session 1:</u> (Number Problem) Three whole numbers multiply to 36. Five more than the sum of the numbers is a perfect square. What is the sum of the numbers?

<u>Videotape Session 2:</u> (Fence Problem) You have 10 boards, each 1 unit by 1 unit by 2 units. (The red rods from sets of Cuisenaire rods were used as a model for this problem.) You want to build a fence 10 units long by 1 unit wide by 2 units high. It is possible to build the fence in a variety of ways. (The side views of 2 possibilities were pictured.) With how many different arrangements of the boards could you build the fence?

<u>Videotape Session 3:</u> (Locker Problem) There are 1000 lockers, numbered from 1 through 1000, and 1000 students. Each student walks by the lockers one at a time. The first student opened all of the lockers. The second student then closed every other locker, that is, the one with the even numbers on them. The third student changed the status of every third locker, that is, opened it if it was closed and closed it if it was open. The fourth student then changed the status of every fourth locker. One by one, the students changed the status of the appropriate lockers. At the end, which lockers were left open and where were left closed?

Videotape Session 4: (Extension of Checkerboard Problem) How many different rectangles are on an 8-by-8 checkerboard? Note, rectangles are considered different if they are different in position or size. So, a 2-by-1 rectangle is considered different than a 1-by-2 rectangle.

Each session included brief discussions before and after the solution in which the subject was asked to reflect on their development of problem solving ability and/or metacognition and to reflect on the just-completed solution. The fourth videotape session included prompted retrospection; that is, the researcher and the subject viewed the videotape of the just-completed solution, with either person stopping the tape at any time for the researcher to probe the subject further about his or her processes or for the subject to voluntarily supply further information about what was occurring at that time during the solution.

The following case study is the story of the development of one of the videotaped subjects, Jackie. Throughout the description, direct quotes are from her written problem solutions, journal entries, or videotape transcripts.

The Case Study of Jackie

Jackie was a "mature" student returning to college after an extended absence. She had completed the two-semester sequence of mathematics courses for elementary teachers and had taken Calculus I and II, earning A's in all of them. She had had no teaching experience or classroom experience. As a problem solver, she described herself as "striving to be creative but this is the most difficult area for me. . . . I would like to be more organized in my mind." She described her previous problem-solving experiences as having "used math in Physics and Chemistry to solve practical problems" and having some brief exposure to strategies "and stuff" in her Math-for-Elementary-Teachers courses.

After the introductory phase of the course

After the introductory phase of the course, Jackie described her approach to solving a problem in the following way.

> I start by reading it and then writing down my interpretation of the problem. Then if I don't know what to do next I just stare at the stuff I wrote down and wait for an idea to come to me. When I get an idea I play with it for a short time and see if it might lead to a solution. If it looks hopeless I abandon it and stare at the problem again waiting for a new idea. I do this until I find a strategy that works or I give up.

She had used this staring technique before beginning this course but "it felt haphazard, out of my control, and I never knew if an idea would come. . . . {Now} I am finding out what the

haphazardness is all about. I stare at problems with more purpose now knowing that metacognitive thinking is taking place on some level." She felt that "the exercise where I keep track of my metacognitive processes on the right side of my work" helped her to access these metacognitive processes. She described her metacognition as "there is more than one 'person' working on the problem."

During her first videotape session (the Number Problem), she used her staring technique and the haphazardness of her attempts was very clear. She tried briefly to guess the numbers but confused the conditions of the problem. She felt overwhelmed with all the possibilities but did not attempt to list them in any orderly way. She briefly tried setting up an equation and then setting up a table but discarded each approach quickly, each one giving her a sense of being overwhelmed with possibilities Finally, she returned to guessing and suddenly found the triple that worked. She said she had "stumbled upon it., . . .I didn't expect it." However, she did feel that trying to talk aloud while solving the problem had interfered "a lot" with her cognitive processes. Upon further probing. she clarified that interference as perhaps more "slowing them down than really changing them."

During the second part of the course

Three weeks after her first videotape session, Jackie felt she had improved as a problem solver, that she was "becoming much more aware of my cognitive and metacognitive abilities." At the same time, she described the changes in herself as "I'm willing to risk making wrong turns more. This class has really improved my confidence and has helped me to think of strategies a little more organized than before."

One week later, in the seventh week of the course, Jackie attempted the Fence Problem in the second videotape session. Before beginning, she described herself as "getting better at realizing what I'm thinking, with the Thinker-Doer stuff. . . . I'm getting a little more sense of control. . . . I'm becoming more conscious of, more aware of what's going on in my head. . . . Writing the metacognitive on the right and the problem on the left, . . .you know once you write it down, you can see what you're thinking." She again referred to "another metacognitive thing that's kind of overseeing the metacognitive part of me." After reading the Fence Problem, Jackie quickly realized there would be many possibilities, became overwhelmed with possibilities, and tried to devise a chart or a way to record them all. She found the chart confusing and returned to generating possibilities. She tried to record them in her chart but found ones that would not fit into the chart. Finally, Jackie said she was ready to quit the attempt. At that point, the researcher intervened to suggest that "sometimes when you are overwhelmed, that's a signal for something," a point that had been discussed in class. Jackie could not accommodate this signal into her thinking. So, the

researcher again prompted Jackie by interjecting that "Frequently making a chart goes with other things. You might consider what other things you frequently do when you make a chart." Suddenly Jackie decided to try solving a simpler problem and began to consider fences of varying lengths. After correcting some errors in the combinations generated for certain lengths, Jackie had to quit the session (to go to a class) on the verge of but without solving the problem. When the researcher would not finish the solution for her, she left with a sense of challenge rather than disappointment. She successfully completed the solution to the problem by the next day.

During the third part of the course

During the next four weeks, Jackie improved significantly in her written problem solutions. She was trying a wide variety of approaches to problems and was beginning to check them in various way. At the beginning of the third videotape session, she commented, "I feel more confident about what roads to take." When asked whether she thought her awareness of her cognitive processes had changed since the course began, she did not let the researcher finish the question but responded enthusiastically, "Very much." When the researcher probed as to whether "your awareness has really changed or developed or do you think you've learned vocabulary to express what you really were aware of before," she responded, "Both." She went on to say that "becoming aware of them has given me more control, I think. . . . Now when I'm stuck or something, . . .I can kind of regulate stuff." She also recognized in herself a willingness to try to do a problem more than one way.

At the third videotape session (eleventh week of the course), Jackie was asked to solve the Locker Problem. After reading the problem, she quickly decided to do a simpler problem (10 lockers) and to record the status changes in a chart. Soon, she saw that the lockers with numbers that are perfect squares would be left open. She felt that this solution was mostly a guess. She then pondered that "I feel like I have an answer, but I haven't got a clue how to check it. But I want to check it." She then extended her chart to include 20 lockers and confirmed her initial solution. "I wonder why that is. It has to do with the numbers." She then tested individual lockers (e.g., locker number 12), now going through their status changes by considering whether or not the locker number was divisible by the student number. "What is there about square numbers? Why are the square numbers left open?" She went on to list all the factors of several numbers, which led her to realize that square numbers have an odd number of factors while others have an even number of factors and thus lockers with square numbers would be left in a different status from other lockers But she still wasn't satisfied. "I wonder why that is. . . . why the square numbers

have odd numbers of divisors? OH! OH! OH! I know! One of them gets counted twice. The middle one. . . . I never knew that before." When asked if she felt "pretty sure of it even though you never went past the 16th locker," she responded confidently, "Yeah, because it's logical." The researcher pointed out to her that "the way you checked it is you understood why the pattern works that way. Now that is a different kind of looking back strategy than you've used before, right?" Jackie agreed, "Yeah, I've never done that before."

During the last part of the course

During the last part of the course, Jackie continued to be a successful problem solver. Her reports of her cognitive and metacognitive processes became more elaborate and sophisticated. However, some of her processes were becoming automatic. After one problem solution, she added a postscript which said, "Oh, I forgot to say in the metacognitive column that I used a chart. It was so obvious, it was just kind of automatic. So my 'meta' self didn't report it. My 'doer' self just did it." Jackie described her own metacognition as "the part of my problem solving thinking that actually directs me in deciding what roads I should and should not take to solve the problem." She reported,

> I now have much more confidence as a problem solver. This has improved my attitude in other problem solving situations outside of here. I'm aware now of my metacognitive processes and they have settled down quite a bit. . . . I am more willing now to keep trying. Looking back has especially improved my learning from problems. . . . My awareness has gone from 0 to 7 or 8. . . . Becoming aware of cognitive and metacognitive processes has improved my problem solving ability. Listening to myself think has made me consider what I am thinking. . . . Watching myself solve the Locker Problem on videotape was particularly revealing. . . . I believe my awareness has actually grown <u>far</u> more than, um, not just my vocabulary to express it. It has grown to such as extent that I feel I have a meta-metacognitive process watching my metacognitions direct my cognitions.

In the fourth videotape session (the Extension of the Checkerboard Problem), Jackie successfully solved the problem by doing simpler problems, making a chart, and looking for and finding a pattern. Initially, she was not very confident of her answer. "I'm not very quick as far as finding patterns. I predict the answer is 1296 rectangles. I don't know why. . . . I see a pattern. The pattern works. It holds as far as I can tell. I really haven't double checked everything. But I think I was pretty methodical about counting squares." She still struggled with why this pattern arose. As she struggled, she eventually recognized it as the squares of the triangular numbers. She used this fact to produce a general formula for the number of rectangles on an n-by-n checkerboard but rejected the general formula as a means of checking. "What can I say? If I get a general formula, it fits my data. It still doesn't

mean it's right." She decided she would quit for then and come back later to trying to figure out how the numbers relate to the picture.

In the discussion after the solution of the problem, when Jackie described the strategies she used to solve the problem, she neglected to report that she solved simpler problems first. When this oversight was pointed out to her, she responded that "You're right. . . . I didn't mention it because it was too obvious to mention. It was just so automatic to do that." Upon questioning about her confidence in her solution, she admitted she had never actually counted the 1296 rectangles on the 8-by-8 checkerboard. Yet she felt "pretty confident" of her solution "if I did the counting right. That's the place where I'm least sure of it." The researcher asked her "How can you tell that you did the counting right?" Though she recognized there were internal patterns in the chart she used to do the counting, she did not at first recognize them as a means of checking. Her suggestion to check was to redo the problem. However, she still hoped to understand why the pattern worked.

Later in the prompted retrospection, at a point in the tape when Jackie seemed to be stuck, she again referred to her staring technique, saying she was "trying to bring things in from the outside, . . .waiting for them to come in, float in from where ever they come from." After considerable probing and discussion as to what she might do if nothing "floated in," the researcher finally suggested that "at least what some people do is consciously think of a list of strategies and tick them off one by one. Or they consciously ask themselves those questions that Polya suggests at each stage of the problem-solving process. Or. . . ." At this point, Jackie had a sudden insight and excitedly interrupted with " Oh yeah! Of course! They force things to float into their heads!! Because, indeed, every time something floats in, I suspect that there's a part of my mind that is actively sending it through to float in. That's right! There's a couple of 'me's' in there. And the one who is writing down the stuff, that's the person who is waiting. And the 'metacognator' up there, the thinker, is sending down these things for the "doer" to latch on to and see if they're going to work or not. (Pause, as she obviously ponders.) Yeah. . . . (Pause) Wow! I guess I'm not so passive."

Toward the end of the prompted retrospection, Jackie stated that she thought "maybe the very act of having been a participant in the videotaping might have made a difference" in doing well in the course. [Both students that volunteered for the videotaping did especially well in the course.]

Jackie did very well on the final exam. She felt "confident about all the problems because I tried and found several ways of solving them using different strategies. . . . The problems themselves were fun and interesting." A comparison of her responses on the first and second administrations of the Attitude Inventory showed a clear increase in her persistence, in her enjoyment, and in her confidence in solving problems.

Discussion

It is clear that Jackie made very substantial progress in developing her metacognitions and becoming an expert problem solver. The case study above chronicles that development. Jackie also provided evidence that the development of metacognition can and does lead to at least some metacognitive processes becoming automatic.

The report also includes references to several mechanisms, some explicitly referred to by Jackie, that seemed to spur that development. Those mechanisms all had to do with externalizing her thought processes in some way (either orally or in writing, either during or after problem solving) and in reflecting on or observing herself solving problems. It seems possible to assimilate each of these mechanisms into an ordinary classroom setting.

Note also that Jackie referred several times to another level or aspect of metacognition which monitored her metacognition. It seems reasonable to conclude Jackie was referring to an awareness of metacognitive processes that seems to be separate from the metacognitive processes themselves.

Finally, Jackie referred either directly or indirectly to the effects on her of the development of her metacognitive activity and awareness of that activity. To summarize these effects, she gained a sense of control over her cognitive and metacognitive processes and increased the organization of her processes. As a consequence, she improved her confidence in her problem-solving abilities, her persistence, her willingness to risk error, and her enjoyment—all important characteristics of an expert problem solver.

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The Mechanism of Communication in Learning Mathematics

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The purpose of this paper is to throw light upon the fundamental mechanism of the communication process in learning mathematics. For this purpose, classroom observation, questionnaire and interview to the 8th grade students was done.
The communication process in learning mathematics is a dynamic process for the social interaction between participants, and it develops with alternations of sharing information and making communication gaps. It is an 'autonomous adjustment-process' which has two faces of cooperation and personalization.

Introduction

The importance of social interaction in constructing mathematical knowledge makes us pay attention to the significance of communication in learning mathematics (Good et.,1992; Krummheuer et al.,1990; Pimm,1987; Bishop,1985). But we do not know how we communicate mathematical ideas (Laborde,1990). Von Glasersfeld said,'The underlying process of linguistic communication, however, the process on which their teaching relies, is usually taken for granted, there has been a naive confidence in language and its efficacy (Von Glasersfeld,1987,p.4).' Therefore we should study the mechanism of the communication process in learning mathematics.

The meaning of words to sender or receiver who sends or receives some message depends on his/her knowledge and experiences. We can not transmit meaning to other people, we just can send a message (Samovar ed.,1991). The message does not contain any meaning, the message is just auditory or optical stimulus (Skemp,1982). If we grant that the meaning of message is our own subjective construction and resides in a subject's head, we should analyze the communication process in learning mathematics from the point of view 'the subjectivity of meaning' (Bauersfeld et al.,1988).

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The purpose of this paper is to throw light upon the fundamental mechanism of the communication process in learning mathematics. For this purpose, two studys which consist of classroom observation, questionnaire, interview to the 8th grade students are designed.

[Study 1]

Subjects: One class of grade 8 in Tokyo. It consists of forty students.

Procedure: For these studys, a new japanese word 'Keisan no houhou wo simesu kigou, or a sign which indicates a calculation method' is coined. This coined word is made of well known words 'sign','indicate','calculation' and 'method'. This coined word is not defined. The meaning of the coined word is discussed in the class. Students are asked to classify some mathematical signs by their personal interpretation of the coined word. If students did the same classification, then we regard they could share some common information through the communication. Before the discussion and after the discussion, students answer Questionnaire 1 and 2 respectively. One week later, students answer Questionnaire 3.

Materials: Three questionnaires are prepared.

Questionnaire 1 & 2 : Both of them contain the same question; ' '%' is a sign which indicates a calculation method or not ? Why ? '

Questionnaire 3 : ' 'a^3 (exponent)' is a sign which indicates a calculation method or not ? Why ? '

[Study 2]

Subjects: Another class of grade 8 in the school where Study 1 was done. It consists of forty students.

Procedure: The meaning of the coined word 'a sigh which indicates a calculation method' is discussed in the class. One week later, students answer Questionnaire 2. Two weeks later, students answer Questionnaire 3. And some students are interviewed by the observer. The classroom discussion and interview are recorded by the audio and video tape recorder. Questionnaire 1 is not asked before the discussion, because of avoiding any influence to the students' classroom discussion.

Materials: Questionnaire 2 & 3 are used ,which are the same ones used in Study 1.

Results and Discussions

[Study 1]

[The results of Questionnaire 1 and 2]

Students selected one choice out of the four alternatives, ' '%' is a sign which indicates a calculation method ?', ① Yes, ② No, ③ Neither or ④ Not sure. Table 1 shows us that some students changed their choices after the discussion. In Study 1, students had twenty five minutes' discussion. They discussed the reason why they selected their choices. During the discussion, students expressed their own interpretation. After that, eleven out of forty students changed their choices (Table 1).

Table 1 : The number of students who select each choice for Q1 & Q2.

	① Yes	② No	③ Neither	④ Not sure
Questionnaire 1	13	21	3	3
Questionnaire 2	6	24	6	4

Table 2 shows us that the discussicn had influence upon some students, especially for the students who had selected ① as the answer for Q1. Six out of seven students who had changed their choices from ① to others selected ③ or ④. This result indicates us they were confused by the conflict which was brought by the communication with others (Table 2).

Table 2 : The content of changes of students' choices from Q1 to Q2

Q1→Q2	①→①	②→②	③→③	④→④		Small Sum of unchanged	
	6	20	1	2		29	
Q1→Q2	①→②	①→③	①→④	②→③	③→②	④→②	Small Sum of changed
	1	4	2	1	2	1	11

[The result of Questionnaire 3]

Students selected one out of the two alternatives, ' 'a³ (exponent)' is a sign which indicates a calculation method ?', ① Yes or ② No.

Twenty four students selected ② as the answer for Q2. Twenty students kept their personal interpretation, and four students changed their personal interpretation of the coined word. One week later, they answered Q3. Table 3

shows us they were separated into two groups ① Yes or ② No.

Table 3 : The content of changes for 24 students who selected ② for Q2

Q1→Q2→Q3:①	②→②→①'	9	①③④→②→①	3
Q1→Q2→Q3:②	②→②→②	11	①③④→②→②''	1

From the result of Q3, we know they could not share the common interpretation of the coined word, even if they selected the common choice for Q2. The communication process in learning mathematics develops with alternations of sharing information and making communication gaps.

[Study 2]

In study 2, classroom observation, Questionnaire 2 & 3, and interview had been done and these data were analyzed qualitatively.

[Classroom observation]

Teacher : And then, which signs indicate its calculation method ?
(Some mathematical signs are written on the black board.)

Students : Plus(+). Minus(−). Multiplication(×). ano division(÷).

Student E: Percent(%) ?

Teacher : Percent(%) ?
Do you think 'Percent' also indicates a calculation method ?

Student F: Yes. I think so. For example, the eighty percents of something, we can get the answer with multiplied by 4/5.

Student G: I am sure that we can get the answer with multiplied with 4/5 for the eighty percents of something. But we never write 100 % 80 for the eighty percents of something. So I think it's just a sign.

Teacher : Unn. We never write. as G said. 100 % 80 like 5 + 3.

Student H: We have five plates, and there are two apples on each plate. How many apples are there ? If we solve this kind of problem, we should write the expression like this, two times five equals ten. and adding the unit 'pieces' with parentheses. 2 x 5 = 10(pieces) , so I think % is the same sign which is necessary to be added parentheses or something.

In this episode, students were asked, 'what is the meaning of the coined word 'a sign which indicates a calculation method' ?' This word was coined

by the teacher, and this was an open question. We wanted to know how students try to interpret it. From the classroom observation and personal interview, we know that they tried to interpret it by probable reasoning with using their personal knowledge and experiences. Its illogical uncertainty of probable reasoning brought the necessity of communication with other people. Some student said, 'I want to know others' opinions, and I want to discuss it.'

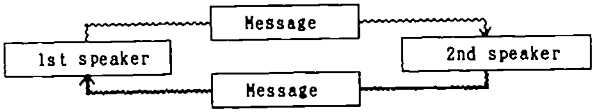
In the episode, three students expressed their personal interpretation. At first, Student F said that % was a sign which indicated a calculation method. And next, Student G sent a message to Student F as feedback. The feedback is defined as an influence which the second speaker has upon the first speaker's thought or attitude by responding to the message that the first speaker sent. In this case, Student G used the method of indirect proof for refuting the first speaker who had used the method of direct proof . They refined their reasoning. The Student G's reasoning is analyzed below.

At first, Student G paid attention to the two things, ''+'is a sign which indicates a calculation method' that had been confirmed by the class and ''+' is a sign which indicates operation'. And he expanded the reasoning for the sign '+' into the case of general signs. Then he generalized the reasoning into 'If the sign 'A' is a sign which indicates a calculation method, then the sign 'A' is used as a sign which indicates operation($P \rightarrow Q$)'. And next, he turned the reasoning $P \rightarrow Q$ into the contraposition $\bar{Q} \rightarrow \bar{P}$. Student G believed the contraposition,'If the sign 'A' is not used as a sign which indicates operation, then the sign 'A' is not a sign which indicates a calculation method', was true. Student G applied this contraposition to the sign '%'. He had finally concluded that his reasoning, 'If the sign '%' is not used in the form of '100 % 80', '%' is not a sign which indicates a calculation method', was true. In the interview, Student G said, 'I do not know such a term 'contraposition'.' He did generate the way of thinking like the indirect proof through the communication with Student F.

The circulation for sending a message and feedback is a basic cycle of social interaction. The communication process develops by linking these basic cycles. Between two communication participants, they respond each other. We call the feedback between a pair of two 'Dyad Feedback'(Figure 1).

Figure 1 : The basic cycle of social interaction

'Sending the first message'

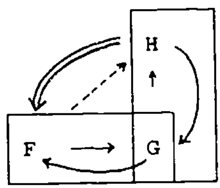


'Dyad Feedback'

Among more than two participants, several sets of dyad feedback link together. In this episode, the third speaker Student H had influence upon Student F and Student G. The message Student H sent acted on Student G as a positive feedback and Student F as a negative feedback. This phenomenon tells us that the same message acts on each one of receivers differently.

Among more than two participants, several sets of dyad feedback link together, and the N-th speaker ($N \geq 3$) could have influence upon the first speaker. Figure 2 indicates us that the third speaker Student H had influence upon the first speaker Student F. We call this phenomenon 'Chained Feedback'. The model of Chained Feedback (Figure 2) consists of two sets of dyad feedback.

Figure 2 : Chained Feedback



In figure 2, two sets of dyad feedback link together. The third speaker has influence upon the first speaker as feedback (the arrow '←' in Figure 2). And this model also indicates us that the first speaker acts on the third speaker (the arrow '→' in Figure 2). The chained feedback is also one of the feedback, it always contains a basic cycle of social interaction (sending a message F→H & feedback F←H). The communication process among more than two is complicated than the communication process between two.

After Student F received the feedback, he changed his personal interpretation. He answered the questionnaire 2 one week later:

Student F wrote : ' '%' is a sign which indicates the unit like 'piece' or 'g' when we write to add it to the next of the answer.

'%' is not a sign which indicates a calculation method, because we can not put it into the middle of numbers like '2 + 2' or '2 × 2'.'

From the answer to the questionnaire and interview, we know Student F has changed his personal interpretation. The first part of the answer Student F wrote is a copy from Student H's opinion, and the second part is a copy from Student G's opinion. This shows us that both of the messages from Student H and Student G acted on Student F as feedback.

And Student G answered the same questionnaire;

Student G wrote : ' I suppose '%' is a sign which indicates quantity. So it is not a sign which indicates a calculation method.'

In the interview, Student G said, 'I think it is better to answer the questionnaire like what Student H said. I should say what kind of sign '%' is rather than '%' is not a sign which indicates a calculation method.' These results show us Student H also had influence upon Student G.

Two weeks later, they were asked Questionnaire 3 and interviewed. Student G and Student H answered, 'a^3 (exponent) is a sign which indicates a calculation method.' But Student F answered, 'a^3 (exponent) is not a sign which indicates a calculation method.' Student G and Student H thought a^3 indicated the process of its calculation a x a x a. On the contrary, Student F thought a^3 indicated the result of the calculation a x a x a. This shows us that their personal interpretation had not completely agreed during the discussion. In Study 1, Table 3 also shows us the same phenomenon as this case (* and ** correspond to Student G & H and Student F respectively).

Conclusion

The result of this paper is summarized, 'The communication in learning mathematics is the social interaction which develops with alternations of sharing information and making communication gaps. This social interaction develops as chained feedback of several basic sets of dyad feedback whose cycle consists of sending a message and feedback. In this cycle, each participant forms his/her personal interpretation based on his/her knowledge and experiences, and he/she assimilates with other people and accommodates his/herself to solving some conflicts. Therefore we can say that the communication process is 'the autonomous adjustment-process' with having two faces of cooperation and personalization.'.

We are involved in the discussion among researchers; ' how to foster

students' competence to communicate mathematically " and " how to solve the problems for communication in the mathematics classroom ". The result of this paper, the communication process in learning mathematicsis the autonomous adjustment-process, tells us both of two problems should be solved simultaneously. We have to take care of the facts that the competence to communicate mathematically is fostered through the everyday practice and the communication diseases in the mathematics class also grow and disappear in the routine activities. The fact that the communication process has two faces of cooperation and personalization indicates us that the communication diseases are the problems for each participant and the whole of community.

This paper gives us one of the perspectives for the future studies. And the next research question is to throw light upon the mechanism of the communication process generating new mathematical ideas when solving a problem in a group.

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MATHEMATICAL ACTIVITY AND RHETORIC: A SOCIAL CONSTRUCTIVIST ACCOUNT

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This paper sketches aspects of a social constructivist account of mathematical activity. Its key feature is that it focusses on mathematical activity within a social context, where the language of school mathematics (including the rhetorical dimension), social relations, etc. are the primary focus. Activity is construed as taking place in a 'frame', and being largely concerned with the goal- directed transformation of signs. Such semiotic activity both presupposes, and also leads to the construction and elaboration by the learner of a meaningful 'math-world' associated with the frame. The theory is illustrated with the analysis of a learner's task and simultaneous interview protocol.

During most of their mathematics learning career from 5 to 16 years and beyond, learners work on textual or symbolically presented tasks. They carry these out, in the main, by writing a sequence of texts (including figures, literal and symbolic inscriptions, etc.), ultimately arriving, if successful, at a terminal text 'the answer'. Sometimes this sequence consists of the elaboration of a *single* piece of text (e.g. the carrying out of 3 digit column addition). Sometimes it involves a *sequence of distinct* inscriptions (e.g. the addition of two fraction names with distinct denominators, such as $1/3 + 2/7 = 1\times7/3\times7 + 2\times3/7\times3 = 7/21 + 6/21 = 13/21$); or it may combine both activities, as in the example below.

A rough estimate of the magnitude such activity is as follows. A child's compulsory (state) schooling in Britain extends for something over 2000 days. Suppose a typical child attempts a mean of between 5 and 100 mathematical tasks per day (an estimate that is quite plausible). Then a typical British school child will attempt between 10,000 and 200,000 mathematical activities in their statutory school career. The sheer repetitive nature of this activity is under-accommodated in many current accounts of mathematics learning, where the emphasis is more often on the construction of meaning. (Notable exceptions include Christiansen *et al.*, 1985, and Mellin-Olsen, 1987.)

There are three levels to the social constructivist theory of mathematical activity presented here. There is the linguistically presented *task*, the *frame* surrounding the task, and the *social context* of the frame (including the classroom, the teacher, learners, and all that they involve).

THE SOCIAL CONTEXT of the classroom is viewed as a complex, organised social form of life which includes the following categories (which are neither disjoint nor exhaustive)

(a) persons, interpersonal relationships, patterns of authority, pupil-teacher roles, modes of interaction, etc.

(b) material resources, including writing media, calculators, microcomputers, texts representing school mathematical knowledge, furniture, an institutionalized location and routinized times.

(c) the language and register of school mathematics (and its social regulation), including:

1. the content of school mathematics; the symbols, concepts, conventions, definitions, symbolic procedures, and linguistic presentations of mathematical knowledge;

2. modes of communication: written, iconic and oral modes, modes of representation and rhetorical forms, including rhetorical styles for written and spoken mathematics.

Thus, for example, teacher-pupil dialogue (typically asymmetric in classroom forms) takes place at two levels: spoken and written. In written 'dialogue' pupils submit texts (written work on set tasks) to the teacher, who responds in a stylized way to its content and form (ticks and crosses, marks awarded represented as fractions, crossings out, brief written comments, etc.)

This theorization of the social context draws on a number of sources which regard language and the social context as inextricably fused, including Wittgenstein's philosophy, Foucault's theory of *discursive practices*, Vygotsky and Activity Theory, Halliday and sociolinguistics. Applications of some of these theorisations to the learning of mathematics have been made by Walkerdine (1988), Pimm (1986) and Ernest (1991).

FRAME This concept is elaborated in a number of different ways by Marvin Minsky, Erving Goffman and others, and applied to mathematical activity by Davis (1984) and Ernest (1987), albeit in an information processing orientation. It bears some similarity to Papert's concept of microworld, as well as to that of 'solution space' in problem solving research. Frames, as used here, concern a specific (but growing) range of tasks and activities, and each is associated with a particular set of representations, linguistic and otherwise, a set of intellectual tools, both symbolic and conceptual (and possibly a set of manipulable tools, such as rulers or calculators).

Frames have a dual existence, both public and private. The public aspect of a frame corresponds to a mathematical topic or type of problem, and the associated language and intellectual tools. It constitutes what is taken-as-shared by a number of persons, although different instantiations of a frame will vary, e.g. with time and social location..

In its private aspect a frame is constructed individually by each person (learner or teacher) as a sense-making and activity performing device (resembling a 'schema'). The meanings, conceptual tools and goal-types make up a 'math-world' which is a subjective construction associated with the frame, at least in outline (specific details may be filled in during particular tasks). Each individual's personal construction of a frame is associated with a body of cases of previous uses of the frame, sets of symbolic and conceptual tools, and stereotypical goals. Social interaction allows some meshing of the individually constructed frames, and a crucial feature of frames in this theory is that they are genetic, continually developing and growing, as a result of interaction and use. (The varieties of frame use and growth correspond to Donald Norman's categories of schema use: tuning, routine use or assimilation, application, restructuring or accommodation).

The process of frame-utilization and growth requires the learner internalizing and

pursuing an activity-related goal (as in Leont'ev's version of Activity Theory). Particularly in the engagement with and performance of non-routine tasks, the learner will be making effort and success-likelihood estimations, and may disengage from the goals and give up the task, or seek assistance from others. The learner may lack confidence and need reassurance; or may not be able to make the transformations unaided (i.e. lack a tool, or not know which to apply), in order to achieve the goal. Then the task lies within the learner's Zone of Proximal Development, and assistance enables the learner to make the symbolic transformations, hence to extend the appropriate frame so that ultimately s/he can undertake this challenging type of task unaided.

TASKS concerning the transformation of mathematical signs are central to this theory Typically a task is a text presented by someone in authority (the teacher), specifying a starting point, intended to elicit a frame (a task in a sequence may assume a frame is in use), and indicating a goal state: where the transformation of signs is meant to lead. The theorization of tasks draws on Activity Theory and semiotic analyses of mathematics (e.g. Rotman, 1988), as well as cognitive science approaches Mathematics education sources include Christiansen *et al* (1985), Mellin-Olsen (1987), Davis (1986), Skemp (1982), Ernest (1987a, b). From a semiotic perspective, a completed mathematical task is a sequential transformation of, say, n signs ('S_k') inscribed by the learner, implicitly derived by $n-1$ transformations ('$\Rightarrow$'). This can be shown as the sequence $S_1 \Rightarrow S_2 \Rightarrow S_3 \Rightarrow \ldots \Rightarrow S_n$. S_1 is a representation of the task as initially construed (the text as originally given, curtailed, or some other mode of representation, such as a figure). S_n is representation of the final symbolic state, intended to satisfy the goal requirements as interpreted by the learner. The rhetorical requirements of the social context determine which sign representations (S_k) and which steps ($S_k \Rightarrow S_{k+1}$, for $k < n$) are acceptable. Indeed, the rhetorical mode of representation of these transformations, with the final goal representation (S_n), is the major focus for negotiation between learner and teacher, both during production and after the completion of the transformational sequence.

Each step $S_k \Rightarrow S_{k+1}$ is a transformation of signs which can be understood on two levels. Drawing on Saussure's analysis, each sign S_k ($= \mathrm{S}_k / \mathrm{I}_k$) is a pair made up of a signifier S_k ('S' for *symbol*) and a signified I_k ('I' for *interpretation* or *image*). So the completed task can be analyzed as in the example shown as Figure 1.

Figure 1: A completed mathematical task as a semiotic transformation

$$\begin{array}{lccccccccccccccccc}
\text{Level of Signifiers:} & S_1 & \to & S_2 & \to & S_3 & \to & S_4 & & S_5 & \to & S_6 & \to & S_7 & \to & S_8 & \to & S_9 \\
 & | & & | & & | & & | & & | & & | & & | & & | & & | \\
\text{Level of Signifieds.} & I_1 & & I_2 & & I_3 & & I_4 & \Rightarrow & I_5 & \Rightarrow & I_6 & & I_7 & & I_8 & & I_9
\end{array}$$

Figure 1 is a schematic representation of the sign transformations in the sample task discussed below. It shows a linear sequence of signifiers with most derived from their predecessor by a symbolic transformation (denoted '$\to$'), it shows each signifier connected vertically to its

corresponding signified; it shows a linear sequence of signifieds, two of which are derived from predecessors by a transformation of interpretations (denoted '⇒'). It parallels Davis' (1986) 'Visually Moderated Sequence', involving symbols and meanings in a goal directed sequence.

To clarify the role of such analyses in the theory it should be noted that, first, Figure 1 illustrates that transformations take place on one of the two levels, or both together. Second, signifieds vary with interpreter and context, and are far from unique. The level of signifieds is a private 'math-world' constructed individually, although in a degenerate activity it may be minimal, corresponding to Skemp and Mellin-Olsen's notion of 'instrumental understanding'. Third, signifiers are represented publicly, but to signify for the learner (or teacher), s/he must relate to them (they have to be attended to, perceived, and construed as symbols). Fourth, Figure 1 shows only the structure of a successfully completed task, represented linearly as a text. It does not show the complex process of its genesis. Finally, the levels of signifier and signified are relative, they are all the time in mutual interaction, shifting, reconstructing themselves. What constitutes a sign itself varies: any teacher-set task is itself a sign, with the text as signifier, and its goal (and possibly frame) as signified

A Case Study

The theory is used to analyze a routine mathematical task carried out by a 14½ year old female 'Nora'. During the Autumn Term 1992, Nora attended a state high school (although absent on a significant number of days) In mathematics class Nora worked from a set mathematical textbook (Cox and Bell, 1986) on a number of topics including trigonometry (first tangent ratios, later sine and cosine ratios). An analysis of N's exercise book shows notes taken from two sessions of exposition, including 3 worked examples on Sine & Cosine and the 'tools' indicated below. Based on what is recorded in her exercise book, during the month or so in which Nora was studying trigonometry (and other mathematical topics) she carried out at least 62 trigonometric tasks (26 Tan and 36 Sin & Cos). Almost all were routine but of increasing complexity, a few were non-routine problem tasks. She had feedback via teacher marking on 22 Tan and 15 Sin & Cos tasks, and was marked correct on all but one Tan task. Her exercise book reveals 2 locations where conceptual and symbolic tools for trigonometry were recorded/developed, notes of a lesson of 17 November, and 4 pages of undated rough notes at the rear of the book. The tools involved were: definitions of trigonometric ratios, 2 mnemonics to assist memorization; review of Pythagoras' rule; relabelling of triangle sides 'O', 'A', and 'H' according to a newly designated angle, inverse ratios, calculator use, cross-multiplying to solve e g Tan P = O/A for A, and similarity of triangles and ratio.

THE RESEARCH TASK The research interview took place out of school on 17 December, based on a routine tangent task from the school text. Nora had available her pencil case, calculator, text and exercise book. She was asked to work on plain paper, to think aloud, and

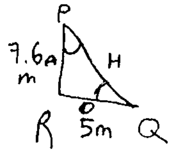
was tape recorded.

THE TASK (*Cox and Bell*, 1986: 58)

4 Sketch each of the following triangles PQR, with $\angle R = 90°$, then calculate both $\angle P$ and $\angle Q$, correct to the nearest 0.1°.

(a) PR = 7.6m; RQ = 5m (b) ...

Figure 2: The first part of Nora's solution to the experimental task (Q only)



$\text{Tan} = \dfrac{7.6}{5} = 1.52$

$56.7° = Q \quad (56.65929265)$

ANALYSIS OF THE WRITTEN SOLUTION

Nora's answer can be analyzed as a sequence of written signs: $S_1 \Rightarrow S_2 \Rightarrow \ldots \Rightarrow S_9$, as follows.

S_1: Figure with labelled vertices

S_2: Figure with lengths of 2 shorter sides marked

S_3: Figure with interior angles P & Q marked

S_4: Figure with sides labelled 'O', 'A' & 'H'

S_5: 'Tan = 7.6/5'

S_6: '= 1.52' added

S_7: [New line] '56.7°'

S_8: '= Q' added

S_9: '(56.65929265)' added

This is a transformational sequence of signs, which can be analyzed as follows.

$S_1 \Rightarrow S_2 \Rightarrow S_3 \Rightarrow S_4$ are transformations of the triangle diagram through additions; stages in the elaboration of the iconic plus symbolic representation. The figure is required by the question (presumably for methodological reasons). It enables Nora to cue and build up a simple math-world of triangles and their properties, and construct a representation of the problem situation within it. (The elaboration of a single drawn figure contrasts with the typical justificatory rhetoric of written school mathematics, where repetition of symbols is often required. The rhetoric of diagrams requires the maximum of 'relevant' information be displayed.)

$S_4 \Rightarrow S_5$ is a shift from figure to written text, indicating the choice of the tangent function to express the required angle in terms of the ratio of known lengths. (Having constructed a task-supporting representation S_4, both as an iconic symbol S_4 and as a mental image I_4, Nora is thus able to retrieve appropriate conceptual tools, and then to represent the linguistic signs that lead via transformations towards the completion of the task.)

$S_5 \Rightarrow S_6$ is the computation 7.6/5 by calculator, at the level of signifiers, with the answer transformed at the level of signifieds (corrected to 1 decimal place; intermediate answer omitted until S_9) and recorded. This is the only dual-level transformation shown in Fig. 1

$S_6 \Rightarrow S_7$ represents the application of the calculator 'inverse tan' function to S_6. (The actual process involved first applying the tan function, and then rejecting it.) The recording of S_7 represents the completion of the main task-goal (the derivation of the answer), but does not yet satisfy the rhetorical requirements of classroom written mathematical language. Thus $S_7 \Rightarrow$

S_8 is the addition of a label ('Q') to the previous answer (labelling answers is a widespread rhetorical demand). Finally, $S_8 \Leftrightarrow S_9$ is the addition of the omitted earlier answer, to show what was actually derived with the calculator, thus completing a perceived gap in the account. (This satisfies Nora's construal of the rhetoric of mathematics as accurately and completely describing the transformational sequence; whereas the out-of-sequence inclusion of S_9 does not correspond to the usual rhetorical demands of school mathematics.) The signs S_5 to S_9 represent the justificatory rhetorical account of the transformation, recorded after the event.

The final written text (Fig. 2) is in abbreviated form. Earlier work of Nora (school and homework recorded in her exercise book) utilizes a rhetorical form as follows, e.g.: Tan Q = O/A, Tan Q = 7.6/5, Tan Q = 1.52, SHIFT TAN = 56.7° = Q, etc. In Figure 2, the initial definition, the argument of 'Tan', the symbolization of calculator use, etc. are omitted.

Figure 1 shows the sign transformations anlyzed into signifiers and signifieds. Most transformations take place on the signifier level of , but in every case these transformations are supported by meaningful interpretations and meaning-relations between them in the math-world.

INTERVIEW PROTOCOL

I. You're going to do number 4a, page 58.

N [timidly] Read the question?

I: It's over to you, I want you to do all of it yourself.

N I can't get any help from you?

I Yes, you can, you can ask me for help, you know, if you need it[1].

N: [Rapidly speaking over the interviewer's last word] I've got to draw it first. PQR, with R angle 90. P, Q, R, [draws triangle] Just doing the triangle...its not to scale. P and Q,..[pause].. oh...alright.[2]

I Alright what?

N: Oh, are you, it just said, I read the first bit and it said sketch each of the following triangles PQR with angle R 90 degrees and calculate P and, um, Q, but I didn't know how long they were, but its (a) and that[3]. P to R, its 7.6 metres and R to Q its 5 metres [writes in side lengths]

I: Speak out loud. what are you thinking about?

N I'm thinking about how to do it, hold on...[marks angles P & Q] I know they add up to 90 degrees together.[4] Do I, do I use sine? ..tan isn't it? No its not, look see, opposite, hang on, where's the angle I want? I shall want this angle here, if I want Q angle, then it's opposite, which is P to R,[5] I've got that one. Is that opposite over adjacent? so it's O A. Tan I need, opposite divided by the adjacent, that'd be 7.6 divided by 5 metres [uses calculator] that is . 1 52 Now I press tan [uses calculater] That's wrong. Maybe it's inverse tan. Tan. [uses calculator] That looks more like it. Is that right?[6] How many significant figures?

I Did the question say?

N: no,...tenth of a degree. That's Q. 56.7. Hang on, I'm going to write out the whole thing. [writes out working][7] Now I must do P, P angle.

INTERPRETATION OF SIGNIFICANT FEATURES OF THE DIALOGUE

Some of the key features of this dialogue from the point of view of the present theory of mathematical activity are as follows (much more could be said):

1. In the preceding three lines Nora is looking for clues to the nature of the roles/positionings for her and the interviewer, and implicitly acknowledging the dominance of latter. (Is it teacher-learner or tester-examinee?) The context is an unusual out-of-school interview with someone who is not the teacher; and thus pertains to some but not all features of the social context of school mathematics, the source of uncertainty. By the end of this mini-exchange, Nora knows she must do as much as she can unaided, before seeking help.

2. Here Nora has internalized the task (and is subserving herself to the textual commands in the task), is beginning to make the initial symbolic representation (sketch). Finally, Nora has cued a frame to carry out the transformation in.

3. When asked to explain "oh.. alright" Nora firsts searches for a way to begin her account, and then constructs a rhetorical sequence to explain to the interviewer what has just taken place in her thought. Normally (in a school presented task) this rhetoric would not be required, unless interrogated by another person - peer or teacher. It ends with the phrase '(a) and that', referring to part (a) following the stem of Q. [Verified afterwards.]

4. At this point Nora has just employed a conceptual tool/item of knowledge, concerning the angle sum of the two smaller angles in a right angled triangle. It is irrelevant here. It suggests that Nora's frame is a math-world based on triangles and their properties, including (but extending beyond) trigonometric properties.

5. At this point Nora has tentatively chosen the 'Tan' function and managed to ignore (or mentally exchange) the labels 'O', 'A' in the figure to construct the correct ratio 7.6/5 for Tan Q.

6. The whole preceding monologue reflects the uncertainties and doubts, the fermes and moves considered and carried out in the math-world, but also involving semiotic representations and tools in the physical world (i.e. keying in the calculations into the calculator). It represents the key thought experiment underpinning the solution and symbolic transformations of the task. Interestingly, it involves self-directed questions, as Nora voices queries and then answers them, regulating her activity meta-cognitively.

7. This represents another shift into rhetorical mode; the representation of the symbolic transformations, after the event, in an acceptable way as required by the teacher in the normal discursive practice of schooling (as construed by Nora) This is followed by a switch of attention to the other part of the question (P) [omitted here]

This analysis reveals some of the multi-levelled complexity involved in a learner carrying out a semi-routine mathematical activity. This included the construction of a math-world, one or

more thought experiments or 'journeys' in it, a monological self-commentary on a 'journey', a rhetorical description of thought processes for the interviewer, andthe construction of a text addressing the rhetorical demands of written mathematics in Nora's social (school) context. Tools developed by researchers in problem-solving and representation theory in mathematics education could take aspects of this analysis further.

Conclusion

This paper suggests that a focus on learner's work on routine (or other) mathematical tasks, bearing in mind both their textual and repetitive nature can provide an illuminating avenue of enquiry for the psychology of mathematics. It sketches aspects of an incomplete social constructivist theory of mathematical activity, which looks at a pupil's learning history in the social context of the mathematics classroom in order to situate their learning activities. It offers a synthesis combining a learner's construction of meaning with their public symbolic activities, situated in the social context of school mathematics. One of the strengths of the approach is that it is able to take account of the demands of the rhetoric of school mathematics, something largely missing in research on learning (although increasingly widespread in the sociology of science and mathematics). Too often the style of mathematics is seen to be logically determined by its conceptual content, when in fact it represents a body of genre conventions, with its own evolutionary history.

Further developments of the social constructivist theory of mathematical activity will also take into account the learner's subjectivity and agency in carrying out tasks, reading texts, working with microcomputers in software including representations of agency (e.g. the cursor and 'Turtle'), a semiotic analysis of mathematical proof, and a social constructivist theory of mathematics in general (Ernest, forthcoming).

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The Change in the Elementary Concepts of Fraction in the Case of the 3rd Grade

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Abstract

The complexity of learning fractions on pupils' side has been underestimated. It causes the traditional approach to fractions to change into rigid and rule-oriented instruction. Then fractions are a troublesome and complex subject.

We pay special attention to the fraction concepts which precede and produce fractions. They are schematized on the base of two aspects: the meanings of multiplication and Vergnaud's relational calculus on the additive structure. We have consequently three kinds of operational schema for the concepts.

One of them is so useful to clarify and analyse research tests and the protocol of 3rd grade pupils as to the change of fraction concepts.

1. Introduction — Two Viewpoints on Fractions —

There are two viewpoints to analyse fractions used in the communication between the teacher and pupils. One is the standpoint based on the fraction concepts which precede fractions and produce them. The other is the standpoint based on the rational number concepts which succeed fractions and make them mathematical.

In this case, the meaning of fraction could be clarified mathematically through the list of rational number concepts made by Kieren, T. E. and Rational-Number Project. Kieren [1975, pp.102-103] lists the following interpretations of rational numbers based on fractions.

1.	Rational numbers are fractions which can be compared, added, subtracted, etc.
2.	Rational numbers are decimal fractions which form a natural extension(via our numeration system) to whole numbers.
3.	Rational numbers are equivalence classes of fractions.
4.	Rational numbers are numbers of the form P/q, where $p.q$ are integers and $q \neq 0$.
5.	Rational numbers are multiplicative operators.
6.	Rational numbers are elements of an infinite ordered quotient field.
7.	Rational numbers are measures or points on a number line.

Kieren's list of rational number concepts

On the other hand Rational-Number Project (Behr, Lesh, Post, and Silver) [1983, pp.99-100] calls fractions "subconstructs" of rational number concepts and proposes the following list about subconstructs.

1. *fractional measure*: a reconceptualization of the part–whole notion of fraction.
2. *ratio*: a relationship between two quantities.
3. *rate*: a new quantity as a relationship between two other quantities.
4. *quotient*: a rational number as an indicated quotient. That is, $^a/_b$ is interpreted as a divided by b.
5. *linear coordinate*: points on a number line.
6. *decimal*: that emphasizes properties associated with the base-ten numeration system.
7. *operator*: that imposes on rational number a function concept; a rational number is a transformation.

RNP's list of rational number concepts

In that case, main issue is the mathematical meaning differentiated neither from human spirit or from problem context. This, however, has not been schematized and analysed fully. Our work is to make it clear schematically and to apply any schematized standpoint to the change of fraction concepts in 3rd grade pupils.

2. Theoretical Framework

2.1 Our Schemata for Fraction Concept

Our target is fractions in the 3rd grade(8 years old) in Japan. These numbers are introduced and used initially for representing some quantity less than a certain unit mass. We call it **quantity fraction** in the mathematics education of Japan. Therefore elementary addition subtraction over fractions can be taught on the base of quantity such as length, area and volume of a particular thing.

What is, then, fraction concept preceding and producing fractions in pupils? If it is not based on quantity but on operation as an activity of dividing object, then addition and subtraction cannot be done over such fractions.

Our aim is to analyse the change of fractional meaning in pupils when we teach them quantity fraction in a class. To fulfill that, it is necessary to make fraction concepts clear as schema in advance. After that we will be able to analyse the protocol of pupils and a teacher in a 3rd grade class of fraction.

Addition and multiplication are taught over natural number before learning fractions. Both of them consequently construct fraction concepts. Multiplication is especially important for them and is taught as repeated addition, times and direct product in this order. These meanings are formalized mathematically as follows:

$$1) \qquad a \times b = c \iff \underbrace{a + a + \cdots + a}_{\# b} = c$$

2) $a \times b = c \iff a \xrightarrow{b} c$

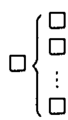
3) $a \times b = c \iff (a, b) \longrightarrow c$

(# means the number of "a")

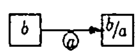
According to the order and meaning itself of the above multiplication, fraction concept first consists of equal division activities which are the inverse operations of repeated addition. Secondly it consists of various images such as shrinking which are opposite to those of enlargement suggested by times. The typical case is the thinking way of "considering a thing as a unit". It thirdly consists of the formal comparison such as ratio between two quantities. This formation is homologous to that of direct product. These three fraction concepts are characterized respectively by concrete operation, inner operation and formal operation.

Each operation makes a cognitive schema possible in mind which corresponds to respective fraction concepts. Vergnaud's relational calculus in additive structure [1982, pp.41–47] suggests that we should schematize fraction concepts. We call them operational schemata in contrast with Vergnaud's schem. .

(1) Schema of equal division



(2) Schema of change



(3) Schema of comparison

☐ refers to quantity of object or object itself
○ means parameter of operation

Figure. 1: Operational schema for fraction concepts

2.2 $\frac{b}{a}$ as Quantity Fraction

Fraction $\frac{b}{a}$ is composed of three kinds of symbol. All possible combination are as follows:

$$\left\{a,\ b,\ -,\ \frac{b}{\ },\ \frac{\ }{a},\ \genfrac{}{}{0pt}{}{b}{a},\ \frac{b}{a}\right\}$$

When we see the teaching context of quantity fraction $\frac{b}{a}$, we know that symbol "b" refers to quantity, symbol "$\bar{a}$" means the operation of dividing "b" and symbol "$\frac{b}{a}$" also refers to quantity. The process of constructing quantity fraction $\frac{b}{a}$ is expressed mathematically by function like this:

$$\bar{a}(b) = \underbrace{\frac{b}{a} + \frac{b}{a} + \cdots + \frac{b}{a}}_{\#a}$$

(# means the number of objects)

It is also schematized by an operational schema like this:

$$\boxed{b}\left\{\begin{array}{l}\boxed{b/a}\\ \boxed{b/a}\\ \ \ \vdots\\ \boxed{b/a}\end{array}\right.$$

The whole of above function and schema is the concept of quantity fraction and a part of them, i.e. b/a and $\boxed{b/a}$, is "quantity fraction". Therefore the concept is the system of producing fractions. In this case b/a is a kind of quantity. This property causes two fractions to be compared and combined.

3. The Content of Research

The purpose of this research is to describe the initial form of fraction concept and the transformation processes from that to the fraction concept based on measures, underlying the interaction and communication between a teacher and the classroom pupils at the 3rd grade. Toward this end the following problems and classroom activities were administrated.

(I) Pilot research
A pilot research was conducted to clarify the initial form of fraction concept on 17 November 1992 with 38 pupils in a class just before teaching quantity fractions. They were asked to describe the length of 4 fish with suitable number or words under the fictitious unit "papua".

(II) Classroom activities and discourse
A teacher's teaching activities and pupils' learning activities were observed and recorded in eight classes which began on 27 November and finished on 7 December 1992. A teacher managed to develop the classroom activities into three phases as follows.

- 1st phase: fraction as division of an object and combination of the parts
- 2nd phase: reconceptualization of above fractions to construct the concept of quantity fraction
- 3rd phase: compare, addition, and subtraction using fraction based on quantity

(III) Postresearch

The postresearch tests were conducted three times at first, midst and last in the 2nd phase of classroom activities. They aim at the analysis of the change of fraction concepts in pupils. So as to do it, following problems were provided to them.

(i) First problem

"*Now is here a piece of ribbon. Take a piece of a quarter meter ribbon* ($^1/_4$ *m*), *bring it to me*," said a teacher. Children are not informed that the length of a original ribbon is 120 *cm* length.

(ii) Midst problem

Compared with the first problem, the midst problem had different numerical values. Pupils were asked to take a one fifth meter ($^1/_5$ *m*)ribbon from a piece of 60 *cm* ribbon.

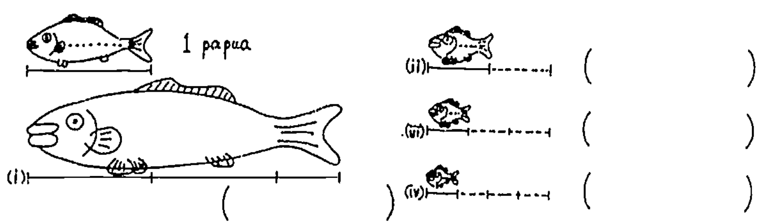
(iii) Last problem

It was the same with the first problem.

4. Results and considerations

4.1 Pilot research — Elementary Concept of Fraction —

A pilot research aims at knowing the way of pupils how to express some length less than a certain unit. Questions and answers are as follows:

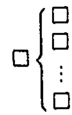


The way of expression \ Que.	Answer (i)	(ii)	(iii)	(iv)	The num. of Ans.
Ordinary language A	2.5	big	middle	small	5
Ordinary language B	2 and half	half	half and half	half and half and half	4
Natural number	3	1	2	3	2
Decimal A	2.5	1.2	1.3	1.4	2
Decimal B	2.5	0.2	0.3	0.4	6
Decimal C	2.5	0.5	0.3	0.2	3
Fraction	2.5	$^1/_2$	$^1/_3$	$^1/_4$	13
Both decimal & fraction	2.5 $2^1/_2$	0.5 $^1/_2$	0.33 $^1/_3$	0.25 $^1/_4$	3

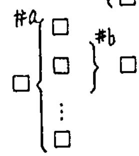
Table 1: Result of pilot research

According to the answers in table 1, 14 pupils (ordinary lang. B, natural num., decimal A and decimal B) seem to have the same operational schema fig. 1- (1) though they showed various expressions.

In this case, □ does not refer to quantity but to object itself. We can divide an object equally but cannot divide quantity concretely because quantity is one of the properties of a thing.

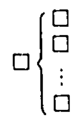


The above operational schema develops into a right schema easily. A fraction b/a produced by this schema refers to the combination of 2 operations. This is not quantity. We call this kind of b/a division fraction in the mathematics education of Japan. Denominator "a" is always larger than numerator "b" in a division fraction.



On the other hand, 3 pupils in the group of both decimal and fraction have another kind of operational schema.

In this case, □ does refer to quantity because 3 pupils use decimal besides fraction. Therefore a fraction b/a produced by this schema refers to quantity. We call this kind of b/a quantity fraction. Denominator "a" is not always larger than numerator "b".



13 pupils in the group of fraction show correct answers. It is difficult to distinguish their answers into a division fraction or a quantity fraction. We had better understand that they form gray zone between two kinds of fraction. This will be shown in the later observation and research of the class on fraction.

4.2 Postresearch — Transformation Process —

The postresearch was conducted three times, separately at first, midst and last, in the 2nd phase of classroom activities. The data can be classified by the length of ribbon and the result is summarized in Table 2.

First		Midst		Last	
*25 *cm*	(2)	12 *cm*	(28)	*25 *cm*	(24)
30 *cm*	(37)	*20 *cm*	(6)	30 *cm*	(7)
Other	(0)	*Other*	(5)	*Other*	(8)

Notes: * indicates correct answer, ( ) is the number of pupils

Table 2: Result of postresearch

In addition, according to the operational schema, above data is classified into two groups as below:

Operation-oriented group: Pupils in this group interpreted fraction $1/a$ as division fraction as we said. All of them in this group folded a piece of ribbon in four or five and cut it off at a crease through a teacher managed to tell them quantity fraction in the 2nd phase. They tor . a piece of ribbon in 30 *cm* length at the first test, 12 *cm* at the second test, and 30 *cm*

at the third test. Then we call this group "operation-oriented group". Generally speaking, division fraction $^b/_a$ consists of two operations, i.e. division and combination.

Quantity-oriented group: Pupils in this group interpreted fraction $^1/_a$ as quantity fraction. Then we call this group "quantity-oriented group". Therefore they cut a ribbon of correct length. They first measured a piece of 1 *m* ribbon from given one and divided it into four or five parts equally. Then the pupils taking a piece of ribbon in 25 *cm* at the first test, 20 *cm* at the second test, and 25 *cm* at the third test are included in this group.

Table 3 indicates the number and the percentage of pupils included each group.

Type of schema	First		Midst		Last	
Quantity-oriented	2	(5%)	6	(15%)	24	(62%)
Operation-oriented	37	(95%)	28	(72%)	7	(18%)
Other	0	(0%)	5	(13%)	8	(21%)

Table 3: Post-test data classified into two types of cognition

The 2nd phase of classroom activities aimed at making pupils to understand "$^1/_a$ *m*" as quantity fraction. As for this attainment, a teacher often asked pupils about the difference between division and quantity fraction regarding to "$^1/_a$ *m*", based on the ribbons cut in different length by pupils. They finally named "the changeable fraction" for a division fraction by themselves in the last class of the 2nd phase. They also named "the unchangeable fraction" for a quantity fraction by themselves. They came to use such words as the technical terms in a classroom discussion as follows. A quarter piece of paper, a half glass of milk and etc. are all the changeable fraction. On the other hand, one-fifth meter, three quarter *dl* and etc. are all the unchageable fraction. There is the case that a quarter glass of milk is more than three quarter glass of milk because these fractions are changeable.

	1st post-test			
Midst post-test	Quan Ori. Gr.	Ope Ori. Gr.	Total	
Quan Ori. Gr.	2	4	6	(15%)
Ope. Ori. Gr	0	28	28	(72%)
Other	0	5	5	(13%)
Total	2	37	39	(100%)
	(5%)	(95%)	(100%)	

	Midst post-test				
Last post-test	Quan Ori Gr.	Ope. Ori Gr.	Other	Total	
Quan Ori Gr.	2	20	2	24	(62%)
Ope Ori Gr.	2	4	1	7	(18%)
Other	2	4	2	8	(21%)
Total	6	28	5	39	(100%)
	(15%)	(72%)	(13%)	(100%)	

Table 4: Cross table of post-test data

Table 4 indicates us that the change was very slowly and complex. First, it is only 4 pupils that moved from operation-oriented group to quantity -oriented group between the 1st test and

midst test. Although 22 pupils have moved to quantity-oriented group at last, the change was very slowly against our expectation. Secondly, although most pupils have moved from operation-oriented group to quantity-oriented group, 2 pupils moved back to operation-oriented group from quantity-oriented group. In fact some pupils never admited the case in which numerator was larger than denominator in the 3rd phase when discussing about $^3/_5 + ^4/_5$. They insisted that $^7/_5$ was quite nonsense because we can divide one thing into 5 pieces but cannot combine 7 pieces from them. Therefore $^3/_5 + ^4/_5$ equals to $^7/_{10}$. They came back to the operation-oriented group in the 3rd phase although they had been in the quantity-oriented group in the 2nd phase. The fraction concepts has not still been comfirmed at the last stage of fractions teaching in the 3rd grade.

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PROBLEM SOLVER AS A REALITY CONSTRUCTOR: AN ETHNOMETHODOLOGICAL ANALYSIS OF MATHEMATICAL SENSE-MAKING ACTIVITY

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ABSTRACT

The purpose of this ethnomethodological study is to construct a description and interpretation of MATHEMATICAL ACTIVITY of a problem solver. By MATHEMATICAL ACTIVITY I understand an act of sense-making, an act of rendering a given situation sensible or rational. A theoretical framework which is grounded on the theoretical underpinnings of ACTIVITY THEORY and ethnomethodology is presented. Transcripts of video-audio records of problem solver and interview data are analyzed in the light of the framework to generate a description of the MATHEMATICAL ACTIVITY. Results from the interpretation of the data reveal that the problem solver implicitly but deliberately construct mathematical reality not by logical and rigorous reasoning but by ones commonsense background knowledge of doing mathematics.

INTRODUCTION

There has been a growing consensus among research community about the necessity for developing theories and methodologies that encompass affective and cognitive aspects of learning mathematics. This article is based on a framework which is grounded on the theoretical underpinnings of ACTIVITY THEORY and ethnomethodology, relatively new to the field of mathematics education, which studies the method people use to create and sustain particular sense of reality.

This article investigates MATHEMATICAL ACTIVITY of two graduate students while engaging collaborative problem solving. The investigation focuses on the ways that problem solver constructs meaningful mathematical reality. The following discussion consists of two parts. First part involves an indication of the theoretical framework and justification for the methodology used. Second part involves description and interpretation of sample data of a case study.

A THEORETICAL FRAMEWORK FOR ANALYZING MATHEMATICAL ACTIVITY

By MATHEMATICAL ACTIVITY I understand an act of sense-making (Schoenfeld,1991;1992), an act of rendering a given situation sensible or rational. If one wishes to render subject's sense-making sensible, one must take into account the following "units" (theoretical constructs) of analysis. First unit is "motive and belief" that are thought to govern mathematics practice. Second unit is "subjective meaning" or "sense" that the learner gives to task and situation. Third unit is "mathematical work" to attain specific goal set under specific conditions. Hierarchical structure of these units is called "a framework for analyzing MATHEMATICAL ACTIVITY"(Fig.1, See Ohtani (1993) for detail).

The first unit involves such currently accepted category of affective domain of mathematics practice (Cobb,1985; McLeod,1992;Nicolls,1989;Schoenfeld,1991,1992) as task-involvement, ego-involvement, belief about mathematics, about self, about mathematics teaching, and about social contexts. Third unit involves knowledge about mathematical facts and procedures that are purely cognitive aspects in mathematics practice. Second unit involves "situation definition" (Wertsch,1985;159): the way in which objects and events in a situation are represented or defined. The unit mediates and bridges first and third units.

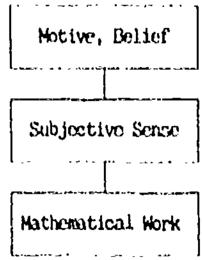

Fig. 1
Framework for Analysing MATHEMATICAL ACTIVITY

In the framework, "commonsense knowledge and reasoning" or "commonsense rationalities" (Garfinkel,1967;262) play a significant role. MATHEMATICAL (sense-making) ACTIVITY renders a given situation sensible by "commonsense rationalities" as opposed to scientific rationalities. Resent research (Lave,1988;Resnick,1988;Schoenfeld,1991) have a common and persuasive vision of mathematics as situated social practice, ill-structured discipline, and sense-making activity. There seems to be considerable agreement regarding the role " commonsense rationalities" play in mathematical practice.

Research traditions implicitly and explicitly use a wide range of theoretical conceptions as wide-reaching scheme of interpretation. They rendered subject's acts sensible or rational by noting the ways in which subject's acts are rule governed and directed toward achieving goals that were specified by the theoretical constructs. Subject's acts are reasonable in the sense that they had no choice but to behave in the fashion that they do. The framework affords the author a device to make sense how subject thinks and acts during problem solving, and why they act as they do. Thus the framework isn't intended to impose from "outside" certain scheme on acts, rather try to understand from "inside".

THEORETICAL UNDERPINNINGS: ACTIVITY THEORY AND ETHNOMETHODOLOGY

HIERARCHICAL STRUCTURE OF ACTIVITY

Concept of MATHEMATICAL ACTIVITY is grounded on the Vygotskian perspective on the relationship between a thought and an utterance (Выготский, 1983) and ACTIVITY THEORY elaborated by Leont'ev (Bartolini Bussi,1991;Christiansen & Walther,1986; Леонтьев,1965,1975). Their basic tenet of research philosophy is that in order to explain the highly complex forms of human consciousness one must seek the origins of conscious activity in the external processes of social life.

Given this perspective, Leont'ev distinguished three interwoven "units" of higher, psychologically mediated aspects of human ACTIVITY; First, "particular activities", using their energizing "motives" as the criterion. Second, "actions", the processes

subordinated to conscious "goals". Finally, the "operation", which depends directly on the "conditions" under which a concrete goal is attained. These units of human activity form its macrostructure of ACTIVITY. Human conscious activity occur with the active interaction between human and external reality. It is precisely these interaction that give rise to human motive. Thus are formed those specifically human activity. This activity can give rise to goals which, in turn, lead to specific acts. These acts are carried out by the appropriate set of operations. These "units" are mutually independent in the sense that same actions can be contained in many kinds of activity: an act can be carried out by many ways of operations relative to the objective conditions of external world. These different levels of "units" allow us to examine a single segment of act from a variety of viewpoints.

SENSE AND MEANING

The distinction between "meaning" and "sense" was introduced in Soviet Psychology by Vygotskii in his classical work "Thought and Language".

> The sense of a word,...,is the aggregate of all the psychological facts emerging in our consciousness because of this word. Therefore, the sense of a word always turns out to be a dynamic, flowing, complex formulation which has several zones of differential stability. Meaning is only one of the zones of the sense that a word acquires in the context of speaking. Furthermore, it is the most stable, unified, and precise zone. As we know, a word readily changes its sense in various contexts. Conversely, its meaning in that fixed, unchanging point which remains stable during all the changes of sense in various contexts (Выготский,1983;346).

Thus, "sense" is the signification of an individual instance of a word, as opposed to the stable, objective system of relations (meaning).

Vygotskii gained insight into the significant role "sense" plays in human communication activity. According to his hypothesis on the relationships between a thought and an utterance, a thought motivated by certain problematic situation or intention passes through several stages in the process of becoming clear and expanded chain of speech. The initial thought is mediated by inner speech which is only comprehensible to the subject. Then it is transformed into a system of meanings formulated in words comprehensible to others. Inner speech retains all of the analytic, planning, and regulative functions. "Sense" dominates "meaning" semantically in it.

Leont'ev thought that the hypothesis is also true for human conscious ACTIVITY. According to Leont'ev, "sense" and "meaning" correspond second and third units of macrostructure of ACTIVITY. Leont'ev puts it:

> We distinguish two aspects of ACTIVITY. First, the processes subordinated to conscious goals which is internally connected with "unit" of consciousness called "personal meaning". Second, we distinguish content of action which depends on the conditions. We call the aspect "operation". It is also connected with specific "unit" of consciousness that is called "meaning" (Леонтьев,1965;479).

Given these theoretical perspectives, the author also named the second unit "subjective

sense". The third unit, the objective properties and possible means of action is, analogous to mathematics, named "mathematical work".

BASIC CONCEPTS OF ETHNOMETHODOLOGY

Ethnomethodology (Cicourel et al.,1974;Garfinkel,1969;Leiter,1980;Wieder,1974) is based on the assumption that reality is implicitly but deliberately constructed by knowing subject. One of the aim of ethnomethodology is to study the processes of sense-making and the practices of commonsense reasoning that members of society use to create and sustain the factual character of the social world. In educational research, Voigt(1989a.1989b) shows ethnomethodological analysis of social interaction in mathematics classrooms. In these researches, Voigt "reconstructed" specific patterns of social interaction and showed how are patterns made meaningful, how are patterns applied to concrete situations, and how do teacher and students use patterns to sustain what is called "participation structure"(Lampert,1988;8).

The facticity of the sense is maintained by interpretive work. However, expressions are vague and equivocal, lending themselves to several meanings. The sense of these expressions cannot be decided unless a context is supplied. This contextual nature of objects and events is called "indexicality". Even though any act and utterance have indexicality, how people make the event meaningful and how they secure its objectivity? Garfinkel argues that through the use of "the documentary method of interpretation" (Garfinkel,1967;78), the facticity of the social world is created and sustained. The use of the method produces a sense of social structure by providing objects and events with consistency, the sense that they are the same over time. An equivocal act then becomes clear in the way that it obtains its sense as typical, repetitive, and more or less uniform, i.e., its sense as an instance of the kind of action which is already familiar.

The situated appearances of objects and events and the transcendent scene are neither independent nor definitive. They mutually elaborate each other. Ethnomethodologists call this character "reflexivity". Reflexivity underscores the idea that the social world is a product of the very way we look at it and talk about it.

A CASE STUDY

PROBLEM USED AND DATA COLLECTION

Purpose of the case study (Ohtani,1991) is to investigate MATHEMATICAL ACTIVITY of problem solver, that is, to investigate the way of rendering a given situation sensible or rational while engaging mathematical problem solving. Subjects are two graduate (master's program) students majoring in algebraic-geometry and number theory. They are asked to work the following problem (Revised version of Hiatt, 1987).

— Problem

Consider any three digits from 0 to 9. With these three numbers, form all possible

two- digit numbers. A digit can be repeated. Find the quotient whose dividend is the sum of all the two-digit numbers and whose divisor is the sum of the three original numbers. Think about natural extension of the result.

Let a, b, and c be the original numbers. Then all possible two-digit combinations of these three numbers are 10a+a, 10b+a, 10c+a; 10a+b, 10b+b, 10c+b; 10a+c, 10b+c, 10c+c. The sum of these combinations is 33(a+b+c), which, when divided by the sum of the three original numbers yields 33. By natural extension I mean to consider different n digits from 0 through 9 and to find the quotient whose dividend is the sum of all r-digit numbers formed from n digits and whose divisor is the sum of the the n digits. The result is $n^{r-1}(10^{r-1}+10^{r-2}+\cdots+10^{0})$. This result is true for any base and holds for $r > n$.

Guided by the experimental design developed by Balacheff(1991), they work the problem with one pen and as many papers as needed. Collaborative problem solving process was audio and videotaped for later analysis. They were asked to work the problem until they had finished it. I was not with them for not collapsing their ACTIVITY (on the problem of collapse of ACTIVITY see Рубинштейн,1940;468-513). The data are transcribed in order them to interpret their experiences, acts, and thoughts while working the problem.

RESULTS: MATHEMATICAL WORK

They worked the problem for 56 minutes. Eventually they reached the correct answer. In the following, description and interpretation of their "mathematical work", third unit of the framework, are presented. For convenience "MO" and "UE" denote the subjects.

At the beginning MO chose 0, 1, and 2 and got 32.[1] Then MO chose 3, 4, and 5 and got 33. In the meantime, MO proposed UE to choose digits that aren't consecutive, taking 3, 5, and 7 to get 33. MO found that she would get 33 for the first case if 00, 01, and 02 are included.[2] MO presented a proof in algebraic form.[3]

After that they proceeded to natural extension. First, MO considered two-digit numbers from four different digits from 0 to 9 and gave an answer 44 without any overt computation. Then UE considered three-digit numbers from three different digits and inferred the result as 333. MO verified the conjecture in algebraic form.[4]

Given these results, MO formulated a problem: "Consider different n digits ($0 \leq n \leq 10$) from 0 through 9. Find the quotient whose dividend is the sum of all m-digit numbers ($1 \leq m \leq 10$) formed from n digits and whose divisor is the sum of the the n digits". Representing 33 and 333 as 3*11 and 3*111 respectively, they derived formula $n(10^{m-1}+10^{m-2}+\cdots+1)$.[5]

Now the episode starts. In this episode, MO found that the formula doesn't hold for m=1 because the answer should be 3 for m=1 and n=3. Then MO introduced additional condition "$m \geq 2$". UE, however, seemed to be uncertain.

MO: If we consider 1-digit numbers,... Then, m must be not less than 1. Is that right? In case m equal to 1..

[1] 0, 1. 2

10, 20, 12. 11, 21. 22 = 96

96/3 = 32

UE: In case m equal to 1, then it's just n.
MO: For 1-digit,...The sum is x+y+z. So dividing it by the sum of the original digits, we get 1.
UE: I got you mean. ...Hum...wrong?
MO: What. 1 and 11 then 111, and if chose 3-digit...,then 33 and 333 and...[6]
MO: How can I represent? "m" is not less than 2.[7]
UE: I've not come to the point....In case m equal to 1, how, ..,In case m equal to 1,...
MO: Then,...
UE: Hum, it's the same.
MO: Yah. the result should be 1 .
UE: Here may be some mistake.
MO: Anyway, it should be 1. It's true even for four-digit numbers.
MO: Therefore m-digits number must be, here must be 2. There had been no question. Don't you think so?
UE: Well,... However, isn't it strange?
MO: Why?
UE: Is it really true?
MO: Why don't you think so? 'Cause,... in case we choose two-digit,...
UE: For three,...For three-digit number, are you sure, for three-digit?
MO: What's wrong?
UE: Is it really OK?
MO: Yah,...'cause, in the case,...here are "x+y+z" for hundreds, and "x+y+z" for tens, isn't it.
UE: That's right.
MO: The point is why the formula doesn't hold for one-digit numbers.
UE: I can't realize what's the reason.
MO: Why this is not equal to 3.[8] Reasonably,it is equal to 1.MO: I don't know the reason.
UE: Me too. (Smiling.)
UE: One moment,...Let's check for the case of three-digit, consider in concrete case.

[2] 1. 2 00. 01. 02
3 01

[3] [illegible]

[4] $100(x+y+z)+10(x+y+z)+3x \ldots = 333(x+y+z)$

[5] $n(10^{m-1}+10^{m-2}+\cdots+1)$

[6] 1, 11. 111. 1111
33 333 3333

[7] $n(10^{m-1}+10^{m-2}+\cdots+1)$ $m \geq 2$

[8] 1, 11. 111. 1111
3 33 333 3333
1 22 222

ANALYSIS OF MATHEMATICAL ACTIVITY: MOTIVE, BELIEF, AND SUBJECTIVE SENSE

Generally, it is difficult to make sense of problem-solver's motive, belief, and subjective sense. Having these constraints in mind, the author interviewed with them and asked them to give an interpretation of their acts and utterances. Interview shows that their activities seem to be guided completely different anticipations. Referring to the transcripts they interpreted their situation definitions. In the following, "IN" denotes interviewer.

MO: When I consider the ranges of m and n, I noticed that m must be not less than 1 writing the formula and seeing its item 10^{m-1}. I'm accustomed to do so. Whenever we make formulae and raise the powers, it's necessary to consider some cases. Because we are always thought to do so. This is the reason why I was at a loss what to do.
IN: What were you thinking about when you introduced the condition?
MO: Hum,..., sometimes, it happens that for m=1 a formula doesn't hold but for m≥2 it does. ... For m=1 we define one answer and for m ≥2 we define the another..., it's often the case for recurrent sequences. So I understand that it's possible. ... I was confident that it was true for not less than 2. However, I didn't realize why it didn't the case for m=1.

IN: How about you? (Asked to UE.)
UE: At the beginning, I didn't notice at all. ... I thought it strange when MO said "the answer is 1 for one-digit number". It really strange that m=1 doesn't hold the formula. I think it's wrong.

As envisioned, they defined the situation quite differently. MO anticipated that the result was absolutely true. UE anticipated that the result might be wrong, because answer should hold without exception. According to her utterance (underlined above) and her own interpretation, MO seems to have rendered wrong result rational. The methods MO used were founded neither by logical reasoning nor by mathematical facts and procedures. Rather, MO used her everyday commonsense background knowledge and belief about doing mathematics. An equivocal event becomes clear in the way that it obtains its sense as typical, repetitive, and more or less uniform, i.e., its sense as an instance of the kind of action which MO was already familiar. MO portrayed an event as instances of a familiar pattern of behavior, she made them parts of an already known pattern. Thus MO normalize the uneven results to sustain faciticity. On the other hand, UE's account, "It really strange that m=1 doesn't hold the formula. I think it's wrong", shows that he also uses his commonsense knowledge on mathematical experience.

