DOCUMENT RESUME

ED 373 965 SE 054 556

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TITLE Algebra Word Problems: A Review of the Theoretical Models and Related Research Literature.
PUB DATE Apr 94
NOTE 55p.; Paper presented at the Annual Meeting of the American Educational Research Association (New Orleans, LA, April 5-8, 1994).
PUB TYPE Speeches/Conference Papers (150) -- Information Analyses (070) -- Reports - Research/Technical (143)

EDRS PRICE MF01/PC03 Plus Postage.
DESCRIPTORS *Algebra; *Cognitive Style; Models; Problem Solving; *Schemata (Cognition); Secondary Education; *Thinking Skills; *Word Problems (Mathematics)

ABSTRACT

Research indicates that students have great difficulty solving algebra word problems and that few high school seniors have mastered the fundamentals of algebra, let alone algebra problem-solving skills. Improving students' algebra problem solving skills is considered to be critically important by those who have worked on reforming mathematics education over the past 10 years, as algrebra has become almost an entry level skills for most scientific, business, and technical jobs in western economies. This paper reviews the current models and theories of algebra word problems and algebra word problem solving and integrates these models into a more comprehensive view and model of algebra word problems and problem solving behavior. The empirical research literature is then reviewed in terms of the models presented and summarized, and the implications of each line of inquiry is discussed as well as the types of studies that needs to be done to advance knowledge in this area. Contains 97 references. (Author)

Algebra Word Problems: A Review of the Theoretical Models and Related Research Literature.

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ABSTRACT

Research indicates that students have great difficulty solving algebra word problems and that few high school seniors in America have even mastered the fundamentals of algebra, let alone algebra problem solving skills. Improving students' algebra problem solving skills is considered to be critically important by those who have worked on reforming mathematics education over the past 10 years, as algebra has become almost an entry level skills for most scientific, business, and technical jobs in western economies. This article reviews the current models and theories of algebra word problems and algebra word problem solving, and integrates these models into a more comprehensive view and model of algebra word problems and problem solving behavior. The empirical research literature is then reviewed in terms of the models presented and summarized, and the implications of each line of inquiry is discussed as well as the types of studies that need to be done to advance knowledge in this area.

Paper presented at the annual conference of the American Educational Research Association, New Orleans, April, 1994.

Overview

A number of studies have found that few high school seniors in America have mastered the fundamentals of algebra (e.g., Dossey, Mullis, Lindquist & Chambers, 1988). Other studies have found that students' tend to have inordinate difficulties with certain types of algebraic problems, particularly those involving the translation of proportions and relations inherent in word problems to algebraic formulae (e.g., Lochhead & Mestre, 1988; Mestre & Gerace, 1986 and Niaz, 1989a). These same studies, moreover, report that these translational difficulties are most likely due to a number of different factors (e.g., key contextual features, aptitude, cognitive development, and cognitive style), and that a detailed review of the literature in this area needs to be done to better understand the nature of the difficulties students have solving algebra word problems so that their performances may be improved (see Caldwell, 1977; and Sims-Knight & Kaput 1983a for details).

Improving student performance on algebra problems and the fundamental skills associated with solving algebra word problems is considered to be critically important by most math ducators (see NCTM, 1989), as algebra has become almost an entry-level skill for most scientific, business, and technical jobs in Western economies. Further, people's need for basic algebra and problem solving skills is going to increase significantly and not lessen in the future.

Introduction

The purpose of this article is to present a detailed review of the research and theoretical literature on solving algebra word problems. In particular, the propositional relational algebra word problem will be focused on in this review as this is the type of problem that the most work has been done on and students find most difficult.

A propositional relational algebra word problem relates two variables (minimally) and their relative proportions together in an equation or propositional statement (Mayer, 1982 and 1987; Kintsch and Greeno, 1985). A propositional relational algebra word problem has a "deep" or "core" structure (i.e., the underlying formal relational equation) and a "surface" layer, or syntactical structure or form (as it is called by researchers in this area), which is the "clothing" for the underlying structural equation. The "clothing" or syntactical form of the problem is referred to as the contextualized features of the problem by both theorists and researchers in this area, and there has been a great deal of work directed at understanding how key context features affect subjects' problem solving behavior.
The key contextual features that can be identified from the literature are familiarity (familiar and unfamiliar) and imageability (readily imageable and not readily imageable), and variable type (discrete and continuous quantities). For example, the propositional relational statement that "there are six students for each professor" is relevant to and

typical of a real life situation; i.e., usually one professor lectures several students. Such a propositional relational statement is considered to have familiar, readily imageable and discrete contextual features.

In addition to the key contextual features of the propositional relation problem, there are other factors that influence problem solving and must be considered. One of these factors whether the problem is presented in a pictorial, symbolic or verbal form, and whether it is to be (cross) translated to a pictorial, symbolic and verbal form. Problem presentation and cross translation format, moreover, are constrained by another factor, called the response mode may be generative or passive. In a generative response format, the subject must generate the correct answer in the correct form, whereas in the passive response format the subject just has to select the correct answer in the correct form.

In addition to the above stimulus related factors there are also factors associated with the problem solver. Most prominent among these factors are level of cognitive development (i.e., level of formal reasoning), cognitive style in the form of degree of field dependence and independence (Witkins and Barry, 1975), gender, ethnicity, and native language. These variables and the interactions of these variables are the major organizing theme of this review.

The next section of this review presents the conceptual

and theoretical basis of problem representation and problem solution variables. It also gives a brief description of the knowledge requirements, summarized by Mayer (1982; 1987), necessary for solving propositional relational algebra word problems. In addition, a constructivist view of problem representation and solution variables is given, which is derived from the Kintsch and Greeno study (1985). This view will be used to explain how key contextual features are related to and influence the errors subjects make in the process of solving algebra word problems. Subsequent sections will then review the research on (1) the translation and cross translation processes in algebra word problem; (2) errors made by subjects in the solution of the propositional relation problem; (3) how subjects use of the key contextual features in arithmetic and algebra word problems, and (4) the relationships between cognitive style and formal logical reasoning (cognitive development), and aptitude to algebra word problem solving behaviors.

