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ABSTRACT

This two-volume document provides the proceedings of a conference on the psychology of mathematics education. Plenary session themes were "Diversity and Equity," (with papers by Gilah Leder; Walter Secada; and Ubiratan D'Ambrosio), "Teacher Education," (with a paper by Thomas Cooney), and "Technology" (with papers by Jere Confrey; Sharon Dugdale; Celia Hoyles and Richard Noss; and Carolyn Kieran, Maurice Garancon, Lesley Lee, and Andre Boileau). The 2 volumes also contain 59 research papers, 45 short orals, and descriptions of 6 posters and 14 discussion groups presented at the conference. The research papers are organized into the following categories: (1) Advanced Mathematical Thinking; (2) Algebraic Thinking; (3) Assessment and Evaluation; (4) Epistemology and Cognitive Processes; (5) Functions and Graphs; (6) Language and Mathematics; (7) Modeling; (8) Number and Proportion; (9) Probability and Statistics; (10) Problem Solving; (11) Social and Cultural Factors Affecting Learning; (12) Students' Beliefs and Attitudes; (13) Teacher Education; and (14) Teachers' Beliefs and Attitudes. A list of authors and committee members is included at the end of volume 2. (MKR)

Proceedings of the
Fifteenth Annual Meeting

North American Chapter
of the International Group
for the

Psychology of Mathematics Education

Volume 1: Plenary Sessions and Research Papers

October 17-20, 1993

Pacific Grove, CA, U.S.A.
Asilomar Conference Center

SAN JOSE STATE
UNIVERSITY

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Editors:
Joanne Rossi Becker
Barbara J. Pence

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History and Aims of PME

PME came into existence at the Third International Congress on Mathematical Education (ICME 3) held in Karlsruhe, Germany in 1976. It is affiliated with the International Commission for Mathematical Instruction.

The major goals of the Group and the PME-NA Chapter are:

1. To promote international contacts and the exchange of scientific information on the psychology of mathematics education.

2. To promote and stimulate interdisciplinary research in the aforesaid area with the cooperation of psychologists, mathematicians, and mathematics teachers.

3. To further a deeper and better understanding of the psychological aspects of teaching and learning mathematics and the implications thereof.

Editors' Preface

The Conference Committee decided to select three themes for this conference: Diversity and Equity, Teacher Education, and Technology. We also chose to organize the plenary sessions around these three themes in different ways. The Diversity and Equity strand will begin with a panel of Gilah Leder and Walter Secada, with a response by Ubiratan D'Ambrosio. We are indeed pleased to have these distinguished scholars, especially our colleagues from abroad, provide some theoretical frameworks for our investigations in this critical area. This panel will be followed on Monday morning by equity discussion sessions in which you will have the opportunity to explore the ideas brought up in the panel in more depth and in smaller groups. Some of these discussion sessions have a focus, while others are more open in nature. However, all of them should provide substantial time for participant discussion.

The Teacher Education plenary consists of a lecture by Tom Cooney with a response by Alba Thompson. Cooney raises questions about the direction descriptive research is taking us and suggests a way to look at teacher education that might improve both theory and practice.

The Technology theme also has a unique format. The Plenary lecture by Jere Confrey is followed by invited lectures on the theme; then Alan Schoenfeld synthesizes the information presented in all seven lectures. The papers printed in this volume provide a major focus on the learning of fundamental concepts of variables and functions and the role of multiple representations in that learning. Confrey's paper also provides a connection between the technology and equity themes as she explores how the teaching of algebra needs to change tb reach a goal of "algebra for all."

The research papers represent a variety of interest and have been organized in these volumes in the following categories:

1. Advanced Mathematical Thinking
2. Algebraic Thinking
3. Assessment and Evaluation

4. Epistemology and Cognitive Processes
5. Functions and Graphs
6. Language and Mathematics
7. Modeling
8. Number and Proportion
9. Probability and Statistics
10. Problem Solving
11. Social and Cultural Factors Affecting Learning
12. Students' Beliefs and Attitudes
13. Teacher Education
14. Teachers' Beliefs and Attitudes

Each research report proposal was reviewed by three reviewers with experience in the speciality using the criteria established by PME-NA as guidelines. In cases of disagreement, the Conference Committee members studied the reviewers' comments and carefully considered the proposal. This procedure resulted in denying about one-fourth of the proposals.

In addition to the 59 research reports, there are 45 short orals, 6 posters, and 14 discussion groups in the program.

We would like to thank the other members of the Conference Committee for their valuable assistance, and all of the reviewers for their generous contribution of time and expertise. Special thanks go to Laurie Dengel for her invaluable assistance throughout the summer in keeping all our records straight.

Joanne Rossi Becker
Barbara J. Pence
September, 1993

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DIVERSITY AND EQUITY: INEVITABLE OR A PARADOX?

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Four major areas have shaped the way I have reflected on the topic set for discussion: the links between diversity, equity, and inequity; selected paradigms for addressing gender differences in mathematics learning; the importance of the social context in which mathematics is learnt; and revisiting previous research. Each theme suggests directions for future research. The main arguments and topics to be addressed are outlined below under the relevant headings. As requested, the focus in this paper is on gender.

Diversity: desirable or a cause for concern?

It has been argued (e.g., Passow, 1960) that diversity is essential if equity is to be achieved:

> Democracy is not the fruit of uniformity. Diversity within unity does not equal a single program in every classroom. The ideal is best achieved where arrangements include guidance ... to cultivate individual differences. (Passow, 1960, p. 55)

Yet diversity can also lead to, or be the result of, inequitable outcomes, practices, opportunities or resources.

More than seventy years ago Terman and his colleagues (e.g. Terman, 1960; 1965; Terman & Oden, 1947) embarked on an ambitious project: documenting the lives of a group of students who scored very high on certain IQ tests. Identifying nonintellectual factors likely to facilitate life success[1] was an important part of the work. Comparisons of those ultimately judged successful, and those not (arguably a value-laden rather than absolute judgement), led to a number of provocative conclusions.

Both groups appeared to be equally successful during the

[1]defined as 'the extent to which a subject had made use of his (sic) superior intellectual ability, little weight being given to earned income' (Terman, 1960, p. 5)

elementary school years, began to draw apart in the early high school years, and differed substantielly by the end of high school. Various explanations for the behaviour and performance patterns among the two groups were put forward. Some differences in the home backgrounds of the two groups were noted, for example.

> But the most spectacular differences between the two groups came from three sets of ratings ... on e dozen personality traits.... These were 'persistence in the accomplishment of ends', 'integration towards goals, as contrasted with drifting', 'self-confidence', and 'freedom from inferiority feelings'. (Terman, 1960, p. 55)

In this case the divereity identified wae linked with negative rather than positive outcomes.

The influence of affective factors on fulfilment of potential, identified by Terman and his colleagues, is noteworthy and consistent with persistent conclusions drawn in the mathematics and gender area. Affective variables (including confidence, beliefs about sex-role congruency, attributional style, willingness to take risks, ...) are often cited as partial explanations for gender differences observed in mathematics learning. The data in Table 1 offer e convenient summary of such factors. More detailed information can be found in Leder (1992).

<u>Table 1:</u> Selected models of mathematics learning

Author(s)	Year	Relevant components in the model
Deaux & Major	1987	beliefs about the target, about self, social expectations, effect of context, response to actions selected
Eccles	1985	persistence, self-concept of ability, attitudes, expectations, attributions
Ethington	1992	self-concept, expectations for success, stereotyping of maths, difficulty of maths

Author(s)	Year	Relevant components in the model
Fennema & Peterson	1985	confidence, willingness to work independently, sex-role congruency, attributional style, engeges in high level cognitive tasks
Leder	1990	confidence, attributional style, learned helplessness, mastery orientation, sex-role congruency
Reyes & Stanic	1988	societal influences, teacher attitudes, student attitudes

Terman's recommendations for future research ectivities remain timely and applicable as well to research on mathematics and gender:

> Intellect and achievement are far from perfectly correlated. To identify the internal and external factors that help or hinder the fruition of exceptional talent, and to measure the extent of their influences, are surely among the major problems of our time. (Terman, 1960, pp. 55-56)

Paradigms for addressing gender differences in mathematics

It is convenient to focus on the various theoretical models of gender differences in mathematics learning which were discussed during the opening session of the *International Organization of Women and Mathematics Education (IOWME) Study Group* (one of the strands organised as part of the *7th International Congress on Mathematical Education (ICME-7)* held in Québec in 1992). The distinction drawn in the previous section between functional and dysfunctional diversity continues to be en implicit theme.

Four theoretical perspectives were identified at IOWME: the intervention perspective, the segregation perspective, the discipline perspective, end the feminist perspective. These categories are not mutually exclusive. Rather, there is inevitable overlap.