The case study indicates that problem solver constructs reality not on the strict rational and logical bases but rather on the commonsense knowledge for conducting their everyday affairs. MO's commonsense knowledge also structured her environment by connecting a given act to its possible goal or to some specific consequence of the act among its many consequences. These data show that commonsense knowledge normalize uneven situation by giving sometime positive and sometime negative meaning to a given result.

CONCLUDING REMARKS

Analysis of MATHEMATICAL ACTIVITY shows that subject's presupposition is the bedrock of his/her practical activity and that commonsense knowledge is not just an inferior version of scientific knowledge. It is an important part of mathematical activity. It does not hinder mathematical activity rather it is produced through the use of commonsense rationalities. Once doubts do arise, they are resolved in such a way as to sustain the factual properties of the world. The use of background knowledge, then, not only provides results with their specific sense, it also sustains the subject's sense of mathematical structure through the normalization of discrepancies which would destroy the congruence of the answer as one particular kind of mathematical object. While some researchers and educators imply that the use of background commonsense knowledge undermines the mathematical development of plain folks, we have shown that it is through the use of background commonsense knowledge that the objectivity of the result is secured.

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THE CONSTRUCTING MEANINGS BY SOCIAL INTERACTION IN MATHEMATICAL TEACHING

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ABSTRACT

The routine is comparatively clear. But the meaning is not necessarily clear. It varies according to contexts and perspectives. First of all pupil's worlds must be clear. I recognise four worlds such that ① Living world, ② Representative world, ③ Computational world, ④ Mathematical world. Thus meaning in mathematical teaching is defined as correspondence among the pupil's worlds.

Here I show S. Miura's lessons whose content was "positive and negative number" at first year of junior high school in Japan. It consisted of three topics, positive and negative thinking in every day life, cards game, computation. It is a example of constructing meaning by social interaction in which one of factor is that teacher and pupils elaborate to approach the question which some pupils have, collaborating.

1. Introduction

Most mathematics teachers have a trouble that although pupils can compute or apply any formula, they can't use it to solve problems. This tendency is remarkable in Japanese pupils. They can get high points to computing, but not to solving word problems or applied problems. This is caused by the fact that they seldom understand meaning of fundamental computation.

Most textbooks are constructed according to Hankel's (1839-1873) 'principle of permanent formula', to show meaning after routine procedure. Consequently pupils are used to learn procedure without understanding the meaning, which is apart from routine procedure after all. Of course they have some attempts to teach meaning but don't succeed sufficiently.

Here I shall consider what is the meaning in mathematical teaching, and on this point of view, how to make pupils understand it, showing the example of lessons.

2. Routine and meaning in mathematical teaching

Philosopher of language, Gilbert Ryle [1949] differentiate 'knowing how' from 'knowing that' as a criticism to intellectualism. In mathematics education, relational and instrumental understanding which R.R. Skemp [1976] described are very famous. These discussions are concerning to difference of routine and meaning in knowledge.

But those relation is not necessarily clear. J.S. Brown, & R.B. Burton [1978] make the computer system "BUGGY" to diagnose systematic student errors and analyse cognitive process of "bugs" for place-value subtraction. And Brown, & K. VanLehn [1982] suggest a generative theory of bugs. These researches are based on the view that

arithmetic bugs are caused by procedural ones.

And yet L.B. Resnick [1982] points out that arithmetic errors are due to not only procedural bugs but underlying semantic errors, that is insufficient or incorrect understanding of meaning. Thus she closely observed 4 children (three second-graders and a third-grader) for 7 months to see whether it is useful in understanding routine arithmetic procedures to understand semantics using learning representations, for example Dienes blocks, color-coded chips, bundles of sticks, or pennies and dimes.

Consequently the data makes it clear that the familiarity only with the semantics using learning representations need not lead the acquiring procedural routine, i.e. understanding syntactic knowledge of the base-ten system. Thus she taught three children linking syntax and semantics with correspondences, "mapping" which consists of code mapping, result mapping, and operations mapping between the written algorithms and the concrete materials, so that all could acquire correct skills. This means that syntax and semantics i.e. routine and meaning should be bound very closely and are not separable in mathematical teaching.

3. What is meaning in mathematical teaching

The routine is comparatively clear, it is "knowing how", syntax, or procedure, algorithm, rules, which we can find anywhere in mathematical teaching. But the meaning is not necessarily clear. If it's "knowing that", it's definition in mathematics. Otherwise, it's knowledge to treat concrete materials. However, when pupils sometimes says "I can't understand meaning", this meaning doesn't only restrict the definition, but contains a property or even a way to use formula.

Thus meaning varies according to contexts and perspectives. Nik Azis Nik Pa [1986] discusses meaning in arithmetic from four different perspectives. He argues that "the constructivist perspective is a potentially fruitful framework within which to recast the issues involved in the analysis of meaning in arithmetic" [p.11]. Thus, he describes that:

> From the constructivist perspective, the meaning of a symbol like " $\frac{1}{4}$ " consists of the child's interpretation of the symbol based on the schemes that are available to the child. The primary task of the development of meaning in arithmetic is viewed as the construction of such schemes (or mental activity). [p.13]
>
> The constructivist characterizes meaning as a signifier-signified relationship [de Saussure, 1959] established by the child where both the signifier and the signified are themselves constructed by the child. Thus the child creates meaning by establishing a relationship between items isolated within the stream of experience. [p.14]

This perspective explains that it is difficult to teach meaning to pupils directly, for they can't construct the signifier or the signified which are isolated within the

stream of experience, or can't relate them. But their items are to be systematized as each knowledge, domain, or world which is not independent for individuals. As they communicate each other, they have common intersection which constructivist calls "consensual domain". [von Glasersfeld, INTRODUCTION, xvi]

Generally speaking, the pupil's world can be classified into some micro worlds. Magdalene Lampert [1986] divide ways to know multidigit multiplication for fourth-grade learners into four categories as follows:

> Intuitive knowledge is thought to be derived from and bound to the context in which the knower is confronted with personally relevant problems to solve. …
> Another kind of mathematical knowledge about multiplication that is more familiar in school settings is computational. Using computational knowledge entails doing things with numerical symbols according to a set of procedural rules. [p.308]
>
> A third category used to describe a type of mathematical knowledge that can be applied to multiplication is concrete. This involves knowing how to manipulate objects to find an answer. [p.309]
>
> The possession of principled understanding is thought to enable the knower to invent procedures that are mathematically appropriate and to recognize that what he or she knows can be applied in a variety of different contexts. …
> We might call the laws that apply to multidigit multiplication the principles of multiplication and thus call the ability to invent procedures that conform to these laws principled knowledge. [p.310]

I calls these pieces of knowledge as worlds to apply general mathematical teaching, such as

① Living world - intuitive knowledge.
② Representative world - concrete knowledge.
③ Computational world - computational knowledge.
④ Mathematical world - principled knowledge.

In traditional mathematical teaching, these worlds are emphasized separately and treated independently. But meaningful mathematical teaching must connect the worlds, as Lampert asserts that:

> …, "understanding" is thought to be increased by increasing competence in any one of the four areas outlined earlier. …
>
> …, doing mathematics involves making connections among activities in all three domains [Noddings' three categories], …, Davis (1984) also attributes the acquisition of "meaningful" mathematical knowledge to making explicit connections among different ways of knowing. …, [p.313]

Therefore, meaning in mathematical teaching is defined as correspondence among the pupil's worlds which they must select appropriately. This definition is extension of the Resnick's mapping and consistent with Lampert's and Davis's assertion.

4. Case study of constructing meaning by social interaction

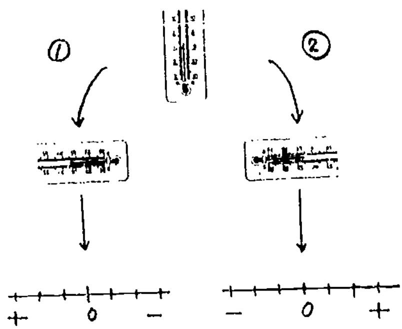
The model of constructing meaning by social interaction is Lampert's experimental lessons [1986, 1990]. Here I show another example. This is usual lessons which Shouji Miura gave to general pupils at Chiryu municipal junior high school in Aichi prefecture Japan, independently of Lampert's, and whose content was "positive and negative number" to be taught at first year of junior high school in "Course of Study". It consisted of three topics, positive and negative thinking in every day life, cards game, computation.

(1) Positive and negative thinking in every day life

The teacher took up a centigrade thermometer in classroom, and asked pupils what they knew about minus sign and number,as the degree on it. They presented examples.

- minus point for error in quiz
- minus volume on television
- +,- on an electric battery
- minus (under) and plus (over) in golf
- - sign on calculator
- +,- in an electric current
- water level of a dam

And to introduce number line containing negative number, he placed thermometer horizontal, narrow the width to be line on a blackboard. But a pupil suggested that:



C 1. Teacher, may I turn it inverse?

T. Oh, good.

C 2. I learned it in elementary school. As moving to the right from 0, number is larger and larger.

T. So, is the right plus or minus?

C 2. It is plus.

T. In the interest of C 1's opinion, please consider the reason why ① or ② is better.

C 3. I am used to write toward the right since elementary school.

C 1. In stead of my opinion, I am also more used that the right is plus.

T. Do you have anything whose right is plus and left is minus?

C 4. By increasing image, the left from a point is minus, the right is plus.

T. What is a point?

C 4. Of course, it's central point, original point.

T. Where in this line?

C 4. It's 0.

T. Ok, the number means the position from 0.

C 5. Teacher, I think about the quantity increasing more and more. But to do so, it is easy to place the line vertical. The high low image in such a line brings water level.

C 6. On quantity, is the heavy·light ok?
C 7. Oh! they seem opposite words.
After this, they presents opposite words as follows:

+	-
small	big
decreasing	increasing
light, low	heavy, high
loss	gain
debt	saving

(2) Cards game
In opposite words they took up the loss (debt) and gain (saving), and played the cards game similar with "Old Maid". Red cards are loss (-), black cards are gain (+). Using cards are 20 cards from 1 to 5 and a Joker as zero. Player takes a card from neighbor. When he judges that his points are highest, he can stop playing a round. A game consists of three rounds. The ranking is decided with the total.

They made strategies to compute by themselves, for example "gathering same color cards, finally subtracting.", "It is interesting that the sum of same number and different color cards is zero. On number line they have same length from 0."

(3) Computation
Replacing cards to expression, they considered meaning and routine of computation. Teacher asked pupils to make all patterns of additive expressions with number 3 and 5. They made that:

① 3 + 5 ② 3 + (-5) ③ -3 + 5 ④ -3 + (-5)
⑤ 5 + 3 ⑥ 5 + (-3) ⑦ -5 + 3 ⑧ -5 + (-3)

These are put in 4 patterns, positive + positive for ① and ⑤, positive + negative for ② and ⑥, negative + positive for ③ and ⑦, negative + negative for ④ and ⑧. First they discussed <positive + positive>, second <positive + negative>.

About ② 3 + (-5)

· Thinking with the cards game · · · Savings are 3 and debts are 5, so -2.
· Thinking with the vertical number line · · The water in dam is saved to line 3, and lost 5 lines.
· Thinking with number line · · ·

C 1. In the same way as positive + positive, 3 + (-5) means to go right from 3, left 5 ···.
C 2. As computing parenthesis first, it means to go left 5 right from 3. The meaning isn't clear like C 1.
T. So it is. But how about the answer?
C. -2.
C 3. It's enough only to go left 5, for as the answer is -2, the way to move from 3 and get to -2 is to go left 5.
C 4. But, it seems distorted.
T. Well, how do you make sure that the expectation of C 3 is right? C. · · ·
C 5. Try with other examples.
T. Very good. Please do it with other expressions.
– They checked 4 + (-9), 7 + (-3), etc., and 0 + (-6) which teacher gave for C 3's way. –
T. Like this it's very important to check many examples so as to certify that your

expectation is right.

T. Before, C 4 said "distorted". Is it clear?

C 6. What I thought checking examples is that to go left we can treat only minus from beginning.

C 7. I agree. I think so.

(Teacher wrote on the blackboard: $3 + (-5) = 3 - 5$.)

C 8. If so, the + has no meaning.

T. How about, C 7.

C 7. As + is to increase, it may be omitted.

C 8. But in case of positive + positive, the + meant right hand in number line.

C 7. Uh-uh. · In 'Juku' [private class to help pupils get ahead] I learned + (-) = -, also in textbook.

C 8. Uh-hun. Though C 7's routine or method may be right, I can't understand the meaning of + as right hand goes away. Well even if you say it is a formula, it is so, After all, I want to understand the meaning.

T. Good talking. (And he put the point at issue in order.) Now how do we think?

T. If we add positive plus sign + to 3 + 5, positive + positive form,

+ 3 + (+ 5).

C. How strange!

C10. I think the + (plus) in + 3 and + 5 and addition + of two numbers have different meanings.

C11. The additional + is usual one, positive and negative number are opposite words.

C12. The numbers have meaning such as saving debt.

C13. Really going left is regarded as the reverse of the right.

C14. If so, - is the reverse and what is +?

C13. It may be unchanging. These satisfies meaning of opposite words.

T. (He wrote in blackboard as the right.)

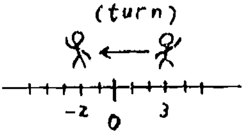
Ok. Do you agree to this?

3 + (-5)
5 to the reverse of the right from 3
5 left from 3

C. (Almost all raised the hands)

C 7. I can understand, but the meaning of only number, more · · .

T. C 7 don't understand clearly. Others, how about it? So I use model. Here is runner. Let's explain using him. Please, C 7.



C 7. He is at 3, toward right, so turns right, and to the reverse, so he turns left, goes by 5.

C. It's easy to understand.

T. How about you, C 7?

C 7. It is useful to see a person moving. Nevertheless I feel it's uneasy to turn the person.

T. "Feel uneasy" is good expression. As C12 said, if a number itself has meaning you

want to move without turning. When not turning at 3 and moving to -2, how do you see [moving a person]?

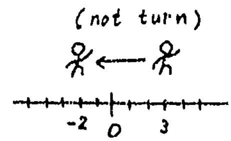
C. Moving left.

T. How this person?

C 7. Moving back.

T. In the case of 3 + 5?

(He moved it in blackboard.)



C. Moving ahead.

T. Yes, + is moving ahead in other wards progress. How about - ?

C. Regress.

T. C 7, how do you feel?

C 7. As progress and regress are very opposite words, I understand clearly.

After this they progressed to meaning and routine of multiplication and division. In the lesson some pupils noticed the commutative law of multiplication and used it to compute <positive × negative> regarding it equal to <negative × positive>.

5. Discussion and Conclusion

In first topic, pupils were introduced from living world which consists of knowledge in every day life and in other subjects to mathematical world in which one of the object was number line in this lesson that was used as important means to think in other topics. And they made correspondence between positive and negative number in mathematical world and opposite words in living world.

In second topic, they were introduced to representative world whose features are concrete and fictional. In Resnik's experiment the objects are Dienes blocks. In Lampert's lessons pupil's posing of word problems, in this lesson the cards game which corresponds to computational world.

In third topic, some pupils were satisfied with knowing the routine, others inquired into the meaning. Miura lead them to discuss their question. They tried to solve the question using the means which they got in their worlds. He did not answer, but help pupils to solve it. He never teach meaning directly, but teach what is to understand meaning and how to do it.

As Lampert insists, pupil's four worlds are all important. And meaning is correspondence among these worlds, correctly speaking something that each pupil constructs by it. Classroom is a society which consists of pupils and teacher, in which many types of social interaction could occur according to teacher's ability to educate. Therefore the social interaction must be so plentiful that each pupil can construct meaning.

The common sense spreads widely that teacher is to pose problem, pupils are to answer. It is not necessarily right, because such interaction is one-sided. In mathematics learning questioning is more important than answering. That teacher and

pupils elaborate to approach the question which some pupils have, collaborating, is one of major factors in social interaction.

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THE COMPLEXITIES OF REFUTATION IN THE SOCIAL INTERACTION: AN ETHNOGRAPHIC INQUIRY IN A MATHEMATICS CLASSROOM

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The present study investigated the social aspects of refutation in the mathematics classroom by an ethnographic method. One high school geometry class in the United States was observed and videotaped half a year, the teacher and students interviewed. From analysis of the data, seven methods of refutation were found, appealing to either an authority, a condition, an experiment, a counterexample, a contradiction, an alternative framework, or a rule for working. Its various properties were identified: problematic interpretations among the participants, the degree of explicitness, interactive construction with other participants, retrospective and evaluative meanings, hierarchical structure, focus, and context dependency. Refutation constituted part of the processes of instruction, sense-making, communication, and negotiation in the classroom. It is argued that research on refutation should be reconceptualizd to reflect the face-to-face social interaction in the classroom.

This paper is an attempt to explore the nature and roles of refutation appeared in the mathematics classroom. The importance of refutation in mathematics was demonstrated in Lakatos's historical case studies (1976). He showed that mathematical knowledge is social construction, that it is always refutable, and that refutation provides an important opportunity to improve mathematical knowledge "Informal, quasi-empirical, mathematics does not grow through a monotonous increase of the number of indubitably established theorems but through the incessant improvement of guess by speculation and criticism, by the logic of proofs and refutations" (1976, p. 5). This Lakatos's view of mathematics seems to encourage mathematics educators to incorporate critical discussion into classroom processes.

Research on refutation is closely related to that of counterexample; indeed, presenting counterexamples is a popular type of refutation in mathematics. Several research studies report that children, including some secondary students, do not always eliminate conjectures when counterexamples are provided and do not always consider that one counterexample is enough to reject a conjecture (Balacheff, 1991; Galbraith, 1981; Williams, 1979). Balacheff (1991) then identified six types in students' treatment of counterexamples: rejecting the conjecture, modifying the conjecture, considering the counterexample as an exception, attaching a condition to the conjecture, modifying the definition, rejecting the counterexample.

Unlike most of those previous studies, which used questionnaires or laboratory settings, the present study took place in the ordinary classroom setting. As Lampert's classroom study (1990) and recent research on situated cognition (e.g., Lave, 1988) inform, the setting where a refutation is situated seemed to have significant influence in shaping the refutation and its treatment. This paper is based on my dissertation study (Sekiguchi, 1991), which explored the social nature of proofs and refutations in the mathematics classroom. The present paper will discuss some of the results concerning refutation only, focusing on types, properties, and roles of refutation in the classroom. (Results on proof were reported in Sekiguchi, 1992.)

Methodology

The study used an ethnographic approach. It enabled the investigator to obtain data on the actual practice of refutation in the classroom, the context where the refutation was situated, and the participants' perspectives One Advanced Geometry class in a high school in the United States was chosen for the study The participants were the teacher, the student teacher, and the students in the class The class was observed every day throughout the second half of the school year 1989-1990. Interviews of the participants and collection of their writings were often conducted to supplement the observational data. The lessons were audio- and video-recorded every day. As a main strategy of data collection and analysis, the constant

comparative method (Glaser & Strauss, 1967) was used. Categories and hypotheses relevant to the research questions were developed by applying this method during and after the data collection.

For Lakatos (1976), refutation means a critical--often very harsh--argument among mathematicians against a conjecture, proof, or theory. When that idea is put into the context of classroom instruction, however, it seems a little unrealistic to expect that a teacher and students would engage in a critical argument like professional mathematicians Classroom and mathematics communities have rather different goals, interests, organizational structures, sizes, interaction patterns, and so on. Because this study was exploratory and aimed at contributing to the improvement of classroom instruction, I define *refutation* rather loosely so that it not only encompasses "criticism" in the sense of Lakatos but also is feasible in the classroom.

In common usage, the verb *refute* is used to refer to the act of arguing that an argument uttered, written, or suggested (by another person or oneself) is false or wrong. In this meaning the act of refuting seems to consist of (1) a person who initiates the act, refuter, (2) an argument that the refuter is aiming at, (3) the refuter's act of claiming that the argument is not valid, and (4) the refuter's argument to support that claim. Though the specific term *refute* itself was not in the vocabulary of the classroom participants, the refuting act was commonly found in the classroom. *Refuting* in this study was a construct I abstracted to capture a broad range of activity in the argument among people when they are expressing disagreement, objection, opposition, denial, rejection, and so on. The noun *refutation* may either refer to the act of refuting or the product of that act. I use the term *refutation* mainly in the former sense. In speech, refutation can be expressed in various linguistic forms. Most explicit forms of refutation begin with phrases like "I don't think so," "I disagree with you," "You're wrong," "It doesn't work," "But . . ." In a less explicit form of refutation one simply says something contradictory to a statement. Refutation can also be expressed in the imperative like "Never do that!" or the interrogative like "How can it be true?"

Results

Types of Refutation

There is no unique way of classifying refutations. Lakatos (1976), for example, studied refutations by counterexamples. He classified counterexamples based on their targets. He proposed two dimensions for classification, global and local. A counterexample is global if it contradicts a main conjecture. A counterexample is local if it contradicts a subconjecture of a proof of the main conjecture (pp. 10-11). Then he found three kinds of counterexamples. It is possible to generalize Lakatos's classification of counterexamples into that of refutations by simply replacing the term *counterexample* by *refutation*. That is, a refutation is global if it contradicts a main conjecture A refutation is local if it contradicts a subconjecture of a proof of the main conjecture.

The present study, on the other hand, focused on the ways of refuting rather than the targets of refuting. According to my conceptualization, refutation accompanies an argument to support the refuter's claim (though the argument may not be always explicit). When a refuter wants to convince other people of his or her refutation, he or she has to do more than just make a claim. For example, in Lakatos's (1976) book, refutations were made mostly by presenting counterexamples. Refutation needs a strategy for convincing other people of its validity I call the strategy the method of refutation The method of refutation was not limited to presenting a counterexample. Various methods of refutation were observed in the classroom. The classification of those methods presented in the following was developed through the analysis of many instances of refutations in diverse contexts and situations. There are problematic aspects in identifying and classifying the methods of refutations. These aspects will be discussed in the next section.

Authority Method: "That's not correct." The Authority Method is a refutation which includes the least elaboration as to what is wrong with the statement to be refuted. A refuter denies or rejects a statement and

simply gives an alternative statement that is considered to be authorized on a certain basis, for example the teacher, the textbook authors, able students, or long-established facts. For example, careless mistakes in calculation were often observed in the classroom. They were usually just replaced by correct calculations. The basis of the correctness was simple arithmetic facts (long-established facts).

When the class was checking the answer to a homework problem, the teacher often said to a student, "That's not correct." The teacher was ordinarily expected to know the correct answer, an authorized answer. When a student's answer was different from it, the teacher sometimes just said, "That's not correct. The correct answer is . . . ." The basis of the correctness was the teacher's experience or the textbook's authority. This was a common method of refutation in the classroom.

Condition Method: "You can't apply that." The students had to learn many definitions, theorems, and formulas and to use them in solving problems. When solving problems, they often used definitions, theorems, or formulas in inappropriate ways. For example, one student used a tangent ratio for a triangle that was not a right triangle. The teacher looked at it and said, "You can't use this. This is not a right triangle." Another student applied the area formula for regular polygons to calculate the area of a nonregular polygon. The teacher said, "You can't do that. One half times apothem times perimeter is the area formula for a regular polygon." We assume that for each definition, theorem, or formula, except a tautology, there are prescribed conditions under which we can use it. When we use any of them outside that condition, one can make a refutation by pointing out that the prescribed condition for its use is not met.

Experimenting Method: "Check it out." Though not frequent, there were several occasions in which students engaged in exploring empirically, that is, drawing a conclusion from data obtained by actual work. For example, one day, the class was doing a small group activity on constructions. Some students eventually concluded that a construction procedure did not produce a desired construction from the results of constructions they actually performed, though their constructions were not very accurate.

Counterexample Method: "Look at this." Showing a counterexample is a common method of refutation in mathematics. The students in the geometry course studied the concept of counterexample in the section on conditional statements. Nevertheless, the use of counterexample seemed to be very limited in this classroom. In the episode below the teacher falsified students' statements by using a counterexample.

[Episode 1:] One day, the class was going over a homework problem in coordinate geometry: "The point (*a*, *b*) is equidistant from (-2, 5), (8, 5), and (6, 7). Find the values of *a* and *b*." When solving it, the class got an equation "$d = \sqrt{9 + (b - 7)^2}$." Pete suggested moving the 9 under the square root sign to the left side.

Pete: "You could do that. [Carl: "I don't understand."] You could do that. Move the 9 over there with *d*."

T (Teacher): "Move the 9 over here [moving her finger from '9' of the right side to 'd' of the left side]?" (The movement of T's finger is indicated by an arrow in Figure 1.)

$$d = \sqrt{9 + (b - 7)^2}$$

Figure 1. The movement of the teacher's finger.

Pete: "Yeah. Couldn't you do that [Luis: "It's under the square root sign."]?"
T: "No. You can't do that."
Pete: "Why not?"
T: "Well, first of all, it's under the square root, and it's been added to something."
Pete: "So?"

T: "Well, look at this. [T writes $\sqrt{9 + 16}$ at the right margin of the OHP screen] Pete, what is 9 plus 16? What is 9 plus 16? [Alan: "Square root of 25."] Not a hard question."
Pete: "25."
T: "What's the square root of 25?"
Pete and Alan: "5."
T: "Well, you wanna take the square root of 9 over here and pull it to the other side [pointing at the equation '$d = \sqrt{9 + (b - 7)^2}$']. If I take the square root of 9, what do I get?"
Pete: "3."
T: "What's the square root of fou-, 16?"
Pete: "2, [inaudible]."
T: "What's the square root of 16?"
Pete: "4."
T: "Is 3 + 4 equal to 5?"
(T's writing on the OHP is shown in Figure 2.)
Pete: "Yeah."
Bill: "No."
Pete: "No?"
T: "[Slashing out the writing shown in Figure 2] So, you can't pull this out [pointing at '9']."

$3 + 4$
$\sqrt{9 + 16}$
$\sqrt{25}$
5

Figure 2. The teacher's writing on the OHP.

A counterexample appeared to reject a statement in a clear and definite way in the classroom. I did not observed any effort to further explain why the statement was wrong at a conceptual level. For instance, though a counterexample showed that $\tan A + \tan B$ is not generally equal to $\tan (A + B)$, no discussion followed as to why it is not true by returning to the definition or meaning of tangent.

Contradiction Method: "That's not possible." Pointing out an inconsistency or contradiction in an argument is a popular form of refutation in mathematics. An argument may be refuted by either internal or external inconsistency. The first method points out an inconsistency between assumptions used in the argument. The second method points out an inconsistency of the argument with statements which are considered to be true but not necessarily used in the argument. The students actually studied this method of refutation when they were learning indirect proof. In the following episode a student Jack used this method to argue against the teacher's explanation:

[Episode 2:] The class studied the trigonometric ratios of the right triangle. Those ratios were defined for acute angles only. Mary found the values of the trigonometric ratios of angles 0 and 90 degrees by calculator and asked the teacher to explain why the calculator gave those values. The teacher tried to explain it in class by using infinitely thin right triangles, which had an interior angle of 0 degrees.

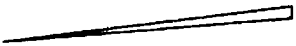

Figure 3. A right triangle used to explain cosine of zero.

T: "All right. If this angle, right in here [pointing at an angle of the left end of the triangle], were to measure zero degrees, wouldn't the hypotenuse of this little, itty-bitty right triangle fall on top of the adjacent side?"
Ss (Students): "Yes."
T: "Okay. So, the adjacent side and the hypotenuse would have exactly the same length. Do you understand that? This was zero. Wouldn't this fall directly on top of the adjacent side? Okay? Do you understand that?"
Ss: "Yeah."
T: "All right. What do you suppose the sine--. So the cosine of zero degrees is one because we've just discussed that the hypotenuse and the adjacent side, would have the same length."
The teacher continued similiar explanations on other trigonometric ratios. After a while, Jack argued against her explanation.
Jack: "If we have an angle of zero, wouldn't the two lines be in the same space, and you can't have that."

[Jack's main point was this: If an interior angle had 0 degrees, then two sides of the triangle would share the same space. But this contradicts a postulate of the Euclidean geometry that through any two points there is exactly one line.]
To counter Jack's argument, the teacher tried to show "cos 0 = 1" by introducing a definition of trigonometric functions, which had not been studied in class, and then appealing to empirical justification by calculator. But Jack's argument was persistent and the teacher failed to convince him.

Alt-frame Method: "You're blind." A truth in one framework may not be a truth in another framework of knowledge. For example, the Euclidean postulate "Through a point outside a line, there is exactly one line parallel to the given line" is not generally true in non-Euclidean geometry. Thus, we can refute an argument by using an alternative framework and contradicting an assumption of the other framework. In the classroom, for example, when the students had a different understanding of a concept or problem from the teacher or an unexpected solution of a problem so that they got different results, this type of refutation happened.

[Episode 3:] In a test on coordinate geometry, Pete wrote a proof, but the teacher refuted all the steps of his proof and graded it as wrong. Pete persistently claimed that his proof was correct except for a wrong choice of unit at the beginning: "The proof is right, except one thing." After a period of argument, the teacher looked closely at his proof. She found that though Pete had made a wrong assumption in the initial step, his proof used the same idea that she had. She gave additional credit to his proof.

Rule Method: "You're not allowed to do that." There were rules in the class on how to work on a problem. When someone violated some of those rules in solving a problem, one could refute the solution by citing the violation. Because those rules were applicable to the limited context, so was this type of refutation. Some of the rules observed in the classroom were as follows:

- Do not skip necessary steps in writing a proof.
- When you are working on a proof, you cannot assume what you are trying to prove.
- When you are working on a proof, you cannot assume more than is given in the hypothesis in the problem.
- In construction you can use only compass and straightedge. You cannot use a ruler for measuring purposes.
- In proof problems in the chapter on the coordinate geometry, you have to use coordinate geometry.

Properties of Refutation

Recognition and interpretation of refutation. Recognizing a refutation and determining which method the refutation belongs to were problematic for both the participants and the researcher. As symbolic interactionism suggests, every human action can be interpreted differently among different people. For example, when an utterance takes a form of interrogation, it can be understood either as a refutation, or as a plain question, or as something else.

Thus, an utterance that was not intended by a speaker to be a refutation might be interpreted as such by a listener, and vice versa. In addition, even if an utterance was interpreted by a listener as refutation, the understanding of which method the refutation employed might differ between the speaker and the hearer.

Explicit and implicit refutation. The degree of explicitness was another source of problematic situations in recognizing a refutation. For example, there were numerous occasions when the teacher or students ignored suggestions made by others. In a lesson on constructions, the teacher elicited a theorem for justifying a student's construction. The students proposed various theorems: "Side angle side," "Side side postulate," "Angle side side," "Side angle side," "Angle angle similarity," "Side side side." The teacher then picked up on the last suggestion "Side side side" and ignored the others. This ignoring indicated her rejection of those suggestions: "They are not correct."

Communication and negotiation within a refutation. Refutation was not necessarily a monologue by one refuter. It might take the form of a dialogue. The dialogue was a process of communication and

negotiation. For example, in Episode 1, the teacher refuted Pete's idea by using a counterexample. The teacher did not complete the refutation in a monologue. She segmented the refutation into small and easy steps and made Pete and other students participate in each step in the form of a dialogue, thereby building up agreement between herself and them, guiding them to an obvious contradiction, and concluding it by her evaluative comment. The refutation itself is thus "collaboratively constructed by the teacher and children" (Newman, Griffin, & Cole, 1989, p. 125).

Retrospective and evaluative meanings of refutation. An action or a series of actions may not initially have the status of a refutation. It may be assigned the status later by participants in consideration of later events. When the teacher was initiating a dialogue expecting that it would lead to a refutation of a student's argument, the dialogue itself might not have the full status of a refutation either to the teacher or to the students. The students especially might not have been sure of how the dialogue would conclude. When they saw a contradiction and looked back at the whole dialogue, they might reinterpret it as a refutation and evaluate the target argument. Also, when students were performing an experiment on a construction problem, there were two possibilities: The experiment might or might not support the claim that a given construction procedure worked. When the experiment concluded, they interpreted the result in the context of the whole experiment, evaluated the initial claim, and labeled the experiment either confirmation or refutation. Here, a series of actions was retrospectively assigned the meaning of refutation (Mehan, 1979, p. 64).

Hierarchical structure within refutation. When the argument of a refutation consisted of several subarguments and had a complex structure, the refutation would contain another refutation within it. In a lesson on constructions the teacher compared the length B'C' with the length BC in two constructions and denied that B'C' = BC. Because she relied on the appearance of the actual construction, her objection was considered to be an Experimenting Method refutation. However, this objection was only a part of the "analyzing" process, a piece-by-piece comparison of a student's construction procedure with the correct procedure. The conclusion B'C' ≠ BC became the grounds for the rejection of the student's procedure by the Authority Method.

Focus of refutation. Refutation negates an argument The negation may target a specific part of the argument. That is, the refutation has a focus that the refuter is questioning. For instance, in Episode 2 Jack's argument focused on the legitimacy of conceiving angle zero and applying cosine to the angle zero, whereas the teacher was interested in the value of the cosine of angle zero. That episode seems to suggest that the focus of a refutation needs to be understood in order for an interaction to reach an agreed conclusion.

Context dependency of refutation Just as the way one talks depends on the context of the conversation, so the way one makes a refutation depends on the context of the interaction. For example, in Episode 1 when seeing Pete's lack of acceptance, the teacher changed her refutation from the Condition Method "It's under the square root, and it's been added to something" to the Counterexample. In a different occasion I observed the teacher using three different Contradiction Method refutations in responding to students' refutations.

As mentioned above, in a small group activity on constructions, some groups used Experimenting Method refutations. Experimenting is an empirical approach. The empirical approach was given a less important status than deductive argument in the geometry classroom. In fact, the use of the Experimenting Method was very limited in the class. However, for students, the empirical approach may be often the easiest method to check and support their argument. When the teacher's assistance was not available, the empirical approach was used as the last resort to arrive at their position.

Roles of Refutation

The classroom is considered to be a very complex place where people are engaged in various practices simultaneously. Talk in classroom discourse is a means and process to achieve various goals: "Any one utterance can be, and usually is, multifunctional" (Cazden, 1988, p. 3). Refutation was also part of classroom discourse. Refutation had different functions when viewed in different contexts of practice: instruction, participants' sense making, communication, and negotiation.

The teacher had the authority to decide what understandings and procedures were acceptable in class. The students' understandings and procedures had to fit the standard and lesson plan implicitly or explicitly set by the teacher. Refutations were often used to inform students about unacceptable understandings and procedures, which were often labeled by the teacher with negative evaluations such as "mistakes," "errors," "wrong," or "inappropriate."

The students did not always accept what the teacher told them and asked them to do. They were not simply content to fit into the teacher's standard and lesson plan. They also wanted to make sense of what they were doing. When they were puzzled by conflicting situations, they often expressed their puzzlement and asked the teacher or other students to resolve the situations. In Episode 2 Jack persistently refuted the teacher. He found a conflicting situation in her explanations and expressed his puzzlement. The teacher's explanations did not help him to resolve it: He was not interested in whether cosine of zero degree is one or not. He was struggling to make sense of the concept of angle zero in his own framework.

In discussing a problem, each person had his or her own framework for understanding and solving it. When the teacher and students exchanged long arguments in the classroom, a close analysis revealed differences in the frameworks among them. The differences were especially distinct when the Alt-frame Method was used. The participants' exchanges then could be considered to be efforts to resolve disagreements and achieve shared understanding by communicating and negotiating the differences in their frameworks.

Even though the teacher had the strongest power to control communication and negotiation in the classroom, she had to sometimes concede part of that power to students The curriculum, lesson plan, and standard of correctness were not definitely predetermined by the teacher There was plenty of room for the teacher to change or improvise them through communication and negotiation with the students. In Episode 2, Mary's question led the teacher to discuss an extra topic in the lesson. Jack's persistent refutation led the teacher to explain the material beyond the course and reduce the time she wanted to spend for other topics. She also had to use an empirical justification, which was not acceptable in the course She needed to let the students have a voice and to incorporate their ideas into her lesson.

Discussion

The traditional view of mathematics seems to have stressed its logical aspects and have led mathematics educators to conceive counterexample and contradiction as main tools of refutation in mathematics. The study indicated that the method of refutation in the classroom is not limited to them. Frequent use of Authority, Rule, Alt-Frame, and Condition Methods seems to illuminate the social aspects of classroom learning of mathematics: exercising authority, following social conventions and rules, negotiating between different frameworks, and so on

In his case studies Lakatos (1976) identified "concept-stretching" refutation as an important tool for mathematical creativity. It was a method of generating counterexamples of a claim by expanding the extension of a concept used in the claim. That method may be considered to be a special case of Alt-Frame Method refutation, which is part of a negotiation process between different frameworks (cf. Bloor, 1976, p. 135). This may suggest a potential link between mathematical creativity and social interaction.

For treatment of counterexample in mathematics, Lakatos (1976) and Balacheff (1991) identified several types of the treatment. The present study indicates that identifying the treatment of counterexample could be problematic. Even when a students' treatment of the counterexamples appeared to belong to the method of "surrender" (Lakatos,1976, p. 13)--a rejection of the proposition, the presentation of the counterexamples involved within it a communication and negotiation process between the teacher and the students. When a refutation is situated in a face-to-face interaction of classroom, it seems necessary that attention should be paid not only to how a person deals with a counterexample--a mathematical object, but also to how the person (refutee) interacted with a refuter who argued that the object was a counterexample.

For Lakatos (1976), the dialectic of proof (or justification) and refutation was the means for the growth of mathematics. The role of the dialectic was mainly on the improvement or advancement of mathematical knowledge (p. 5). In the classroom as well, the dialectics of justification and refutation may be conceived as the medium for the improvement of students' mathematical knowledge--the construction of viable knowledge. When the exchange of justification and refutation was located in the classroom interaction, it seemed to play another important role in the classroom community: construction of shared mathematical knowledge. For a claim to be accepted in the classroom, it had to be justified in accordance with the standard set by the textbook and the teacher, and at the same time the claim and their justification had to make sense to the participants. The participants communicated their ideas and negotiated their frameworks through the exchanges of justification and refutation. Those exchanges in the discussion then led to the formation of shared claims and shared justifications of them in the classroom community (Voigt, 1989).

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METACOGNITION: THE ROLE OF THE "INNER TEACHER"(5)

Research on the process of internalization of "Inner Teacher"

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ABSTRACT

The nature of metacognition and its implications for mathematics education are the main concerns of our investigations. We argued in the last four papers that "metacognition" is given by another self or ego which is a substitute for one's teacher and we referred to it as "inner teacher".

In this paper, we investigate more deeply the concept of the inner teacher through the analysis of the process of internalization of metacognition using two Questionnaire 1, 2 and a stimulated recall technique specially designed for this research in elementary school students in 5th grade.

We found that there are some items of Questionnaire 1 in which the students responded quantitatively to be very helpful in solving the problem influenced by the teacher's utterances.

In the case study, the student implied the existence of specific steps of the process of internalization of metacognition.

AIM AND THEORETICAL FRAMEWORK OF THE RESEARCH

1. Definition of "metacognition" and "inner teacher"

We are often inclined to emphasize only pure mathematical knowledge in education. And we fail to enact it in students. Consequently, they fail to solve mathematical problems and forget soon after paper and pencil tests.

Recently, "metacognition" has come to be noticed as an important function of human cognitive activities among researchers of mathematics education as well as among professional psychologists. But even so, the definition of "metacognition" is not yet firmly settled, and results from the research have been of little use to the practice of mathematics education.

The ultimate goal of our research is to develop clear conceptions about the nature of "metacognition" and to apply this knowledge to improve methods of teaching mathematics. This paper is one of a series of studies in pursuit of this goal.

Roughly speaking, we could regard "metacognition" as the knowledges and skills which make the objective knowledges active in one's thinking activities. There are a few proposals on the categorization of "metacognition" in general, but here we will

follow the suggestion of Flavell and adopt four divisions of metacognitive knowledge of:

(Metaknowledge)

1. the environment 2. the self
3. the task 4. the strategy

and three divisions of metacognitive skill:

(Metaskill)

1. the monitor 2. the evaluation
3. the control

Our unique conception is that this "metacognition" is thought to be originated from and internalized by the teacher him/herself. Teachers cannot teach any knowledge per se directly to students but teach it inevitably through his/her interaction with students in class.

We start from a very primitive view that teaching is a scene where a teacher teaches a student and a student learns from a teacher. In the process of teaching, a phenomenon which is very remarkable from a psychological point of view will soon happen in the student's mind; we call this the splitting of ego in the student, or we may call it decentralization in a student, using the Piagetian terminology. Children, as Piaget said, are ego-centric by their nature, but perhaps as early as in the lower grades of elementary school, their egocentrism will gradually collapse and split into two egos: the one is an acting ego and the other is an executive ego which monitors the former and is regarded as the metacognition. Our original conception is that this executive ego is really a substitute or a copy of the teacher from whom the student learns. The teacher, if he/she is a good teacher, should ultimately turn over some essential parts of his/her role to the executive ego of the student. In this context, we refer to the executive ego or "metacognition" as "the inner teacher" .

The advantage of this metaphor is that we could have the practical methodology to investigate the nature of metacognition; that is, we may collect many varieties of teachers' behaviors and utterances in lessons and carefully examine and classify them from some psychological view-points.

2. Positive and Negative Metacognition

For Metacognition, we think that there are two types. One is a positive metacognition that promotes positively students' problem-solving activities. The other is a negative one that obstructs their activities. For example, most students believe that statements like questionnaire item III.19 "When you get lost while solving the problem, please think of other strategies." help them and have a positive effect on problem-solving. This item shows metacognitive knowledge of strategy for problem-solving. This works according to the monitor "I have lost my ideas for the

next step". A metaskill of control "Please think of other strategies." works successfully according to a logical conclusion of modus ponens from two premises; item III.19 and the monitor. (See Hirabayashi & Shigematsu, 1987, for more detail.) Other statements, like item IV.18 "Can't you solve this easy problem?", are believed to make students do worse and to have a negative influence on problem-solving.

METHODOLOGY OF THE RESEARCH

Hypothesis of the Process of Internalization of Metacognition

At first, we specify the process of internalization of metacognition (inner teacher) as follows;

1) A student is very much aware of the current problem-solving.
2) Before or while the student solves the problem, a teacher gives him/her a suitable metacognitive advice when he/she wants to get it.
3) The student remembers the teacher's metacognitive advice tentatively at this time.
4) The student can solve the problem referring the teacher's metacognitive advice and has a good affective feeling.
5) The student wants to remember it permanently.
6) The student can solve similar problems referring the teacher's metacognitive advice.
7) The student acquires the metacognition as the inner teacher.

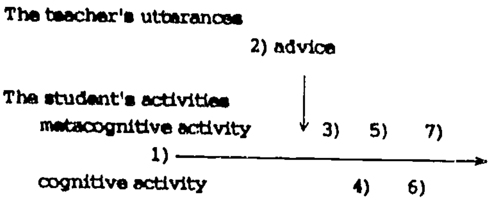

Fig.1 Process of internalization of metacognition

We assume the internalization of metacognition is mainly originated from teaching-learning communication between a teacher and students in a classroom lesson as this hypothesis.

Teaching-Learning Process of Experimental Lesson in Class

1. The teacher introduced a topic in the form of problem-solving and students understood the goal of problem by working on some examples given by a

teacher using an over-head projector:

Material and purpose of this lesson:

1) The teacher posed a situation which the students found out the patterns to open a square of door.

2) In order to open the door, the students were encouraged to find out the way which divided into two congruent parts using a dividing line as many as possible.

3) The students were given some 4 × 4 cross striped squares drawn on a piece of paper.

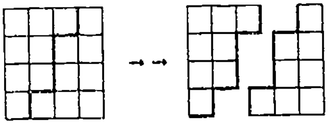

Fig. 2 An example of the solutions

2. The students solved the problem individually.
3. The students discussed their solutions by working with their class mates.
4. The teacher summarized the day's mathematical idea by using or referring to the students' solutions.

Method of Analysis of the Process — Stimulated Recall Technique

It is very difficult to record the student's real time cognitive and metacognitive activities. Therefore, we used the modified stimulated recall technique(Yoshizaki, S. et al. 1992) to record students' metacognitive activities after the lesson finished;

1) We used Questionnaire 1 to analyze the students' metacognition before the lesson.

2) We videotaped the lesson from the back seat in the class.

3) After the lesson, we gave to the students Questionnaire 2 to analyze students' cognitive-metacognitive activities during the lesson:

The students watched the video-tape for about 2 ~ 3 minutes working at four specific times: (a) when the students were given the problem, (b)when they began to work on the problem individually, (c)when they began to work on the problem working with their class mates, and (d)after they finished working on the problem in the lesson.

4) We used Questionnaire 1 to analyze the students' metacognition after the lesson.

Questionnaire 1 (See Shigematsu, 1992, for more detail)

1) Categories of Items

I. explanation 19 items, II. questioning 16 items, III. indication 25 items

2) Responses to Each Item

For each question, students indicated which of the following two-part responses best reflects their experience.

My mathematics teachers have made this comment often   Yes   No

a b c   a b c

This kind of comment: a. helps me.
b. doesn't help me.
c. makes me do worse.

Questionnaire 2

This questionnaire is conducted about 100 minutes after the lesson finished.

1) The students asked to answer the questionnaire after each review of the video-tape four times in all as explained in the above.
2) The main question items are as follows;
   (1) What kind of activities did you do while you were watching the video-tape? (Cognitive activity)
   (2) What kind of ideas occurred to you while you were watching the video-tape? (Metacognitive activities)
   (3) Did you remember what your teacher said while you were watching the video-tape?
   (4) What did the teacher say, if so? (Teacher's utterances)
   (5) Did you find the teacher's advice helpful?
   (6) Did you have a will to remember the teacher's advice?

Data collection

We collected the data from students of elementary school 5th students in Nara, Japan. The numbers were 31.

RESULTS AND DISCUSSION

Difference Between Teacher's Utterances and Students' Responses in Questionnaire 1

In order to examine how the lesson would change the students' reactions, we conducted Questionnaire 1 twice as we said above.

We compared the results between the teacher's utterances and the students' responses.

1. Items indicating teacher's utterances in the lesson

I. 3, 8,13 II. 4,12,14,16,17,21,25 III. 2, 4, 6, 9,12,19,27,32,37,40

2. Items where significant change was observed in the students' responses

I. 3 II. 4,12,21 III. 6, 9,19,27,32,40

We are interested in the items in which the students responded that the teacher uttered positively in the lesson and in which the students chose Yes-a. These items are those in which the students thought their mathematics teachers had made this comment often and helped them, that is, it proved to be very helpful in solving the problem.

III. 6. If you can solve the problem by one strategy, try to solve it by another one.

19. When you get lost while solving the problem, please think of other strategies.

27. Write your solutions as if you were to explain your ideas to others.

These items are certainly the positive metacognition(metaknowledge of the strategy) which are important for students to solve mathematical problems and to communicate with each other.

Process of Internalization — Case Study

In order to examine the process of internalization, we concentrate on the response from one subject. Here below, we coded the response to six questions in Questionnaire 2.

Case 1: The stage of problem-solving by individual student

A female who found out nine patterns

(1) What kind of activities did you do while you were watching the video-tape?

She was solving the problem through trial and error.

(2) What kind of ideas occurred to you while you were watching the video-tape?

It was good to check several patterns of dividing the square.

(3) Did you remember what your teacher said while you were watching the video-tape?

Yes.

(4) What did the teacher say, if so?

I. 8 This problem might be slightly more difficult than the previous one.

(5) Did you find the teacher's advice helpful?

Yes.

(6) Did you have a will to remember the teacher's advice?

Yes.

Case 2: The stage of problem-solving by class

A female who found out nine patterns

(1) What kind of activities did you do while you were watching the video-tape?
She was solving the problem through trial and error.

(2) What kind of ideas occurred to you while you were watching the video-tape?
There were some other strategies which she could not use.

(3) Did you remember what your teacher said while you were watching the video-tape?
Yes.

(4) What did the teacher say, if so?
Ⅱ. 12 Did you have other strategies to solve the problem?

(5) Did you find the teacher's advice helpful?
Yes.

(6) Did you have a will to remember the teacher's advice?
Yes.

Her answer implies the existence of the specific steps of internalization of metacognition.

But we know that this is not enough to analyze the process of internalization of metacognition. In order to identify these steps, we need more experimentation on this issue.

CONCLUSION

In this paper, we investigate more deeply the concept of the inner teacher through the analysis of the process of internalization of metacognition using two Questionnaire 1, 2 and a stimulated recall technique specially designed for this research in elementary school students in 5th grade.

At first, we proposed the hypothesis of the process of internalization of metacognition as seven steps. According to this hypothesis, we implemented the experimental lesson that students solved the process-problem. After the lesson, students answered the questionnaire 2 which analyzed the process of internalization using video tape recorder as a stimulated recall technique.

We obtained several findings as follows:

1. Comparing the results between the teacher's utterances and two Questionnaire 1, there are some items of Questionnaire 1 in which the students responded quantitatively to be very helpful in solving the problem influenced by the teacher's utterances. For examples,

   Ⅲ. 6 If you can solve the problem by one strategy, try to solve it by another one.
   These items are certainly the positive metacognition which are important for

students to solve mathematical problems and to communicate with each other.

2. In the case study, a student who is over-achiever answered Questionnaire 2 to imply the existance of the specific steps of internalization of metacognition.

But we know that this is not enough to analyze the process of internalization of metacognition. In order to identify these steps, we need more experimentation on this issue.

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CHILDREN'S PERCEPTIONS OF THE USEFULNESS OF PEER EXPLANATIONS

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When 26 students in the author's own Grade 5 classroom were asked to respond regarding the usefulness of the explanations in both small group interactions (4-5 children) during mathematics problem solving sessions, and large group presentations (large group size 12-14), the overall consensus was that the peer talk and the explanations were helpful. Qualifications: (1) in order to be helpful, the explanations had to be "clear"; (2) a number stipulated that explanations were helpful if you had gotten the wrong answer, or if you did not understand the problem, but felt it was not useful to hear alternate approaches if the individual or group had arrived at what turned out to be the correct answer; indeed some stressed that it was confusing; (3) the (social) climate must be seen to be conducive, i. e. accepting, before many will venture to ask for or to give explanations.

As a teacher-researcher in my own elementary school classroom (10-11 year olds), I have been engaged in pursuing a long-standing interest begun in my doctoral work (Zack, 1988) in the ways in which peers interacting together can contribute to the construction of their knowledge. The thesis work dealt with the talk aspect of the peer interaction (supported talk, including justifications, and explanations). The focus in the NCTM Standards (1989) on the aspect of problem-solving as central to the mathematics curriculum, and on the aspect of communication (both spoken and written) as a vital component meant that my return to the classroom in 1989 after 10 years of teaching at the university level was auspiciously timed.

Problem-solving is at the core of the mathematics curriculum in the classroom; non-routine problems are drawn from various sources (Charles & Lester, 1982; Meyer & Sallee, 1983, and others). My goals are that the children see that (1) they have the ability to arrive at a solution, (2) there are many ways to arrive at a solution, and (3) that I am very interested in how they arrive at a solution, as interested in it if the procedure is wrong as if it is correct. I feel that their talking together and thinking through, play a vital role in the construction of their knowledge (see summary of constructivist view of mathematics learning in Wood, Cobb, & Yackel, 1991, citing Clements & Battista, p. 591).

In regard to the specific domain of talk and explanations, I would point out briefly that as a teacher I look for instances of (1) clear explanations, (2) diverse approaches to solutions from amongst the children, and (3) encountering an elegant explanation (it is a rare occasion but constitutes a delicious surprise (see Duffin & Simpson, 1991)). I feel that the children might learn more about the structure of mathematics if they can see a solution from diverse perspectives.

As you will see below, based on the children's reponses to the questions I posed, I would say that the children (1) value clear explanations (their criteria for what makes a clear explanation will follow), and (2) they find explanations helpful primarily if they got the wrong answer or didn't understand the problem.

Classroom set-up

St. George's is a private, non-denominational school, with a middle class population of mixed ethnic, religious, and linguistic backgrounds; the population is pre-dominantly English-speaking. Total class size is 26; however I always work with half-groups (12, 14 children in each group) of heterogeneous ability. Mathematics class periods are 45 minutes each day. Problem-solving is the focus of the entire lesson three times a week. In class the children often work in Groups of Four teams: each teacher-selected heterogeneous team has 4-5 members, a problem is given, is either worked through by the team members together at their table, or begun by each child and then discussed at their table (the group selects the approach it favours). When the three teams are ready, they gather together. (They consider themselves ready when all team members have "understood" the solution. Any member can be called upon to present, although they usually present on a rotation basis. I change the composition of the groups every three weeks.) Presentations of the solutions to the whole group of 12 or 14 take place at the chalkboard. The children also work individually on one challenging problem at home (Problem of the Week), and are expected to write in their log about all that they did as they worked the problem (Zack, 1991). The children present their Problem of the Week solutions to the class. I videotape each of the teams (Groups of Four) on a rotating basis, and videotape all the presentations done at the board (Groups of Four team discussions, and Problem of the Week discussions), and observe and take notes during the sessions. Much of the class session is conducted by the children. Data sources are: anecdotal observations, videotape records, student artifacts (copybooks), teacher-composed questions eliciting opinions (written responses), and class discussions regarding research topics.

The Role of Talk and Explanations in Problem-Solving

The focus of the study was on the aspect of helpfulness of working with peers. I asked two questions at the end of last year (Questions #1, #3 below, May 26, 1992):

Question #1: Does talking with a partner or with your group help you in your problem-solving?

Question #3: Does listening to a classmate explain at the front of the class help you in understanding the Problem of the Week?[1] the Groups of Four Problem?

which evoked some interesting responses. Thus I addressed the same questions to my current Grade 5 class during the first term this year (November 2, 1992), with a view to analyzing the responses and documenting my reflections upon them for the PME meeting in Japan. The findings were as follows:

[1] I will deal in this paper only with the Groups of Four problem solving situation. The Problem of the Week situation is different, since in the latter case the children have not worked through the problem together.

Question #1: Does talking with a partner or with your group help you in your problem-solving ?

	Yes	No	Y and N (It depends)	Total
1991-1992 (May 26, 1992)	11	8*	4	23
1992-1993 (Nov. 2, 1992)	16	4	6	26

* Of those who said "No", 5 mentioned they felt time was wasted in arguing and disagreeing, and 2 others said that the talk interfered with their thinking.

Question #3: Does listening to a classmate explain at the front . . . help you in understanding. . . ?

	Yes	No	Y and N (It depends)	Total
1991-1992 (May 26, 1992)	12	1**	10	23
1992-1993 (Nov. 2, 1992)	2	3	21***	26

** The one written comment reflecting "traditionalist" (Gooya, 1992) leanings, i. e. the desire to know only whether the answer was right or wrong, was: "No, because if I had an answer book I would see my mistake and relieze it."

*** Provisos: The explanations were helpful if they were **clear** [6 children], if one's solution was wrong or one was stuck [2], if one was wrong, yes the explanation helped, however if one's answer was correct, an explanation that was different confused the listener [4], the explanation helped if the listener was sure ahead of time that the explainer's answer was correct [2], and other [7].

In pursuing further their proviso of clarity, I asked the children: "What makes for a clear explanation?" (Question #4, November 3, 1992). The children's definitions addressed the *mechanics* of presenting to a group (i.e., loud voice, use the board, face the audience not the board, kneel so that everyone can see the work on the chalkboard), as well as the aspect of *meaning*,, the notion of **making something intelligible** and **making known in detail** (Random House College Dictionary, 1975, p. 466): 'The person explains everything in detail without skipping steps, but <u>not</u> complicated'. Two children insightfully tied the notion of good explainer to the person's having a good grasp of the problem: "If the person really understands the problem, they should have a good explanation" (Michelle B., Question #4).

There were a number of aspects of interest to me which emerged in the November responses (a) regarding differences between "at table small group talk" and "front of the room large group presentation", and (b) regarding alternate approaches, and so I posed Questions #7 (a, b, c) and Question #8 (discussed later) to both to my current Grade 5 class, and to my last year's Grade 5 class (referred to at times as Grade 6, their present status). Question #7 (shortened here, posed December 15, 1992) was as follows:

Question #7: Think of explanations (1) during discussions at the table and think of explanations (2) during presentations to the whole group.

(7a) Are the explanations different? (7b) Which do you prefer? (7c) Are the explanations helpful? Which?

When I posed questions #7b and #7c I thought that the majority of the children would prefer explanations at the table and would find explanations at the table more helpful because more interaction was possible. Those who chose **table talk/ explanations** did indeed point out that there was more interaction at the table, there was a personal touch, the possibility to work through to the answer together, accommodation to the individual, option to keep at it until all the team members understood (patience, persistence), and it was more conducive to asking questions (although as I will show below, the children stressed that much was contingent upon who the group members were). Hence the talk was exploratory: more halting, more open to interruption, and more informal.

However, I was also given to understand what the children perceived to be the advantages of the explanations given in the **presentations to the whole group**. All the children were intent upon finding out what the correct answer was. Some noted that after the possibilities had been `sifted down' at the table during small group session, you were more likely to be closer to the correct answer, even though there was also an `opening up' of possibilities when the three teams met together, in that there might well be options and answers not thought of by one's own group. In the large group, there was a tighter focus; it was more teacherly, more formal. There was more product, less process--working together to find the answer had happened in the small group. Two of the children even pointed out the differences (Q #7a, Grade 5):

Rebecca: Usually when the presenter presents for the whole 14 of us, they make the problem seem so easy, but back at the table, we're presenting for the first time, and it's much more harder.

Margaret: The explanations are different "because at the table people are testing their answers and on the board they're pretty sure of their answer."

The Affective Aspect of Peer Interaction

Wood, Cobb & Yackel have aptly stressed how necessary it is that the teacher promote an environment in which the children feel safe, so that each child can

know that all answers will be considered and respected, and that no effort or answer will be denigrated as silly or incompetent (1991, 599). What I would add is that a safe environment is what the children must cultivate for each other as well. Their responses to my questions indicated that they were much more comfortable asking questions at their table in their small group, than while in the large group at the chalkboard. In addition, the children were cautious at the table about asking for explanations and were selective about whom they approached. In answer to Question #7b, Lindy (Gr. 6) said: "Sometimes I feel more comfortable at the table saying I don't understand it. It depends on who it is. If it's people who get the answer right, if they're sure of themselves . . . I feel stupid. Let's say it's a person who will accept, and say I'll explain to you, I'll feel comfortable." Some people were deemed "good explainers" not only because they could make themselves understood, and `knew their stuff', but also because they were patient and did not make the listener/tutee feel stupid. One would think that in a group of 4-5, the setting would be conducive to having one child explain to the others in the group. However, most of the comments referred to dyadic interaction between two peers at the table rather than to one person explaining to the team. (I would have to return to the tapes and analyze to see the comparable number of dyadic, triadic, or one-to-4 or 5 interactions; it appears to me that much of the interaction at the table is indeed dyadic.)

The children need to feel a valued part of the working group. Feelings of inclusion/exclusion play an important role in the lives of children. As Christopher said in his self-evaluation of his work in his Groups of Four group for his November report: "I usually have trouble here because, sometimes most of the people don't listen to you, and then somebody says the same thing, and everybody listens to them, and that makes you feel left out!" (November 9, 1992).

The children's own comments have provided the foundation for further classroom discussion for us on how the children can make the climate as comfortable as possible for all in the group.

Alternate Approaches

I was spurred to look at children's reaction to alternate approaches due a lively interchange I had with Lo (1992) in preparation for the Working sub-Group dealing with The Role of Language in the Formation of Elementary Concepts by Young Children for the ICME conference (August 17-23, 1992). In response to a comment in Lo & Wheatley's paper (1992) that "when asked to state a goal for mathematics class discussion, some students indicated (1) to learn a more effective way from other students, and (2) to find out what the correct answer was", I responded to Lo in a letter that I was not sure that the same was true of the children in my 1991-92 class. In looking back at their responses to Questions #1 and #3 done May 26, 1992, I found that there were two children who had indeed stated that it was of interest to see other approaches. I subsequently asked all the children (Grades 5 & 6) a direct question concerning alternate approaches (December 15, 1992):

Question #8: In cases where your Groups of Four team and another team have the same answer to a problem, but have used different ways to solve it, do you find it interesting, or helpful, to see another solution?

I was struck by the difficulty the children felt was entailed in following another child's explanation when it diverged from the person's own. A number [10] mentioned that trying to understand an alternate solution was confusing. David (Gr. 6), for example, said he found hearing another approach neither interesting nor helpful: "Actually, I [find] it complicating." Lindy (Gr. 6) said that listening to an alternate approach was confusing when she had an answer: ". . . sometimes I feel frustraded [when] what I think is added on with other people's understandings."

Four children stated that they could not see the point of hearing another approach since they already had the answer and were not going to do the same problem again.

I had thought that perhaps the children would value not just other approaches, but rather the better approaches. This aspect did arise in the comments of two of the children who are quite adept. They spoke of the benefit of seeing "an easier or faster way" (Bryan, Gr. 6, Moustafa, Gr. 5). I was surprised that none of the most adept qualified it in this way.

Nineteen children of the 48 polled, i. e., close to 40% and thus more than I had expected, indicated that they appreciated hearing alternate approaches. Their reasons seemed to fall into two categories: **pragmatic** (12 out of 48 responses) and **aesthetic** (7 out of 48). What I have termed **pragmatic reasons** would relate to the children saying that they would/might make future use of the alternate solution in problems, whereas **aesthetic reasons** would refer to the children saying that they are better able to understand the problem itself or their own thinking due to hearing the alternate solution. Marc (Gr. 6), for example, said that "yes" it was interesting and helpful "because I could see a relationship between them and I could understand the problem better." Margaret (Gr. 5) said: "I find it helpful because I'm usually able to relate and sometimes I can understand what I did better."

To date I have seen no studies which deal with the importance for growth of mathematical understanding of appreciation of alternate approaches, or studies of its relationship to mathematics achievement, i. e., Are the more adept students better able to follow and appreciate alternate approaches? Also, there is the question, hard to investigate, of whether the child does actually apply the approaches seen. (See one intriguing report by Maher & Martino (1992) in which a 7-year-old child is seen to adapt an approach used by a working partner a full five months after he first saw it although he seemed in the first instance not to even take it into account.) Although some of the children did state that seeing other approaches is important because one can use "it" in a future problem, given the nature of the non-routine problems, I would expect it would be very difficult for the child to apply. Delving into what sense the children make of other children's explanations, including diverse approaches to a problem, is a challenging sociocognitive component to pursue.

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Psychology of Mathematics Education

PME XVII

JULY 18-23, 1993

PME 17

Proceedings of the Seventeenth International Conference

UNIVERSITY OF TSUKUBA
TSUKUBA, IBARAKI, JAPAN

e III

Psychology of Mathematics Education

PME XVII

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Volume III

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PME 17 PROCEEDINGS

Edited by Ichiei Hirabayashi, Nobuhiko Nohda, Keiichi Shigematsu and Fou-Lai Lin

PREFACE

The first meeting of PME took place in Karlsruhe, Germany in 1976. Thereafter different countries (Netherlands, Germany, U.K., U.S.A., Belgium, Israel, Australia, Canada, Hungary, Mexico, Italy) hosted the conference. In 1993, the PME conference will be held in Japan for the first time. The conference will take place at the University of Tsukuba, in Tsukuba city. The university is now twenty years old. It is organized into three Clusters and two Institutes. There are about 11,000 students and 1,500 faculty members. The Institute of Education at the University of Tsukuba has a strong commitment to mathematics education.

The academic program of PME 17 includes:

- 88 research reports (1 from an honorary member)
- 4 plenary addresses
- 1 plenary panel
- 11 working groups
- 4 discussion groups
- 25 short oral presentations
- 19 poster presentations

The review process

The Program Committee received a total of 102 research proposals that encompassed a wide variety of themes and approaches. After the proposers' research category sheets had been matched with those provided by potential reviewers, each research report was submitted to three outside reviewers who were knowledgeable in the specific research area. Papers which received acceptances from at least two external reviewers were automatically accepted. Those which failed to do so were then reviewed by two members of the International Program Committee. In the event of a tie (which sometimes occurred, for example, when only two external reviewers returned their evaluations), a third member of the Program Committee read the paper. Papers which received at least two decisions "against" acceptance, that is a greater number of decisions "against" acceptance than "for", were rejected. If a reviewer submitted written comments they were forwarded to the author(s) along with the Program Committee's decision. All oral communications and poster proposals were reviewed by the International Program Committee.

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Monbusho [The Ministry of Education in Japan]
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This PME seventeenth conference is supported by The Commemorative Association for the Japan World Exposition (1970.).

We also wish to express our heartful thanks to the following local committee and local supporters who contributed to the success of this conference:

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HISTORY AND AIMS OF THE P.M.E. GROUP

At the Third International Congress on Mathematical Education (ICME 3, Karlsruhe, 1976) Professor E. Fischbein of the Tel Aviv University, Israel, instituted a study group bringing together people working in the area of the psychology of mathematics education. PME is affiliated with the International Commission for Mathematical Instruction (ICMI). Its past presidents have been Prof. Efraim Fischbein, Prof. Richard R. Skemp of the University of Warwick, Dr. Gerard Vergnaud of the Centre National de la Recherche Scientifique (C.N.R.S.) in Paris, Prof. Kevin F. Collis of the University of Tasmania, Prof. Pearla Nesher of the University of Haifa, Dr. Nicolas Balacheff, C.N.R.S. - Lyon.

The major goals of the Group are:

- To promote international contacts and the exchange of scientific information in the psychology of mathematics education;
- To promote and stimulate interdisciplinary research in the aforesaid area with the cooperation of psychologists, mathematicians and mathematics teachers;
- To further a deeper and better understanding of the psychological aspects of teaching and learning mathematics and the implications thereof.

Membership

Membership is open to people involved in active research consistent with the Group's aims, or professionally interested in the results of such research.

Membership is open on an annual basis and depends on payment of the subscription for the current year (January to December).

The subscription can be paid together with the conference fee.

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THE CONCEPT OF FAIRNESS IN SIMPLE GAMES OF CHANCE

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ABSTRACT:

This study examines the mathematical concepts of "fairness" and "expectation" in probabilistic situations. The subjects were 40 high school students in Semester I, Year 11, Maths in Society classes in three Queensland high schools. Twenty "gamblers" were identified by questionnaire and subsequent interview. A control group of similarly achieving "non-gamblers" was selected. The research compares the ability of each group to construct a working definition of the concept of mathematical expectation and to use this concept in determining the fairness of a number of games of chance.

This study examines the mathematical concept of "fairness" as it applies to simple single-event games of chance involving coins, dice and cards.

In the determination of fairness, two aspects are examined:

- the misuse of an heuristic of "representativeness" or "availability"
- the use of an intuitive understanding of the concept of "expectation".

Misconceptions in probabalistic reasoning involving the use of "representativeness" and "availability" heuristics have been well documented by researchers including Shaughnessey (1977. 1981. 1983). Scholtz (1986). Tversky and Kahneman (1982), and Peard (1991a, 1991b).

The use of "representativeness" to determine the fairness of a coin or game is illustrated when in situations the subject takes a short term sequence of events as being "representative" of the long term situation and erroneously concludes bais or unfairness.

"Availability" is used to come to the same conclusion by reasoning that such short term sequences are readily recalled. More "balanced" results are more readily "available".

Bright, Harvey and Wheeler (1981) in a study of fair and unfair games claim that "fairness" is best described by calling attention to an intuitive understanding of "unfairness". In refering to students in years 4-8 they claim that "Helping students recognize when a situation is fair or unfair is a reasonable expectation of the school curriculum." (p 50). Research by Anderson and Pegg (1988) also reported difficulties primary school pupils encountered with the determination of fairness.

The mathematical concept of fairness, as opposed to a merely intuitive understanding, relies on the concept of "expectation". A game is "fair" if all participants have equal mathematical expectation. This in turn requires an understanding of mathematical expectation which is defined as the product of probability and return.

These concepts are clearly beyond the elementary level but require the application of only basic probabilistic reasoning. For simple games involving only two players, one need only determine the probabilities for each to win and then calculate the required amounts for each to be a constant product

(or inverse proportion). This constitutes an effective concept of equal mathematical expectation for both players.

Bright et. al. note.

> In complex situations it may be difficult to determine mathematically whether a situation is fair. (p 50)

Lovitt and Clark (1988) questioned whether pupils about to leave school had realistic ideas about the outcomes of gambling and concluded that "there is a huge gap between perception and reality... in which pupils demonstrated misconceptions of the concept of expectation."(p.77)

The inclusion of basic probability and its applications in the general school mathematics curriculum, both elementary and secondary, has been a relatively recent development. Pereria and Swift (1981) writing in the N.C.T.M. Yearbook made a strong arguement for probability to be part of every students education. Since then considerable progress has been made world wide as is evidenced by the N.C T.M. statement of Standards in the U.S.A. and the inclusion of "Chance and Data " in the Australian National Statement which makes specific reference to "fairness" and "expectation".

However numerous difficulties with the implementation of such programs have been reported. In Australia, teacher unfamilarity with much of the content is recognised by, for example, Peard (1987). Pedagogical problems with the teaching of probability are also well documented.

(Garfield and Ahlgren (1986, 1988), Kapadia (1984), Brown (1988), Pegg (1988), Green (1982, 1986), del Mas and Bart (1989).)

Thus it is reasonable to assume that at the present time very few students will have had formal instruction in the topics of fairness and expectation prior to the Senior Secondary grades and that only some will gain this knowledge in these years.

Objectives:

The subjects in this study were 40 high school students in Semester 1, Year 11, Maths in Society classes in three Queensland high schools. Two of these schools were in a lower socio-economic region, close to horse racing,dog racing and trotting tracks. Many senior students in these schools followed the races.

The study is part of a larger study investigating the construction of various probabilistic concepts within a social context by students whose background includes a familiarity with the phenonomen of gambling, particularly in relation to "track" betting These are subsequently refered to as "gamblers". Interview questions established that all of these subjects were familiar with betting in track situations, the use of "odds", and methods of calculating payouts.

The objectives of the study were to determine:

(1) The pupils ability to recognise fairness in simple games of chance.

(2) Whether or not an heuristic was misused in incorrect identification.

(3) Whether or not there was any difference in this ability between "gamblers" and "non-gamblers"

(4) The ability of the students to recognise or construct a concept of expectation in simple games of chance in which players have unequal chances.

(5) The ability to use the concept of expectation in determining fairness.

(6) Whether or not these abilities were related to: social background (gambling), school achievement, and gender.

Methodology:

The "gamblers" were identified by questionnaire administered with the help of either the regular classroom teacher or a special needs teacher. A subsequent interview was given to validate responses. Only those indicating a "great deal" of interest in at least one form of track racing were considered as "gamblers". A control group of "non-gamblers" was selected from those responding negatively to all forms of gambling and games of chance.

All schools were coeducational and an approximately equal number of male and females responded positively to interest in gambling. Thus a balance of subjects by gender was easily obtained A balance of subjects by achievement was also obtained

The research methodology employed was that of the structured clinical interview as described by Romberg and Uprichard (1977).

The interview asked open-ended questions relating to.

Category 1 - Representativeness and Fairness.

- the subjects' ability to recognise when a simple game of chance is "fair" and whether or not a heuristic of representativeness or availability was used in the decision making.

Questions:

The first questions asked were of the type:

1. (a) 'You and I play a game of chance in which a coin is tossed. Heads I win. Tails you win. Of the last 15 people who played this game with me 10 lost. Is this a fair game?"

Similar questions relating to rolling a single die and drawing cards from a deck followed.

These questions are similar to those asked by Shaughnessey (1981). He reported a high incidence of the use of availability to conclude that the coin tossing game was not fair.

Those believing the games to be unfair do so by either using the short term results, for example, of 15 tosses to be "representative" of the long term probability of the coin or reply that they expect the next person to lose since "people tend to lose at this type of game" (availability).

Thus the next questions asked in this study were:

(b) "Is the coin/die/card game fair?"

(c) " Why?" or "Why not ?", depending on response.

Follow-up questions in the structured interview were of the type:

To those who responded affirmatively to (b)

(d) "How many tosses would you need to conclude that the coin was unfair?"

Those who recognised that a very long run was required before bais could be suspected were considered to be free of the misuse of the representativeness heuristic.

2. (a) "You and I play a game of chance which involves throwing a single die. We each bet $1, winner takes the $2. If the numbers are 1, 2,or 3, I win.If they are 4, 5,or 6 you win. Is this a fair game?"

(b) "If we change the rules so that if they are 1. 2. 3,or 4 I win; 5. 6. you win. Is this a fair game now?"

(c) "Why or why not?"

(d) "Can we change the amounts each player puts in to make this game fair?"

This last question then leads in to the concept of "expectation"

<u>Category 2 - Expectation and Fairness.</u>

Questions: (following from above)

3. (a) "Since I have the better chance of winning can we make the game fair by increasing the amount I put in?"

Those who responded negatively to this were considered to have no concept of expectation. Typical responses were:

"You will still have a better chance than me and that's not a fair game."

Those who responded affirmatively were then asked

(b) "How much should I put in?"

To demonstrate a basic understanding that expectations can be made equal. it was not required that the subject use formal mathematical language. A typical response was:

"Well you have four chances to my two, that's twice as many. So if you put in twice as much, that would be fair."

The extent of understanding was investigated further:

(c) "What if I chose five numbers and left you with only one? How much should I put in now?"

For those who were able to answer this correctly different situations were then investigated.

e.g.

(d) "If we draw cards from a deck and I choose any Ace leaving you the rest, how much more than me should you put in to make the game fair?"

(e) "If I choose just one card such as the Ace of Spades,how much now?"

(f) "If I choose the 16 "coloured cards" - ace, king ,queen,jack of each suit, leaving you the 36 remaining cards and I put in $1, how much should you put in?"

Those who were able to demonstrate consistently in all of these situations that "fairness" can be established by each contributing an amount in inverse relationship to the probability (i.e. an equal product of probability and return or equal expectation) were considered to have a complete understanding of the basic concept.

An exact answer to the last question was required for this. It was not sufficient to reason along the lines (as did some):

"I have more than twice your chances so I should put in more than twice as much."

A "complete" understanding required reasoning that resulted in the calculation of 36/16 x $1 = $2.25.

Results and Analysis of Data:

From the responses to these questions subjects were classified:

Category 1 - Representativeness and Fairness

(1) Recognises a fair simple game

33 of the 40 were able to recognise that in all situations the game/coin/die were fair and that deviations were not unreasonable.

(2) Uses an heuristic to misjudge a fair game

5 of the 40 were classified in this category.

Of these 3 responded using the "representativeness" heuristic and 2 using an "availability" heuristic.

2 reponded that they were unable to make a decision.

None of the 5 used the heuristic in questions of the type of 1(e),(f)-very short sequences.

5 of the 7 were non-gamblers but due to the small size of this category no test of significance was performed.

Rather, we note that the majority of both gamblers and non-gamblers were able to recognise that the situation itself was in fact fair.

(3) Free of the "representativeness" misconception

(correct response to Q.1(d))

Of the 33 who recognised fairness 23 were able to conclude correctly that a much longer sequence than that given would be required to infer bias or unfairness.The others were unsure or undecided.

Category 2 - Expectation and Fairness

(1) No knowledge of mathematical expectation.