General Processing in Problem Solving

Understanding how individuals organize problem information and how they go about constructing procedures, storing information, and retrieving from memory their problem-solving steps is important to researchers in cognitive psychology and mathematics education who are involved in modeling problem-solving procedures and characterizing the types of difficulties student face when

solving a specific algebra problem. Our understanding of these processes can also help researchers to investigate what measures can be taken to remediate the errors or flawed conceptions a learner has. Therefore, adapting models of problem-solving developed by cognitive psycholrgists to a specific domain in algebra or arithmetic has been seen to be critical to understanding many of the difficulties in the presentation and solution of algebra word problems.

In the area of probl m-solving, several math educators (e.g., Jerman, 1973; Hayes, 1981; Paige and Simon, 1966 and Mayer, 1982) describe a general two stage model of problem solving that consists of a representation stage and a solution stage. In the representation stage, students encode the problem into an internal meaning or representation. In the solution stage, students translate the representation into another form in which operators, relations and equivalences are formulated which culminates with the solution.

Based on this two-stage model, Mayer (1982, 1987) organized five types of knowledge requirements that related to both phases. They are linguistic, factual, schema, algorithmic and strategic. Of particular impor:ance, it should be noted, is schema knowledge.

The representation processes of the two stage model require that the problem solver possess linguistic knowledge of the English language, and about mathematical terms and words in the English language used in special ways in the

problem-solving situation, which includes factual knowledge about objects or events in the problem. Schematic knowledge represents the template-like structure of one problem as it relates to another problem. This schemata knowledge deals with the problem-solvers required knowledge structure of the components of the problem. In the solution stage, representation processes are highly dependent on how the schematic knowledge integrates linguistic and factual knowledge with the structural template of the algebra problem.

The solution stage can be divided into two components: namely, planning/monitoring and execution. The planning and monitoring process require strategic knowledge about how to break the problem into sub-parts and store these sub-parts in memory in order to recall them during the procedure. The execution process requires algorithmic knowledge, such as knowing the algebra procedures and operations involved; e.g., students formulate an equation or compare quantities, and then evaluate and assess the values of unknowns in the algebraic form.

Greeno (1978) also has proposed a two stage model that several important theoretical features . In Greeno's model, the conditions required to solve a problem are three in number. First, a coherent internal representation of the problem is needed. Second, the representation of the problem must correspond to the problem being solved, and third, the problem representation formed must be connected to the

problem solver's types of knowledge.

These stage models guide our understanding of how subjects solve a problem. Therefore, the limited literature on the functional processes of how to solve a problem can be used as theoretical basis for understanding the influences of the key contextual features in problems, and as a basis for analyzing the nature of the errors students make in solving algebra word problems.

Solving a Problem

This section reviews a problem-solving model proposed Kintsch and Greeno (1985) in order to produce a prescriptive model of problem presentation features and a method for assigning key contextual features to the problem components. Of particular importance, this prescriptive model will specify where most of the faulty conceptions or errors are generated in solving algebra word problems, and how the key contextual features of the problem contribute to the production of the correct or incorrect solution.

Greeno and Kintsch (1985) have produced a model of problem presentation and knowledge structures, drawing upon the general theory of text comprehension proposed by Van Dijk and Kintsch (1983). The model they have developed deals with the problem's representation in a student's memory as a result of the various processes involved in the reading of the information that is in the text and developing a model of the situation described by the text. The situation model is derived from the story-line, or information made by

inferences using the student's general knowledge of the topic area (factual and linguistic knowledge).

According to Greeno (1983), early literature on the function of knowledge structures in problem-solving have defined structures as discrete constituents of the overall problem-solving process. Greeno states that cognitive processes of understanding described in the older literature have been entirely tacit; that is, the "how" processes of understanding in problem-solving have not been explicitly described. In recent years, however, Greeno (1983) and Kintsch and Greeno (1985) have provided some analysis of the conceptual representation upon which problem-solving processes can operate. Kintsch's model addresses types of arithmetic problems, where the conceptual relations of two or more quantities are transcendent in their similar schematic representation, to the propositional relational problem. The generalizability of Kintsch and Greeno's model is an important characteristic of the model. As we outline the Kintsch and Greeno problem-solving model, we will refer to the general knowledge structures described by Mayer as they seem appropriate.

Kintsch and Greeno's Model

Kintsch and Greeno (1985) introduce their model by describing the two basic and macro processes of problem representation and solution:

> "Problem representations are built in several information-processing steps, which do not necessarily occur in a strict sequence. The verbal input is

> transformed into a conceptual representation of its meaning .... The propositions are organized into a task-specific macrostructure that highlights the general concepts and relations that are mentioned in the text." (p.111)

In the overall process described above, quantities which refer to objects in the pictorial, symbolic and verbal propositional relational structure are organized concisely into a schema or connected to an already existing schema from (semantic) long term memory. Corresponding to this propositional relational structure is what Mayer considers to be the linguistical and factual knowledge needed to solve the problem. Second, there are also schemata that represent properties and the relations of sets. The overall process described, therefore, is seen as one in which students make the relation between quantities in qualitative manner; namely, all the stimulus information is integrated in a general overall approach. Third, during the execution stage, there are schemata used in constructing (solution) operations and retrieving (solution) algorithms.

Two global process elements, therefore, are involved relative to the knowledge structures and reading/decoding processes needed to construct appropriate problem representations. The first these two process elements is the set schema, and the second is the propositional frame. The set schema is the major process element or component relative to developing a representation of the problem as it deals directly with the structure of the algebra problem; i.e., the general form of the story line. The story line itself has

four major attributes; namely, <u>object</u>; <u>quantity</u>; <u>specification</u> and <u>role</u>.

The <u>object</u> contains the set of all of the noun referents relevant to the problem; i.e., the specific type of elements or noun objects in the problem. If an object is immediately familiar in working memory during process (i.e., it is unfamiliar to the processor), then the object frame makes a schema that calls on resources from long term memory during the process of comprehension to supply appropriates knowledges and facts. The <u>quantity</u> component stores and maintains the cardinality of the set; e.g., "6 professors" or "10 students." This cardinality is considered a qualifier. The <u>specification</u> component holds information connected to the quantity; e.g., owner, location and time case. The <u>role</u> explains the relation between the object and its modifier; e.g., information about which is the larger set or smaller set in "6 professors" and "10 students."

In the <u>propositional frame</u> component, all five of the frame types Kintch and Greeno propose are important to the propositional relational problem, and used to derive a story-line schema. The five frame types are: <u>existential</u>, <u>quantity</u>, <u>possession</u>, <u>compare</u> and <u>time</u>. Thus, the frames should be viewed as the functional processes that utilize the "primitive" template structure or set schema of an arithmetic problem or propositional relational problem. In each function of the frame, certain elements are eluded to; e.g., noun referents or qualifiers.