The intervention perspectiva

Those working in this paredigm assume that intervention programs can end should be mounted to ensure thet more females engage in mathemetics end related pursuits. *Thus equity is*

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equated with equal participation outcomes. Examplas of possible initiatives are outlined economically in the strategies put forward by Amot and Weiner (1987). Long as well as more readily achievable short term goals are included.

Table 2: Strategies for enhancing gender equity

ACTION	STRATEGIES/OUTCOMES
Persuading girls into math, science and technology	Exploring what is 'herstory' or girl-and woman-centered science and technology
Requiring a coepulsory core of subjects: math and science for girls; humanities for boys	Providing both groups with skills and knowledge to challenge the traditional work and home environment
Reducing stereotyped choices and models	Broadening girls' horizons
Reviewing school organisation, e.g., uniform, discipline	Changing the nature of schooling
Producing in-service courses and policy guidelines	Exploring sexual harassment in school and the workplace
Creating posts for equal opportunities	Facilitating consultation and collective working

Adapted from Amot and Weiner (1987, p. 356)

However, initiatives taken to increase females' participation in mathematics are sometimes overtaken by events which occur in the wider context. For example, as unemployment increasas, many disillusioned youth are sceptical about arguments which stress that studying mathematics will laed to increased employment opportunities. In such a climate, previously successful intervention strategies which focus on the usefulness of mathematics become irrelevant.

The segregation perspective

Thosa working within this framework assume that curricula and methods of teaching are geared to the needs of males rather than females. Many of the lattar, it is argued, will respond better in a learning environmant sensitive to the needs and preferences of females. Organising learning in single-sex settings is sean as an attractive and viable alternative to

the far more common co-educational setting.

Yet single-sex schooling as the preferred option for improving schooling for girls has as many advocates as critics (see e.g., Leder & Forgasz, in press; Lee, Marks, & Byrd, in press). The latter noted that

> U.S. single-sex Catholic secondary schooling has been shown to produce benefits, especielly for young women, on a range of outcomes, including academic achievement, academic attitudes and aspirations, less stereotypic views on sex-roles in family and professional life, and political activism.

Their recent study (Lee, Marks, & Byrd, in press) explores and documents effectively the complex interaction of the many - often subtle - factors that need to be considered when different educational settings are compared. A number of Australian researchers are also cautious about the supposed advantages of the single-sex strategy. In a comprehensive review of research on single-sex and coeducational schooling Gill (1988) concluded:

> Not all girls who are educated in single sex classes or schools do well. Simply grouping girls together in single sex classrooms is not going to bring about the desired changes in educational outcomes. However when girls are seen to achieve highly or enrol in higher numbers in mathematics and science it seems that single sex learning experiences are frequently involved (p. 14)

Willis and Kenway (1986) have been particularly critical of the single-sex strategy as a means of redressing gender inequities in education.

> The fundamental problem with the single-sex strategy is that it refuses to acknowledge the complex ways in which schools interact with the gender and class structure of society.... (T)he single-sex strategy appears naive, partial and particulerly inappropriate for the majority of girls who are rejected by the competitive academic curriculum. Even within its own terms, however, the single-sex strategy in its most populer manifestation is unlikely to change the educational opportunities of girls in any fundamentel way because it focuses almost exclusively on changing the behaviour of girls. It neglects the much more difficult problem of changing the attitudes and behaviours of boys and teachers, and the nature of the curriculum itself. (pp. 137-138)

It seems premature to conclude that segregated schooling per

se will necessarily minimise inequities.

The discipline perspective

Various writers have argued that the nature of the mathematics (and science) taught in schools (and universities) needs to be questioned and possibly reassessed. Some of the strategies outlined by Amot and Weiner (1987), and summarised in Table 2, challenge assumptions implicit in traditional mathematics curricula. Willis (1989) maintained that a mathematics curriculum embedded

> in real-world social concerns and in people-oriented contexts; presenting it as making 'humansense', non-arbitrary, non-absolute and also fallible; and presenting a social-historical perspective to help students become aware of the 'person-made' quality of mathematics (p. 38)

is possible and more likely to be inclusive of girls. What such a curriculum might look like has been shown by Barnes (1989) who was asked to produce 'gender inclusive material for teaching calculus in Australian schools'. Among the guiding principles she adopted were the following:

> Present mathematics as a human endeavour, changing and developing over time, and not, as students often perceive it, something that has been handed down to us complete and perfect, all worked out by 'superbrains' who were very different from ordinary mortals.
>
> Present mathematics in context - as a socially relevant activity, helping us to understand the world we live in and to solve problems of concern to people today, especially young people.
>
> Value the life experiences and cultural background of all students and the mathematics that arises out of them.

Given the paucity of curriculum materials written with such a focus it could be argued that under current circumstances, inequity is inevitable.

The feminist perspective or more appropriately: feminist perspectives

It is beyond the scope of this paper to present a detailed discussion of this vast and complex literature. Instead, I will continue the link with the IOWME session by focusing on

the three broad feminist theories discussed by Mura (1992): the feminism of equality, radical feminism, and feminism of difference. Some of the key elements of the different theories are summarised, all too simplistically, in Table 3.

Table 3: Overview of three feminist theories

Theory	Important elements
Feminism of equality	Is action-oriented, demands actual and legal equality between males and females, and considers political action and the socialisation and education of females as effective tools for change.
Radical feminism	'refuse to define themselves through equality with men. They identify patriarchy as a social, political and economic system that oppresses and exploits women individually and collectively, sexually and economically' (Mura, 1992, p. 3)
Feminism of difference	Do not wish to eliminate gender distinctions. On the contrary, they insist on the recognition of difference, they see women as possessors of specific knowledge, culture and experiences and the feminine as an affirmation of life. They assert that women have their own ethics, their own way of knowing and their own language. They wish to reclaim and revalue all things from a feminine perspective. (Mura, 1992)

There is an interesting correspondence between the first three perspectives discussed and the feminist paradigms summarised in Table 3. Those relying on strategies described under the intervention perspective heading seem to be working within the feminism of equality paradigm, those adopting the arguments

put forward under the discipline perspective heading apparently identify most readily with radical feminism, while those advocating strategies described under the segregation perspective heading seem to be working within the feminism of difference paradigm.

The following dialogue, reported in Phillips and Soltis (1991), is sobering and serves as a suitably cautious conclusion to this second theme.

> A: Perhaps you can help me. I've been trying to make sense of something, but I can't quite do it.
>
> B: I'll try. What are you trying to understand?
>
> A: I've been taking a course on learning theories and I've tried to use them to help me understand how I learned to juggle last semester.
>
> B: That should be easy. I took that learning course last year. What's the problem?
>
> A: There are two problems really. First, some theories don't seem to work at all, but if they are valid theories they should explain all cases of learning, shouldn't they?
>
> B: Well, perhaps ... which theories don't seem to work?
>
> A: Plato's theory for one, I didn't 'recall' how to juggle did I?
>
> B: How do you know you didn't? If you didn't have an inborn ability to juggle you could never have mastered it, right? I've tried to learn how and I can't....
>
> ... ...
>
> B: Try the Gestalt theory. Didn't you just keep on trying until it all came together in a rhythm, a pattern, a smooth flowing continuous action as a whole? One minute you couldn't juggle and the next you could! That's Gestalt!
>
> ... ...
>
> A: OK, you've convinced me that I could use Lockean, Platonic, and Gestalt theory to explain my learning to juggle when I only thought I could do it with behavioral, problem-solving, Piagetian, and information-processing theories. But that gets me to

> my second problem. If all these theories can explain the same phenomenon, how do we know which is right?
>
> B: Whoever said theories need to be right or wrong?
>
> A: Don't they?
>
> (Phillips & Soltis, 1991, p. 98)

Our challenge is to distinguish between the two sets of theories.

Learning mathematics in a social context

A common element (either implicit or explicit) in the various approaches for explaining/reconciling/applauding differences in mathematics learning is the importance of the broad social context in which learning occurs. Material found in the print media can serve as a convenient indicator of this environment.

A case study: election campaigns
(November 1992, U.S.A.; March 1993, Australia)

Some recent excerpts from the popular (Australian) press (Feb 1993) illustrate the conflicting messages used to woo female voters. For example:

Under the headline: *Policy launch raises debate on notion of 'women's vote'*

the Prime Minister was quoted as saying that

> Women were concerned about unemployment and losing their jobs, and saw the issues of pressing concerns to be child care, violence against women and women's health. (The Age, Feb 11, p. 8, 1993)

The article continued:

> Paul Keating (the Australian Prime Minister) finally realised he had a problem with women ... His tone was patronising, his perspective out of touch.... By June, the well-known expatriate feminist Dr Anne Summers was in his office on a ... contract to clean up his act. (The Age, Feb 11, p. 8, 1993)

Part of Dr. Summers' approach was to arrange for Mrs Keating to step 'into the campaign spotlight with a modelling appearance in *Vogue Australia* magazine'. Interviews which

might focus on her opinions, values, or intellectual pursuits, rather than her appearance, were not encouraged. 'Her media relations during the election are being ruled with an iron fist by her special adviser (with her) media exposure (kept) to a minimum' (The Age, Feb 17, p. 1, 1993). *Focusing primarily on women's appearance, it seems, is still considered a vote winner.* The many syndicated articles that regularly appear in the popular press reveal the continued prevalence of stereotyped reporting in other countries as well.