These subjects were unable to answer Q3(a) correctly and would tend to reason: "A game can only be fair if each player has the same chance of winning" Two "non-gamblers" admitted to having no basis on which to make decisions of fairness.

Total: 21 Gamblers : 8 Non gamblers: 13

(2) Some intuitive knowledge of the use of expectation in determining fairness. These subjects answered questions 3 (b) and (c) correctly but were unable to answer all of the more complex questions 3 (d) - (f)

Total: 13 Gamblers :10 Non gamblers: 3

(3) A thorough knowledge of the basic concept of mathematical expectation as demonstrated by their responses to all parts of question 3

Total: 6 Gamblers : 5 Non gamblers. 1

The Null hypothesis

Ho: "There is no difference between the gamblers and the non-gamblers in their knowledge of mathematical expectation." was tested using a Chi-squared test of statistical significance and rejected at the 5% level.

Table 1

	Observed				Expected Under Ho			
	None	Some	Thorough		None	Some	Thorough	
Non-Gamblers	13	3	1	18	9.45	5.85	2.7	18
Gamblers	8	10	5	22	11.55	7.15	3.3	22
Total	21	13	6	40	21	13	6	40

Implications:

Category 1

Since the misuse of an heuristic to conclude unfairness was not common amongst either group we cannot compare groups. These misconceptions were not as frequent as is reported in the literature. Shaughnessey (1981). for example, found the misuse of availability to imply unfairness widespread even amongst college entrants Tversky and Kahneman (1982) noted that "misconceptions are not limited to naive subjects" (p.5) However Kapadai (1984) has questioned much of this research and suggests that some of the misconceptions may actually refer to misinterpretation

of the question. The results of this study which were obtained from a structured clinical interview rather than questionnaire or test items would seem to support Kapadia in this.

Category 2

The fact that the gamblers were significantly better at using expectation to determine fairness has a number of important implications.

First, the concept is not part of the regular school curriculum - they do not use the term "expectation" but construct what is essentially an equivalent procedure.

Second, since all of the gamblers were familiar with track betting, the use of "odds" in betting situations and the calculation of resulting payouts, it is hypothesised from the results of this study that this mathematical knowledge may be attributed to the prevalence of gambling within the social background of this group.

The fact that this ability did not relate to school achievement or gender would tend to give support to the hypothesis. As such, the knowledge may be considered as a form of "ethnomathematics" as defined by D'Ambrosio (1985):

> ..mathematics which is practised among identifiable cultural groups (whose) identity depends largely on focuses of interest and motivation. (p. 45)

This has implications for the classroom teacher As Clements (1988) says "It needs to be remembered that often in Australia there are unique factors influencing how children learn mathematics." (p.5)

With the concepts of fairness and expectation now specifically within the curriculum, the teacher must be aware of the knowledge that pupils bring with them to the classroom.

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WHO CAN BENEFIT FROM PEDAGOGICAL DEVICES AND WHEN

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Two computerized microworld environments are examined regarding their effectiveness in solving two-step word problems.
One of these environments is S.P.A., based on schematic analysis while the other microworld is A.P., based on operations and dimensional calculation.

We are witnessing growth in using computerized microworlds as learning environments for the teaching of mathematics.
This development was accompanied by questions related to the pedagogical value of such environments. It is characteristic of computerized microworlds as described by researchers in this field that they are instructional representations which:
"(1) represent the concept or idea to be acquired in a veridical, if simplified way; (2) be "transparent" to the learner (i.e. represent relationships in an easily apprehended form or decompose procedures into manageable units); (3) map well onto expert models of understanding skill" (Resnick, 1976)
Reusser (1991) suggests that such environments should "foster and encourage user's conscious and constructive efforts after meaning, i.e. their acts of understanding, problem solving, planning, and reflection." They can be used as tools for reasoning. "Pedagogically useful representational formats permit students to organize a task around salient properties and invariants of its deep structure... Among their most important dual function is thus to provide bridges from ordinary language, or the learners' everyday conceptions, models and intuitions of objects, to canonical scientific conceptualizations

and formalisms, as for example, mathematical notations.... Finally, externalized representations supply teachers and students with a language to talk about what is to be learned. Good representations give referential meaning not only to students' thinking, but also to the instructional dialogues between learners and teachers. Given all the above considerations there is still an empirical question: are there more or less effective environments?"

We should note that since acquaintance with any new pedagogical device is time consuming it should therefore be weighed according to its effectiveness.

We think that the effectiveness of the pedagogical device should be observed at the most difficult tasks. For other tasks a more direct approach might suffice. Moreover, the definition of a difficult task is related to the level of the population.

We have selected the domain of two-step word problems in arithmetic as a complex domain, notorious for student failure. We have examined its teaching in a special computerized environment: S.P.A. software (Hershkovitz, Nesher and Yerushalmi 1991) which provides a learning environment that requires schematic analysis of the problem, and supplies feedback related to the schematic representation.
This was compared to other software - A.P. (Schwartz 1987) which provides a learning environment for problem solving, but does not promote schematic analysis, and provides feedback related only to the dimensional analysis. The solution proccess involves detection of the given parameters in the problem and searching for the operations needed to solve it.

Two research questions were derived from the above consideration:

1. In which tasks is S.P.A. most instrumental?
2. Who benefits most from S.P.A.?

The variables in this study were as follows:

Independent Variables:

A. The type of instructional program, with two values (S.P.A. and A.P.).

B. The type of problems included in the study.
All word problems included in this study were chosen on the basis of a previous study (Nesher and Hershkovitz, in press, 1992). In that study 21 types of problems were given to about 2000 students in grades 3 to 6. All 21 problems were used in the instructional phase and five of these served as the criteria (test) problems indicating the difficulty continuum for 2-step word problems. Table I presents the percentage of students correctly solving each problem from the easiest to the hardest.

Table I: Success Percentage for the Criteria Problems

Problem	1	2	3	4	5
Success percentage	89	81	65	56	40

C. Student Levels:
Students were divided into two levels: high and low, where the high level students were those who obtained a score above 70% in an independent math test administered by the Ministry of Education. All the rest were considered low achievers.

Dependent Variables:
Level of success in each of the criteria problems.

The Population:
The students were 6th graders from a middle class school in a small town in Israel. Students in the two classes were randomly divided into S.P.A. (n=26) or AP (n=31) treatment groups. The mathematics achievement level of the two groups was measured by an independent

test administered by the Ministry of Education and no significant differences (t,56 = .49) were found.
Each treatment group was divided into another two subgroups: high achievers and low achievers.
Table II presents the population distribution.

Table II: Distribution of High and Low Achievers in the Two Programs:

Students \ Program	A.P.	S.P.A.
Low Achievers	14	13
High Achievers	17	13
Total	31	26

Procedure

The experiment consisted of two parts. The first involved tutoring and in the second, students were examined for criteria. The tutoring phase took place in two different classes, each learning with one of two computerized programs (S.P.A. or A.P.) at a frequency of twice a week for about four months. The tutoring included acquaintance with the computerized program and learning to solve word problems with the aid of the computer. Both groups worked on the same set of problems for the same length of time. Both were tutored by the same two teachers attending each lesson. Each student worked at his own pace and was examined individually at the end of the tutoring stage.
Two different tests were administered to the students. Both included the same type of problems chosen according to their difficulty level. One was administered on the computer as a natural continuation of the tutoring stage, and the second was administered as a pen and paper test. The pen and paper test consisted of slightly different contexts and smaller numbers.

Findings:

Tables III and IV present the success percentage for each problem in the criteria test for the entire population.

Success Percentage for the Entire Population

Table III: In computerized test

Problem	1	2	3	4	5
S.P.A.	92	85	77	92	84
A.P.	91	84	62	55	45

Table IV: In pen and paper test

Problem	1	2	3	4	5
S.P.A.	96	88	88	80	77
A.P.	87	87	87	68	34

From comparing tables III & IV to table I, we can see that students treated by S.P.A. benefited more than students treated by A.P. Moreover S.P.A. students were more successful, especially in problems 4 and 5 which were the most difficult. Taking problem 4 for example, 92% of the S.P.A. students solved it correctly compared with only 55% of the A.P. students. For each of the five criteria problems a Chi square test for independence between success with the problem and the type of treatment was performed. On the computerized test for Problems 1 to 3 (the easy problems) there was no association between these two variables. However, for problems 4 and 5 (the difficult problems) there was a significant association between success on the criteria test and the type of treatment (X^2,(1) = 8.06, p<.01; and X^2,(1) = 7.84, p<.01, for Problems 4 and 5, respectively). Similar results were obtained in the pen and paper test.

The above results for the difficult problems for the entire population are even more dramatic in reference to the low achievers.

Tables V and VI present the success percentage of low achievers on each of the five problems in both the computerized and pen and paper criteria tests.

Success Percentage of Low Achievers

Table V: In computerized test

Problem	1	2	3	4	5
S.P.A.	92	77	77	92	77
A.P.	80	66	33	36	21

Teble VI: In pen and paper test

Problem	1	2	3	4	5
S.P.A.	92	77	85	69	54
A.P.	80	73	73	53	13

As can be seen in Tables V and VI, low achiever students in the S.P.A. treatment succedded in solving all problems better than low achiever students in the A.P. treatment.
Again, the most significant effect is observed in the difficult problems. This effect is stronger than what was observed for the entire population. For example, problem 5 of the computerized test: 77% of the S.P.A. low achievers solved this problem successfully, while only 21% of the A.P. low achievers managed to solve it.

Tables VII and VIII present the same results for the high achievers.

Success Percentage of High Achievers

Table VII: In computerized test

Problem	1	2	3	4	5
S.P.A.	92	92	77	92	92
A.P.	100	100	88	70	64

Table VIII: In pen and paper test

Problem	1	2	3	4	5
S.P.A.	100	100	92	92	100
A.P.	94	100	100	82	53

As seen in Tables VII and VIII, the difference in success percentage between the S.P.A. treatment and the A.P. treatment, is much smaller for the high achievers and not significant, except for Problem 5 (the hardest) in the pen and paper criteria test.

Interactions between Type of Programs, Problem Difficulty and Student Level.

We can now summarize the overall effect of the two programs in the special conditions we hypothesized as significant, i.e., hard vs. easy word problems, and high vs. low achievers. Table IX presents the mean score on the hard and easy word problems and their interaction, according to programs (S.P.A. and A.P.), and student level. The range of possible scores is 0 to 2, as we have pulled together the two hardest and the two easiest problems.

Table IX: Mean Score on Easy and Hard Problems

by Programs and by Student Level

Program	Students	Easy Problems (1,2)	Hard Problems (4,5)
S.P.A.	Low	1.69	1.69
	High	1.85	1.85
A.P.	Low	1.5	0.57
	High	1.0	1.35

A 3-way variance analysis (program X problem difficulty x student level), yielded a significant main effect for all three variables ($F=10.47$, $p<.01$; $F=17,27$, $p<.01$; and, $F=9.68$, $p<0.1$, respectively). There was also an interaction effect for program X problem difficulty ($F=17.27$, $p<0.01$); and program X student level ($F=3.64$, $P<.05$).

Summary of Findings:

Question 1:

It can be seen that there was no significant difference in the level of success for the easy problems. For the hard probelems, however, students who worked with S.P.A. succeeded more in both criteria tests (the computerized and the pen and paper tests).

Question 2:

Low achievers working with S.P.A. succeeded more than low achievers working with A.P. This was most apparent and significant in the hard problems. As to high achievers, all succeeded in both programs to the same degree.

Discussion and Conclusions

The following explanations are suggested by the above findings:

We regard S.P.A. and A.P. as two computerized programs that represent different pedagogical approaches. The attempt in S.P.A. was to let students operate only within a 3-place relation, representing the mathematical additive and multiplicative schemes. These basic schemes can subsequently be composed for more complex problems. It was assumed that if the student is engaged in identifying schemes underlying given

problems, it will assist her to construct her own schematic view, and schemes in the given domain. The developers of S.P.A. regard the task of solving word problem, to be a task of constructing a scheme.

S.P.A., therefore, demands that in working on a word problem the solver is to use all the information given in the text including the question component that is necessary for determining the roles of each component within the given relation and what operation is to be performed. This bears very important consequences for S.P.A. which can provide complete feedback based on schematic analysis of the operation chosen by the student.

The other program, A.P., cannot provide such feedback. Although it monitors the dimensional calculation, it directs the students whether they are on the right or the wrong track.

It seems that future research should concentrate on the specifics of building a computerized microworld environment to achieve the greatest effectiveness.

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III-17

VALUE-LADEN, CONTEXT-BOUNDED AND OPEN-ENDED PROBLEM SOLVING IN MATHEMATICS EDUCATION

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In mathematics education, we have mainly depended upon the scientific context and emphasized upon the teaching of mathematical knowledge and skill. Recently, re-examination of learning mathematics in social context has been stressed. In connection with this tendency, the most important component of problem solving has been changing from real world to social context. Emphasizing social context instead of scientific context means the shift from content-oriented education to value-laden education. In this paper, I do propose that the use of value-laden, context-bounded and open-ended problem solving is effective for fostering and evaluating learners' value recognition through their mathematical activities.

1. INTRODUCTION

In 1992, the first Japanese space pilot, Mamoru Mori boarded the space shuttle "Endeavor" , and gave a lesson to Japanese children. I think this episode shows typically that teaching content as well as teaching method can be changed according to the environment of the lesson. Mathematical facts as well as scientific facts are recognized by learners who are the creators of mathematical culture in the classrooms which are microcosms of the culture.

Recently, under such a view of mathematical cognition, re-examination of learning mathematics in social context, has been frequently stressed. Schoenfeld (1989) has argued that mathematical behavior of learners are determined by the school mathematics they are being taught, by means of analyzing students' solutions of nonsensical but interesting problems. Balacheff (1986) has reported that students' conception of "polygon" and "diagonal" have a crucial effect on their solving of problem requiring the number of the diagonals of a polygon. These mathematical activities through problem solving can not be said completed mathematics but quite meaningful processes in which the mathematical truth would be approached, and the context of discovery would be characterized. Moreover, in these activities, it is implicitly suggested that the students have not yet reached the mathematical maturing level at which they can accept completed mathematics in the context of justification. On such a point of view, I think it is necessary to consider the process in which the mathematical truth would be approached in the context of discovery.

2. FROM REAL WORLD TO SOCIAL CONTEXT

Since the beginning of 1980's, problem solving has been emphasized in mathematics education. Many researchers have formulated problem solving processes variously. The following is one of them, quoted frequently, formulated by Krulik who was the editor of the well-known NCTM yearbook "Problem Solving in School Mathematics"(Krulik et al 1980) and characterized by placing real world and mathematical model as the two poles of the process.

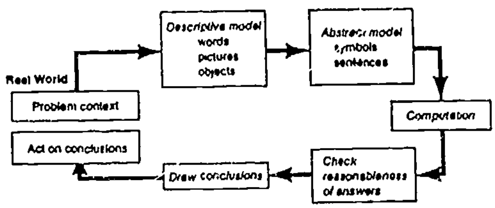
(1)Confronting the real-world problem

(2)Translating the problem into a suitable workable mathematical model

(3)Working out the solution

(4)Translating the solution to the mathematical model back into terms that reflect the original problem (Krulik 1977, p.649)

Though this formulation is brief and effective, it is necessary to consider that descriptive model such as word problems plays an important role in problem solving usually tackled by all learners in class. With such a point of view, "application cycle" proposed by DeVault (1981) is very noticeable, since it is suggestive of the connection between problem solving and representational mode.



By means of using this application cycle, I have identified three types of problem solving, and corresponded them to the three levels of the linguistical study i.e. syntax, semantics and pragmatics.(Iida 1985) Unreal problem solving which starts from the abstract model can be characterized by requiring appropriate computation and its result, therefore, this type is a routine problem solving and can be corresponded to the activity at syntactical level. Quasi-real problem solving which starts from the descriptive model can be characterized by requiring abstraction from word problems, pictures or objects to symbols or sentences, therefore this type can be corresponded to the activity at semantic level. Real problem solving which starts from the problem context can be characterized by requiring situational value judgement or decision-making. Because of such human intervention, this type can be corresponded to the activity at pragmatic level.

It is impossible to introduce real problem solving tackled by all learners in

mathematical classroom, since the degree of reality in real problem solving are different by individuals. In Japan, mainly at the first of the 1950's, we experienced such a dilemma concerned with reality in mathematical classroom under the core curriculum influenced by progressive education. Then, we have a tendency to make light of real problem solving. But, I think, we must not ignore learners' context in mathematical problem solving, if we consider that school context is an important aspect of learners' reality.

Such a tendency has been notable in American mathematics education for this about ten years since "Agenda"(NCTM 1980). Brown (1992) overviews:

> "The theme of problem solving thus begins to acquire social context--frequently referred to as "communication" in addition to a continued focus upon "real world" application."(p.1)

The most important component of problem solving has been changing from real world to social context. Taken this recognition into consideration at the pragmatic level of problem solving, solving of context-bounded problem rather than realized one would be highlighted.

3. CONTEXT-BOUNDED PROBLEM SOLVING

In order to acquire certain mathematical knowledge through semantic problem solving, it is necessary for users to have common code. But, the idea of common code for users is no more than a hypothetical one as a matter of convenience. In most of human problem solving, we usually do solve problems by depending too much upon the context by which the tense relation between user and code can be maintained. This context-bounded property is the most important one at the pragmatic level of problem solving. At the semantic level of problem solving, learners can understand the problems by means of reading or decoding. On the contrary, they must interpret the problem in order to understand them at the pragmatic level of problem solving.

The following statement (Brown 1992) is very noticeable:

> "Now while this kind of activity reflects a view of rationality there are reasonable alternatives which are left out. In particular such a model neglects to take into consideration both problematic aspects of the nature of mathematical thought and a realization that matters of education per se have a humanistic dimension that are not captured by a concept of problem solving as scientific. Thus in viewing problem solving as a component of *mathematics education*, the field is unfortunately driven by a limited view of the former (mathematics) rather than a robust view of the latter (education)." (p.8)

Emphasizing social context instead of scientific context means the shift from content-oriented education to value-laden education and ,as a result, means the removal of demarcation between formal mathematics and informal mathematics. Until now, mainly by the demarcation under the absolutist view of mathematics, we have tended to believe that mathematics is neutral and value-free. In addition, all that satisfy the absolutist value are admitted as bona fide mathematical knowledge, and anything that does not is rejected as inadmissible. Ernest (1991) has stated,

> "Once the rules of demarcation of the discipline are established in this way, then it can legitimately be claimed that mathematics is neutral and value-free. For in place of values there are rules which determine what is admissible. Preferences, choices, social implications and all other expressions of values are all eliminated by explicit and objective rules. In fact, the values lie behind the choice of the rules making them virtually unchallengeable. For by legitimating only the formal level of discourse as mathematics, it relegates the issue of values of a realm which is definitionally outside of mathematics." (p.260)

Moreover, from the viewpoint of social constructivism, he has proposed that school mathematics should reflect the following features of mathematics.

(1)Mathematics consists primarily of human mathematical problem posing and solving, an activity which is accessible to all.

(2)Mathematics is a part of human culture, and the mathematics of each culture serves its own unique purposes and its equally valuable.

(3)Mathematics is not neutral but laden with the values of its makers and their cultural contexts, and users and creators of mathematics have a responsibility to consider its effects on the social and natural worlds.(Ernest 1991,p.265)

Falsificationism as well as constructivism has gradually taken on as social-oriented character especially through the work of elaborating by I.Lakatos. Balacheff (1991) has stated as follows and positively analyzed students' problem solving behaviors in mathematics classroom from the viewpoint of the situational and social restrict.

> "The historical study presented by Lakatos shows the importance of the social dimension of this dialectic. This dimension plays also an essential role in the learning process taking place in the mathematics classroom." (p.89)

In new mathematical epistemology, there has been a tendency to emphasize on social-cultural context of mathematics in which value-judgement by the creators of mathematics would be a indispensable component.

4. VALUES IN CONTEXT-BOUNDED PROBLEM SOLVING

Under our content-oriented course of study, we have mainly depended upon the scientific context and emphasized upon mathematical knowledge and skill. I think, we must notice the epistemological suggestion shown in the preceding section. In Japanese mathematics education, the following view (Bishop 1991) would be considered to be natural and desirable.

"A mathematical education needs both pupil and environment to play a strong role. The pupil is not to be thought of as a *receptive* vessel for mathematical knowledge. On the contrary the pupil is the person who must decontextualise and reconstruct the mathematical knowledge from the contextualised situation offered in the classroom." (p.206)

But, the following criticism would also be considered to be reasonable.

"As one example, it is very possible for a teacher to choose three pupils' methods of solving a particular problem, to have the pupils write these on the board, and to encourage a discussion on which method is better and why. ---- This is not a natural thing for pupils to do, nor is it at present a typical thing for teachers to do, but, I firmly believe that class or group *reflection* following an activity offers an excellent for dealing specifically with values." (Bishop 1991, p.208)

If we emphasize upon social context as well we scientific context and upon value as ell as content, the aim of mathematical activities must be considered as humanistic and inquiring problem solving consistently to the last.

In most of mathematical activities tackled for acquiring certain knowledge or skill, we can find that the aim of them is transformed under the influence of the situational and social restrict. The aim in the first half of them would be focused on contextualized problem solving in which learners are encouraged the creation of various solutions. But, in the last half of them, the various solutions presented are examined in the classroom situation, so called *mathematical* solution is selected, it is fixed as a knowledge. It is natural that such a solution is gradually decontextualized, becomes meaningless and on the contrary is related to a mysterious knowledge which has general and arbitrary property. With regard to the phenomenon of transforming the activity of problem solving into that of acquiring knowledge, Balacheff (1986) has pointed out it as one of the effects of didactical contract as follows:

"One of the well known effects of this contract is to transform the problem of establishing the validity of a proposition (i.e. to "show that") into the problem of establishing that the pupil knows something about mathematics." (p.12)

I do worry that this transforming phenomenon might be an obstacle to foster the learners' critical thinking and value recognition. If so, it is a crucial problem. At least in Japan, I think the learners' value recognition through mathematics education have not fostered intensively. Nevertheless, problem solving is a fruitful teaching method by which we can foster the learners' value recognition of mathematics. I think, the important meaning of problem solving in mathematics education is to make learners be able to pose valuable problems from contexts and also recognize the values by solving of the problems. We must note that problem solving is not only a teaching method of knowledge and skill but also an inquiring activity from contexts.

The following is the six clusters of mathematical values especially of Western Mathematics enumerated by Bishop (1991):

1.rationalism 2. objectism 3. control 4. progress 5. openness 6. mystery

Though every value is important, the values closely related to contextualized problem solving are openness and mystery. Because solutions and propositions created in a context are surely open. Moreover this openness through the free thought would be indispensable for the creation of mathematics. On the other hand, mystery which is a complementary cluster of values to openness can be recognized as the solutions and propositions are decontextualized and become meaningless. We can sometimes appreciate the humanistic and mysterious mathematization through the free thought. In order to make learners recognize such values, I do propose context-bounded and open-ended problem solving as an example of humanistic activities.

5. CONTEXT-BOUNDED AND OPEN-ENDED PROBLEM SOLVING

"Open-ended Approach" has been proposed since the 1970's in Japan. It is very important that by this approach we pay attention to the openness which is a key point of the context of discovery in mathematical activities. Moreover, it is also important that we give thought of fostering and evaluating the affective aspects which tends to be neglected in Japanese mathematics education. In Japan, open-ended problems mean "the problem conditioned by being able to have many kinds of right answers".(Shimada 1977) And it have pointed out that open-ended problems can be classified into the following three types i.e. how to find, how to classify and how to measure. Though by the open-ended problems which have been proposed we can expect learners to think mathematically, such problems have little relation to their humanistic aspects. What I want to emphasize is value-laden and humanistic problem solving.

Let's think about the following problem.

"There is a bag of grass seed covers 4 square meter. How many bags would be needed to uniformly cover 30 square meter?"

This is an open-ended problem which implies the need of judging the validity of the solutions from the viewpoint of values. The critical discourse from the viewpoint of values would be the heart of solving this problem. If the learners answer that 8 bags are needed and grass seed covers 2 square meter (i.e. half of a bag) remains and do not propose any refutation, I cannot help judging that they seem to have solved problems mechanically in daily classroom and their teacher has not succeed in fostering the learners' critical thinking and encouraging their development of value recognition.

Such a problem which implies the need of interpreting the context and judging the values tends to be regarded as an inappropriate problem or noise in the content-oriented curriculum. For example, the teaching theme: "Division with remainder" shows the tendency typically. Because of the restriction by teaching content. learners rarely interpret the context, rarely recognize the value and never suspect mathematics. But, most of the problem in real world can not be solved by formal mathematics and they are usually context-bounded and open-ended. Brown (1984) has stated :

"The "real world" applications seem to be narrowly defined in terms of the scientific rather than the humanistic disciplines. In particular questions of value or ethics are essentially nonexistent. ----- I know of essentially no "real world" problems that one decides to engage in for which there is not embedded some value implications." (p.13)

Let's consider the following problem[1] which is context-bounded and open-ended.

"A party of thirty-one guests put up at a small hotel. The hotel has nine rooms. The each room has a capacity of five guests. How do you divide them in order to have them put up at the hotel? Find dividing ways as many as possible! "

this problem has relation to the following two ways of "division with remainder".

(a) 31 ÷ 5 = 6 ... 1 (b) 31 ÷ 9 = 3 ... 4

Nevertheless, the purpose of solving this open-ended problem is not acquiring the cognitive aspect of such computations but recognizing the affective aspect of the relation between mathematics and human beings.

Pupils in the fifth and sixth grades responded very variously. Especially, each of the following three responses was given by more than half of them.

(a)' 5 5 5 5 5 5 1 0 0 , (b)' 4 4 4 4 3 3 3 3 3

and (ab)' 4 4 4 4 4 4 4 3 0

(Each number shows the number of guests who put up at each room.)

The way of (a)'can be identified as the result of the division by grouping and as the convenient way on the side of the hotel. On the contrary, the way of (b)'can be identified as the result of the division by sharing and as the thankful way on the side of the guests. In addition, the way of (ab)'can be identified as the result of substituting 4 for the divisor of (a) or as the result of substituting 8 for the divisor of (b). After various responses were presented, an interesting discourse was conducted concerning the value judgement of the existence of vacant rooms, a room for one person, many rooms for the same number of guests and rooms for various number of guests. If various responses are not accepted, the learners can not recognize the values of the relation between mathematics and human beings.

6.BY WAY OF CONCLUSION

The important meaning of problem solving in mathematics education is to make learners be able to pose valuable problems from contexts, and also recognize the values by solving of the problems. From the linguistical point of view, the pragmatic level of problem solving has close relation to the pupils' learning contexts in classroom. The use of context-bounded and open-ended problem solving has proved to be effective for fostering and evaluating learners' value recognition through their mathematical activities.

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FOOTNOTE

1. I am grateful to Prof. Yamashita A., Prof. Hashimoto Y. and their colleagues to quote this problem.

EFFECTS OF EXPLICIT TEACHING OF PROBLEM SOLVING STRATEGIES: AN EXPLORATORY STUDY

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This study investigated the effects of explicit teaching of problem solving strategies for improving the problem solving performances of sixth graders, in consideration of their achievement levels. Two treatments were compared: one emphasized the acquisition and application of strategies explicitly, and the other emphasized practice in solving problems. Problem solving tests were performed before, during and after instruction.
The following was found: 1)The expiicit teaching of strategy was effective when students received application lessons in addition to acquisition lessons for strategies. 2)All kinds of strategies need not necessarily be taught explicitly to all achievement levels.

Many Japanese students have difficulty with problem solving in mathematics. When problems are given, many students tend to simply choose numbers and select an operation mechanically. Many problems in school mathematics can be solved, even if not effectively, by using strategies. But problem solving strategies are usually not taught explicitly in school mathematics. Many Japanese teachers think that problem solving strategies develop incidentally as students solve many problems.

Purpose of the Study

The purpose of this exploratory study was to examine the effects of the explicit teaching of strategies on the achievement levels, by comparing the performance of students who received instruction focusing on problem solving strategies with the performance of students who practiced solving problems with the explanation of the solution not foucusing on strategies.

METHOD

Subjects

The subjects were 60 students of two sixth-grade classes. One class of 30 students was treated as the strategy group and another class of 30 students was treated as the practice group. On the basis of a mathematical achievement test, each class was divided into three subgroups: High, Average and Low. Each subgroup consisted of nine students.

Table 1 gives the results of the mathematical achievement test for the strategy group and practice group. An ANOVA revealed the achievement level only had a significant effect ($F=119.08$, $p<.01$).

Table 1

Means, and Standard Deviations for Achievement Test

Achievement level		High(N=9)	Average(N=9)	Low(N=9)
Strategy Gr.	Mean	57.3	48.8	40.4
	SD	3.77	2.39	2.79
Practice Gr.	Mean	55.4	48.5	39.7
	SD	2.27	2.32	3.92

Procedures

The strategy group was taught five problem solving strategies: Trial and Error, Draw a Figure, Make a Table, Look for a Pattern, Consider Special Case. This group had twelve teaching lessons of forty five minutes each over a period of eight weeks. Nine non-routine problems were used in the first session to acquaint students with the strategies(six lessons) and to teach the application of strategies in the second session(six lessons). The first teaching session emphasized acquisition of the five strategies. During the first session students were encouraged to name strategies they used, based on characteristics they found in their process. The second teaching session emphasized selection of appropriate strategies and applications. The teacher asked the students which strategies were better or best to solve each problem.

For the practice group, the focus of the treatment was on providing students with practice in solving problems and explanation of the typical solutions. They had nine teaching lessons. During each lesson two problems were used, one of them was the same problem used for the strategy group, and the teacher explained the solution after they worked individually. The practice group solved a total of eighteen non-routine problems over a period of eight weeks.

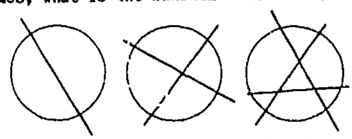
Problem Solving Test

Before the first teaching session, the first problem solving test(pretest) of five problems was given to both groups. After the first session, the second problem solving test was given, and the third problem solving test(posttest) was given after the second session.

Following five problems(45 minutes for problems 1-3, 30 minutes for problem 4-5)were included on the third problem solving test:

Problem 1 When Taro opened his book, the product of the two page numbers was 600. Which pages did he open his book to?

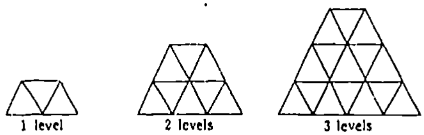
Problem 2 When one piece of a Japanese pancake is cut once, the piece will be divided into two parts. When the piece is cut twice, the maximum number of pieces will be four. When the piece is cut three times, the maximum number of pieces will be seven. When the piece is cut five times, what is the maximum number of pieces into which the pancake can divided?



Problem 3 Masao is eight years old and his father is thirty eight years old. How many years from now the father will be twice as old as Masao?

Problem 4 Two electric light poles are sixty meters apart. How many trees could be planted at five meters intervals between these poles?

Problem 5 Using equilateral triangles of 1 cm on each side, you can construct a trapezoid as below:



When you construct a trapezoid of seven levels, what is the perimeter of the trapezoid?

Scoring of the Test

Each problem solved in the problem solving test was scored by awarding 2, 1, or 0 points in each, the planning a solution and getting an answer phase. The scoring schema is shown in Figure 1 and was adapted from Charles, et al.,(1987).

Planning a Solution	2: Plan could have led to a correct solution if implemented properly
	1: Partially correct plan or inefficient plan
	0: No attempt, or totally inappropriate plan
Getting an Answer	2: Correct answer
	1: Computational error, careless mistake
	0: No answer, or wrong answer

Scoring Schema

Figure 1

RESULTS

Total Score on the Three problem Solving Tests

The total score is the sum of the points for "Planning" and "Getting an Answer". Table 2, Table 3 and Table 4 shows the mean scores for the achievement levels on the problem solving tests. An ANOVA of total scores on the three problem solving tests was calculated to compare the performance of the two groups.

There was a significant effect for the achievement level only in the first test($F=12.04, p<.01$)and second test($F=5.21, p<.01$). But the achievement level($F=16.96, p<.01$) and teaching treatment($F=20.38, p<.01$) had a significant effect on the third test. This result from the third test suggested that each achievement level in the strategy groups showed a significantly better performance on the problem solving test than the corresponding level in the practice groups.

Table 2

Means, and Standard Deviations for the First Test

Achievement level		High(N=9)	Average(N=9)	Low(N=9)
Strategy Gr.	Mean	11.22	8.00	4.66
	SD	4.02	4.13	2.31
Practice Gr.	Mean	10.22	6.77	4.44
	SD	3.39	3.04	4.09

Table 3

Means, and Standard Deviations for the Second Test

Achievement level		High(N=9)	Average(N=9)	Low(N=9)
Strategy Gr.	Mean	12.11	10.11	8.66
	SD	1.29	3.09	5.19
Practice Gr.	Mean	12.22	9.44	6.66
	SD	4.61	4.00	3.77

Table 4

Means, and Standard Deviations for the Third Test

Achievement level		High(N=9)	Average(N=9)	Low(N=9)
Strategy Gr.	Mean	17.44	15.55	11.33
	SD	1.17	1.77	5.07
Practice Gr.	Mean	13.77	11.11	5.88
	SD	2.24	4.25	4.35

Strategies used on the third Test

Table 5 lists the strategies used in each of the third test.

Table 5

Strategies Used for Each Problem on the Third Test

		Problem 1		Problem 2		Problem 3		Problem 4		Problem 5	
Strategy	Level	S	P	S	P	S	P	S	P	S	P
Trial	H	9(9)	7(6)	5(0)	7(1)	2(1)	3(3)				
&	A	9(9)	8(7)	8(0)	8(0)	2(1)					
Error	L	8(6)	6(4)	8(0)	8(0)	1(0)	3(1)				
Draw	H							6(6)	7(7)		2(1)
a	A							8(5)	6(5)	1(1)	5(2)
Figure	L							8(6)	1(0)		3(1)
Make	H					4(4)	5(4)				
a	A					6(6)	3(3)				
Table	L					4(1)	1(1)				
Look	H		1(0)			2(2)				9(9)	6(8)
for a	A					1(1)				7(7)	3(3)
Pattern	L									8(7)	3(2)
Consider	H			4(4)	1(0)			2(2)			
Special	A			1(0)	1(1)						
Case	L			1(0)				1(0)			

Note: The numbers in parenthesis indicate the number of students who got the correct answer. S and P indicate Strategy Gr. and Practice Gr.

While the range of strategies was similar for each group, strategies were used more frequently by students of the strategy group. The strategy group used strategies on 91% of the problems; the practice group, on 75% of the problems.

"Trial & Error" strategy was used frequently to solve problem 1, 2, 3. In problem 1, all levels of each group selected this strategy more often and almost all succeeded. In problem 2, average and low levels of the strategy group and all levels of the practice group selected this strategy more often. In contrast with problem 1, almost all of the students failed, because this strategy was inefficient in solving problem 2. Four students of the high level of the strategy group selected the "Consider Special Case" strategy and they found a pattern. It seems that "Consider Special Case" strategy is quite difficult to use for average and low levels of strategy group.

The "Look for a Pattern" strategy was used often in problem 5 by students in the strategy group and in the high level of the practice group. This strategy is more efficient than "Draw a Figure" strategy on problem 5. Four students of the average level of the practice group selected "Draw a Figure" strategy while this level of the stratgy group selected "Look for a Pattern" strategy. Six of twelve students who selected "Draw a Figure" strategy in this problem failed.

In problem 4, most students except those in the low level of the practice group selected "Draw a Figure" strategy and most succeeded. In problem 3, most of high level students of both groups and the average level students of the strategy group used "Make a Table" or "Trial & Error" strategies. Low level students of both groups failed more than the high and average level students of both groups.

DISCUSSION

The result of this study confirm that teaching a set of strategies to students explicitly improves their problem solving ability, which is the finding of previous studies(Ishida,1990; Charls & Lester,1984; Lee,1982). This result also suggests the importance of the application session(Krulik & Rudnick,1980). During the application session students were askedto choose an appropriate strategy. Even if they could not select a good strategy for a given problem, they had a chanse to learn a better strategy

used by other student and to learn the reason why it was suitable for the problem, through discussion. Choosing an appropriate strategy, those which were named during the first session, and appreciating strategy used by comparing several solutions seem to be important points included in the application session. In fact, on the second test most of errors were due to the inappropriate choice of strategies.

The frequent use of "Look for a Pattern" by each level of strategy group indicates that explicit teaching of strategy can increase the selection of sophistcated strategy in solving problems. How students select and use strategies depends on their achievement levels. For example, in the strategy group, high level students could solve problem 2 by "Consider Special Case" strategy, but others not. High level students of the practice group could solve problem 5 by "Look for a Pattern", but the average level could not.

By comparing the strategies used by the strategy group with the strategies used by the practice group, it appears that the kind of strategy which needs to be taught explicitly is different according to the achievement level. That is, some strategies can be learned without explicit teaching of problem solving strategies like in the practice group. From table 5, the following tentative hypothesis was found: "Look for a Pattern" strategy and "Consider Special Case" strategy need to be taught explicitly for all levels . "Make a Table" strategy needs to be taught explicitly for average and low levels. "Draw a Figure" strategy needs to be taught explicitly for low level.

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MATHEMATICAL PROBLEM POSING: THE INFLUENCE OF TASK FORMATS, MATHEMATICS KNOWLEDGE, AND CREATIVE THINKING

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The purpose of this study was to examine the influence of task format, mathematics knowledge, and creative thinking on a type of mathematical problem posing, in which a sequence of problems were formulated from a given situation described in a story form. The Test of Arithmetic Problem Posing (TAPP) was developed to quantitatively measure posed problems. Results of ANOVA tests indicated differences in the production of mathematics problems and problems with sufficient data for subjects using the task format containing numerical information and for subjects with high mathematics knowledge; the latter group also produced more problems that had plausible initial states. Overall, there was no observed difference between the high and low creative thinking groups nor any interaction between task format with each of the other variables.

Statement of the Problem

Recent reports dealing with reform in mathematics education have advocated an increased role for problem-posing activities in the mathematics classroom. For example, the *Curriculum and Evaluation Standards for School Mathematics* (NCTM, 1989) advocated the inclusion of activities emphasizing student-generated problems in addition to having students solve already formulated problems Before the implementation there is a need for research on factors affecting problem posing and evaluation methods on problem posing. In this study, mathematical problem posing refers to the formulation of a sequence of mathematical problems from a given situation.

Although problem posing has been viewed by Pólya (1954), Brown and Walter (1983) and Dillon (1988) as an inseparable part of problem solving, it has received far less research attention and systematic study (Kilpatrick, 1987). There have been some studies of mathematical problem posing in Japan and in the United States. In Japan, Hashimoto (1987) had extensively studied the impact of including problem posing in elementary mathematics instruction. In the U.S , Silver and Mamona (1989) studied the problem posing of in-service middle school teachers and categorized the posed problems. However, the research conducted to date has not considered the impact of inter-subject and inter-task difference on problem posing. In addition, the investigation on the problem-posing ability of prospective elementary school teachers is important because of their vital role in the implementation of posing activities in their classroom This study involves a cognitive evaluation of the problem posing of prospective school teachers with respect to two task formats and simultaneously explores the relationship of problem posing to subjects' mathematical knowledge and creative thinking.

Theoretical/Conceptual Framework

Since problem posing is an inseparable part of problem solving the research background on problem solving helps to bring suggestions for systematic study on problem posing. This study borrowed from the design of problem solving studies in the way that it also aimed at end products. While prior solving studies looked at solutions the current posing study analyzed posed problems. In addition, the classification on variables in prior solving studies (Kilpatrick, 1978) suggested that task and subject are two promising avenues for investigations of posing.

Task variable in problem solving was extensively explored and representative pieces of work were compiled (Goldin, & McClintock, 1979). In particular, the amount of numerical information content was being manipulated as a task variable (Goldin & McClintock, 1979). Numerical information content can also be treated as a task variable in a study on problem posing. The presence or absence of information content may constitute an effect on problem posing, as Guilford (1967) said, "[o]n the need for a good supply of information for successful creative production, there is practically unanimous agreement" (p 319).

Subject variable in problem solving was also heavily researched upon and mathematics knowledge has been a consistent variable among the many subject variables for affecting problem-solving performance The importance of mathematics knowledge in problem posing was discussed by great scientists "[T]he formulation of a problem is often more essential than its solution, which may be merely a matter of mathematical or experimental skill. To raise new questions, new possibilities, to regard old questions from a new angle, requires creative imagination and marks real advance in science" (Einstein & Infeld, 1938, p 92). In the discussion Einstein and Infeld also purports the importance of creativity, another subject variable. It seems reasonable to explore both subject variables in a study on problem posing.

The exploration on problem posing with the above three variables cannot be done without an instrument that measures problem posing. The Test of Arithmetic Problem Posing (TAPP) was developed to quantitatively measure the posed problems according to four important aspects prompted from the problem solving literature. The feasibility as well as interrater reliability of the instrument was checked in two pilot studies.

Methodology

The Instrument: Test of Arithmetic Problem Posing (TAPP). TAPP consisted of two problem situations the House problem (Problem 1) and, the Pool problem (Problem 2). The problem situations were modified tasks from the instrument on creativity used by Getzels and Jackson (1962). Each problem situation was presented in two formats: With numerical information content (Format A), and Without

numerical information content (Format B). Each format B is identical to the corresponding format A except that the pieces of numerical information are purposely taken out. The two problem situations in the same format are presented in a paragraph of equal number of words. Figure 1 shows one item of TAPP (Problem 1, Format A).

Figure 1

Example Item In Test of Arithmetic Problem Posing

Mr. Smith decided to purchase a house whose cost was $150,000. He made a down payment of $50,000 and agreed to pay the rest with monthly payments. Each monthly payment included a portion of the principal, an interest charge computed at the rate of 8 percent per year, plus a charge for insurance which amounts to $1295 per year. Mr Smith found by talking to the former owner that the average cost to heat the house was $200 per month. Later Mr. Smith added insulation to the house which cost him $4000, but which the contractor who installed it guaranteed would reduce his heating costs by 15 percent.

TIME: 20 minutes
INSTRUCTIONS: Consider possible combinations of the pieces of information given and pose mathematical problems involving the purchase and operation of the house. Do not ask questions like "Where is the house?" because this is not a mathematical problem.

- Set up as many problems as you can think of.

Think of problems with a variety of difficulty levels. Do not solve them.

- Set up a variety of problems rather than many problems of the same kind.
- Include also unusual problems that your peers might not be able to create.
- You can change the given information and/or supply more information.

When you do so, note the changes in the box with the problem to which they apply.

- Write only one problem in each box.

If you think of more problems than the number of boxes provided, write the others on the back of the sheet.

According to TAPP, the evaluation of posed problems includes four independent aspects: problem type (Rescher, 1982; Krutetskii, 1976); logical/linguistic structure (Caldwell & Goldin, 1987; Mayer, 1985), characteristics of solution structure (Newell & Simon, 1972); and, semantic structure (Marshall, Barthuli, Brewer, & Rose, 1989). Specific codes in each category will be given under the section on data-coding

Procedure and Task Administration. A pilot study was done in Summer 1991 to examine the appropriateness of tasks for prospective elementary teachers and the feasibility of using the proposed evaluation scheme on problem-posing products. Results of the task validation and the check of inter-rater reliability confirmed the feasibility and appropriateness; some minor revisions of data coding and task format were made. Another pilot was done in Fall 1991 on five items and two were included in this actual study, which took place in Spring 1992. The subjects were 49 students who enrolled in two sessions of

"Mathematics for Elementary Teachers" offered by an American university for prospective elementary school teachers. Each subject did three tasks in a whole class setting on three separate days

Subjects took a pre-test on mathematics during the first meeting of the course. The pre-test was the sample test of the mathematics component of the Pre-Professional Skills Test (PPST) developed especially for prospective elementary teachers and consisting of 40 multiple choice items and the answer key was given (Educational Testing Service, 1986). The scores represented the subjects' mathematical knowledge.

Subjects then took TAPP after the mid-term examination of the course. The test required them to pose a sequence of problems that can be attached to the two given problem situations in story forms. Subjects were instructed to pose as many problems as they could; as many kinds of problems as they could, and to include problems that they thought their colleagues might not be able to create. Subjects were instructed that they could supply additional information or change the given information. They were also told not to solve the problems. The investigator read the instructions out loud to the class and answered any query concerning the task before distributing the booklets. Written instructions were attached to each problem situation so that subjects could refer to them again as needed. The four combinations of both problem situations were randomly assigned. The four sets, in order, for random sampling were: 1A then 2B, 1B then 2A, 2A then 1B, and 2B then 1A. The first and the second format in each set were color-coded for the experimenter to assure that the subjects worked in the assigned order. The time allowed was 20 minutes for each format.

Subjects did the third task, on creative thinking one week after they did the second. The verbal component of the Torrance Test of Creative Thinking (TTCT) was purchased from Scholastic Testing Services. It used six word-based exercises to assess fluency, flexibility, and originality. The time for administration was 45 minutes. The scores on TTCT represented subjects' level of creative thinking

Data Coding and Analysis. Scores of the pre-test (Pre-Professional Skills Test) were obtained from the course instructors and the scoring on TTCT were done by Scholastic Testing Services. The experimenter did rating on problem-posing. Problem-posing responses were coded according to the four aspects specified in TAPP. The first aspect is on problem type which includes Content: Math/Non-Math,

Feasiblity of initial state: Plausible/Implausible[1]; and, Data required in solving Sufficient/Insufficient/Extraneous. The second aspect is on logical/linguistic structure of question or problem statement[2] which includes Assignment, Relational, and Conditional. The third aspect is on characteristics of solution structure: number of objects, operators, and steps involved in their solutions The fourth aspect is on the semantic structures: CHANGE/GROUP/COMPARE/RESTATE/VARY. The inter-rater reliability was checked by having the additional rater did 20% of all responses.

Scores from PPST-mathematics (Mathematical Knowledge) and TTCT-verbal (Creative Thinking) were arranged in thirds. The scores that appear in the middle one-third were removed and the rest were entered in blocks of High or Low accordingly. The scores on positive measures[3] of problem posing, together with scores on mathematical knowledge (PPST) and creative thinking (TTCT) were recorded Differences and interactions of means on the measures were tested by t-tests and two-way ANOVA.

Results and Discussions

Regarding task format, the difference between task formats was statistically significant for two categories Math ($p=0.0031$) and, Just Sufficient ($p=0.0012$). Though not statistically significant ($p>0.005$), the means of all but one positive measures on problem posing for task format A (with numerical information content) are higher than the respective means for task format B (without numerical information content) This results regarding task format indicated that more positive response category (e.g math rather than non-math) were produced when subjects were using task with numerical information in problem posing The higher means for task format A over task format B suggested a preference of task containing numerical information by all subjects.

There was also an investigation into the mathematics knowledge variable. The difference between the high and low mathematics knowledge groups was statistically significant for three categories: Math ($p=0.0019$); Plausible ($p=0.0034$), and Just Sufficient ($p=0.0011$). Though not statistically significant, the means of all other measures on problem posing of the high mathematics knowledge group were higher than those of low mathematics knowledge group. results in relation to the mathematics knowledge variable showed that more positive response category (e.g multi-step problems rather than single step) were produced by subjects with high mathematics knowledge.

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1 An implausible problem consists of invalid pre-suppositions (Rescher, 1982) that make an initial state of problem impossible. Empirical example: "What is the percentage of girls to boys in class?"

2 Empirical examples: Assignment--"How much does Smith still owe?", Relational--"How much over $150, 000 does he end up paying for the house?" and Conditional--" If the insurance is not included how much is he paying?"

3 Positive measures are the measures on problem posing that are generally accepted to be more mathematically important. There are five positive measures altogether. They are math problems, problems with plausible initial state, problems with sufficient data required in solving, problems presented in conditional forms, and multi-step problems.

The relationship with creative thinking was explored on the same measures in a similar manner. Results of the statistical tests concluded no significant difference on all positive measures between groups of high or low creative thinking. The ANOVA tests were conducted also to determine, if any, relationships of task format with each of the other two variables. Results of ANOVA tests indicate that there was no observed significant interaction of task format with either mathematics knowledge or creative thinking. However, the high mathematics knowledge group who did a format B produced more responses in the math, plausible, and just sufficient categories than the low mathematics knowledge group who did a format A, even though in general the mean number of responses in these categories for format B was lower than the corresponding means for format A.

Subsequent analyses involving correlation tests suggest interesting relationships of subscores on creative thinking and productivity of products and the relationship of two other positive measures on problem-posing. First, subjects with a higher score in fluency[4] tended to produce more problem-posing products. Second, subjects who scored high in flexibility[5] did not necessarily scored high in the multiplicity of response categories in problem posing (Logical/Linguistic Structures; Semantic Structures). Third, subjects who scored high in the multiplicity of response categories concerning Logical/Linguistic Structures also scored high in the multiplicity of response categories concerning Semantic Structure. The results of correlation tests indicated that the requirements of performing TTCT, the test that measure general creative thinking, were different from the requirements of performing TAPP, the test of problem posing.

The counter-balanced design on task format yielded two interesting findings regarding order effect. First, differences between formats or groups were generally lower than the corresponding differences in the second posing activity than in the first; the means of the lower groups increased while those for the high mathematics knowledge group stayed about the same. Second, subjects who did a format A in the first posing activity produced fewer responses in math, plausible, and just sufficient categories than when they did a format B during the second posing activity. On the other hand, those who did a format B in the first posing activity produced more responses in math, plausible, and just sufficient categories than when they did a format A during the second posing activity. These two findings suggest some "learning" of problem posing during the first activity; and more so for the low mathematics knowledge group.

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4 Fluency is a subscore of TTCT(verbal) which refers to the number of responses (Torrance, 1966).
5 Flexibility is a subscore of TTCT(verbal which refers to the number of response categories (Torrance, 1966)

A Summary of Results. The aim of the quantitative analyses on problem-posing was to find relationships of positive measures of problem-posing products with three variables: task format, mathematics knowledge, and creative thinking. Overall, the results of statistical tests on positive measures suggest a relationship of products with task format and mathematics knowledge but not with creative thinking. In addition, there was no observed interaction between task format and mathematics knowledge nor between task format and creative thinking. Subsequent analyses involving correlation tests suggest interesting relationships of subscores on creative thinking and productivity of products and the relationship of two other positive measures on problem-posing.

Conclusions and Implications

Prospective elementary school teachers were likely to be inexperienced in mathematical problem posing. If teachers, whether high or low in mathematics knowledge, were more "successful" in problem posing by using a more structured task format that containing numerical information this finding would suggest the use of these tasks at early stages in instruction on problem posing. Coincidentally, researchers in Japan expressed difficulties in early instructional use of open-ended tasks[6] for in-class problem posing and switched to "developmental problems" which were more structured to open-ended tasks comparatively (Nohda, 1984).

Results of ANOVA tests indicated differences in the production of mathematics problems and problems with sufficient data for subjects using the task format containing numerical information and for subjects with high mathematics knowledge; the latter group also produced more problems that had plausible initial states. Overall, there was no observed difference between the high and low creative thinking groups nor any interaction between task format with each of the other variables. The findings here suggested a higher influence of task format and mathematics knowledge over creative thinking.

This study addresses one aspect of task (task format) and its interaction with two subject characteristics (mathematical knowledge, creative thinking) in relation to mathematical problem posing. It yields an instrument for the evaluation of arithmetic problem posing and simultaneously renders several suggestions on task preparation for a problem-posing pedagogy in the mathematics curriculum.

[6] Open-ended tasks are open in that students can formulate their problems according to own perspective

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ON ANALYSING PROBLEM POSING

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The paper will describe a new theoretical framework to analyze results of problem posing activity according to the independence from the original problem and to the suitability for a satisfying mathematical solution. Fieldwork is undertaken and the data are analysed under this framework. A greater dependence on the original problem than desired was shown.

Much concern has been shown in presenting Mathematics as a creative activity. This encourages independent thought, and, it is hoped, lessens Mathematical phobia. Problem solving may be considered as a first step towards this aim. A theoretic framework for problem solving was developed in the 1950's by George Polya. Problem solving has a creative element but not much freedom. Creativity needs freedom. To give more freedom the only natural course is to allow students some room to form their own questions. Investigations (as described in [3]) permits this, but the freedom is restricted by having to keep to a given mathematical framework. Recently another approach has been developed to maximise freedom; "Open Problems". Open problems are basic vehicles for the open approach style of teaching as described in Nohda (1985,1987). (A study of the nature of Open Problems can be found in [10]). A problem might be open by inviting the generation of other problems; the resultant activity is termed problem posing. Problem posing has the restrictive influence of the original problem, but except of this the activity is free of any bounds. Brown and Walter have given an exposition on strategies for problem posing in their seminal work [1]. Previous research (e.g.[9]) has considered aspects of problem posing. In this paper we concentrate on forming a new theoretical framework which will enable us to comment (amongst other things) on how students use the extra freedom offered by problem posing.

The usual form of Problem Posing tasks

A specific mathematical problem (termed the prompt) is given to the subjects and they are encouraged to try to solve it on paper. Afterwards the subjects are invited to write down any questions that they wish guided by "free-association" with the prompt.

Note 1 Some kinds of investigation involve problem posing but the stress is on the fact-finding. For a problem posing task the situation is reversed; the stress is on the question, not on the answer. (This difference might seem artificial to a mature mathematician, as he would recognise that problem posing is dependent very much on facts found by answering other problems).

Note 2 The prompt ideally is only an initial base (or indeed a catalyst) for the posing; in the end we hope the posing is effectively independent of the prompt. However in practice the prompt may be an undue influence on the posing. This concern will motivate the first categorisation of the problems posed.

<u>Note 3</u> For this paper, solution of the prompt itself are of secondary interest (though they may inspire consequent problem posing). No analysis of the solutions will be given here.

The Prompt

The prompt should be designed such that there should be a recognisable broad "context" or "environment" or "topic" where the prompt belongs. Let us keep to the word <u>context</u>. The choice of the context of a prompt may be very subjective, but it should act as a general background not involving explicit parameters or assumptions. Further to the context, we want to identify <u>the structure</u> and <u>the question</u> of the prompt. The structure is a specialised framework which describes parameters and assumptions.

<u>An example</u> (which is the task used for our fieldwork described later).

<u>The task</u>: The Bank of Cyprus decided to mint new coins of different values (we do not know the values). If someone has only one coin of each denomination then the total value would be 100; also if someone picks up the proper coins (s)he is able to pay the cost of anything up to 100 (without needing change). The Bank wants to mint as few coins as possible. Try to find a solution. Then write down any other questions or problems that occur to you.

<u>The Context</u>: Simple transactions with coins.

<u>The Structure</u>: The different denominations of the coins are $a_1, \ldots, a_r$. We have the following conditions

(a) $\sum_{i=1}^{r} a_i = 100$

(b) $\forall$ natural numbers k less than 100 $\exists$ a subset of $(a_1, \ldots, a_r)$ such that the summation of the elements is k.

(c) $\nexists$ a set of natural integers satisfying (a) and (b) with fewer than r elements.

<u>The Question</u>: Find values of $a_1, \ldots, a_r$ (satisfying the above cond^{ns}.)

First Categorisation of Problems Posed : "Deviation from the Prompt"

The prompt has three components, context, structure and question. For each problem P posed we can ask whether the context (the first component), the structure (the second) and the question (the third) are the same as (or "parallel" to) the prompt. Hence we may associate to P an ordered triple for which each component has value Y (yes) or N (no). In this way we have a very natural categorisation into 8 classes. A ninth class (termed category 0) is needed for questions that cannot be otherwise placed : we term these as non well-structured. They constitute

(i) non-questions (without any obvious implicit problems involved)

(ii) questions with no realistic hope of being modelled in terms of mathematics.

(iii) questions for which the use of language or the content makes the meaning of the question obscure.

We make notes on some categories, and then remark on the use of the categorisation. A problem in category (Y,Y,Y) is said to be essentially the same as the prompt; the only difference is that some of the values of the parameters may have changed. The category (N,N,Y) will always be empty as it is not sensible to have the same question if both the context and the structure differ. However we do allow, for example, category (N,Y,Y) because we can imagine a parallel structure and a parallel question in a different context.

Having N in the first component means that the context has been broken, N in the second that the structure is changed, and N in the third that the question is changed. We regard each N as a deviation from the prompt, where an N in the first component is of the greatest degree of deviation and so on. Hence we have a hierarchy (Y,Y,Y), (Y,Y,N), (Y,N,Y), (Y,N,N), (N,Y,Y), (N,Y,N), (N,N,N) where we regard (N,N,N) as the category most deviant from the prompt. Any analysis using this categorisation gives a good feel how dependent the work of the subjects was of the separate three components of the prompt, and the hierarchy gives a less reliable but still useful overall picture.

First categorisation (Cat. 1): How does it fit with the categorisation in [9]?

In [9], a categorisation was designed to analyse responses to a particular problem posing task. This categorisation was developed on the same lines as Cat. 1 "deviation from the prompt". In [9] there are five classes all of which correspond well with classes in Cat. 1.

Special Goals and Initial Conditions	(Y,Y,Y)
General Goals	(Y,Y,N)
Implicit Assumptions	(Y,N,Y)
Others	O

The categorisation in [9] did not allow for problems posed to break the context (called in that paper "the task environment), and it did not allow for (Y,N,N). A longer paper [2] by the same authors anticipated the need of an extra class which would correspond to (Y,N,N); it just happened that no problems posed for that particular experiment fall into that category, so it was ignored.

We see that two classes in [9] correspond to (Y,Y,Y). In fact the class "Special Goal" is perhaps best regarded as not being relevant to this framework; in [9] the particular prompt is itself an investigation of an empirical type; the class

"Special Goal" comprises conjectures (not problems posed) about the overall solution of the prompt. We contend that Cat.1 is easier to use and it is more general in application that the one found in (9).

A Need for more information?

Certainly. We should like to know

(i) the originality

(ii) the suitability for a satisfying mathematical solution (SMS)

and (iii) the style of presentation

of each question posed. The originality has been at least partly covered by the categorisation of "deviation from the prompt"; a question deviating from the prompt may not be original because the subject has prior acquaintance with that question, but possibly it could be said to be original in linking it (by "free-association") with the prompt. We will not say anything more about this. Also we will not attempt to form a theoretical framework to analyze (iii); we feel an overall impression is most appropriate. However (ii) is amenable to analysis; this leads us to a second categorisation:

Second Categorisation of Problems Posed: "Suitability for SMS"

Category A - the same as category (Y,Y,Y)[1]

Category B - the same as category O. Termed "non well structured"

Category C - questions which need modelling requiring extra data or further assumptions to make the question amenable to mathematical argument. Term such questions as "incomplete"

Category D - questions whose answers require only a substantially lower level of mathematical attainment than that expected of the subjects. Term such questions as "elementary"

Category E - questions whose answers require a substantially higher level of mathematical attainment than that expected of the subjects. Term such questions as "intransigent"

Category F - questions which are

(i) not in category A

(ii) not elementary

(iii) may be answered using mathematical skills which lie within the attainment expected of the subjects. Term such questions as "workable"

Commentary on the Second Categorisation

This is a new categorisation of those questions which are well-structured and not essentially the same as the prompt. (This commentary then is directed only to

[1] We wish to "isolate" problems of type (Y,Y,Y) because their "suitability for SMS" is automatically bound with the one of the prompt. (We assume that the prompt is workable!). We feel that only questions with some independence of the prompt should analysed.

classes C, D, E, and F). The classes in this new categorisation depend on the feasibility and the sophistication of the modelling and the mathematics needed in obtaining a satisfactory answer of each question posed. Also the classification is dependent on the mathematical maturity of the population tested. For instance a question posed by a primary child may be considered workable, whilst if the same question is posed by a secondary school student it might be considered elementary. Judgements such as "feasibility", "the level of sophistication", "satisfactory answer", "mathematical maturity" are made on a subjective basis, but the classes are so broad that it is expected to have a good correlation with any independant check on analyses based on the categorisation.

If problem posing is regarded simply as an exercise of producing questions then a classification according to the feasibility and difficulty of mathematical solutions would seem inappropriate. However we contend that if the process of making questions is to be useful there should be some motivation in making them; hence there should be interest in the answer and in turn the way the answer is reached. Given this interest we are biased towards mathematical solution (and so questions which are unlikely to be modelled in terms of mathematics are regarded as non-well-structured (category B)). If a question seems to be amenable to mathematical argument, it is then natural to want to know whether a question is incomplete, elementary, intransigent or workable. To decide whether a question is incomplete, intransigent or workable might take a long time thinking about the answer (rather than the question itself). As a rough guide, if the researcher thinks the subject knows <u>how</u> to solve the question whilst writing it, then that question is elementary.

Combination of the two categories

In the obvious way we may make an <u>array</u>; we place any question posed in the appropriate space in the array according to its category in categorisation 1 (the columns) and each category in categorisation 2 (the rows). We can then gauge whether suitability is affected by deviation. This is only useful with a large sample.

A Test

We tested our theoretical framework by fieldwork. Both categorisations proved workable.

The Fieldwork

Two separate groups both of 15 subjects were given the problem posing task already stated. The session was of 45 minutes. The task was done individually and on paper. one group consisted of student teachers at pre-primary level, the other of student teachers at primary level. All subjects were in the first year of tertiary education, but the "primary level" group had on average a more "formal" Mathematical background.

Analysis according to Category 1 : The Deviation from the Prompt

The entries on rows 1 and 3 correspond to the number of problems posed in the appropriate category and population.

	O	Y,Y,Y	Y,Y,N	Y,N,Y	Y,N,N	N,Y,Y	N,Y,N	N,N,N	Total
Preprimary	6	13	15	5	18	0	0	1	58
Percentage	10	22	26	9	31	0	0	2	
Primary	12	25	5	4	1	0	0	1	48
Percentage	25	52	10	8	2	0	0	2	

Analysis according to Category 2 : The Suitability

	A	B	C	D	E	F	Total
Preprimary	13	6	6	19	14	0	58
Percentage	22	10	10	33	24	0	
Primary	25	12	6	4	1	0	48
Percentage	52	25	12.5	8	2	0	

Notes on the Analysis

1) As the sample was small, we did not attempt to present an array to relate the two categorisations. It was noticed though that the more deviant question [categories (Y,N,N), (N,N,N)] tended to be incomplete or elementary. There were some exceptions but these questions were probably previously learned. The least deviant questions [categories (Y,Y,Y) and (Y,Y,N)] tended to be workable, reflecting that the prompt itself is workable.

2) Only 2 out of 106 problems posed broke the context.

3) The preprimary group were less dependent on the prompt than the primary; also the preprimary gave fewer non well-structured questions.

4) It was noticed that the style of presentation tended to be most imaginative, satisfactory and clear for elementary questions.

Discussion

Our results suggest that the subjects' background can affect very much the behaviour in posing problems and that a more formal training may be a hindrance rather that a help in imaginative posing. Also, remembering that the subjects were student teachers, it was remarkable how more lively was the presentation when a question was on the level of their future students. Perhaps it is always helpful to imagine that

you are addressing questions to a second person in this kind of activity. Whom this second person is will influence the type of questions posed; for instance questions may be made elementary deliberately to be suitable for a young child.

Our main concern is that the questions written down very rarely broke the context. Let us propose likely ways for the context to be broken, and deduce these "ways" are largely absent in the subjects' work. Firstly, probably in common with most problem posing tasks, the context of the prompt is physical; mathematical abstraction then would break the context. Secondly, by imagination or by analogy with a previously met problem, a question with a parallel structure to that of the prompt but with a different context could well be formed. However the categories (N,Y,N) and (N,Y,Y) were empty in our analysis, so we had no evidence of this happening. Thirdly, a question may come from an aspect of an attempted solution of another question; it is very likely that the new question will not fit in with the original context. This third way of breaking the context is perhaps the most significant. We see from it that few generative processes were inspired by the task, and we think this will be the same for any task of the same style. In the end, then, we feel that subjects do not use the extra freedom offered by problem posing.

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III-49

PROSPECTIVE STRUCTURES IN MATHEMATICAL PROBLEM SOLVING

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Summary

The aim of this study is to explore problem-solving processes and find some characteristic activities by expert problem solvers. The video-taped and audio-taped records are analyzed and interpreted from the viewpoint of the solver's structures of the problem situation. *From this analysis, the following characteristic acitivities are found; (1)During the problem solving process a solver constructs prospective structures of the problem situation; (2)even if the prospective structure proves to be inappropriate, trying to elaborate the prospective structure can produce the useful information to the solver. Existence of prospective structures is also justitied by the recent views in philosophy of mathematics.*

Some researches on problem-solving processes have characterized experts' solving processes in terms of schemata (e.g., Kintsch & Greeno,1985; Owen & Sweller,1989). Even from the schema-theory side, however, the opinion that such a characterization may not be adequate especially for the genuine problem-solving is presented (Greeno,1991).

While the episodes-analysis (Schoenfeld,1983) gives us another characterization of experts' solving processes, the problem used in the experiments was rather easy for the expert (even though the problem has two parts, it takes only 20 minutes to complete the solution) and the solving process is organized very neatly (see the figures in Schoenfeld,1985 or 1992). Furthermore, he characterizes experts' solving processes in term of metacognition, not cognition.

So, it would be worthy to explore the congnitive aspects of the solving processes in more genuine problem-solving settings.

1. The Solver's Structures of the Problem Situation

If Polya's four phases are no longer the stages which the solver takes in a certain order (e.g., Wilson,1991), a new viewpoint is needed to describe the progress in the problem solving process. The notion of the solver's structures of the problem situation is introduced for that purpose.

In the problem solving process, the solver gives a certain structure to the problem situation on which the qeustion of the problem is asked. This structure consists of elements the solver recognizes in the situation, relationships (s)he establishes among elements, and the senses (s)he gives to the elements or the relationships. It is called *the solver's strucrure of the problem situation* (Nunokawa,1990). The solver's structure does not necessarily remain the same throughout the problem solving process. Rather it is usually expected to change as the solving process proceeds. If the solver can make sense of the problem situation by means of mathematical knowledge to the extent that (s)he can make the decision about the question, his/her solving attempt succeeds. This activity is directed not to making sense of mathematics with respect to the situation (e.g., Silver & Shapiro,1992), but to making sense of the problem situation with respect to the knowledge the solver has.

2. Method

The series of experiments I report here is the part of the attempt to explore characteristics of the experts' problem-solving activities. In the sessions mentioned here, the subject was a graduate student who studies mathematics education. When a undergraduate student, he studied college-level mathematics, especially modern algebra. So it is possible to consider him a expert problem solver, at least, as far as school mathematics is concerned. To make the problems used in the experiments challenging enough, they were selected from Klamkin(1988).[1] In every session, problem solving activites continued for 45 to 100 minutes. This implies that these problems were challenging enough and could be genuine problems for him.[2] This subject participated in nine sessions, in each of which he solved one problem.

In each session, the subject is asked to solve the problem in the think-aloud fashion. Even if he remains silent rather long, however, the researcher does not intervene. The researcher does not respond the qeustion concerning the problem. When the subject reports the finish of his solution, the solving part of that session becomes ended. If the subject has spent too long time(ca. 90 to 100 min.) and he seems to be stuck, the researcher find the appropriate moment and intervene to stop his solving activity.

It takes 60 to 140 minutes to implement the experiments, including the interviews. The whole session are audio-taped and video-taped. The description of the audio-taped records, *i.e.* protocols, are made, which then are complemented by the video-taped records.

In the following, because of the limitation on the length of the paper, only the unique phenomena of this subject's problem-solving activities will be presented, focusing on the activities observed in a certain session.

[1] In the experiments, Japanese-translated versions are used.

[2] The time spent by other subjects, all of whom are graduate students, were almost the same.

3. The Outline cf the Subject's Activities

In the seventh session, the subject tackled the following problem;

> PROBLEM: If A and B are fixed points on a given circle and XY is a variable diameter of the same circle, determine the locus of the point of intersection of lines AX and BY. You may assume that AB is not a diameter. (Klamkin,1988,p.5)

This problem was presented without any diagrams.

In the beginning, the subject drew a circle and two diameters. And he found that if AB is a diameter there is a case in which AX and BY do not intersect. Then he decided to draw a diagram more precisely;

(3:00) Introducing the coordinate axes may make it easier...
(3:04) If I introduce the coordinate axes...
(3:11) Hmm, introducing the coordinate axes may make it easier, OK, I'll try it, I'll try using the coordinate axes. [3]

He introduced the coordinate axes, assigned coordinates to each points (e.g., $A(1,0)$, $X(cos\theta, sin\theta)$). Then he represented two lines AX and BY by expressions and tried to solve the equation system representing the poin of intersection. But this calculation got rather complicated. At this moment, he decided to change the approach;

(29:56) Hmm, calculating it, I don't want to do it.
(30:12) I should think about it better before jumping into calculation.
(30:20) That calculation seems complicated...
(30:25) First, I try to get a plan or something...

The subject drew new diagrams without the coordinate system. First, he drew a diagram with the diameters placed at ordinal positions. Next, he drew a diagram of the special case, i.e. the case in which the point X is at the point A. In this case, the subject considered, as the line AX, the tangent line of the circle at A. Then he added another diameter, which is at normal position, and drew two lines using this diameter.

(32:44) So, perhaps I can take the tangent line...
(32:51) If I move it to this point..., here, yes, here.
(32:58) It moves in a rather wide range.
(33:10) Maybe, this is a circle, like this...
(33:40) Ah, I see...
(33:48) A circle...
(33:57) OK, I see, I see, well...

[3] Here, (3:11) denotes that this is the utterance talked after 3 min. 11 sec.

The subject drew a diagram once again, which has two diameters at ordinal positions. On this diagram, he pointed out that he should show that two angles appearing at the points of intersection are congruent. He drew a diagram with point X placed at point A. He continued investigation of the problem situation for about eight minutes. Then he said the following;

> *(41:31) Mmmm?*
> *(42:00) Ah, OK, OK, Maybe I've completed...*
> *(42:05) So, try to write down, well...*
> *(42:11) Maybe I can show it in this way.*

He continued to write his solution for a while, and reported spontaneously that he had completed the solution of the problem. The solution given by the subject, even though different one from that given by Klamkin(1988), can be considered correct.

In these activities, some characteristics indicated by other researchers can be found; appropriate monitoring or controling (see (29:56)-(30:25) above); using problem-solving strategies (e.g. drawing diagrams, thinking about the special case); trying to relate the situation with the domain-specific knowledge he has (e.g. introducing the coordinate axes to use the knowledge concerning the locus of two curves); some hypothetical reasoning (Ferrari,1992).

Here we should notice that it took about 43 minutes to complete the solution after saying "Maybe, I've completed," which is more than the half of the whole time. During these minutes, the subject was engaged in "implementation" phase (Schonfeld,1985). Furthermore, the basic idea of his solution —showing that every angle appering at the point of intersection is congruent, which can be considered a inscribed angle of a certain circle— did not change during this phase. None of the above-mentioned characteristics can explain this long period to complete the solution. To analyze the activities in detail, another approach might be needed.

4. Prospective Structures and Elaborating

Analyzing the protocol closely reveals that one of the reasons why it took so long time is the fact that in the midst of this period the subject had recognized the following; There are some distinct cases concerning the positions of the points of intersection, and the basic idea must be modified slightly according to each case.

First, he thought that the basic idea always applied to every point of intersection without modification (fig.1). In applying it, he found some cases to be distinguished. But he did not know how to formulate those cases (fig.2);

> *(53:19) What is the matter is how to take the point X...It is OK if X is on the arc AY.*

(53:49) ...*Well, How should I express it?*
(54:16) *The opposite of P, express it as the opposite of P.*
(54:42) *It is OK in the case of the opposite side. Maybe I can show it in the same way.*
(57:48) *Really? Can I show it in the same way in that case?*
(58:42) *If X is on the longer arc AB, it is OK.*
(59:10) *No, that's not, in what case should I consider?*
(59:26) *What case, what case sould I distinguish?*
(59:39) *How sould I express it?*

Consequently, he differenciated the cases according as the point of intersection existed inside or outside of the given circle (fig.3). It took eight minutes to formulate the cases.

Even after he found other cases, he believed for a while that he could apply the basic idea to other cases without any modification. As the solving process procceded, he realized that some kind of modification was needed. Then the subject started to modify the basic idea in order to apply it to the case in which the point of intersection existed inside of the circle. It took almost 25 minutes to complete this modification.

In other words, just when he started to write down his solution after saying "I've completed," he wrote down it according to his structure of the problem situation which was homogeneous and allowed him to use the basic idea everywhere.

This fact can be supported by the result of the interview. When asked whether some distinct cases were recognized from the outset, the subject said "No" and "During the solution I have realized that I had to distinguish some cases, but I didn't know how to formulate the cases."

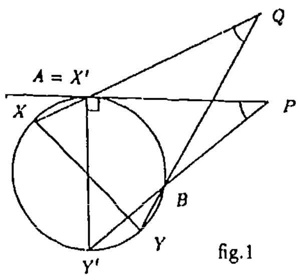

fig.1

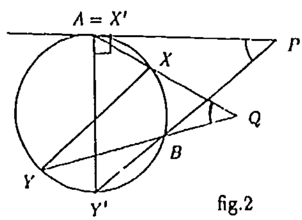

fig.2

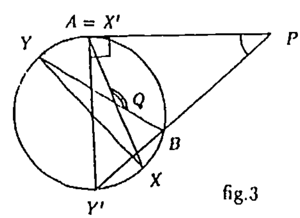

fig.3

The similar tendency of the activities can be found during the application or modification of the basic idea. For example, it took more than 10 minutes for him to write down the

solution to the first case (fig.1). And it took also 10 minutes to write down the solution to second case (fig.3), after saying "Aha, I get it."

Analyzing the protocol of those parts reveals that in writing down his solution the subject checked the conditions or elaborated the relationships between angles he would use. And to check the conditions or elaborate the relationships, he drew new diagrams besides the ones used in explaining the solution.

Here it can be said that when the subject said "I've completed" his structure of the problem situation did not have the enough information to support his solution. Furthermore, the structure included the information the subject expected or took for granted. And his structure was enough for him to think that he had completed.

Let us call this vague structure *the prospective structure*. While it includes information the solver expectes or takes for granted, it has enough information to support the solver's feeling of achivement. It took rather long for the subject to complete the solution because he had to elaborate the prospective structure in order that he could explain the solution logically.

The prospective structures does not always get elaborated enough to support his solution. Indeed, in other sessions this subject failed to elaborate it, and the information which had been taken for granted before became suspicious to him. He investigated the information in detail. As a result, he recognized new elements or relationships in the problem situation, or slightly changed the conditions he had used. These elements, relationships, and changed conditions played important roles in the rest of the solving process.

For example, in the second session he faild to elaborate the prospective structure. In that elaboration, however, he found that the height of a certain triangle changed as a certain point moved. He introduced a parameter expressing the height, and the parameter appeared in the solution he achived at last.

In short, even if the elaboration of the prospective structure fails, it can have the positive influence on the solving process.

5. Justification of the Prospective Structure

Lakatos'(1978a) Methodology of Scientific Research Programme is the notion to explain the progress of the science. There are, however, some attemps to modify and apply it to mathematics (Hallett,1979; Koetsier,1991). In these attempts, the program takes smaller scale than that of Lakatos(1978a), and a series of studies done by one mathematician can be considered his program.