The existential frame refers to the component that states a subject or an object exists (e.g., professor or students). In the propositional relational problem, this type of frame considers the noun referents in the problem statements (e.g., apples, oranges, bongadas, etc.) which refer to other noun referents. The quantity frame occurs with the noun referents (i.e., existential frame) and is considered to be the "adjectival connective;" i.e., it connects the adjectives with the object. The possession frame assigns the specification of sets or noun referents to the setting or to quantities. This frame of representation is the most important component because it is the major source of student "malfunctions" (i.e., errors) in problem solving. Because the possession frame assigns the specification of sets (e.g., "S stands for 6 students in a classroom"), it relates the "adjective connective" and the quantity to the noun referent (the students), and it is then placed in the time frame. This component creates the meaning dimension in and for the problem; i.e., that the context may be irrelevant to the mathematical content of the problem but nevertheless important to the overall process of problem-solving.

The compare frame considers the qualitative comparison between two specifications, and relates the quantities involved in the problem (i.e., there are more students than professors). The time frame indicates a proposition for a time argument; e.g., "the mule walks at 5 miles per hour,

slower than a sports car whose speed is 100 miles per hour." The time in the domain or event occurs concurrently. The time frame proposition makes use of noun referents, qualifiers, quantities and relations, to set the sequence of events for an arithmetic problem.

The set schema and propositional frame provide a framework to describe the functional organization and process of solving two-step arithmetic problems. Because of the specificity of Kintsch's constructivist model to two-step addition and subtraction problems found in the K-8 curriculum (Riley, Greeno and Heller, 1983), we have adapted and extended some components of the model to handle the propositional relational algebra word problem because of the theoretical and practical importance of this type of problem.

Propositional Relational Word Problems

In propositional relational algebra word problems, a conceived story-line schema is proposed. This story-line schema (and problem component) is based on problem analysis and adaptation of the problem's template as analyzed from the pertinent literature (e.g., Mayer, 1982). Furthermore, the schema is based on the structural properties of propositions in the problem. This schema will not provide the functional frames needed to solve the problem.

There are four structural elements pertinent to algebra word problems with similarities to Kintsch and Greeno's set schema in arithmetic problems. These elements are noun referents, qualifiers, quantities and relationships. The

noun referents refer to the objects in the problem statement, i.e., apples, oranges, professors, and so on. Qualifiers function as determiners as well as adjectival modifiers; e.g., "stupid professors or 2.2 students." Other examples are "speed of a car" or "length of a box" which function as determiners of nouns. Thus, single noun referents cannot function independently of their determiners. Quantities are the adjectival modifiers which hold the cardinality of the objects, e.g., "2.2 professors or 6 students," where 2.2 and 6 are the quantities. This definition of quantities is similar to Kintsch and Greeno's definition of quantities in arithmetic problems. Relationships connect the quantities of the noun referents into a proportion; e.g., "there are 20 students for one professor."

The set schema is always used as the basic unit of structure for the information in the problem, and it is functionally processed by the frames. According to Greeno, in the solution phase, the processor tries to solve the problem by using "solution patterns" called <u>make-set</u>, <u>transferset</u>, <u>difference</u> and <u>superset</u>, which again are specific to two-step arithmetic problems.

As described by Kintsch et al., the solution patterns are processed in certain steps and sequences. The make-set is elicited by the quantitative proposition about some kind of noun referent or object. The make-set forms a data structure for representing a particular set; e.g., 6 professors. The transfer set is used to remove larger or

smaller quantities from the whole set, based on the data requirements specified in the problem as it applies to the addition and subtraction problems. The difference set appropriates the correct quantity from the larger quantity or from the superset.

Adapting the above solution patterns to include a way to solve the propositional relation problem would not be very easy to do within Kintsch and Greeno's model for addition and subtraction. However, by adapting and extending some components of the Kintsch model, a generalized problem-solving procedure for the propositional relation problem may be devised.

Theoretically, a schema for solving a propositional problem can be described in three steps or stages. First, a schema is activated which interacts with information already stored in memory; i.e., prior knowledge. In this process, objects, quantities, and modifiers are elicited, then integrated to form a compounded knowledge structure. In the second stage, frames are used to functionally process the information. In this process, quantities are assigned to the proper noun referents; i.e., six to students and one to professor. This quantity itself is represented in the schema pertaining to the number of students or number of professors. When functionally compared, the possession frame assigns a coefficient to the second object in the proposition to be related. In the third and last stage, a <u>result set</u> is created through the use of two make-sets which are

dynamically related to each other, by proportional reasoning processes, through use of the possession frame. Then objects are compared using the comparison frames. Attachments, such as the equivalences, are formed by associated procedures. They include subprocedures to identify the arguments as one professor lectures six students, and to form an abstraction by first forming a representation, and then abstractly translating and representing the problem.

We have attempted to provide a prescriptive problem-solving schema for the propositional relational algebra word problems. This model was derived in part from Kintsch and Greeno's (1985) set schema and propositional frame model for two-step arithmetic problems. The model we have developed can be used to assign key contextual features to the propositional relational algebra word problems. The model we have developed can also be used to analyze theoretically how students make errors in solving the propositional relational algebra word problems, particularly the reversal error.

Errors in Solving Algebra Word Problems

The previous section presented a problem-solving process and procedure for propositional relational algebra word problems. The first major macro element of the model presented was a set schema which define the components of the problem's story-line. The second major macro element was the propositional frame that applies the set schema the problem to proceduralize the solution. This problem-solving model should help reveal the sources of and reasons for solution

errors. This section of the review, therefore, focuses on studies that examine the errors students make in solving algebra word problems. The studies have several relevant findings. The most well-studied algebra word problem solving error is the reversal error.

Several studies have attempted to explore the cognitive underpinnings of the reversal error by studying problem-solving steps through the interview method. Among these Clement (1982); Clement, Lochhead and Monk (1981); Wollman (1983); Rosnick and Clement (1980); Gerlach, (1986); Gerace and Clement (1986); Niaz (1989a) and Mestre (1989), have reported on the robust nature of errors when translating word problems to algebraic equations. Rosnick and Clement (1980), Clement (1982) and Mestre and Gerace (1986) found that the majority of students (for the most part first-year engineering students and/or students who have taken two semesters of calculus) committed a "variable reversal error," such as the type of error made on the "professor and students" problem. They explained the causes of the error as a result of the static correspondence of word-nouns to their adjectives in the word problem, or in Kintsch's model, as a failure to assign the objects to their respective quantity.