Whatever our own values - and I have concentrated intentionally on aspects of the Australian cultural climate to personalise this section - it is unrealistic to ignore the context in which students make educational and eventual career and life choices. Both short and long term solutions need to be considered if diversity and equity are to be reconciled.

Revisiting previous research

Elsewhere (Leder, 1992) I have argued the importance of using multiple methodologies to examine gender differences in mathematics education. Large scale studies served, and still serve, a useful purpose but need to be supplemented with intensive observations of a small group of students. I will draw on two recent studies from my own work to illustrate why previous research findings should not be accepted uncritically.

Previous assumption 1: *females prefer to work (do mathematics) cooperatively in groups and do better in such a setting.*

Intensive observations, over a sustained period of time, of five students (three females and two males) working on a group project during mathematics lessons revealed considerable stereotyping of tasks. This occurred even though one of the females took on the organisational leadership role, was accepted by the others in that role, and often dominated the group discussion.

The females typically took on (or were assigned) the

recording, colouring in, and presentation of the final product; the mathematical explorations and calculations were carried out primarily by the males. Two representative excerpts from the field notes taken during the observation period are reproduced below:

> *Girls coloured bar graphs, labelled graphs and axes; boys carefully plotted graphs* (Extract from summary of main activities carried out in lesson 4)

> *Girls worked on writing up report; boys worked on mathematical calculations related to projected profits of new canteen* (Extract from summary of main activities carried out in lesson 6)

The students' assessment of the work done in class was realistic and reflected the different activities in which they had been engaged. According to the three girls in the small group, the purpose of lesson 6 was to 'write up the report', 'finishing graphs, working together', and 'writing up the results'. The two boys in the group considered that the main purpose of the same lesson was 'working with calculators, cooperating' and 'figuring out the results'. *Small group work, per se, does not ensure that females are more engaged in mathematical tasks.* What are the features - group composition, task set, classroom climate ... which facilitate diversity and equity?

<u>Previous assumption 2</u>: *females are disadvantaged on multiple choice questions, particularly if a penalty is imposed for wrong answers, because they are less likely to guess/take risks than males.*

Three statistical measures of risk taking were applied to data from the Australian Mathematics Competition, which currently attracts approximately one-third of Australia's secondary school students. In line with findings reported in the literature, the results obtained indicated that males were more likely than females to take risks on multiple choice questions. Yet when asked whether they 'would guess the answer', 'leave a blank', or were 'not sure what I would do'

under different conditions females consistently indicated that they were both more likely to guess than males and less likely to omit questions of which they were uncertain. This interesting paradox warrants further investigation.

Collectively, the two examples illustrate the importance of reassessing earlier findings and of using different research strategies to explore issues which have already attracted research attention.

> Education is a complex social phenomenon. Any research in it, quantitative or qualitative, will involve a good deal of interpretation about what has to be controlled or taken into account to understand the results.... we cannot even presume that the phenomena we study are the same over time or in different countries, that what holds in one situation ... will apply elsewhere. (Yates, 1993, p. 107)

Concluding comments

Mathematics and gender issues have attracted much research attention in recent years and are likely to continue to do so. It is a far cry, conceptually, from sex differences in mathematics learning (a popular topic two decades ago) to our current considerations of equity and diversity with respect to gender. The diversity of research paradigms, research techniques, and settings in which relevant research has been carried out has heightened our appreciation of the myriad of subtle factors that may help explain when diversity is desirable and when it is a cause for concern. As I have argued in a somewhat different context (Leder, 1982) a common language and respect for alternate positions are needed if constructive debate is to occur between those grappling with the same issues from different positions or within different theoretical frameworks.

Educationally, females are not a typically disadvantaged group. They often do well at school and in many countries now have higher retention rates than males. They are certainly capable of doing mathematics, although there are some differences in the ways they participate in mathematics and

related subjects, both in school and beyond. Acceptance of this as a reasonable synthesis of research findings to date inevitably shapes decisions about future research. The agenda for future research most likely to facilitate diversity and equity needs to be set collectively. In this paper I have set out some of the variables and constraints which are a high priority for me. Discussions during the conference should clarify further the most constructive path ahead.

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EQUITY AND A SOCIAL PSYCHOLOGY OF MATHEMATICS EDUCATION

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In previous work, I have critiqued much of the current research in mathematics education for its uncritical use of psychology. That use has resulted in mathematics education accepting a world view in which social and public events have become private and part of an individual's inner world, in which particular cultural norms are taken for granted, and in which social issues are transformed into problems residing within individual students. Based on those critiques and since equity is an issue of social justice, I have questioned the utility of much of mathematics education research for developing an agenda for equity in mathematics education.[2]

Some Legacies of Psychology and Education

Critiques of the uncritical uses of psychology and of structural inequities in education are not new. Women, minorities, immigrants to the United States, and people from lower social class backgrounds have repeatedly been subjected to psychological analyses and research that have defined them as fundamentally deviant and deficient when measured against dominant norms. The common school was created, in part, to Americanize American Indians and the children of immigrants, to prepare young ladies for their roles in society, and to train people from the lower social classes for manual labor and other menial jobs. The purposeful denial of educational opportunity to African Americans, Asian Americans (especially in California), and Hispanics (especially in the Texas and the Southwest) and the use of psychological theories and testing to grant legitimacy to those practices are part of the story of the common school up to a scant 30 years ago.

Over the past thirty years, social and educational policy and research have moved away from accepting beliefs about social-group inferiority and the sorting functions of schools. And, the most blatant actions supported by such beliefs seem to have disappeared. Yet many practices and organizational structures found in schools, which are the legacies of those beliefs, have not disappeared; rather, they have become covert, transformed for other (seemingly more benign) uses, and affect our ongoing efforts to improve schooling for children. What is more, beliefs about non-dominant groups that can be traced to these earlier times reappear with frightening frequency and derive new support from psychological research. Finally, efforts to re-create and improve the sorting functions of schools continue to reenter the public policy and research arenas.

Mathematics educators who work in equity should not be too surprised that progress has been slow. For example, over those same three decades (and even longer if we go back to John Dewey and other progressive educators), educators have invested large amounts of intellectual, financial, and material resources in studying and trying to change how children are taught their school subjects--mathematics being one of the most visible examples. Yet study after study and report after report have documented our limited success in changing the pedagogical legacies of the creation of the common school. Why then, should we assume that in the case of equity--an area in which the intellectual resources have been fewer and policy support often has been one of malign neglect if not outright antagonism--we would have made substantially more progress in changing those same legacies?

Nor should we be surprised that beliefs and inequitable practices, which can be traced to the early years of psychology and the common school and which should be obsolete, reappear with distressing regularity. Many policy makers are the products of those

earlier times. And just as there have been ebbs and flows in research, policy, and practice, between (a) progressive positions and (b) basic skills and traditional forms of teaching, so too should we expect similar sorts of ebbs and flows for the case of equity.

The Current Context

These observations and criticisms should not be taken to mean that I believe that nothing can be developed which would support a mathematics-education equity agenda based on psychological constructs and research. On the contrary, mathematics education is consolidating its hard earned gains in research (viz., the *Handbook*) and policy (viz., the NCTM *Standards* documents). Those gains rely, in large part, on advances in our understanding how people reason and learn mathematics, and how teachers learn to teach mathematics. That work could help in the development of an equity agenda.

What stands in the way of an equity agenda being developed, , however, is that much of this work fails to question the boundaries of current practice in ways that incorporate what is known about the social contexts of teaching and learning. For example, the psychological evidence that is cited as supporting the *Standards* documents revolves around carefully defined ideas of understanding: understanding has coherence, is constructed and develops slowly, is linked to pre-existing knowledge, and can involve the re-organization of knowledge. Absent are the psychology of affect and analyses of (a) the social and cultural origins of understanding and (b) how understanding develops in context.

An Analysis of Reform

For a moment, assume that the case that is built in the various reform documents is compelling. Assume that the school-mathematics curriculum contains trivial facts unconnected to the mathematics that people in the real world will need in order to function.

Assume the curriculum lacks coherence, ignores students' informal knowledge, sacrifices depth for superficial coverage of content, and interferes with the construction of knowledge--in other words, the curriculum actively interferes with how people reason mathematically. Assume that teaching as telling dominates over students constructing their own knowledge. Assume that a single person tries to exert control over the day-to-day lives of students in ways that are problematic and seem unconnected to the purposes of schooling. Finally, assume that the same elementary-school content is covered yearly with little hope of students encountering new and potentially more interesting content. From these claims--that something is pathologically wrong with current practice in school mathematics--have come the calls for change.