How the solver sees the problem situation, *i.e.* what kind of the structure of the problem situation (s)he constructs, guides the solution. It also determines questions the solver can ask about the problem situation and new problems the solver can make based on the

original one. In short, the solver's structure of the problem situation influences the solver in the similar way as the Research Programme influences scientists (Nunokawa,1992).

The Research Programme is allowed to include facts which cannot be explained by itself. It can suspend those facts and proceed further. Similar tendency can be found in the progress of mathematics (Lakatos,1978b; Koetsier,1991; see also Lakatos,1976, especially Appendix I). Sometimes theories include explicit or implicit assumptions which cannot be explained fully.

Noticing the similarity between the Research Programme and the solver's structure of the problem situation, the solver's structure may be allowed to include explicit or implicit assumptions and expected information. Those assumption or information will be investigated when needed (e.g., in writing down the solution to show it to someones else; in discussing the solution with other students). Because of keeping the solver from exploring the details of the problem situation, the prospective structure can facilitate achiving tentative results extending the problem or the solution. While the prospective structure may, indeed, have the risk of solving unlogically, it can play a positive role in the problem solving process.

According to some philosophers, making these assumptions or expected information explicit facilitates the progress of mathematics (Lakatos,1976; Koetsier,1991). It is similar to the fact that in elaborating processes the subject established new elements or relationships in the problem situation. This similarity also suggests that the prospective structure can play the positive role in the similar way as the Research Programme plays.

6. Conclusions

Analyzing the protocols from the viewpoint of the solver's structures of the problem situation, it is found that the solver constructs the prospective structure, which includes expected information or assumptions. Though it is not enough objectively to complete "correct" solutions, it can support the solver's feeling of achievement, and can facilitate getting tentative results. Even if the elaboration fails, useful information can be found durint the elaborating activity. The prospective structure plays the positive role in the solving process.

According to some results of philosophy of mathematics, using the information or assumptions which cannot be explained at that time, has occured in the dicipline of mathematics and played some heuristic roles. This supports the idea that constructing the prospective structure should be accepted as the acitivity of the expert problem solver.

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MATHEMATICS - 'A BUNCH OF FORMULAS'? INTERPLAY OF BELIEFS AND PROBLEM SOLVING STYLES.

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Affective and cognitive aspects were evident in the problem solving styles and in the beliefs about the nature of mathematics of six students enrolled in two high school teacher education courses in Florida. In this paper, it is suggested that there is an interplay of styles and beliefs which arises not only in the context of individual differences such as preferences for visual processing, but also under the influence of mathematical experiences in courses the students have attended. Evidence is presented that content knowledge cannot be separated from the experiences in which it was constructed.

A VIGNETTE

"Gee, it's kind of frustrating because, even like a couple years ago, there's formulas for this stuff, y'know, where I'd be able to go back to it and now, I've forgotten the formulas. It's kinda like Spanish, you know, something like that. My friend used to be really fluent in Spanish and now that she doesn't use it very much any more she's starting to lose it."

Jake (pseudonym), a prospective high school mathematics teacher enrolled in a university education program, spoke these words while attempting to solve a mathematics problem relating to a sequence of numbers. The part of the problem Jake was working on involved finding the fiftieth term and the sum of thirty terms of the sequence 5, 8, 11, ...

Yes, Jake is correct that there are formulae for the general term and the sum of n terms of such an arithmetical progression. But it was not necessary to remember these to solve the problem in question. Jake's remark illustrates a belief that appears to be widespread among students at school and as they begin their teacher education courses, that mathematics consists of "a bunch of formulas" (as another student put it), and that once these have been forgotten, like the vocabulary and grammar of Spanish, they can no longer communicate their ideas or solve the problem.

The study described in this paper was part of a wider ongoing research project which began in Fall, 1991 and which investigates, *inter alia*, the interplay between content and professional knowledge as students progress through certain teacher education courses at The Florida State University. This paper highlights the influence of experiences while constructing mathematical content knowledge on the beliefs and problem solving styles adopted by six students as they "solved aloud" two mathematical problems in audiotaped interviews. As Brown (1991) suggested, content knowledge cannot be separated from the experiences in which it was constructed. In these interviews there was evidence that the experiences these students had, both at school and in their mathematics teacher education courses, influenced their cognition and affect as they solved these high school level mathematics problems.

THE STUDY

Jake was one of the six students who were interviewed while they were enrolled in either or both of two courses for prospective high school mathematics teachers taught by the writer in the Fall of 1991 at The Florida State University. The courses were as follows:

MAE 4332: Teaching Secondary School Mathematics. (A general introduction to teaching mathematics in grades 7 - 12; usually a first course taken by mathematics education majors in the program.)

MAE 4816: Elements of Geometry. (A course which introduces students to geometry other than Euclidean, and which is suitable for middle and high school; this informal geometry course has practical components involving extensive use of manipulatives.)

In both these courses the pedagogical and epistemological assumptions were constructivist (Davis, Maher and Noddings, 1990). The methods of instruction involved mathematical problem solving in small groups as well as whole class sharing. Certainly in neither of these courses was there an emphasis on mathematical formulae: the stress was on construction of conceptual rather than procedural content knowledge (Hiebert, 1986). However, the traditional teacher centered mode of instruction which these students had experienced at school - still widespread in the U.S. (National Research Council, 1989) - had in many cases left the legacy of a belief that mathematics is "a bunch of formulas". Changing beliefs and metaphors of prospective teachers in this complex context is a research focus of the ongoing study. The present paper concentrates on beliefs and styles in just one problem solving interview with each of six students. The methodology is of necessity interpretive (Heshusius, 1992); the study is qualitative (Eisner and Peshkin, 1990). The six students were audiorecorded as they "thought aloud" (Krutetskii, 1976), in an interview in which they solved two mathematical problems, viz., the number sequence problem Jake commented upon in the vignette, and a problem which assumed some content knowledge of Euclidean geometry as taught in high school.

Mathematical problem solving, then, is the focus of this paper. One important determinant of an individual's problem solving style in any instance is the individual's personal preference for visual methods of solution (Krutetskii, 1976). The six students were chosen on the basis of their scores on the writer's test and questionnaire for mathematical visuality (Presmeg, 1985), which had been validated for use with Florida students in December, 1990. A student's mathematical visuality is the extent to which that student uses visual methods (such as diagrams, charts or visual images) when attempting nonroutine mathematical problems which may be solved with or without such visual methods. According to their mathematical visuality scores, two of the chosen students were visualizers, i.e., they preferred to use visual methods, two were nonvisualizers (seldom needing an image or diagram) and two had scores which were "in the middle", i.e., close to the median score for these two classes. The students were given pseudonyms as follows:

visual group: Jem and Mart
middle group: Jake and Ellen
nonvisual group: Pam and Del.

It just happened that the three male students, Jem, Mart and Jake, fell on the visual side of this continuum of scores, while female students Ellen, Pam and Del fell on the nonvisual side. This phenomenon was not typical: in previous research, the writer found no significant differences in mathematical visuality between the sexes (Presmeg, 1985).

INFLUENCES IN PROBLEM SOLVING

The two nonvisual students, Del and Pam, both experienced their lack of tendency to visualize as a hindrance in mathematical problem solving. As Del expressed it, "If you can't see it, you can't draw it!" And in many College mathematics courses (e.g., Calculus III, which includes three-dimensional work) she felt distinctly hampered because she could not "see" the mathematics or do the drawings. She spoke as follows:

> "And so, like towards the end of the course, what it went out to be, I would understand or I would be able to do all the algebra part and I could do all the mathematical part of it, but I didn't understand how it fit in. I didn't understand *why*. And so, just if you come to having to explain why I got it, then I was, you know, in trouble."

Del had succeeded in her mathematics courses largely by remembering formulae. This is her implicit definition of "mathematics" when she speaks of "the mathematical part of it".

Del and Pam, and also Jake, were of the opinion that not just elementary teachers but also teachers of higher mathematics courses should "draw more pictures", even in abstract mathematics where at all possible, because they believed such diagrams would be an aid to their understanding or making sense of the more abstract concepts. By way of contrast, visualizers Mart and Jem did not express a need for their mathematics teachers to draw any diagrams: as visualizers, they could, and did, create their own images and diagrams in the process of making sense even of abstract mathematical concepts. Their mathematical processing suggested that they experienced other difficulties (see later). Ellen used nothing that she could designate "imagery" in the process of solving these mathematical problems, and she did not express a need for imagery or diagrams at all.

The consensus (of all the students except Ellen) was that the presence of an image or diagram in mathematical processing aids conceptual knowledge and obviates the need for total reliance on formulas or procedural knowledge.

(1) The sequence problem.

Logic is a necessary ingredient in all mathematical processing (Krutetskii, 1976). Logic was evident in all successful solutions of the sequence problem in these interviews, whether imagery was reported as having been used or not. It was logic, without the textbook formula (which none of the students remembered), that eventually led all students except Ellen to a pattern which gave them the fiftieth term of the sequence 5, 8, 11, ... Ellen wrote down all the terms up to the fiftieth one to find a solution; after that, extensive questioning and prompting by the interviewer led her to a shorter way of arriving at the same answer.

The influence of content knowledge of mathematical methods used in the course MAE 4332 (Teaching Secondary School Mathematics) was apparent to a greater or lesser extent for Del, Pam, Jake and Jem, but especially in Mart's processing. Mart commented, "This is very typical of something we've worked on [in MAE 4332]. And this, I'm just applying some of the principles that I've learned in there, I think." These principles influenced Mart to proceed as follows:

* First, as he said, "I would take S sub 1 equal to 5, S sub 2 equal to 8, S sub 3 equal to 11."
* Then he clarified, "S is the term, right. First term, second term, third term. Now we're looking for S sub 50. That would be equal to .... If we can find a relationship between these, then come up with a general equation and test it, it will probably just plug in, hopefully the sub number 50, and come up with the answer."
* Imagery involving jumps of 3, together with logic, gave him a formula, i.e., $S_n = 5 + 3(n-1)$, which led to the correct answer when he substituted $n = 50$.

In Mart's case, a formula was used; but it was his own formula, not one that he learned by rote and attempted to store away for future use. Content knowledge of a method or principle had given him something far more powerful than mere memorization of a formula for the general term of an arithmetic progression (although if it is not simply memorized for tests and subsequently forgotten, it may be useful to have such a formula available for use). The formulae constructed by these students were diverse and idiosyncratic: for instance, Del used the patterns of numbers to obtain an equivalent formula $2(n+1)+n$.

This principle, viz., taking simpler cases or the first few terms of a sequence, finding a pattern, generalizing, testing and proving, then substituting to find a required case, was less successfully used in the second part in which the sum of 30 terms of the sequence was required. None of these students constructed a solution unaided. So the interviewer told all but Jem (who stopped at that point) the "story" of Gauss who added the first and last terms of such a sequence, then the second and last but one, and so on. All five students rapidly solved the problem using this hint. It is noteworthy that the girls reported no imagery in their solution processes, while Jake and Mart reported a "kind of dome" image with lines linking the outer terms, then successive pairs of terms within. (Kaput, 1991, has done extensive research on various kinds of mathematical representations using a similar problem.) Jake indicated that the linking lines of his image were prompted by the FOIL method (First terms, Outer terms, Inner terms, Last terms) often used in high school algebra classrooms as a procedure for multiplying two binomials.

(2) The geometry problem.

The kinds of content knowledge which influenced processing in the geometry problem were, understandably, largely associated with high school Euclidean geometry and the Elements of Geometry course (MAE 4816) in which four of the students were enrolled. However, in the third section of the problem which involved area, the theme of feeling paralyzed if one cannot remember a formula was again painfully evident, in three of the six cases.

The problem was as follows.

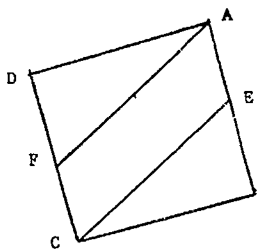

Figure 1.

ABCD is a square and E and F are the midpoints of AB and DC respectively. CE and AF are joined.
(a) Prove that triangles ADF and CBE are congruent.
(b) Prove that AECF is a parallelogram.
(c) If the area of the square is 4 square units, calculate the area of the parallelogram.

In the first and second parts, memories of theorems about congruent triangles and parallelograms (or the absence of such memories) appeared to be the overriding influence. Only Del and Mart remembered these theorems sufficiently to facilitate their attempts at proof. Solutions to these two parts will not be examined in detail here; just two points of interest will be mentioned.

Firstly, there was a tendency for students to go by the visual appearance of the diagram in constructing proofs. For example, Del, Pam, Ellen and Jem all wanted to take lines FA and CE to be given as parallel because they looked parallel in the given diagram. This intrusion of visual appearance in proofs is widespread amongst visualizers such as Jem (Presmeg, 1985). A second, related, phenomenon is the constructing of geometrical prototypes (Hershkowitz, 1989; Presmeg, in press). Several of the students redrew the square so that it was "straight", because their prototypical image of a square involved horizontal and vertical line segments. The visual prototype is compelling for many students.

The tendency for students to take FA and CE to be given as parallel lines introduces a further point. Del explained another influence at work in her thinking. In reply to the interviewer's question, "Now, you say, by *looking* at it. Is that because it *looks* parallel?" she exclaimed, "Oh, that's right! I can't assume that it's parallel, I forget, because of whenever, um, I've worked with *proofchecker*, and whenever you work with proofchecker all you have to do is draw a parallel line and of course, if it goes with it then you can say it's parallel. So that's why I just wanted to look at it and say, um .... Okay!" Here Del's case points to a possible disadvantage of extensive use of computer software such as she described, in Euclidean geometry.

The third part of the geometry task, i.e., finding the area of the parallelogram, turned out to be a very rich activity which was handled in diverse ways by these students.

In three cases (Pam, Jake and Mart), the influence could be seen of tessel-

lation (tiling) activities in which the students had been involved in their Elements of Geometry class (MAE 4816) with the writer. Two other students, Del and Jem, were not enrolled in this course. In class, the students had been constructing "Escher-like" tiling patterns based on works of the Dutch artist Escher. They had learned that by taking a simple shape which will tessellate, such as a square or equilateral triangle, they could create intricate tiling patterns by removing a piece of their template shape and taping it on to a different side appropriately. Pam's reaction to the area task was to remove triangle CBE from the bottom of the square and in her mind "tape it on" to the top of the square so that CB coincided with DA, as follows.

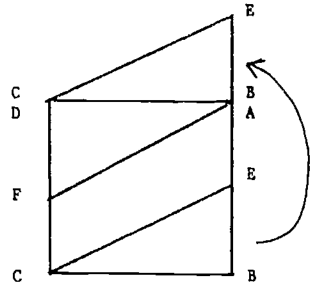

Figure 2.

She saw immediately that the area of parallelogram AECF is half the given area.

Jake and Mart used dynamic (moving) imagery in their solutions, both of which showed the influence of rotations which they had used in the tessellation tasks in class. The students in MAE 4816 had constructed a principle that any quadrilateral will tessellate by rotating the shape through 180 degrees around the midpoint of any side. The following figure illustrates their method of solving the area problem.

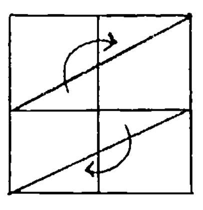

Figure 3.

Rotating through 180 degrees the small triangles at the centre of the parallelogram, as shown by the arrows, they concluded that the area of the parallelogram is two square units. They used logic to validate this movement, e.g., Mart commented, "I bisected this [by drawing the cross in the center of square ABCD] so I split the hypotenuse [i.e., AF] in half." It is

noteworthy that not one of the 54 visualizers interviewed using this task in a previous study (Presmeg, 1985) solved the problem by these methods.

Those students (Del and Pam) who tried to recall a known formula for the area of parallelogram AECF and then attempted to work with CE as the base, struggled with the problem and were unsuccessful using this method. Jem, on the other hand, successfully used a formula in conjunction with a visual analysis. He saw that the required area is the area of the whole square less that of triangles ADF and CBE; each of these triangles has an area of one square unit, as he calculated using the half-remembered formula for the area of a triangle (which he at first thought was base times height).

What was striking in the protocols for the area problem, was how effectively visualization and logic can be combined to construct novel, accurate, and in some cases, elegant solutions.

SOME RECOMMENDATIONS

In summary, the interviews described above suggest to the writer the following recommendations for mathematics educators at high school and college level.

* The belief that mathematics is "a bunch of formulas" to be memorized is detrimental to effective problem solving, and educational practices which foster construction of this belief should be avoided (Del, Ellen, Jake and Jem).

* To facilitate understanding of concepts rather than rote, procedural knowledge, teachers should draw and encourage diagrams whenever possible (Del, Pam and Jake).

* Teachers should avoid always drawing diagrams in a standard orientation in geometry, since inflexible prototype images may hinder student cognition (Jake).

* Creative, dynamic imagery should be encouraged in the solution of problems in geometry, including Euclidean geometry. The rigorous proofs, based on students' processes, can be worked out later, after they have experienced the excitement of a "visual solution" (Pam, Jake and Mart).

The overriding impression which remained with the writer after the interviews described in this paper, was the helplessness engendered by the belief that mathematics is "a bunch of formulas". With this belief, when no formula could be remembered, the paralyzing negative affect which followed went hand in hand with a state of mind in which little mathematical problem solving cognition was possible.

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MATHEMATICAL PROBLEM SOLVING IN COOPERATIVE SMALL GROUPS: HOW TO ENSURE THAT TWO HEADS WILL BE BETTER THAN ONE?

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Instructional methods where students work in cooperative small groups are popular, in particular for teaching and learning about mathematical problem solving. However, some recent studies of problem solving have shown that results obtained by students working in small groups are not necessarily better than individual results. In this paper we review some theoretical reasons for expecting that work in small groups should be beneficial for students learning to solve non-routine mathematics problems, and we analyze three recent studies of mathematical problem solving in groups. Finally, we suggest possible directions for future research and development designed to help teachers increase the chances that two heads will be better than one when their students solve mathematical problems in groups.

Instructional methods that involve students working in cooperative small groups have become increasingly popular, and such groups are seen by many teachers as particularly appropriate for teaching and learning about mathematical problem solving with non-routine problems (Good, Grouws, & Mason, 1990). However, the research on the learning of students working in groups as opposed to working independently has been equivocal, and some studies of non-routine mathematical problem solving in groups (Stacey, 1992; Treilibs, 1979) have shown that students working in groups do not always perform better than students working individually. Before analyzing three recent studies of mathematical problem solving in groups, we briefly present some theoretical reasons for expecting that working in small groups should be beneficial for students learning to solve non-routine mathematics problems.

Theoretical Foundations

Webb (1982) has provided a wide-ranging review of research on learning in cooperative small groups. Her review, includes consideration of theoretical mechanisms bridging the interactions within groups and the achievement outcomes. According to Webb, researchers who have considered *how* participating in a group might help students learn have hypothesized two main kinds of mechanisms that could relate interaction in groups to student achievement: (1) mechanisms directly affecting cognitive processes, and (2) mediating variables thought to create an emotional or intellectual climate conducive to learning. The cognitive processes discussed by Webb include mere verbalizing (vocalizing), cognitive restructuring, and conflict resolution, the socio-emotional ones motivation, anxiety, and satisfaction.

Noddings (1985) considered a number of social theories of cognition in synthesizing a framework for her research on mathematical problem solving in small groups. She concluded that the most useful theoretical position is that of Vygotsky, who claims that children's individual mental functions are internalized from relations among children in groups, and that reflection is induced by the need for each child to defend his views against challenges brought by other children. The following are some specific

hypotheses about group processes being internalized by students learning to solve non-routine mathematics problems in cooperative small groups:

- Useful heuristics should begin to spread through the group; less useful ones should fall away.
- Challenge at the group level should appear as reflection at the individual level.
- Requests for explanation in the group should be reflected in an internalized heuristic, "Why is this?"
- The procedures of the group should reappear as an orderly attack on problems by the individual. ...
- The exchange of peripheral information in the group should result in vocabulary development in individuals.
- References to other contexts in the group should produce contextual sophistication (less contextual dependence) in the thinking of individuals.
- Conversation in the group should be reflected in a greater volume of problem-oriented inner talk in individuals. (Noddings, 1985, pp. 351-352)

In the light of these promising and seemingly reasonable predictions, we now turn our attention to two studies of mathematical problem solving in cooperative small groups which seem to suggest that "two heads may not be better than one."

Previous Research on Mathematical Problem Solving in Groups

Stacey (1992) has reported a two-part study in which a written test of non-routine problem solving was given to Year 9 students under two conditions: individual work and small group format. Since the result of this initial testing was that students working in groups scored slightly *worse* than students working individually, a follow-up study was carried out focusing on observations of groups as they worked on three of the tasks contained in the written test. All three tasks have similar mathematical structure and involve the detection and extension of patterns. One task, referred to as the Sequence Task, is shown in Figure 1. Its underlying mathematical structure is $S(x) = 6x - 2$. A second task involved ladders made from match sticks; in this task two specific ladders were shown, one made from 8 matches having 2 rungs, and one made from 11 matches having 3 rungs. The relationship between the number of rungs x and the number of matches $M(x)$ in this situation is $M(x) = 3x + 2$. In this task students were asked to find the number of matches needed to make a ladder with 20 rungs, and the number of matches needed to make a ladder with 1000 rungs. In the third task, diagrams of Christmas trees of sizes 1, 2, and 3, having 3, 7, and 11 lights respectively, were shown. The mathematical structure of this problem is $L(x) = 4x - 1$, where x is the size of the tree and $L(x)$ the number of lights. Students were asked to find the number of lights on a size 20 Christmas tree, and the number of lights on a size 100 Christmas tree. In earlier research, Stacey had established that students tend to make a small number of prevalent errors in problems like these. One such error incorrectly using direct proportion, the student arguing, for example, that $L(100) = 5 \times L(20)$.

Fill in the blanks in this number sequence which continues on and on in the same pattern. 4, 10, 16, 22, 28, __, __, __, __, __, __ What is the 10th term of the sequence? What is the 100th term of the sequence?

Figure 1: Sequence Task (Stacey, 1992, p. 265)

In the follow-up study, seven single-sex groups of three students each, ranging from Year 7 (age 12) to Year 9 (age 14), were observed and videotaped as they solved two or three of these tasks. The groups of students were formed by their teachers, who selected above-average-ability students who would work well together. The observer provided the problems and encouraged the students to discuss them but took no other part in the discussion. In order to promote cooperative working, only one pencil was provided to each group.

Analysis of the observations showed that the decision to use or not to use the simple but wrong method identified in previous research was critical to the success of the group. All but one of the seven groups proposed using the common erroneous approach at some point, and all but one of the seven groups proposed other ideas that could have been used to solve the problem correctly. Most of the groups, however, exhibited little checking or relatively ineffective checking of their work. Often, queries went unanswered or were answered by mere repetition of a proposed calculation. Instead of discussing each others' proposed methods, new ideas were put forward, and even when correct methods were given with convincing explanations, they were sometimes dismissed or abandoned. Stacey concluded that it was not the getting of ideas that was difficult for these students, but choosing which ideas to implement.

Recent Research Using a Task-Based Interview

Lin (1992) invented a card game for four players and used it in an interview with a group of students in Grade 11 (age 16) to investigate whether and how the students would apply probability concepts in analyzing practical situations. Relevant probability concepts such as equally likely outcomes, probabilities of joint events, etc. were included in the curriculum of the mathematics course these students had taken in Grade 9. To play the game, one of the players shuffles six cards numbered 1 to 6 and deals one card face down to each of the four players, leaving the remaining two cards face down. Then the players each pick up their card and place it on their forehead, so that each of the players can see the cards of all the other players, but not their own card. The player holding the highest card wins. Betting begins with the dealer and ends after two rounds; players may raise the bet, call, or fold.

Lin interviewed four students (referred to as Students X, Y, Z, and T) as they played this game. The interview was audio-taped and transcribed, and students' written work was collected. Students were told that after the betting ended they would be stopped before revealing their cards and asked to write down the ideas that led them to call or drop out of the betting. Then all the cards would be revealed and the students' ideas would be discussed in the group. The first time the game was played, Student X was dealt the 1, Student Y the 4, Z the 2, and T the 5. The following comments show that each student initially had a

different approach to establishing the probability of winning. Only Student Z gave a correct analysis of this situation.

Y: *Well, it's pretty obvious that the highest card out there was 5. That means the 6 was still around. I saw a 1 and 2 as well. That gave me a 3/6 or 50% chance of having a 6 as my card. That's almost like flipping a coin or answering a yes or no question.*

X: *I saw 2, 4, and 5. This meant that 6 was still around. There are six cards in the pile so I should have 1/6 a chance of having the 6 card. That's around a 17% chance of winning. That's not too bad. At least it's not like 0% or something.*

T: *There are six cards in the deck. If only four are in use, then I had a 1/4 chance of getting a specific card. I saw that 1, 2, and 4 were used up, so I knew that 5 and 6 were still around. Using the "or" rule of probability, the chance of having a 5 or 6 was 1/4 + 1/4 = 1/2.*

Z: *I saw that there were 3 cards being used. These were 1, 4, and 5. This meant that I could have 2, 3, or 6. This really only gave me 1/3 chance of winning. That wasn't bad.*

The students played many more rounds and commented on their thinking each time. The following exchanges (we hesitate to characterize them as interactions) are taken from one of the last games played, when Student X had the 4, Y had the 3, Z the 1, and T the 2.

X: *I saw that there was no way of losing since the 1, 2, and 3 were revealed. This meant I had a 4, 5, or 6. The rest of my explanation doesn't make sense.*

Int.: *What do you mean?*

X: *I know there are 6 cards in the deck. If I am to get 4, 5, or 6, then I should have o 3/6 chance of getting any of these. That means I have a 50% chance of winning. But, I know I should win 100% of the time this game.*

Y: *The highest card I saw was 4. This meant I needed to have 5 or 6 to win. I still had a 3/6 chance to get 5 and a 3/6 chance to get 6. Huh?*

Int.: *You look puzzled. What's wrong?*

Y: *If I have 3/6 chances to get 5 and 3/6 chances to get 6, then don't I have 6/6 chances to get either one?*

Lin decided not to help Student X or Y with their apparent confusion. Rather, he encouraged them to work on coming up with explanations for their results, stressing that he was interested in *their* thinking.

Z: *I saw the 2, 3, and 4. I must have a 1, 5, or 6. I had a 2/3 chance of winning.*

Int.: *2/3 chance? I'm curious. ... How did you get that?*

Z: *Well, if I could have 1, 5, or 6, this meant that only 2 of the 3 possible cards allows me to completely win.*

T: *I think I had a good chance because I only saw 1, 3, and 4. That means I have a 2, 5, or 6. Using probability: Pw (2 or 5 or 6) = 0/6 + 1/6 + 1/6 = 1/3 or 33.3 %* [Pw is the "probability of winning"].

By this time in the interview, the betting had become somewhat unimportant, but all of the students were very involved in analyzing various aspects of the game and trying to synthesize a strategy, if any, to win further rounds. Students were then asked if the game was fair; the following are their conclusions.

X: *The game must be unfair, at least at some points. If there is a 6 at any time, then you have 0/6 to win. Also, if I see 1, 2 and 3, then I know I have won "for-sure." The rest of the time you have 1/6, 2/6, 3/6 chances of winning, I think.*

Y: *I think the game is confusing. When I see 1, 2, and 3, I must have 4, 5, or 6. I have 3/6 chances of getting each. I'm not sure that makes sense. Math makes things so difficult sometimes.*

Neither Student X nor Y seemed to have any more to contribute; they seemed rather frustrated. Lin decided to not press the issue.

T: *I decided to try out some possibilities* [shows the written work presented in Figure 2].

I see 1, 2, 3 --> I must have 4, 5, or 6 --> Pw (4 or 5 or 6) = 1/6 + 1/6 + 1/6 = 50%
I see 2, 4, 6 --> I must have 1, 3, or 5 --> Pw (1 or 3 or 5) = 0/6 + 0/6 + 0/6 = 0%
I see 4, 5, 6 --> I must have 1, 2, or 3 --> Pw (1 or 2 or 3) = 0/6 + 0/6 + 0/6 = 0%
I see 1, 2, 4 --> I must have 3, 5, or 6 --> Pw (3 or 5 or 6) = 0/6 + 1/6 + 1/6 = 33%
I see 2, 3, 4 --> I must have 1, 5, or 6 --> Pw (1 or 5 or 6) = 0/6 + 1/6 + 1/6 = 33%

Figure 2: Student T's Calculations (Lin, 1992, p. 4)

Int.: *Your first possibility ... you say you have 50% chance of winning?*

T: *I know, ... it should be 100%, but I'm not quite sure how to explain it.*

Student Z's analysis was similar to Student T's, but different in two respects. First, he carried out his listing systematically, accounting for all 20 possibilities that a player may face, and second, he used 3 rather than 6 as the denominator, recognizing that the relevant sample space consists only of the three unseen cards. Figure 3 gives an abbreviated version of Student T's work; the actual work was systematic, complete, and correct.

Option	If I see these	I could have any of these	My probability of a guaranteed win is
1	1, 2, 3	4, 5, 6	3/3
2	1, 2, 4	3, 5, 6	2/3
3	1, 2, 5	3, 4, 6	1/3
⋮	⋮	⋮	⋮
20	4, 5, 6	1, 2, 3	0/3

Figure 3: Table constructed by Student Z [abbreviated] (Lin, 1992, p. 11)

Referring to the table above, Student Z went on to produce an argument proving that the game is fair.

Z: *This game cannot really be called fair or unfair. Option 1 shows that I have no way of losing. However, Options 4, 7, 9, 10, 13, 15, 16, 18, 19, and 20 show that it is impossible for me to win "for-sure." These results do agree with the game statistics. But, overall the game is fair because the probability of getting a winning card is* $\frac{1\times\frac{3}{3} + 10\times\frac{0}{3} + 6\times\frac{1}{3} + 3\times\frac{2}{3}}{20}$ or $\frac{1}{4}$. *This means that any of the four players has a 25% chance of getting a winning card.*

The students in Lin's study showed relatively little change from the beginning of the interview to the end. Student Z, who was correct from the first round of the game, was a bit tentative at first, but seemed to become quite certain of his correctness. He appears to have convinced himself, but not the others in the group, of the validity of his arguments. Student X recognized situations where the probability of winning is either 0 or 100%, but his analyses of all other possibilities were wrong, as were Student T's, because they were based on a sample space of size 6. None of the other students seemed to learn much from Student Z, whose work at the end of the interview is exemplary in its clarity and orderliness. Therefore, we conclude that in this case, too, two heads were not better than one.

Although Lin was a student-teacher undertaking a practicum at the time he carried out this project, he took it as his assignment to learn from students, rather than teach them. In order not to bias his results, Lin chose not to intervene, e.g., by providing information to Students X and Y, or by asking the group to resolve the obvious difference between the first line in Student T's work in Figure 2, and Student Z's analysis of the same situation in the first first line of Figure 3. However, he did identify a number of opportunities to "orchestrate discourse" (NCTM, 1991, p. 35) which arose during the interview, but which he did not take advantage of, due to his understanding of his role.

Recent Research on Metacognition-Based Instruction

Gooya (1992) investigated the effects of metacognition-based teaching in a mathematics course for undergraduate students, many of whom were intending to become elementary school teachers. The instructional strategies she used included journal writing, cooperative work in small groups, and whole-class discussions. Gooya worked hard to establish a social norm in the small groups and whole-class discussions where meaning-making, alternative problem solving strategies, and explanations based on mathematical evidence were valued. Her report includes a great deal of evidence documenting the progress students made in clarifying their thinking, becoming more reflective, and monitoring their progress.

For example, Gooya describes an incident early in the course in which a student, Nina, presented her group's correct solution to a problem but immediately erased it when another student asked her why she did what she did (Gooya, 1992, p. 78). This situation contrasts sharply with another incident late in the course where nine students, including Nina, took part in an extended discussion of two different interpretations of the meaning of a problem, of different approaches to the problem, and of different solutions to the problem based on the different interpretations. The problem being discussed was *If someone offers you a job that pays one penny on the first day and doubles the money every day after that for 30 days, would you accept the offer?* The following are some excerpts from the interactions.

Clara: *I will just tell you what we've done in our group. But we haven't got the answer.*

Zahra: *That's fine. We don't want a finished product. We just want to hear from each other.*

Clara: *People worked it out on a calculator as 2 ×2 ×2 ×.... The second way that we did was ... and then we took like for the first day it was only one penny and then the second day it doubled. It was 30 days. The second day was 2, and that was for 29 days. So we put 2^{29}, and it worked out on calculator, and both answers were the same.* [What Clara's group did first was to multiply by two 29 times and then add one. The second way was to use the x^y function on the calculator and add one at the end.]

Zahra: *So did you get the sum of the money the person got at the end of the 30th day?* Clara: *Yes.*

Melissa: [Raising her hand] *What we* [another group] *did was make a table having three columns. The first column was a penny and the second column was doubling that money and the third column was accumulated pay. What we noticed was, for example, on the 11th day, you earned \$10.24 a day and the sum of the first 10 days which was for previous days, so ... like the 30th day, you'll be actually earning* 2^{30} *but minus one cent. That would be the sum of what you've earned from the first day to the 30th day.*

Zahra: *Thanks . . . so what about you* [pointing to another group]?

Shirley: *I didn't quite understand what the 10th day has anything to do with it.*

Melissa: *Oh, just to give an example.*

Patrick [from Melissa's group]: *Well on the 10th day you earn \$5.12 and the accumulative pay on the 10th day was \$10.23 and on the 11th day you make \$10.24. You make one penny more than you've already made.*

Melissa: *That was accumulative pay from the 1st day to the 10th day which is equal to the pay on next day. But you actually make one more penny.*

Patrick: *So, on the 10th day, you make* 2^9. *That's how you earn \$5.12. The total pay was \$10.23 which was* $2^{10} - 1$. *So, on the 16th day, I used* 2^{15}. *The accumulated pay was \$655.35 which is* $2^{16} - 1$. *So the formula is* $2^x - 1$ *penny.*

Jack [from another group]: *We did it as: on day 1 you make 1 penny, on day 2 you make 2 pennies, day 3 you make 4, so we noticed the pattern on the right hand column. The 1st day is* 2^0, *2nd day is* 2^1, *3rd day is* 2^2, . . . , *so after 30 days, that's equal to so many cents . . . is equal to* 2^{29}. *When you punch that into calculator, that was how many cents you would earn.*

Nina [same group as Patrick]: *So more like what you guys did* [pointing to Clara's group].

Patrick: *That's how much money you made on the 30th day.*

Jack: *The problem we had and I've still seen is that the power is one less than the day, and originally we had* 2^{29}. *But because you start at 0, you don't start at one, that's where the difference comes in.*

(Gooya, 1992, pp. 80-85).

As in many verbatim transcripts of classroom interactions, the exact meaning of these utterances is not always completely clear, but what we think is clear is the great extent to which the students are listening to and responding to one another, explaining their methods, comparing various methods, and seeking and giving clarifications; they are not ignoring or dismissing one another as the students in the Stacey and Lin studies seemed to do. In these studies, the role of the researcher was as an observer who, for objectivity's sake, did not not intervene in ways intended to influence the group activities; in fact, the observer in one study "took no part in the discussion, feigning lack of interest" (Stacey, 1992, p. 264). Gooya, on the contrary, adopted the role of a participant-observer who deliberately attempted to communicate to students her expectations for the roles they should take in the discourse during the course. A key feature of Gooya's work is the social norms that she attempted to establish. The means she used to establish these norms cannot be discussed adequately in this paper, but those norms and means and their implications are much more significant and much more complex than the "norms" in some research on

small groups. For example, a study by Egerbladh and Sjodin (1986) used "group norm" as one factor in a four-factor ANOVA, meaning that students were given brief instructions that they were to cooperate with one another, or compete with one another, or they were given no such instructions. Webb has described such research as work which has "sought to predict achievement from a few characteristics of the individual, group, or setting. Without data on students' experience in groups [and we would add what teaching, if any, they receive about how to work in groups], these studies present incomplete pictures of the influences of group work on individual learning" (Webb, 1982, p. 422).

Directions for Future Research and Development

The evidence presented in Gooya's study shows some results of a planned and documented intervention, rather than characteristics of students' "natural" interactions in groups as presented in Lin's and Stacey's reports. We see value in both types of research, as they characterize *what is* in some (perhaps many or even most) classrooms and also what *could be*. The *Professional Standards for Teaching Mathematics* (NCTM, 1991, pp. 35-51) contain standards for teachers' and students' roles in classroom discourse that put the responsibility on teachers to establish classroom discourse that is much more like what Gooya has reported than what Stacey and Lin describe. The comment in the *Standards* that "the kind of discourse described ... does not occur spontaneously in most classrooms" (NCTM, 1991, p. 35) is borne out by Stacey's and Lin's research. We believe that further research and development is needed to show what kinds of progress teachers can make towards the *Standards*' vision, and what sorts of teaching would lead to students interacting productively with one another and learning from one another in cooperative small groups. Results like Stacey's and Lin's may be common, but they are not inevitable.

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THE DEVELOPMENT OF COLLABORATIVE DIALOGUE IN PAIRED MATHEMATICAL INVESTIGATION

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This paper explores the aspects of collaborative dialogue in paired mathematical investigation. A transcribed protocol of problem solving sessions, in which two sixth grade children were videotaped when they were working on an investigational task in pairs, was analyzed. The protocol was selected for it had been obtained as the data of the most successful pair out of six worked on the same task. By the analysis of the protocol, a hypothesized model of successful collaborative dialogue in paired mathematical investigation was proposed. Based on this hypothesized model, some possible teachers' role to urge children to engage in collaborative dialogue were discussed.

Several studies on dialogue in paired problem solving suggest that, strange to say, we cannot always observe what is called "collaborative" processes, which is usually implied by the term "dialogue". Yackel et al.(1991), for example, reported such an example of second grade children working in pairs as follows.

> *The two boys develop different solution methods and do not attempt to achieve consensus in the sense of having a mutually acceptable solution. Nonetheless, their activity is collaborative in that each child's construction is influenced by comments made by the other child. (p.402)*

Similar behaviors were observed by the author when he analyzed the protocols of three pairs of junior high school students working on a construction problem in plane geometry (Shimizu, 1992).

> *Although they worked on the problem together, they proceeded into the different directions. They often asked the partner to explain the approach, listened to partner's comments and exchanged some ideas.(p.333)*

Forman & Cazden(1985) identified the following three types of the interaction pattern in paired problem solving of fourth to fifth grade children ; "parallel", "associative", and "cooperative". According to their study, the interaction pattern that one of the pairs *"seemed to prefer was either predominantly or entirely parallel in nature"(p.335)*. Kroll(1988) reported the similar pattern of behaviors when she observed three pairs of college women.

> *Each woman proceeded with the task with which she felt comfortable, and by diversifying their attack the pair may have doubled their chances of finding a solution to the problem. (p.131)*

These studies seem to suggest two matters. First, when they work together with peer,

the individuals in pairs do not always share the ideas and they may obtain different solutions although they may exchange their ideas and comments with each other. Second, more importantly, we can see the value of having students work together in pairs to solve problems although they may obtain different solutions. From a pedagogical perspective, therefore, it seems to be of significant that we explore how dialogue may aid the problem solving in such situations, and further, when the dialectical processes will take place.

This paper explores the aspects of collaborative dialogue in paired problem solving focusing on the "structure" of dialogue. For this purpose, one of the transcribed protocols of problem solving sessions, in which six pairs of sixth grade children were videotaped when they were working on problems in pairs, was selected and analyzed. The protocol was selected for it had been obtained as the data of the most successful pair out of six worked on the same investigational task.

Method

Subjects: Twelve sixth grade students were selected for this study. Six of them were males and other six were females. The pairing was done by their teacher in the same sex. By considering their achievements in regular school tests and their personal characteristics, it was intended that the individuals in each pair were on an equal footing.

Procedure: Following instructions were given to each pair ; "solve the problem together talking freely. Call me when both of you think your solution to the problem is completed." After these instructions, there was no intervention by the experimenter, until they declared their completion of problem solving. This entire process was videotaped. Follow up interviewing was conducted and audiotaped for clarifying some details. These records were transcribed as verbal protocols to submit for the analysis.

Task used: Three tasks were used in this study. One of those was the following investigational task in arithmetic (Figure 1). This task was adapted from Whitin(1989), who taught a fifth grade class using this task. This task was selected by the following reasons. First, we could expect the elapsed times of the students' pairs would be pretty long and we would be able to observe some mathematical processes (making conjectures, showing counter-examples, and so on) in the students' activities. These would give us the opportunities to explore the aspects of collaborative dialogue. Second, to those children

who work on this problem, the correct answers seem to appear gradually as their investigation proceed. Thus, children would have to determine by themselves when they should, or had better, stop their solution process.

(page 1)		
YOSHIKO took a four-digit number 8532 and reversed its digits.	8532	9643
Then she subtracted this new number from the original.	-2358	-3469
Next, she tried the same operation on 9643.	6174	6174
Using the four-digit number 7861, try the same operation like YOSHIKO. What happens by your operations ? (go to page 2)		
(page 2)		
Can you find any other number that works ? Find these numbers as possible as you can.		

<Figure 1>One of the tasks used in this study

Results

The results of each session are given in the Table 1 and Table 2. Table 1 shows the elapsed time (min.) of each pair. Table 2 shows the number of correct answers found by the students in each pair.

Pair	A	B	C	D	E	F
Time	45m.50s.	33m.50s.	42m.10s.	32m.50s.	47m.20s.	66m.40s.

<Table 1>The elapsed times(min.)

Table 1 indicates that the elapsed time of pair F is as twice as many than pair B and pair D. The mean of six pairs is 44 minutes and 50 seconds.

Pair	A		B		C		D		E		F	
Student	A_1	A_2	B_1	B_2	C_1	C_2	D_1	D_2	E_1	E_2	F_1	F_2
# of answers	2	0	0	1	2	2	13	10	2	0	2	7
Total	2		1		3(1)		21(2)		2		9	

<Table 2>The number of correct answers

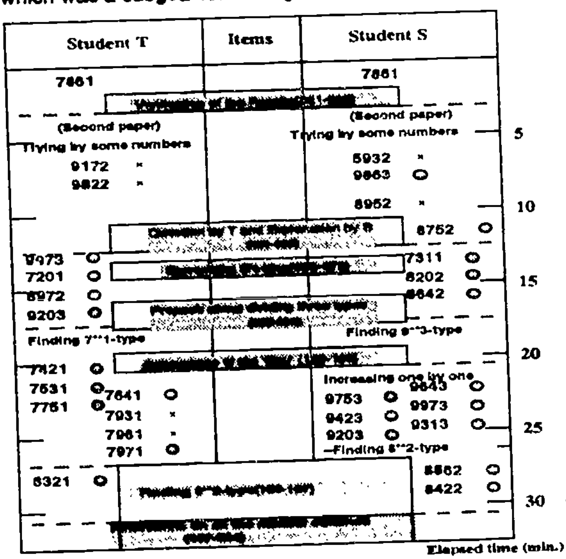
* *( ) shows the numbers of the same correct answers found by two students individually*

As table 2 indicates, the numbers of correct answers of pair D (21) and pair F(9) were larger than the other four pairs. These four pairs could find only 1-3 answers as follows ; Using 1111, which are obtained by subtractions between two numbers in the problem statement(e.g.9863－8532), they got 7421 (8532－1111), 6750 (7861－1111)<pair A, C>.; Exploring the odd and even numbers, they got two answers<pair E>.; Using trial and error approach, repeat the calculation on some numbers<pair B, C>.

An analysis of dialogue in paired mathematical investigation

The protocol of each pair was submitted for the analysis focusing on "problem transformation" (Shimizu,1992). By the term "problem transformation", we mean the phenomena that solvers, successfully or not, transform (reformulate) the problem at hand to easier one or to "related" one in solution processes. In other words, we say that a problem transformation occur, when we observe a change of the "problem" to the solvers in the sense that they restate one goal they are trying to attain to other goal. By examining the verbal items of solvers at these points, we can identify the turning points, where dialogue played an important role in solution process of individuals in each pair.

Figure 2 shows the graph of problem transformations of pair D. In this figure, each dashed line indicates the point where problem transformation was observed. Student T at about 18 minutes past, for example, began to work on a new "problem", "finding the four-digits of type 7**1", which was a subgoal for solving the original problem.



<Figure 2> The solution process of pair D

The protocols of the other five pairs were analyzed in the same manner. The results of analysis showed that, showing "parallel" interaction pattern basically, each student in these pairs tried to find the numbers individually although they exchanged many comments.

As Table 1 and Table 2 show, the pair D (student T & S) solved the problem most successfully. Their success is outstanding for their many correct answers but their shortest elapsed time. And, as shown in Figure 2, while each student had worked alone in several times, they shared in the work. In the followings, to explain their success, we will see briefly the solution process of them and discuss a striking characteristic of their dialogue.

After each of them tried the four-digit number 7861 given in the problem statement, they confirmed the result of each calculation together, which showed the number 7861 worked. Then they began to try other numbers individually.

At the point about ten minutes past from the start, student T, who was in an impasse, asked to student S; "*What are you doing now ?*"(Item 051; this number shows the placement of the utterance in the complete protocol.) At this point, student S had already found that 9863 worked and that all the numbers that worked had such a characteristic as follows; the difference between two ends of the four-digit was six, and the difference between inner two numbers was two. She had not understand, however, the reason why this characteristic made the numbers work. Indeed, because of the 'moving down' on the way of calculations, she "*could not understand the setup of the inner numbers*"(Item 056). Then, she began to examine by herself whether each digit of 9863 could be divided by 3 or not.

Student T, on the other hand, after she listened to student S's explanation, tried some numbers using the characteristic described above. Then she proposed an idea to student S as follows (Note : Each slash " / " corresponds to a pause.).

088. T:ah,this /in turn/by decreasing or/ by increasing/ can we find all of the answers ?
089. S:uhm? / what do you mean ?
090. T:so/ the problem says 'as possible as you can'/ thus/ by increasing one by one
091. S:one by one?/ and all answer?
*092. T:...cause/ nine-three (she is saying about four-digit 9**3)/ this is two-zero, isn't it (saying about the inner two digit)/ three-one/ and four-two/ by doing like this / and the using eight-two(8**2)/ and then using seven-one(7**1)*
093. S:what ?
094. S:uhm? / I don't understand what you mean

At this point, student T had already noticed that they could find the answers by fixing two ends of four-digits like 9**3, 8**2, and 7**1, and by increasing the inner two digits one by one as follows; *20*, *31*, *42*..... . And then, student S also noticed the method by "increasing one by one". She explained her idea to student T, using 8752 as an example.

In this situation, however, although their dialogue seems to hold seemingly, they are misunderstanding the partner's idea with each other.

098.S:If we take this (8752) /and add 1 to each digits we will get 9863/ this works like this
*099.T:yeah/ like that / first, seven-one(7**1),seven-one/ and then two-zero(*20*), here, and three-one(*31*), and four-two(*42*)*
100.S:yes, like that/ just like that/ thus we can got answer by moving one by one

Indeed, the method to find the answers by "increasing one by one" was interpreted differently by each of them as follows. Student T thought of that method as adding 1 to the inner two digits, like *31*, *42*, and so on. Student S, on the other hand, thought of this method as adding 1 to all the digits. The difference between these interpretations appeared in the following conversation. Student T said, *"I'll try the 'seven-one'/ which do you like to try, 'eight-two' or 'nine-three' ?"* (item108). But, student S couldn't understand what T said, responding as *"what do you mean by 'eight-two' or 'nine-three'?"*(item109). After this conversation, they returned to try some numbers individually.

129.S:this...one by one/ you increase this side too?
130.T:no/ this and this (two ends) are not changed/ this is only for nine-three and eight-two
131.S:did you say we couldn't use ?
132.T:in this case/ firstly two-zero/ and then we add to each digit 1/ so, two-zero/ then three-one, just like this
133.S:ah! / is that what you have been said ?
134.T:we can find answers gradually
135.S:yes
136.S:but, we can get the answers by adding 1 to all four-digit, can't we ?
137.T:uhm? / don't know/ but/ you can't use in the case of nine-three/because the sum will yield 0
138.S:that's right

Student S noticed at this point that she had been misunderstanding the idea proposed by student T (item 133). It appeared in her utterance ; *"ah! / is that what you have been said ? "*. However, after this utterance, student S also said like this ; *"but, we can get the answers by adding 1 to all four-digit, can't we ?"* (item 136). Although she approved of student T's explanation, she thought that her own idea was right too.

Student T presented a counter-example to student S's idea. (If we add 1111 to the 9**3-type number like 9643, the sum is not a four-digit number.) By this counter-example, student S abandoned her idea. Then they shared in the remaining work and found all the answers by examining 7**1-type (student T) and 9**3-type (student S) respectively.

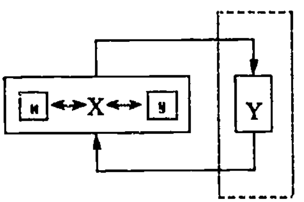
Discussion

The analysis of dialogue between the students certainly suggests some lessons to be considered here. However, for this is an exploratory study, we will focus the "structure" of collaborative dialogue and propose a hypothesized model of it in the followings of the paper.

As described above, in the solution process of pair D, the discrepancy of their interpretations as to the "one by one" had appeared rather lately. The dialogues, however, had been continued to hold. This was partly because student S had been thought of student T's idea about "increasing one by one" as the same to her own. Thus, we can hypothesize that she had been imaged a model of "T's interpretation of 'my' idea" in her head. In this sense, student S supposed herself "in her partner". On the other hand, the interpretation by student S of student T's proposal was different from those intended by student T. It can be interpreted as student S had been imaged another model of student T's idea.

Thus, we hypothesize two mental models supposed by student S. One of which is about "what is the partner(T) thinking of the problem". The other is about "what is the partner thinking of 'my' idea". In dialogue during paired problem solving process, we are both thinking of the problem and imaging the state of understanding by the partner both on the problem and ourselves.

Based on considerations here, we propose a hypothesized relationship between the speaker X and the receiver Y as in Figure 3. The lines with an arrow at end in Figure 3 indicate verbal and non-verbal information and dashed line surrounding Y means that Y is regarded as a "receiver" here. The term "x" shows "speaker X's idea supposed in the partner", that is, mental model of what the partner thinks of "my" idea. Speaker X constructs this "x" and will modify in the process of dialogues if some conflicts take place. The term "y", on the other hand, shows "the partner's idea supposed by speaker". Namely, "y" shows the mental model of what the partner thinks of the problem.



<Figure 3> The hypothesized relationship between X and Y

This model suggests the importance of constructing "internal partner" and "oneself in the partner" and modifying them in response to the comments, suggestions, and critiques from the partner. The successful solution process by pair D seems to confirm this point.

Using this model of collaborative dialogue, we can discuss some possible teachers' role. If we want to make children's dialogue be collaborative in their problem solving

processes, it will be effective to urge children to construct both the "internal partner" and the "him/herself in the partner" and to modify those constructed mental models so as to fill the gaps between their interpretations and partner's utterances.

Thus, it seems to be effective that we guide children based on this hypothesized model. As to this point, it may be of interesting to compare this approach to that of by Yackel et al.(Yackel,Cobb & Wood,1991 ; Wood & Yackel, 1990). Yackel and her colleagues have been shown the possibilities of making children learn the social and classroom norms during paired problem solving and of raising learning opportunities. For example, for the mutual construction of classroom norms, children are urged to explain their solution methods to their partner and to try to make sense of their partner's problem-solving attempts. The construction of social norms seems to correspond to the construction of "x" and modification of "y". For example, making children explain their solution methods to their partner seems to correspond to guiding them to construct 'internal partner' with each other.

It should be noted here that the hypothesized model of collaborative dialogue have to be submitted to further studies. And issues about some possible teachers' role described in this section also have to be examined more closely.

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A FIRST PROBLEM SOLVING COURSE: WHAT STUDENTS SHOW FROM THEIR PREVIOUS EXPERIENCE

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This study examines the work shown by students who received a first problem solving course at the college level. The learning activities used during instruction included the use of nonroutine problems, small group discussions, and the use of diverse cognitive and metacognitive strategies. Seven students were asked to work on two problems (task-based interviews). The analysis focused on discussing dimensions related to the understanding of the problems and the use of cognitive and metacognitive strategies while solving the problems. The results indicated that although the students showed disposition to work on nonroutine problems, they often failed to use basic problem solving strategies. This lead to the conclusion that it is difficult for students to forget what they are used to do to solve problems and that it takes time for students to conceptualize problem solving strategies and use them on their own.

Background to the Study

Problem solving is an issue that has permeated the mathematical curriculum for several years and during the last 15 years has been the dominant model influencing the teaching of mathematics in North America. Stanic & Kilpatrick (1988) pointed out that "problem solving has become a slogan encompassing different views of what education is, of what schooling is, of what mathematics is, and of why we should teach mathematics in general and problem solving in particular" (p. 1). Schroeder and Lester (1989) distinguished three possible interpretations that characterize and differentiate courses based on problem solving. Although there may be common characteristics among these approaches, the main focus of the course and the organization of the material are components that differentiate one approach from another. For example, Schroeder and Lester (1989) identified one approach as "teaching *about* problem solving". This approach emphasizes Polya's four stages (understanding the problem, designing a plan, carrying out the plan, and looking back) identified during the process of solving mathematical problems. There is explicit discussion about these stages when solving the problem and discussion about basic heuristics for solving the problems. The second approach, identified as "teaching *for* problem solving", focuses on the use or application of mathematical content. Therefore, the initial understanding of mathematical content is prerequisite to applying it in various contexts. As a consequence, problem solving emphasizes the applications rather than the understanding of the mathematical content. The third approach to problem solving is that in

which mathematical content emerges from a problem solving situation and in which this situation actively engages the students in the process of making sense of content. This approach is identified as "teaching *via* problem solving". It is suggested that this approach has similarities with the process of developing mathematics. Schoenfeld (1989) suggested that it is possible to provide a class environment in which the students not only understand mathematics via problem solving but also develop mathematical content. Schoenfeld (1985) provides a model to analyze the process used for the students while solving problems. This model identifies four dimensions that influence the ways that students solve problems: domain knowledge, cognitive strategies, metacognitive strategies, and belief systems.

Domain knowledge includes definitions, facts, and procedures used in the mathematical domain. Cognitive strategies include heuristic strategies, such as decomposing the problem into simpler problems, working backwards, establishing subgoals, and drawing diagrams. Metacognitive strategies involve monitoring the selection and use of the strategies while solving the problem, that is, deciding on the types of changes that need to be made when a particular situation is deemed problematic. Belief systems include the ways in which students think of mathematics and problem solving.

Aim of the Study and Methods

The aim of the study was to analyze the students' approaches to two mathematical problems. The subjects were students who took a calculus course at the college level with emphasis on problem solving. Thinking aloud was the method used to gather information about the students' processes involved in solving the problems. Seven students (volunteers) were asked to work on the problems at the end of course. The problems were:

1. Find values of a and b so that the line $2x + 3y = a$ is tangent to the graph of $f(x) = bx^2$ at the point where $x = 3$ (line problem). From Selden et al (1989).
2. Find all rectangles with integer sides whose area and perimeter are numerically equal (rectangle problem).

Results from the Interviews

The students initially read each problem once and immediately started to recall similar problems that could help them solve the problems. For the first problem which involved finding two constants a and b that would determine a tangent line to a parabola, the direction that the students took was to make some

calculations, such as working on the derivative and trying to relate it to the statement of the problem. For the second problem which did not involve current terms, the students spent more time trying to understand the statement before getting involved in getting the solution.

It seemed that the context in which the problem was set often provided them with the elements to design a plan. For example, Lydia expressed: "...it is easy if you have done a hundred in a book that are exactly the same kind; but when you get any of these without any reference it's harder to start." It was observed that self-monitoring processes which were sometimes initiated by the students themselves helped them to analyze the information provided in the problem more carefully. For example, reflecting on "What do I want to do?" led some students to transform the representation of the line into the "slope-point representation" and to analyze the relationship between the slope of the line and the derivative of the parabola.

The students' analysis of the information was focused on data that they could easily transform. For example, in the rectangle problem, the students rushed to write down the formulae for the area and perimeter of the rectangle. For the line problem, the students calculated the derivative of the parabola. It seemed that the familiar terms involved in the statement were used by the students as the initial point to calculate or add more information about the problem, even though the students often struggled using that information. For example, Alex after having calculated the derivative of the parabola asked whether "b" was a constant. This suggested that he got involved with several calculations but without being clear about the role of key information given in the statement of the problem.

The students' exploration of ways for solving the problems relied on first using the familiar information either to represent the problem or to process the information through the use of specific concepts familiar to them. For example, for the rectangle problem, the students wrote down the expression of the area and perimeter and equated them. Up to this point, the students transformed the statement of the problem into the corresponding symbolic representation, that is, $ab/2 = 2(a + b)$. Although they isolated one of the variables included in the equations, they did not know how to interpret this result or finding. At this stage of the problem, they tried to reflect on or look for more information that could help them use this information. Some of the students even went back to the original equation and substituted the obtained value again. This may suggest that for the first exploration the students tried to fit the information given in the problem in some sort of frame that they were already familiar with instead of

focusing on a possible new frame that could emerge from the information given in the problem. For example, they kept working on finding a solution for a and b in the rectangle problem from the equation, even though the transformations that they made to the equation did not show any progress towards the solution.

It is suggested that the students did not use a systematic exploration to consider plausible ways for solving the problems. They did not consider various alternatives before deciding which direction to take. It seemed that the students' only alternative in pursuing the solution was to relate the familiar given information to another familiar problem. Mike for example, found the derivative of f(x) and experienced difficulty in using it. He expressed: "I am not sure how to relate this to the equation of the line. I guess I need to substitute this value somewhere but I do not see where...."

New considerations for approaching the problems often came after having worked on the calculations that were related to some terms involved in the statement of the problem and not being able to use them to solve the problem. For example, "tangent to the graph" suggested to them to get the derivative of the function; "at the point 3" suggested to the students to evaluate at 3; "the straight line tangent to" suggested the same slope. However, they failed to recognize the role of the parameters a and b and the relationship with the information that they obtained from the statement. It was evident that they knew the content involved in the problem but were unable to make the necessary connections for using it.

It is suggested that the students made sense of the statement of the problem in parts without considering the problem as a whole. For the rectangle problem, the students also explored the familiar terms until they got an expression that related one side of the rectangle as a function of the other side. The students at this stage did not know how to deal with this expression and struggled to relate it to the statement of the problem. Maria after having represented the rectangle problem symbolically expressed, "I am not sure what it wants me to say, does it want me to say how long the length and the x's have to be? What would I do?"

While working on the line problem, the interviewer asked them to change the values of the constants a and b and to explain what happens to the line and the parabola; the students then realized that they were dealing with a family of lines and parabolas. For the rectangle problem, the students initially hesitated to try specific examples in the expression in which one of the sides of the rectangle was isolated. It may be suggested that the students did not consider that the use of guess and test could help them solve the problem. They focused

on searching for an algebraic approach that finally could provide the lengths of the sides of the rectangle. The lack of success in finding that algebraic approach led the students to explore some cases for one of the sides $a = 2b / b-2$) and they were able to identify some numbers for the sides. It was then that they started to use this strategy to analyze the relationship. Finally, the students were able to limit the domain of the expression and consequently solve the problem.

Students' Beliefs about Mathematics and Problem Solving

To examine to what extent the students' views about mathematics changed after instruction, the students were asked at the end of the task-interview to reflect on the activities in which they had been engaged during instruction. The results showed that the students considered mathematics to be an important subject that could be applied in various areas. They found calculus a more interesting course compared to their previous courses in mathematics. They found the assignments challenging and interesting because there was much discussion about each of the problems. Lydia, at the end of the interview, stated:

There is a lot of people in our class who are enthusiastic, people who argue their points; we argue the points when we think it's the right point. ... I think we are learning more because we understand what is going on, because we are thinking instead of just doing exercises from the textbooks.

It is suggested that the class interaction was of benefit for the students. For example, Don stated that there were many concepts that came out from the class discussions without his even realizing that they were there. For example, he stated:

... once in class, just before we were doing derivatives, where actually we were doing derivatives and nobody knew it . That was interesting because after when we started doing the actual...like we got the name for it, we already understood what was going on and then it was very fast after that...

Another student Linda, in the same vein, when referring to the development of the class expressed:

I would like to say that this class is different from the ones that I had before, and I like this one the most. This is the best class so far, and I guess it is because [there are] more assignments and more thinking about math than just doing it.

In general, the students commented at the end of the task-interviews that as a result of the calculus course, they were motivated in discussing

mathematical ideas with their classmates. They also indicated that it is important to understand the underlying concepts associated with the problems and not only to master the content. Although the students experienced some difficulties while working with the problems, it seems that their willingness and motivation showed during the course could be the initial point to construct a more integral way for problem solving.

Discussion of the Results

Although during class instruction the students spent time discussing the importance of representing a problem, the students showed no intention of representing the first problem graphically. For the problem that involved finding a and b , the students did not recognize that they were dealing with a family of lines and parabolas. It may be suggested that although the students may identify the general equations of the line and parabola they experience difficulty interpreting the meaning of the parameters involved in the equations. For the rectangle problem, the students drew a rectangle; however, it did not provide useful information to approach the problem. That is, the students did not use the representation to work on the solution.

Polya (1945) recommended the use of guess and test to interpret the problem, to represent the problem, and to monitor the process while solving the problem. The students did not seem confident in using these strategies. They might use guess and test as the last attempt. For example, in the rectangle problem, they stuck to the algebraic approach for a long time trying to solve the equation. Although the students noticed that no progress was made when dealing with the algebraic expression, it was only when the interviewer asked them to try some particular cases that they realized that the testing method was useful in this problem.

The students also had some expectations about the types of results that they could get at different stages when solving the problem. If the results did not match their expectations, they often changed the approach or spent quite a long time looking for an error that often did not exist. For example, Lydia, after she obtained the derivative and evaluated it, stated, "First thing, I think this looks like a wrong number; I must have done something wrong,....It does not look right. You know immediately. You think that is not like five, or four. Let me check if I have done something wrong...." Schoenfeld (1985) pointed out that the type of examples presented during mathematical instruction and the type of assignments that the students work on determine the students' ideas about problem solving.

There is indication that the students normally monitored their processes as a response to a certain type of anomaly that they detected while working on the problem. For example, if they experienced some difficulty in accepting what they were getting or if the approach used did not show progress, then they asked themselves some kinds of metacognitive questions. However, monitoring their processes not did always lead to a better approach to the problem. For example, realizing that they could not solve the relationship representing the equality of the area and perimeter of the rectangle, some of the students substituted the isolated value in the original equation which was exactly the same equation that they used to isolate one of the parameters.

Although the students did not follow a systematic plan in order to solve the problems, they showed several transition points at different stages of the process while solving the problems. For example, the initial interaction with the problems was to read the problem and explore or transform the familiar terms involved in the statement. The next stage was to become involved in calculations or representations of the problem in accordance with the terms that were familiar to the students. At this stage, the students often used metacognitive strategies as a response to some type of difficulty found when working on the calculation. As a result, they often went back to the original statement of the problem and checked for additional information or other alternatives that could help them to overcome the difficulties. It seemed that at this stage the students identified the important information of the problem and its relationships. The students actually understood the problem. The final stage involved getting the solution and checking the process involved in solving the problem.

The students were asked to reflect on some of the difficulties that they experienced while solving the problems. They indicated that their first attempts at solving the problems were based on using their understanding of the problems that were discussed during the class. For the line problem, they mentioned that the term "tangent" triggered the use of derivative, whereas for the rectangle problem, the conditions of equating the area and perimeter suggested the use of an equation. Perkins (1987) identified two mechanisms for transfer: "low-road transfer" which may occur when the students solve a problem in one context and this triggers the students to solve another problem which resembles the previous one and "high-road transfer" which occurs when the students are able to apply the content in other contexts which may not be similar to the context studied in class. It is suggested that the students spent time exploring for "low road transfer" when they were asked to solve problems.

Conclusions

Santos Trigo (1990) indicated that people initially resist the use of activities that are different from the ones that they normally carry out. The process of assimilating and implementing problem solving strategies in mathematics instruction should be seen as an ongoing process in which there is always room for improvement and adjustment. There was evidence that students need to discuss their ideas with other students and the instructor in order to clarify, defend, or use what may help them solve the problem. It was observed that the students often had ideas that were useful to explore the possible solution but they did not know how to use them. It is recommended that problem solving instruction should encourage students to discuss their ideas and work on problems with their classmates. The instructor should also challenge and discuss the students' ideas while solving problems. This type of activity could help students develop a frame for using and judging mathematical ideas.

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III-19

A STUDY OF METACOGNITION IN MATHEMATICAL PROBLEM SOLVING: THE ROLES OF METACOGNITON ON SOLVING A CONSTRUCTION PROBLEM

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ABSTRACT

This paper reports the roles of metacognition on solving the construction problem of a quadrangle. The subjects of this research are elementary school pupils of third graders and fourth graders.

Results suggest that metacognitive knowledge and metacognitive skill play a vital roles on problem solving. Especially in this research, we can confirm through protocol analysis that metacognitive knowledge about the strategy dominated the orientation of pupils' solution method.

Moreover the effects of paired problem solving are discussed. As a result, we can observe that some pairs work on a problem cooperatively, but the other work on cooperatively at first and work on individually on the way. Furthermore, although paired problem solving inclines to evoke metacognitive activities in general, results suggest that it doesn't always produce the positive effects.

INTRODUCTION

For the past decade, much attention has been paid to the roles of metacognition on mathematics education as well as on pure psychology (ex. Garofalo and Lester, 1985; Schoenfeld, 1985,1987). Among children who posses the same knowledges to be needed for solving a problem, we often observe that some children can solve , but the other cannot. Why does these differences reveal? In other words, it seems that having knowledges to be needed for solving a problem is one thing, and it's quite another thing to be able to use that knowledges actually. The latter, that is, being able to use knowledges actually, is related to aspects which involve monitoring a state of one's own knowledge base and performance of problem solving, controlling one's own solving behavior and evaluating the process of problem solving and so on. These aspects are related to the issues of "metacognition". Inspite of many preceding researches about metacognition, the features and roles of metacognition have not been elucidated enough and the concept of "metacognition" is still a ambiguous and fuzzy concept.

In this paper, we report the roles of metacognition on mathematical problem solving by analyzing the third graders' and fourth graders' performances as to the construction problem of a quadrangle. Moreover the effects of paired problem solving are discussed.

THEORETICAL FRAMEWORK

1. Classification of Metacognition

Roughly speaking, metacognition refers to something like "thinking about thinking", "reflection on cognition" or "cognition about cognition". As to a classification of metacognition, a few views have been proposed. For example, according to Flavell who is pioneer of this area, he distinguishes between "metacognitive knowledge" and "metacognitive experience" (Flavell, 1979, 1981). And he points out that "metacognitive knowledge" has three kinds of variables, that is,

person variable, task variable and strategy variable.

Brown who is one of vigorous researchers of metacognition has emphasized the dynamic aspects of metacognition such as monitoring, self-regulation, executive control, planning and checking (Brown, 1983, 1987). Furthermore, in the developmental psychology, Sternberg refers to metacognitive aspects as his word "metacomponents" which are higher-order control processes used for executive planning and decision making in problem solving from the view point of information processing approach (Sternberg, 1980, 1982). In Sternberg's componential approach, two types of circulating mechanism are distinguished. One of two is the circulating mechanism of knowledge system or concept and "metacomponents". The other one is the circulating mechanism of "performance components" related to execution and "metacomponents". In Sternberg's model, metacomponets play a vital roles on the intellectual performance and metacomponents control the other components such as acquisition components, retention components, transfer components, and performance components.

On the other hand, as Schoenfeld described, research on metacognition has focused on three related but distinct categories of intellectual behavior: (1) your knowledge about your own thought processes, (2) control, or self-regulation, (3) belief and intuitions (Schoenfeld, 1987). In Schoenfeld's framework, belief systems are emphasized as a important aspect related to metacogniton as well as metacognitive knowledge and metacognitive skill. Indeed, it seems to me that belief system have a great influences on mathematical problem solving performance. For example, the belief such as "*I feel that it's difficult to solve a word problem.*" will have a negative effect for solving a word problem, and the belief such as "*I like mathematics*" will have a possitive effect and would promote learning of mathematics.

Although there are many proposals about metacognition considered above, the conception of metacognition is not still integrated enough. By synthesizing views described above, we represent a basic classification of metacognition according to Flavell and Brown and represent the relation of metacognition and belief system shown as Fig.1 in this research.

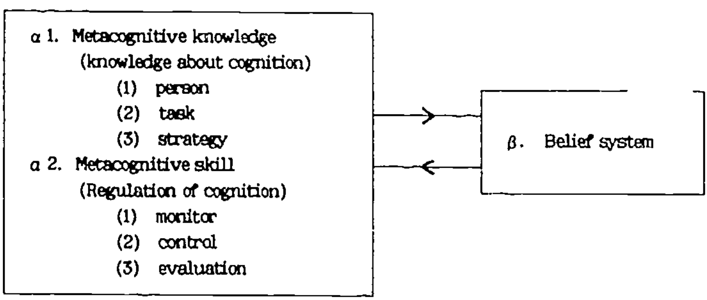


Fig.1
Classification of metacognition and the relation of metacogniton and belief system

Some explanations will be needed about this framework. α 1 and α 2 represent two aspects of metacognition, that is, metacognitive knowledge and metacognitive skill. And β represents belief system. As to the relationship of metacognition and belief system, if a strong impression were given to some metacognitive knowledge or metacognitive skill and that metacognition was strengthened, it

seems that such metacognition will be stored as a kind of belief.

In this paper, the roles of metacognition based on Fig.1 will be discussed through the empirical research as below.

2. Framework for Analysis

As one criterion of a classification of student's performance, we adopt the types of control which is one of the important factors of metacognition. For the analysis by using the types of control is easier for us to grasp the change of student's performance than the analysis by using the other factors. In other words, the effects of control incline to reveal on the student's constructions on papers directly because it seems that students solve the given problem under some controls ultimately after monitoring, checking, and self-evaluating. That is, the transition on the paper represents the effects of some controls. Thus, in this research we will classify the student's performance based on the types of control, and will discuss the relation of other metacognitive factors and student's beliefs.

As Schoenfeld (1985) points out, there are four types of control: 《(Type A) Bad decisions guarantee failure: Wild goose chases waste resources, and potentially useful direction are ignored. (Type B) Executive behavior is neutral: Wild goose chases are curtailed before they cause disasters, but resources are not exploited as they might be. (Type C) Control decisions are a positive force in a solution: Resources are chosen carefully and exploited or abandoned appropriately as a result of careful monitoring. (Type D) There is (virtually) no need for control behavior: The appropriate facts and procedures for problem solution are accessed in long-term memory.》

In this paper, we classify the student's problem solving performances by using these four types of control because of the reasons described above, and consider the differences of problem solving performance.

METHODOLOGY OF THE RESEARCH

We asked third graders and fourth graders to solve the construction problem of a quadrangle as shown below. All they had learned the concepts of a quadrangle and have known how to use a compass.

Subjects

Four third graders and four fourth graders are asked to solve a problem in pairs. For reasons of convenience, we represent pupils by assignning numbers to them as following. Fourthermore, pupils were divided into two groups, that is, Group A and Group B. In Group A, each pair was given one problem sheet written a problem. On the other hand, in Group B, each pupil was given one problem sheet. Discussion and working on a problem in pairs were permitted in both Group A and Group B.

《Grouping》

(Group A) Each pair has one problem sheet.
Third Graders-----Pupil ①, ②
Fourth Graders----Pupil ③, ④

(Group B) Each pupil has a problem sheet.
Third Graders-----Pupil ⑤, ⑥
Fourth Graders----Pupil ⑦, ⑧

Method

The entire process of problem solving were videotaped, and which were used for analysis. After pupils solved a problem, the researcher asked some questions about their solution process for following. Pupils' responses to questions were useful to pursue the effects of metacognition and paired

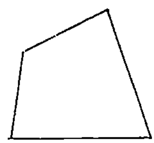
problem solving.

As to the adequacy of their construction, they could ascertain the adequacy of one's own construction by overlaying the transparent sheet which was written the given quadrangle on a problem sheet.

Problem

Task used in this research was the following construction problem of a quadrangle.

Draw the same quadrangle in both in size and in shape as a quadrangle shown below. In construction, you are permitted to use only a compass and a straightedge.



RESULTS AND DISSCUSSION

The outline of pupils' constructions and their solution processes are shown as table 1 and table 2 at the end of this paper (Although some detailed comments about pupils' performances are needed, they are ommitted here on account of space consideration). As we desrribed above, we adopt the Schoenfeld's framework of "the types of control" for the classification of pupils' performances. According to table 1 and table 2, we can not say in general that fourth graders are superior to third graders on the performances in this research, which means that the difference of the grade are not influenced to their performance of problem solving. Here we will suggest some implications from the results of this research.

The Roles of metacognition on solving the constrction problem

(1) The effects of metacognitive knowledge about the strategy

As tables show, we can find that metacognitive knowledges about the strategy play a vital roles on problem solving.

First of all, metacognitive knowledge about the strategy ,that is, "*We will be able to solve this problem if we translate a given quadrangle.*" influenced the performance of pair ⑤ and ⑥. Moreover they adoptted the strategy raised by this metacognitive knowledge from beginning to end and failed ultimately.

Fourth graders ③, ④ possessed such metacognitve knowledge that "*We will be able to solve if we focus on the diagonal of this quadrangle.*" at the begginig of their solving and could make a correct construction using the method raised by this kind of metacognitive knowledge.

In fourth graders ⑦, ⑧ especially, metacognitve knowledge about the strategy dominated their performances strongly. In this pair, they tried to work on the problem cooperatively at first. However they began to solve individually on the way because their metacognitve knowledge about the strategy were different. Pupil ⑧ had such metacognitive knowledge about the strategy that "*We will be able to solve this problem if we focus on the diagonal of a quadrangle.*" On the other hand, pupil ⑦

rejected pupil ⑧'s proposal and began to draw a rectangle which surround a given quadrangle. After that, he tried to make a construction by measuring the length of each vertex of the rectangle and each vertex of a given quadrangle. Pupil ⑧ had proposed his idea to pupil ⑦, but they didn't reach at agreement finally. We can catch this phenomenon from pupil ⑦'s utterance which is "*We won't solve even though we draw a diagonal of this quadrangle.*" (Pupil ⑦ told ⑧.).

In third graders ①, ②, we can point out that metacognitive knowledges about the strategy play negative roles. For example, as pupil ① said "*Although it seems to me that we have to take the angle, how can we know how to take the angle?*", they thought that taking the angle would become the first step to the success on problem solving. After all, this led to the failure.

Consequently, the differences of metacognitive knowledge about strategy dominated the pupils' performance. Furthermore, we can find that many metacognitive knowledges about the strategy incline to reveal at the planning stage of problem solving in general.

<u>(2) The effects of monitoring on matehmatical problem solving</u>

In this paper, we can confirm that monitoring which is one of the important factors on the performance of problem solving and also plays a vital roles on pupils' performances.