Clement (1982) elaborated extensively on the reversal error, and explained its causes in terms of the verbal format of the problem. He said students approach the problem in a syntactic surface correspondence approach, placing the number six, the "adjective connective," in front of students, using

the noun as a label rather than a symbol standing for a variable. Similarly, they place the number <u>one</u> adjacent to the literal "P."

Clement discovered from his interview protocols that students can descriptively provide comparisons of the quantities involved. However, they lack the ability to properly use the possession frame in order to attribute (link) the object to the quantity involved in a relational manner; in other words, individuals lack the necessary ability to think in a proportional manner.

Furthermore, the pervasiveness of this error has shown itself in the translation from pictorial representation to algebraic equations (Clement and Konold, 1985), which places less emphasis on the syntactical decoding theory of the verbal structure. Another view suggests that students do not lack the ability to represent the problem, but err in the way they make use of the procedural attachments; e.g., the coefficient and symbol representing the quantity and object (6 and students respectively).

Some studies associate the reversal error with language related factors and mathematics abilities. For example, Mestre and Gerace (1986) and Mestre (1985) used Hispanic students and native speakers of English to compare their performance on the algebra problem-- there was no difference among these groups. However, ability was the most important factor contributing to student performance. In a similar study, Mestre, Gerace and Lochhead (1982) pointed to an

important association present between grade point average, language proficiency (with three subparts: vocabulary, speed of comprehension, and level of comprehension) and translation tasks among Hispanics and native speakers of English. The strong correlation indicates that students' linguistic and factual knowledge may be influenced by the particular medium of instruction, which may have allowed the students to comprehend the vocabulary better in English, in some way facilitating the translation and solution to the correct or incorrect algebraic formulae, with little or no differences between ethnic groups. This is substantiated by other studies that identified the reversal error across nationality lines (Lochhead, Eylon, Ikida and Kishor 1985, and Bernardo and Okagaki, 1992).

A different theoretical perspective has been offered regarding the nature of the reversal error and other congruent types of errors resulting from the propositional relation problem. Sims-Knight and Kaput (1983a) noted the importance of the cognitive information processing model. In particular, these researchers emphasized that errors occurring in the translation of word problems are connected to their representation and execution. They pointed to the semantic aspect of the error by relating the process of problem-solving to visual abilities versus the syntactical process of translation. They suggested that the difficulty in translating and solving the propositional relation problem is influenced by the imageability and familiarity of key

contextual features connected to the problem.

Niaz (1989a), who approached the problem from a developmental perspective, supported the view that students who lack formal operational reasoning perform the reversal error more frequently. He associated the reversal errors with the inabilities to reason in a proportional manner. By administering a translation test in the generative mode and passive mode, Niaz found most of the reversal errors produced by the generative translation were of the reversal type. The most important findings in Niaz's study were that those students who performed the reversal error had low scores on the proportional reasoning tasks. This finding was supported by a Pearson correlation coefficient of $r=+.57$ between total scores on the translation tasks and formal operational reasoning. Thus, those students who lack proportional reasoning abilities and/or skills have not fully developed the notion of a variable as it covaries in the algebraic formulae.

Although Niaz's result highlighted the importance of formal reasoning in solving the propositional relational problem, Sims-Knight and Kaput (1983b) stated that proportional reasoning problem tasks are not all the same, and do not elicit similar cognitive demands as there are different types of propositional relational problems. Nevertheless, they indicated that proportion problems that require a numeric solution do correlate moderately with the proportional reasoning tasks.

A number of studies using the interview method to understand the underlying cognitive processes leading to the reversal error, showed that students shift between schemas when solving the algebra problem (Rosnick and Clement, 1980; Clement, 1982; Wollman, 1983; Gerlach, 1986; Gerace and Clement, 1986; and Mestre, 1989). One important finding made by these researchers in the use of the verbal format in the propositional relation problem is that students decoded the linguistic structure syntactically by a naive approach to the problem. Some students, therefore, approach the problems using only factual and linguistical knowledges, and making no reference to or use of algorithmic knowledge and strategic types of knowledges (or strategic knowledges). These students may be compared to other students who access and use defective strategic knowledges and thus make metacognitive errors.

Translation from One Mode of Representation to Another

Translation is the act of recognizing and connecting related quantities, functions, and structures in two modes of representation. Cognitively, translation can be described as the unconscious operation of a schema that makes conscious the content of the mind (Brewer and Nakamura, 1984). From a different perspective, translation is explained by two mental spaces (Fauconnier, 1985) in which the mapping procedure from one mental space to another is the translation between them. Fauconnier states the following:

> "Mental spaces [are] represented as structured, incrementable sets- that is, sets with elements

(a,b,c...) and relations holding between them (R1ab, R2a ....)." (p.16)

Within these spaces, other subspaces exist, which represent the represented world, such that each subspace has a representing "sub-subspace," *ad infinitum.*

Kaput (1987a) and Palmer (1977) name the two corresponding worlds or mental spaces in which the translation is made between them as the representing world and the represented world. Kaput's (1987b) particular specification of a representation is described as follows:

> "(1) the represented world, (2) the representing world, (3) what aspects of the represented world are being represented, (4) what aspects of the representing world are doing the representing and (5) the correspondence between the two worlds." (p.23)

Correspondence can be thought of as the translation from the representing to the represented. The correspondence between the different formal structures (either pictorial, verbal, or symbolic) can be isomorphically represented in different modes, such that any true statement in one system can be correspondingly represented as a statement in another.

Translation Studies

Empirical studies have shown that multiple cross translation between different forms have helped problem-solving (Khoury and Behr, 1982). A number of studies that investigated student performance propositional relational algebra word problems have suggested the importance of translation in problem-solving (e.g., Rosnick and Clement, 1980 and Clement, 1982). None of these studies, however,

investigated propositional relational word problem with respect to the different representation (verbal, pictorial, or symbolic) or cross translation (e.g., verbal to symbolic) modes. Bruner's original idea that the cognitive development is related to information presentation and responding modes is equally important and in need of study, particularly in relation to algebra problems.