For the moment, however, let's put aside those calls and instead inquire about what would constitute a psychologically healthy and adult-like response to such a sad state of affairs. As I suggest in my *Handbook* chapter, the healthy response by any adult would include disengagement (broadly construed) in those practices with a concomitant lower performance on outcome tasks. Among the indicators of disengagement are time-off-task, increased school absence, failure to persist in course taking, and dropping out of school.

Next, consider the student populations who fit this profile. Historically, they have been labeled disadvantaged, at risk, or in other ways that suggest deviance or pathology. They include minorities, students from society's lower socio-economic strata, and the children of immigrants. But, the above analysis suggests that, in fact, these students are reacting in a mature, adult-like manner to educational experiences whose characteristics have remained constant since the formation of the common school. Moreover, this analysis

suggests that, from an affective point of view, the pathology lies in the students who accept such a sorry state of affairs.

Social Evidence

Evidence in support of this analysis can be found in a range of areas. First, there is evidence concerning the low-quality instruction that minority students and students from lower SES backgrounds receive. This includes studies on ability grouping and tracking, on categorical programs such as Chapter 1 (the old Title I) and special education, and comparisons between urban and rural schools on the one hand versus suburban schools on the other. These studies all point to an inescapable conclusion: the worst quality education is provided to those students who fit the profile of being disengaged and low-performing.[3]

Moreover, there is social science evidence to suggest that minority students, students from lower SES backgrounds, urban and rural students, and children of certain immigrant groups who are being socialized according to their traditional norms are all socialized to take adult-like responsibility for themselves at a very early age. This includes caring for younger siblings, performing large and meaningful tasks, and accepting responsibility in and helping to support an extended family. One characteristic of adult-like learning is that people judge the utility of what they are supposed to learn; such judgements are critical for these children.

An Example

As an example of how this analysis might work, consider the case of how one middle-class family confronted a common crisis in the learning of third-grade mathematics. A common third grade objective is the memorization of basic multiplication facts and an all-too-common method for achieving this objective is through the use of timed tests.

Shortly after the start of the second term, a third grader returned home very upset. (S)he began saying that (s)he hated mathematics and was no good in the subject. After much prodding, this child told its parents that the teacher had begun to use two sorts of timed tests. The first was written, in which children were given 3 minutes to complete a worksheet of all 100 facts (up to 10 x 10). The second test was oral; the teacher would call out multiplication problems in a non-stop monotone and children were to write the number sentence with the correct answer. The speed at which the teacher read off the problems meant that the children could not stop and reason through their answers; they simply had to react. On top of that, the teacher posted each child's performance so that he or she could trace improvement during the course of the term; the goal was for each child to score 100% by May.

The father determined that his child actually agreed with the teacher's goal of memorizing the number facts: "You may not think that this [number facts memorization] is important, dad, but I want to get an A." The timed tests were so stressful that the child could not perform well; and also, (s)he was embarrassed to compare her/his performance against the rest of the class. With this information, the father called the teacher and set up a conference.[4]

The meeting between the parent and teacher was uneven, to say the least. The teacher insisted on the validity of the academic goal (memorizing the number facts) and that the method of timed tests was benign: "Like a sprinter who uses time trials to help chart her progress." The father insisted that the objective was not at issue, but that the problem lay in what the timed tests and grade posting were doing to his child . Finally, both agreed that the child had bought into the objective of memorizing the basic facts, that the parents would

help the child memorize them, and that the teacher (who recalled that another student, who had scored a 0 on the exam, cried upon seeing the results) would take steps to reduce the children's anxieties.

The teacher did seem to alleviate the third graders' anxieties, though the timed tests continued through the end of the school year. As children got perfect scores they were given rewards and encouraged to try to complete two or more worksheets in the given three minutes. The parents and child created a ten by ten grid with the number facts on a large poster that was hung in the child's room. The third grader had to figure out what went in each of the squares before the answer was written in. On request, the parents also drilled their child on the multiplication facts.[5]

This particular vignette shows how school mathematics is co-constructed by a child, the parents, and the teacher. In this example, the parents intervened with both the child and the teacher in an effort to repair a rift in how the child was experiencing school mathematics and to minimize the likelihood that the student would disengage from the class because of testing.

All the participants agreed, after some negotiations, to the eventual objective--the memorization of number facts. The distinction between goal and how it was being achieved (timed tests and out-of-school practice) proved to be subtle, but important since it allowed for that agreement to take place.

Interpretation is at the heart of this event. These middle-class parents interpreted their child's reactions as anxiety. Other parents might interpret it in a different way, for example, as an indicator of ability or as another example of the meaningless things that schools ask students to do (and that the child should ignore). The teacher, on the other

hand, saw the timed tests as benign events whereby where children could trace their own growth in "learning the basic facts." The child saw the tests as stress producing, demonstrating a personal lack of ability, and humiliating. One can imagine the teacher's reaction to the student had (s)he, whose interpretation of the test differed so radically from the teacher's, begun to withdraw from mathematics and had parents, who also did not share the teacher's interpretation, sanctioned the child's doing so.

The repair was also socially negotiated. Other children might not be as forgiving of events such as these. Indeed, though this particular student saw that the timed tests were meaningless since the mathematics that they contained did not help in solving real problems, (s)he bought into the teacher's agenda because (s)he wanted to get an A.

Other parents might have reacted differently based on their interpretations of this event. Some middle class parents might provide different kinds of support to their children, intervene with the teacher in a different way, or they might force their child to memorize the facts because they learned the facts in the same way. Other parents might not have done anything, instead leaving it to their children to make some judgements about what is being asked of them and to take appropriate actions. For example, 9- and 10-year old Hispanic children often help to take care of their siblings, may work in the fields if they are migrants, and decide whether or not to go to school because their parents had to go to work early; in comparison, this event seems like a minor one that does not require adult intervention. Finally, other parents, who may have had similar experiences in their own schooling, may encourage their children to resist the teacher's efforts.

There are many possibilities, but two points remain:

1. the memorization of number facts and the use of timed tests were a socially constructed event. In another setting, it may have been constructed differently and the results may have come out differently.
2. psychological theories of learning are inadequate for explaining events such as this. The meanings that were attached to timed tests and to the memorization of number facts, the negotiations among the participants, and their subsequent actions cannot be explained by accounts based solely on the individual. A full account requires cultural and social analyses.

Towards a Social Psychology

One way of reading the above vignette is as a story of how middle-class parents short-circuited what could have turned into an adult-like reaction by their child and to treat *the intervention* as the social pathology which requires psychological explanation.[4] Another reading could come from an equity point of view. In this case, one could look at variability in how similar events are constructed by students, parents, and teachers across different settings. An equity issue would be asking on whom the burden falls for repairing the child's educational experiences.

Regardless the purpose, we need are ways of understanding how variation in setting can lead to variation in meaning, the possibilities of learning, and the learning which actually takes place. It is for this that mathematics education needs to move beyond narrowly construed, psychological accounts to a broader, social psychology.

Ideas from ethnomathematics, situated cognition, social contructivism, and neo-Vygotskiian theories of learning show promise in this regards. All seem to aver that the settings in which mathematics is learned deeply affects how that mathematics is structured,

recalled, and applied in later life. If mathematics is learned as meaningless and done by brute force, then those features (and their consequent affective states) will become part and parcel of a child's basic understanding of mathematics. If the adults who are supposed to support students' learning do not help that learning to make sense or if they do things which lead to student resistance in mathematical contexts, then it should be hardly surprising that students' meta-cognitions and beliefs re-create these same characteristics (recall that meta-cognition is thought to grow out of social interaction).

Hence, the kinds of social psychological efforts that I am calling for are those which combine social and cultural analyses, the psychology of affect (beliefs and feelings), and the psychology of learning.

Equity and Social Psychology

Equity includes a strong value position that we should try to distribute resources fairly and create a fair society. Though there may be disagreement on all of the details of how such an agenda might be enacted, there is broad based agreement among equity advocates that destereotyping of people is an important first step. Unfortunately, even among people who work in equity, there are beliefs about student abilities, upbringing (e.g., cultural deprivation, disadvantagement, at-risk), and the like which have resulted in a discourse of social-group deviancy and deficiency. I would hope that my analysis of the reform's rhetoric makes plausible the possibility of mathematics educators unlearning those beliefs and avoiding the resultant discourse; there are other ways of telling the story of differential student achievement and course taking.

But also, I would hope that we avoid going to the other extreme. Though I may have alluded to a social pathology that resides in the middle class because it prevents its children

from behaving in an adult-like manner, my intent was not to create a new stereotype. My goal was to drive home the point of how value-laden and socially constructed are the ways by which we are interpreting the phenomena that are at the heart of equity.