In third graders ①, ②, we can identify the monitoring behaviors of their solution method from protocol. For example, pupil ①'s utterances such as "*It seems that this is something wrong.*" or "*Is this construction adequate?*" reflects on the results of metacognitive skill about monitoring.

But this pair ①, ② could not make a adequate construction finally, which means that it is not always a sufficient condition although it is a necessary condition for success on ploblem solving to be evoked metacognitive skill about monitoring. Thus, it seems that it's important how pupils should act after monitoring, which would be issues of "self-evaluating" and "control".

The effects of paired problem solving

In general, it seems that working on the problem in pairs promotes problem solving. We can obtain some implications about the effects of paired problem solving.

(1) We can identify from protocol that the peer in some pair plays the roles of the metacogntive behaviors to the other. For example, in fourth graders ③, ④, although pupil ④ tried to adopt the method which won't lead to the success at the beginning, pupil ③ proposed him the strategy of focusing on a diagonal. After this proposal, they seized an opportunity to promote their solution and modified their strategy. On the other hand, in pair ⑦, ⑧, pupil ⑦ offered opposition to pupil ⑧ although pupil ⑧ tried to propose the method of focusing on a diagonal some times. These reveal at the following pupil ⑦'s utterances, that is, "*We won't be able to solve this problem because we can't take the angle of a given quadrangle.*" After all, pupil ⑦'s idea led to restrain pupil ⑧'s performance. In this sense the utterance of pupil ⑦ work on as the negative role of metacognition for pupil ⑧.

(2) In this research, we try to extract metacognition as a form of verbal data as well as promoting pupil's performances by paired problem solving. Although we think that we can reach at the aim of this research described above as a whole, the following issues has come clear at the same time.

As keiichi Shigematsu (1992) points out, metacognition has two aspects which are possitive one and negative one. In this research, paired problem solving evoke the possitive metacognition at one time (ex. pair ③, ④), but that evoke the negative one at the other time (ex. pair ⑦, ⑧). Consequently it seems that paired problem solving don't always evoke the possitive metacognition.

Moreover we can confirm cases in which it seems as if they had worked on the problem

cooperatively. However, as a matter of fact, there were some cases in which each pupil solve the problem individually. In this paper, each pair was provided one problem sheet in group A. On the other hand each pupil was provided one problem sheet in group B. As a result, pupil in group A inclined to work on the problem cooperatively by using the same method. But, because each pupil in group B provided one problem sheet, pupil in group B inclined to solve individually when the idea of each pupil in pair was different each other. It seems to be one of issues that we need to develop more adequate methodology for promoting pupils' positive interaction.

BY WAY OF CONCLUSION

In this paper, we have considerd the roles of metacognition, especially, metacognitive knowledge about the strategy and metacognitive skill about monitoring on mathematical problem solving. Furthermore, we have discussed the effects of paired problem solving.

As to the methodology, we adopt the method of paired problem solving to extract metacognitive behaviors. And we videotaped pupils' solution processes and try to identify the roles of metacognition through the protocol analysis. In this sense, we have assume that the change of pupils' utterance and behaviors reflects on the effects of internal metacognitive activities. Although it seems to me that this protocol analysis is adequate to some extent, it will be open to question. Thus, one of issues to be considered will be about the adequacy of verbal reports as data.

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Table 1. Constructions of Group A

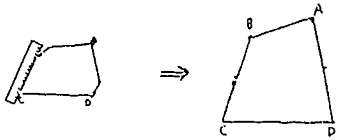
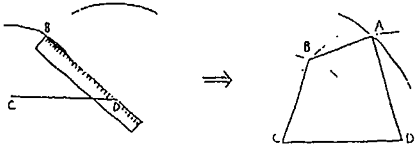
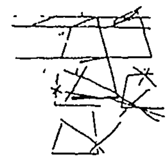
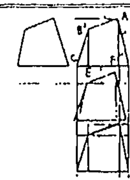
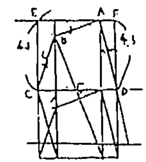
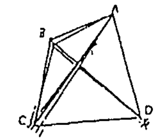
Type of control	C	
Pupil	Third Graders ①. ②	Fourth Graders ③. ④
Construction		
Solution Process	They constructed a middle point of AB, DA and their first construction method was not useful for getting a correct answer. After some trials, they focused on a diagonal and adopted a new method of measuring a length of diagonal BD. However, when they constructed point A and B, they used a straightedge insted of a compass. This construction method wasn't correct complete-ly.	They constructed a quadrangle adequately from the beggining although they troubled with the place of point B. After that, they began to notice of the existence of a daiagonal and finally could make a correct construction.
Features of Solution Process	Although there were some problems in how to use a compass, their construction method was almost adequate.	When they started their construction, pupil ④ told his partner ③ that we could solve this problem by focusing on the diagonal and using a compass. Moreover pupil ④ had a powerful confidence about her solution method.

Table 2. Constructions oı Group B

Type of control	A		B	
Pupil	Third Grader ⑤	Third Grader ⑥	Fourth Grader ⑦	Fourth Grader ⑧
Construction				
Solutiuon Process	After they discussed about their solution method, they began to measure the length of each side of a given quadrangle. Moreover, after some trials and errors, they draw parallel lines and tried to construct by translating a given quardrangle. They couldn't make an adequate construction ultimately.		He draw rectangle CDFE and then measured the distance from each vertex of rectangle CDFE to each vertex of a given quadrangle (ex. the distance between point E and point A. He couldn't make a adequate construction ultimately.	He measured the length of diagonal and then paid much attention to triangle ABC and ABD. However his construction of triangle is not adequate and he couldn't reached to the correct construction.
Features of Solution Process	They thought impressively that we could make a construction by tanslating a given quadrangle. Although they failed in the first challenge, they used the same method as first one in the second challenge.		He possessed the belief such that he could make a construction by measuruing the length of each side and the distance from each vertex of a rectangle CDFE to each vertex of a given quadrangle.	He possessed the poweful belief such that the existance of a diagonal of a given quadrangle was related to the construction method. But that belief was [illegible]t exploited ultimately.

APPENDIX 2

Teacher Students' Use of Analogy Patterns

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Analogy plays an important role in our intellectual life, and especially teacher students should get an impression of its importance by their own experience during their studies. An investigation has been carried out, part of which was an analysis of the patterns teacher students used in order to obtain statements about "elementary" geometrical figures by modifying given statements. The latter refer to comparisons of lengths (using the relations = and <) in figures composed of line segments. The used patterns belonging to a special statements are illustrated by examples.

Analogy is recognized as an outstanding principle in the historical development of science, in learning and understanding (obviously not being restricted to mathematics or science learning), in problem solving, in teaching processes, and many other intellectual activities (f.i. language, tales). So, mathematics education also has to contribute to future teachers' understanding of the role analogies play in human thinking.

In several publications, G. Polya shows, by convincing examples, the role of analogies in discovering new properties of mathematical objects or in trying to solve problems, especially to find proofs. The importance of analogical reasoning, particularly in mathematical thinking, could be confirmed by psychologists, who f.i. found that especially students highly gifted in mathematics use analogies in extraordinarily skilful forms, when storing information in memory and processing information (cf. Klix, van der Meer).

According to G. Polya, going back to Thomas of Aquino, analogy is a kind of correspondance between relations; if we consider two different domains of objects, one of them being more familiar than the second to a problem solver, a transfer of knowledge may consist in finding corresponding relations (or transformations) between these two object domains and specific relations in each of them (cf. Polya, 1954, p. 35; a more formal analyse of the analogy concept f.i. in Klix 1979, p. 163) Of course, an analogy transfer usually will not be of cogent character; we always have to check whether or not a result obtained by analogy transfer is valid.

So, applying an analogy consists in two steps quite different from one another:

- identifying correspondences between two different topic areas, one of them already known or more familiar to a learner than the other, and formulating a new property assumed to be valid within the latter, obtained by some kind of transfer of relations holding in the better known topic area
- checking whether or not the new result obtained by analogy transfer actually does hold.

Students have to understand both steps as components of its own, which, thus, can be carried out seperately.

Analogy transfer may consist in one or more of the following patterns for modifying an initial property or statement:

- reducing or extending complexity of a topic under consideration (concept, theorem, argument) or a task (with respect to the formulation or to the means to be utilized)
- reduction or increasement of the number of parts determining the topic (number of variables, of vertices, edges, faces, or other parts of a geometrical figure)
- increasing or reducing dimensions of parts or of the whole figure or exponents in equations, function terms, inequalities
- changing number or kinds of the parameters determining the topic, of construction or proof steps
- exchange of premise and conclusion of a theorem, forming new combinations of partial premises or partial conclusions.

Teacher students (elementary and secondary level) in a beginners' course (comprising elementary geometry and elements of linear algebra) in fall 1992 were asked to find further statements by modifying initial statements, using analogy principles. These concerned quite "elementary" figures, namely line segment and combinations of line segments, defined f.i. by any inner point of a triangle, the properties being restricted to the relations = and < between lengths of line segments; the initial list of properties thus contained the triangle inequality and additivity property of length function for line segments, as well as conclusions drawn from these. The context was an attempt to establish a logical order within the system of the statements which leads to seperating some of them as a basis for argumentation and the others as deduced theorems (according to H. Freundenthals program of "ordering a field" of topics (Freudenthal, 1973, p. 128).

During the process of logically ordering areas of properties of elementary figures (also with examples different from the topic "Inequalities") arose several opportunities to show the importance of analogy transfer and to illustrate its role by examples. Especially these examples manifested the need to check statements obtained by analogy transfer.

The mentioned beginners' course comprises weekly exercise activities, one special exercise was given to the students asking but generating new state - ments by applying analogy principles to a list of initial properties.

The topic "Inequalities" contains the following statements:

For $C \in \overline{AB}$: $L(\overline{AC}) \leq L(\overline{AB})$ (1)

For $C \in \overline{AB}$: $L(\overline{AB}) = L(\overline{AC}) + L(\overline{BC})$ (2)

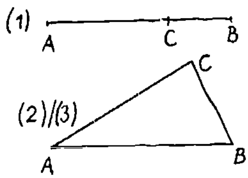
For $C \notin \overline{AB}$: $L(\overline{AB}) < L(\overline{AC}) + L(\overline{BC})$ (3)

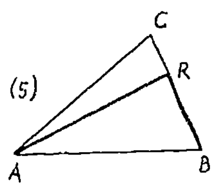
For any inner point P to Δ ABC: $L(\overline{AP})+L(\overline{PB}) < L(\overline{AC})+L(\overline{CB})$ (4)

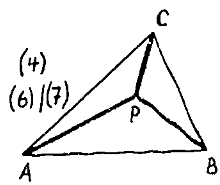
For any point $R \in \overline{BC}$ ($R \neq B, C$): $L(\overline{AR})+L(\overline{RB}) < L(\overline{AC})+L(\overline{CB})$ (5)

For any inner point P to Δ ABC: $L(\overline{AP})+L(\overline{BP})+L(\overline{CP}) < L(\overline{AB})+L(\overline{BC})+L(\overline{AC})$ (6)

For any inner point P to Δ ABC $\frac{1}{2}(L(\overline{AB})+L(\overline{BC})+L(\overline{AC})) < L(\overline{AP})+L(\overline{BP})+L(\overline{CP})$ (7)







The present enumeration follows the sequence in which these statements were originally "discovered" and listed according to the actual starting situation: "ordering" a field of properties of elementary figures.

The first part of the exercise demands a transition from line segments to angles, the second from plane objects to spacial ones. A detailed explanation in the task sheet repeats some topics about analogy transfer, mentions some illustrating examples (an oral explanation adding a hint to the analogy between circumscribed and inscibed circle of a triangle), suggests to write a "word-book" for the transition from one topic domain to the other, and stresses upon the "ambigous" role of analogy transfer, especially directing to the fact that a correctly applied analogy not necessarily provides a valid statement.

In order to separate consistency in applying analogies from (obvious or guessed) validity of a statement, the students were asked to assess whether or not the statements they had found were also valid.

50 students (among approximately 70 participants) gave written answers. They need a certain amount of correctly solved exercises to obtain a certificate about the success of the course, but usually have enough correct solutions altogether. Of course, it is difficult to decide whether an answer in this context "deserves" the assessment "solved correctly", but the special situation of necessarily vague answers suggests a high degree of "generosity".

The written answers were analyzed according to the patterns of analogy transfer, and to the kind of objects which the obtained statements refer to;
- for the transition line segment --> angle:
angles with the same vertex, angles in a triangle and in partial triangles, angles belonging to a figure "similar" to the picture of a fan blower (in the plane),
- for the transition plane --> space
lines, faces of solids, or prisms ("placed" on the plane figures) in the space.

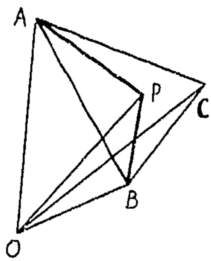
Among the answers there were on the one side fairly poor ones, on the other side those which turned out to be remarkably original.

A concept reported earlier, then connected with forming geometric proofs (Becker, 1986), is also applicable to analogy transfer. If a given statement is "transformed" in a new one, by drawing logical conclusions, or by using analogy patterns, the process of transformation can be described by applying an operator, the admissibility of which has to be checked in advance. The difference is that in tranfer by analogy the "result" is by far not determined as rigidly as in proof tasks, due to vagueness in selecting operators and incertainty about whether or not objects described in the initial statements have to be replaced in order to enable a meaningful analogy transfer.

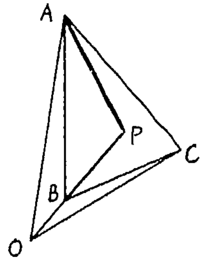
Here is a selected sample of observable answer categories, belonging to line 4. (above):

a) constituting a transition "line segment --> angle":

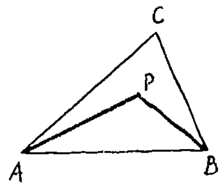
(1) a set of 4 rays with the same endpoint O, the rays running through the corresponding points forming the triangle and one of its inner points; the line segment with endpoints A and B corresponds to the angle AOB or BOA, and so on



(1)



(2)



(3)/(4)

(2) the same as before, only 3 rays (such that the obtained statement holds)

(3) comparison of the sums of angle measures in the "whole" triangle ABC and the "partial" triangle ABP, the angle vertices being points A and B

(4) comparison of the sums of angle measures in the "partial" triangle ABP (as in (3)) and the "outer" quadrilateral APBC, the angle vertices being A and B

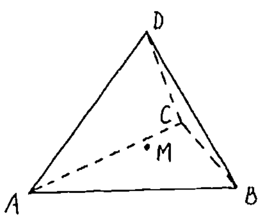
(5) possibly originally the same idea as (1), but "corrected" in order to obtain a valid statement, which now seems to be an application of the addition postulate of inequality (one letter actually is cancelled and replaced by another)

(6) no sum (of angle measures) occurring in the statement

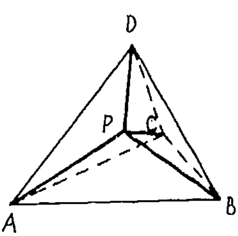
b) constituting a transition "plane --> space":

Here, of course, a bigger variety of different categories can be found, because of several opportunities to increase the dimension.

(i) similar to the corresponding statement for the plane triangle, but especially for the centroid as point P



(i)



(ii) - (vi)

(ii) the inequality formulated for the three edges of tetrahedra, having points P and D respectively as vertices (P being any inner point of the original tetrahedron with vertex D - as in the following categories -)

(iii) the inequality formulated for 2 lateral faces of tetrahedra, with points P and D respectively as vertices

(iv) the inequality formulated for the 3 lateral faces of tetrahedra, with points P and D respectively as vertices (increasement of the number of "lateral elements" under consideration)

(v) the inequality formulated for 3 lateral faces of pyramid, with points P and D respectively as vertices (increasement of the number of vertices of the "basis", and of the number of "lateral elements" under consideration)

(vi) the inequality formulated for the 3 angles of tetrahedra, adjacent to points P and D respectively, as vertices.

The answers to 5., as well as to other statements, reveal all types of analogy principles listed in advance.

General results:

Only few answer sheets showed a rigid consistency using the very same analogy transition for all the 7 statements.
Among them : rays with one common endpoint (category (i))

The whole list of initial statements can be more or less divided into pairs of successive "similar" ones (1. and 2.; 3. and 4.; 4. and 5.; 6. and 7.) Thus, most of the answers use seperate analogy transitions for these pairs.

Some probands gave more than only one answer to single statements, especially in the case of spacial objects (part b): the transitions
line segment --> (spatial) line segment (edge and so on) in a tetrahedron,
line segment --> face of a tetrahedron or a pyramid
were used.

In the spatial case (part b) certain probands did not realise that a mere repetition is not an analogy. Of course, since line segments belonging to one triangle, are also spatial objects, lying in a special plane, 1. - 7. also hold for the space (in a trivial sense).

A special statement corresponding to initial statement 6. was recognized as being obviously not valid. Many of those who gave the (correct) judgement "wrong statement" added the correct answer, interchanging the signs < and >, subsequently "copied" the ">" in statement 7. and wrote down a line which then was no longer consequently analogous to the original statement.

Although probands appreciate the difference between a statement obtained by correctly carrying out an analogy transition and a valid statement it seems that the assessment of validity of a statement could have influenced the chosen analogy pattern (cf. f.i. the preceding answer categories to 4., (2)).

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CHILDREN'S DEVELOPMENT OF METHODS OF PROOF[1]

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As a component of a longitudinal study of children's thinking, approximately 250 third, fourth and fifth grade children from three project sites (urban, working class, and suburban) were asked to build a solution to a counting problem and then to provide a convincing argument for their proposed solution. Videotape data from these problem solving sessions at each of the project sites indicated certain progressions in the children's problem solving behavior. Students who invented methods of proof did so in attempts to build a justification for their solution.

Whereas earlier views of "learning mathematics" found it acceptable to tell the ideas of mathematics to students, today's constructivist approach requires something else. If students gradually build up in their minds representations of the key ideas of mathematics, it becomes important for research to study the process by which these representations are constructed. While one does not do this by literally looking inside brains, one can make reasonable inferences from behaviors that can be observed and studied by careful use of videotape, especially when one follows the same students closely over an extended period of time.

Background and Setting

The work that is reported here is a component of a longitudinal study, now in its fifth year, of the development of

mathematical ideas in children. Analysis of the videotaped data and of the children's written work have made possible assessments of what students can do as they are engaged in problem-solving situations that are appropriate for their level of development and within the range of their inventiveness (Davis, Maher & Martino, 1992; Martino, 1992; Maher & Martino, 1991; Maher, Davis & Alston, 1991). In each of the three project sites, there are several settings for this study: the regular mathematics classroom in which children may work with a partner; in a whole class discussion following the paired problem solving session; in individual task-based interviews following the whole class discussion; and in a small group discussion following the individual interviews.

The Development of Children's Ideas

Our interest is in studying how ideas develop in children. We do this by presenting children with a problem that involves building a solution. We then challenge the children to create a convincing argument for the correctness of that solution. Our attention is focused on how the child represents the problem and justifies it; on what strategies the child employs in building the argument; and on what notation the child invents.

We are also interested in knowing whether the methods invented by children continue to be used by them. That is, are these methods, whether right or wrong, stable -- or do they change over time? If the methods that students invent are subsequently refined or modified, we are interested in tracing how the students may have been influenced to make these changes. In particular, we study whether, and how, students make use of the ideas of others.

In pursuit of gaining insight into children's thinking, our observations for this report center on the following:

(1) Children's building and representing their initial ideas;
(2) Children's modifying and refining their ideas;
(3) Children's developing justifications for their solutions.

The data for our initial understanding of the development of children's thinking come from analyzing the problem-solving behavior of approximately 250 third, fourth, and fifth-grade children from three communities (urban, suburban, and working class) in several settings.

The Problem Task

Each pair of children was given unifix cubes of two colors, dark and light. Their task consisted of two parts: the first was to build all possible towers (that is, a vertical array of adjoining cubes) of a particular height when they could select from the two available colors; the second task was to convince each other that there were no duplicates and that none had been left out.

In the initial administration of the problem (third or fourth grade), the towers were to be four cubes tall. For a second administration of the problem, the towers were to be five cubes tall. In subsequent small group discussions in which the children were asked to convince each other of their solutions, towers of height three were used. In this case, the emphasis was on studying children's justifications of their solution.

Procedures and Data Source

As described earlier, some variant of the problem-solving task was posed in a natural classroom setting, using small group instruction, to all of the 250 students in the study. After the small-group problem-solving session, all of the children were invited to share the representations of their solutions during a whole class discussion. The class discussion was videotaped.

Also, children from each of the sites who were subjects in the longitudinal study were videotaped during the classroom session as they worked on the problem task with a partner. These children were later individually interviewed after the classroom session. These interviews were videotaped. A subset of the children from the longitudinal study were again videotaped in a small group discussion format for assessment purposes.

Hence, the data for this study come from several sources: observations of the classroom episodes; transcripts and analyses of videotapes of children working in pairs or small groups during the problem-solving sessions; the written work which students produced during these taped sessions; researcher observations recorded on-site; and two individual written assessments of the tower task. Preliminary analysis of these data are reported in this paper.

Results

The children's problem solving generally moved sequentially through three distinct stages.

Stage 1. In the first stage, they began by finding particular towers and by checking to see if they were building duplicates. Initially they appeared to be using a "guess and check" strategy.

Examples from Stage One

1. Creating individual towers and checking each new arrangement with those previously made by comparing it to others;
2. Inventing names for repeated towers such as "duplicates" and "doubles";
3. Inventing names for certain patterns *within* a single tower. For example, a tower with alternating colors was sometimes referred to as having a "checkerboard" pattern.
4. Discovering a pair of towers with opposite colors in the corresponding positions, and naming them, frequently, as "opposites" or reverses."

Stage 2. As the children generated more towers, they came to recognize certain *collections* of towers and typically began to use the word "pattern" to describe some, but not all, of the collections. In the second stage we see examples such as the ones illustrated in Figures 1 and 2 where two local organizational schemes have a potential conflict when used together.

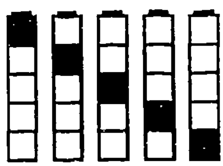

Figure 1. Elevator Pattern

Figure 2. Staircase Pattern

Examples from Stage Two

1. Discovering other ways of generating groups of towers such as inverting a tower. These, for example, were referred to as "cousins" or "inverses;"
2. Organizing sets of towers according to a particular relationship. For example, a tower, its opposite, its cousin, and the opposite of the cousin.
3. Organizing sets of towers according to particular schemes, such as the elevator and staircase patterns that are illustrated in Figures 1 and 2, respectively.
4. Finding the opposites of the *groups* and discovering duplicates by examining the intersection of the sets; In this stage the checking process becomes more elaborate and is conducted with sets of towers.

During the second stage, considerable disequilibrium evolves as children begin to realize that some local organization scheme is inadequate for organizing all possible towers of a given height. The emergence of duplicates from the intersection of the local organizations often leads to uncertainty. As one fourth grader, Stephanie, indicated:

> You always have to think there's more...because you can't go...you never know if there's gonna be...you can't say I found two that's enough...cause you always have to think there's more...cause you never know if it's enough or not..you know what I mean?

And later, in response to the instructor's question as to whether she thought there was any way that she would know if she found all possible towers, Stephanie replied:

> No, because you could buy like the biggest...you could have like reds and blues all...reds and yellows all over this room and people could still get ideas...you would not know that...one person could have 44 [different towers] and the other person could have...be having 58 and still going for more because you don't know until you're finished, until you're absolutely, positively sure.

Stage 3. As students come to see conflicts between different local organizations, they began to realize the need for a single global scheme. Analysis of student work shows two kinds of successful global organization schemes, one leading to proof by cases and the other leading to proof by mathematical induction. These ultimate solutions are not reached immediately but only as a result of gradual refinements.

Examples from Stage Three

1. Attention to certain patterns such as exactly no, all, and one dark cube encouraged the reorganizing of former sets into "elevator" subsets such as exactly two dark cubes together, or exactly three dark cubes together, etc. This rearrangement naturally directed student attention to the placement of other towers in which the dark cubes were separated and stimulated the necessity to make new groupings.
2. Consideration of how patterns might work when "referring to simpler problems," such as towers that could be built that were three cubes tall and two cubes tall. Consideration of these situations led to building a proof by mathematical induction.
3. Consideration of how some patterns related to the patterns of isomorphic problems already encountered and successfully solved, such as making outfits with shirts of two colors, pants of two colors, hats of two colors, etc.

During this last sequence children were more receptive to input from others. Although an ultimate solution did not always surface at this time, observations several months later often revealed refinements of some of the more firmly held earlier methods (Maher, Davis, Martino, in progress).

Note

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THE ROLE OF CONJECTURES IN GEOMETRICAL PROOF-PROBLEM SOLVING
--- -FRANCE-JAPAN COLLABORATIVE RESEARCH ---------

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Abstract: The purpose of this research is to clarify differences between French and Japanese students' conjectural activities in geometrical proof-problem solving, and to find out its instructional implications.

For this purpose, this research makes investigations to clarify for students' thinking processes in geometrical proof-problem solving.

In conclusion, the instructional implications of geometrical proof-problem solving are as follows : (1)activating students' thinking processes towards "actions for figure" and "dynamic viewpoints for figures", (2)helping the students in the use of the characteristics of triangles and of those expanded characteristics, (3)helping the students in the applications of numerical values and algebraic processes.

1. Introduction

"France-Japan collaborative research"(*) of "Mastering space and learning Geometry" has been carried out since 1987. Our subgroup in this research has the theme "The Role of Conjectures in Problem Solving in Geometry". The common aim of our group is to discover the conditions which lead to conjectural activities for students, to study the relationships between the types of conjectures and mathematical contents and environments (with or without computer), and finally to analyse the possible productions of proofs. (Balacheff, et al., 1989)

From this point of view, we had carried out researches on students' difficulties in geometrical proof-problem solving in each country (Gallou-Dumiel, E., 1988, Harada. K. & Nohda, N., in press).

For our theoretical background of researches, we had employed Balacheff's didactical theory (Balacheff, 1990) and Piaget's theory of cognitive development and equilibration theory (Piaget, 1975).

Also we will clarify the differences between French and Japanese

students' activities in geometrical proof-problem solving in this research.

The purpose of this research is to clarify differences between French and Japanese students' conjectural activities in geometrical proof-problem solving, and to find out its instructional implications.

For this purpose, it is necessary to clarify students' thinking processes and undertake common methods of investigations in France and Japan.

In geometrical proof-problem solving, there are two types of conjectures in students' thinking processes depending on two types of problems (one wherein the conclusion is given or one it is not). The first type of conjectures involves only processes of the proof while the second type not only involves processes of the proof but also a conclusion of proposition. In this research, we will employ the first type of conjectures.

To conjecture a process of proof, it is necessary for students to discover "procedures for proof". It will be discovered through "actions for figure". Here the "procedures for proof" mean "means of proof-problem solving" and the "actions for figure" mean "actions for drawing geometrical characteristics from the figures in the problem".

In this investigation, we will consider students' thinking processes from these two viewpoints.

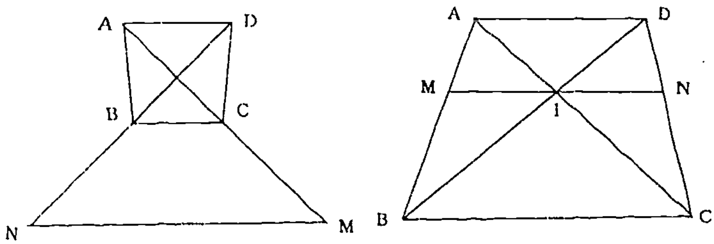
2.Method

(1)Problems of Investigations

The problems of investigations employ two problems in "France-Japan collaborative research". These problems can be generally solved using Thales' Theorem and can not be solved by applications of simple knowledge. We intend to draw various students' activities in problem solving processes.

Problem 1. ABCD is a trapezoid. M is the image of A in the point symmetry with C as center. The intersection of line BD and line parallel to AD including M is a point called N. Show that N is the image of D in the point symmetry with B as center.

Problem 2. ABCD is a trapezoid. The intersection of (AC) and (BD) is a point called I. The parallel to (AD) including the point I is secant to (AB) in a point called M and secant to (CD) in a point called N. Show that I is the middle of (MN).



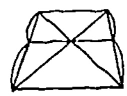
Problem 1 Problem 2

Figure 1. The figures of Problems

(2) Viewpoints of Investigations

We take account of the difference between France and Japanese curriculum in secondary school mathematics in geometry and then give some elements of "actions for figure" and " procedures for proof" as follows.

(a)actions for figure

① modification of the figure (auxiliary lines, translation,...)
② resolution of the figure (segments, lines,....)
③ resolution of the figure into triangles
④ marking segments of proportional relations ()

(b)procedures for proof

① using congruence conditions of triangles
② using similar relationships of triangles
③ using theorem about midpoints of two sides of triangle
④ using Thales' Theorem
⑤ using numerical values and algebraic processes
⑥ using formula of area of triangle
⑦ using visual characteristics of figures

(3)Subjects

(a)Investigation 1

France : six pairs of the 9th grade students

Japan : six pairs of the 8th grade students

(b)Investigation 2

France : six pairs of the 10th grade students

Japan : six pairs of the 9th grade students

The subjects in the "Investigation 1" studied Thales' Theorem (In Japan, it is referred to as "Theorem about parallel lines and ratio of line segments".) just before the investigation. The subjects in the "Investigation 2" studied Thales' Theorem one year before the investigation.

(4)Method of Investigations

Each pair was given two geometrical proof-problems. At the same time, each pair was given one pencil and sheets of papers as much as they want. When they finished the problems, we interviewed the students in each pair individually. Their problem solving processes and our interviewing processes were recorded by using video and audio tape recorders.

3.Results and Discussions

(1)Results of Investigations

(a)Investigations in France

Pairs S&C, P&F, L&A, S&B, L&P, and B&F in "Investigation 1" are named A, B, C, D, E, and F respectively. Pairs A&C, C&I, M&F, D&K, P&L, and S&A in "Investigation 2" are named G, H, I, J, K, and L respectively.

The results are shown in "Figure 2".

(b)Investigations in Japan

Pairs K&T, T&K, Y&N, B&F, M&K, and S&K in "Investigation 1" are named M, N, O, P, Q, and R respectively. Pairs T&K, S&T, K&Y, G&S, T&N, and S&W in "Investigation 2" are named S, T, U, V, W, and X respectively.

The results are shown in "Figure 2", in addition to the results of investigations in France.

	actions				procedures						
	1	2	3	4	1	2	3	4	5	6	7
Investigation 1		A	A	A			A				
		B	B	B			B				
	C	:	C								C
	D								D		
		E	E				E				
		F									F
Investigation 2		G	G				G				
	H										H
		I	I	I			I				
		J						J			
		K	K				K				
		L		L				L			

Problem 1 (France)

	actions				procedures						
	1	2	3	4	1	2	3	4	5	6	7
Investigation 1	A	A									A
				B					B		
			C	C				C			
	D										D
	E		E								E
		F	F	F				F			
Investigation 2		G		G				G			
			H								H
		I	I					I			
	J										J
		K	K	K				K			
		L		L				L			

Problem 2 (France)

	actions				procedures						
	1	2	3	4	1	2	3	4	5	6	7
Investigation 1	M	M	M	M	M	M	M				
		N	N	N		N			N		
	O	O	O	O		O			O		
	P	P	P	P		P		P			
	Q	Q	Q		Q*						Q
	R	R	R		R*						R
Investigation 2		S	S	S		S	S				
	T	T	T	T	T	T	T				
	U	U	U	U	U	U	U				
		V	V	V	V	V	V				
		W	W	W	W				W		
	X	X	X		X*	X	X*				X

* : using the procedure by mistake
Problem 1 (Japan)

	actions				procedures						
	1	2	3	4	1	2	3	4	5	6	7
Investigation 1	M	M	M	M	M	M				M	
		N	N	N	N	N		N			
		O	O	O	O	O				O	
	P	P	P	P	P	P					
	Q	Q	Q		Q	Q					
	R	R	R		R*						R
Investigation 2		S	S	S		S		S			
	T	T	T	T	T*						T
		U	U	U		U		U*			
		V	V	V		V		V*			
		W	W	W		W			W		
	X	X	X		X*						X

* : using the procedure by mistake
Problem 2 (Japan)

actions for figure

1. modification of the figure
2. resolution of the figure
3. resolution of the figure into triangles
4. marking segments of proportional relations

procedures for proof

1. using congruence conditions of triangles
2. using similar relationships of triangles
3. using theorem about midpoints of two sides of triangles
4. using Thales' Theorem
5. using numerical values and algebraic processes
6. using formula of area of triangle
7. using visual characteristics of figures

Figure 2. Results of Investigations

(2) Discussions

By considering the results, we will clarify differences between French and Japanese students' conjectural activities.

(a) For "procedures for proof", the French pairs used only one procedure, whereas Japanese pairs used a lot of procedures.

We think that in France, the secondary school curriculum in geometry deals with these problems within the narrow context of "Theorem about midpoints of two sides of triangle" or "Thales' Theorem", whereas in Japan, the secondary school curriculum in geometry deals with the wide context of "congruence of triangles", "similar relationships of triangles", etc.

Therefore, French students could pinpoint only one strategy of problem solving, whereas Japanese students have many strategies.

(b) French pairs could apply "Thales' Theorem" for these problems, whereas Japanese pairs could not apply the theorem even when they know about it. Especially, for "Problem 2", they must integrate two propositions of "similar relationships of triangles" and one proposition of "Thales' Theorem" to solve this problem.

In this case, "equilibration of between differentiation and integration" is necessary for students (Piaget, 1975). We consider that French students have high level of cognitive development and are used to these kinds of operations in this problem.

(c) "Problem 1" can be solved using either "Theorem about midpoints of two sides of triangle" or "Thales' Theorem".

In France, in "Investigation 1" three pairs used only "Theorem about midpoints of two sides of triangle" as "procedure for proof". In "Investigation 2", three pairs used "Theorem about midpoints of two sides of triangle" and another two pairs used "Thales' Theorem".

In Japan, in "Investigation 1" one pair used "Theorem about midpoints of two sides of triangle" and another one pair used "Thales' Theorem". In "Investigation 2", five pairs used only "Theorem about midpoints of two sides of triangle".

We consider that students in both countries tend to use the theorems of triangles and have knowledge based on characteristics of triangles as a root of their knowledge.

(d) In France, "using numerical values and algebraic processes" as a "procedure for proof" was used by two pairs in "Investigation 1".

In Japan, this procedure was used by two pairs in "Investigation 1" and one pair in "Investigation 2".

This is not a procedure in geometrical proof-problem solving. However, we think that this procedure is a useful means in general problem solving.

(e)The "using visual characteristics of figures" was used by students in both countries. We think that actions with reference to this procedure are "type α action" of compensations (Piaget, 1975). Students who used that procedure has low level of cognitive development. It is necessary that these students have rich intension of figures and expand their extension of figures.

Therefore, for example, to give dynamic viewpoints for figures to those students will be useful in geometrical problem solving.

(f)French pairs used a little "actions for figure", whereas Japanese pairs used many kinds of actions.

We will note that in France, the pairs who used some "action for figure" could solve these problems.

In France, the secondary school curriculum in geometry deals with translations, rotations, reflections, dilatations, etc. based on the transformation geometry.

Whereas, in Japan, the secondary school curriculum in geometry deals with "congruence of triangles", "similar relationships of triangles", etc. based on Euclidean geometry.

We think that French students can use "modifications of figure" and "resolutions of figure", etc. which are restricted for geometrical proof. Whereas, Japanese students can use a lot of "actions for figure" such as "modifications for figure" (drawing auxiliary line, translating figures, etc.) and "resolutions of figure" (lines, segments, triangles, parallelogram, etc.) which are taught as strategies of geometrical proof.

(g)Many "actions for figure" which were used by many pairs depend on characteristics of triangles in both countries. This tendency is the same as in "procedures for proof".

4. Conclusions

By our investigations, we clarified the differences between French and Japanese students' conjectural activities in geometrical proof-problem solving. We can point out that there are differences in "actions for figure" and "procedures for proof"

between French and Japanese students' conjectural activities, and those differences reflect the structure of knowledge in each country.

For both French and Japanese students, the instructional implications in geometrical proof-problem solving are as follows:
(1)Activating students' thinking processes towards "actions for figure" and "dynamic viewpoints"for students are important means of helping students' geometrical proof-problem solving.

(2)The use of characteristics of triangles and those developed characteristics are effective in geometrical proof-problem solving.

(3)The use of numerical values and algebraic processes are useful in general problem solving.

However, to demonstrate the instructional implications mentioned above remains a problem.

Notes

(*)French representative is Professor C. Laborde (Université Joseph Fourier, Grenoble) and Japanese representative is Professor Y. S[illegible]ma (Tokyo Gakugei University, Tokyo).

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The Effects of Elaboration on Logical Reasoning of 4th Grade Children

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We investigated the effects on retrieval and logical reasoning of students having involved in instruction that required them to engage in elaborative processing. The 4th grade students(N=40) were randomly assigned to two treatment groups: Elaborative group and Non-Elaborative group. Students were taught geometrical knowledges for 7 days by computer. Retrieval of knowledges which were taught in instruction was tested, and the ability of logical reasoning which was combination of mathematical knowledges and logic itself was tested. Our results showed that the elaborative process had significantly effects on the reasoning ability as well as retrieval of knowledge.

The importance of logical thinking is always emphasized in learning of mathematics and everyday life. Moreover, it is no doubt that the importance of logical thinking will be a matter of concern in the education at the later. To draw out the suggestion in educating logical thinking, in this study, the nature of logical thinking was reviewed, Piaget's perspective and information-processing theorists' perspective about the logical thinking were compared, and the relation between logical thinking and the structure of knowledge was reviewed.

There are two perspectives about the nature of logical thinking, i.e., Piaget's perspective and information-processing theorists' perspective. According to Piaget, children in the preoperational stage are egocentric and concrete operational stage are limited to logical thinking. On the other hand, information-processing theoriests suggest that every people at any level of logical ability is capable of logical thinking, because he(she) has production systems.

According to information-processing theory, the sources of errors in logical thinking are due to the errors in the process of logical thinking rather than the lack of logical abilities. Thus, information-processing theorists assume that if the errors in the process of logical thinking can be prevented, and then logical thinking ability will be improved. Specifically, they believe that the capacity of working memeory should be extended to prevent errors from occurring in processes of logical thinking. Information-processing theorists suggest elaboration theory as one of the methods which could extend the capacity of working memory. Elaboration is the process of adding related knowledge to the new knowledge. This elaboration provides alternate retrieval routes and improve limited-capacity of working memory during retrieval.

Anderson(1985) and Gagné(1985) stated that elaboration facilitated retrieval in two ways - by means of alternate retrieval routes through the propositional network and by inference. For example, assume that a problem solver should recall the proposition P1 to prove a geometry problem, but he or she can't remember the proposition. In such a case, he or she would

retrieve the proposition P2 and P3 as alternate retrieval routes, and then he or she would construct the proposition P1 with inference.

P1: The measure of an exterior angle of a triangle is equal to the sum of the measures of the two remote interior angles.

P2: The sum of the measures of the angles of a triangle is 180.

P3: If the exterior sides of two adjacent angles are opposite rays, the angles are supplementary. (Jeon, 1988, pp.23-24)

Stein et al.(1982) demonstrated that some fifth grade children spontaneously generate elaborations and insisted that in general, learning gains are associated with spontaneous generation of precise elaborations. Pressley et al.(1987) concluded that elaborative interrogation(such as answering why questions) was a potent strategy, in part because it produced as much learning as occurred in imagical coding condition. And more recently, Wood, Pressley, & Winne(1990) supported the studies which demonstrated that elaborative processes facilitated acquisition of facts or knowledges.

Stein and his colleagues(1982) studied the effect of different types of elaboration on recalling sentences. The results showed that for any group(successful, average, less successful), a student who gave a precise phrase to connect two given sentences was more likely to correctly recall the sentence than one who gave an imprecise phrase. They also found that more successful students were more likely to generate precise elaboration than were less successful students. This results suggest that precise elaboration can aid better recall than imprecise elaboration.

Most of researchers interested in elaboration have been in research for the effects of elaboration in retrieval of knowledges or facts. But recently, Siver(1982) recommanded the need for application of elaboration theory in mathematical problem solving.

The purpose of the present study was to investigate the main factor which influences on the children's logical reasoning, and to analyze the effects of elaborative learning in retrieval of mathematical knowledge and in logical reasoning.

For this purpose, three research questions were attempted;

(1) What is the main factor which influences on the logical reasoning of the 4th grade children? Is it logic itself or knowledge for logical reasoning?

(2) Is there any significant difference between Elaborative group and Non-elaborative group in the retrieval of mathematical knowledge at the first day and at the 7th day after instructional treatment?

(3) Is there any significant difference between Elaborative group and Non-elaborative group in the KoReT(knowledge+reasoning test)?

METHOD

Subjects

Ninty three 4th grade students were selected from an elementary school in Chongwon,

Chungbuk. The subjects were divided into two subgroups; R_0 group (scored high at the reasoning test, but low at the retention test), R_1 group(scored low at the reasoning test, but high at the retention test). Twenty students in R_0 and twenty students in R_1 were assinged to two kinds of instructional treatment groups; Elaborative group(10 in R_0 and 10 in R_1) and Non-elaborative group(10 in R_0 and 10 in R_1), and the remainders were not included in this study.

Instruments and measure

Five kinds of tests were administered to the subjects: Retention Test(RetT), Reasoning Test(ReasT), Retrieval Test 1 and 2(RevT_1 and RevT_2), Knowledge+Reasoning Test(KoReT). The pretests(RetT and ReasT) were administered for 20 minutes before the regular class at 17th, 18th day of June, in 1992. Three posttests(RevT_1, RevT_2, KoReT) were administered for 20 minutes before the regular class at 4th, 14th, 15th day of July, in 1992, respectively.

Retention Test(RetT)

Retention Test(RetT) was used to measure the degree of understanding of the basic concepts of geometry. The purpose of this test was to divide the subjects into two subgroups: by score at RetT. The test covered the basic concepts of triangle, right-angled tirangle, isosceles triangle, equilateral triangle, quadrangle, rectangle, square. The students were given fifty items and required to answer by yes or no.

Reasoning Test(ReasT)

Reasoning Test, designed to assess the student's ability to understand logic forms for logical reasoning, was used to divide the subjects into two subgroups by score at ReasT. This instrument used in the present study was the translated form of TCS(Test of Cognitive Skills) which was developed by CTB/McGraw-Hill, Del Monte Research Park. The original version of this instrument was divided into five levels that are related to grade range and each level of TCS included four subtests: Sequences, Analogies, Memory, and Verbal Reasoning. The original version of this instrument was modified slightly to make them appropriate for 4th grade students. In the modified version of the test used in the present study, only reasoning category among the four categories was tested and the items of the test were fifteen.

Retrieval Test 1 and 2(RevT_1 and RevT_2)

Retrieval Test 1 and 2 were designed to measure student's ability to retrieve geometrical knowledges taught in instruction. RevT_1 was administered at the first day after instructional treatment, and RevT_2 was administered at the 7th day after instructional treatment. The items of two tests were same except only the order of the items. The test covered basic concepts of the rectangle, square, trapezoid, parallelogram, and rhombus learned in instruction. In the both RevT_1 and RevT_2, the subjects were given 25 items and required to answer the statements by yes or no.

Knowledge+Reasoning Test(KoReT).

Knowledge+Reasoning Test, designed to investigate factor which influences on logical reasoning, was used to find the difference in logical reasoning ability after instruction between Elaborative group and Non-Elaborative group. This test was developed by combining logic itself tested in ReaT and geometrical knowledges learned in instruction. The KoReT of 20 items contained 12 items requiring inductive reasoning and 8 items requiring deductive reasoning.

Insturctional treatments

The instructional treatment was to instruct two subgroup in different way. Geometrical knowledges were given directly to Non-Elaborative group(NELAB), but to Elaborative group(ELAB), was given elaborative processes. Seven lessons were provided in morning class before regular classroom during the 30 minutes every day(23, Jun - 3, July).

The contents of instruction are as follows:

① right-angled triangle and isosceles triangle and equilateral triangle
② rectangle and square
③ trapezoid and parallelogram
④ rectangle and trapezoid and parallelogram
⑤ square and trapezoid and parallelogram
⑥ trapezoid and parallelogram and rhombus
⑦ rectangle and square and rhombus

All of the instructions were proceeded by personal computer. Every students participated in this instructional treatment was given by personal computer.

Instruction for Non-Elaborative group

Non-Elaborative group was instructed the statements such as "The lengh of the four sides of rectangle are same" directly. For example, In the instruction of 「RECTANGLE AND SQURE」, Non-Elaborative group was provided basic concepts such as "The lengths of the four sides of rectangle are same", "The four angles of the rectangle are all right-angle", "The squres have four sides", "The four angles of the squre are all right-angle", "The rectangles have four sides". After provided these statements, they were only given the relation of the knowledge such as "We can say that squres are rectangles", "We can not say that rectangles are squres".

Instruction for Elaborative group

Elaborative group was instructed the same knowledge that Non-Elaborative group learned, but the way of instruction was not same. Instruction for Elaborative group emphasized the elaborative processes of the knowledge. That is, in order to elaborate the knowledge, Elaborative group was required to decide whether the statements(basic concepts) provided by personal computer were true or false.

For example, in the instruction of 「RECTANGLE AND SQURE」, Elaborative group was provided basic concepts such as "The lengh of the four sides of rectangle are same", "The four angles of the rectangle are not all right-angle", "The squres have four sides", "The four

angles of the square are not all right-angle", "The rectangles have three sides", and was to required to decide whether the statements are true or false. Then if their responses were wrong, they were given feedback. After provided these statements, they were also given the relation of the knowledges such as "We can say that squres are rectangles", "We can not say that rectangles are squares". However, they were expected to think the validity of the statements. That is the diffenrence between two treatments.

RESULTS

1. Mean Score of two pretests and classifying subgroups

The mean scores attained in the two pretests are shown in Table 1. By the results of the two tests, subjects(N=93) were divided into two subgroup: R_0(the students who scored 7 or more at the ReasT and 37 or less at RetT) and R_1(the students who scored 7 or less at the ReasT and 37 or more at RetT).

Tablo 1
Mean scores and Standard Deviations of two protosts

Tosts	N	Moan	SD	Min	Max
RetT	93	37.914	5.793	20.00	49.00(50)
ReasT	93	7.796	3.198	1.00	14.00(15)

' The scores of parenthesis represent maximum which can be attained.

2. Factor that Influences on Children's Logical Reasoning

Multiple regression analysis(independent variables were ReasT and $RevT_1$, dependent variable was KoReT) was carried out to check the main factor which effects on children's logical reasoning. The results are shown in Table 2. The RetT was significant variable which influenced logical reasoning(Sig T=.00, p<.05), but ReasT which tested logic itself was not significant variable(Sig T=.176, p<.05). The results showed that retention of mathematical knowledges was more significant variable in logical reasoning than logic itself.

Tablo 2
Multiple Regression Analysis on Logical Reasoning

Ind Var	Dep. Var	B	Beta	T	Sig T
ReasT	KoReT	.842	.189	1.380	.176
$RetT_1$		2.143	.570	4.153	.000**

** p<.05

3. *The Effects of Elaboration* on Retrieval *at first day*

The T-Test was carried out to analyze the effects of elaboration on retrieval of geometrical knowledges at the first day after instruction treatment. In the Table 3, there is no significant difference in $RevT_1$ between Non-Elaborative group(NELAB) and Elaborative group(ELAB), between NELAB-R_0 and ELAB-R_0, between NELAB-R_1 and ELABR_1($p<.05$). So, there is no effect of elaboration on retrieval immediately after instruction in all subgroups.

Table 3
T-Test for effects of elaboration at first day

Groups	N	Mean	SD	t	df	Pro.
NELAB	20	18.15	3.990	-1.67	38	.104
ELAB	20	19.95	2.724			
NELAB-R_0	10	19.2	3.521	-1.16	18	.263
ELAB-R_0	10	20.9	3.035			
NELAB-R_1	10	17.1	4.332	-1.25	18	.228
ELAB-R_1	10	19.0	2.10[illegible]			

4. *The Effects of Elaboration* on retrieval *at seventh day*

The T-Test was carried out to analyze effects of elaboration on retrieval geometrical knowledges at the seventh day after instruction treatment. As shown in the Table 4, there is significant difference in the mean scores of the $RevT_2$ between NELAB and ELAB group, between NELAB-R_0 and ELAB-R_0 at level .05, between NELAB-R_1 and ELAB-R_1 at level .1. That is, at the seventh day after instruction, the effects of elaboration were significant.

Table 4
T-Test for effects of elaboration at seventh day

Groups	N	Mean	SD	t	df	Pro.
NELAB	20	17.40	3.152	-3.47	38	.001**
ELAB	20	20.30	2.003			
NELAB-R_0	10	17.9	3.143	-3.12	18	.006**
ELAB-R_0	10	21.4	1.647			
NELAB-R_1	10	16.9	3.247	-1.97	18	.064*
ELAB-R_1	10	19.2	1.751			

* $p<.1$ ** $p<.05$

5. *The Effects of Elaboration on Logical Reasoning*

T-Test was used to check whether there is any difference between Non-Elaborative and

Elaborative group in KoReT. The results Table 5 showed that Elaborative group socred significantly higher than Non-Elaborative group($p<.05$). And there are signficant differences between ELAB-R_0 and ELAB-R_0($p<.05$) and between NELAB-R_1 and ELAB-R_1($p<.1$). But, it should be noted that for any groups, elaborative processes significantly improved the logical reasoning ability.

Table 5
T-Test for effects of elaboration on KoReT

Groups	N	Mean	SD	t	df	Pro.
NELAB	20	33.03	11.584	-3.19	38	.003**
ELAB	20	44.93	12.061			
NELAB-R_0	10	33.50	14.347	-2.41	18	.027**
ELAB-R_0	10	47.13	10.624			
NELAB-R_1	10	32.55	8.773	-2.00	18	.061*
ELAB-R_1	10	42.75	13.550			

* $p<.1$, ** $p<.05$

CONCLUSIONS AND DISCUSSION

First, toward to main factor which effects on children's logical reasoning, Piaget emphasized logic itself or form of reasoning. But recently, cognitive psychologist and information processing theorists(cf: Henle, 1962; Revilis, 1975; Braine, 1978; Bryant & Trabasso, 1971; Riley & Trabsso, 1974; Johnson-Laird, 1985) insisted that knowledges(contents) of reasoning more influenced on logical reasoning than logic itself.

According to the result of the present study, mathematical knowledge was more significant variable than logic itself. That is, the factors that effect on logical reasoning might be logic and mathematical knowledge, but more important factor was knowledge. This fact is consistent with the suggestion of information processing theorists, and supports the results of their study.

Second, there are many studis that elborative processes enhance retrieval of knowledge. But these studies analyzed the effects of the elaborative processes at the day after seventh day since instructional treatment. In the present study, the effects of elaborative processing was compared by analyzing knowledge retrieval ability of Elaborative group and Non-Elaborative group at the first and at the 7th day after instructional treatment.

The results suggested that at the first day, there was no significant difference in the retrieval of knowledge between Elaborative group and Non-Elaborative group, and between subgroup Elaborative R_0 and Non-Elaborative R_0. The reason of this fact might be that forgeting or decaying of knowledge was almost equal in two groups.

However, Elaborative group was better than in the retrieval of knowledges at the 7th day

after the instruction than Non-Elaborative group. In particular, the effect of elaboration on retrieval of gemetrical knowledge was high in subgroup R_0(scored good at the reasoning test, but poor at the retention test).

These findings support studies(cf: Stein & Bransford,1979; Stein et al.,1982;Bradshaw & Anderson,1982; Palmere et al.,1983; Pressley et al.,1987; Swing & Peterson,1988) who insisted that elaborative processes dramatically improve memory and retrieval of knowledges.

Third, the students who scored good at the retrieval test was successful in logical reasoning. In particular, the effect was higher in subgroup R_0(scored good at the reasoning test, but poor at the retention test) than in subgroup R_1(scored good at the reasoning test, but poor at the retention test). In other words, elaboration improves logical rasoning as well as retrieval of knowledges. This result support Bradshaw & Anderson(1982) who insisted that elaboration provided redundant retrieval routes and then facilitated inference, and demonstrated the effects of elaboration in logical reasoning.

The results drawn from the present study mean that because every children have at least logic itself(by information processing theorists, production systems); if the students have structured knowledge(by elaboration)m then they are capable of logical reasoning.

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TRANSLATING A SEQUENCE OF CONCRETE ACTIONS INTO A PROOF

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The study assumes that a sequence of concrete actions shifts to a proof, and aims to find the effective instruction to the shift. Using teaching experiment as methodology, three 7th grader 's responses and the instructions were analyzed, then the following conclusion was drawn. If the 7th grader satisfies three conditions: 1: she or he has generated a sequence of concrete actions in which we can see deductive reasoning; 2: she or he has reproduced the sequence in other cases; 3: she or he has verbally expressed a critical action for a translation into a proof, then two instructions are effective to translate the sequence of concrete actions into a proof: one of the instructions establishes thematic correspondence between the theme of view of the concrete object and the literal expression through translating the critical action for a translation into a proof, and the other requires explanation in the large number context after translating the sequence of concrete actions into numerical expressions.

Introduction

The formal approach to proofs[1] in mathematics education has been causing undesirable reaction to proofs in student's mind: just a school game (Schoenfeld, 1982), they have no relation with the universality (Fischbein & Kedem, 1982). Contrary to this approach, an other approach to proofs has been progressing in mathematics education. The approach attempts to shift students from an empirical explanation to a formal proof. Although some researchers have partly examined the possibility of the shift (Balacheff, 1988), we have not found specifiable instructions[2] effective to make the shift yet.

I also take the latter approach, especially in shifting an explanation through concrete actions to a proof. There seems to be three steps in the approach. I found 6th graders can generate a sequence of concrete actions in which we can see deductive reasoning, and reproduce it in other cases [the first step] (Miyazaki, 1991). I operationally defined the action necessary to translate a sequence of concrete actions into a proof (Critical action for a translation into a proof). Then, I identified the instructions effective to make 6th graders express the action verbally [the second step] (Miyazaki, in press). Thus, I would like to answer the research problem related to the final step as following.

> If a 7th grader satisfies three conditions: 1: she or he has generated a sequence of concrete actions in which we can see deductive reasoning; 2: she or he has reproduced the sequence in other case; 3: she or he has verbally expressed a critical action for a translation into a proof, what kinds of instructions are effective to translate the sequence of concrete actions into a proof?

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[1] The word "proof" in the paper means that in order to show the generality of a property of a number or a figure a person represents deductive reasoning between the assumption and the conclusion in formal mathematical language, and the products of the representation.

[2] The word "instruction" in the paper means that in order to change the students' thinking and behavior toward the achievement of an aim, the observer encourages the student to do something.

Method

Procedure

I use "Teaching experiment" (Steffe,1991) as my method. The subjects are three 7th graders of a public junior high school. In the prefecture, the achievement level of the school in mathematics is intermediate. From September to October, the three students learned a literal expression which involves only one kind of letter.

Name	Grade	Sex	Achievement in Mathematics	The date of experiment
ADV	7	Male	Advanced	31/10/1992
INT	7	Male	Intermediate	7/11/1992
LOW	7	Male	Lower	21/11/1992

Table 1. The characteristics of the subjects.

The task required the student to induce the property that the sum of five contingent numbers is five times the center number, and to explain it. In explaining it, the observer required the student to first use colored magnets on the whiteboard, and then to use a numerical or literal expression.

The observer prepared two levels of instruction which seemed to be necessary to translate concrete actions into a proof by analyzing the task. The observer first observed and instructed the advanced student. When the observer couldn't get the intended responses from the student, the observer improved the instruction for the intermediate student as the need arised. In the same way, the observer observed and instructed both the intermediate and the low students individually. All activity of the students and the observer were videotaped.

The instruction prepared to translate a sequence of concrete actions into a proof

The concrete action refers to two situations: a view of the concrete object and a change of view[3]. If these situations are carried out in turn, the concrete actions become a sequence. For example, when we explain, by using marbles, the property that the sum of five contingent numbers is five times the center number, we can consider the sequence of concrete actions as following.

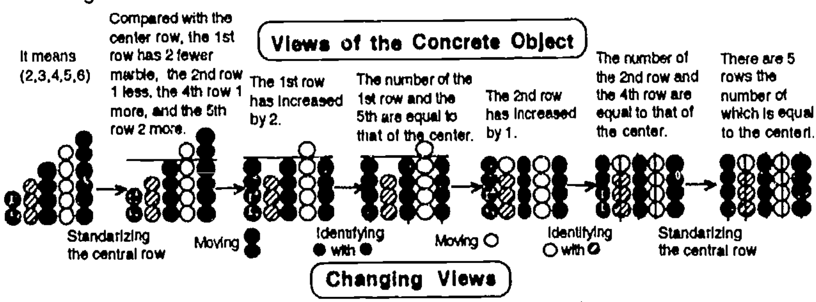


Figure 1. The sequence of concrete actions.

[3] The word "a change of view of the concrete object" involves two kinds of changes: a change of view of the same object, a change of view caused by the transformation of the object.

A proof roughly consists of two elements: a statement and the transformation of that statement. For example, the proof of the previous property also consists of them.

$$
\begin{aligned}
&(X-2)+(X-1)+X+(X+1)+(X+2)\\
&= \{(X-2)+2\}+(X-1)+X+(X+1)+X\\
&= X+(X-1)+X+(X+1)+X\\
&= X+\{(X-1)+1\}+X+X+X\\
&= X+X+X+X+X\\
&= 5\cdot X
\end{aligned}
$$

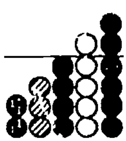
Figure 2. The proof.

Translating a sequence of concrete actions into a proof consists of two parts: translating a view of the concrete object into a statement, and translating a change of view into a transformation of a statement. We call it the translation between the corresponding elements.

Compared with the center row, the 1st row has 2 fewer marble, the 2nd row 1 less, the 4th row 1 more, and the 5th row 2 more.	View →	**Statement**	$(X-2)+(X-1)+X+(X+1)+(X+2)$	
Moving ↓	**Changing** View →	**Transforming Statement**	↓	Moving +2
The total number of marbles is the same, and the 1st row has increased by 2.	View →	**Statement**	$=\{(X-2)+2\}+(X-1)+X+(X+1)+X$	

Figure 3. The translation between the corresponding elements.

Analyzing more minutely, a view of the concrete object consists of three themes of view. The first theme is related to parts of the object, the second theme is related to relations between the parts, and the third theme is related to relations between the concrete object and the previous object. Then, translating concrete actions into a proof means to discern three themes of view, to correspond each theme with the syntax of statement, and to represent views as statements. We call it the thematic translation.



	Themes of View		**The syntacs of statements**
The 1st row has 2 fewer than the center. The 2nd row has 1 less than the center. The 3rd row is central. The 4th row has 1 more than the center. The 5th row has 2 more than the center.	Parts of the object	↔	The 1st term: X-2 The 2nd term: X-1 The 3rd term: X The 4th term: X+1 The 5th term: X+2
The object consists of 5 rows.	Relation between parts	↔	(X-2)+(X-1)+X +(X+1)+(X+2)
The number of marbles is equal to the previous arrangement.	Relation between the object and the previous	↔	Equality Sign "="

Figure 4. The thematic translation.

The observer prepared two levels of instruction. Level 1 is a translation between the corresponding elements, and Level 2 is a thematic translation. In Level 1 of the instructions, the observer used the paper on which the views translated into statements were illustrated

and the changes of view were represented as arrows. We call it a translation paper. In Level 2 of the instructions , the observer used, as an example, the concrete action which changes a view of five contingent numbers into the view of figure 4: compared with the center, the 1st row has 2 fewer marbles, the 2nd row 1 less, the 4th row 1 more, and the 5th row 2 more. The action is a Critical action for a translation into a proof[4] (Miyazaki, In press).

It seemed very difficult to translate a sequence of concrete actions into a set of literal expressions directly for 7th graders who had just learned a literal expression. On the other hand, it is numerical expressions that 7th graders are familiar with and they had experience translating concrete actions into such. Therefore, the observer decided to require the student to translate a sequence of concrete actions into numerical expressions, then to rewrite those numerical expressions into literal expressions[5].

Results and Discussion

Three students individually induced the property that the sum of five contingent numbers is five times the center number. They represented (1,2,3,4,5) as the arrangement of magnets on the whiteboard, and explained the property through the transformation as in figure 1. They could reproduce the transformation in cases other than (1,2,3,4,5). Receiving the instructions Miyazaki(In press) had identified, they individually verbalized the critical action for a translation into a proof: if we represent five contingent numbers as the arrangement of magnets, compared with the center, the 1st row has 2 fewer marbles, the 2nd row 1 less, the 4th row 1 more, and the 5th row 2 more. Then, the observer asked them to explain that the property held good in any case. Although they tried to explain by means of literal expressions, the descriptions are much removed from the proof.

The instruction given to the student ADV and his response

The observer presented the translation paper and required ADV to translate a sequence of concrete actions into numerical expressions: the translation between the corresponding elements [Level 1 of the instructions]. Numerical expressions ADV wrote (see figure 5) has three problems to be rewritten into literal ones. The first problem was that ADV didn't correctly translate the critical action for a translation into a proof: the second line '2+3+4+4+4+1+2' instead of '(4-2)+(4-1)+4+(4+1)+(4+2).' The second problem was that ADV didn't translate the concrete object for every row (see the second line, the third, and the fourth). The third problem was that there was no equality sign between numerical expressions.

The observer didn't especially cope with the first problem. As to the second problem, the observer encouraged the student to translate the concrete object into numerical expressions for every row. ADV changed the second line, the third, and the fourth (see figure 5). As to the third problem, the observer recommended that ADV inserted an equality sign, for the reason that the answer for each numerical expression was equal to each other.

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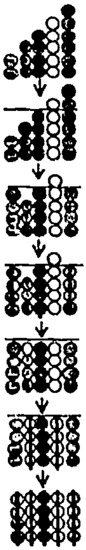
[4] If we translate the explanation without the critical action into numerical expressions:2+3+4+5+6 = (2+2)+(3+1)+4+(5-1)+(6-2) = 4+4+4+4+4 = 4·5, it isn't clear why the number "+2" and "+1" must appeared, the literal expressions which the numerical ones were translated into, and can't be a proof.

[5] If numerical expressions represent the same deductive reasoning as literal expressions, we can rewrite numerical expressions into the literal ones.



	The first version	The second version
	2+3+4+5+6	2+3+4+5+6
	2+3+4+4+4+1+2	2+3+4+4+1+4+2
	2+2+3+4+4+4+1	2+2+3+4+4+1+4
	4+3+4+4+4+1	4+3+4+4+1+4
	4+3+1+4+4+4	4+3+1+4+4+4
	4+4+4+4+4	4+4+4+4+4
	4X5	4X5

Figure 5. The translation by ADV.

In order to rewrite numerical expressions into literal ones, the observer required ADV to explain the induced property by numerical expressions for the first time. Then, ADV wrote numerical expressions in (3,4,5,6,7) instead of (2,3,4,5,6) (see figure 6). Moreover, ADV could rewrite the numerical ones into literal expressions with no help from the observer.

3+4+5+5+1+5+2	$(x-2)+x-1+x+(x+1)+x+2$
~~3+1+4+5+5+5+2~~	
3+2+4+5+5+1+5	$=x-2+2+x-1+x+x+1+x$
5+4+1+5+5+5	$=x+x-1+1+x+x+x$
5+5+5+5+5	$=x+x+x+x+x$
5X5	$=x\times 5$

Figure 6. The numerical expressions and the literal expressions.

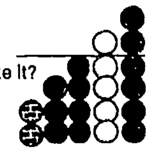
ADV couldn't completely translate the sequence of concrete actions into numerical expressions. It shows that Level 1 of the instructions isn't enough even for ADV. However, translating the concrete object every row is a part of Level 2 of the instructions . Therefore, through the translation, ADV could grasp the correspondence between the concrete object and the syntax of numerical expressions which was the same as literal ones. So, he could rewrite the numerical ones into the literal ones.

The instruction given to the student INT and his response

When the observer did Level 1 of the instructions, INT translated the second picture of the

translation paper into '-2+-1+0+1+2.' Then, the observer did the second level of instruction as following in order to translate the second picture every row. As the result, INT wrote "4+-2+4-1+4+4+1+4+2. Thus, the first and the second problem of ADV were resolved.

Observer: (Pointing to the center row) This is 4, isn't it? (Pointing to the fourth on the left) This, the white circles, how do you write them?
INT : Four.
Observer: Four and ?
INT : one.
Observer: (Pointing to the fifthon the left) How do you write it?
INT : Four and two.
Observer: As the same, (Pointing to the second on the left) How do you write it?
INT : Three and minus one.
Observer: (Laughing) OK. You've got a good position.
INT : Four and minus one.
Observer: I see. (Pointing to the first on the left) How about this?
INT : Four and minus two.



4-2 4-1 4 4+1 4+2

After translating all pictures on the translation paper, the observer required INT to explain the induced property by numerical expressions for the first time. Then, INT wrote a set of numerical expressions with equality signs (see figure 7). However, INT couldn't rewrite them into literal expressions by himself. So, the observer had to equate a numerical expression with a view of the concrete object. In fact, it took more than ten minutes to do so.

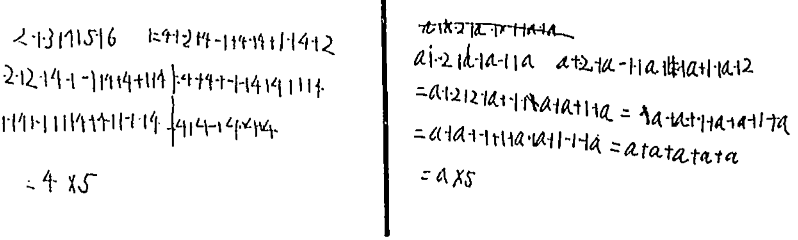


Figure 7. INT's numerical expressions and literal ones.

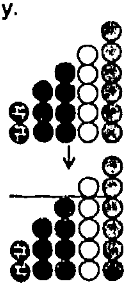
The observer met the fourth problem, it being that it was very difficult to rewrite numerical expressions into literal ones. Actually, the observer had to equate a numerical expression with a view of the concrete object. So, it is clear that INT didn't know what the numerical expression refereed to in the concrete object. We can consider one of the reasons being that because there was no parenthesis "( )", INT couldn't discern the operation symbol "+" as being between a view of each row and a view of the combination of rows. In fact, Level 2 of the instructions for INT didn't correspond with the theme of a view of the relations between parts, with the syntax of numerical expression.

Although INT couldn't move from the numerical expression to the concrete object, if INT had recognized the embedded algorithm independent of each case, then INT could have rewritten numerical expressions into literal ones. Therefore, after translating a sequence of concrete actions into numerical expressions, the observer changed the instruction to one were the observer requests the student to rewrite the numerical expressions in the large number context (Hiebert, 1988), that is, (258,259,260,261,262). This is because, by means of

the changed instruction, a student seemed to recognize the numerical expression as variant, and to find the embedded algorithm.

<u>The instruction given to the student LOW and his response</u>

LOW translated the second picture in the translation paper into "-2+-1+0+1+2" (Level 1 of instruction). The observer first summarized the view of each row of the picture in order, then LOW translated it into '(4-2) (4-1) (4-0) (4+1) (4+2).' Next, the observer summarized the combination of rows, and said "what would you like to write between them?" Then, LOW said "plus," and inserted the symbol "+" between them: (4-2)+(4-1)+(4-0)+(4+1)+(4+2). Finally, the observer summarized the whole number in changing a view, then LOW wrote "2+3+4+5+6 = (4-2)+(4-1)+(4-0)+(4+1)+(4+2). As the same in the other pictures, LOW first wrote the equality sign, used the parenthesis '( )' and wrote the sign "+" between them. Thus, the third problem related to the lack of an equality sign was resolved by doing Level 2 of the instructions completely.



2+3+4+5+6=20

-2+-1+0+1+2

(4-2)+(4-1)+(4-0)+(4+1)+(4+2)

2+3+4+5+6=(4-2)+(4-1)+(4-0)+(4+1)+(4+2)

<u>Figure 8</u>. LOW's translation into numerical expressions (a part).

The observer required LOW to explain the induced property by means of numerical expressions in (258,259,260,261,262) for the first time, then LOW wrote the following numerical ones (see figure 9). Next, the observer required LOW to rewrite it into literal expressions, then LOW said "it isn't necessary," returned the translation paper, and, in less than four minutes, rewrote it into literal ones only by seeing the numerical ones in (258,259,260,261,262). All the while, LOW needed no help from the observer.

=258 + 259 + 260 + 261 + 262	
=(260-2)+(260-1)+(260-0)+(260+1)+(260+2)	=(a-2)+(a-1)+(a-0)+(a+1)+(a+2)
=(260-2+2)+(260-1)+(260-0)+(260+1)+(260+0)	=(a-2+2)+(a-1)+(a-0)+(a+1)+(a+0)
=(260-0)+(260-1)+(260-0)+(260+1)+(260+0)	=(a-0)+(a-1)+(a-0)+(a+1)+(a+0)
=(260-0)+(260-1+1)+(260-0)+(260+0)+(260+0)	=(a-0)+(a-1+1)+(a-0)+(a+0)+(a+0)
=(260-0)+(260-0)+(260-0)+(260+0)+(260+0)	=(a+0)+(a-0)+(a-0)+(a+0)+(a+0)
= 260×5	= a×5

<u>Figure 9</u>. LOW's numerical expressions in the large number context, and literal ones.

INT took more than ten minutes to rewrite numerical expressions into literal ones, and INT needed much help from the observer to equate the expressions with the concrete object. On the contrary, LOW took less than four minutes to rewrite them, and needed no help. Therefore, we can consider that the fourth problem related to the difficulty in translating numerical expressions into literal ones was resolved by corresponding the themes of view of the concrete object with the syntax of literal expressions (Level 2 of the instructions), and by rewriting numerical expressions in the large number context.

Conclusion

We can draw the following conclusion:

If a 7th grader satisfies three conditions: 1: she or he has generated a sequence of concrete actions in which we can see deductive reasoning; 2: she or he has reproduced the sequence in other cases 3: she or he has verbally expressed a critical action for a translation into a proof, then two instructions are effective to translate the sequence of concrete actions into a proof: one of the instructions establishes thematic correspondence between the theme of view of the concrete object and the literal expression through translating the critical action for a translation into a proof, and the other requires explanation in the large number context after translating the sequence of concrete actions into numerical expressions.

The conclusion suggests that the theories on the representational system and the translation between them (Kaput, 1987; Hiebert, 1988) are useful to develop the instruction in the complicated context: the shift from an empirical explanation to a formal proof

The students' final literal expressions aren't simple. In order to write them more simply, students have to disassociate the expressions from the concrete object completely. However, in order to create ideas through the proof, students have to come and go between the concrete object and the expression. Therefore, it is how to deal with this ironic relation between the symbol and the referent that presents a problem worthy of future research.

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THE MATHEMATICAL MODELLING PERSPECTIVE ON WOR(L)D PROBLEMS

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Word problems in text-books often make little sense. An experiment is reported in which students' reaction to such problems was gauged. For some of the problems used (but not others), teenage students predominantly answered in an unthinking way. The implications and directions for further research are discussed.

If 17 men build 2 houses in 9 days, how many days will it take 20 men to build 5 houses? (Treviso arithmetic, 1478; cited by Saljo, 1991)

Laura and Beth started reading the same book on Monday. Laura read 19 pages a day and Beth read 4 pages a day. What page was Beth on when Laura was on page 133? (Lester, Garofalo & Kroll, 1989)

These two problems, spanning more than 500 years, are not particularly atypical. There would be no difficulty in listing many other examples for which the expectation is that calculations based on exact proportionality will be applied, despite the imperfections of such a model that a moment's reflection would reveal.

Saljo (1991, p. 262) unfolds some of the implications of the first problem. This unpacking uncovers implications such as that the productivity of one man, whether working in a group of 17 or a group of 20, or in building 2 houses or 5, is the same; even less realistically, it would seem to be implied that all the men are equally productive. Moreover, the exact answer given in the original text (19 days and 3 hours) appears to entail that each man works at a steady rate 24 hours a day. The reader is invited to carry out a similar analysis on the second problem cited above.

In contrast to the examples, it is not difficult to construct problems for which exact proportionality is reasonably appropriate. In general, students are given no training in making such discriminations, in exercising judgment, or in examining the assumptions implicit in the routine solutions of routine paper exercises.

Davis and Hersh (1981) analysed a number of situations for which addition

is ostensibly appropriate, beginning with this example:

One can of tuna fish costs \$1.05. How much do two cans of tuna cost? Their grocer, it turns out, sells two cans for \$2.00. The point is that having the price of something proportional to the quantity is a reasonable convention that is useful in many cases, but by no means all. As Freudenthal (1991, p. 32) said:

> Mathematics has always been applied in nature and society, but for a long time it was too tightly entangled with its applications for it to stimulate thinking on the way it is applied and the reason why this works. . . . money changers, merchants and ointment mixers behaved as if proportionality were a self-evident feature of nature and society.

Results are reported here for an exploratory experiment in which the degree to which students would adjust assumptions of direct proportionality suggested by the surface structure of word problems was assessed.

Another very striking observation was reported by Davis (1989). Pairs of children were each given five balloons to be shared. One boy cut the fifth balloon in half. Davis (p. 144) put the question: "Was this boy really thinking about solving the actual problem (i.e. effectively sharing the five balloons) or was he trying to accommodate himself to the peculiar tribal culture of the American classroom?"

Experiment

The findings summarized here are from an exploratory study carried out in 1992. A total of 100 13- and 14-year-olds from two classes from each of two schools in Northern Ireland was tested. In one school the students in the classes could be estimated as being at roughly the 98th and 80th percentiles for the population; corresponding estimates for the other school were 70th and 50th. For simplicity, data for the two classes within each school have been collapsed, so the two figures cited in each case are percentages for the two schools (with the percentage for the school with the more able students appearing first in each case). The instructions given were minimal and non-directive; in practice, there was no perceived difficulty in completing the tests. It was made clear that calculators could be used, and they were. Results are reported here for 6 of the 8 item pairs used. In each case a table is presented containing the item pairs and percentages of responses in various categories.

Pair 1: Pizzas and balloons

	If there are 14 pizzas for 4 children at a party, how should they be shared out?	If there are 14 balloons for 4 children at a party, how should they be shared out?
3 each		
with comment		71 54
without comment		19 21
3.5	100 88	8 21

Relatively few students gave an answer implying cutting balloons in half (see Davis, 1989), and a variety of sensible suggestions for equitably disposing of the two extra balloons was offered. One student wrote: "Children each get 3 balloons and 1 balloon between 2 OR each child gets 3 balloons and the others are not given out OR the other 2 could be used in a game.

Pair 2: Pieces of rope

	A man cuts a piece of rope 12 metres long into pieces 1.5 metres long. How many pieces does he get?	A man wants to have a rope long enough to stretch between two poles 12 metres apart, but he only has pieces of rope 1.5 metres long. How many of these would he need to tie together to stretch between the poles?
8 pieces		
answer only	96 100	77 88
comment		8
9 pieces		
explanation		15

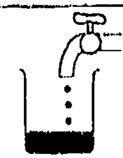
Most students gave 8 as the answer to the second item in the pair, without comment (for an interesting discussion of a more complex but related example, see Kilpatrick, 1987).