Nelson (1975) used three groups of students who received different treatments with respect to verbal presentations, verbal and pictorial presentations and verbal to pictorial and their converse pictorial to verbal cross translations. No significant differences among the different groups was found. Clarkson (1978) conducted the first empirical study that investigated cross translation among all of the different representational systems: i.e., verbal (V), symbolic (S) and pictorial (P). She established the limitations to cross translation as it relates to convergent and divergent response modes or formats. Convergent responses on the representational problem were considered to require generative translations (students had to supply the correct answer themselves) and the divergent problems were by definition passive translations (students selected the correct answer from alternatives provided). Among the passive translations, Clarkson found the symbolic-verbal and symbolic-pictorial translation to be the most challenging to the high school students. On the generative or active translation (i.e., pictorial-symbolic, verbal-pictorial and

verbal-symbolic), the pictorial-symbolic followed by the verbal-symbolic, was the most challenging to the secondary school students.

Shoecraft's (1971) study on translations from verbal representations to equations used the medium of concrete materials (i.e., high imagery) to aid in the translation. Most of the students in his study were inclined to translate without referring to the medium presented with the exception of low-achievers who tended to use concrete material.

Evidence from the literature cited above indicates that the most challenging problem seems to be the verbal to symbolic cross translation. This particular difficulty seems to indicate that the problem solvers' ability to generate or activate a response seems to be hindered by those problems that require translation to symbolic form. This type of translation specifically requires the solver to activate tacit and symbolic knowledge not readily available to the problem-solver.

Key Contextual Features of Problems

The term key contextual features in algebra problems is defined as information that is embedded in a problem that might be necessary or unnecessary for the solution of the propositional relation problem, but that remains separate logically from the problem's structure and syntax. These features in problems, which are the problem's "clothing," are sometimes implicit and elusive (Cadwell, 1984). Further, they may provide information which may be nonsensical or

inconsistent with reality and might effect the total process of problem solution. For example the following problem was presented by Paige and Simon (1966): "The number of quarters a man has is seven times the number of dimes--how many has he of each coin?" The information which the problem presents is contradictory and unrealistic. Context is, then, an attribute that describes the non-mathematical meanings present in the problem statement.

Empirical Studies

Key contextual features are the type of information that may help to induce meaning to the mathematical content (Kulm, 1984, p. 17). Contextual information carried by words and grammatical structures in a problem statement directly relate to its depth of encoding that students undertake to relate elements in the problem. Paige and Simon (1966) showed that when auxiliary information was embedded in a problem, the information was not used as part of the translation process. They gave this type of problem, "there are seven times as many quarters as dimes." Most of the students interviewed failed to recognize meaningless information. In an informal study we conducted, we administered a translation test to a group of college trigonometry students containing one item with auxiliary information adapted from Simon and Paige's exploratory tests. All the students (n=27) who took the pilot algebra test failed to use the auxiliary information (i.e., noun referents) in the problems.

The elaboration on the referents in problems such as

Paige and Simon's "quarters and dimes" problem requires knowledge of what these quantities represent as indivisible nominal entities in their contextual domain. According to (1987b), "the elaboration [on problems] is done by using the features of the reference field of the symbol system rather [than] using its symbol scheme syntax (p.177)." This view and statement means that by "sieving" out redundant information (i.e., contextual attributes connected to noun referents and objects), the problem itself becomes much easier to handle as its elements are clearly organized and understood.

In the research literature, one finds key contextual features investigated mostly in verbal problems, despite the various format options that can be applied to the context dimension in problems. However, it is possible to assign key contextual features to pictorial and symbolic presentations of the problem. In this way, researchers can also employ key contextual features to study other problem-solving processes. Furthermore, in the literature key contextual features (KCFs) are found associated with arithmetic (one-step) and algebraic (two-step) word problems. The KCFs tend to be contrasts such as concrete-abstract, real world-fictitious, and familiar-unfamiliar. In addition, those studies considering KCFs associated with the every day reality (e.g., Washborne and Morpett, 1928 Houtz 1973, Caldwell, 1977 and Quintero, 1980) used contexts based on children's experiences with concrete materials (regularity of the stimulus) moving on a continuum

that ended in abstract or hypothetical modes. Thus, none of the aforementioned studies have used KCFs as they relate to the pedagogical events in students' common instructional experiences (e.g., the use of discrete and continuous quantities).

Some of the key contextual features found in the literature are: abstract, real-life, concrete, continuous and discrete quantities, familiar, unfamiliar, and imageable. One finds these key contextual features inconsistently defined, and when defined, they are defined in relatively vague ways. Some studies view context in terms of experiential reality. For example, White (1934) used contexts based on children's experiences. Houtz (1973) used the notion of realism along the continuum from abstract to increasingly concrete; i.e., experienced situations suggesting a continuum where performance increases inversely from concrete to abstract. Houtz also found effects in which students' performance was higher on those problems that were concrete in their representation than those that were abstract.

Caldwell (1977), Caldwell and Goldin (1978), and Caldwell (1984) described abstract and concrete problems as containing factual and hypothetical components. The abstract word problem "involves a situation which describes abstract or symbolic objects while concrete word problems describes a real situation with real objects (Goldin and Caldwell, 1984, p.238)." The factual type describes an event and the

hypothetical type describes a situation which undergoes a change.

Caldwell devised four types of problems: concrete-factual (CF), concrete-hypothetical (CH), abstract-factual (AF) and abstract-hypothetical (AH). According to the developmental model she used, Caldwell suggested that concrete problems should be easier to solve than abstract ones. Caldwell hypothesized the following succession from easier to harder: concrete-factual, concrete-hypothetical, abstract-factual and abstract-hypothetical. Elementary students had the least difficulty with concrete-factual and concrete-hypothetical problems, followed by abstract-hypothetical and abstract-factual. However, the difference in performance on all types of problems diminished with older students, supporting the developmental view of performance on abstract and concrete problems, which suggests that when individuals reach formal operational thinking they perceive problems as being connected with reality, and can no longer ignore part of the hypothesized problem.

As we operate from an information processing perspective, we hypothesize that concrete attributes in the individual's cognitive structure should either entice (i.e., intrinsically motivate) the solver to process the problem, or arouse well anchored and familiar ideas which will aid in the production of a direct representation of the problem. In terms of this hypothesis, we have identified two particularly important key contextual features (KCFs) from

the experimental literature. These 2 KCFs are: familiarity, (familiar and unfamiliar) and imageability, (readily imageable and not readily imageable). Attached to the imageability key contextual feature is the variable type (discrete and continuous quantities). The familiarity feature will be reviewed first.