Nor do I advocate a wholesale abandoning of notions of knowledge deficiency. Deficiency should not be measured against some dominant-group norms; instead, deficiency should be measured against each individual's stated goals. It is a fact that some people cannot accomplish their stated goals because they do not know something that is needed in order to accomplish those goals; in this sense, their knowledge is deficient.

I began and ended my *Handbook* chapter with a call for scholarly inquiry on the nature of differential student achievement. I have sketched out, in some rather broad terms, how such inquiry might need to create a social psychology of mathematics education. The goals of such an effort should be to question the boundaries that have been placed around the traditional uses of psychology in mathematics education and to draw synergistically on fields of inquiry that help us understand the social contexts in which mathematics is taught and learned.

There is one final imperative with which I would end this paper. If my analysis is right--i.e., that the achievement and course taking patterns of students who have traditionally been labeled as disadvantaged and at-risk has been the result of their experiencing the worst cases of schooling--then reform, if it is to matter, needs to begin with those students populations. We need to take seriously their educational conditions and to create curriculum, teaching, and assessment practices which support reasoning, are meaningful to students against their cultural identities and lived realities, and that prepare them for the reality of the world that they will inherit.

There are four reasons for this. First of all, we owe it to them. All along, their performance has been an early-warning signal of what the reform has recently come to understand. Second, they are the most critical test of assertions about reform and teaching for understanding. Other student populations are likely to be more forgiving of our errors and hence their responses to reform efforts will not be a valid tests of its efforts. Third, the history of educational innovation has shown that genuine reforms move from populations that are labeled as economically disadvantaged to their more affluent cohorts, but not in the reverse direction because of institutional and social barriers. And fourth, the history of reform and innovation is one in which the rich get richer: those who are situated to profit from a innovation get a disproportionate amount of those benefits.[7] We need to take affirmative action to avoid re-creating stratification of opportunity along new and more potent lines of enhanced mathematical programming.

1. The preparation of this paper was supported, in part, by a grant from the Office of Educational Research and Improvement, United States Department of Education (Grant No. R117G10002) and the Wisconsin Center for Education Research, School of Education, University of Wisconsin-Madison. The opinions expressed herein are mine and they do not necessarily express the views of either OERI or WCER.

2. Some mathematics educators who write from cross-cultural or international perspectives have criticized American writers for limiting themselves to work that has been carried out in the United States and to American issues or perspectives. Though I have sympathy for that criticism, there is a countervailing danger when writing about social issues: superficial coverage and stereotypes may replace more subtle, nuanced analyses. In the case of equity, I prefer to write about contexts with which I am most familiar and to work in coalition with people who work in other countries or from different perspectives. Hopefully, as we find common ground in our work, we will be able to generate broader analyses that avoid the pitfalls of stereotyping cultures or countries.

3. I am not suggesting that the quality of education received by middle class or white students is wonderful. What I am rejecting, however, is the simplistic assertion that things are uniformly bad everywhere. There are important distinctions to be made in the quality of education that students receive; and we serve no one very well by ignoring those distinctions, as do many of the reform documents.

4. Prior to that meeting, however, was scheduled another series of timed tests. The night before these weekly tests, upon realizing that (s)he could not avoid them, the child spent a full half hour breathing into a pillow, unable to go to sleep until the parents said that (s)he did not have to go to school the next day.

5. Multiplication flash cards were sold out at the local teacher store. The sales clerk noted that they did a brisk business from third grade parents throughout the school system at that time of year.

6. Indeed, given the complaints about parental resistance to new mathematics curricula and evolving ways of teaching, such a reading may not be too far off the mark.

7. For example, in our study of school-level reform in the teaching of mathematics, Lisa Byrd and I have surveyed of schools that were nominated as engaged in efforts to substantially enhance their mathematics programs. Over 70% of the 200 schools that responded to our surveys described themselves as suburban.

DIVERSITY AND EQUITY: AN ETHICAL BEHAVIOR

Ubiratan D'Ambrosio

Comments motivated by the papers of Gilah C. Leder and Walter G. Secada presented at the 15th Annual Conference of the North American Chapter of the International Study Group for the Psychology of Mathematics Education, Pacific Grove, CA 17-20 October 1993.

Although the main concern of this meeting is Mathematics Education, I believe I will be allowed to subordinate my comments to a higher objective: the survival of civilization on Earth with dignity for all. This is not merely jargonizing. The world is threatened, not only by aggressions against nature and the environment. We are all equally concerned with increasing violations of human dignity. We face more and more cases of life under fear, hatred and violations of the basic principles upon which civilization rests. A return to barbarism and even the inviability of our species are possibilities for the future. This has everything to do with Education, in particular with Mathematics Education. We recognize that Mathematics has so much to do with the modern world, with everything that is identified with advanced science and high technology, that Mathematics indeed created the possibilities of such development. Besides, Mathematics has understood standards of precision, rigor and truth which permeate every fact of modern thought. Mathematics is a body of knowledge which grows from early concepts taught in the schools and popularly, rightfully, identified as a single mode of thought. Are we ready to risk a return to the Middle Ages' clamor against mathematicians? Or to some literary views of Mathematics as having nothing to do with love and tenderness?

I see our mission as educators as the generators of new directions towards allowing each one to develop and to fully realize his/her creative potential and towards a more just relationship between individuals, escaping from vicious behavior. We as Mathematics Educators use Mathematics as an instrument to carry on this major mission. I hope everyone agrees, Mathematics has much to do with this role, and I hope no one will object to these major, someone may call visionary, objectives underlying my comments. If we are unable to identify in Mathematics a role compatible with this, why Mathematics in schools?

I like to see Mathematics Education in these broad terms, particularly when related to diversity and equity. I can sympathize with Gustave Flaubert when I read his pungent defense of the right of the people to read *Madame Bovary* and the almost despair of D.H. Lawrence when *The Lover of Lady Chatterly* was banished. All this has to do with diversity and equity. We now face similar problems of a similar nature as Mathematics Educators.

The two major objectives I see for Education rely upon diversity - each individual is different from any other individual - and equity - just, fair and impartial behavior not imposing on anyone's right to develop and realize his/her creative potentials. Although somewhat unusual in discussions about Mathematics Education, I will begin with some reflections on ethics.

Diversity and equity are implicit in what is called the ethics of diversity, which calls for an individual behavior of human beings based on i) respect for each other, as individuals with all their differences, and for a social behavior of human beings showing and acting with full respect for cultural diversity, both essential for the creativity and preservation of the dignity of the species; ii) for each and every human being to be in solidarity with fellow human beings in the satisfaction of their needs for survival and transcendence; iii) and human beings sharing their responsibility in the protection of the common good and of bio-diversity, essential for the preservation of life in its various forms. Love, solidarity, and responsibility are the essence of this ethics of diversity.

Some may be questioning what does this have to do with Mathematics Education. I say, everything! And the papers of Gilah C. Leder and Walter G. Secada, in which special attention is given to research in Mathematics Education, focus on these same issues. I may say I am echoing the concerns of both, maybe in a different style. I hope these thoughts will not be merely discarded as jargonizing, a classifier which has been entering some educational circles in the USA.

The title of Leder's paper reveals in itself the main difficulty of the issue. It touches the ideological issue. To be more explicit in bringing ideology to the discussion, I would add the word "desirable" to the title. And by accepting the desirability of diversity together with equity, an ethical behavior would be implicit in the entire discussion.

Leder focuses on gender. The first and major part of the paper is a well-organized search of the literature and a very convenient system for the analysis of this literature. Indeed gender is in the root of the theme diversity and equity. Nothing is more evident and desirable and needed with respect to diversity, and nothing is more violated throughout history with respect to equity. And we see a much broader perspective on gender based inequities in several sectors of societal behavior, particularly scientific knowledge. It is clear that in every culture we have seen the growth of a class which expropriated knowledge generated by the people as a whole. And for various reasons, the main one is the "golden egg metaphor," that is preserving the women as producers of the main factor in work. Women are excluded from ruling classes, hence from having influence over this production. While shocking to some analysts, this seems to be the root of the inequities. The prevalent idea was: let us preserve women to do what can not be done by men. Males could not produce new workers. This is clear in the Middle Ages figure of the "pregnant male" and the conflicts involving Mariolatry. Of course, the growth of the importance of Mathematics in societal affairs naturally implied the undesirability of having women mathematicians. This is absolutely the same behavior as relating to slaves, to immigrants, and to other classes of individuals which were an important factor in production: that each one be placed in the societal machinery as cogs.[1]

According to Leder, this was a "popular topic two decades ago." Indeed, we can trace these concerns much earlier, explicitly back in the eighteenth century, when models of democratic societies were taking shape. The comparative study of Marc-Antoine Jullien, in 1817, has a special section devoted to "Education of Women," which discusses the "influence of women which, well directioned, becomes the complement of education for men."[2] It is clear, in these as in other proposals, that classes were educated with specific purposes to serve a hierarchical social structure.