Pair 4: 3-mile times

	A barge travels a mile in 4 minutes and 7 seconds. About how long would it take to travel 3 miles?		An athlete's best time to run a mile is 4 minutes and 7 seconds. About how long would it take him to run 3 miles?	
Modification to direct proportionality				
Explanation			4	
Estimate only			4	4
Direct proportionality				
Correct	85	58	73	67
Time/decimal confusion	15	42	19	21

Most students responded as if direct proportionality were appropriate for the athlete (errors in calculation, through failure to interpret minutes and second correctly, were made by those who used calculators e.g. entering 4.7 and multiplying by 3).

Pair 5: Filling flasks

	The flask is being filled from a tap at a constant rate. If the depth of the water is 2.4 cm after 10 seconds about how deep will it be after 30 seconds?			
				
Modification to direct proportionality				
Explanation			4	
Direct proportionality				
Explanation			8	
Answer only	100	96	88	92

Pair 6: 3 minutes

	A girl is counting cars going past her house. In one minute she counts 9 cars. About how many will she count in the next 3 minutes?		A girl is writing down names of animals that begin with the letter C. In one minute she writes down 9 names. About how many will she write in the next 3 minutes?	
Modification to direct proportionality				
Explanation			4	
Estimation only				
Direct proportionality				
Answer only	62	88	92	96
Estimate indicated	35	12	4	4

For the first item, multiplying by 3 gives an appropriate estimate (few students commented that it was an estimate). Only one student realized that direct proportionality is unreasonable for the second item.

Pair 7: Sales of cards

	A shop sells 312 birthday cards in December. About how many do you think it will sell altogether in January, February and March?		A shop sells 312 Christmas cards in December. About how many do you think it will sell altogether in January, February and March?	
Modification to direct proportionality				
With explanation			30	21
Estimate			15	63
Direct proportionality				
Explanation			19	
Estimate	38	21	12	17
Answer only	46	71	4	
Modification for lengths of months	15			

In this case, for the Christmas cards, the performance by the pupils in the second school was better. For Christmas card sales in January-March a variety of estimates with supporting explanations were given. Interestingly, some students hedged by stating that sales would be low, but gave the answer that would be obtained if the level of sales remained constant.

Discussion

As expected, hardly any errors were made on the "straightforward" items. For four of the more complex items, namely those concerning the pieces of rope being tied together (from Pair 2), the time to run 3 miles (from Pair 4), the filling of the flask of diminishing cross-section (from Pair 5), and the number of animal names beginning with C generated in 3 minutes (from Pair 6), a large majority of responses showed no adjustment for realistic constraints.

On the other hand, most students recognized that sharing balloons by dividing some in half is not appropriate, and that the sale of Christmas cards is not uniform across the year. The results confirm that students are liable to respond to word problems according to stereotyped procedures assuming that the modelling of the situation described is "clean". To counteract this tendency, it is suggested that the straightforward use of one or more arithmetical operations should be viewed as only one of a number of candidate models. Instead of being able to assume simple and unproblematical modelling, students should have to consider each textually represented situation on its merits, and the adequacy and precision of any mathematical model they propose. From an educational point of view, this means that careful attention has to be paid to the variety of examples to which students are exposed.

Such an approach offers several important advantages:

1. It attacks the problem that the stereotyped nature of word problems means that apparent success can be achieved through superficial methods rather than actually thinking about the situation described (Reusser, 1988).

2. It addresses the concerns raised by what Freudenthal (1991, p. 5) termed "the poor permeability of the membrane separating classroom and school experience from life experience", part of which is due to the unrealistic nature of school word problems -- which led to the following *cri de coeur* (Freudenthal, 1991, p. 70):

> In the textbook context each problem has one and only one solution: There is no access for reality, with its unsolvable and multiply solvable problems. The pupil is supposed to discover the pseudo-isomorphisms envisaged by the textbook author and to solve problems, which look as though they were tied to reality, by means of these pseudo-isomorphisms.

Wouldn't it be worthwhile investigating whether and how this didactic breeds an anti-mathematical attitude and why the children's immunity against this mental deformation is so varied?

3. The modelling perspective is pervasive throughout mathematics. Introducing students to it at an early stage is part of the process of enculturation into a community of mathematicians. Freudenthal (1991, p. 123) includes as one of five "big strategies" for acquiring a mathematical attitude:

> Identifying the mathematical structure within a context, if any is allowed, and barring mathematics where it does not apply.

Moreover, there are dangers in applications of mathematics of which people should be aware, which is only possible if they understand the nature of modelling (Davis & Hersh, 1986).

The minimal suggestion arising from this paper is that nonsense should be filtered out of school word problems. A more radical suggestion is to take seriously the nature of mathematical modelling; this implies a major shift in the conceptualisation of school arithmetic. One example of such a radically different approach is the Realistic Mathematics Education approach to arithmetic, that begins with contextual problems and develops procedures progressively, always keeping in mind the characteristics of the situation being modelled (Treffers, 1987, chapter 6). Another is through "authentic mathematical activities" (Lesh & Lamon, 1992), involving the identification and exploitation of rich model-eliciting activities, and the distinction between "real" and "mathematical" problems.

Various lines of investigation for further research suggest themselves:

• What are the characteristics that differentiate items which almost all students answer inappropriately from those for which they recognize the constraints of reality?

• In interview, how much awareness would students show of the degrees of appropriateness of direct models in relation to the realistic characteristics of the situations described?

• How important is the social/educational context in which the items are presented? Are students aware of the implicit rules of the school word-problem

"game"?

• How easy would it be to change the beliefs and conceptions underlying students' responses by appropriate teaching?

• How aware are teachers of the issues raised in this paper?

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CHILDREN'S UNDERSTANDING OF FRACTIONS IN HONG KONG AND NORTHERN IRELAND

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This investigation assessed the fraction knowledge of children in Hong Kong as compared to their counterparts in Northern Ireland. A pencil and paper measure was given to children in ten Northern Irish schools and four Hong Kong schools. The Hong Kong children performed considerably better overall, with notable success on items which proved difficult for their peers in Northern Ireland. Possible factors for the differences are discussed.

> Fractions are without doubt the most problematic area in Elementary Mathematics Education. (Streefland, 1991, p. 6).

Fractions are among the most complex and important mathematical concepts which a child encounters in their years in Primary Education. They are the "seedbed" (Hunting, 1983) of more complex mathematical ideas including notions of equivalence, decimals, probability, ratio, and proportion. It is acknowledged that many of the problem areas in mathematics such as algebra are extensions of fraction based knowledge. There is a great deal of documentation which shows that children experience considerable difficulty with fractions (Bright, Behr, Post & Wachsmuth 1988; Carpenter, Matthews, Lindquist, Montgomery, & Silver 1984; Kerslake, 1986). On this basis it is important that research attempts to assess children's difficulties and develop approaches which improve performance on rational number items so that the link can be successfully made to more complex areas.

In research over the past twenty five years which has compared the performance in mathematics of Asian children with those from other countries, tne former have been found to consistently out-perform their peers.
The International Association for the Evaluation of Educational Achievement (IEA) has carried out two major cross-national studies which revealed

considerable differences in achievement by children in different countries. In the second study, Japanese children were ranked first out of fourteen economically advanced nations with England 10th, and the United States 11th. Other extensive cross-national investigations involving Japanese, Taiwanese, and American children by Dr.Harold Stevenson and his colleagues have produced similar results with American students being shown to lag behind both Taiwanese and Japanese children in performance on mathematical tests. In their studies, the Taiwanese children consistently scored between the Americans and the Japanese.

The suggestion that Asian children's success is limited to rote learning of algorithms for mathematical tasks was addressed by Stigler & Perry (1988) who found that "the Asian advantage in mathematics, at least at the elementary school level, is not restricted to narrow domains of computation but rather pervades all aspects of mathematical reasoning" (p. 32). This suggestion is also emphasised by Stigler, Lee, and Stevenson (1990).

It is clear from the research involving Taiwanese and Japanese children that they outperform their counterparts in the United States and other Western countries. If Asian children are successful over a wide range of items such as those included in these tests, it seems possible that an assessment of their approach to the specific topic of fractions might help to clarify an area which causes so many problems for children in Britain. In order to investigate this possibility, the cross-national comparison in this study employed children in Hong Kong. This represents a new addition to the existing body of work, and similarities in curricular content and school set up between Northern Ireland and Hong Kong allow comparison with minimal cultural bias.

Experiment

The pencil and paper measure was designed to include items assessing children's understanding of five areas involved in rational number concepts. These were as follows:

1. Fractions as part of the number continuum.
2. Fractions on the number line.
3. Different embodiments of fractions.
4. Equivalence of fractions.
5. Operations involving fractions.

The test was translated into Chinese by a Hong Kong resident fluent in both Cantonese and English.

Subjects

Fourteen hundred children participated in the test and came from a total of fourteen schools. These children were either in their last year of Primary school or their first year of Secondary school. For the purposes of this paper I will be discussing the results from the children at Secondary level.

In Northern Ireland, subjects were from two High schools, and three Grammar schools. (The High school children are differentiated from the Grammar school children by verbal reasoning tests in their last year of Primary school.)

In Hong Kong, subjects came from an Anglo-Chinese Grammar school, a Chinese Middle school, and an International school run under the English schools Foundation in Hong Kong (the results from the International school are not included in this report since the national make-up of the school represents a distinct educational environment). Children enrolled in the International school follow the National Curriculum as set in the United Kingdom and work towards GCSE (General Certificate of Secondary Education) and 'A' Level (Advanced Level) examinations as set in the United Kingdom. The Anglo-Chinese Grammar school and the Middle school follow the curriculum as set by the Education Department in Hong Kong which leads to the Hong Kong Certificate of Education Examination (HKCEE). This is roughly equivalent to the 'O' level

which was used in the United Kingdom prior to the GCSE. In all but the Middle school the language of instruction is English.

Procedure

The test was given during the children's mathematics classes in January 1992 for the Northern Irish children and in May for the Hong Kong children. The Hong Kong schools were given the choice of either the English or Chinese version. The tests were followed up by semi-structured clinical interviews with a small group of children who were classed as average, above average, and below average based on their performance on the test. Eighteen children from Northern Ireland and twelve from Hong Kong were interviewed. The interviews provided useful insight into the thought processes which children adopted to solve the problems.

Results

The children in Hong Kong performed considerably better on the test than their Northern Irish counterparts; the mean scores out of 36 were 25.8 for Hong Kong and 18.5 for Northern Ireland.

While the Chinese children performed better overall than their Northern Irish peers, some items are of particular interest because of the extent of the difference in understanding shown by the two groups. Figure 1 shows the type of item used and the total percentage of children who answered correctly in Hong Kong and Northern Ireland.

Figure 1: Examples of items from the five categories, with percentages correct for Hong Kong (H.K.) and Northern Ireland (N.I.)

1. Fractions as numbers :

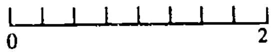
Put rings around the NUMBERS in this set:

A 4 x 1.7 16 $\frac{2}{5}$ 0.06 47.5 $\frac{3}{9}$ $1\frac{4}{5}$ 100

H.K. 63.9 N.I. 12.9

2. Fractions on the number line :

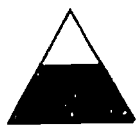
Show where $\frac{3}{4}$ would be on this line:



H.K. 42.4 N.I. 51.2

3. 'Part of a whole' embodiments :

What fraction of the diagram below is shaded?



H.K. 96.7 N.I. 87.9

4. Equivalence:

Tick the one that is the same as $\frac{2}{3}$:

$\frac{3}{4}$ ☐ $\frac{8}{12}$ ☐ $\frac{12}{13}$ ☐ $\frac{3}{2}$ ☐

H.K. 94.9 N.I. 63.7

5. Operations :

The answer to 25 + $\frac{1}{4}$ would be :

MORE THAN 25 ☐ LESS THAN 25 ☐

H.K. 78.4 N.I. 36.6

Discussion

The findings of this study have mirrored previous cross-national studies, with the Asian children fairing considerably better than their counterparts in the United Kingdom. Apart from the general implications, the specific points of interest with regard to rational number concepts and their development are those items in which the Chinese children experienced minimal difficulty, while many of the Northern Irish children struggled. It seems that some of the areas which we have come to expect as inevitably troublesome need not necessarily be so. The interviews with the children stressed two areas in particular where the discrepancy between the two groups was very noticeable. These are the children's recognition of fractions as numbers in their own right, and their understanding of the effect of operations involving fractions.

Fractions are not thought of as numbers in their own right. The results on the embodiment items which incorporated the 'part of a whole' model (item 3) represented the greatest success for the Northern Irish children. It was evident however that they did not extend this notion of a fraction as a part of something to recognise that it is also a number with its own specific value. This was also evident in the item asking children to circle the "numbers" in the set (item 1). This lack of acceptance of fractions as numbers is well documented (see for example Kerslake, 1986) but the results indicate that this is not a universal phenomenon. When children were asked directly in the interviews whether or not they thought that fractions were numbers, the overwhelming response by Northern Irish children was that they were not numbers but parts of some thing else. This contrasted with the Chinese children who seemed more aware of the broader connotations of fractions. Typical responses from the Northern Irish children were:

> No, fractions aren't proper numbers.
>
> Sort of like a piece of a pie or something.

This lack of recognition of fractions as numbers in their own right is a factor which limits children's progression to more complex ideas.

"*Multiplication makes bigger, division makes smaller*". Although relative performance on the equivalence items was also weaker by Northern Irish children than by their Chinese peers, the data on operations (item 5) is of particular interest in assessment of the children's underlying concept of what fractions are. Both groups followed similar curricula and should have been able to recognise that division by a fraction does not result in an answer smaller than the subject of the operation. Those items using multiplication revealed similar misconceptions. The Northern Irish children seemed to be influenced to a greater extent by their understanding of the result of the operation in the whole number domain, than by appraisal of the specific numbers used in the item. This phenomenon has also been well documented (see for example Behr & Post 1987), but again the responses of the Chinese children show that this need not always be the case.

This paper highlights a few of the findings from the investigation. Cross-national studies of this kind must be firmly rooted in the broader educational and social context of the countries involved. In recognition of this, further work considers the following:

1. Cultural norms including educational policy, and society's attitude to education.
2. The school environment including timetables, teaching and curricula.
3. The role of the family.

Preliminary results from parental questionnaires suggest a replication of Stevenson's (1986) finding that the Asian parents believe that effort is the main influence on academic success while the Northern Irish parents choose natural ability as the primary predictor of success. This may be one of the factors which contributes to the Chinese children's achievement. Northern Irish parents seem more inclined to believe that some children simply are not good at Mathematics, and as a result of this they are less inclined to stress effort as a means of progress in Mathematics to the child. The Chinese parents' belief that effort enables the child to improve in turn influences their contribution to the child's learning process. Asian parents also seem to be less willing to rate their child as above

average, and are less likely to be satisfied with what they consider to be an average performance. Clearly these factors are also of interest and need to be considered as potential influences on the child's progress.

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ITERATES AND RELATIONS : ELLIOT & SHANNON'S FRACTION SCHEMES

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In using shemes of operations in fraction and rational number contexts, children can modify and build on successful whole number schemes. They can also devise new schemes that are idiosyncratic to fraction and rational numbers, particularly those that grow out of a developing relational understanding between parts and wholes. In this article we detail the actions of two grade 3 children throughout a 5 month teaching experiment. We pay particular attention to their unit iterative schemes and their development of relational thought.

By the time fraction instruction commences, children have already considerable knowledge of whole numbers and how they behave. It has been observed that children's whole number schemes interfere with the acquisition of fraction knowledge (Behr et al, 1984; Streefland, 1984, 1991; Hunting 1986). Recently attention has turned to investigating ways in which the whole numbers might assist, and indeed form a basis for, developing concept of rational numbers (Hunting & Davis, 1992; Steffe & Olive, 1993). We discuss examples of the modification of two children's operational schemes in the context of a computer-based teaching experiment using fractions as operators. Earlier work on fractions as operators in teaching experiments was carried out by Kieren, Nelson, and Southwell (Kieren, 1976; Kieren and Nelson, 1978; Kieren and Southwell, 1979; Southwell, 1984).

Throughout 1992 we worked with 10 grade three children in a study that emphasised fractions as operators. Some of this work is described in Davis, Hunting and Pearn (in press). We implemented a computer version of fractions as operators, which we call the CopyCat (the name is due to John Bigelow). This model is described in some detail in Davis (1991) and Hunting, Davis and Bigelow (1991). Initially the application was written in structured Basic for Atari machines; now it has been re-written in Hypercard for Macintosh machines.

The methodology used was that of the constructivist teaching experiment (Cobb & Steffe, 1983; Steffe, 1984). In this method a pre-determined curriculum is not followed. Rather, the activities of a session are decided on the basis of observations and interpretation of children's behavior in prior sessions. A brief overview of the teaching sessions is given in the table below. We conducted teaching sessions twice weekly for periods of three weeks. Each session lasted approximately 20 minutes. In a morning we would conduct from three to four sessions; each session working with groups of two or three children. The children were grouped initially according to performance on a set of interview tasks administered prior to the teaching sessions. All sessions were video-taped for later analysis.

DATE	BRIEF SUMMARY OF SESSION CONTENT
1. FRI MAY 29	First session was an introductory one, using SuperPaint, sharing designated pieces of chocolate to determine the number of people, when given the number of pieces per person. E.g., If 12 pieces were shared between 3 then 4 pieces to each,
2. TUE JUNE 2	Children were introduced to the CopyCat program with the fractions hidden. They were instructed as to the use of the buttons on the machine and allowed to experiment with various inputs. They were able to predict 2 for 1, 1 for 2, and used small numbers to predict 1 for 3.
3. FRI JUNE 5	CopyCat was used this time with the fraction revealed. The following fractions were used: 2 for 3 using inputs of 4, 39, 30, and 27, and 3 for 4. Using a 3 for 6 machine they realised that although 3 for 6 was equivalent to 1 for 2 they would need to be able to divide the inputs by 6.
4. TUE JUNE12	A worksheet was given which asked for them to list the types of machines that would work on an input of 20 balls. They didn't have to use the machine but could if they wanted to.
5. TUE JUNE16	Revision of 1 for 2 then onto 3 for 2. Shannon realised fairly quickly that you need to split the number in half and add it to the original number. Went on to try 2 for 1, 4 for 3, and 6 for 10.
6. FRI JUNE 19	Children completed worksheets containing three columns -- Number In, Number Out, and Fraction. Two entries were given and on was to be found. The CopyCat program could be used for confirmation or assistance. Worksheets involved fractions $\frac{1}{2}$, and $\frac{1}{3}$. Shannon was given a sheet of mixed problems with more difficult fractions.
7. TUE JUNE 23	CopyCat program was used. Session commenced with a review of previous session and children were allowed to complete previous sheets. Two groups were given sheet asking for the output given the input of 6 balls, then 15 balls, using the fractions $\frac{1}{3}, \frac{2}{3}, \frac{3}{3}, \frac{4}{3}$...... Another group investigated problems involving $\frac{2}{1}, \frac{1}{2}, \frac{1}{1}$, and $\frac{2}{2}$.
8. TUE SEPT 1	CopyCat used with fraction $\frac{2}{3}$ hidden, then $\frac{3}{4}$ and $\frac{2}{4}$. Experiments were conducted to discover the fractions.
9. FRI SEPT 4	All students were given the same worksheet that listed the fractions $\frac{1}{4}, \frac{1}{2}, \frac{1}{3}, \frac{2}{3}, \frac{3}{4}$ horizontally, and inputs of 2, 3, 4, 5, 6, 7, 8 vertically. All students except Shannon used the machine to either prove predictions or solve the problem.
10 TUE SEPT 8	Different problems were planned for three groups. Alisha and Elliot worked on sheets as before but the fractions were fifths and inputs ranged from two to 15. Diane, Nadia, and Tammy used the CopyCat to see what inputs would work with $\frac{1}{2}, \frac{1}{3}, \frac{1}{10}, \frac{1}{5}$. Sukey, Nick, and Sandy tackled the following problem: "Put a circle around the fraction that makes the CopyCat go for the number of balls in the box". The first number chosen was six and the fractions had unit numerators with denominators from one to 10.
11. FRI SEPT11	Brief exploratory session using the CopyCat and SuperPaint. The key problem was: "Would the $\frac{2}{3}$ CopyCat machine work if we put 30 chocolates on the In tray?"
12.TUE SEPT 15	Discussed worksheet which asked children to compare various pairs of fractions and to give written reasons for choices. CopyCat used to confirm answers.
13. FRI SEPT 18	Both SuperPaint and CopyCat programs were used. Sukey and Alisha in one group, Nick and Sandy in another group, and Tammy, Nadia, and Diane in another used SuperPaint to solve problems such as $\frac{1}{3}$ of 24, $\frac{1}{5}$ of 30 where both input and output could be seen together. Elliot and Shannon work on problems of comparing fraction pairs.

14. TUE OCT 13	A large cardboard voting booth was modified to simulate a CopyCat machine and the children had to design problems for each other using fractions given by the teacher. One child was stationed inside the booth and operated on the inputs. Problems arose when children put out incorrect outputs.
15. FRI OCT 16	Inputs and outputs from the previous session were recorded on a board and discussion took place. Worksheets were given in which various input-output pairs were displayed, and the appropriate fraction required. The sheets encompassed a range of difficulty.
16. TUE OCT 20	The SuperPaint program was used to make boxes $\frac{3}{2}$ times larger, $\frac{3}{4}$ times larger. Alisha and Sukey, together,and Elliot by himself did this. Tammy and Nadia continued working problems in the same vein as in Session 15. Diane and April experimented with $\frac{1}{2}$ and $\frac{1}{3}$ machines using various inputs.
17. FRI OCT 23	Work on CopyCat set to the fraction $\frac{5}{10}$ which was hidden. The first input was 10. All thought the CopyCat was a $\frac{1}{2}$ machine. Other inputs attempted did not produce expected results.
18. TUE OCT 27	Following on from Session 17, the children in the various groups were asked to predict the numbers that would make a $\frac{5}{10}$ machine work. There was some discussion about other machines that would work like a $\frac{1}{2}$ machine.
19. FRI OCT 30	Children in the various groups were asked to explain what they thought was happening inside the CopyCat, and to explain how a new machine could be built.

As a computer environment in which children can act, the CopyCat is quite restricted. There are severe limitations on what it is that children can do in this environment. They can press an up or down button to add or subtract sprites from the "in-tray" of a particular fraction machine; they can press a "go" button to see if the machine will or will not work; and they can change the numerator or denominator of the machine so as to get a new fraction machine. Eventually we want to be able to string machines in parallel and series, but this is not yet possible. So in this sense the fraction machines provide a distinctly pictorial implementation of fractions as operators which corresponds to the construction used in the complete ring of quotients (Davis, 1991) but that is all they do. They are a pictorial tool which quickly and accurately allows a teacher to implement a particular fraction as operator. Used in conjunction with other graphic tools we have found them to provide a strong relational model for children's thinking about fractions.

An important additional graphic tool we use is Aldus SuperPaint. The combined draw and paint, copy, paste, cut, fill, and rotate capacities of SuperPaint make it an excellent action environment for children. If the CopyCat provides us with a pictorial and animated implementation of fractions as operators, then SuperPaint provides us with an environment in which to prepare objects so that fractions as operators can operate on them.

The two students we report on from this study are Elliot, 8 years 4 months at the start of the teaching experiment at the end of May, 1992, and Shannon, 8 years 11 months at that time.

INITIAL INTERVIEWS

All children in the study were interviewed individually using a comprehensive set of tasks. The tasks included partitioning of non-continuous items, basic fraction knowledge, verbal counting, counting composite units, quantification of arrays., and ratios. Shannon and Elliot were remarkably similar in their performance on these tasks.

The partitioning tasks involved distribution of items equally between dolls: 12, 28 and 56 items between four dolls, 19 between three, and a task in which 20 items were to be shared equally in as many different ways as possible. For the task of distributing 28 items between four dolls, Shannon showed his facility with whole numbers by allocating seven items to the first doll, seven to the second doll, and so on, indicating that he used numerical operations to anticipate the result. For 56 dolls he placed 14 items in front of the first doll saying he was "doing it by fours". Further questioning indicated Shannon first considered the results of placing 10 items for each doll. This would have left 16 items, which when divided by four, meant four more items would be needed for each doll. Shannon was able, fairly easily using 20 items to manipulate, to determine the numbers of dolls which could share the items equally to be five, four, two, one, 10, and 20.

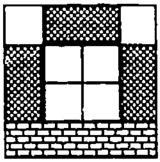
Elliot like Shannon, also placed 10 items out for each doll in the 56 item task. However this result was likely an estimate. He placed out one, two, two, and two in the next round. He was observed to count each share and make adjustments in a non-systematic trial and error fashion. For the task of determining the different number of possible shares that could be made with 20 items Elliot made five groups of four, split these groups into 10, and then split the groups into 20. His final successful result was four groups of five. He unsuccessfully attempted to make seven equal groups.

Counting tasks. Shannon and Elliot were fluent in counting composite units where cumulative totals were to be determined as rows of objects were progressively uncovered. Units of two, three,...,ten were displayed. Verbal counting skills were also assessed. Both Shannon and Elliot were able to count forwards and backwards by twos, threes, fours, five, sixes, and tens, beginning at any number. Elliot displayed verbal behavior reminiscent of a tables sing-song. He was slower with nines, where he segmented each unit into four and five.

Array tasks. For the first task rectangular arrays of small square regions were displayed with instructions to evaluate to number of visible squares. Shannon used the strategy of relating multiplication facts with the relevant number of items in each row and column, or added or subtracted a row or column from a previously computed display. Elliot was observed to do likewise. For the second task a large base array (10x12) was displayed. The interviewer placed different sized rectangular pieces of card over the base array to cover certain sub-arrays. The problem was to calculate how many items were covered. Both children were quite fluent with this task type. They seemed able to visualize and count each non-visible row in the covered array.

Ratio task. A question typical of the sequence asked was "There are six dolls and I want to give five counters for each doll. How many counters will there be?" Counters were available, and six dolls were placed in a line across the table. Shannon gave the answer to these questions without reference to counters or physical contact. Elliot used multiplication facts for simple tasks but he also used serial group counting as evidenced by his head movements.

Fraction tasks. Three types of fraction tasks were given. A square arrangement of cuisenaire rods adapted from a task used by Saenz-Ludlow (1992) showed rods arranged thus:



The children were asked to show one-half, one quarter and one eighth of the unit pattern. Shannon responded appropriately for the fractions one-half and one-fourth. He divided the unit into two equal parts for one-eighth.. Elliot's first attempt to show $\frac{1}{2}$ was unsuccessful, but on a second attempt he reorganised the rods appropriately. The second task involved coins with denominations of 5, 10, 20, and 50 cents. Fractions of one dollar were required, including $\frac{1}{2}$, $\frac{1}{5}$, $\frac{1}{4}$ and $\frac{1}{10}$ Shannon did not know the fractions $\frac{1}{5}$ and $\frac{1}{10}$ in this context. Elliot was successful with these fractions. Small chocolate Easter eggs were used to pose problems about an Easter egg hunt for the third task type. Fractions involved were $\frac{1}{2}$, $\frac{1}{3}$ and $\frac{3}{5}$ Shannon did not succeed with $\frac{1}{3}$ or $\frac{3}{5}$. Elliot succeeded with $\frac{1}{2}$ only. Overall Shannon was able to respond appropriately to tasks involving the fractions $\frac{1}{2}$ and $\frac{1}{4}$. Elliot was successful with $\frac{1}{2}$ tasks.

UNIT ITERATIVE SCHEMES

Iterative schemes in children's numerical mathematics are fundamental (Steffe, Cobb and von Glasersfeld, 1988). It is no exaggeration to say that the physical and mental iteration of units and comparison with other units is an essential part of the psychological basis for the development of number. Steffe and Olive (1993) describe iterative fraction schemes in which:

> "children easily established their own fraction language by iterating fractions so many times. The term 'two-thirds' was an abbreviation for 'two-one thirds', etc. In this way, the children established any fraction as a multiple of its unit fraction, and thus as a modification of their 'times-as-many' multiplying schemes for whole numbers."

Although Steffe and Olive refer to the unit that is being iterated as a "fraction", it is not clear to us what the evidence is that the children conceive of those units in relation to the wholes of which they were a part. Nevertheless, the iteration of a continuous unit in this way is a clear use of a whole number scheme in the context of fraction tasks, and does appear to lead children to an understanding of $\frac{2}{3}$ as twice $\frac{1}{3}$, for example. A similar example of this sort of iteration of a unit fraction is described in Davis, Hunting and

Pearn (in press). So it appears that the ability to iterate unit fractions is an important step in children's construction of fractions as numbers, and that their whole number unit schemes may play a major rôle in these constructions. Basically it seems that if students have constructed the explicitly nested number sequence (Steffe, Cobb and von Glasersfeld, 1988) then they may be able to successfully modify those schemes in the context of fraction tasks.

An example of the power of whole number iterative schemes occurred in our last teaching session in 1992. We asked the children to tell us what they thought might be happening inside the CopyCat. The teacher had drawn a rectangle in SuperPaint to represent the CopyCat and asked Elliot and Shannon what sort of machine it should be:

> Teacher: "What sort of machine should we make it? What sort of machine? What sort of fraction? A 1 for 3, 1 for 2?
> Shannon: 1 for 16.

The teacher then copied 16 small squares in a scrambled bunch and copied that 5 times to put 80 items on the fraction machine model in SuperPaint.

> Teacher: "So what might this 1 for 16 machine ... what might be happening inside it? What do we draw?"
> Shannon: "The man might be stealing 1 and throwing away ... throwing back 16."

Elliot then took 1 one small square from the fraction machine and placed it on the right of the screen. Shannon said to get 16 more, and Elliot did, placing them on the left of the screen.

> Teacher: "Yeah, what do you do now? You just do it again do you?"
> Elliot: "Yeah, we move this." (He indicates more counters from the machine).
> Shannon: "You better put the 1 back now."
> Teacher: "Oh! you put the 1 back now? Oh, really?"
> Shannon: "Yeah, cause then you have to have another - then you have to have 16."

Elliot proceeded to repeat his actions and Shannon pointed out that there wouldn't be enough small squares left on the last turn to make 16. He estimated there would be about 12. Elliot did not seem to understand Shannon's remarks: he continued until, to his apparent surprise there were only 11 squares left. Both children decided to copy the 5 squares situated at the right of the machine.

It seems to us that Shannon was able to imagine the repetition of the act of placing one square to the right of the screen and 16 to the left. If he was simply imagining the repetition of placing the 16 to the left then he would presumably not have realised there would not be enough counters left at the final step in his mind he had to carry the placing of the 1 square to the right as part of the action to be repeated. We believe Shannon conceived of a tally of 1 for every 16 - where the "1" was a mental tag not tied to

the physical quantities upon which he was operating. This modification of a whole number iterative scheme was particularly powerful for Shannon because it enabled him to realise that his proposed model for the working of the machine could not be correct. Elliot, on the other hand, was absorbed in the physical actions of carrying out Shannon's model and it was not until the last step that he realised there was a problem.

RELATIONAL THOUGHT

Elliot had many ways of guessing or figuring an answer to our fraction problems. Often his strategies were additive in that he would count on by a multiple of a given number, by 2's for example. What was striking about Elliot was his active, agile mind and his exuberant attitude. This mental activity seemed to us to manifest itself in wonderful rules: patterns of number behavior inferred by Elliot. As the teaching episodes progressed Elliot exhibited more and more examples of subtle relational thought. For example, on June 19, in a teaching episode with two other children, Alisha and Shannon, Elliot was able to guess that Alisha would get an output of 16 from an input of 24. He had previously input 9 and stated correctly that 6 would come out of the $\frac{2}{3}$ - machine. Then he predicted that 14 would come out when 21 was put in, and he confirmed this prediction. His reason for this was as follows:

> Teacher: "What's your reason?"
> Elliot: "Cause ... umm .. I thought like this way." (He holds the teacher's hand and shakes it up and down). "If it's over ... umm ... 20 it's 7. If it's over 15 but under 20 it's 6, and if it's over 10 and under 15 it's 5."

Elliot's agitated expression during his explanation indicated to us that he had a sudden insight. We believe his insight was connected with the common factor that one multiplies 3 and 2 by, respectively, to get the input and output for a $\frac{2}{3}$ - machine. In other words he seemed to believe he had figured out when a whole number changed from the form 3n+k to 3(n-1)+k, with k small. Then, we infer, in order to find the output all he had to do was to multiply n by 2.

REDUCED FRACTIONS AND QUANTITIES

If a young child habitually represented fractions in reduced form, one might suspect that they had a quantities model for fractions. That is to say they might interpret fractions as operators on quantities to produce other quantities, rather than operators on whole numbers. There were many episodes in our 1992 study in which children interpreted fractions in reduced form: indeed, few in which they did not. The simplest example is the interpretation of $\frac{2}{4}$ as $\frac{1}{2}$. This phenomenon seemed to be very persistent and resistant to suggestions from us. All the children did it, from the slowest to the quickest. This seemed rather unusual for Elliot and Shannon, because they both had considerable adroitness with counting numbers, and episodes with them indicated that they were thinking relationally, in terms of inputs and

outputs, about quite complicated fraction machines. Yet they regularly put fractions in reduced form and seemed reluctant to give what was, in our terms, an "obvious" answer in terms of the input and output of the fraction machines. For example, Elliot and Shannon were asked what sort of machine might produce an output of 4 for an input of 25. Elliot guessed 6, and then realised that was not correct. Shannon gave no answer that session, but the next session told us that he had thought about the problem in class, and the answer was $\frac{1}{7\frac{1}{2}}$. Allowing for his calculational error, what internal scheme prompted Shannon to express the answer this way, rather than the (to us) obvious way as $\frac{4}{25}$? Episodes like this raise the possibility that many children, perhaps most, do indeed have a model of fractions based on quantities.

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CONCEPTUAL BASES OF YOUNG CHILDREN'S SOLUTION STRATEGIES OF MISSING VALUE PROPORTIONAL TASKS

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Abstract
Analyses of young children's problem solving activities on proportional tasks revealed that the build-up strategy is much more complicated than some have suggested. One specific version of their strategy, using the relationship "X quarters for Y candies," was further analyzed. It appears that construction of this strategy requires children to understand the unit-price as rate-relationship. Furthermore, the ability to coordinate two counting sequences contributes to the degree of sophistication of such strategy.

Introduction

Proportional reasoning plays a crucial role in formal mathematics curriculum (Lesh, Post & Behr, 1988). The ability to recognize structural similarity, and the sense of co-variation and of multiple comparisons are at the core of algebra and more advanced mathematics. Examples involving proportional situations are amply present in a wide variety of daily problem-solving situations, for example, shopping, cooking, scale drawing. As the mathematics curriculum shifts attention from computation to problem solving, proportion-related tasks provide a good source of meaningful situations for students to construct their mathematics.

Because of the importance of this topic in school mathematics, children's concepts of proportionality have long been a focus of mathematics education research. Studies have shown that proportion is a difficult concept for students of middle and upper grades (Hart, 1984). Researchers suspected that the difficulty of conceptualizing proportions might result from an algorithmetic approach to this topic in schools. With tricks like "keeping the like units on the same sides and maintaining the small-to-large comparison" it is possible for students to create the "correct equation" for a missing value task without understanding the invariant relationship across relative pairs of quantities (Kaput & West, in press).

Furthermore, researchers have also pointed out that many factors such as "the physical principles underlying the problem situations" and "multiplicative relationships between problem quantities" affect student reasoning patterns in solving missing value tasks (Harel, Behr, Post, & Lesh; 1991). Based on the constructivist epistemology, we take such findings as evidence that proportion is not something a student either has or not. It seems imperative that researchers understand children's proportional reasoning instead of measuring children's thinking against their own proportional reasoning schemes.

The exiting studies on students' informal methods provide strong evidence that children can and do develop their own understanding of proportion through making sense of their daily experiences, without being explicitly instructed. Two types of successful strategies were reported

for tasks comparable to the following (Lin & Booth, 1990): "To make soup, we need 6 potatoes for 8 people. How many potatoes do we need for 6 people?" The first approach, "build-up using halving," was reported by Hart (1984). With this approach children would find out the amount of potatoes needed for 4 people by noticing 4 is half of 8, thus halving 6 to get 3 potatoes. Then they halved the amount of potatoes for 4 people again to obtain the amount of potatoes needed for 2 people. Finally they either added these two amounts (for four and for two) or multiplied the amount of potatoes for two people by three to obtain the required amount for six people. Even though researchers were encouraged by the sophisticated thinking behind this approach, there were concerns about the apparent lack of generality when the numerical relationship could not be derived by combinations of a series of 'halving' or 'doubling' approaches.

A second, "unit-factor approach," was reported by Lin and Booth (1990). Children would first attempt to find the amount of potatoes for one person by a variety of sharing/partitioning strategies, then find the amount of potato for six people either by multiplying or repeatedly adding. This approach has been recommended as the basis of initial instruction on proportion because studies have shown that even young children can understand the notion of 'unit-factor' and because this approach can be easily generalized to different tasks (Schorn, 1989).

In summary, much is known about factors which influence students' performance and the "informal methods" which deal with certain types of proportional tasks successfully, yet we know very little about how those difficulties or successes were developed as a part of the individual's mathematics knowledge. Lesh, Post and Behr (1988) pointed out that concepts like the part-whole relationship, composite units and unit coordination, representation-related abilities, and measurement-related abilities are all critical to the development of proportion concepts. We concur that it is important for studies on children's understanding of proportion to include these constructs in their analyses whenever appropriate. Because an individual's concept of proportion is a result of an individual's accommodation and re-construction of existing knowledge. One purpose of this paper is to present such an analysis on young children's solution strategies of missing value proportional tasks.

The Study

As an initial step to understand the constructions young children make when they attempt to make sense of a variety of proportional tasks and the conceptual basis for those constructions, we interviewed 7 second graders and 8 fourth graders on a variety of proportional tasks, including sharing, pricing, cooking, linear measuring and identifying similar shapes. All the tasks were comparable to those tasks used in previous studies with older students, with the exception of two features, 1) all the tasks were presented verbally in story-type settings and with physical objects to represent the initial conditions of each task, 2) all the numbers in the tasks were natural numbers and were kept small. Students were encouraged to manipulate those materials, draw pictures or write on the papers if they wished to do so. They were also encouraged to verbalize and reflect

their solution strategies during and after their problem solving process. The purpose of this study is to establish some baseline data on young children's strategies of different proportional tasks, since almost none of this type of information exists.

Because of the exploratory nature of this study, we did not feel the need to conduct these interviews with pre-set tasks following a pre-set sequence. Rather, we made changes and modification as we actively constructed our initial models of individual children's concepts of proportions during each interview. In many ways, our interview sessions resembled the individual sessions of a teaching experiment. Our goal was to construct viable accounts of the meaning these children gave to each proportional task. We fully realize that all conclusions we made should be treated as tentative and subjected to further investigation.

In this paper, we will present our initial analysis of students' solution strategies of one particular type of missing value task. For example, "Yesterday, I bought C_1 candies (pointing to the C_1 cubes) with Q_1 quarters (pointing to the Q_1 coins). Today, I go to the same store with Q_2 quarters, how many candies can I buy?" Because of the nature of this task, our analysis was influenced by the work on composite units and unit coordinations in the setting of counting (Steffe, von Glasersfeld, Richards & Cobb, 1983: Steffe & Cobb, 1988), multiplication and division of the natural numbers (Steffe, 1990), and fractions (Saenz-Ludlow, in press; Watanabe, 1991). This analysis, although not yet complete, has challenged our own notions of proportion, provided us with paradigm cases to reflect on, and raised questions to guide subsequent investigation. We would like to offer our tentative analysis as a basis for discussion.

Data Analysis

Because of space limitation, we will present our analysis on two students' solution strategies in detail, referencing the analysis of other students' solution strategies when appropriate.

<u>Bruce's Solution Strategies:</u>

The first task Bruce was given had the condition of ($Q1=4$, $C1=10$; $Q2=6$). The first thing Bruce did was to re-arrange the 10 cubes to form groups of two. Then he counted the number of two-group, with finger pointing to each one of them and found out he had five groups. He did not seem to like the result. He thought for a while, then started putting the 10 cubes into groups of three. After making two groups of three, he realized that he could not arrange 10 cubes into groups of three evenly. He abandoned that action and attempted to search for other grouping. His attempt resulted in two groups of five, then five groups of two again. Neither of these results seemed to meet his expectation. He looked puzzled. The interviewer then asked Bruce , "Can you find out how many candies two quarters can buy?" Bruce answered this question quickly, "five" because "that's a half of a dollar (2 quarters) so you split these (cubes) up and you can get 5." With this new information. Bruce solved the original task by adding 5 to 10 because 6 is 2 more than 4.

We interpreted Bruce's initial actions of making equal groups of certain numbers as an action to find the number of candies he could buy with one quarter. That was why he felt puzzled about not being able to form a one-to-one match between candy groups and quarters. Nevertheless, Bruce did construct the the relationship between 4 quarters and 10 candies needed to be preserved for a given number of quarters or candies. This notion enabled him to answer the question, "How many candies could you buy with 2 quarters?" Because he needed to maintain the rate relationship, one-half of the money could buy one-half of the candy.

Bruce's success in solving the above task relied on identifying a useful 'X quarters for Y candies' relationship. In the subsequent questions, Bruce consistently demonstrated that he had the intention to find ways to group candies and quarters so that a one-to-one correspondence could be formed between each group of candies and each group of quarters. For example, given the condition of (Q_1=12, C_1=28; Q_2=15), he was able to identify the equivalent relationship '3 quarters for 7 candies' which he then used to solve the question for 15 quarters. He re-conceptualized 15 as 12 and 3 more, thus adding 8 to 28 to get 35 candies.

Bruce's strategy of finding a useful 'X quarters for Y candies' relationship was trial-and-error based. For this particular task, Bruce first divided 28 candies into 7 groups of 4. When asked why he answered that because 4 could be divided into 28. Then he started to divide the quarters into groups of 2, counted the number of groups, found that there were 6 groups rather than 7 groups. Then he proceeded to find other ways to group the quarters. When none of those groupings were satisfactory he then re-grouped the candies.

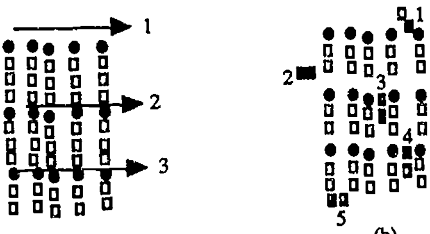
After several tries, he made 4 groups of 7 candies. Then he started to divide 12 quarters into groups of 4, rather than trying to form 4 groups. We were not clear why Bruce did not use his multiplication facts to help him decide whether it is possible to divide 12 quarters into 4 equal groups. It appeared that Bruce had a rather rigid view of the divisor as the number of elements in each group. He did not seem to recognize that the number of groups and the number in each group were interchangeable when the total number is pre-determined.

Bruce's strategies of finding the useful 'X quarters for Y candies' relationship, although not widely documented in the existing literature, was also used by four other fourth graders and one second grader we interviewed. Two of those fourth graders used their multiplication facts to help them identify the possible ways of grouping, and seemed to be able to interpret the divisor both as the number in each group and the number of groups. Additionally, two fourth graders and two second graders could identify the useful 'X quarters for Y candies' and carry out the necessary coordination of two counting sequences when the situations required only halving and/or doubling.

Martha's Solution Strategies:

Martha's solution method was unique in our study. We will illustrate it with a task of (Q_1=15, C_1=40; Q_2=21). Martha started by re-arranging the 15 quarters into three rows of five. Then she proceeded to distribute cubes (candies) one by one beside each quarter until there was no

more cube left (Figure 1a). After examining the physical arrangement of the quarters and cubes, Martha removed those cubes which seemed to be extra. She the counted what she took away and found out that there were 10 cubes.



Note: These cubes were virtually non-differentiable. The patterns were used to aid comprehension.

Figure. 1

After figuring out the number of cubes still needed to be distributed, Martha wrote 15 divided by 10 in the traditional written format and figured out the answer was 1 r 5. It appeared that Martha did not find this information to be useful. She put down the pencil, and started to point in the air with two fingers of her right hand. After pointing 8 times, Martha stopped the pointing and did not seem to like the result. She started over. This time she pointed five times and appeared to like the result. Then she started to place 5 cubes one at a time. She counted what she had left, thought a while, then placed the remaining 5 cubes one at a time by the previously placed 5 cubes (Figure1b).

To solve the question, "How many candies can I buy with 21 quarters?" Martha drew 6 circles on the paper, examined her arrangement of cubes and quarters, pointed 8 times, then wrote '8' by each of the three circles on the paper. She then figured out 15x8, added 16, and announced "136." Yet immediately Martha changed her mind, she added 40 and 16, then indicated the answer should be "56."

Martha's actions reminded us of the dealing scheme (Hunting, Pepper & Gibson, 1992). She had a clear idea of what dealing could accomplish, formation of equal groupings. With the help of visual re-presentations, she was able to keep track of the cycles of her dealing without paying much attention to either the number of the cubes that needed to be distributed, or the number of cubes in each group.

Unlike some students, Martha did not abandon what she had accomplished with dealing after realizing that dealing alone was not enough to help her achieve her goal, distributing all the cubes equally among all the quarters. Her gestures, pointing with two fingers in the air systematically (from right to left and top to the bottom), indicated she was trying to figure out if one candy for

two quarters would work. These gestures served the function of "segmenting" (Steffe, 1990). The eight-pointing indicated that she was attempting to segment 15 quarters into groups of two's.

After successfully distributing 10 candies among 15 quarters, Martha was able to give meaning to the result of her actions. She constructed the '3 quarters for 8 candies' relationship as she reflecting on the whole dealing process. In order to figure out the number of candies 21 quarters could buy, Martha re-conceptualized 21 as 15 and 6 more, then as 15 +3+3. Then she calculated the number of candies by uttering '40 candies, 8 more, and 8 more.' She then computed 40 +16 with paper and pencil to get the answer '56.' In a sense, the '3 quarters for 8 candies' became a countable unit for her, and the counting of this particular unit required the coordinations of two counting sequences.

Discussion

There were similarities and differences between the conceptual bases of Bruce's and Martha's solutions. First of all, both their actions seemed to be based on their preliminary notions of *unit-price* as a rate relationship. When asked to solve the task of (Q_1=4, C_1=10; Q_2=6), some students we interviewed who had not constructed 'candies per quarter' as a rate relationship were not perturbed by their solutions in which some quarters could buy more candies than the other quarters. Even though Bruce did not set out to find the *unit-price* when solving these tasks, he recognized the existence of a relationship between the number of quarters and the number of candies in a given condition and the need to preserve this relationship within a task. This observation was based on Bruce's inferred intention to search for a one-to-one match between candy groups and quarter groups. At that time, we did not know enough about Bruce's construction of fraction concepts to further analyze his construction of *unit-price* as a rate relationship.

Compared to Bruce, Martha's notion of *unit-price* was more sophisticated. Her action was clearly guided by the intention to give every quarter an equal number of candies. Martha was the only student we interviewed who was comfortable about the idea of breaking a whole candy into parts. When asked how she solved the task of (Q_1=9, C_1=21; Q_2=6), Martha said, "Per quarter you had 2 and 1/3, so you just, you took away 3 quarters and all their candies, and you counted up the rest of them and came up with 14." This statement indicated the emerging sense of the equivalent relationship among '9 quarters for 21 candies,''one quarter for 2 1/3 candies,' and '3 quarters for 7 candies.' Still, additional tasks are needed to further investigate Martha's construction of *unit-price* as a rate relationship.

Second, both Bruce's and Martha's strategies seemed to rely heavily on the manipulation of cubes and coins. It was not clear what they would do if there were no cubes or coins available to manipulate nor how they would deal with situations involving larger numbers. The way tasks were constructed did not provide us with opportunities to answer either of these questions.

Martha relied much less on the physical manipulations of cubes and quarters than Bruce did. For the task of (Q_1=15, C_1=40; Q_2=21), she formed a mental picture of 15 quarters arranged in a linear faction, even though they were physically arranged as three rows of five. This observation was based on the sequence and position in which Martha placed the five cubes (Figure 1b). She was able to mentally segment this array of 15 quarters into groups of 3's without making a physical arrangement of them. More amazingly, Martha was able to construct the '3 quarters for 8 candies' relationship as she reflected on her own dealing actions in order to form her plan to solve the given task. This relationship appeared to be her mental construction, even though the presence of the physical objects might have aided in this construction.

Third, both Bruce and Martha constructed useful 'X quarters for Y candies' relationship as the basis of their solution strategies, so did other students who could solve this type of tasks successfully. This approach seemed to share a similar root with the halving or doubling schemes in the sense that they both made the use of build-up process possible for a particular task. One thing we concluded from this study is that the build-up process is not as primitive as some researchers may suggest. For example, when asked how many candies could 6 quarters buy given that 2 quarters could buy 6 quarters, Simon (a second grader) answered '24.' He explained that he added 6 more candies for 2 more quarters, and he got 12 candies. Then he added two more quarters, and 12 more candies to the 12 he already had. In this instance, Simon successfully formed 6 quarters by re-presenting two more groups of two quarters, yet he was unable to maintain the '2 quarters for 6 candies' relationship. We believe that the scheme to coordinate two counting sequences is a necessary construction for understanding proportionality.

However, the effectiveness of both Bruce's and Martha's approaches could be greatly facilitated by using the concepts of divisors as the possible number of equal groups formed with a given number. Even though Martha's approach appeared to be more systematic, it was still trial-and-error based. Consider the case of (Q1=18, C1=46). For this particular task, Martha would have 10 cubes left after she distributed 2 cubes to each of the 18 quarters. Using her approach, she might have to go through distributing candies among 9 groups (as a result of segmenting by 2's), 6 groups (as a result of segmenting by 3's),3 groups (as a result of segmenting by 6's) as likely candidates until she reached 9, which would segment 18 into two groups. During the interviews, neither Bruce nor Martha indicated any use of of multiplication or division in their search for equivalent relationships. We are currently developing additional tasks to learn more about Martha's and Bruce's concepts of multiplication and division.

We are surprised by the lack of documentation of this "X quarters for Y candies" approach in the literature. It is yet to be determined what mental operations can be constructed through this strategy. Nevertheless, both the process of identifying a 'X quarters for Y candies' relationship and its equivalent relationships from a given condition, as well as the nature of coordinating two counting sequences in a build-up process are everything we image when considering proportional

reasoning as "a sense of co-variation and of multiple comparisons, and the ability to mentally store and process several pieces of information" (p.93, Lesh, Post & Behr, 1988). We expect our on-going work with Bruce and Martha will help to answer some of these questions.

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EARLY CONCEPTIONS OF FRACTIONS
A PHENOMENOGRAPHIC APPROACH

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In this study, pupils from grade 2, 3, 4 and 6 (N=41) were given problems intended to reveal their conceptions of 1/2, 1/4 and 1/3. The answers expressed six conceptions of these fractions – 'fair shares', 'parts', 'form', 'size', 'one of a number of parts' and 'ratio' – and three conceptions of how fractions can be created: through 'measuring', 'halving' or 'estimating and adjusting'. The 'ratio' conception was more often expressed by the younger than by the older pupils. Yet, practically all these younger pupils, in parallel to the 'ratio' conception, could in other contexts express the conception that 'fractions are parts of a certain size'. A comparison between the answers of the younger and the older children indicated that in the attempt to change the 'size' conception, teaching stresses the idea that the denominator denotes the number of parts. This seemed to have formed the older pupils' conviction that 1/3 cannot be 2/6, impeding their understanding of fractions as 'ratio'.

Vygotskij and his followers Leont'ev and Luria have stressed the social and cultural influence of the ways in which we understand the world around us, illustrating that the words we use have different meanings for people belonging to different societies and cultural groups. Further, within one and the same society the meanings of the words as experienced by one single individual are changed from time to time, depending on new situations in which they are used.

For teachers it is important to know about pupil's experiences of the subject matter dealt with in teaching. Only then can they set out from this informal knowledge, trying to extend it through communication and through confrontation that can bring existing misunderstandings to light.

Research related to the phenomenographic approach, within which the study presented here is carried out, has as its aim to ease communication – not least, communication between teacher and learner – through revealing experiences or conceptions of different phenomena.

The phenomenographic approach

Phenomenography considers knowledge to be relations between man and world created through experience. World here means physical and social world. Learning is seen as 'change of conception' towards a more functional knowledge. Development concerns development of conceptions, not developmental levels of children in the Piagetian sense.

Phenomenographic investigations are often – like the study presented here – cross-sectional studies carried out through 'clinical' deep interviews, which are tape-recorded and transcribed. The analysis of the transcriptions is made with the intention to reveal different experiences of the phenomena concerned, and the description of those experiences, or conceptions, are the result of the investigation.

Design of the study

In the study presented here nine pupils in grade 2, ten in grade 3, eleven in grade 4 and eleven in grade 6 from six classes in a lower and middle school in the south of Sweden, were interviewed. One pupil in each grade was a 'high achiever', one a 'low achiever' and the others were 'average'

according to their teachers. The pupils were given confronting problems aimed at revealing their conceptions and possible misconceptions of 1/2, 1/3 and 1/4. The 2nd and 3rd graders had got some instuuction of how to read and write 1/2, 1/4 and 1/3, but had not been introduced to fractions in other ways, while the 4th and 6th graders had been taught fractions.

The following problems were given (orally):

1. If you were to give me one half (one fourth, one third) of something, what would you then give me? Draw (the thing/s/ you would give me) and encircle the half (the fourth, the third) of it.
2. Give me a half (a fourth, a third) of this string.
3. Give me a half (a fourth, a third) of these blocks (12 cubes: 3 rows with 4 cubes in each).
4. Fill a half (a fourth, a third) of this glass with water.
5. Here is a picture of a jogging-track (fig 1). You are supposed to start at 'start' and run all the way round, finally coming back to 'start' again. When you have run half way you will meet me. Put a mark on the jogging-track at the point where we will meet. (The same task but with the request to put a mark at 1/3 and 1/4 of the track).

Start

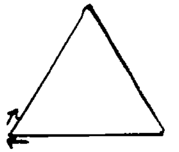


Fig 1

6. Mr Johnson had bought six bushes. He had three garden plots and he wanted to have 1/3 of the bushes on each plot. Here are the three plots (three rectangles on the pupil's protocol sheet). How did he plant his bushes?
7. Mark all figures where 1/3 is shaded (fig 2)

Fig 2

Results

Six experiences or conceptions of 1/2, 1/4 and 1/3 were expressed:

'fair shares' *'parts'* *'form'* *'size'* *'one of a number of parts'* *'ratio'*

Three conceptions were expressed of how these fractions can be concretely formed.

through 'measuring' *through 'halving'* *through 'estimating and adjusting'*

The younger children often used a 'finger span' or just their eyes for measuring, while the older ones also could use part of their pencil or rubber as a measure. Some pupils also measured in this way to get 1/2, 1/4 or 1/3 of the string when they solved problem 2. Yet most children measured the string – or the 1 cm broad band by which this was sometimes replaced – by folding it. Mostly they then used 'halving', but some older pupils 'estimated and adjusted' when they cut 'thirds'.

Examples of the conceptions

The numbers of answers of different kinds are reported for 1/4 in table 1 and for 1/3 in table 2 at the end of this presentation. Only a few children expressed 'fair shares', 'parts', and 'form'. Only one of these three conceptions, the conception 'form', has been reported under a heading of its own in the tables. Otherwise these conceptions are reported under the heading: 'Single categories'.

Examples of how fractions were concretely formed will be given in the contexts where the six conceptions of fractions are described below.

Fair shares

A few younger children, and even one 6th grader, did not use up the whole when dividing the string. They compared the parts after having cut the string, found that they were not of the same size, cut off the surplus and threw it away. For them parts meant 'fair shares', and the whole was of no interest.

Parts

In the drawings it was difficult to get exact halves, fourths and thirds. Yet, when asked, most pupils said that the halves had to be equal. The equality of fourths was less important and the equality of thirds even less. As many as six of the eleven 6th graders expressed the idea that the three parts in figures 2a and/or 2c were thirds (the answers to problem 7 are not accounted for in tables 2 and 3, which explains the absence of a specific column for 'parts'). One 2nd, one 3rd, one 4th and one 6th grader also illustrated in their pictures for problem 1 that they disregarded the equality of thirds (fig 3 a, b, c). Yet, the 6th grader made three drawings because she wanted her thirds to be more equal.

Fig 3a (grade 3)

Fig 3b (grade 4)

Fig 3c (grade 6)

Form

Pia and three further children in grade 2 thought of 'one third' as 'form'. Their 'third' was a triangle. This is often called 'trekant' by Swedish children, The part 'tre' – the Swedish word for 'three' – is also the first part of 'tredjedel' (a third), and the small children seem to mix these two words. Thus, when asked to give 1/3 of something to the interviewer, Pia said:

P: Then I'd give you a sandwich ... which is like this ... three sides .. ... (fig 4)
There're sandwiches with three sides ... ... Dad likes them (writes 1/3 and explains what 3 is)
It means this ... If there's a little triangle (circles a '
little triangle') ... there ... I can break it up ... There ...
(draws the 'little triangle') you've got a little ... If it isn't enough ...

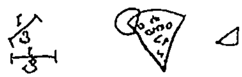


Fig 4

What is in Pia's mind is not the 'thirds' the interviewer thinks of. It is 'trekantiga' (three sided) sandwiches and 'sharing'. The last part 'del' in the word 'tredjedel' is close to the Swedish word 'dela' (share). When asked to give the interviewer 'one third' of the band Pia cuts it like fig 5, saying:

P: Now, I've got a little third (tredjedel)!
I: A little piece with three sides (tre kanter)?
P: It's called a third (tredjedel)

Fig 5

Åsa, another second grader, drew this 'third' of a piece of sugar as a gift to the interviewer (fig 6):

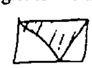


Fig 6

Size

The 'size' conception could be expressed in two different ways:

1. There must be four in 'one fourth' and three in 'one third'

2. One fourth is bigger than one third

These ideas are also observed in earlier research (see for instance Hunting, 1989). They could appear simultaneously and both seemed to be related to the children's experiences of whole numbers. Further, the denominators in 1/2, 1/3 and 1/4, which were known by many of the children might also have contributed to these conceptions, even if the problems were not given in written form.

There must be four in 'one fourth' and 'three in 'one third'

Even if the first way in which pupils expressed the 'size' conception was related to all answers it was most often observed in problem 3 with the 12 cubes. When the children should take 1/4 of these, many of them took four cubes instead of three, explaining that this was 1/4 'cause it's four'. The same sort of explanation was given when three cubes were taken for 1/3.

In problem 2, this conception could be expressed by giving all the four pieces in which the string was divided to the interviewer, in order to give 'one fourth', and all three for 'one third'. Some children gave only 1/4 of the string to her, but did divide up this 'fourth' first into four small pieces.

Matilda, grade 4, expressed the idea of 'four in a fourth' when she solved problem 5, the 'jogging-track' problem, putting a mark at the first 'corner' of the track:

M: It has to be up at the top .. Then you can start going round again

I: How do you know there's a fourth there, then?

M: First, the third, that's a two, three (points to the first, the second and the third side of the triangle) It was there (the starting point of the 'track', where she had put a mark to show where she has run 'one third' of the track). If you add that bit on you get four

I: So then you can start again?

M. Yes

(The third mark is for half the track)

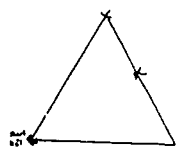


Fig 7

The same idea was also expressed by other pupils when the 'track-problem' concerned 1/2. Two children, Thomas and Pia grade 2, illustrate that 'halves' partly mean just 'pieces', partly also 'something with two pieces in it'. Both children put the mark for 'one half' at the second corner of the track when they solved problem 5. After that, the following dialogue took place between Thomas and the interviewer:

I: Ahah, how do you know you've run half way, when you get there, then?

T: Well, there's one (points to the first side of the triangle), and two more there (points to the second and third side of the triangle) ... and if you want another half, you can put a mark here (points to the corner of the track, and marks it, Fig 8), and another half (points again to the triangle's third side).

I: How many halves have you got now, then?

T: Three halves: one two three (points to the three sides of the triangle)

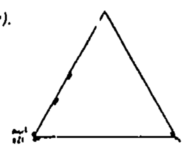


Fig 8

The reason why the mark by both the children is put at the second and not at the first corner, might be that - even if 'one half' could be either of the 'three halves' – there should be two pieces rather than one in 1/2, since the denominator is 2.

Frederick, grade 6, expressed several times the idea 'three in the third'. In the 'string-problem' he folded the string into four parts, cut it and gave three of the parts to the interviewer in order to give her 1/3. His drawing of 1/3 in problem 1 (fig 9), where he encircled 3/4 of the rectangle he had written and gave this part as 1/3 to the interviewer also expresses the idea of 'three in a third' (says: 75%).

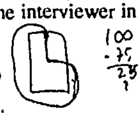


Fig 9

When the bushes were planted this same idea of 'three in the third' was expressed by many children. Even two 6-graders planted 3 bushes on 2 plots, letting the third plot stay empty.

2. One fourth is bigger than one third

Exactly as the first idea related to the 'size' conception, the second one was also expressed in all problems, yet, most often in problem 5, the problem with the three glasses. Margaret, grade 6, for instance, measured the first glass with help of her finger-span, to find its middle, before she half filled it with water. After that she put the second glass beside the first one measuring its lower half and filling it to a fourth', meaning 'half of the half', as 'fourths' were called by the 2nd graders. She even poured out some water in order to have exactly half as much in the second glass as in the first one. Yet, when she had later, just as carefully, measured the third glass using her finger-span three times and poured water into it up to the first mark, she looked at the glasses in astonishment, saying:

M: No, it should go down in steps – one, two, three! This one (the third glass) is a fourth (she swaps the last two glasses, so they are ranked according to her idea of 'one', 'the glass with the most water in it, 'two', the 'middle' one, and 'three', the glass with the least water in it) (fig 10a).

Sometimes the ranking was extended to include even 1/2. The 6th grader Frederick, for instance, expressed an idea which was somewhat different from Margaret's when he solved the same problem. After he had filled the first glass half full, he filled the second one to the brim, saying:

F: A fourth is a whole!

To have the third glass filled to 'one third' he poured a little more water in it than there was in the glass filled half full. When asked about how he knew that this was 1/3, he answered:

F: It's like steps (moves one glass a little, as in Fig 10b). This one's full, this one a bit less ... ...

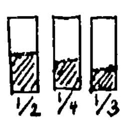


Fig 10a

Fig 10b

This idea simultaneously expresses the conception that there should be three in 1/3 and four in 1/4 and even two in a half: two fourths in 1/2, three fourths in 1/3 and four fourths in 1/4 (for Frederick 100% or 'the whole'). Frederick here expresses the same idea as the one he expressed when he gave the interviewer 3/4 (talking of 75%) of the rectangle and 3/4 of the band, when asked to give her 1/3.

The idea that 1/4 is larger than 1/3 is of course implicit in the idea that there should be three in 'one third' and four in 'one fourth'. Both ideas express the 'size' conception.

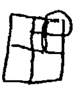
Mostly 1/2 and 1/4 were created by 'halving', through folding the string once or twice before cutting it in problem 2, the 'string-problem'. Yet, a lot of pupils expressed the conception that dividing a continuous quantity always has to be done through halving. These pupils struggled hard to get 1/3 of the string through folding it in the middle again and again, before they finally gave up. Anna, a 3rd grader, who folded the string many times 'in the middle', in vain, finally decided to fold it into four parts. She then cut it and threw away 1/4 giving the interviewer one of the three remaining parts, complaining:

A: There's no other way! I can't make it three

The 6th grader Margaret, folded the string into 'lots of parts', and gave one of the very small pieces to the interviewer. Asked about why she thought this piece was 1/3 of the string, she answered:

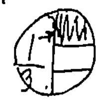
M: I folded it so many times that it became a third

Many young pupils 'halved' the string repeatedly to get thirds, in the way Margaret did, and ended up with very small parts. This way to form thirds might support the idea that 1/3 is less than 1/4. The idea of halving as the only way to divide, even for getting thirds, appeared also in drawings (fig 11).



Grade 3 Fig 11a

Grade 3 Fig 11b



Grade 4 Fig 11c

Yet, most of the older pupils divided the band through measuring, not through folding. And if they folded they did not think as Margaret and the younger children: 'If you fold enough times, then you finally must get 1/3!' They manipulated the string in different ways, began for instance as in fig 12a and moved the string until they had formed it as in fig 12b. They estimated and adjusted. Yet, that the string was finally formed as in fig 12b often seemed to happen by chance.

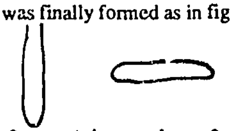


Fig 12a

Fig12b

One of a certain number of parts

A conception often expressed by the older pupils was that 2/6 cannot be 1/3. They had problems with the six bushes which should be planted on the three plots, 1/3 on each plot. Three of the eleven 6th graders put one bush on each plot, explaining that it could not be two, because then it would be 2/6 on each plot, not 1/3. Asked if 2/6 could not be 1/3 they gave very convinced answers of the type: 'No, thirds can't be sixths!' In contrast, all but one of the 4th graders correctly put two bushes on each plot.

The idea that 2/6 cannot be 1/3 was also revealed in problem 7, figs 2c and d. While 6 of the 11 pupils in grade 6 thought that in these figures 2/6, and thus not 1/3, were shaded – one of them later changing her mind to 2/3 – 8 of the 10 pupils in grade 3 found it self evident that 1/3 was shaded, even explaining: 'Two, two, two ... three twos ... one of them is shaded'

Thus the 'ratio' experience seems to exist at a very early stage, but then context-bound and in parallel with other conceptions, for instance the 'size' conception. What seems to happen is that when teachers want to help children understand that the denominator means neither the number of objects within each part, nor the size of the part, they stress that it means the *number* of parts. If this idea is not related to the idea formed earlier that '1/3 can be one of three twos', it impedes further development of the embryo of a 'ratio' conception of fractions, that is intuitively expressed by younger children.

Ratio

If problem 6 and the figures c and d in problem 7 are not correctly solved, this illustrates that the conception of fraction as 'ratio' is *not* developed. Yet, the fact that these problems are correctly solved cannot alone illustrate that the 'ratio' conception *is* developed. The young children who solved these problems correctly only illustrated a context-bound 'ratio conception' existing in parallel with the non-functional 'size' conception. A 'ratio' conception which is not context-bound presupposes that the denominator is understood neither as denoting the number of objects in each part, nor as denoting the size of the fraction, nor as denoting the number of parts. It must be understood in relation to the numerator. A 'ratio' conception that is not bound by context is only expressed if all seven problems are solved correctly. None of the pupils expressed this kind of conception of 'ratio'.

Concluding remarks

In a study earlier presented (Neuman, 1991) concerning children's conceptions of division it was illustrated that most second graders could solve partitive and quotitive division problems in the same way through proportional thinking. They thought of the divisor as expressing the objects shared out: in partitive problems one object to each part in each round, and in quotitive problems all objects to one part in each round. Yet, this was only the first step in a longer or shorter 'ratio table' which they used in order to solve the division problems, for instance by saying: 'Seven marbles', one to each of the seven children, 'fourteen marbles, two to each child', 'twenty-eight, four to each child'... ... .

This proportional idea also has to be related to the understanding of fractions: 2/6, 3/9, 4/12 ... must be understood as other ways of expressing 1/3. In order to be able to bring about this understanding teachers have to become aware of how children understand, and misunderstand, fractions. Some knowledge of this kind has hopefully been revealed through the study presented here. Its results underline what Streefland (1991) has strongly emphasized in his book on 'Fractions in realistic teaching': fractions should not be taught in isolation from division and proportionality.

The teaching of fractions would probably more often end up in learning of fractions if all childish misunderstandings related to early experiences were brought to light through confrontations, and if all early powerful and functional understanding was taken care of and further elaborated through communication in the classroom.

Table 1: Number of answers (number of pup'ls within brackets) in each grade, who solved the 5 problems concerning 1/4 in different ways.

	SIZE							
	Larger than 1/3	Larger than 1/2 and than 1/3	Four in the part	Correct	Not given or not categorized	Single categories	Don't know	N
Gr2 Answers	3(1)	1(1)	9(6)	25(9)	5(5)	2(1)*	0(0)	45(9)
Gr 3 Answers	8(5)	0(0)	9(7)	22(7)	10(2)	2(1)**	0(0)	50(10)
Gr4 Answers	10(6)	1(1)	13(5)	21(8)	5(3)	5(2)**	0(0)	55(11)
Gr6 Answers	2(2)	1(1)	6(4)	44(10)	0(0)	0(0)	2(2)***	55(11)
Total Answers	23(14)	3(3)	37(22)	112(34)	20(10)	8(4)	2(2)	205(41)

*) 'Fair shares' **) 'Parts' (4 answers: '1/4 is a rather small part') ***) 1/4 of the jogging track.

Table 2: Number of answers (number of pupils within brackets) in each grade, who solved the 6 problems concerning 1/3 in different ways.

	SIZE		FORM	ONE OF A NUMBER OF PARTS				
	Less than 1/4	Three in the part	Triangle	2/6 is not 1/3	Correct	Not given or not categorized	Single categories	N
Gr2 Answer	17(7)	13(7)	4(3)	0(0)	13(4)	6(3)	1(1)*	54(9)
Gr3 Answer	14(7)	7(5)	2(1)	1(1)	24(9)	10(2)	1(1)**	59(10)
Gr4 Answer	16(9)	14(7)	0(0)	1(1)	28(10	3(3)	4(1)***	66(11)
Gr6 Answer	3(2)	10(5)	0(0)	3(3)	47(10)	0(0)	3(3)****	66(11)
Total Answer	50(25)	44(24)	6(4)	5(5)	112(33)	19(8)	9(6)	245(41)

*)Very small part **) 1/3 of 3/4 ***) 1/3 is a rather large part ****) 'Parts'

Acknowledgement

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CONCEPTUALIZING RATE: FOUR TEACHERS' STRUGGLE

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The concept of rate is important to the development of many areas of mathematics. Yet teachers have little experience with rate beyond using the distance-rate-time formula. Using speed as an instance of rate, Thompson has shown that the development of the concept of rate contains stages where the individual thinks of speed as distance, then speed as a ratio of total distance to total time, and finally as proportional accumulations of both distance and time, that is, as an abstracted ratio. In this study, four middle school teachers worked through a series of problems dealing with rate over a period of several weeks. The point of the problems was to lead the teachers to understand rate conceptually rather than as a mathematical formalism, and how children develop this concept. The difficulties encountered by both the teachers and researchers are chronicled.

Mathematics educators agree that teachers' knowledge is one of the most important influences on what transpires in their classrooms and consequently on what their students learn (Ball & McDiarmid, 1990; Fennema & Franke, 1992). Although many consider content knowledge to be important, there is little known about how teachers' content knowledge impacts on teachers' instructional decisions. Even in areas that have been richly researched, such as rational numbers (Behr, Harel, Post, & Lesh, 1992), there is still a need to "build a comprehensive picture of teachers' understanding of rational number topics and then relate this understanding to teachers' decision making and classroom instruction in a systematic way" (Brown, in press, p. 7).

This paper will describe four middle school teachers' understanding and reconceptualizations of the concept of rate and the instructional implications for the teachers' conceptualizations. The four teachers described here had been selected to participate in a long-term research effort to investigate the role of teacher knowledge of middle school number and quantity concepts and reasoning on instructional practices and on student learning. (See Sowder, 1991, for a detailed framework guiding this research). These teachers were all recognized within the local mathematics education community as exemplary teachers and leaders and were well-known by our research group. The general conceptions and practices of these four teachers are addressed elsewhere (Philipp, Flores, Sowder, & Schappelle, 1992).

Background: The Concept of Rate

We chose to investigate teachers' conceptions of rate because of the role of this concept in the development of many areas of mathematics (Lesh, Post, & Behr, 1988; Thompson, 1992). We agree with these scholars: The concept of rate is too important to leave until the study of algebra, especially because by that point, students are usually taught a procedure by which to solve problems of rate instead of a way of thinking about the important concepts.

Our conceptualization of the concept of rate is based on Thompson's theoretical framework (1992; in press-a). Thompson's distinction between rate and ratio, unlike other distinctions described in the literature (Lesh, Post, & Behr, 1988; Vergnaud, 1983), is grounded in the mental operations by

which people constitute situations, and not upon the situations per se. That is, one cannot know whether the relationship is a ratio or a rate for an individual until one knows how that individual conceives the relationship.

For Thompson, a ratio is a multiplicative comparison of two quantities, whether the quantities are of like nature (10 years old compared to 6 years old) or unlike nature (10 miles traveled in 2 hours). These quantities may involve a comparison of the two collections as they are, or they may involve a comparison of one as measured by the other. For example, the quantities 3 apples and 2 oranges may be compared as they are (3 apples to 2 oranges), or as 1 1/2 apples to every 1 orange (one measured by the other). As long as the multiplicative comparison is between two specific, non-varying quantities, the comparison is a ratio. A rate is a reflectively abstracted constant ratio. That is, when one conceives two independent, static states, such as 3 apples to 4 pears, one has made a ratio. When one reconceives that situation as being that the ratio applies outside of the originally conceived phenomenal bounds, then one has generalized that ratio to a rate (3 apples to 4 pears might become 3/4 apples to one pear).

Thompson considers speed is an instance of rate, and as such, they develop similarly. Children's first image of speed is as a distance, with the time unit being omitted and only conceived implicitly. For example, if asked to determine how long it would take someone traveling 30 miles per hour to travel 100 miles, the child would conceive of lengths of 30 miles (in one hour) covering the 100 mile distance. For the child, each iteration of the distance-speed implies a time-unit, and he may be able to see that it would require 3 1/3 time units to cover the distance. If a more difficult problem is asked involving finding the speed one must travel to cover the distance in a given time, the child, being unable to see a multiplicative relationship between speed, time, and distance, will have to resort to a guess-and-check strategy of finding a speed-length that will cover the distance in the given number of units of time. This is exactly what occurred in Thompson's research during an interview a teacher held with one of his 6th-grade students. When asked to determine the speed at which a turtle might run 200 ft in 8 seconds, the student was able to draw upon her previous work with speed-lengths to determine that 8 speed-lengths of 25 feet (in one second) would work. However, when she asked to determine the speed at which a turtle might run 200 feet in 7 seconds, she was unable to find a solution because she could not conceive of speed-lengths of a size such that exactly 7 would fit into 200 feet.

In between the stages of conceptualizing speed as a distance, which was just described, and speed as a rate students perceive of speed as a ratio of total distance to total time. It is only in the final stage that the students conceptualize total accumulations of both distance and time growing simultaneously with accruals of each.