Familiarity

Key contextual features, such as familiarity, have been investigated as early as 1926. Washborne and Osborne (1926), Washborne and Morpett (1928), Brownell and Stretch (1931) cited in Webb (1984), Sutherland (1942), and Lyda and Franzen (1945), indicated the superiority of the familiarity key contextual feature (as experienced on a daily basis) on one-step and two-step arithmetic and algebra probl . Lyda and Franzen (1945) provide strong support for the developmental model, finding age as a major factor affecting performance. Their findings suggested that as students get older, their performance differences on problems with familiar and unfamiliar key contextual features diminishes. Quintero's (1980) study provided support for the developmental model among middle school students. She used key contextual features of low embeddedness and high embeddedness on arithmetic problems. The higher the level of embeddedness, the higher the level of familiarity. Her middle school students performed higher on the high embedded problems than on the low embedded ones, which supported her hypothesis.

Sims-Knight et al. (1983a) found high performance on

problems with the unfamiliar key contextual features contained in the relationship of a propositional relation problem versus the familiar one. Chipman, Marshall, and Scott (1991) studied context as it related to the familiarity of the structure of the three-step algebra problems. In factoring context-familiarity out of the problem, they found that familiarity structure had significant effects on the problems. Unfamiliar contextual features, therefore, seem to alert the problem solver that she is dealing with a real problem and not something routine and known, whereas a familiar problem <u>structure</u> evokes well established problem solving routines which efficiently produce correct solutions. Consequently, these two types of familiarity (or unfamiliarity) need to be clearly distinguished at all times.

<u>Imageability</u>

Imageability as a key contextual feature in algebra problems has not been investigated extensively. Further, imageability as a key contextual feature has not been defined in the mathematics education literature, and those studies that attempted to provide a definition have been rambling and inconsistent with one another (e.g., Quintero, 1980 and Sims-Knight and Kaput 1983a).

In some instances, researchers have confused spatial ability with the concept of imagery as it relates to mathematics learning. Clements (1981) stated that these two components (spatial ability and imagery) of cognition have

qualitatively different processing strategies. Imagery is considered to be a mental picture that is formed in the mind, prior to identification of a new image. According to this view, the mental picture analogy describes a "registration" of previously stored sensory patterns (Kosslyn and Pomerantz, 1977). Hence, imagery is conceptual in nature, and the process of imagery is a by-product of internal abstract representations which are encoded in any form.

Operating from an information processing perspective, imageability as a key contextual feature as defined here, is based on the propositionally based theories proposed in the early 1970s (see Simon, 1972; Anderson and Bower, 1973 and Pylyshyn, 1973). The definition suggests that knowledge can be represented by a set of propositions, and that verbal, pictorial or symbolic representations are transformed into propositions. As a student solves a problem with salient imageable attributes, she/he searches for the proposition which represents the imageable component. This proposition is then transformed into verbal information that is imageable. In a case where the abstract representation system does not contain a proposition relevant to a required piece of information, a person may deduce this proposition from those which are available. This latter type of situation would indicate an analogy with a not readily imageable context as defined here.

Belmore, Yates, Bellak, Jones and Rosenquist (1982), like Gagne (1976) agree that an imaginal representation is helpful

for language processing and for comprehension and retention. Schank (1976) and Begg (1972) argued for the use of imagery in sentence processing. They suggested that if a problem is represented in an implicit and elusive way (i.e., if its construction from natural language transforms and recombines the problem's discrete elements into a general and coherent knowledge structure which solves the problem), then imagery is more advantageous for organizing and schematizing the new knowledge created.

Key contextual features, such as imageability, have rarely been assigned to algebraic or arithmetic problems. The earliest works by Bramhall (1939) investigated two context dimensions in arithmetic problems. The first one was the conventional dimension of only the basic information required to solve the problem. The second dimension was "imagery" information that included nonessential information. No significant difference was reported between either dimension in Bramhall's problem.

Recently, Sims-Knight and Kaput (1983a; 1983b) investigated three imagery contextual features, which they called unimageable, imageable-patterned and imageable-unpatterned. These three differing imagery conditions were assigned to propositional relational algebra word problems. In their tasks, these two researchers asked students to translate verbal problems to algebraic formulae. Problems with imageability attributes had discrete elements (e.g., pigs and horses), and the unimageable problems used such

examples as months and years. Two basic types of problems were constructed: one patterned (five fingers to a palm), and one imageable but non-patterned (six hens for each pig). Their results indicated that the imageable-patterned elements of problems influenced students performance by causing student to make non-equivalence errors, whereas nonpatterned images and familiar attributes caused students to make the equivalence or reversal type of errors. The latter errors, it should be noted, tend to be made in the initial (decoding) phases of the problem solving process, whereas the former error tends to be made in the later phases of the process. The effects of imagery, therefore, are differential and interact with other key contextual features as well.

In summary, key contextual features such as familiarity and imageability have been used as attributes in arithmetic and algebra problems. However, most of these studies have been inconsistent in their definitions and classifications of these features. Evidence is sparse but most of the studies point to the effects of high imageability on obtaining the correct answer and indicate that this particular effect diminishes as students get older. The familiarity key contextual feature, moreover, may have a larger effect than the unfamiliar, imageable or unimageable features.

Further studies are needed which take the studies summarized above further by presenting the student-solver with a combination of key contextual problem features in problems that have pictorial, verbal or symbolic formats and

cross translations. There is no study in the literature to date that has systematically varied the key contextual features of familiarity and imageability, and variable type, or varied these features along with the pictorial, symbolic and verbal problem representations and cross translations identified earlier in this articles.

Discrete and Continuous Variables

A system of representation in mathematics may mean very little without the quantification of its referents (i.e., variables). One can talk about abstract concepts such as freedom, anarchy and democracy, but they are all mathematically meaningless without the quantification of its referents. Although attributes can be collected for these concepts and one can reason about them in mathematical ways by constructing classifications and categories, one can not quantify these elements easily. As Rosnick, Cauzinille-Marmeche and Mathieu (1987) state, "there are no denotable objects in mathematics (p. 170)," such as democracy, anarchy, and so on. Those denotable objects that are quantifiable are usually embodied within the structure of the problem; namely, noun referents and obje ts. These quantities appear to our consciousness through the direct metric operation or quantification.