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[1] See the interesting collection of essays The "Racial" Economy of Science: Toward a Democratic Future, ed. Sandra Harding, Indiana University Press, Bloomington, 1993.

[2] Marc-Antoine Jullien de Paris, Esboço de uma obra sobre. A Pedagogia Comparada, trad. Joaquim Ferreira Gumes, Atlantida, Coimbra, 1967 (ed. orig. 1817); p. 40.

Thus research in this area has much more to do with social and political factors than with cognition. Every and each individual learns differently from each other: John and Thomas, identical twins, react differently in the same situation, as well as John and Jim, neighbors, so why not John and Mary, and why not Mary and Kun-Teal and Sigrid and Ogunlade? People are different -- period! A broad theoretical framework allows for the recognition of this kind of behavior. Regrettably, much of current research in cognition by Educators is narrow and thin. I disagree with Secada's claim that criticism of American writers comes from their emphasis on American issues and perspectives. It is correct to do this since it refers to the country's most immediate interest. My view is that the criticism comes from an intellectual behavior similar to what in another context Senator James W. Fulbright labeled *The Arrogance of Power* (1967). But it is absolutely unfair to say that this is characteristic of USA academe. This is typical of scholarly chauvinism so common in the academic circles. This is manifest all over the world, above all in the choice of editorial boards of journals, in the blind referring system, in the citation game in which "dwarfs stand on the shoulders of dwarfs" and in several other indicators which are, indeed prevalent in the USA.[3]

What does this have to do with the topic of this paper? Clearly, this attitude of academia is carried on to Education, particularly to Mathematics Education, as a form of pedantry. As a result, the public image of the mathematician is built on this, the attitude of the teacher reflects this and school systems act based on this perception. This arrogance of knowledge, intrinsic to mathematical knowledge, is in the root of our discussion on diversity and equity. I hope to make this clearer as we proceed.

The importance of research is undeniable. But how does research affect practice? Research results may help the future work with the subjects of research, that is students plus the researcher(s). But change where and when or change one individual and we have a different group, an entirely different situation.

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[3] This is not easy for me to say and I anticipate a possible resentment. This is not new. When he was asked to make a study of the Swiss educational system, back in the early nineteenth century, Marc-Antoine Jullien of Paris anticipated the reactions of the Swiss: "He is a foreigner, they will say, a Frenchman in charge of a study of comparative pedagogy in the several cantons of Switzerland." See footnote 2, p. 33.

Let me give an image. A good actor the moment he enters the stage to perform Hamlet reviews the audience and recognizes that they all, his faithful fans who acclaimed him three years ago, are there. But he also recognizes in the audience Mr. Peter Claudius, who a few weeks ago had been considered not guilty by a popular jury - on a tight vote - on a passionate murder case. He must realize that if he performs Hamlet with the same accent as he did three years ago, this time instead of applause he may incite a riot!

Rhetoric is as yet a major tool in education even if we concede that the role of the teacher is changing to a managing role in a joint educational venture.[4] To know the learners is essential for the teacher. And research is an instrument for knowing. Thus the teacher as a researcher comes as a most attractive proposal. These holistic views of education and the role of research in educational practice do not differ in essence from the position expressed by Leder in her paper. Indeed she quotes Yates (p. 12) to stress the fact that there are differences in time as well as geography when using research results. The same situation may give origin to different strategies. Again, the teacher as a researcher is a possible direction to follow.

From the very beginning of his paper Walter G. Secada recognizes that equity is an issue of social justice. He touches the main point when he says that "school was created, in part, to americanize American Indians and the children of immigrants, to prepare young ladies for their roles in society, and to train people from the lower social classes for manual labor and other menial jobs." (p. 1). And he proceeds to recognize that efforts to change this are met with reactions which practically annihilate these efforts. In the midst of recognizing these difficulties a message of hope is implicit. An anthological assertion in Secada's paper is the following: "consider the student populations who fit this profile [of time-off-task, increased school absence, failure to persist in course taking, and dropping out of school!]. Historically, they have been labeled disadvantaged, at risk, or in other ways that suggest deviance or pathology. They include minorities, students from society's lower socio-economic strata, and the children of immigrants. But, the above analysis suggests that, in fact, these students are reacting in a mature, adult-

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[4] As proposed in Ubiratan D'Ambrosio "Environmental Influences," Studies in Mathematical Education, vol. 4, UNESCO, Paris, 1985; pp. 29-46.

like manner to educational experiences whose characteristics have remained constant since the formation of the common school. Moreover, this analysis suggests that, from an affective point of view, the pathology lies in the students who accept such a sorry state of affairs." (p. 4). This goes straight to the point. The system labels them misfits.

A similar approach is central in Michel Foucault's analyses of systems such as health, justice, knowledge, and in his analysis of sexuality. Why not of Mathematics Education, where the filtering tools are so subtle and at the same time so strong? It is very difficult to go to the essence of the subject without raising the issue of the importance of history and epistemology. Except for the brief initial remark of Secada quoted above, both papers avoid these discussions.

We need an holistic approach to knowledge, such as the one in the Program Ethnomathematics, to face all the complexities of the theme, which bring together cognitive, epistemological, historical and political analyses.[5] Secada provides examples which have correspondents in several other cultural environments and which are in the heart of these considerations and his two points on the top of page 9 can be regarded as a research program. Indeed he proposes this as a broad social psychology. I believe research on the theme diversity and equity has to follow these paths.

But then Secada becomes compromising in his proposals, allowing for some commonalities that interfere with diversity. The first paragraph under the heading "Towards a Social Psychology" (p. 9) points to some basic issues. He recognizes that parent reaction to innovation could be interpreted as an important form of pressure. I am bold in saying among Mathematics Educators that this may be the result of an over-valuation of Mathematics in general education. I know this is a very sensitive question, but it has to be considered if we want to face seriously the matter of diversity and equity.

This is in fact slightly implicit in Leder's paper. I will rephrase what she writes at the end of page 12, enlarging the class of "they" - females in the writing of Leder - to the totality of the species:

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[5] See the historiographical proposal in the paper, Ubiratan D'Ambrosio, "The cultural dynamics of the encounter of two worlds after 1492 as seen in the development of scientific thought" in Impact of science on society, vol. 42. n.3. 1992; pp. 205-214.

Everybody is certainly capable of doing mathematics. Surely, but at what expense to their individuality - hence creativity? Is it worthwhile? Does it have a justifiable purpose? This brings to my mind the educational proposal in Anthony Burgess' A Clockwork Orange. I discuss these matters with some detail in another paper.[6]

Summing up, I claim that diversity stands for creativity as biodiversity stands for the environment: absolutely necessary. And that equity is the essence of morality. No one disagrees. Indeed, the two papers reaffirm this. Then, why this meeting and the papers? Just to reassure us of what no one disagrees?

Of course, everyone is waiting for proposals on how to preserve diversity while at the same time achieving equity.

My radical - ideal - proposal calls for an inversion of roles: the teacher learning from the students and the students learning from their environment (nature, artifacts, family, community, peers) and all together - students and teacher - moving into critical reflection. Practical proposals on how to bring this into effective classroom practice is contained in several projects developed in the course of these last twenty years. Of course, compromising has always been necessary; fortunately, at a decreasing rate. My analysis is that eventually the dream - nightmare for others! - will turn into reality, in spite of a few backlashes.

6 Ubiratan D'Ambrosio, "Mathematics and Literature" in Essays in Humanistic Mathematics, ed. Alvyn White, MAA (to appear).

ISSUES IN ACHIEVING EQUITY IN MATHEMATICS TEACHING AND LEARNING

Nadine S. Bezuk and Frank A. Holmes

San Diego State University

Our aim is to achieve equity in mathematics for all students, particularly those from diverse ethnic and cultural groups. We believe that several issues are important in attaining this goal. This discussion session on equity will focus on three broad issues: How to facilitate collaboration among different groups of people involved in students' education, how to facilitate change in teachers, and how to impact students.

How to facilitate collaboration among different groups of people involved in students' education?

Our thesis is that many different groups of people (such as teachers, parents, students, university professors, and district administrators, for example) must work together toward achieving the goal of helping all students succeed in mathematics. What contribution can this collaboration make? What types of collaboration are important? What factors facilitate or impede this collaboration?

How to facilitate change in teachers?

Some factors that we have found to be effective in changing teachers' approach to teaching mathematics include creating a sustained period of interaction between teachers and mathematics educators, with workshops and discussion groups year-round rather than just for a few weeks in the summer; as well as providing opportunities to "try out" new techniques in a less structured setting, in our case, in before- or after-school mathematics enrichment sessions with children.

How to impact students?