Data Collection and Procedures

A series of seminars was held with the four teachers who were subjects of this study. During the seminars, we provided teachers with opportunities to discuss their thinking about carefully selected tasks designed to encourage a reconceptualization of mathematical concepts about which they already

possessed some understanding. This approach was based on Piaget's notion of generalized assimilation, explained nicely in a recent paper by Thompson (in press-a):

> We recognize situations by the fact we have assimilated them to a scheme. When features of that situation emerge in our understanding that do not fit what we would normally predict, we introduce a distinction, and the original scheme is accommodated by differentiating between conditions and subsequent implications of assimilation (p. 4)

One roadblock to introducing distinction when working with teachers is that they possess formal, symbolic skills that enable them to successfully solve problems without relying upon the underlying conceptual foundations. Yet teachers must explore and understand the role of these conceptual foundations if they wish for their students to acquire deep understanding. For example, although most middle school mathematics teachers understand how to combine ratios and how to combine fractions, they may not have thought deeply about how it is that ratios and fractions differ in terms of the role of the unit. Therefore, in order to orient the teachers towards this important idea, they must be asked a question or exposed to a situation which can not be solved or reasoned within their currently held rules or conceptualizations. For example, the following questions instigated a long discussion because of the difficulties it presented to the teachers:

> A student says that 3/8 + 5/12 is 8/20 and justifies her reasoning as follows: "If I made 3 out of 8 free-throws in the morning, and 5 out of 12 free-throws in the afternoon, then altogether I made 8 out of 20 free throws." How would you respond to that student?

The teachers could not rely upon a formal solution to this problem, because this problem does not fit into any stereotypical mold. It is only in the process of reexamining their understanding of the relationship between ratio and fractions that teachers could make sense of this problem.

Three of the seminars included discussions dealing with rate. The first session involved using the computer microworld OVER & BACK (Thompson, in press-b), involving a turtle and a rabbit who run over and then back along a number line of length 100 feet (Figure 1). Both animals can be assigned speeds at which to run, with the turtle having the capability of running at one speed over and a different speed back, whereas the rabbit must run both directions the same speed. The animals may either be run together, as in a race, or separately, and as they run a timer shows elapsed time. The race may be interrupted by pressing pause, at which time the distances traveled by both animals will be displayed on the screen.

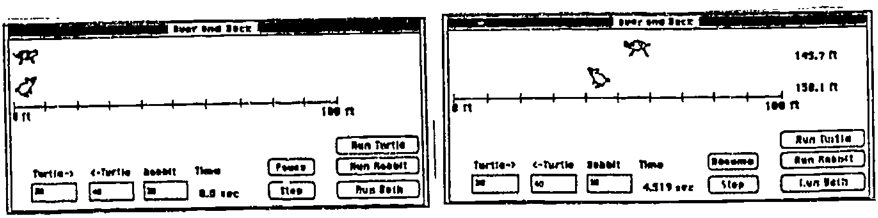

Figure 1.

Between the first and the second seminars the teachers completed a written test of matnematical content knowledge. Two of the items on this test deait with rate. The tests were returned to the teachers, unmarked and unscored, and discussed in detail with the teachers during the second seminar. During the third seminar two rate problems were presented to the teachers and discussed in detail. Finally, the teachers completed a "homework" assignment several weeks after the discussions on rate. This assignment included two rate problems together with several other questions.

Results

This section will combine the description of our work with the teachers and the results.

In their work with the microworld, the teachers were instructed to think about each of the following questions before running the animals. The questions were:

1) Predict the time over and back (100 feet for each direction) for the rabbit if the speed is:
   a) 50 ft/sec b) 40 ft/sec c) 30 ft/sec
2) What speed would you have to set in order to go over and back in
   a) 8 seconds b) 2 seconds c) 6 seconds d) 7.5 seconds

After working with these questions using the microworld, the teachers discussed what they found. The first question seemed to be easy for all four teachers. They quickly solved each of the three parts to question 1 and explained their reasoning as either seeing speed-lengths, or by dividing 200 by each number. The discussion of the second question was a little more interesting. The first teacher reported using information from the first question when answering the second question. For example, she realized that since 50 ft/sec took 4 seconds, it would require half that speed to take twice as long (8 seconds) and twice the speed to take half as long (2 seconds). Then, she knew that it would require 1/3 the speed of the 2 second trip to make the 6 second trip. She used a calculator to divide 100 by 7.5 on the last question. A second teacher solved 2a, b, and c by thinking of speed as a distance. She explained this as picturing chunks. On part d, she used a calculator, but reported that she was not sure whether to multiply or divide. A third teacher solved question 2 by dividing 200 by each number, using a calculator for part d. He then looked at the numbers to see if they made sense, but he too was not "$20 sure." The fourth teacher solved the problems correctly and said that she was one million dollars sure of her work.

When asked what they might do to help students understand these problems, the teachers responded in a variety of ways. The first teacher suggested relating each of the parts in question 2 to question 1. She explained this for parts a, b, and c, but did not attempt to explain how the same reasoning might be applied to part d. The second teacher reported that she was confident she understood until she solved problem 2d, at which time she relied upon a formula. The third teacher expressed concern with "instantaneous start-up, etc...", versus average speed. He suggested that one way to solve these kind of problems would be to find a pattern. One of the researchers directed the teachers' attention toward the conceptualization held by students who perceive speed as a distance. The fourth teacher responded by stating that she saw chunks as rate, involving distance and time, instead of just distance. This response indicated that she understood how her view differed from the speed-as-distance view. The other three teachers did not appear to comprehend the significance of her statement.

At this point the teachers viewed Thompson's videotape described earlier in this paper of a teacher interviewing a sixth- grade student whose conception was of speed as a length. After successfully solving the question requiring her to determine the speed at which the rabbit must run over and back in order to complete the 200-foot trip in 8 seconds, the student in the videotape was unable to determine the speed at which the rabbit might complete the same trip in 7 seconds. The teacher in the videotape, who did not understand the student's conception of rate, was at a loss as to how to assist the student. We asked the four teachers to consider the following questions while watching the videotape:

1) What does the student understand and what does she not understand?
2) Why was the subject so confusing to her?
3) What was the teacher doing that contributed to the student's confusion.

After viewing this videotape, we immediately watched another videotape of Thompson himself working with the same student. Thompson, realizing that the student possessed a conception of speed as a distance, provided the student with a sequence of tasks designed to orient her toward developing the initial understanding of speed as a ratio. He accomplished this by first drawing a distance segment labeled 523 feet and then asking the student to answer questions directed toward understanding how the distance might be partitioned into a different number of segments. After the student seemed to understand that, for example, dividing the length up into 5 sections would result in each section being the same length (523 ÷ 5), he introduced another segment labeled time. After asking a series of questions, the student was able to see that partitioning the time segment into 5 sections must be accompanied by partitioning the distance segment into 5 sections, whereby each section of distance 523÷5 would be run in 1/5 seconds. The student was then asked to solve a problem similar to that which had caused her so much trouble the previous day, except for this new problem she was asked to determine the speed at which the rabbit must run in order to go over 100 feet in 7 seconds. She solved this by drawing two segments, one labeled distance and the other labeled time, and dividing both segments into 7 equal sections. It was then very clear to her that the speed at which the rabbit must run would be 100/7 feet (in one second). Notice that although the student had developed the initial concept of speed as a rate, distance was still predominant. At this point we ended the videotape and discussed what we had seen.

The teachers had difficulties understanding the student's conception of the problem. The researchers tried explaining to the teachers how the student's conceptualization of speed as a distance evolved into an initial conceptualization of speed as a rate, but the teachers did not seem to follow the explanation. The comments of two of the teachers indicated that they themselves possessed the speed-as-distance conceptualization, and therefore could not understand how more advanced views of rate could be built upon this view. As a result, the significance of what Thompson was able to accomplish with this student seemed lost on these two teachers.

After this group discussion the teachers took the content test which included the following two rate problems:

19) A biker rides at a speed of 20 km/hr for half and hour and then jogs at a speed of 15 km/hr for half an hour. What is her average speed?
20) A biker rides at a speed of 20 km/hr for a few km, and then turns around and walks home (same route) at a speed of 5 km/hr. What is his average speed?

The following seminar was devoted to discussing the teachers' responses on the test. A substantial amount of time was spent discussing the two rate questions. Teacher 1 had not responded to either of the two questions and teacher 2, 3, and 4 correctly answered #19. None of the three teachers who responded to question #20 answered it correctly. Two responded 12.5, and the third responded that she could not say.

Three of the four teachers did not understand how time played a role in problem #20. Teacher 2 began by showing how she had divided the sum of 20 and 5 by 2, resulting in 12 1/2. Teacher 3 commented that he'd done the same, adding, "We don't know the number of kilometers so only the speed is important, not the distance." It was pointed out that in problem #19, the runner spent the same amount of time running the two rates, whereas in the problem #20, the runner spent four times as long running at one rate than the other. The teachers then began to conceptualize the problem as total distance divided by total time, resulting in the correct answer 8. However, they were still not seeing rate as the proportional accumulation of distance with respect to time. This session ended with the teachers still confused about problem #20.

During our next meeting, at which only three of the four teachers were present (teacher 1 was absent), we discussed two rate problems involving different contexts. The problems follow:

1) Tom brought his class to a movie. He bought 30 chocolate bars at 80 cents a bar and 30 taffy bars at 50 cents a bar. What was the average cost per candy bar?
2) Sue brought some friends to a movie. She spent $48.00 on food, half of it on hot dogs and the other half on popcorn. If the hot dogs cost $4.00 each and the popcorn cost $2.00 each, what was the average cost per item of food?

There was immediate concern about problem #2. Why would anyone ever want to determine the average cost of different items. The discussion revolved around this issue until one of the faculty proposed the following "real life" problem:

> Suppose some parents and children go to a movie, but not all of the parents of all of the children attend. Altogether $48 is spent on tickets, with half being spent on adult tickets costing $4 apiece and half being spent on children's tickets costing $2 apiece. It is decided that everyone will pay the same amount for each ticket. How much should parents not attending send with each child to pay for his or her ticket?

Even after a lengthy discussion of this problem, two of the three teachers had difficulty seeing how this problem was similar to problem #20 from the content test.

The last exposure to rate problems with the teachers occurred during an "assignment" (we use this term loosely, because our sessions with the teachers were informal and they never really had any homework) given to the teachers toward the end of the semester during which time we asked the teachers to reflect on what they had learned. Although this assignment primarily focused on the impact they felt the sessions had had on them, we included the following two rate problems:

1) A store owner mixes 12 kg of peanuts worth \$4.00 / kg with 12 kg of cashews worth \$9.00 / kg. At what price shall she sell the mixture?
2) A store owner mixes 12 kg of peanuts worth \$4.00 / kg with 8 kg of cashews worth \$9.00 / kg. At what price shall she sell the mixture?

All four of the teachers correctly solved both problems. Teacher 1 found the arithmetic mean on the first and a weighted average on the second (12/20 x \$4 + 8/20 x \$9). Teacher 2 explained that she went back and tried to find an analogous problem we had previously solved. Although she answered both correctly, she reported that these problems led to feelings of frustration and intimidation, and she was not really sure what she had done. Teacher 3 reported similar doubts about his work. He solved both the these problems in the same manner: by determining the total cost and dividing that value by the total weight. Teacher 4 expressed confidence solving the two problems. She solved the first by calculating the arithmetic mean of 4 and 9, and she solved the second by thinking of them as weighted averages. She explained, "In my thought processes I initially visualized the two quantities and knew that my 'answer' would be closer to the \$4 amount and thought about what was reasonable before computing [(12x4 + 8x9) / 20]. I am confident in my result."

Discussion

Only one of the four teachers in this study possessed a deep conceptualization of speed, with two of the other three teachers conceiving of speed as a distance and relying upon formal symbolic manipulations to solve problems. During the discussion of the content assessment, which occurred after the teachers had already worked with and discussed the microworld, at least two (and perhaps three) of the teachers did not clearly understand the constant ratio accumulation of distance with respect to time. Furthermore, although they realized the problem could be solved using weighted averages, they did not focus on the fact that it is the time during which distant accrued, and not the relative rates, that must be weighted.

It might be asked why do we need to bother helping the teachers reconceptualize the concept of rate? Thompson provides one answer to this question:

> To tell students that speed is "distance divided by time" with the expectation that they comprehend this locution as having something to do with motion, assumes two things: (1) they already have conceived of motion as involving two distinct quantities - distance and time, and (2) that they will not take us at our word, but instead will understand our utterance as meaning that we move a given distance in a given amount of time and that any segment of the total distance will require a proportional segment of the total time. In short, to assume students will have any understanding of "distance divided by time" we must assume that they already possess a mature conception of speed as quantified motion. This places us in an odd position of teaching to students something which we must assume they already fully understand if they are to make sense of our instruction. (Thompson, in press, p. 37)

Why didn't these teachers reconceptualize the complex concept of rate? We will focus on two related reasons. First, we believe that the teachers did not see a need. Even those teachers who were aware that they possessed less developed notions of rate were eventually able to solve the problems that were posed. But this leads to the second reason that the teachers did not reconceptualize the concept of rate: the researchers had not completely reconceptualized the concept themselves. When reviewing the

transcripts of our group sessions, it became clear that although the researchers understood the students' misunderstanding, our attempts to orient the teachers toward the students' thinking fell short for two reasons: First, we did not understand our teachers' conceptualizations, and second, even as we began to see how they were thinking, our own conceptualization of the development of the concept of rate was insufficient for devising meaningful tasks through which we might have helped reorient the teachers' thinking. Although we had a sense during the study that something interesting was occurring, it was only when we sat down to think through this paper that the story became clear. We hope that this new understanding will enable us to be more successful with the next group of teachers.

This study provides a microcosm for what we believe must take place in order for a deep reform of the teaching and learning of mathematics to occur. In order for our teachers to learn to listen to their students, they must develop a deep understanding of both the mathematical concepts as well as how these concepts relate to the students' emerging conceptions. But teacher educators can not facilitate this process until they have done likewise, and more - that is, teacher educators must understand the content, the students perspectives, the teachers' perspectives, and the relationship between the three.

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NEW FORMS OF ASSESSMENT BUT DON'T FORGET THE PROBLEMS

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Without good problems it is impossible to improve assessment. This paper addresses the development of better forms of assessment from the view of the problems. It focuses on the assessment of some key concepts and abilities on percentages. A series of problems has been developed for assessing these. The problems, which are rather different from the traditional kind, have been tried out in two American grade 7 classes. A report is given on the results collected from these problems and some recommendations are given for their improvement.

1. Introduction

Assessment is in fashion[1]. For almost one hundred years now assessment has been a matter of great concern to anyone involved in education. It was the ability testing of Thorndike that brought assessment into schools at the turn of this century (see Du Bois, 1970). Since then various waves of different trends in assessment have swept past, and are still appearing (see, among others, Kilpatrick, 1992; and Ernest, 1991). Again recently there is strong increasing interest for assessment. Voices are being raised everywhere saying that assessment must change. According to Ernest (1989) the introduction of new forms of assessment is one of the key areas of innovation for the 1990s. The general reason for this interest in assessment is two-fold. The first being that new ideas on the teaching of mathematics have emerged that require new forms of assessment. It has often been argued that innovation of education cannot succeed without new forms of assessment (NCTM, 1989; Romberg, Zarina and Williams, 1989). The second reason for the recent interest in assessment stems from the need of policy makers to have a wide scale overview of the output of education. The NAEP in the USA is an example of this. Assessments like these, intended for measuring the quality of education, are considered as strongly influential. Again the danger lurks that poor test items will result in the attenuation of education.

Characteristic for recent attempts to change assessment is that most of the attention is devoted to both the formats of assessment and the organizational aspects of assessment. As alternatives for class administered written tests consisting of multiple choice questions or short answer items one pleads for a variety of assessment tools and methods such as portfolio, performance assessment, project assessment, group assess[illegible] classroom observations, etcetra (see NCTM, 1989). Another hot issue is the shift from st[illegible]ized tests to forms of assessment for which the teachers themselves are responsible (see Graue, in press). Less attention is however paid to the problems used for assessment. And these are precisely the most crucial. Without good problems one can forget any effort to achieve better forms of assessment. It was Freudenthal who stressed time and again that the problems we offer the students should always be meaningful. Later this was elaborated more explicitly for assessment (De Lange, 1987 and 1992; Van den Heuvel-Panhuizen, 1990; Van den Heuvel-Panhuizen and Gravemeijer, 1991). The members of the Freudenthal Institute (formerly the IOWO and OW&OC) are not the only ones who paid attention to the type of problems. Bell, Burkhardt, and Swan (1992) and Sullivan and Clarke (1987) did the same, although the latter's plea for good questions does not focus specifically on assessment but rather on teaching in general.

This paper means to address the development of better kinds of assessment from the view of short task problems and their presentation. In consequence it will not deal with assessment of mathematics in general, but with the assessment of a specific topic. The specific topic at hand is percentages. A short overview is first given of ways to improve the quality of problems designated for assessment. This is followed by examples of problems dealing with percentages to serve as illustration. Besides some background information about the development of the problems there is

also a report about the information collected from these problems about students' understanding and abilities. Also some recommendations are given for improvement of the problems at issue.

2. Improvement of assessment by improvement of the problems

Both the format of the assessment and the organization of the assessment are important aspects to take into account when assessing. The goal of collaboration in mathematical problem solving, for instance, cannot be assessed by means of an individual written test. Notwithstanding, the format and organization as such do not primarily determine the quality of the assessment. What determines it most is the type of problem that is used. In other words, it is not the short task written test format as such that makes this method of assessment unsuitable for a new approach to teaching mathematics, for instance one like realistic mathematics education. In Van den Heuvel-Panhuizen (1990) and Van den Heuvel- Panhuizen and Gravemeijer (1991) an overview is given, complete with examples, of what qualities short task written tests and the corresponding test items should include in order to meet the requirements of realistic mathematics education: a) covering the entire mathematical area in width (all chapters of the subject) and in depth (problems on each level: from basic skills to higher order reasoning); b) allowing students to show what they are capable of; c) providing for a maximum of information on students' knowledge, insight and abilities, including their strategies, and d) easy for the teacher to administer. Several steps can be taken to insure that these requirements are met: asking for own productions, using test sheets with a piece of scrap paper drawn on it, using problems with more than one correct answer, giving choice-problems, giving problems with auxiliary problems, and varying the presentation of the problems (from context to bare problems and vice versa). All of these steps can reveal a great deal about the ability of the students. According to one of the basic ideas of realistic mathematics education (see Treffers and Goffree, 1985; and Treffers, 1987). contexts play an important role in the construction of items. The use of contexts serves two purposes. Firstly, the use of contexts, often accompanied by a pictorial presentation of the problem, allows students to grasp the intention of the problems immediately, without an extensive or oral explanation. The problems often relate to everyday-life situations, at least situations which the student can imagine[2]. The latter does not only contribute to the accessibility of the problems but also towards achieving the second purpose, one that is even more important than the first. The context gives students more latitude to display what they are capable of. If selected properly the context can serve as a kind of model or can give the students a context-connected way of tackling the problem. In other words, the context provides the students with solution strategies. Moreover, built-in stratifications can provide the possibility of solving a problem at different levels. The principles mentioned in the foregoing and the related steps for improvement of assessment problems also played an important role in answering the question: how to assess percentages?

3. How to assess percentages

Before this question can be answered three other questions must first be answered: what capabilities related to percentages must be assessed, at what stage of the learning process are the students, and what is the purpose of the assessment? All of these questions, in particular the first two, are strongly related to both the content and the didactics of the curriculum that is used. In this case the assessment corresponds with a unit on percentages called "Per Sense" (initially designed by Van den Heuvel-Panhuizen and Streefland, 1992) developed for the Math in Context Project of the National Center for Research in Mathematical Sciences Education of the University of Wisconsin, Madison. This is a NSF funded Middle School Project. The aim of the project is to improve the mathematics curriculum of the middle school. The Freudenthal Institute of the University of Utrecht is involved in this project. It is their task to develop draft materials for the project, including student books and teacher guides. The 'Per Sense" unit is intended for grade 5 students. Its goal is to help students to make sense of percentages. Because dealing with the contents of the unit would exceed the limits of this paper, attention will only be devoted to the assessment part. The unit contains three different assessment parts: the unit starts with evoking the informal knowledge of the students (see Streefland and Van den Heuvel-Panhuizen, 1992) a summary activity at the end of each chapter can serve as an intermediate assessment, and at the end of the unit there is a final

assessment. It is this final assessment that is the subject of this paper. Its main purpose is to document the achievements of the students in order to make decisions about further instruction. The "Per Sense" test (Van den Heuvel-Panhuizen, 1992) that was developed for this purpose concerns several goals. Again a selection is necessary for this paper. Attention will only be paid to a few key concepts and key abilities on percentages. An exclusion that has not been made is the all too frequent mistake of leaving out higher order insight goals. The examples will therefore include computational goals as well as higher goals concerning understanding. Taking all that into account, instead of the more general question 'How to assess percentages?' the following questions will be answered: How to assess whether

- the students understand that percentages are relative numbers, and that, in consequence, a percentage is always related to something, and that one cannot compare them without taking into account what they refer to (1), and that a percentage is the same if the ratio is the same (2a) and that the percentage changes if the point of reference changes (4);
- the students are able to compute the part of a whole if the percentage is given (2b) and to compare different parts of different wholes by using percents (3).

The numbers in parentheses refer to the four problems (see Figure 1) that have been developed. They are discussed in the following.

4. Four problems on percentages as an example

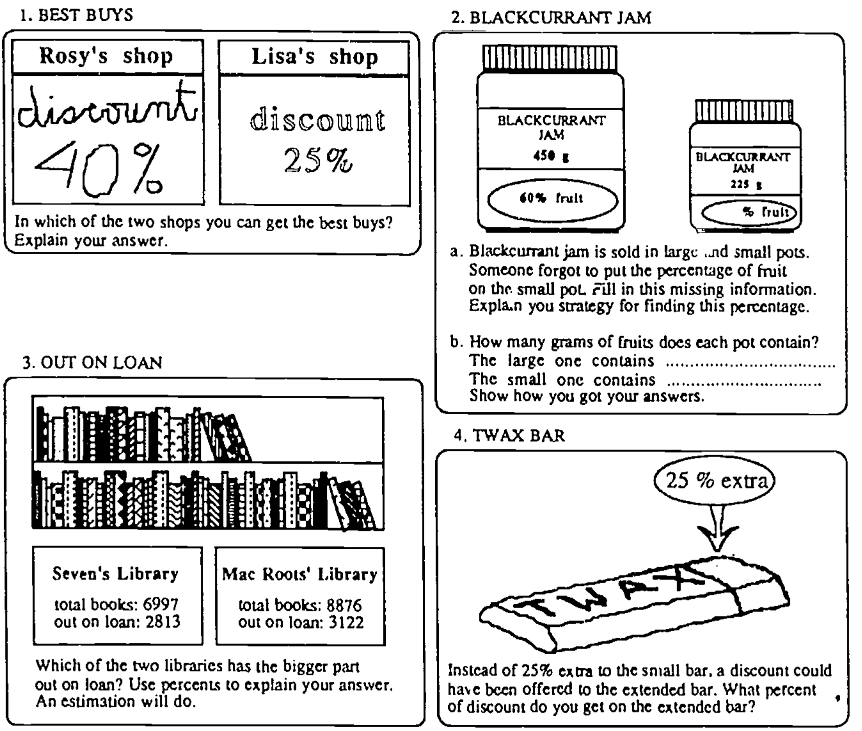

Figure 1 Four problems for assessing percentages

To assess whether students understand that percents are relative numbers, and that, in consequence, one cannot compare them just like that, the familiar situation of a sale was chosen. Two shops have a sale on. In the first shop one can get a discount of 25% and in the other a discount of 40%. Both have put up a big poster in the shop window. The manner in which the two shops advertise their discount suggests that the two shops do not sell wares of the same quality This is done on purpose to alert students to consider what the percents refer to when comparing them.
The contents of foodstuffs was one of the topics chosen by which to assess whether students understand that a percentage does not change if only the absolute amount or numbers change and not the ratio. One problem is about a quality jam. It contains 60% fruit. The jam is sold in large and small jars. The question is, are the students aware that the size of the jar does not matter for the percentage of fruit? Or in other words, is their understanding strong enough for it to withstand this visual distracter. The same jam context is used to assess whether the students can compute a part of a whole if the percentage for that part is known. In other words, can they compute 60% of 450 g and what strategies do they use in doing so?
Although this last problem gives an indication of whether students can do computations with percentages, it does not reveal whether students can apply percentages if a problem situation calls for it. The 'books loaned out' problem is used to assess this. Here students must compare different parts of different wholes. This library context has been chosen because of the bookshelves one finds there. The model of bookshelves can be helpful to the students by using it as a bar on which they can first mark the number of books out on loan and then convert this to a percentage.
Another situation from everyday life outside school has been chosen to assess whether students understand that the percentage changes if the point of reference changes. Giving parts of something for free is a common advertising gimmick. A small candy bar plus one fourth of its length for free costs as much as a large candy bar minus one fifth of the price. Hence, the shaded part refers to two different percentages. Depending on the chosen point of reference it is either 25% or 20%. In this problem the pictorial presentation plays an even more important role than in the previous problem. It is on purpose that the bar is given a name of four letters because this gives structure to the bar and can help the students to organize the problem.

5. The try-out of the "Per Sense" test

The "Per Sense" test was tried out in two grade 7 classes of a school near Madison.[3] The two classes can be classified as regular classes. In total they consist of 39 students. The test was administered at the end of May 1992. The class did the "Per Sense" unit before the test was administered. This took slightly more than three weeks. Yet the results cannot be considered as an output of this unit. The conditions under which the unit was tried out were less than ideal. Because there was no advance teacher training and the teacher guide was not completed on time the unit was not tried out as was intended. As far as the test is concerned, it was the first time that a test such as this was administered in these classes. The students were used to tests consisting of bare problems. Problems like those in Table 1.

6. What grade 7 students know about percentages

Items on percentages, 4th NAEP grade 7			
Concept items	**% correct**	**Calculation items**	**% correct**
A. Express .9 as a percent	30	A. 4% of 75	32
B. Express 8% as a decimal	30	B. 76% of 20 is greater than, less than, or equal to 20?	37
		C. 30 is what percent of 60?	43
		D. 9 is what percent of 225?	20
		E. 12 is 15% of what number?	22

Table 1 Percentages items and scores 4th NAEP grade 7[4]

Before coming to the results of the "Per Sense" problems, first a quick look at what more traditional problems on percentages tell us about the students' abilities. As is shown in Table 1, according to the 4th NAEP most of the grade 7 students in the USA are not yet very proficient at percentages. Only when it concerns common percentages for which they know fractional equivalents are they somewhat more successful (Kouba, Brown, Carpenter, Lindquist, Silver, and Swafford (1988). The question, however, is, what the results and conclusions would be if other problems had been used instead? In a way this question is answered in the next section.

7. What can the "Per Sense" problems reveal about the students' abilities

Firstly, they reveal more than only whether or not students were capable of finding 'the' correct answer. Elicited by the problems, students frequently gave responses which do not allow one to confine oneself to the criterion of correctness. This would do injustice to the richness of the answers. Instead of the criterion of correctness it is better to use the 'is the answer reasonable' criterion. Above all it is important to take the standpoint of the student: what could she or he mean by the response, what can his or her reasoning have been? Apart from the problem of the point of cut off (what is reasonable and what is not) the responses often varied so much that more than two categories are needed. Moreover, categories alone will not suffice, are not informative enough. Therefore each category is illustrated with examples of students' responses. The categories as such were determined a posteriori, based on the responses of the 39 grade 7 students who took the test.

Problem 1. Best buys

Analysis of the responses (shown in table 2 [5]) show that at least half of the students (20 out of 39) understood that one cannot compare percentages without taking into account what they refer to. The other half compared the two percentages absolutely. This does not however mean that the latter group lacks understanding of the relativity of percentages. To be sure of this the problem can be extended with an additional question. A question that serves as a safety net. By way of this extra-chance question one can pick out those students who understand the relativity of percentages, but still need extra help in expressing this. In this case the safety net question could be: "Is there any possibility that your best buy could be at Lisa's? If yes, give an example."

BEST BUYS		
Answering categories	**N**	**Examples**
a. Taking into account the original price	15	- "It depends on the original price of the objects they are selling" - "Both, I mean how much does the items cost, nobody knows" - "Lisa's, because if you buy something that's already been used, you will have to fix it up or ..."
b. Taking the same price as an example	3	- "Rosy's, if something at both stores was $30.75. At Rosy's it would be $12.33, at Lisa's it would be $28.95"
c. Taking the same price as an example; wrong conclusion	2	- "Lisa's, because for example a shirt cost $50; 40%=$20 and 25%=$12.50; with Lisa's deal you're paying less"
d. Comparing the percentages absolutely	18	- "Rosy's, 40% is better than 25%" - "Rosy's, because it is closer to one hundred percent, so there would be more off"
e. No answer	1	

Table 2 The responses to problem 1, Best buys

Problem 2. Blackcurrant jam

Only 10% of the students (4 out of 39) knew that the percentage of fruit is the same in the two jars (see table 3). This does not however mean that the students have no insight at all in this aspect of percentages. It might also be the case that their understanding is still unstable and cannot yet withstand the visual distracter of the problem. To check whether the latter is the case this problem

also needs a safety net question. The question that could be asked next is: "Let's take a look at the taste of the jam in the two jars. Will they taste the same or not"?

BLACKCURRANT JAM (a)		
Answering categories	**N**	**Examples**
a. Correct answer (60%) explanation indicates insight in 'same ratio, same percentage'	4	- "They are the same, except one is at a smaller scale, both pots contain 6/10 of fruits" - "Big got 60%, little got 60%"
b. Correct answer (60%) without reasonable explanation	1	- "Guessed"
c. Incorrect answer (30%) 'halving g-halving %' explanation	25	- "You look at the bigger bottle and its half g, so you take half of the %" - "225 is 1/2 of 450, so 30% is 1/2 of 60%" - "450 devide by 2 is 225, so you divide 60 by 2 and you get 30"
d. Incorrect answer (30%) other or no (3) explanation	5	- 450+60=7.5 and 225+7.5=30
e. Incorrect answer (others)	3	- 25%: "Both divided by 9, got my percent" - 22%: "First subtracted the grams to get the difference (450-225=245), then 22 by 10, and put a decimal, is 22%"
f. No answer	1	
BLACKCURRANT JAM (b)		
a. Correct answer (270 g) based on a standard strategy	5	- 450 x .60 = 270.00 - "0/450 = 60/100; 270/450 = 60/100"
b. Correct answer (270 g) based on an informal strategy	1	- "10% of 450 is 45; 45 x 6 ("from 60%") = 270"
c. Correct answer (270 g) no information about strategy	3	
d. Resonable answer based on a standard strategy	2	- 275: 450 x 0.60 = 275.00
e. Resonable answer based on an informal strategy	4	- 263: Bar, approximated 60% by repeated halving - 200:"450+2=225 and you have to take a little more away to make 60%"
f. Resonable answer no information about strategy	1	- 250
g. Incorrect answer (450 g) not able to work with percentages or no information about strategy (1)	12	- "It says on the bottle" - Bar divided in parts of 15% - 450 + 225 = 2; "1/2 of 60% is 30%"
h. Incorrect answer (others) not able to work with percentages or no information about strategy (3)	10	- 390: 450 - 60 = 390 - 7.5: 60 + 450 = 7.5 - 13.3 or 13: "60/450 x ?/100"
i. No answer	1	

Table 3 The responses to problem 2a and 2b, Blackcurrant jam

The second question about the blackcurrant jam is, what is 60% of 450 g? About 40% of the students (16 out of 39) gave a reasonable answer. Of those whose answers are reasonable and strategies are obvious, almost half used a rather informal indirect strategy. Contrary to them there was also a large group who did computations which make no sense. It seems as if they were trying to remember what the procedure was exactly instead of using common sense.

Problem 3. Out on loan

Although estimation problems are not very common in school, quite a few students were able to solve this problem. Some 60% of the students (24 out of 39) arrived at the correct answer and another 20% who failed to get the right answer did at least show that they are capable of working with percentages to some degree. Only 20% of the students failed entirely. Most of these again did impressive computations which made no sense.

OUT ON LOAN		
Answering categories	**N**	**Examples**
a. Correct answer (Seven's) estimation with rounded off numbers	12	- 7000, 3000, 45%; 9000, 3000, 33 % - 7000+3000=1/2, 50%; and 9000+3000=1/3, 30%
b. Correct answer (Seven's) estimation by means of bar or line	2	- Number line, repeated halving to find the percentage
c. Correct answer (Seven's) other strategies, or no information about strategy (2)	9	- "S has 40% out an R has 34% out" - The numbers on loan have been doubled - The numbers not on loan have been computed - "Because R has 8876 total and S has 6997. When S loans out books it hardly has any left. R still has a big selection"
d. Incorrect answer (both (3) or Mac Roots'(3)), but strategy indicates able to work with percentages to a certain degree	8	- "They both have the same amount out on loan" - Number line, both 40% out on loan - "R has about 35% of his books out on loan" - "He has about 33%, S has 30%" - The percentage not on loan has been computed
e. Incorrect answer (Mac Root's) not able to work with percentages, or no information about strategy (2)	7	- 6997+2813=9810; 8876+3122=11998; "98% for S and 119% for R; R has a greater %" - 6997-2813=4184; 8877-3122=5754; "4184/5759=?/100; R has a bigger percent, about 73%"
f. No answer	1	

Table 4 The responses to problem 3, Out on loan

Problem 4. Twax bar

This problem was the most difficult one of the series. Only 20% of the students (7 out of 39) succeeded in solving this problem. More often than not their answers contained excellent explanations such as: "Without the 25% extra, each part (meaning bar) is divided into 4ths. But if you have 1/4 more, it becomes so that each part (meaning bar) into 5ths. So if you divide 100 by 5, you get 20. So 20% is the discount." Obviously some used the picture to solve the problem (see figure 2). Others drew a bar or a number line to find the discount.

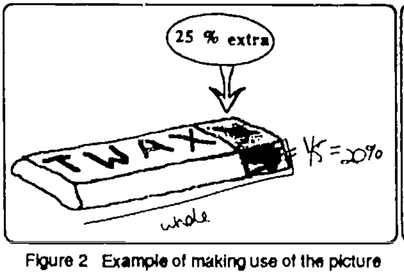

Figure 2 Example of making use of the picture

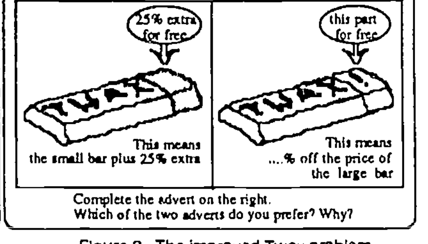

Figure 3 The improved Twax problem

These explanations and drawings could be first-class teaching materials for teaching others who are not as advanced. Among the students who were unable to solve the problem there were at least

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three who did not understand the question. This was probably due to the wording of the problem. The presentation of the problem can be improved by placing the two manners of advertising next to each other (see figure 3).

8. Concluding remarks

The "Per Sense" problems turned out to be most suitable to document the achievements of the students. They can be very helpful for making decisions about further instruction. As was illustrated by the last problem, they even yielded good teaching materials. Therefore written tests as such are not all that bad. It depends on which problems are used. Different kinds of problems will evoke different results. You will get what you deserve. The richer and more open the problems, the more they will reveal about the students' abilities. A consequence is however that the responses may be more difficult to interpret than those to closed and bare problems. It is better to confine oneself to a few good problems. In the end they will reveal more than a large number that are easy to grade.

Notes

1. No such luck for those doing research on this topic. It could be considered as jumping on the bandwagon.
2. In Dutch, instead of 'to imagine' the word 'to realize' is used. Because offering students the means to imagine oneself in the problem situation is a key aspect of the new Dutch approach to mathematics education, this approach is called 'Realistic Mathematics Education'.
3. Actually the test was administered to three classes. The third class was a special class with low attainers. Their scores are not included in the results described in section 7 of this paper.
4. The items and scores are derived from V. Kouba, C. Brown, T. Carpenter, M. Lindquist, E. Silver, and J.Swafford (1988). At this moment I do not have the 5th NAEP items and results at my disposal.
5. The dotted line indicates the possible cut off between reasonable and not reasonable answers.

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Construction and Coordination of Units: Young Children's Fraction Knowledge

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This article presents a theoretical analysis of relationships between children's whole number and fraction concepts. Three different interpretations of fractions -- part-whole, operator, and quotient -- were analyzed with the focus on children's notions of units. It is suggested that children's schemes to coordinate units are closely related to their construction of fraction interpretations and further research with this focus may be fruitful.

Introduction

Children's understanding of fractions has long been a focus of mathematics education researchers. Fraction concepts, and understanding of rational numbers in general, seems to be one of the first major shifts in children's mathematical world. In the past two decades, much has been learned about children's notions of fractional quantities. Investigators involved in research projects such as the Rational Number Project have contributed much to our efforts to make sense of this issue.

In the meantime, the mathematics education community has also experienced a shift in the way it conceives knowledge and learning. With a wider acceptance of the constructivist epistemology, knowledge is no longer viewed as an entity to be conveyed to the learners. Rather, learning is perceived as the act of accommodation to one's knowledge structure motivated by perturbations.

If we take the constructivist's view that knowledge comes from within an individual, then it is natural to ask, "what are the roots of fraction concepts?" It is often assumed that one possible such root is children's whole number concepts. This assumption has been, and should be, challenged (Confrey, 1990). However, the discussion of the viability of this assumption is much beyond the focus of this paper. Rather, I would like to accept this assumption and try to analyze relationships between children's whole number and fraction concepts, with a specific focus on children's notion of units. (I believe that Confrey would

agree that children's whole number concepts do play a role in their construction of fractions; however, she argues that the whole number concept alone is not sufficient to construct a true mulitiplicative structure, of which fractions are a part of.)

Before I begin, I would like to stress that the analysis presented here is not that of mathematics *of* children (Steffe, 1988). In fact, it is not even an analysis of mathematics *for* children, rather, it is an analysis of mathematics of the author. However, it is presented with a hope that there may be sufficient *fit* to make this analysis provocative.

Notion of Units

Steffe et al. (1983) investigated young children's counting, and their findings suggest that construction of units is one of the most fundamental processes in children's construction of whole number concepts. Furthermore, their analysis revealed that children's unit concepts play significant roles in their construction of arithmetic operations (Steffe and Cobb, 1988), and place value concepts (Cobb and Wheatley, 1988; Steffe and Cobb, 1988). More recently, Wheatley and Reynolds (1991) found that a focus on children's units was fruitful in their investigation of children's tiling activities. Thus, it appears reasonable to conclude that the notion of unit plays a fundamental role in children's construction of mathematical knowledge. For this analysis, I will use the following definition of units: *a unit is an individual's mental construction with which s/he can perform a certain mental operation repeatedly*. It should be noted that a unit is not simply a unified whole, but it must be an object (for the individual) to operate with.

In young children's construction of whole numbers, the unit of one plays the most fundamental role. It is a building block all other numbers are made of. Furthermore, children construct different units, which reflect their cognitive sophistication (Steffe et al., 1983). However, as children continue to develop their number world, they start constructing more complex units, such as the unit of ten. As the children's notions of units expand, so do their needs to coordinate those units. My earlier analysis of second grade children's problem solving activities showed that there are four types of schemes to coordinate two units: one-as-one, one-as-many, many-as-one, and many-as-many (Watanabe, 1991). These schemes were then hypothesized as the bases for three levels of cognitive sophistication. In Level One, a child's coordination scheme is limited to one-as-one. In Level Two, a child also uses one-as-

many and/or many-as-one. Finally when a child has constructed the many-as-many scheme, s/he is said to have reached Level Three.

These three levels of sophistication were found to correlate with the children's notions of ten as unit as described by Steffe and Cobb (1988) and Cobb and Wheatley (1988). Furthermore, there were evidences that seem to indicate that these unit coordination schemes may also be related to the children's notion of the fraction 1/2, using Bigelow et al.'s model of children's understanding of fractions. The model proposed by Bigelow et al.'s include three levels, qualitative, quantitative, and abstract. Only the child with the many-as-many scheme was judged to have reached the abstract level while the child in Level One had the qualitative understanding of 1/2. This model was used for the analysis mainly because the ages of the participants were comparable. However, much of the existing research base for children's fraction concepts involve children from the middle grades, and in the remainder of the paper, I would like to discuss how this notion of unit coordination may relate to the models of fraction concepts presented in those investigations.

Fractions

Several researchers have analyzed fraction concepts and proposed lists of different subconstructs/interpretations of fractions (e.g., Behr et al., 1983; Kieren, 1980). Although there are some minor differences, most seem to agree that there are four major ideas related to fractions: 1) part-whole, 2) multiplicative relationship (i.e., ratio/rate/proportion), 3) operation, and 4) quotient. The multiplicative relationship is an important research focus in itself. Furthermore, it appears inaccurate to describe the relationship between fraction concepts and ratio/rate/proportion as one being a *sub*construct of the other. The multiplicative relationship between two quantities is often expressed as a fraction, but what it signifies is a relationship, not a quantity. Although an analysis of this notion with the focus on units and unit coordination schemes may be fruitful, it is far more complicated than what can be included in this paper. Therefore, in this analysis only the other three interpretations will be analyzed.

Part-Whole

Steffe and Cobb (1988) describes the part-whole operation as a part of a child's

construction of whole numbers. One of the important ideas is that a child's number sequence must become inclusive, i.e., a number, say 7, includes all previous numbers. A simple diagram may look like this:

[[[[[[[*] *] *] *] *] *] *]

Therefore, the number 3 is embedded in the number 7. This notion of embeddedness is important in order for a child to construct a fraction as a part of a whole. In a typical region model of the fraction 3/7, the three shaded regions must be counted both as the number of parts in the fractional quantity and the number of parts in the whole.

However, this embedding alone is not sufficient to construct fractions because 3 in 7, or 3 out of 7, does not signify a single numerical value. As a result, it is impossible to compare the magnitudes of two fractions, for example 3/7 and 4/9, with this embedding alone. In order for a fraction to have a numerical value, a child must construct units for fractions, i.e., 3/7 must be three "sevenths." Construction of this unit, a seventh, seems to involve sophisticated relationships between two units, the unit of one (whole) and the unit that describes the number of partitions (parts). The relationship between these two units seems to require a complex coordination between them.

Although the unit of one may be considered as consisting of seven units of a seventh like the unit of ten consists of ten ones, the construction of this unit of a seventh is qualitatively different from that of ten as a unit. Construction of ten as a unit is through the integration operation (Steffe and Cobb, 1988). The process is a building-up process. Therefore, ten can be thought of as a unit *of* units. On the other hand, construction of the unit of a seventh involves decomposition of the unit of one. Thus, the unit of a seventh is a unit *for* a unit (one). This is where an apparent paradox occurs. The unit of one must be decomposed to a specified multiplicity; however, construction of the unit (seventh) that gives rise to the multiplicity (seven) is the result of the construction. Thus, the result of the construction is needed in the process. Confrey's (1990) notion of splitting appears to play a significant role in this construction process; however, it does not address this paradox. My current conjecture is that the nature of relationship between these units is metaphorical (Lakoff and Johnson, 1980). The individual, then, metaphorically extends one specific

characteristic of the multiplicity, that is, it is composed of units, onto the unit of one. This conjecture needs much more careful analysis and refinement.

Operation

Behr et al. (1990) describes two ways this fraction as an operator idea may be realized. For example, the fraction 3/4 can be thought of as a sequence of 3-for-1 (3/1) and 1-for-4 (1/4) exchanges (or vice versa). This is equivalent to interpreting a fraction, p/q, as the sequence of multiplying by p then dividing by q (or dividing by q then multiplying by p). On the other hand, the fraction 3/4 can be thought of as a direct 3-for-4 exchange. As you can see, both of these are based on exchange/substitution idea.

Although this interpretation of fractions does not assign numerical value to a fraction, it is still possible to compare fractions quantitatively. Since the effect of operation is independent of the numbers to which the operations are applied, one can compare two fractions (operators) by simply applying the operations to the same number and comparing the resulting numbers. For example, 3/4 applied to 8 gives 6, but 1/2 applied to 8 is 4; thus, 3/4 is "greater" than 1/2.

When a fraction is perceived as an operator, the unit of the numerator and the unit of denominator are the same. In other words, the operator 3/4 signifies the exchange of three of a unit with four of the same unit. The exchange relationships fraction operators signify appear to closely resemble the unit coordination schemes. In Behr et al.'s discussion that the fraction operators with either the numerator or the denominator of 1 (such as 3/1 or 1/4) are more primitive than the operators where both the numerator and the denominator are numbers other than 1, for example, 3/4. Similarly, it was found that the unit coordination schemes of one-as-many and many-as-one were more basic than the many-as-many scheme.

In my previous analysis, I suggested that a further investigation between the one-as-many and many-as-one schemes was needed. Behr et al.'s assertion that the order in which 3-for-1 and 1-for-4 operations are applied results in different complexities appears to support my suggestion. Furthermore, I have also suggested that it is necessary to subdivide the many-as-many scheme into x-as-x and x-as-y schemes. In the operator interpretation of fractions, is there any cognitive difference between 3/3 and 3/4? What about the notion of "equivalence?" Is there any cognitive difference between 1-as-2 and 2-as-4? How about the

operators 3/4 and 6/8? These questions must be carefully investigated.

Quotient

The quotient interpretation of fractions says that the fraction p/q indicates the result of p divided by q where both p and q are whole numbers. Since both p and q are numbers, the result also has numerical significance. However, since the division operation is not closed with the set of whole numbers, construction of numerical meanings from this interpretation is not straight forward.

Although this interpretation involves an arithmetic operation of division, unlike the operator interpretation, the focus is on the result of the operation, not the operation itself. Furthermore, a fraction from this interpretation is the result of a specific division situation; thus, this interpretation seems to be less abstract and more context bound than the operator interpretation. One of the common context in which this interpretation may arise is the experience of sharing. For example, when 3 pizzas are shared equally by 4 people, determining the amount of pizza each person eats can be signified as 3/4, or 3 pizzas divided by 4 people. (This brings up the issue of different measure fields, but the analysis of this issue is beyond the scope of this brief article.)

In my research with four second graders, there were two different strategies for this type of setting. (The task was to share 3 clay "cakes" fairly among 4 people.) The child who had only constructed the one-as-one scheme randomly sliced the clay cakes without much regard to the equivalence of the sizes of the pieces. For him, the most important factor was that everyone gets the same number of pieces. The two children who had constructed both the one-as-many and the many-as-one schemes sliced each of the three cakes into four equal parts and gave one to each person. Finally, the child who had constructed the many-as-many scheme sliced the first two cakes into halves and the last cake into fourths.

On the surface, neither one of the successful strategies reflect the interpretation of 3 divided by 4. The strategy employed by Level Two children indicated (1/4)+(1/4)+(1/4), while the last child's action shows (2/4)+(1/4). However, I believe that the last child's action of sharing 2 cakes among 4 people reflects the division interpretation (2 divided by 4). It is hypothesized that when the child share 2 cakes among 4 people, she first united the two

cakes into a whole, which, in turn, was divided into four parts. (Here, again, is the paradox I have discussed earlier comes in.) I believe that she was able to do this because she was able to coordinate 2 and 4. On the other hand, the other two children who divided each cake into four parts, could not take the three, or two, cakes as a whole to be coordinated with four. Thus, it appears that the coordination scheme of many-as-many is needed to construct numerical meanings of fractions interpreted as quotients.

Concluding Remarks

The brief analysis presented here is meant to be a beginning for a much more careful analysis. However, I believe that children's notion of units and their schemes to coordinate units are important factors as they construct meanings of fractions. Although there may be other factors influencing children's construction of fraction concepts, such as Confrey's splitting notion, a focus on children's unit concepts should be one of the main foci in future research.

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CULTURAL CONFLICTS IN MATHEMATICS LEARNING: DEVELOPING A RESEARCH AGENDA FOR LINKING COGNITIVE AND AFFECTIVE ISSUES

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ABSTRACT

In trying to link cognitive and affective aspects of mathematics learning, the idea of 'cultural conflicts' appears to have much merit. Developing from the recent research on ethnomathematics, it offers a focus for a research agenda which could probe issues of great significance in many countries. In this paper a framework for developing such a research agenda is presented.

Introduction and background

Up to ten years ago, Mathematics was generally assumed to be culture-free and value-free knowledge, explanations of 'failure' and 'difficulty' in relation to school mathematics were sought either in terms of the learners' cognitive attributes or in terms of the quality of the teaching they received; there were several attempts to make mathematics teaching more affectively satisfactory to the learners, with few long term benefits, and 'social' and 'cultural' issues in mathematics education research were rarely considered.

Within the last ten years, there has been an increasing move to make mathematics accessible to all learners, there has been an increasing questioning of the relevance of ex-colonial models of education in developing countries, and in countries with indigenous 'minorities'; the social dimension has come into greater prominence in research in mathematics education and the cultural nature of mathematical knowledge has become clearer to many mathematics educators.

However, within current educational practices we also have to recognise that efforts to develop multi-cultural mathematics education have produced violent

hostility in many education quarters, many schools, and teachers, claim "we don't have that problem in our school", and 'real' mathematics is felt to have 'powerful' connotations within many educational systems.

In addition, many learners continue to find mathematics difficult, threatening, anxiety-provoking, boring, meaningless etc. Therefore it seems that some fresh thinking is required, in order to develop some fresh avenues for research. The area of cultural conflicts appears to have great potential, particularly for linking the cognitive and affective aspects of learning mathematics.

Cultural conflicts

Three important research directions are being shaped by the recent work on ethnomathematics, with the following foci:

a) mathematical knowledge in traditional societies (Anthropology) e.g. Ascher (1991), Zaslavsky (1973), Gerdes (1985), Harris (1991), Pinxten (1987).
b) mathematical developments in non-Western cultures (History) e.g. Ronan (1981), Joseph (1991), Gerdes (1991).
c) the mathematical knowledge of different groups in society (Social psychology) e.g. Lave (1984), Saxe (1990), de Abreu (1988), Carraher (1985).

Underlying all this work is the fundamental epistemological question of: is there one mathematics appearing in different manifestations and symbolisations, or are there different mathematics being practised which have certain similarities? However from an educational perspective the concerns are generally focussed by the implications of the differences between cultures, and with the cultural conflicts caused by different conceptions of mathematics.

School mathematics is tending towards a universal form, for various reasons, and is often therefore wrongly assumed to be culture-free and value-free. Particular social groups for whom conflict with, and alienation from, this

school mathematics has been documented are:

- ethnic minority children in Westernised societies
- second language learners
- indigenous 'minorities' in Westernised societies
- girls in many societies
- Western 'colonial' students
- fundamental religious groups, often of a non-Christian nature
- children from lower-class and lower-caste families
- physically disadvantaged learners
- rural learners, particularly in Third World communities

The documented conflicts vary but concern some or all of the following:

- language
- geometrical concepts
- calculation procedures
- symbolic representations
- logical reasoning
- attitudes, goals, and cognitive preferences
- values and beliefs

In the face of such documentation, it is very unclear what the teaching and learning task should be. The established theoretical constructs of mathematics education, developed through a research history which has failed to recognise cultural conflicts, are at best misleading and at worst irrelevant and obstructive. To separate 'cognitive' from 'affective' issues seems counterproductive, and the task of exploring the implications of cultural conflict seems to require some fresh thinking.

One way to make a start on this is to search for <u>similarities</u> between situations of conflict, and between the similar experiences of <u>different</u> alienated groups. Hitherto mathematics educators have been reluctant to do that, with their research focussing on, and remaining within, the problem-space of any one

group (for example, ethnic minority students, girls, or second language learners). Not only have mathematics educators not looked across different groups <u>within</u> their field, they have failed to look across at similar conflict situations experienced in history education, TESOL education or religious education, to name but three areas. Therefore, it is helpful, I believe, to begin to recognise and analyse similarities between the responses of educators to those different situations of conflict. The following table represents an attempt to do this:

Approaches to culture conflict	Assumptions	Curriculum	Teaching	Language
Culture-free Traditional view	No culture conflict	Traditional Canonical	No particular modification	Official
Assimilation	Child's culture should be useful as examples	Some child's cultural contexts included	Caring approach perhaps with some pupils in groups	Official, plus relevant contrasts and remediation for second language learners
Accommodation	Child's culture should influence education	Curriculum restructured due to child's culture	Teaching style modified as preferred by children	Child's home language accepted in class, plus official language support
Amalgamation	Culture's adults should share significantly in education	Curriculum jointly organised by teachers and community	Shared or team teaching	Bi-lingual, bi-cultural teaching
Appropriation	Culture's community should take over education	Curriculum organised wholly by community	Teaching entirely by community's adults	Teaching in home community's preferred language.

A Possible Research Agenda

Various research questions are now raised by this kind of analysis, and here I will refer to just three areas of questions, which relate to the three levels of curriculum: intended, implemented, attained. This structuring has been chosen to expose the three significant levels of cultural conflict recognition, and thus potential resolution.

1. Regarding the mathematical knowledge as represented in the intended curriculum.

Here we are becoming more aware of the need to consider three very different educational structures, and among the potential research questions, the following seem the most promising:

a) Formal mathematics education

What theories could influence the 'culturalising' of the formal mathematics curriculum?

What values are developed within the current school mathematics curriculum? What other values can be emphasised?

What criteria should be used to evaluate an appropriate intended mathematics curriculum in a culturally pluralistic society?

b) Non-formal mathematics education

What roles are non-school alternatives fulfilling in relation to cultural conflicts?

Are these alternatives on the increase?

Is an increase a measure of the communities' satisfaction, or their dissatisfaction, with formal mathematics education?

c) Informal mathematics education

In what sense are informal societal and community influences on mathematics learners educational?

2. Regarding implementing a mathematical knowledge environment in schools and classrooms

Here there are three main research avenues which in my view are worth exploring further:

a) Implementing a culture-blind intended mathematics curriculum

   To what extent can a culture-blind intended mathematics curriculum be made less of an obstacle to learning in the classroom?

   Can mathematical learning activities be usefully characterised as more-or-less 'open' in relation to their cultural framing?

b) The mathematics teacher as social anthropologist

   What outside-school mathematical knowledge do teachers recognise as legitimate inside the classroom?

   What knowledge about the learners' cultures can help mathematics teachers with their classroom decision-making?

   How do teachers recognise cultural conflict in their classrooms?

c) Multi-cultural mathematics classrooms

   What teaching strategies do mathematics teachers adopt in if they recognise their classrooms as being multicultural?

   What values exist in the knowledge environment created by mathematics teachers in their classrooms?

3. Regarding the mathematical knowledge attained by the learners

   What outside-school mathematical knowledge do learners recognise as legitimate inside the classroom?

   What cultural conflicts are actually experienced by mathematics learners and how do they cope with them?

   How does the 'cultural distance' of their home mathematical culture from the school mathematical culture relate to the quality of their mathematical learning in classrooms?

   How does bi-cultural mathematical learning differ from bi-lingual mathematical learning?

Finally, what are the implications of all these questions for determining the appropriateness of any mathematical assessment procedures?

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Linking the Cognitive and the Affective: a Comparison of Models for Research

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We consider four different types of model for relating the affective and the cognitive in the study of mathematical performance and problem solving. The models are: (A) models with a "macro" perspective, aiming to explain individual differences in performance scores or participation; (B) "micro" models focusing on an individual's problem solving episodes; (C) micro models, drawing on psychoanalytic insights; and (D) models,informed both by psychoanalytic and post-structuralist ideas. We compare them in terms of: theorisation of the effective and the cognitive, methodology (how both of the latter are explained), methods used to study the relationship, and recommendations for practice and pedagogy.

The importance of affective factors in the learning of mathematics is reasserted periodically. This occurred in the 1970s, given the need to study barriers to females' participation and performance in maths (e.g. Fennema and Sherman, 1976), and also barriers to "second chances" for university students and adults generally (e.g. Richardson and Suinn, 1972). More recently, the need to account for blocks in mathematical problem solving episodes has seemed to require a different level of analysis, e.g McLeod and Adams (1989). Other researchers, such as Walkerdine (1988) and Taylor (1989), have sought to harness psychoanalytic and/or poststructuralist ideas.

A. Macro "differential" models

These models aim to explain individual differences in performance scores, participation (taking maths courses), etc. using measures of affect such as "attitudes towards mathematics" (e.g. Fennema and Sherman 1976). Affect in this approach tends to be represented, not by "hot" emotion, but by "cool" attitudes; in fact, there is a tendency here to see as characteristics both the cognitive ("performance levels", "skills", if not innate abilities) and the affective ("personality", "traits", as in trait anxiety).

In the "differential" models, exemplified by Fennema's (1989) generic model (see Fig.1), the links are produced by what are seen as "causal" relationships. The "external", the social, the cultural, socialises the individual, so that values and affect are "internalised"; affect in turn influences cognitive outcomes in the individual. In these studies within educational psychology, the process of socialisation is, to a greater or lesser extent, bracketed as largely the province of sociology, anthropology or social

psychology - whereas the causal links assumed between cognitions and affect are considered accessible to analysis using correlations and statistical modelling[1].

Fig.1 Fennema's generic model

Social Influences → Affect → Mediating Learning Activities → Outcomes e.g. Performance Participation

Social Influences → Mediating Learning Activities

In these models, affect is seen as having an influence on cognitive outcomes, causal and (presumably) one-way. The ultimate explanation for both comes from social influences - outside the individual - which have effects on affect through beliefs, etc. In Fennema's model, social factors also have effects on cognitive outcomes through "mediating learning activities"[2] .Affect is measured by scores on attitude scales or ("trait") anxiety scales such as the MARS (Richardson and Suinn, 1972); outcomes by number of maths exams passed, scores on standardised tests, etc. The recommendations for pedagogy which follow include: removal of the causes of problems, e.g. maths anxiety programmes; or compensatory programmes for disadvantaged groups, e.g. "remedial maths".

There are advantages and disadvantages in using this type of model. First, it can be made comprehensive by including the affective factors of interest, and a range of social and cultural variables, as well. Second, the outcome variables are those of interest to teachers, parents and policy makers, viz. differences in maths performance, maths course-taking, etc. (perhaps gender-related, see Fennema, 1989). On the other hand, the comprehensiveness of the model may lead to over-complexity. Moreover, the nature and operation of the many social "influences" is unclear, e.g in the attempt to include the social through taking account of "socializers" aspirations, expectations, etc. and the students' perception of these. That is, the cultural transmission referred to in these models is not at all straightforward: for example, children do not apply their parents' (or teachers') values independently of the context (Mandler, 1989b).

1 Another example is the "academic choice" model which uses an "expectancy-value" approach, so called because the "perception of task value" and "expectancies of success" are seen as operating together to produce the probability of choosing to do another maths course (or that of other achievement behaviours). Some argue that this approach allows us to subsume most of the "affective" factors previously discussed, under "expectancy" (confidence, perceived difficulty, attributions) or under "value" (enjoyment, perceived usefulness of mathematics) (Chipman and Wilson, 1985, pp.294-95).

2 These include thinking independently about problem solving in maths, choosing to do it, persisting at it, achieving success.(Fennema, 1989).

B. Micro"cognitive-constructivist" models

We can distinguish "macro" approaches such as the differential models from "micro" approaches; the latter focus on the process of an individual attempting a particular task or problem, as follows:

(i) a discrepancy (or interruption) between the student's expectations and the demands of ongoing activity leads to visceral arousal;

(ii) the psysiological arousal, on the one hand, and the person's evaluation of the situation, on the other, lead to the "construction" of emotion; and

(iii) emotion may lead to a reduction in the conscious capacity available for problem-solving. (Mandler, 1989a; McLeod, 1989a). Here emotion is more "hot", more intense, than affect in model A above.

Since emotion is "constru(ct)ed" as a result of the cognitive evaluation of a physiological arousal, the links between the cognitive and the affective are produced by assimilating affect to some extent under the cognitive umbrella. These models do not necessarily assume a one-way causality, of the form "(negative) affect is debilitating for thinking"; for example, Cobb et. al. (1989) illustrate a freedom from such an assumption.

Although questionnaire measures are perhaps appropriate for the measurement of repeated emotional reactions to a category of (say mathematical) tasks, more process-sensitive methods are seen as necessary for describing reactions which are not yet so automatised (McLeod, 1989b). Thus, the methods used here are: (i) description of particular episodes of mathematical problem-solving; and (ii) "cross-subject" comparisons, e.g Mcleod and Metzger (1989), who compared problem-solving "experts" and "novices".

Recommendations for practice: The problem-solver needs to learn how to "manage emotions", e.g. Mcleod and Metzger (1989). Also, the teacher can show the pupils how to construe their own (or others') actions, as a basis for emotional acts,. e.g. Cobb et al. (1989). The rationality required for maths might be intersubjectively constituted on the basis of emotional acts which bind the pupil to the subject of maths (or her/his classmates and teacher). In this account, the rational human subject remains a basic presupposition, and the affective is appropriated by the cognitive.

The advantages of this model are its relative parsimony, and the relative clarity of the operation of its effects. On the other hand, it is focussed purely at the individual level or at the intersubjective[3]. level whose starting point is a pre-formed individual subject.

[3] See Cobb, Yackel and Wood (1989) which discuss the mutual dependence of emotional acts, belief and social norms.

Summary so far

Both models A and B display an hierarchical opposition between cognitive and affective. In A, the affective is put forward as an influence on, an explanation of, the outcomes of interest, performance, participation, which are basically cognitive. In B, affect may "interfere with" clear thinking, again devaluing the affect. Both models are "cognitivist"[4]. A takes account of the social as separate variables, i.e. it is dependent on a society vs. individual split. With B, it is difficult to take account of the social at all. Neither A nor B fully take account of the particularity of the subject's history.

C. Micro models, informed by psychoanalytic approaches

A number of researchers have studied affect around mathematics, using psychoanalytic approaches e.g Nimier (1978), Legault (1987)[5]. These approaches start from the Freudian position that affect can be thought of as a "charge" attached to particular ideas. Ideas with strong negative charges, e.g. anxiety, or which mobilise intrapsychic conflict, tend to meet defenses, and to be repressed into the unconscious. Therefore, much thought and activity takes place outside of conscious awareness; everyday life is mediated by unconscious images, thoughts and fantasies - which sometimes appear as jokes, slips, dreams, etc. These unconscious meanings are linked to complex webs of meaning. The affective charge can move from one idea to another along chains of associations by displacement, and can build up on one particular idea through condensation. Thus any product of mental activity, including interview talk, may, upon deeper investigation, reveal hidden aggression, suppressed anxiety, forbidden desire, and defences against these wishes. (Hunt, 1989)

For example, Nimier (1978) shows how a student's "setting myself against" algebra, though she was "excited" by geometry, might be explained by anxiety displaced from the sounds of her parents' arguments in the next room, that kept her awake when she was young, to the "purring" sound of the algebra teacher's voice, that "got on my nerves" (p. 169; Evans' translation).

Here the affective is apparently privileged: it provides the charge for ideas, it "powers" thinking (Buxton, 1981). However, anxiety apparently relating to maths may result from anxiety displaced onto mathematics, as in the above example.

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4 For a critical reading of cognitivism and the problems inherent in any attempt to move beyond it, e.g. to challenge hierarchical oppositions, see Tsatsaroni (1991).

5 Buxton's (1981) model is basically B, though he acknowledges the possibilities in C.

The methods used for model C include clinical/semi-structured interviews, but also questionnaires including open-ended questions (Nimier, 1978). They can be treated as case studies, or used in cross-subject comparisons; see Legault's (1987) comparison of schoolgirls "strong" and "weak" in maths. As indicators, we must not consider only expressions of anxiety, but also what are likely defences against anxiety, i.e. exhibiting of it. Indicators of defences against anxiety (or other conflict in the psyche), include Freudian slips, denial, "behaving strangely", impatience, dreams and fantasies.

Recommendations for practice are difficult to formulate clearly. But they include the teacher's being aware of multiple (and suppressed) meanings of ideas used in supposedly making maths meaningful, e.g "shopping with mother" (Adda, 1986).

Many of the conclusions drawn from using models A and B describe the relationship between cognitive and affective in terms of affect "supporting" cognition, or "interfering with" it. This would be in line with modernist discourses which produce subjectivity, i.e. the identity of a rational unique self, only by marking its difference from the affective as the "other" of the cognitive. Now, in the psychoanalytically informed view of the affective in terms of charges attached to (or infusing) ideas, and as related thus to the cognitive, the affective is not entirely "other" to cognition. Moreover, in psychoanalysis, affect can be displaced onto ideas different from those to which it was originally attached. This means that, though affect is not entirely "other" to cognition, neither is it completely "at one with" cognition.

Thus, the advantage of model C is the depth possible in the treatment of affect. However, opening up the problematic of affect has as its effect the conceptualisation of the field of mathematics knowledge as an open system with an inability to distinguish neatly between its cognitive structure and the social context (see below).

D. A psychoanalytic approach, informed by post-structuralism

The work of some post-structuralists radicalises model C by questioning some of its assumptions. Walkerdine (1988, 1990) has shown the need to understand thinking in terms of social difference and deprivation, as well as in terms of pleasure, anxiety and defences, likely to be seen in model C as related to early family dynamics. Taylor (1989) has argued for the importance of socially available discourses and dreams in understanding both mathematical" problem solving and motivations such as career hopes.

These researchers develop the basic psychoanalytic approach by emphasising signification. Rather than considering the movement of the affective charge to take place along chains of associations, they adopt the (Lacanian) view that affect can be displaced along chains of signification, where the linking of signifiers to produce meaning may be

determined, or contingent and based on contiguity. Thus, they draw on theories of signification to analyse the elements and structures of discursive practices, in particular signifier / signified relations and devices such as metaphor (linked with condensation) and metonymy (linked with displacement); see Henriques et al. (1984, Sec.3). This allows the analysis of meanings, both at a general level and for particular subjects; it thus provides the basis for study of how subjects are examining and thinking about specific problems to be solved, and of the emotions they feel. At the general level, studies of how specific discursive practices have their effects in education are done; see e.g. Walkerdine (1984) on "child-centred pedagogy".And for particular subjects, semi-structured interviews are used, with either a problem solving, or a clinical / life-history focus - or both (Taylor, 1989; Evans, 1991, 1993).

In this work, the context of cognition is constituted by the discursive practice(s) in which the subject is positioned. It is necessary to specify the practice(s) within which the subject is addressing the problem. Evans (1993) has attempted to produce a synthesis of previous answers to the problem, that would avoid tendencies to overemphasise either the determination of human action - as in some of Foucault's work, or alternatively its freely chosen character. His approach involves describing the <u>positioning</u> of a subject confronting a problem as a "resultant" of the practice(s) in which all subjects in that situation are <u>positioned</u>, and the practice(s) which the particular subject <u>calls up</u>. His analysis of a set of interviews with adult students (e.g Evans, 1988, 1991) shows that it is often (though not always) possible to describe a particular subject's positioning in a particular situation - and to understand their thinking and emotions in this context.

Thus, for example, one episode from the case of "Ellen" (Evans, 1991) can be reformulated for the analysis here. When asked to calculate a 15% tip for a meal she has chosen from a restaurant menu, she hesitates, then makes a "slip" (dividing by 15, rather than multiplying). In response to a "contexting question" about how often she does this sort of calculation, she admits that she doesn't usually pay, but nevertheless, she habitually adds up the cost of her meal - since she doesn't "want to be an expense". It is possible to read "expense" as a signifier on which meanings are <u>condensed</u>: it would signify for Ellen both the cost of, say, her meal obtained by summing the individual dishes, <u>and</u> her being a burden within a relationship - and these two ideas are <u>metaphorically</u> linked in her history. Also, in this episode, the anxiety, guilt, pain of being an expense would be <u>displaced</u> onto the idea of the cost of her meal including any tip, and these two ideas are <u>metonymically</u> linked through the idea of summing. Her response may look like "maths anxiety". But, because of her multiple positioning, the signifier "expense" is located at the <u>intersection</u> of two (at least) discourses, and this linkage allows the strong feelings around her relationship and eating out, to be displaced

onto the calculation problem - which at first seemed so obviously to be simply mathematical!

For another problem [reading a graph of changes in the gold price], "Fiona" seems to make two errors or "slips", especially given her assertion that "My father's a stockbroker, so I do understand a little about opening and closing [prices]". Her answers to the contexting questions show that this problem has called up family discourses charged with disappointment and anger around her relationship with her father, and anxiety about his work. When asked to describe her father's work, Fiona responds as follows:

"capitalist,...corrupt,...business-like,...mathematical,...
calculating,...devious,...unemotional..."

This is a chain of signifiers particular to her own "history of desire". A chain which describes the activities that make up her father's work, and the way the father is perceived to treat other members of the family, especially her. These activities and relationships are constructed by available discourses. Further, in these discourses there are echoes of a corrupt capitalism, and of popular discourses about maths. At this point of condensation, there is again - as with Ellen - an intersection of discourses, of which the word "calculating" speaks.

In these examples, the term "mathematics" shows up in unexpected ways, and what seem to be terms of mathematics are sometimes shared, at intersections with other discourses. The consequences are that what appears to be "mathematical" activity, or "maths anxiety", may be read quite differently.

Concluding Remark

In our analysis we refer to a set of intersecting discourses which produce a heterogeneous chain of signifiers. The term intersection is provisional because the present paper has considered only two moves concerning the cognitive / affective relationship. The first two are on a methodological level - from causation to interpretation - and on a theoretical level - in terms of conceptualising the cognitive / affective relationship. These need to be complemented by a third. In models A and B, the canon of rationality, which sees mathematics as rule-governed, guarantees the unity of the mathematical field and its pre-constituted quality. It thereby asserts the existence (and unity) of a rational, cognitive, human subject. That is, these models are linked with an "absolutist" philosophy of mathematics (cf. Ernest,1991).

With model C, this unity can no longer be guaranteed: as Freud himself once remarked, the "discovery" of the qualities of the affective was the last blow, namely to the

psychological subject, that scientific research dealt to "the universal narcissism of men".[6]. This, however, means that Freud presupposes an affective order in the way that mathematics constitutes itself as a field - and in the way that the student relates to the subject of maths. The implications of this last point for understanding the field of mathematics have not yet been fully considered in our version of model D.We need to direct our attention to a view of the way in which the specificity of mathematics is produced and delimited. Different answers to this question will entail different positions on the question of the boundary between the cognitive and the affective.

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[6] The other two are: the cosmological, associated with Copernicus; and the biological, associated with Darwin (Freud, 1917/1953-, p.137).

INCONSISTENCY IN LEVELS OF INTERACTION
- MICROSCOPIC ANALYSIS OF MATHEMATICS LESSON IN JAPAN-

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Abstract

The aim of this research is to discuss the construction of interaction between a teacher and students in mathematics lesson in Japan. Thirty-six mathematics lessons in fifth grade were observed and recorded with two video cameras and interviews were conducted with a teacher and 8 students. Then the data are transcribed and analyzed from both the viewpoint of collected action and of an individual participant. As a result, we identified three levels of interactions, 1.basic level, 2.mathematical knowledge level, and mathematical rationale level. And we proposed model of interaction which is called Kumagai model in terms of the level of interaction. According to this model, interaction in mathematics lesson has inconsistency in deep, even though it has consistency in surface.

In Japan there are 40 students to a teacher in a mathematics classroom So we can find various ideas in students' solution and thinking. But, some researchers suggested that Japanese mathematics lesson has an introduction, conclusions, and a consistent theme, that is mathematically emphasized theme (Becker et. al.,1990; Kumagai,1988; Stigler, 1988,1991). For example at the beginning of the lesson, a teacher pose a problem, "25000 ÷ 800 = , How do you calculate?", on the board. And then students solve this problem and a new problem , "Why do we delete the same number of zeros?", is formulated through interaction between a teacher and students. And they discuss solutions of the new problem. And at last the teacher and students share a solution of the new problem, mathematical rationale for procedures. At the end of lesson, the teacher summarize their discussion

The aim of this research is to discuss the construction of interaction between a teacher and students in mathematics lesson as we described above We will analyze interaction between a teacher and students from the viewpoint of social interactions

Method

We observed and recorded thirty-six mathematics lessons in fifth grade with two video cameras from the beginning of the school year, from 11th in April to 3rd in July And interviews are conducted with the teacher (28th in June) and 8 students (7th in May).

Teacher; The teacher Mr. Yama.G. has experienced elementary school teacher in 14 years.

Students, There are thirty-six students (15 boys and 21 girls) in a classroom. nineties of them (8 boys and 11 girls) have never experienced Yama.G.'s teaching. Particularly two of them came from other schools. And the teacher has been teaching mathematics other 17 students (7 boys and 10 girls) for two years We distinguish these two group of students. One group R consists of pupils who have studied under the teacher Yama.G. And the other group N consists of pupils who have not studied under the teacher Yama.G.

Contents; The teaching content follows the course of study in Japan. For example, in fifth grade, it contains numeration system of the whole n mber and the decimal fraction, multiplication and division with decimal fractions.

Data ; We transcribed and analyzed the following data

-transcription of verbal interaction by the teacher and students

-transcription of blackboards

-transcription of interview with the teacher and students.

-whether a student belong to group R or group N

Three levels of interaction

To investigate the interaction, we focused on interactions in a problem formulating activity from the view point of implicit rules (Bauersfeld,1980; Cobb,1992, Lampert,1990, Voigt,1985,1989). At first we analyzed the teacher's interventions and identified three types of interventions. Two of them changed as time went on The first type of intervention could be observed only in the beginning of two weeks. And the second one had been observed in the beginning of one month. And the third one was observed through all observation periods From students point of view, most of students met the teacher's expectations correspond to the first and the

second type of interventions. Particularly, new students who belong to group N could come up to the teacher's expectation with two types of interventions But students could not easily interact with the teacher according to the third type of intervention.

We found three levels of interaction, that is basic level, mathematical knowledge level, and mathematical rationale level, in a mathematics lesson. (Kumagai, 1991).

Level.1 Basic level A teacher expects students to pose problems actively, in particular, and in general participate interaction with others and a teacher Students are obliged to pose questions, problems, explanations, opinions, and so on actively. The teacher is under the obligation that he must accept pupils active expressions And, in general, this level of interaction includes classroom managements

Level.2 Mathematical knowledge level A teacher expects students to pose questions or problems with mathematical knowledge. Students are obliged to pose questions and problems with mathematical knowledge they have learned A teacher must accept students' questions and problems based on mathematical knowledge. And a teacher attempts to develop these questions to mathematically valued.

For example, students classify two or more solutions and identify the differences of them, whether they have already learned or not. And students identify the differences of them as problems.

Level.3 Mathematical rationale level A teacher expects students to pose mathematically valued questions and problems. Students must pose mathematically valued questions and problems that is suitable for the situation in interaction. A teacher is obliged to accept mathematically valued questions and problems posed by students.

We can identify three levels of interaction according to three levels of implicit rules, in particular, problem formulating process.

In consistency in levels of interaction

Inconsistency in level of interaction from the view point of collected process

Explicit inconsistency In some problem formulating situation, most of the students interact according to the third level of interaction, and a few according to the second level. But in other problem formulating situation (for example, it was observed 30th in April), some students interact according to the second level, and others do in the first level. And a few students interact with the teacher in the third level. When we closely analyze this situation, the teacher's intervention observed in this situation is different from interventions in other situations. The teacher did not follow the typical sequence of interventions, requesting various solutions, comparing these solutions, and posing questions and formulating a problem. He intervened in an open way in this situation.

We could observe like this inconsistency in levels of interaction in problem formulating situations evidently.

Implicit inconsistency In the first situation in one lesson (observed 17th in April), students and the teacher formulated a problem, "Why do we delete the same number of zeros?" When we analyzed only this situation, it seemed that they interacted in the third level.