For instance, a proposition states that on the Merrimack River an observer noted that there are at least six boats for every one duck. This propositional statement may be taken subjectively to mean that ducks are becoming extinct as a

result of the boats' constant interference in their ecosystem, or it may have a completely different meaning. The concept of a duck in a mathematical sense is an abstract representation, even though they may be seen on a daily basis. The "mathematization" of the environment endows these objects with the concept of "number" which itself is a representation of some referential entity. "So we have in mathematics a domain in which from the very beginning, people must reason about objects that exist only as abstractions" (Resnick, Cauzinille-Marmeche and Mathieu, 1987, p.170). From this framework, it is concluded that real objects and events are considered to be easier to encode. These objects are discrete and countable into wholes. But those propositions that have continuous events are considered to have continuous quantities, which makes them harder to image.

Discrete and continuous variable attributes found in the domain of the problem are more pedagogically specific than the imageable and familiar features which could be experienced on a daily basis. There is relatively little literature considering the use of variable forms in algebra problems, with the exception of Horwitz (1981), who from an information processing perspective used discrete and continuous variables to indicate the level of visualizations in problems. She found higher error rates produced on the continuous variables because of the unimageability of their attributes.

In defining discrete and continuous quantities, Horwitz

indicated that discrete quantities are those which can be counted; continuous quantities are those that can be measured. In some instances problems dealing with intensive quantities such as velocity are not measurable, though their units are measured. Continuous quantities are considered as quantitative elements in problems that could not be imageable (e.g., the weight of a box, versus the discrete quantity), such as "two boxes" or "two oranges." Furthermore, continuous quantities are variables that can take on numerical values, such as the units of currency, whereas apples and horses are cardinal values that have no numerical value. Discrete quantities represent a simple, perceptual system.

Studies are needed not only to assess the effects of discrete and continuous variables (quantities) on student performance, but also to assess the influence of student field dependence/in dependence on performance, because field-independence is in part the ability of individuals to disembed discrete components for a more complex structure. We believe that field-independent individuals will perforr better on problems with discrete attributes than field-dependent subjects. Including students' degree of field dependence/independence in studies is, we believe, very important, as currently researchers have only considered level of cognitive development (i.e. logical reasoning) as the chief causal agent of students problem solving abilities, whereas there is both data and theory that suggests that

field dependence/independence may be a powerful intervening variable that needs to be considered and investigated. Both of these points are considered more fully below.

Cognitive Development and Cognitive Style

It has been suggested formal operational reasoning as discussed above may be an important factor influencing students' success on the propositional relation problem (see Niaz, 1989a). In addition, Niaz (1989b) has suggested that operational reasoning is not the only factor operating in this relation, but field-dependence may be a second factor in student success on the translation of the propositional relation problem.

The intellectual abilities, of operational reasoning and cognitive style, are empirically related. However, their structural conceptions, nature, and influence are viewed differently. Logical reasoning is a developmental construct (Inhelder and Piaget, 1958), and changes in students' intellectual development is strongly related to students' age and training. On the other hand, the construct of a given cognitive style of intellectual functioning is argued to be relatively constant throughout development.

Essentially, operational reasoning refers to the content of cognition or the question of "what", while cognitive style is concerned with the manner of the behavior. In addition, cognitive development maintains a unipolar dimension in that development progresses in a one-directional continuum, while cognitive style is a bipolar construct where field-dependence

and field-independence are at either ends of a non-continuous dimension (Messick, 1976). In addition to differences between formal operational reasoning and cognitive style, there are also some commonalities between the two constructs which may be broadly classified as analytic abilities. Those individuals who are at the formal operational level may show similar analytic abilities as field-independent subjects. It has also been suggested from the empirical data that the ability to think in a formal operational manner is positively correlated with the cognitive style of field-independence (Saarni, 1973; Lawson, 1978; and Lawson and Wollman, 1977).

In this section, no attempt is made to review the literature dealing with cognitive style and operational reasoning separately. However, an attempt will be made to join some of the theory dealing with these two intellectual functions as they relate to the propositional relational problem and to general mathematics problem-solving aptitude. In the published literature available, few studies have empirically related cognitive style and proportional reasoning using the propositional reasoning task. Several studies, however, have attempted to connect the two constructs as they interact to influence mathematics achievement and problem-solving. The propositional relation problem has been characterized as a task within proportional reasoning requiring thought patterns that individuals use to interpret variability or covariation between two or more variables. The view states that success on the propositional

relational problem as hypothesized may reflect individuals' formal operational reasoning and field-independence abilities and skills.

Roberge and Flexer (1983) found a substantial number of studies that empirically supported the position that field independent students scored higher on standardized mathematics achievement tests than field-dependent subjects. Further, these two researchers found no studies that related operational reasoning to mathematics achievement. Therefore, Roberge and Flexer (1983) compared cognitive style, performance and operational reasoning on the Mathematics Metropolitan Achievement Tests. This achievement test included computation, mathematics concepts and problem-solving tasks. Significant main effects were reported for operativity and cognitive style. However, no significant interactions were found among the two factors. *Post hoc* tests indicated that field-independent students with formal operational reasoning scored higher than their counterparts; namely, concrete operation thinkers and field-dependent students.

Some studies have attempted to predict several factors affecting operational reasoning. Adi and Pulos (1980), for instance, in their stepwise regression analysis found the cognitive style factor of field dependence/independence entered first in the equation and sequential reasoning entered second on the dependent variable of operational reasoning. Field dependence/independence accounted for 29%

of the variance in formal operational reasoning, and when it was removed as a factor, sequential reasoning accounted for 11% of the variance in formal operational reasoning.

Collings (1985), in order to understand the stability of cognitive style over time, designed educational material to increase field-independence. Those pupils who were trained with the material improved their field-independence scores, and a significant improvement was also shown on the measure of operational reasoning. Thus, training with simple material designed to give practice in restructuring affected scores on both field-independence and formal operational reasoning abilities.

Several detailed descriptions and definitions of cognitive style have been given that differentiates between field-dependent and field-independent subjects. According to Pascual-Leone (1977), pupils who are field-dependent tend to see information differently than field-independent subjects. Field-dependent subjects tend to select and focus on cues that are irrelevant to the overall problem-solving process. This particular view, however, is not supported by Linn's (1978) finding that cognitive style was uncorrelated with levels of operational reasoning. Linn in his tests, on the other hand, presented conflicting information and her results may be due to an ordinal interaction between cognitive style and question context. This specific potential interaction and other potential interactions between context, structure, cognitive style, logical, reasoning, solution errors, and

other variables needs to be empirically explored more fully. The findings of a number of studies in this area are somewhat moot because of this interactional factor, and this point needs to be kept in mind when reviewing and evaluating studies involving the aforementioned variables.