There are several components of our philosophy regarding improving students' mathematics understanding, achievement, and attitudes. These include the following: (a) "beyond the bell" activities:--in our case, before- and after-school mathematics enrichment sessions, which increase the amount of time students devote to mathematics each week; (b) enrichment rather than remediation, aimed at helping students become interested in mathematics, confident in their ability to succeed in mathematics, and viewing mathematics as more than just computation; (c) focus on developing conceptual understanding with students constructing knowledge rather than on drill and practice aimed at memorizing procedures; and (d) starting at an early age--in our case, in second grade, providing a solid foundation and feeling of success.

Our discussion session will consider these issues aiming at establishing a dialogue among persons of different experiences and viewpoints and expanding and elaborating on these critical issues.

Title

Feminist Perspectives on Equity in the 1990"s

Presenter Lyn Taylor & Charlene Morrow
University of Colorado-Denver, Mount Holyoke College
Institution

We propose to facilitate an equity session focusing on: reactions to the plenary panel; avenues to "connected" teaching; equity implications from the NCTM Standards (both the *Curriculum and Evaluation Standards for School Mathematics* (1989) and the *Professional Standards for Teaching Mathematics* (1991)); recent Women and Mathematics Education (WME) equity contributions; issues raised by the AAUW report (February 1992) about how poorly girls are served by our nation's schools; and international perspectives on gender equity presented at IOWME in Quebec (1992). We believe that there are common themes that weave together and connect these topics.

In *Women's Ways of Knowing* (1986), Belenky, Clinchy, Goldberger, and Tarule give a compeling description of the ways in which our nation's educational approach alienates women from most academic areas. As women described meaningful educational experiences, it was clear that these instances allowed the weaving together of multiple aspects of their lives. These have been called connected learning, meaning that students are encouraged to build on their entire knowledge base rather than leaving all personal experience at the classroom door. A teacher who facilitates this kind of learning is engaging in connected teaching. The following avenues describe the journey from disconnection to connection that we have observed in our students:

Confirmation of self in the learning of mathematics
Learning in the believing mode of communication and questioning
Taking on challenges with support
The development of voice
Becoming a constructor of knowledge
Redefining and connecting mathematics in meaningful ways

We will describe and facilitate discussion among the participants in this session about the approaches used in our classrooms (at SummerMath and the University of Colorado-Denver) that facilitate our students' journeys to become connected knowers.

As President and past-President of WME we would like to share and discuss WME's gender equity contributions (Think Together sessions at NCTM meetings, resource bibliography, newsletters, annual program at NCTM, and our current proposal for producing a new interactive video series updating the *Multiplying Options and Subtracting Bias* intervention program). We look forward to facilitating lively interactive discussions about these new frameworks for thinking about gender equity in the 1990's.

ON THE NOTION OF AUTHORITY APPLIED TO TEACHER EDUCATION

by
Thomas J. Cooney
University of Georgia

There has been a dramatic change in how we see research contributing to the improvement of teaching and teacher education. We have come to appreciate the value of descriptive studies that yield insightful stories about teachers and life in the classroom. But the question arises as to what role theory should play in the telling of these stories. One particular theoretical orientation that has potential involves the notion of authority. Scholars such as Green, Rokeach, Perry and Kelly all discuss aspects of beliefs and the role authority plays in conceptualizing beliefs. By considering the nature of authority, we can conceptualize the process of teacher education from a perspective that can enhance our potential for conducting research and for designing teacher education activities.

We have come a long way in terms of thinking about what constitutes science in the field of mathematics education and how research can inform the art of teaching. Even a brief comparison of chapters in the first three Handbooks of Research on Teaching reveals profound differences in how the field has changed. I would make the same claim with respect to teacher education. Compare the review written by Brown, Cooney, and Jones (1990) with that written by Cooney (1980) a decade earlier. Research has moved from an analysis of what teachers *were* to what teachers *did* to what teachers *decide* to the more contemporary emphasis on what teachers *believe*. It would limit our characterization of change, however, to focus only on what is studied. What has also changed is our orientation and expectations regarding the kinds of outcomes we value from research on teaching and teacher education. We have clearly moved away from deterministic methodologies toward more descriptive ones.

The seminal work of Kuhn (1970) laid the foundation for us to think of research less as searching for answers and more as understanding context and meaning. Concomitantly, the Platonic view of mathematics that often underlied the positivist approach to research on the teaching of mathematics has been challenged and essentially discarded. Further, the innovative work of Thompson (1984) and Ball (1988) have emphasized the human experience in teaching. Mitroff and Kilmann's (1978) personifications of different research types highlight a crisis of belief about science and allowed us to appreciate the role of telling stories as a means of conveying research findings.

Although I am very sympathetic with respect to the many wonderful stories and descriptions we have about life in the classroom and about why teachers believe and behave as they do, I have a certain

uneasiness about this line research. Certainly careful descriptions of teaching can better enable us to understand and focus on what is really important. Further, descriptive studies do not have to serve as the step child for experimental studies. But it does seem reasonable to challenge ourselves to ask the question, "For what purpose are the descriptions intended?" This is not to pass judgment on what has been done, but rather to bring into question the relevance of our orientations toward teaching and teacher education. How is it that we can orient our research so that we can collectively develop the kind of descriptive power that propels us forward as a field? I am concerned that our current emphasis on description is, in some sense, for its own sake and creates the potential of us entering a new era of dust bowl empiricism in which the dust consists of endless descriptions that fail to make us collectively wiser as a profession. As a partial response to this concern, I would like to address a possible perspective on how we can orient ourselves toward teacher education so as to improve both its theory and practice.

The Importance of Orientation

It is difficult to imagine a quality mathematics teacher education program that does not embrace a significant amount of mathematics albeit connections between teachers' knowledge of mathematics and their teaching of mathematics is not well documented. There is evidence (See Brown, Cooney, and Jones, 1990) that many elementary teachers lack the mathematical sophistication necessary to promote the kind of reform being called for by the National Council of Teachers of Mathematics. While the documentation that elementary teachers lack an understanding of topics such as ratio and proportion, geometry, measurement, and number relationships is not unusual, it begs the question of how this lack of understanding influences instruction or how it should be addressed. So long as we hold the view that "something" is lacking, (e.g., mathematical knowledge), we run the risk of conceiving solutions that emphasize the "giving" of something--more mathematics, pedagogy, or psychology.

As a profession, we have distanced ourselves from thinking of a child's mind as a pot to be filled. It is less clear, however, that we have collectively discarded this metaphorical perspective with respect to teacher education We are filled with wonderful ways of telling teachers what it is that they should know about how students construct knowledge. If teachers are asked to learn mathematics through a process of transmission, then we should not be surprised that their teaching consists of basically this same means of delivery. I recall an interview with a mathematician who maintained that his lectures could help

students see mathematics come alive. He failed to realize the incongruity that exits in trying to make something come alive through a passive medium. In short, the medium belied the message.

Studies cited by Thompson (1992) reveal that many middle school and secondary teachers communicate a limited view of mathematics. Although it is not clear whether the teachers held a limited view of mathematics or whether the ethos of the classroom encouraged or even dictated the communication of a limited view, the question seems moot when you consider the effect on students. Other studies (See Brown, Cooney, and Jones, 1990) suggest that what a teacher thinks about mathematics is influenced in fairly substantial ways by their experiences with mathematics long before they enter the formal world of mathematics education. Further, these beliefs do not change dramatically without significant intervention. What seems crucial is that we develop a way of thinking about how teachers orient themselves to their students, to the mathematics they are teaching, and to the way that they see themselves teaching mathematics.

Considering an Orientation Toward Teachers' Beliefs

How is it that we can think of teachers' beliefs in a generative way? Green (1971) offers the following three metaphorical ways of thinking about beliefs.

> We may, therefore, identify three dimensions of beliefs systems. First there is the quasi-logical relation between beliefs. They are primary or derivative. Secondly, there are relations between beliefs having to do with their spatial order or their psychological strength. They are central or peripheral. But there is a third dimension. Beliefs are held in clusters, as it were, more or less in isolation from other clusters and protected from any relationship with other sets of beliefs. Each of these characteristics of belief systems has to do not with the content of our beliefs, but with the way we hold them. (p. 47-48)

The constructs of quasi-logical and psychological strength are quite different. A person can believe that technology should be used to teach mathematics and therefore believe that students should be allowed to use calculators--a sort of primary/derivative kind of belief. But the psychological strength of this belief may not be very strong. That is, when a teacher is faced with the usual classroom impediments, his/her commitment to use calculators dissipates.

Rokeach (1960) also talks about beliefs being psychologically central and the notion of primary beliefs. He writes, "The concept 'primitive belief' is meant to be roughly analogous to the primitive terms of an axiomatic system in mathematics or science" (p. 40). From these primary beliefs stem other

beliefs that are more peripheral. Like Green, Rokeach discusses the notion of beliefs that are in isolation from each other. Thus, according to both Green and Rokeach, it is possible for a teacher to hold simultaneously that problem solving is the essence of mathematics and that students best learn mathematics by taking copious notes and mastering every detail. Isolation occurs when the contradictory beliefs are not explicitly compared or when beliefs are held from a nonevidentual perspective, that is, they are immune from rational criticism (Green, 1971).