But when we consider the other situation following the first situation in this lesson, there is some question whether the interaction in the first situation occur in the second or the third. In the following situation, students explained the procedures of division or of deleting the same number of zeros They did not mentioned about rationale for procedures. The teacher explained that students' explanation is not a proper reason for deleting the same number of zeros From this observation, we can identify that interaction in the first situation did not occur according to implicit rule on the third level but in the second level

In the first situation, the teacher intended to interact with students in the third level But students interact with the teacher in the second level It seems that there is inconsistency in levels of interaction between the teacher and students But it is difficult to identify like this inconsistency in only one situation for the teacher and observers.

Inconsistency in level of interaction from the view point of the individual students

We have mentioned before, there are various inconsistency in ievels of interaction To closely analyze these inconsistency, we will attempt to prepare another perspective, that is individual student's perspective.
We focused on three students, Sai,T.(gr.R) Toku,N.(gr R) , and Furu,Y.(gr.N)
Type A Furu,Y (gr N) can interact with others in the second level if others are in the second level But, through the all periods we observed, Furu,Y. interacted with others in the second level even if the teacher expected to interact in the third level and some other students interacted in the third level For example, when the teacher expected to discuss the reason for validity of procedures of calculation, she said "Because the result of calculation, answer, is correct, the procedure of calculation is valid"
There are some students who interact with others same as Furu,Y. does. He/she interact with other in second level in every situations. We represent such interaction as follows (Fig.1).

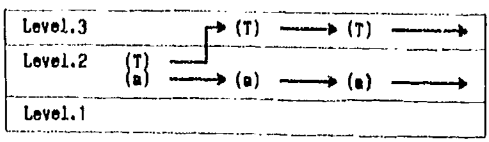

Fig 1

Type B Toku,N.(gr R) interact with others in various level. He does not necessarily meet with the teacher's expectation. Even in one situation (observed 30th in April), he interacted in various levels, the first, the second or the third But the teacher expects explicitly, for example with a typical sequence of interventions, Toku,N. can meet with the expected level
He/she, like Toku,N. interact with others in various level. In some situations, they meet the teacher's expectation and can interact with others in the third level In other situations, they can not interact with others in the third level. We represent like this interaction by the following model (Fig.2)

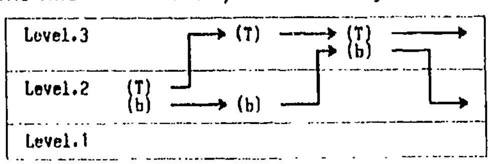

Fig 2

Type C Sai,T.(gr R) can interact with others according to the teacher's expectation Sometimes, he takes an opportunity of initiating the third level's interaction and the teacher and other students follow him (for example, it was observed 27th in April)

There are students, like Sai,T., who can meet the teachers' expectation in every situation. Sometimes they can lead appropriately the mathematics lesson We represent such interaction as follows (Fig.3).

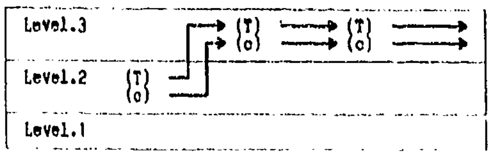

Fig.3

We can find three types of interaction between a teacher and students from the view point of inconsistency

Model of interaction in mathematics classroom

As we described before, a teacher expected to interact with students in various level We suppose that a teacher expected to interact with students at first in the second level, then in the third level and at last, in the third level And some of students (a)/belong to type A, (c)/belong to type C, (b)/belong to type B participate interaction directly in each situation in this order At first, a student (a) can interact with others in the second level. And when the teacher expected to interact in the third level in the following situations, students (c) and (b) can meet the expectation We modeled such interaction as follows (Fig 4) We named this kumagai model of interaction type S (KMIS)

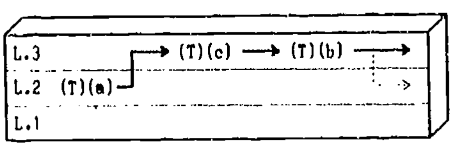

Fig 4

When we focus on individual students, we will present an alternative

model of interaction The analysis, from the view point of individual students, suggests that there is some differences of levels of interaction between the teacher and students and between students. When the teacher expect to interact in the third level, some students ,for example type A, can not meet with the teacher's expectation and they continue to interact in the second level We can discuss in the same way for students type B and type C.

Many inconsistencies in levels occur between a teacher and students Some of them can be observed in various situations easily, but others can not. Because, in a mathematics lesson, all students do not have opportunities to participate verbal interaction directly, express their idea, solutions, and so on But a teacher and students interacted every moments.

We present a model of interaction in a mathematics lesson from the viewpoint of individual students, we call this Kumagai model of interaction type D (KMID) (Fig 5)

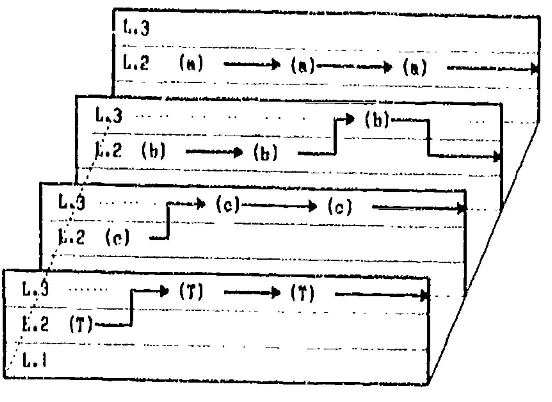

Fig.5

According to model of interaction type S, it seems that there is no inconsistency in level between a teacher and students But according to model type D, we find inconsistencies between a teacher and students , and students and students These model represent surface and deep phase of interaction in mathematics classroom

Conclusions

We proposed the model of interaction type S and type D in a mathematics lesson in Japan. According to this model, interaction in mathematics lesson has inconsistency in deep, even though it has consistency in surface. From the viewpoint of interaction level, these inconsistency in deep is the character of interaction in mathematics lesson.

We need to pay attentions to learning of an individual student. If a student interacts in the second level on every situations, whether he/she learn mathematics as sets of procedures or not Because he/she explain procedures of solution, do not explain rationales, and can interact smoothly, he /she believes that explanation without mathematical rationale is valid explanation.

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Oral Communications

A STUDY OF EVALUATION IN RELATION TO MATHEMATICAL PROBLEM SOLVING

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Students exhibit a variety of reactions when presented with mathematical problems. We realize that these are affective reactions.

In recent years, there have been a number of studies made on the relation between affect and mathematical problem solving.

On the basis of the study made by McLeod (1989, 1992), we want to pursue an enquiry into the evaluation of affect as it should be taken into account in the mathematics teaching in Japan.

In our study we limit affective domain to "belief" and "emotion". In analyzing the mathematics classroom instruction situation, we have limited the question of "attitude" to that of the impartial observer. We have made that limitation because, in Japanese, "attitude" [taido] presumes a third personsubject; and also because the affective response of the student manifests itself by physical and verbal expression.

In an analysis of a classroom situation, we note that the changes in an individual student's facial expression and his words and gestures show clearly when something is at variance with his own understanding. Thus we saw that, in large part, the student's words and gestures and strongly influenced by his confidence in his relationship with those around him, i.e. by social context.

We feel that this should be the object of further study.

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CONSISTENT THINKING IN THE PRIMARY SCHOOL

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Teachers of primary school classes frequently perceive that the children are qualified to think logically, if the process of solving the task or of doing the calculation is based on their personal experience of own actions. The items which are used in a consistent argumentation can be assigned to the following categories (cf. [2])

1. facts
2. conclusions
3. evaluations (opinions, norms)
4. combination of arguments
5. explanations, supplements
6. restrictions
7. conditions
8. hypotheses, suppositions
9. examples
10. proposals to act
11. to abstract
12. agreements
13. refusels, opposition
14. doubts
15. attacks
16. to insist
17. to compromise
18. repetions
19. challenges
20. questions

The pupils' competence of using arguments and of logical thinking will be promoted with the increasing age of the children, and the question is, how to describe this development of the pupils' cognitive abilities. Using some former results (cf. [1]) we taught the same problem to pupils of three different classes: a second, a third and a fourth class. We took a problem which is not a part of the official curriculum, therefore the progress in learning arithmetical skills could not influence the result of the childrens work.

Analysing the pupils' process of solving the problem one may observe several details concerning the development of using arguments, for instance: the decreasing number of suppositions corresponds with an increasing number of explanations from the second class up to the fourth class.

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CROSS-CULTURAL STUDY ON TEACHERS' ACTIVITIES IN MATHEMATICS LESSONS AT ELEMENTARY SCHOOLS

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Teachers' activities in American and Japanese mathematics lessons concerning same lesson units were compared in terms of the linkage of teachers' and children's activities. We considered the connection of three activities: initiative teachers' activities, children's responses and subsequent teachers' activities, as a linkage.

Through the analyses of video-taped lessons, which were collected by the joint project of US and Japan, the differences of the teachers' activities between two countries were made clear.

American teachers gave the children much more questions and instructions than Japanese ones. As for the linkage, American teachers' initiative activities were almost the "ask with the simple questions" requiring yes, no, or simple answers, and their activities after the children's responses were "evaluate the children's responses(right or wrong)" and "reconfirm children's responses". Japanese teachers also showed many "simple questions" but they gave more "process questions" requiring children to answer their thinking processes than American ones. Japanese teachers' activities after the children's responses were "ask with the simple questions", "reconfirm children's responses" and "explain". In addition, we found the tendency of Japanese teachers' asking again "process questions" after the initiative "process questions".

American teachers gave the children so many easy "simple questions", and gave the evaluation and re-conformance after children's responses. On the other hand, Japanese teachers much more intended to draw out children's ideas about their thinking processes than American teachers, and after the children's responses, they performed activities mentioned above to make children examine and clarify their thinking processes with the whole members of the class.

FOCUSING ON SPECIFIC FACTORS OCCURING IN CLASSROOM SITUATION THAT LEADS THE TEACHER TO CHANGE HIS PRACTICE AND MAKE HIM MODIFY HIS ORIGINAL PLAN

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The work presented here come from a study carried out by a team of researchers from different disciplines, who set out to build up methodological and theoritical implements that could help describe and explain math teachers' practice in classroom situation. Thus, the classroom is no longer a place in which theories are applied but becomes a place for the researchers to work out theoritical tools.

Hypothesis :

What comes into account when teachers design their course planning is their beliefs of their discipline, the type of mathematics they have to teach and other phenomenon such as the conditions under which knowledge is transmitted and learning is acquired by their students. This preplanning activity can be seen as a whole complex field that we could analyse in terms of macrodecisions. When these "macrodecisions" are implemented, gaps between course planning and its application in classroom situations come out. We'll analyse these gaps in terms of "microdecisions". These "microdecisions" are triggered off by different factors some totally independant from the learning situation and others tightly linked to it.

We wish to focus more specifically on the processes at work when the teacher takes the decision to alter his original plan when it occurs that the practice in classroom situation differs from what was previously planned. We wish to underscore the nature of these situations, the specific factors coming into play, the change they entail and how these new situations are handled.

Case Study

The observation was carried out in three different forms of the same age group 14-15 years'old who were taught the notion of square root.

During the observation we focused on the following elements of the classroom situation such as :

- the language used by the different actors and specially the teacher
- the pre-teaching tasks and the tools used
- the students'reactions and the teacher's answers
- the contents relevance i.e. what students have to learn
- the students'production

In the presentation of our study we'll try to underscore

- the different types of factors occuring in classroom situations that lead teachers to alter their original plan
- the different types of situations in which these factors come into play
- the presence of reccuring elements in the observed teachers'practice
- the relations existing between the math teacher's «microdecisions» in his classroom and his beliefs of the notion he introduces and between his students'response and the institutional constraints.

Research category : Teachers' beliefs and attitudes (Secondary level, E, N)

The role and function of a hierarchical classification of quadrilaterals

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This paper argues that some students' problems with a hierarchical classification of quadrilaterals is not that of a lack of relational or logical understanding, but rather of a lack of functional understanding (De Villiers, 1987). The viewpoint is taken that a partitioning of quadrilaterals is not mathematically wrong, but simply less useful than structuring them hierarchically. A theoretical analysis is therefore made in this paper of the role and function of a hierarchical classification, and why in this case it is preferable to a partition classification. A distinction with examples is also made between two different types of classifications, namely, *a priori* and *a posteriori* classification.

Lastly it is argued from a constructivist viewpoint that students should be allowed to formulate, compare and choose their own definitions and classifications, even if they are partitional. By now specifically discussing and comparing the relative merits of a hierarchical classification with a partitional one, students will eventually realize the advantage of the former, and make a voluntary transition towards it (De Villiers, 1990).

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GRAPHIC REPRESENTATION OF REAL-LIFE SPACE: FRENCH AND JAPANESE CULTURAL ENVIRONMENTS

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1. Statement of the problem

Study of the development of the mastery of space occupies a privileged position as it involves direct interaction of children with their material, physical and social environment. This requires the use of cultural codes.

Psychologists have been very interested in the origins of spatial concepts of young children. But much of their work has been outside of the school system.

In France, this topic is listed as an objective in the early grades of elementary school, but after the age of 8 years, linking between space and its representations are not teaching objectives. After this age, students are taught only changing from one graphic representation to another (scale drawings) and using the reading of a map for geographical purposes.

The systematisation of this knowledge has been largely haphazard (M-G. Pêcheux).

And the reading and use of maps remains a source of difficulty even for a number of adults.

2. Experiment. Interest in use of French and Japanese environments

The idea of presenting students with a real problem relating real space and its corresponding representation has been used by geographers and also by several mathematics education researchers, René Berthelot and Marie-Hélène Salin in French frameworks and by Grecia Galvez in urban areas in Mexico.

In this experiment, children's representation of space while travelling between home and school is a means to study the manner in which the children structure their space. This is the only example of school learning in which certain kinds of geometric knowledge can be implemented implicitely or explicitely and in which the urban macrospace can occur, a space on which one cannot only use local control.

This activity has been proposed for both French and Japanese elementary school students in the context of Franco-Japanese cooperative research.

Analysis of student productions will permit identification of geometric acquisitions on which one can later build specific learning: treatment of graphic representation, reference points, encoding.

Is the use of two environments, as different as Japan and France, a means for identifying effects of a teaching system and of socio-cultural learning ?

TESTING THE LIMIT CONCEPT WITH A CLOZE PROCEDURE

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The *cloze procedure* (W.L.Taylor) is used in language ability tests of students. A *cloze unit* is an attempt to reproduce accurately a part deleted from a text by deciding from the context that remains, what the missing part has to be. The method was used in a mid term test to check understanding of limits of functions of a real variable. The test group consisted of 39 first year undergraduate students majoring in mathematics, physics or computer science who had been introduced for several weeks to the formal theory of limits, including the (ϵ, δ) definition. The students were given an incomplete proof of a limit argument; the method had been changed such that not only missing parts of a proof had to be filled out, but also false statements had been inserted which they had to detect and correct, with a total of 16 problematic spots in a one page text. Only structural elements had been omitted or changed, not purely lexical ones (in that respect the design of the test was different from what is usual in language teaching where ommissions are systematic, e.g. after every five words).

From the answers resulted a subdivision of the problems (*cloze units*) in three groups. A first group might be qualified as "easy" (more than 80% answers were correct) and was mainly related to memorisation; answers could be derived from exercises the students had been working on (e.g. that any $\epsilon > 0$ in all circumstances) . A second group consisted of "more difficult" problems (from 55% to 70% correct answers) and was related to insight into the mutual relationship between parts of the limit definition (such as if $x \to +\infty$ then $x > M > 0$ for arbitrarily large M). A third group of cloze units was "very difficult" (less than 45% correct answers). The hard problems seemed to be due to two different causes. (i) Conceptual obstacles: the problems required the ability to formulate properly the problem, to translate it into verbal *and* into formalised statements, and (ii) Skills: it became clear that too many students were unable to manipulate formulas involving inequalities and (even worse) involving absolute values.

It is clear that the procedure allowed for cognitive testing. However the question in how far we test understanding versus perception remains undecided. Both factors must be involved in the process, but it is likely that at times a student inserts the right correction because he/she was reminded of what was done in previous classroom sessions, and that was a hint to change or complete the text at hand.

LEARNING MATHEMATICS FOR LEARNING TO TEACH: ANALYSIS OF AN EXPERIENCE

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Conceptualizing the process of learning to teach math as cognitive apprenticeship (Llinares, Sánchez, García, Escudero, 1992) implies defining new characteristics in relation to content, structure and work methods in teacher education programs.

From this perspective, pedagogical performance must be generated to allow the Prospective Elementary Teachers (PTs) to: (i) analyze their epistemological beliefs about the nature of mathematical knowledge/understanding, the teaching of mathematics, the way through learning is produced, the role of the teacher, the pattern of the math class, etc.; (ii) increase their understanding of the different domains of knowledge base for teaching (knowledge of and about mathematics and knowledge of pedagogical content) and (iii) reinforce ways of developing his/her pedagogical reasoning (usage of the knowledge in a teaching situation).

This paper presents the characteristics and first results of the pedagogical experience developed in the Primary Teachers Education program at the Universidad de Sevilla. Specifically the objectives of the modules of this part of the education program which includes:
(a) analyze, and if it is necessary modify, the PTs' conception of the previously mentioned points in (i),
(b) increase the understanding the PTs have of the different characteristics of the activities that generate mathematical knowledge (notion of proof, guess/conjecture, ratification, hypothesis, etc.)

The activity in the working modules was articulated around: (1) problem solving activities in small groups, (2) debates in large groups where procedures, results, etc. obtained in the small groups were/are shared, (3) written diary of the work carried out by the group, (4) session to analyze the process followed in relation to the learning, the features of teaching, the role of the teacher, the nature of mathematic knowledge, what it means to know mathematics, etc.

The information coming from the observations of the work in class, the questionnaires used, interviews, and the documents produced in the work groups. The analysis of this data corpus allow the characterization of the nature of the changes in the conceptions, the generation of pedagogical dilemmas, etc. of the PTs. The results should allow the refining of the initial theoretic outline.

FORMATION OF AN OPEN COGNITIVE ATTITUDE

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Our experience tells us that the process of education becomes more attractive when a student can choose a style of cognition in accordance with his actual abilities. How to teach a child the ability to choose? One of the answers is as follows – an open cognitive attitude should be developed. We shall suggest the conditions which, in our opinion, should be taken into consideration by a teacher wishing to achieve this purpose.

I. Developing the ability to analyze one phenomenon in different ways. We shall offer the following simple example.

It is necessary to solve a quadratic equation. This necessity arose for the first time, the method of solving such equations had not been discussed before. This situation can be analyzed by at least two or three methods. One of them, a consumer's method, consists in finding the information about solving the equations similar to the given one in reference books. Another, a reseacher's method, consists in looking for the answer to the question "Is it possible to rearrange the equation so that the well-known technique of the type $O^2 = m$ might be used?" The third method may be bringing this equation to the form $(x-a)(x-b) = 0$.

II. Designing such situations in education due to which a student discovers the existence of various, sometimes even opposite views on the same problem. Developing respect for such views, for somebody else's opinion.

We arrange studying various branches of mathematics in such a way that a student might look at the problems through the eyes of a physicist, biologist, chemist, and even a businessman. Note that for developing the ability to take into consideration different views on the same problem not a single opportunity for solving problems by different methods should be missed.

We shall cite again the problem of solving a quadratic equation. A student can be "led" to the formula of its solution in a variety of ways: someone will start to separate out the squared binomial, someone will unearth in historical books the method of multiplying the two parts of the equation $ax^2 + bx + c = 0$ by $4a$, someone will bring the equation $ax^2 + bx + c = 0$ to the form $x^2 + px + q = 0$ and then use the substitution $x = y - \frac{p}{2}$, someone will prefer geometry and will start investigating the area of respective figures.

III. Organizing education in such a way that a student might learn about the existence of different styles of cognition, might be able to combine them and eventually work out his own style.

When introducing notions and organizing work aimed at mastering them we try to select such material and use those students' suggestions which may present a situation in different ways, logical, isual, practical, in the form of a play, and so on. If a teacher's activity in this direction is a success not only the emotional background of education but also the quality of learning will improve.

PROFESSIONAL INVESTMENT OR DISCOURAGEMENT

DILEMMAS OF A SECONDARY SCHOOL MATHEMATICS TEACHER[1]

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Among Portuguese teachers, there is a widespread movement of sympathy regarding new orientations for mathematics education. These orientations have been promoted by the Association of Teachers of Mathematics and teacher training institutions and got general recognition in the new mathematics curriculum that is now being generalized in schools. Many teachers that support such orientations, are involved in some sort of innovative activities. In this study we tried to identify which were the reasons that led them to get involved in such activities and how they felt going about them.

Luísa is a secondary school mathematics teacher. In her present school she joined other teachers with similar interests, concerns and views regarding the teaching of mathematics. Just a few months after she arrived to the school, Luísa was elected head of mathematics by her peer colleagues. We interviewed Luísa and talked with her several times. We were able to figure out some of the dilemmas and contradictions with which she carries her professional activities:

1) **Participation in innovative activities.** She likes to be involved in projects with other colleagues but she feels tired because of all the difficulties that need to be faced (lack of conditions in the school, lack of support of the school administration, hostility of most of the mathematics teachers, lack of time, and familiar responsibilities).

2) **Classroom practice.** She would like to make her classes "in a different way", but she has trouble in creating the adequate climate and managing the extensive curriculum.

3) **Views of mathematics.** She indicated that her involvement with this discipline is based in her liking to solve problems and facing challenges but she does not appear to be much involved in mathematical activities other than the necessary for her regular teaching duties.

In this presentation we provide data concerning Luísa's conceptions and professional activities, sketching some tentative hypotheses regarding her dilemmas and contradictions as a mathematics teacher.

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1 This paper reports research made by the Project DIC (Dynamics of Curriculum Innovation and Development Processes), supported by JNICT under contract No. PCTS/P/ETC/12-90. Besides the authors, members of this project include Paula Canavarro, Leonor Cunha Leal and Albano Silva.

Levels of composite unit in single addition

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The purpose of the study is to identify what levels of composite units are formed for the first graders in single addition and to discuss how the tools (concrete objects, fingers, or mental calculation) and the strategies relate to the composite units. In this study the composite levels with respect to the strategies with concrete objects and fingers are mainly discussed. Steffe and Cobb(1988) used three concepts : counting type, integration operation, and strategies, for describing the children's development in solving addition and subtraction problems. In this study, the concept of "composite unit" is applied to the case of single addition by assuming that the composite levels of the 1st addend and the 2nd addend are not always the same.

Through the clinical interview to the first graders, with respect to the levels of composite unit , the following levels were identified for the 1st addend.

(1a) NONEXISTENT LEVEL : Composite unit is missing.

(1b) VISIBLE LEVEL : A collection of visible objects makes a composite unit.

(1c) ERASABLE LEVEL : A collection of erased objects makes a composite unit.

(1d) ABSTRACT LEVEL : A collection of abstract units makes a composite unit.

For the 2nd addend, the following levels were identified :

(2a) NONEXISTENT LEVEL : Composite unit is missing.

(2b) CONCRETE LEVEL : A collection of concrete objects makes a composite unit.

(2c) FINGER PATTERN LEVEL : A finger pattern or a collection of fingers makes a composite unit.

(2d) SPATIAL PATTERN LEVEL : A collection of invisible objects in space makes a composite unit.

(2e) ABSTRACT LEVEL : A collection of abstract units makes a composite unit.

By applying Steffe's concept of composite unit to the analysis of the children's solution processes, we have the following findings:

(1) Children's composite levels of the 1st and the 2nd addends have a relationship with their strategies.

(2) For some children who use the concrete objects, a collection of visible objects makes a composite, but for some children the composite is missing.

For the children's ways of using fingers, however, there also seems to be some influences of their cultural experiences. So more information from that view will be more helpful for interpretation.

LANGUAGE MODES IN MATHEMATICS CLASSROOM DISCOURSE: A WINDOW ON CULTURE

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Ideas come from what we see, hear, feel, taste, smell, and kinesthesize, and from our processing of these: what we imagine, intuit, project, and infer from what we sense. Surely it is the things we have not heard before that constitute fertile input: what then constitutes fertility? New ideas, new situations, new cognitive conflicts, and new language. Explicitly supplying language to the mathematics learner creates potential, creates awareness of language, creates awareness of mathematical knowledge, method and strategies.

The rationale for language-based teaching in mathematics is putting the focus on the active language modes of mathematical behavior, such as describing, comparing, categorizing, choosing, and justifying, rather than on the medium, mathematical behaviors which are largely non-verbal (but not language-free), such as interpreting a problem, transforming, and solving, yet maintaining the same conceptual/procedural component. Would this not enhance metacognitive skill as well as language skill?

Consider four gross modes of oral language used in solving/teaching the solution of a given problem:

explaining- conceptual/procedural information, view of problem, strategy choice

help-seeking- utterance reflects conceptural/procedural/metacognitive need

coaching- agenda negotiation, support for student search for c/p/m data

collaborating- solution-seeking, exchanging progress data, discussing metacognition.

Each of these modes calls for specific language (vocabulary and discourse patterns). In the mature (adult) world of mathematics, a proof will be streamlined, elegant. But does this represent or even suggest the thought that generated the proof? Teacher utterance needs to be evocative, fertile input for the student. Student utterance, especially help-seeking utterance, ought to evoke an impression of student mind-state.

The focus of this presentation, the Macintosh HyperCard stacks Language Modes in Mathematics and Math Language Functions, are tools for exploring the above language issues. Ever narrower classifications of utterance eventually lead to concrete discourse elements. This is of importance in analyzing the roles and responsibilities of both teacher and learner. Once refined, such a tool may be of use in cross-cultural comparison of language use in mathematics teaching/learning.

A STUDY ON A TREATMENTS ABOUT MATHEMATICAL PROBLEM POSING

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It is often said that it is important for students to acquire creative ability. So the importance to enable students to develop their ability to find out more adequate questions in a situation and derive problems from them will be increased. There are some attempts which treat the activities of problem posing (Simizu:1935, Brown and Walter:1983, Takeuchi and Sawada:1984, Hashimoto and Sawada:1984), but I think it should be treated with various approaches. The aims of this paper is to propose various approaches which will contribute the summarization of them.

(1) The approach to derive problems from a given situation.
It used to be done after the world war I in Japan.
I don't treat here.

(2) The approach to derive problems by using some components of a problem.
ex1. Derive problems by using a circle(or string).
ex2. Derive problems by using a right triangle.
ex3. Derive problems by using a cube.

(3) The approach to derive new problems by changing parts of the given problem.
It is familiar with Japanese present elementary textbooks. I don't treat here.

(4) The approach to derive problems by using solving methods or solutions. This approach allows the treatments which are done both after solving a given problem and from the beginning of the classroom teaching.
ex1. Derive problems such that we can solve it with the formula "1+2=3".
ex2. After solving a given problem, a teacher asks the following question. Derive problems that we can solve by using the linear equation with two variables.
ex3. Derive problems that we can solve by using the Pythagorean theorem.

While students might derive problems with passive activities for the first time, but we will be able to expect them to be able to derive problems with positive activities by repeating these approaches.

It has a problem that we have to examine the difficulties between these approches for students, and also consider the methods of evaluating the problems.

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THE CHANGES OF CHILDREN'S NUMBER-COGNITION IN THE LEARNING/TEACHING PROCESS

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We have the over-thirty-year experience of teaching the numbers and calculations based on the Suido-Method.(1)(2) The result is so successful that we can believe strongly that All school-age children, without an exception, are able to understand the basic ideas about the numbers and calculations. This empirical fact is supported by the logical analysis about the teaching materials and contents. But, generally speaking, psychological processes are not necessarily the same as logical ones. So, the interest of this research is in the psychological processes or the cognitive changes that occur there and that are needed to learn more.

The observation was done for the Canadian and Japanese children who had serious difficulties in learning elementary mathematics. For example, a Grade 5 child miscounted the one-digit additions like 6+7 although he knew the way of making an answer by using fingers. And he told that he could't calculate 23+32. But, after three months, he understood the meaning of multi-digit additions and became able to make correct answers for them and to explain them by drawing pictures.

As the Suido-Method is materialized by the "Tiles", the concern was the cognition about them. They have the characteristics as shown in the diagram A and B. The tiles are semi-concrete or semi-abstract and work like a ladder from the concrete to the abstract. The image such as the tiles is called the "Schema."

Abstract
↑
Semi-Concrete (Semi-Abstract)
↑
Concrete
− A −

The result confirmed that the A and B were valid to appreciate the levels of the developmental cognition of children. The children accepted the tiles as concrete things, operated them, and drew them as models. At the beginning, the children drew the single five ones although they operated one chunk five. This level should be distinguished from the cognition of schemas. Because the images of tiles didn't work when the actual or pictured tiles disappeared. The process from the models' to the schemas' was revealed by observing what kind of tiles the children liked to operate/draw and how they operated/drew the tiles. At the level of the schemas', they became able to think in the various ways by operating the images of tiles. The cognitive level of symbols was distinguished from the one of schemas. Because there was a phase where a child could operate the schemas, but couldn't operate the symbols. Eventually the children became able to operate the symbols as if the symbols were concrete.

World of Concrete Things	⟷	World of Models	⟷	World of Schemas	⟷	World of Mathematics
			− B −			

Notes: (1) H. Toyama and K. Ginbayashi: Suido-Hoshiki Nyumon Vol.1,2, Kokudo-Sha, 1971.

(2) The members of the Asociation of mathematical Instruction(AMI) have been doing the study and practice.

ON ROLES OF INNER REPRESENTATIONS IN MATHEMATICAL PROBLEM SOLVING

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The purpose of this study is to make clear the mental mechanism of understanding through observing one's mathematical problem solving activities. When one solves a problem, the person have to make some representation (sometimes inner representation) which is the model of the original problem situation in some sense. So I think that we can explain one's understanding in problem solving by identifying what kinds of representation are employed.

I made some problems sets which have a same mathematical structure and different superficial contexts, and made the investigation using these problems. For example, one of the problem sets is following.

[1] Two baseball players A and B took part in both tournaments in spring and summer. The batting average of A is more than that of B in each tournament. Then is it possible that the total average of B is more than that of A ? Answer by Yes or No.

[2] Each of two racing cars A and B has a low gear and a top gear respectively. The speed of A in each gear is more than that of B. Then is it possible that B wins the race. Answer by Yes or No.

Analyzing the results of my investigation, I found three characteristics of occurring representations in mathematical problem solving.

i)When it is difficult to make the useful representations for given problems, some familiar and similar representation is used as a substitute for that.

ii)When the suitable representation for given problems already exists in ones knowledge, that is used just as it is in the problem solving process.

iii)When the more general representation exists in ones knowledge, in spite of the existence of the simple and useful representation peculiar to a given problem, the general one is ready to be used.

INTUITIVE PROBABILITY CONCEPTS IN SOUTH AFRICAN ADOLESCENTS

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Probability has not been a formal part of any South African curriculum until relatively recently. In 1992 a syllabus was introduced which required that probability be taught to pupils in grade 9 (14 - 15 years of age normally) . Research in progress uses questionnaires adapted from Green (1982) to investigate the understanding of probability amongst these pupils in the Witwatersrand and Transkei areas of South Africa. In order to ascertain differences that may exist between rural, township and urban school pupils in their naive understanding of the concepts involved, schools from such areas were included in the sample even though some of these schools might not have adopted the new syllabus.

The questionnaire has, to date, been applied in 14 volunteer urban schools in the Witwatersrand area. The questionnaires are due to be applied in township and rural schools in the near future. Analysis of data from the 870 pupils who have taken `he questionnaire thus far, indicates that their level of understanding is on a par with that of pupils from the same age group in the United Kingdom who were involved in Green's (1982, 1988) research. Performance of the 360 pupils who took the questionnaire again after tuition is not significantly better although the reasons given for responses are generally more accurate.

Further data and analysis on the performance of township and rural pupils will be presented. Discussion will relate the understanding and misconceptions evinced to the experiences of the pupils concerned (Piaget & Inhelder, 1975).

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(This project is funded by the Centre for Science Development.)

Consider the Particular Case *

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The analysis of particular problems for the application and illumination of principles has long been a central activity in the physical sciences. The attempt to take guidance for the human sciences from the physical sciences has often been unconvincing and subject to criticism. Instead of borrowing notions from the physical sciences, I reflect here on the process of problem solving in a particular case and from that process abstract objectives, methods, and values which can help us identify and solve our own problems and judge the value of those solutions. My aim is not to develop a single, universal method from this example. I present an analysis of how we can proceed to conclusions of interest that we can have confidence in. I begin with a focus on the importance of analyzing particular cases. I make use of a particularly illuminating description by Richard Feynman of a complex physical effect in quantum electrodynamics [1] as a concrete example of a specific form of analysis. I use that worked example to illuminate the meaning of computational models of learning I have constructed. I believe that this comparison is useful in understanding relatiohships between details of particular cases and epistemological analyses based on computational modeling.

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* This presentation will focus on themes from a chapter "On the Merits of the Particular Case" to appear in **Case Studies and Computing**, Robert Lawler & Kathleen Carley, (forthcoming, 1993, Ablex).
1 My primary source for the physics of this comparison is a series of popular lectures by Richard Feynman, published in **QED: The Strange Theory of Light and Matter**, (Princeton; Princeton University Press, 1985).

INNOVATION IN PRACTICE: THE DIFFICULT WAY OF BEING A TEACHER[1]

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João Pedro da Ponte
Universidade de Lisboa

In the last few years, interest by new teaching approaches led Portuguese mathematics teachers to develop innovative experiences in schools. The project DIC was designed to study these experiences, figuring the most important forces behind them, analyzing their inner dynamics, scope, successes, failures, and possible implications. This paper focus in the concerns of Beatriz, a mathematics teacher that is the leader of several activities at her school. In special, she was involved in an extended experience of using the graphic calculator with 10th and 11th grade classes and in the creation of a "Game Room" for the school.

She completed a mathematics methods course in a teacher education program sressing ideas such as applications of mathematics, problem solving, use of calculators, group work, etc. Coming to this school, Beatriz was not well accepted by most of the mathematics teachers, just getting the sympathy of a few of them. In spite of her small teaching experience, she was appointed as the supervisor of successive groups of student teachers. She hoped that her interests and concerns would be shared by these student teachers and that at least some of them would remain afterwards at the school, enlarging the innovative group. We discuss what we feel to be some of her major tensions and personal struggles regarding her participation in these activities. For example:

a) Beatriz does not want to recognize her special role in the group. She prefers to regard herself as one like the others, although for her colleagues she is the unquestionable leader. On one hand she enjoys being part of a group and on the other hand she privileges her personal autonomy.

b) What effort to put in innovative activities in the mathematics classroom and in school activities? How to balance them is a problem she did not solved yet.

c) She defines herself as being more oriented towards action than towards reflection, but the fact is that too much action and little reflection is driving the group towards dead ends and increasing inner conflicts.

[1] The research reported in this paper was made by the Project DIC (Dynamics of Curriculum Innovation and Development Processes), supported by JNICT under contract No. PCTS/P/ETC/12-90. Besides the authors, members of this project include Henrique Manuel Guimarães, Paula Canavarro, and Albano Silva.

Conceptual Analysis of a Mathematical Modelling Episode[1]

João Filipe Matos, Universidade de Lisboa

Susana Carreira, Universidade Nova de Lisboa

Some of the recent new directions for Mathematics Education point out the importance of making mathematical knowledge meaningful to students, namely through the introduction of modelling activities and real life problems in the classroom. However, we found that very little is known about the nature of difficulties involved in such activities. Moreover, there are many questions to be answered concerning the nature of the cognitive processes that take place in mathematical modelling activities.

We had the opportunity to collect data regarding some of these issues during an experience developed in a 10th grade class where mathematical modelling and applications was a major orientation. The problems introduced were related with the study of Functions and Analytical Geometry.

One of the situations presented was the "paper roll" problem. Students were asked to create a mathematical model to relate the length and diameters (inside and outside) of the roll and to explore it in order to get some results such as the length of paper existing in a given roll. The students worked in this activity during a two hours session, in a regular classroom environment where the students had the possibility to use an electronic spreadsheet. One group of four students was videotaped and observed by one of the members of the project team. Students also presented a written report on their work.

In this short presentation we will focus on the analysis of an episode which we believe is a good example for examining some questions under investigation.

[1] This paper presents part of the results of an on-going project (MEM - Modelação no Ensino da Matemática) funded by JNICT and IIE. Besides the authors, other members of the project team are João Ponte, Manuel Saraiva, Graciosa Veloso and Paulo Abrantes.

A COMPARISON OF THE UNDERSTANDING OF MULTIPLICATION AMONG ENGLISH AND PORTUGUESE CHILDREN

Peter BRYANT[1] , Luisa MORGADO[2] , Terezinha NUNES[3]

Parallel studies are currently being carried out in Oxford, England, and Coimbra, Portugal, on 7-9 year olds' understanding of multiplication. British children's school experience with multiplication is mostly focused on problem solving and manipulatives are often used. Little or no attention is given to memorizing multiplication tables at this age level and no teaching of the written algorithm was observed.. Portuguese children receive much drilling in multiplication tables and the written algorithm both as computation exercises and word problem; little emphasis is given to the use of manipulatives. We expected that these different practices would affect children's understanding of multiplication.

Method. Four word problems and four computation exercises were used. The word problems varied in type (one-to-many correspondence, repeated grouping, area, and combination), used small numbers, and could be solved with the support of manipulatives (miniatures of the objects). Two testing conditions were examined, one in which the children received enough manipulatives to represent the situation and find an empirical solution through counting, and a second condition in which only a sample of materials was available so that partial representation of the problem was possible but an intellectual solution was required. The computation exercises involved larger numbers and aimed at exploring children's understanding of commutativity and distributivity.

Results. Condition of testing affected the type of strategy used by children (more empirical solutions were observed when enough materials were given) but not rate of correct responses among the older children and affected both dependent variables among younger children. Problem type affected both children's ability to use the materials to model the problem and rate of correct responses.

The same order of difficulty across problem types was found among English and Portuguese children and no significant differences emerged in their ability to correctly represent the problem situation with the manipulatives. However, Portuguese children were more apt at calculating. English and Portuguese children also differed in their problem solving strategies: English children frequently indicated that two of the problems could be solved either by addition or by multiplication whereas Portuguese children also displayed their undrestanding of the relationship between addition and multiplication though a higher percentage of recognition of the property of distributivity in the computation exercises although they were, once again, weaker in computation skills.

[1] University of Oxford

[2] University of Coimbra; presenting author.

[3] University of London

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CONCERNING THE CHARACTERISTICS OF PROBLEM SOLVING BY STUDENTS IN SCHOOL FOR THE DEAF

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It is noticable that whenever considering the idea of "a good teaching in mathematics", there is a tendency to consider unconsciously such a class only in relation to normal, healthy children. However, were this idea considered in relation to all students, then it would be apparent that such classes for both normal, healthy and handicapped children would need to be contemplated. In accordance with this critical difference and with the results obtained last year (from an analysis of the problem solving characteristics of both normal, healthy children and handicapped childrenwhen given the same mathematical problems), an analysis was made of the characteristics of problem solving by two different classes in different grades of a school for the deaf. One class was given the problem last year, the other this year.

> The problem was to join twenty distinct houses by a direct telephone line, with only one line connecting any two houses, and then to calculate how many separate lines were necessary to ensure that all the houses were connected by the telephone line.

As a result of analysing the solutions presented by the students, the following two points were apparent as their general characteristics.

1) Many students drew twenty detailed houses.
2) All students drew all twenty of the houses, compared to normal, healthy students who generally drew two or three houses only and then represented the remainder by dots only.

These two characteristics suggest that it is difficult for students with hearing impediments or disabilities to acquire or develop an ability for abstraction.

THOUGHTS ABOUT ALGEBRA: TEACHERS' REFLECTIONS AS LEARNERS AND AS TEACHERS

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This presentation examines the thoughts of six elementary and middle school teachers as they study a semester of algebra. In addition, student interviews designed and administered by the teacher will be discussed and related to the reflections of the teacher as a learner.

The information in this paper is based upon data collected from six teachers who took the algebra course as one course in a program of seven math courses and three educatic courses designed to lead to an Elementary Mathematics Specialist Credential. The students of this class consisted of three middle school mathematics teachers, one fourth grade teacher, one second grade student teacher and one student who was in her graduate year preparing to be an elementary school teacher. The algebra course was not a traditional course. Rather, the focus was on functions and the investigation of families of functions. Each function was introduced through an applied situation and the family of functions were studied as needed in developing the mathematical model of the applied situation. The text for the course was entitled *Computer-Intensive Algebra* by Fey and Heid. Through out the course technology was assumed and used as an exploratory tool which encouraged investigations to move across a variety of represen ational forms including manipulatives, pictures, tables, graphs, and symbols. Students worked in groups and presented the results of their explorations in wr tten papers and oral reports. Thus both the content and the classroom environment violated perception of what "Algebra" should be.

Weekly, during the course, the teachers shared their beliefs as they responded to weekly journal questions. In some cases, the teachers were also asked to interview their own students and relate the reaction of their students to the same question. Teacher perceptions of algebra, variables, functions, and modelling were traced across the 15 week semester through responses to more than 30 journal items.

A final project required that each teacher interview at least one student. Of interest is the contrast between the teachers beliefs as they played out in class and the interviews they developed. It is the combination of questions, probes and interpretations of the responses of their own students which begins to examine linkages between the knowledge gained and beliefs challenged in the college algebra class and the mathematics expectations held for their own classrooms.

A PROJECT TO LINK THE ARITHMETIC OPERATIONS AND THEIR USES

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Perhaps the main justification for including the arithmetic operations (addition, subtraction, multiplication, division) in the required curriculum is the need for an informed citizen to function in the many quantitative situations encountered in everyday life. Calculating skills are empty skills if they cannot be applied. Yet, children's performance on the most common school form of applications of arithmetic--the typical "story" problem--is dismaying when any degree of complexity is introduced (e.g., multiple steps, extra information). Indeed, recent international work has suggested that even a fairly good performance on one-step story problems may be tainted by the common use of ad hoc methods which have only limited applicability (e.g., Greer & Mangan, 1986; Sowder, 1988). Although some of these strategies may give success on many one-step story problems with whole numbers, their use with multistep problems or problems involving fractions or decimals is unlikely to give correct solutions. Furthermore, these immature strategies provide a weak background for approaching algebra story problems.

In this project, a team of experienced teachers and a university mathematics educator developed materials to give a greater emphasis to uses of the operations. Nine classes of seventh or eighth graders in two schools were involved in the tryout, five using the materials and four not. One of two forms of a pretest was given early in the school year to each student, with the same form given in May. In addition, two students from each class were selected and interviewed. The test results and the interviews were somewhat disappointing in that the users of the project materials did not clearly outshine the control students. Some students had responded to the thrust of the materials, but others seemed to continue to use the immature strategies. The tryout teachers were surprised, since their perceptions were that the material was "understood" when their classes studied it. Old habits are apparently difficult to change; rather than to try to correct bad habits in middle school, a likely more sensible approach is to emphasize uses of the operations from the early grades on.

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The project described above was supported by the National Science Foundation (MDR-8850566). The opinions are those of the author; the Foundation does not necessarily endorse any assertions in this report.

Directed and negative numbers, concrete or formal?

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Introduction

Directed numbers refer to values of magnitudes or quantities (temperature, above and below water level, possession and debt, and so on), while negative numbers refer to formal mathematical constructs; such numbers are applied as by-products or constructs to support the execution of algebraic operations (cf. Fischbein, 1987).Several famous mathematicians got in conflict with these two manifestations of negative numbers as the history of mathematics shows.(cf. Glaeser, 1981).

Hankel settled the conflict definitely in 1867 by means of his principle of permanence. This principle implied:

- the definition of negative numbers as formal mathematical constructs, having their own operational structure obeying laws like the distributive law of multiplication over addition; and
- the search by mathematicians for concrete models to represent multiplication (and division) was ceased definitely then.

Mathematics programs for secondary education still reflect this historical struggle albeit more often than not unintentionally. On the one hand they show forced attempts to make the operations with negative numbers concrete by having riden negative trains backwards, or guards ....(cf. Chulvers, 1985). On the other hand they reveal the avoidance of any connection with concrete magnitudes or quantities (cf. Liebeck, 1990).

My oral presentation will deal with a teaching experiment in which both learning strands are taken into account:

Concrete magnitudes or quantities

Changes in occupation of buses at busstops serve as a context for a number - pair approach.The numbers of change at the stops can also be negative.

Remark: Water level in locks and (average) temperature will follow later.

Formal constructs

Colums subtraction from left to right (method of shortages).

The underlying priciple is that of plausibility, that means:

- meeting Hankel's permanence principle by appealing on abilities and skills in the learners that will enable them to solve the problems both via the detour of a mathematical framework they know already and within the negative numbers, thus making plausible the rules and laws to be met by the negative numbers.

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CROSS-CULTURAL STUDY ON TEACHERS' INTERVENTION IN TERMS OF INTERACTION BETWEEN TEACHERS AND STUDENTS

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The concern of the present study is differences on teaching behavior in the both countries. Stigler et al (1987) suggested that typical teaching behaviors in the both coutries also differed each other. However, they did not indicate concrete figure in terms of measureable behaviors. Therefore, this research intends to clarify differences on teaching behavior in mathematics classes.

Method

Participants. In Japan, lessons of 7 classes of 2nd grade were filmed. In the US, lessons of 6 classes were filmed. Their lessons were all a unit of subtraction in the second grade.

Subject matter. A subject matter introduced in these classes was subtraction with borroring. In Japan, subtraction of 3 digits was taught, and in US subtraction of 2 digits taught.

Protocol Analyses. All lessons were written up as protocol. Activities which followed after incorrect responses in children were classified into one of the following category. (1)Evaluate student's responses in terms of correct or incorrect ones (2)reconfirm student's responses, (3) ask question which require student's short answer, (4)ask qestion which require children' thinking process, (5)explain, (6)indicate something to students, (7)designate student, (8)instruct, or (9)scold. Further, we set teachers' activities as following; (1)activities which teacher elicited pre-existing knowledge from students,(2)teachers activities which let students indicate similar ideas or compare differenct ideas, (3)teachers activites which let students do common activities in class.

Results and Discussion

Japanese students showed very few incorrect answers in classes. However,the US students indicated more errors than Japanese ones. The US teachers tended to evaluate student's response or ask question which require short answer.

Total frequencies of this activity on eleciting pre-existing were 17 in the US and 33 in Japan.Japanese teachers clearly elicited student's pre-existing knowledge in their classes. This might sigest that Japanese teachers tended to relate current lessons with the pre-existing knowledge.

Total frequencies in comparing and indicating behaviors were 3 in the US and 19 in Japan.The US teachers did not compare among ideas or indicate similar or dissimilar ideas compared to Japanese teacher.

This research is a part of joint research with Jim Stigler of UCLA and Giyoo Hatano of Dokyo University.

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Poster Presentations

[1]PROBLEM SOLVING: TEACHING, ASSESSING AND TEACHER EDUCATION

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António Borralho, Universidade de Évora, Portugal
Domingos Fernandes, Universidade de Aveiro, Portugal
Gertrudes Amaro, Escola Superior de Educação de Castelo Branco, Portugal
Isabel Cabrita, Universidade de Aveiro
Helena Fernandes, Escola Superior de Educação de Bragança

1. MAIN GOALS

- Developing and analyzing heuristic methods of instruction in mathematical problem solving, particulary through the utilization of problem-solving strategies.
- Developing and investigating assessment methods and techniques, coding schemes and rubric scales which take into account the complexity of mathematical problem solving and, particulary, processes involved when students are attacking the solution to a problem.
- Developing and investigating materials which are designed for education of mathematics teachers under a problem-solving perspective.
- Analysing effects of programs of instruction which emphasize problem solving on teacher's attitudes, conceptions and pedagogical skills, and on elementary and secondary student's attitudes, conceptions and performances.

2. PARTICIPANTS

- About 120 pre-service teachers who are in their junior or senior year.
- About 140 in-service teachers from five distinct regions of de country.
- About 700 elementary and secondary students of a sample of 24 teachers.

3. GENERAL PROCEDURES

- First Year - **Teaching step**: heuristc methods of instruction in mathematical problem solving will be investigated in the preservice mathematics teachers context; materials will be developed to be used in the education of in-service teachers.
- Second Year - **Assessment step**: assessment models and techniques will be investigated in the preservice mathematics teachers context; materials will be developed to be used in the education of in-service teachers.
- Third Year - **In-service Teacher Education/Teaching/Assessing Step**: effects of the previously developed materials on mathematics inservice teachers' attitudes, conceptions and pedagogical skills will be investigated; also, the atittudes, conceptions, processes and performances of the students of those teachers will be analyzed.

4. METHODS AND INSTRUMENTATION

- Qualitative methods will be used to analyze teachers' and students' conceptions, attitudes, and processes.
- Quantitative methods will be used to compare students' means on problem-solving tests.
- Instruments: observation grids, interview protocols, checklists, coding schemes, rubric scales, and mathematical problem-solving tests.

[1]This project is supported by JNICT (Junta Nacional de Investigação Científica)

TEACHER PERCEPTIONS OF MATHEMATICS AND PEDAGOGY

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Thompson (1984) drew specific attention to the role for pedagogy that teachers' conceptions of mathematics might play, but argued that this had "largely been ignored" (p.105) in the literature. Further, Chacko (1982) identified teacher attitudes as the most important predictors of affective outcomes, whilst Shaughnessy et al (1983) found that teacher related variables had the strongest relationship with both achievement and attitude, especially in the early secondary years. Lerman (1981) placed perceptions of mathematics within movements described as "quasi-empirical" (problem-solving based, with heuristic progression) and "Euclidean" (knowledge-centred and foundation-based). Both perspectives have been incorporated in curriculum material recently introduced for the beginning years of mathematics instruction in NSW, an Australian state. Of interest, then, in an exploratory study, was whether teachers could accommodate both transmissive and process-oriented views.

An exploratory factor-analytic study piloted a thirty item instrument with five-point Likert scale to incorporate philosophical statements from the NSW "Statement of Principles" underpinning Syllabus documentation. A four factor solution with items loading > 0.45 and interpretable sub-scales yielded factors tentatively described as:

- the product-purpose of mathematics teaching;
- the process-purpose of mathematics teaching;
- the nature of mathematics; and
- the value ascribed to mathematics.

Refinement of the instrument and further field testing suggested a necessity to identify the factors according to a frame of reference incorporating pedagogical considerations such as conservative transmission, higher order aspects of learning, the rationale for the inclusion of mathematics and its value in the curriculum.
Teachers of first year secondary students exhibited greater instructional conservatism in response patterns than other teachers and less orientation towards problem solving. Further instrument refinement is still to be undertaken, with international responses vital to confirm the postulated factors.

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PROPORTIONAL REASONING OF ELEMENTARY SCHOOL CHILDREN

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Proportional reasoning involves both qualitative and quantitative methods of thought. According to Piagetian theories, quantitative proportionality schema does not appear until adolescence (Inhelder & Piaget, 1955). On the contrary, Tourniaire(1986) focused on children's quantitative reasoning, and showed that elementary school children had some understanding of the concept of proportion.

This study examined the relationship between qualitative and quantitative proportional reasoning of elementary school children. Two ratio types, velocity and thickness, were chosen. In the qualitative reasoning task, children were asked to determine the direction in which the numerator of the ratio would change (decrease, stay the same, or increase in value), when the denominator of the ratio changed and the ratio was constant. In the quantitative reasoning task, children were given three components of two equal ratios and were asked to solve for the fourth component. These tasks were individually administered to thirty third-grade, twenty-nine fourth-grade, and thirty fifth-grade pupils. None of the classes had received instruction on proportions before this study.

The results were as follows:

1. Qualitative reasoning preceded quantitative one for each ratio type, and success rate on quantitative reasoning task increased with age.

2. Children's ability to reason qualitatively appeared to have a strong impact on their performance on quantitative reasoning task.

3. Different ratio types, velocity and thickness, showed an effect only on qualitative reasoning of third graders.

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WHEN REFERENTS WERE USED. BETWEENESS PROBLEMS IN RATIONAL LINE. A CASE STUDY.

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Background.

Recent ideas about building knowledge of students (Kieren & Pirie 1992) presented a line of understanding as a dynamic process: knowing, image making... through to inventising. Our last empirical results on rational number knowledge with 10-11 years old students focusing on density preconceptions (Giménez 1991) found a lot of archimedian answers before infinity observations according to Kieren-Pirie's perspectives in number line situations which could be understood according to this scheme. On the other hand, many students didn't have an intuitional knowledge of density because their lack of referents. What will happens when powerful referents appeared (as number line in Streefland 1991)?

Case study presentation. Aim of the study and methodology.

Because of this reasons, we decided to explore some young students (10-11 years old) having number line referents facing density tasks. Emmanuel (10 years old) was in 5th grade class in a regular school in Jerusalem (year 1991-1992) and have a middle level in the classroom. Many situations, referents and meanings of fractions in the topic were used in the learning process: number line context, equivalence, complement to 1, ordering rules of fractions, segment border points as fractions, distances introduced on a metric way... But he never was asked to find distances in fraction neighbors to see if a fraction is greater than other.

The aim of this exploratory study was to find what happens when some betweenness tasks were asked to this student, in order to clarify the links of knowledge trying to solve some density tasks. The basic interview tasks were: (a) finding intermediate points, (b) middle point observations, (c) missing points dividing a segment in ratio-nal line, (d) density questions about how many fractions between two given ones.

Some observations and conclusions.

Observing the answers of Emmanuel (10 years old), we found that the internalization of symbolic algorithmic knowledge of equivalence and number line sense, appeared as a powerful elements to have a middle point strategy, and subsequently an intuitional knowledge of the density property. But the evenness stage in the process act as a distractor to made relevant his knowledge in a more wide sense. The student needed to reformulate their image about equivalence in number line to solve an odd sharing betweenness problem because of anchorage halving ideas. Better strategies did not appeared because he need to relate different meanings of fractions in number line: points and distances.

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RESULTS OF EXAMINATION USING VAN HIELE THEORY -JAPANESE CASE-

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Nobushiro Nohda University of Tsukuba
Nancy C. Whitman University of Hawaii at Manoa

This study is part of our larger study entitled "Comparison of Japan and Hawaii Geometry Curriculum, Instruction and Students Using the van Hiele Theory and the National Council of Teachers of Mathematics (NCTM) Standard" (see "The Attained Geometry Curriculum in Japan and Hawaii relative to the van Hiele Level Theory" (by N.C.Whitman et al.) in "RESEACH REPORTS").

In this report, results of 6 items in the examination of Japanese pupiles and students (4,7,9,11 grade) are further analyzed and discussed by using the cross total from the following viewpoints.

1. Understanding of generality of figures (triangle and quadrilateral) — the 1 level

2. 'The inclusion relation of triangle and quadrilateral' and 'a statement and its convers' — the 2 level

3. Proof-writing ability and understanding of inclusion relation — the 3 level

4. Results of re-examination of 7 grade students after one year

THE SEVEN TOOLS OF THE GEOMETRIC ORIGAMI

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I call "TOOL" the way for obtaining, by ORIGAMI, the cube and each face of polyhedrons. They are: the Cube (C); the Triangle (T); the Square (S); the Pentagon (P) ; the Hexagon (H); the Octogon (O); and the Decagon (D).

They are tools, because making them and linking them, one to others, to form plane figures and polyhedrons is a way for:

1. Developping manual hability;
2. Visualizing spatial forms;
3. Understanding geometric facts, like the concepts of convex, stellated, regular or semi-regular polyhedrons.
4. Studying the facts and theorems of Geometry.
5. Planning ludical activities for class room.
6. Constructing beautiful and criative structures.
7. Calculate areas and volumes.
8. Other applications to be found, as related to Cristallography, for instance.

Triangles and Squares apt to be linked one to others are largely known and presented in ORIGAMI boo'.s.

I have been searching for a way to make Pentagons, Hexagons, Octogons and Decagons as efficient and easily as possible; in this task, I used my experience in teaching Geometry at the Institute os Mathematics and Statistics of the University of São Paulo, from which I retired as Assistant Professor in 1987.

I shall present my work in a poster, to the Seventeenth Conference of the PME at Tsukuba.

My system is based in two facts:

I. ORIGAMI is an approximated process.
(Folding paper has precision of 0.5mm, at best).

II. ORIGAMI is a linear process.
(We link points to obtain straight lines; the intorsection of straight lines gives us a point).

If the way presented in other ORIGAMI books to make polygons is to be compared to a road, I want to call my way a short cut; it is suitable for children and persons not specialized in ORIGAMI.

Activities which can be introduced by use of *Geometric Constructor*

Yasuyuki Iijima (Aichi University of Education, Japan)

1. Outline of the software *Geometric Constructor*.

Geometric Constructor is a software which allows the user to invetigate geometry. (cf. Geometric Supposer, CABRI geometry). Main functions are construction, deformation, zoom in/out, measurement, locus. Ver.1 was developed in 1989 by author, and the newest version is 4.1B. Over 100 schools in Japan use this software.

2. Activities which can be introduced by use of *Geometric Constructor*

From the analysis of problems in junior high school textbooks and analysis of undergraduate student's investigations about them, following activities are identified which can be introduced by use of the software;

(1) Problem making by observation,and testing of it,
(2) Finding of invariants or functionality,
(3) Finding of correspondence of some conditions,
(4) Generalization, specialization, finding of boundary of therom,
(5) Finding of impossibility or outstanding case (monster),
(6) New approach to problem of construction,
(7) Finding the locus as trace of some geometrical object,
(8) Making transformation by construction and use of it.

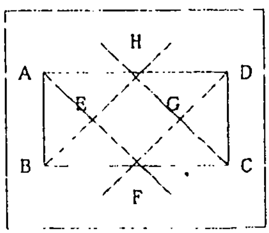
3. In Junior High School Classrooms

In 1992-1993, the author observed over 10 classroom activities, in which this software was used, to test above activites can be done by junior high school students.

For example, to test (5), following problem was shown to 8 graders;

PROBLEM

There is quadrilateral ABCD. For each angle, make angle bisector. And name each cross point of bisectors, E,F,G,H. Now, if we deform ABCD, then EFGH is also deformed. Think ablout it.



Teacher showed them how to investigate and sum up the result in table. After 20 min. investigation, the results was discussed. "EFGH cannot become rhombus or parallelogram" was found. Next, they thought proofs about their results, and finally, they proofed above proposition about impossiblility.

PROBLEMS IN TEACHING MATHEMATICS AT KŌSEN (NATIONAL COLLEGES OF TECHNOLOGY)

Yuji Kajikawa

Yonago National College of Technology

Semon Uehara

Fujitsū Co,Ltd.

Through four years' teaching of Mathematics at the Yonago National College of Technology, I have to some extent shaped my ideas concerning some related problems. This is the problem related to the essence of teaching mathematics at Kōsen. Then I have tried to send out questionnairs among students in the first and third grades on the mathematics lessons at Kōsen. In view of those results I will propose some new ideas how to rearrange the curriculum of mathematics in our school. In short, this outline proposes to use a form of education based on both the level of achievement and the normal homeroom lessons at the same time. In other words, two different types of education are combined into one form, in which they complement each other. By actually using this idea, I hope that mathematics education in our school will improve much and become much more beneficial to our students.

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HOW DO WE MOTIVATE A STUDENT TO LEARN MATHEMATICS?

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A student's mood and consequently the efficiency of his work depend on his motivation and how clear and attractive the purposes of education are to him. That is why writing books for the children from 10 to 15 years of age we give special attention to the motivation of one or other activity. Here we shall try to demonstrate several types of motivation which we use when organizing education.

– Creating such situations in which a student realizes the insufficiency of his previous practical or study experience for the solution of some problem. For further progress he needs new mathematical knowledge.

– Students formulating independently a purpose which becomes the incentive of their activity for a long period of time. All activities are subjected to this purpose.

For example, instead of offering a problem of finding a divisibility sign of a number and a problem of factorization and asking them to do these problems one after another at the very beginning of studying the theme we "suggest" to a student the problem of finding all natural divisors of a given natural number. Some characters' lives depend on the solution of this problem. Every activity (finding divisibility signs, a method of finding divisors) are subjected to the solution of this problem.

– Unusual way of presenting study material.

Our books, among other things, are based on dialogue using different literary genres introducing into the text different games and paradoxical situations.

– Offering such assignments that students have to compare their methods and results with each other.

For this purpose students are asked to become reviewers, find mistakes, continue reasoning, deny the given reasoning, and so on.

– Setting indefinite problems.

We often put students in such a situation in which they must formulate a problem themselves. Depending on their imagination students invent not one but several problems, perhaps a class of problems or form their own understanding of the given problem.

– Discussing with students the ways of controlling the results of their activity or its separate stages.

– Giving a student some knowledge about knowledge and some knowledge about his abilities.

Through special games and talks with a psychologist, through special assignments we tell a student how knowledge is structured and about cognition of the world.

We have given only several ways of motivation. We are sure that a student's activity becomes most effective and efficient only when he together with us sets the goals, plans education and the ways of controlling it, sees the prospects and his place in education.

TITLE MoViL—Teaching Mathematical Concepts Through Moving Visual Language

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ABSTRACT OR DESCRIPTION Over the past five years, we have tested and developed over 250 user-friendly computer programs that allow teachers to demonstrate basic mathematical concepts through moving graphic images, and these have been used successfully in teaching over 15,000 high school pupils and students preparing for university entrance examinations. In surveys, 85% of our students have said the classes are more effective and more enjoyable than those not using computers.

Samples of the Moving Visual Language software (MoViL) that we wish to demonstrate on video have been designed for use in conventional classrooms equipped with display monitors and a personal computer operated by the teacher. They feature computer simulations of semi-concrete objects, or schema, to promote an intuitive and structural understanding of such concepts as permutation integration, the addition theorem and the conservation of area. One example will be a program of conservation of angle by similar magnification and reduction developed after ICME-6.

We will show how, with MoViL, students can follow the gradual deformation, magnification and movement of figures, in both forward and reverse motion, and, learning to transform images in their own minds, can come to understand the components of these processes as parts of continuous flows. We will also discuss MoViL's pedagogical potential, and the way its moving figures form elements of a new CAI language.

CONCRETE CONTEXTS FOR ABSTRACTING ANGLE CONCEPTS

Michael C. Mitchelmore
Macquarie University, Australia

Previous studies of children's angle concepts (reviewed in Mitchelmore, 1989) have almost exclusively investigated abstract diagrams of angles. Researchers have used concrete analogies to "explain" the diagrams, assuming that they were tapping a general concept.

Recent research suggests that such an approach has serious limitations. Children develop mathematical concepts by abstracting the common features of various situations and learning to ignore the specifics (Skemp, 1971). This is not a once and for all process; as more and more dissimilar situations are seen to contain the same common elements, the concept becomes more and more general (Mitchelmore, 1992). Hence, situations which a mathematician would regard as equivalent may be seen by children as completely different. Until they acquire a general concept of angle, children are tied to the specifics of each situation or class of situations and it must be uncertain what, if anything, an "abstract" diagram might represent.

A more fruitful line of research would seem to be to investigate how children go about gradually abstracting the angle concept. What concrete angle situations do young children understand? Which situations do they most easily recognise as being similar? What angle subconcepts develop, and in what order? How are these subconcepts eventually unified into one concept? Children's drawings of angle situations might indicate what common features they have abstracted and what they ignore; another indicator might be the type of abstract physical model (e.g. hinged rods) they can use to represent particular situations.

Such a program of research is currently being planned. The poster will present the first step: a tentative classification of concrete angle contexts on the basis of their physical and functional similarities. The hypothesis is that children abstract the common angle aspects of situations within each category before they see the similarity between situations in different categories. Concrete and abstract physical models intended for use in validating the classification, and a longer background paper, will also be available at the poster session.

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THE UNDERSTANDING OF THE DISTRIBUTIVE LAW

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A form one class of 39 students was chosen from a secondary school. The school mathematics score of the students ranged from 19% to 79% and the class average was 55%. They took a written test and their results were analyzed. Four students were chosen to do written Test II and interviewed.

It was found that most students remembered the distritibutive law as a pattern of distribution after removing the bracket, $c*(a \blacklozenge b)=c*a \blacklozenge c*b$, and hence generalize its application regardless what the operations * and ♦ are. Their generalization might be caused by different factors. The list below gives the phenomenon observed in the interviews:

1. Familiarity with the pattern "$c*(a \blacklozenge b)=c*a \blacklozenge c*b$" by rote memorization and giving a wrong generalization - many mistakes like $c+(a+b)=c+a+c+b$ were made in the written test were due to incorrect generalization from this pattern.

2. Rote learning and failure to generalize to the case $ax(b-c)=axb-axc$.

3. Giving an answer by quoting a rule. Even though students apply the rule in an appropriate situation, they may not be able to explain why they do it.

4. Incomplete view of the role of letter such as a letter is representing a particular unknown rather than a generalized number.

5. Some ad hoc rules are used but the rules are not consistently applied.

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CONCEPTS OF EXEMPLARY PRACTICE VERSUS CONSTRUCTIVISM

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Mathematics is commonly thought of as a body of knowledge and skills which teachers are expected to transmit to the next generation. These ideas seem to be supported by the majority of respondents to a survey of 132 mathematics educators and teachers which was aimed at identifying elements of 'quality' teaching. This poster raises the question of whether such beliefs can be compatible with constructivist notions of mathematics education. If mathematics is "learner's activity" (Wheatley, 1991), we clearly need to confront the view of pedagogy which keeps the teacher and traditional mathematics at the centre of curriculum planning.

This poster describes the methodology used in data collection and analysis then presents the survey results under the subheadings Teaching environment, Lesson aims, Content, Presentation, Class activities, Questions, Aids, Assessment and Closure. It concludes by noting the tensions between common beliefs about mathematics education and constructivist epistomology. It then raises the need to research ways of

- conveying through everyday classroom practice a belief that individuals create social mathematics and also interpret it;
- supporting the creation of communities of understanding (Johnson, 1987) that are based on individual experiences - rather than further refining how to teach "given" concepts;
- using interpersonal processes to turn subjective mathematical knowledge into acceptable objective knowledge (Ernest,1991) without destroying the notion of autonomous learning; and of
- creating processes for the negotiation of learning goals, with a recognition that students actively set and pursue these by selectively interacting with communications from others.

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Conics and Sundials

Ros R.M., Fabregat J. Universitat Politècnica de Catalunya.

Intuition is very important in mathematics. This is an empirical science at the beginning and it is necessary to consider the connection with real experience. For students, this aspect is more attractive than the formal construction and we think it is very interesting to use intuition mixed with problem solving. Teaching through problem solving is now a procedure used in participative education. We propose to present this idea in a specific situation. We want to introduce conics to students by looking for an example in the real world. If students observe a sundial they can see the hour lines and curves for the equinoxes and solsticies. We propose that a group works with sundials.

The Sun has an apparent movement around the rotational axis of the Earth. Basically a sundial is a gnomon (in accordance with the rotational axis of the Earth) and a plane (in accordance with either the Equator, the horizon or the vertical). We can consider the surface determined by a ray of sunlight which leaves the Sun and reaches the end of the gnomon. When the Sun follows a circunference this ray generates a cone. The sections produced by the sundial plane over the cone are: circumferences for the equatorial dial, or hyperbolas and stright lines for the horizontal and vertical dials. At the end, we would like the students to discover the different section curves (circumferences, ellipses, hyperbolas, straight lines and parabolas). The proposal is the following. The teacher presents (20 minutes) the problem with the necessary information about the apparent movement of the Sun and sundials. It is very easy to go to the classroom with a pocket sundial or with some pictures of wall sundials. And finally, the teacher will ask some specific questions to the students. For example, why does the end of the gnomon follow a curve?, what kind of curves are there? ... The teacher or a student will be the secretary for the session. While the others are working (hour long), he or she will write up the most important steps in the process. If it is necessary at the end the teacher can complete the results.

THE DEVELOPMENT OF A TOOL FOR EXPLORATORY LEARNING OF THE GRAPH OF y=Asin(Bx+C)+D

Akihiko Saeki
Kanazawa Institute of Technology

An apparatus for exploratory learning of the function y=Asin(Bx+C)+D was developed. It offers a learning environment where students can explore directly, by hands-on experience, graphs of the sine function.

The apparatus is composed of two disks, two pens, and a printer, and draws the graph y=sin$x$ and y=Asin(Bx+C)+D. The student sets the radius, the rotation speed, the initial phase angle, and the vertical location of the lower disk to draw the graph y=Asin(Bx+C)+D, and compare it with the graph of the upper disk.

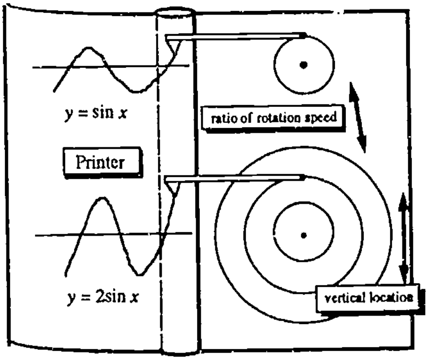
The benefits of the apparatus are as follows :

(1) It encourages students to freely explore, so they can experience the visible embodiment of the concept.

(2) It helps students make hypotheses about the effects they see and test them.

(3) It helps students integrate the concretely experienced concept of transformation in graphs of the sine function with the mathematical concepts of a linear change (x , y) -> (ax+h , by+k).

The future plan is to connect the apparatus to a computer. The computer can monitor students' exploration activity and encourage them to explore aspects they may have overlooked. And, with the aid of the computer, students may be offered hints or suggestions for acquiring the correct concept.



Manipulating the Tool

y = Asin(Bx+C)+D

(A) AMPLITUDE : radius of the lower disk

(B) PERIOD : rotation speed of the lower disk

(C) PHASE SHIFT : initial phase angle of the lower disk

(D) VERTICAL SHIFT : vertical location of the lower disk

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MODELLING IN THE MATHEMATICS CLASSROOM[1]

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To know mathematics does not automatically imply that one is able to apply it to a real life situation. This puts a hard request upon mathematics education, since the world is more and more mathematized. Mathematical concepts, language and ideas permeate our social life and many professional activities. In fact, it is not possible to conceive modern societies without the applications of mathematics.

Modelling is the most important process that enables us to connect mathematics and the real world. From a real life situation we make an interpretation of it and we formulate a problem in mathematical terms, which we try then to solve by all sorts of mathematical methods. Then, the solutions eventually found are contrasted and discussed in accordance with the initial situation. This cycle may be repeated the number of times necessary so that we get a satisfactory outcome.

However. neither students or teachers are used to this kind of work.

Many questions are then in need of investigation, including the difficulties and psychological processes used by the students working in modelling problems. The study of such issues, in the framework of a classroom experiment with 10th grade students, is the object of Projecto MEM (Modelling in Mathematics Education), currently in progress in Portugal.

This poster reports some preliminary results obtained in these issues.

1 This poster reports research made by the Project MEM (Modelling in Mathematics Education), supported by the Instituto de Inovação Educacional. Besides the authors, members of this project also include João Filipe Matos, Graciosa Veloso, Susana Carreira e Paulo Abrantes.

ABSTRACTION AND GENERALISATION IN CALCULUS

Paul White
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Relationships between pieces of knowledge characterise conceptual knowledge (Hiebert, 1986). The degree of abstractness of a relationship can vary in that a relationship is more abstract if it generalises more situations. Investigation of the word "abstract" shows that it can also mean removed from any concrete context. Hence, manipulating mathematical symbols can be abstract, not because the mathematical objects are seen as generalisations of various situations, but because the objects are apart from any context other than that of the symbols themselves.

The poster will highlight a selection of the results from a completed doctoral study (White, 1992). {Other aspects of the thesis were reported in White and Mitchelmore (1992).} The study analysed responses to items involving rates of change and derivative during an introductory tertiary calculus course which focused on derivative as instantaneous rate of change, and employed a method based on examining graphs of physical situations. The items were structured in four versions {a, b, c, d} so that the expected correct response for each version was basically the same. The difference between the versions was that each version successively required less symbolisation. Hence, (a) required symbolising all rates to an appropriate derivative, whereas (d) had all information presented in symbolic form.

The results to be presented in the poster point to the critical role in learning calculus of a concept of a variable which has been abstracted by generalising a wide range of contexts. A concept of a variable which is abstract only in the sense that it involves manipulating mathematical objects apart from any concrete context is totally inadequate.

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THE DEVELOPMENT OF META- AND EPISTEMIC COGNITIVE CUES TO ASSIST UNDERPREPARED UNIVERSITY STUDENTS IN SOLVING DISGUISED CALCULUS OPTIMIZATION PROBLEMS (DCOP'S)

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THE PROBLEM: Many students who are underprepared for university have great difficulty in solving dcop's so that there is a need for the development of learning/teaching strategies to assist them.

CONSTRAINTS	INTRINSIC	EXTRINSIC
THEORETICAL UNDERPINNINGS	Piaget's equilibration, Pascual-Leone & Goodman's (1979) TCO (need to boost executive, operative and figurative schemes), Kitchener's (1983) levels of cognitive processing (first level, meta- and epistemic cognition).	Vygotsky's Zone of Proximal Development (the need for an extrinsic mediator).
EMPIRICAL INVESTIGAT-ION	Analysis of students'(40 students of the Engineering Bridging Unit) attempts to solve dcop's.	Analysis of the dcop section of first year calculus text books - formulation of standard teach steps.
FINDINGS	Students have problems with the two languages (the mathematical and the natural) embedded within dcop's and over-learned rules interfere with their dcop solving.	Text books'standard teach steps inadequate to assist students solving dcop's.
REQUIREMENTS	Revised teach steps to allow for the strengthening of executive, operative and figurative schemes to allow for successful equilibration (metacognition).	Students provided with revised teach steps and conceptual glossary containing words/concepts occuring in dcop's (Craig & Winter 1992) making explicit the epistemic nature of dcop's and dislodging overlearnt rules.

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III-271

MENTAL STRUCTURE ANALYSIS APPLYING FUZZY THEORY

----- Do You Like Mathematics ? -----

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Abstract

It should be more practical to represent our mental structure applying the fuzzy theory rather than the crisp logic.

We would propose a new analysis method of the liked/disliked structre among the school subjects by applying the fuzzy clusterings and orderings, which is based on the simple questionnair data as shown in figure 1.

Here, Y means 'yes' and N does 'no', and the students could mark thier degree how they like it or not.

The author would discuss the approximate analysis method of the mental structure and show its practical effectiveness with a case study as shown in figure 2.

Here, the numbers of the subject are ; ① is English, ② is Mathematics, ③ is Japanese, ④ is Physics, ⑤ is Social Study and ⑥ is Music, and □ implies the very similarly degreed subjects.

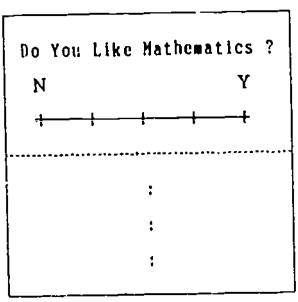

Fig.1 Questionnairs

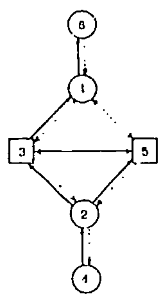

Fig.2 Mental Graph

[REFERENCE]

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