Nummedal and Collea (1981) studied the proportional reasoning task as described by Inhelder and Piaget (1958). Formal operational reasoning is reached in the last of the three stages of operational reasoning. The first stage at formal reasoning begins with the identification of variables, which continues to the last stage of relating two covarying variables. Nummedal and Collea (1981), stressed the importance of ignoring irrelevant information in the total process of problem-solving, which they measured along with the operational reasoning ability. Their results indicated that with the introduction of irrelevant information, the ability to perform at the formal level was related to the degree of students field-independence. The results of this study are in agreement with Linn (1978), who said that the relationship between field-independence and formal operational reasoning is influenced by the information that is embedded in the tasks measuring these constructs. The problems in Nummedal and Collea's (1981) study were specific proportional reasoning problems and may not be generalizable to other algebra problems which have proportions in them; specifically, propositional relational problems. Further, generalizing from these results should be done with caution.

As Sims-Knight and Kaput (1983a) have reported, different types of proportional reasoning tasks produce different results.

Along the same lines, Saarni (1973) explored cognitive development as it related to field independence on a productive thinking type of problem-solving situation. This particular study brought some resolution to these two seemingly conflicting theories of intellectual functioning by stressing the commonalities between them which may be broadly classified as analytic abilities. Saarni's problems required subjects to hypothesize a situation which was not stated in the problem and to make hypothetical inferences to resolve the situation. The results of the study showed no differences in problem-solving performance among field independence subjects within different Piagetian level of operational reasoning. However, from the literature, one could infer that the Piagetian cognitive development factors can predict performance on this problem-solving type. Using student grade level as a surrogate for ontogenetic development, Caldwell (1977) found significant interaction effects between age and the contextual feature of the problem, with lower scores on the abstract problems.

Lawson (1982) used the measure of cognitive style and operational reasoning on several probabilistic and correlational reasoning tasks. He found a significant correlation between field-independence and operational reasoning for both college students and seventh graders. In

a step-down regression analysis the cognitive development factor showed the highest relative contribution to the performance on correlational and probabilistic reasoning tasks, with cognitive style contributing less. It should be noted that these correlations do give further evidence of some of the commonalities between cognitive style and formal operational reasoning.

To measure reasoning levels, Lawson and Wollman (1977) used three sub-tasks. These three sub-tasks were: (1) to test an individual's abilities to isolate and control variables; (2) to test an individual's abilities to balance a combination of weights of objects on a scale similar to the test developed by Adi (1977); and (3) to conserve weights as the shape of the objects were changed. The empirical results for these tasks supported the four levels of logical reasoning which are respectively concrete operational reasoning (IIA), fully concrete operational reasoning (IIB), formal operational reasoning (IIIA), and fully formal operational reasoning (IIIB). Lawson and Wollman (1977) found a high correlation between cognitive style and level of operational reasoning. The correlation between cognitive style and the three formal operational reasoning measures of r=+.66, further lends support to the argument that field-dependence within cognitive style may be a factor affecting formal operational reasoning.

Niaz (1989b) studied the relation of field-independence and proportional reasoning using Lawson's proportional

reasoning tests. Niaz (1989b) found a strong relationship between the ability to solve problems that relate two variables together and field-independence. Of particular note was Niaz's use of the propositional relation problem within proportional reasoning tasks. His objective was to study "subjects' ability to translate sentences into equations [i.e., verbal to symbolic translations], equations into sentences [i.e., symbolic to verbal translations] and a possible relationship between these misconceptions and cognitive style (Niaz, 1989a, p.233)." Niaz found that a large number of students who could not translate verbal algebra problems into equations were field-dependent subjects. Niaz's (1989b) study, therefore, provides some specific information on cognitive style and its relation to student performance on propositional relational problems as compared to proportional reasoning problems. As may be seen above, the results were similar for both types of problems.

The construct of cognitive style provides information on individual differences in analytic ability, as a common factor within and influencing cognitive development. Piagetian notions of operational reasoning concern the evolution of cognitive structures along a continuum from concrete operational thinking to a more hypothetical, formal and abstract point, while cognitive style is a bipolar variable where field-dependence and field-independence lie at the two (logically and conceptually opposing) extremes. A field-dependent person finds it very difficult to disembed

perceptual and conceptual patterns from the organized fields of which they are a part, while the field-independent person is able to overcome misleading and distracting information and recognize significant, perceptual and conceptual patterns in the environment.

The relation between formal reasoning and field-dependence is what seems to restrict the development of formal thought (Pascual-Leone, 1977). Although formal reasoning subjects tend to deal with information in a highly abstract manner, field-dependence in some cases hinders student analysis of the information as well as problem representation construction, and thus appropriate processing of the information. Again the impact of these variables is differential and comes in differing ways at the initial and final phases of the problem solving model presented at the beginning of this article, as did the effects and impact of the key contextual and structure features of problems which have reviewed in this article.

Post Script

No one has investigated the effects of key contextual features (specifically familiarity, imageability, and variable type) on solving algebra word problems that have different presentation and responding formats, in terms of the interactions between these features, formats, and levels of cognitive development and cognitive style. Such studies need to be done not only to answer outstanding questions, but also to create a fuller and more empirically complete version

of the algebra word problem solving model presented at the beginning of this article. The model that we have developed and presented at the beginning of this article, it should be noted, predicts that contexts that are not readily imageable will be most affected by field-dependence which in turn will affect the performance of students who are not formal reasoners. Students who are not formal reasoners should also have a great deal of difficulty with verbally presented problems and problems that require cross translations between the verbal, pictoral, and symbolic problem presentation and responding modes. Pursuing this particular line of research and inquiry, we believe, may generate findings and a great deal of knowledge and theory that may be particularly helpful in assessing and modifying middle school algebra courses and programs where there currently is a dearth of answers and a multitude of problems. The fight to improve students algebra skills and knowledge is being lost in the middle school years, we believe, and not before or after to the extent currently believed. The models, theory, review of research studies and commentary on all of the aforementioned, we believe, establishes the basis and provides the foundation needed for the much needed research program outlined above.

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