For Rokeach (1960) belief systems exist along a continuum of openness /closedness. Briefly, a closed system is one marked by a dogmatic state--where things are right or wrong, black or white, and shades of gray do not exist. It is here that the notion of authority and how one relates to authority plays a central role in determining whether a system is open or closed. Perry's (1970) scheme of intellectual development focuses on an individual's relationship to authority. When a person believes an authority is all knowing, answers to questions are based on that authority and consist of a single voice. This typifies what Perry calls dualistic thinking and reflects what Rokeach calls a closed belief system. Perry's analysis of intellectual development moves from dualism to the recognition that multiple interpretations of events exist without evaluation (multiplistic position) to an evaluation of each position (relativist position) and finally to a commitment to one of the positions (commitment position). Kelly (1955) introduces the notion of "permeable constructs" as a way of describing the extent to which evidence can be incorporated into a person's belief systems. In this sense, a permeable belief is quite similar to what Green calls "evidentially held" beliefs. Kelly's notion of "permeable beliefs" is also related to Perry's scheme of intellectual development in that a dualistic perspective (Perry) that is based on the absoluteness of authority is unlikely to result in permeable beliefs. Reliance on external authority precludes the acceptance of evidence contrary to constructed beliefs since the external authority on which the beliefs are predicated is omnipotent.

The common denominator in Rokeach, Perry, and Green's analyses is that of how people orient themselves to authority. When a teacher believes or teaches from a set of nonevidentually held beliefs the act of "teaching" becomes one of indoctrination (Green, 1971). The notion of indoctrination is particularly relevant as we consider the notion of helping students believe that mathematical truth is a function of reasoning rather than a declaration from authority. Fundamentally, the issue is one of how a person comes to know something. In this sense, there is a certain inseparability between the

mathematics that is taught and the means by which it is taught. This inseparability is often lost in our zeal to "train" or to "give" teachers whatever we deem their "deficiency" to be. It is a common trap for all teacher educators as we fail to see the symmetry between what and how we teach teachers and what and how they teach their students.

The Notion of Authority Applied to Teacher Education

Green (1971), Rokeach (1960), and Perry's (1970) emphasis on authority has considerable import for how we can think about teacher education. It provides a leverage point in helping us consider how teachers can separate authority as a responsibility for classroom management and authority as the determiner of truth. While Maturana's (1978) concept of the individual as a self-organizng entity provides an important perspective for teacher education, the question I am raising is how the person orients that organizing process given the context in which they exist. For example, most of the research with which I am familiar suggests that teachers who are authoritarian by nature also place considerable emphasis on basic computational skills. (See, for example, Donovan, 1990.) In a current project on assessment, I have found that the teacher who is the strongest disciplinarian (whose students are challenging from a management perspective) also has a view of mathematics that is computational in nature. In this project, she has developed expertise in creating and using open-ended questions but always in a computational context, e.g., "How would you explain to Tom that he made a mistake if he thought 1.26 + 3.4 = .150?" Another teacher, who teaches in a relatively high SES, private school, uses open-ended questions in a much more expansive way, e.g., "Pete maintains that an equilateral triangle can not have a right angle. Donna claims that he is wrong? Who is right and why?" The second teacher, who has minimal concerns regarding classroom management, places much more of an emphasis on the students' internal reasoning processes to determine the truth of a mathematical statement. The challenge we face is to help teachers sort out these two distinct responsibilites as they reconsider their role as teachers of mathematics.

In a recent methods course, we were using materials developed as part of an National Science Foundation project with preservice secondary teachers. During the conduction of one of the experiments, one of the preservice teachers proclaimed with a sense of satisfaction, "I finally know the right way to teach mathematics!" It was a moment of both triumph and defeat. Triumph because she conveyed a sense of exuberance and understanding about what that experiment had been about; defeat

because she missed the more general point that the teaching of mathematics is problematic and cannot be reduced to any predetermined "right" way. A relevant question is whether her beliefs about the teaching of mathematics were held evidentially, thus making them more resilient when buffeted by difficulties she would surely face as a beginning teacher or whether they were held nonevidentially being rooted in what an authority had told her. Another question is whether her more currently constructed belief is psychologically central (an unlikely event) or if it is consistent with whatever constitutes her "core" set of beliefs. If the "core" belief comes into conflict with the more peripheral belief (i.e., using experiments to teach mathematics) then it will surely be the case that the peripheral belief will cease to influence her teaching of mathematics.

The case of Fred (Cooney, 1985) presents a teacher who held a relativistic view of mathematics (or at least one in which problem solving was seen as the essence of mathematics) yet held a dualistic notion of pedagogy. When the students failed to appreciate his more experiential way of teaching mathematics, he resorted to teaching by the textbook. He had no pedagogical alternatives to help him realize his problem-solving orientation toward the teaching of mathematics. It appeared that his beliefs about mathematics and his beliefs about the teaching of mathematics were held in isolation from one another.

A Concluding Remark

There is considerable rhetoric at the present time about teachers reflecting on their teaching of mathematics and about helping them to become aware of their teaching behavior. Fundamentally, the notion of reflection is rooted in the constructivist notion of adaptation. (See, for example, Von Glaserfeld, 1989.) The relevance of reflection and adaptation to the preceding analysis is that neither can meaningfully take place from a closed, dualistic perspective. In a study reported at this conference, we found that two preservice teachers initially rejected the use of technology as a means of teaching mathematics. One of the teachers maintained this belief throughout his undergraduate experience while the second teacher's belief was transformed into a conviction that technology was an essential tool for teaching mathematics. Based on observational and interview data, we concluded that the first teachers' belief systems were essentially dualistic, closed, and isolated. They seemed impermeable. The second teachers' belief systems were more permeable and open. That is, the second teacher's peripheral belief about technology was eventually incorporated into his more psychologically central belief that his primary objective in teaching was to help students develop into responsible adults.

There is no merit in using various theoretical perspectives to describe what teachers believe and do unless that orientation enhances our research and development efforts. I believe an analysis of authority and belief systems has the potential to enhance that effort. It encourages us to see teachers' beliefs as systems of beliefs and not as entities based on singular claims. Too, the notion of authority provides us with a conceptual orientation that enables us to create activities that encourage teachers to wonder, to consider what might be, to reflect, and, most importantly, to be adaptive. Such an orientation is essential if we value moving the enterprise of teacher education out of the realm of being simply an activity and toward being a discipline worthy of study.

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THE ROLE OF TECHNOLOGY IN RECONCEPTUALIZING FUNCTIONS AND ALGEBRA

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An invited address for the Group for the Psychology of Mathematics Education -- North American Chapter[1, 2]

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In mathematics education, we see evidence of two distinct orientations to the use of computer technology. The first is oriented towards using the technology to teach the traditional mathematics topics, with a more efficient, dynamic, or appealing presentation. Accordingly, the computer is viewed as an efficient or an effective device for carrying out the standard algorithms or for storing large quantities of information. The majority of the uses of graphing calculators fall into this first orientation---their advocates imply that they necessitate no serious reconsideration of our curriculum. However, if we automate the topics we used to make the basis of our curriculum, (such as long division, symbol manipulation, plotting graphs) without rethinking the curriculum, we will encounter resistance by those who understandably fear the students will learn less. This way of approaching the new technologies, which I will refer to as "unplanned obsolescence," is an intellectual dead-end. It teaches us, however, a critical lesson about our curriculum. Our standard curriculum is not predetermined by some external structure inherent to the discipline of mathematics; it's persistence depends on its inertia, making it inherently conservative. Challenges to it can be successfully blocked by: 1) our own dogged conceptions of mathematics, 2) a self-perpetuating curricular system which is held in check by accountability to pre- and post-requirements, 3) unimaginative assessments, and 4) smug and unjustified assumptions about who can and cannot learn mathematics. These systemic constraints have allowed the curriculum to resist change for too long, in spite of its weak connections to practices outside the school, its generative sterility for its students and discriminatory its practices.

The second orientation to the use of technology entails the expectation that new technologies will fundamentally transform the curriculum. This orientation recognizes that technologies (that is, any significant tools) necessarily alter the character of knowledge. Knowledge, in this sense, is

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not a set of descriptions about the world, but a set of hard-worn realizations about how human beings interact with the world (including with each other) through the use of tools (including language). One can distinguish two broad categories of tools--those that help accomplish some task, working tools, and those that assist us in communicating to others, communication tools. Both types have significant impact on what we consider to be knowledge and how we share that knowledge with others.

In 1985 at Cornell University, I designed and implemented a new precalculus curriculum designed around the use of contextual problems, families of functions and their transformations. In 1987, my research group and I designed and implemented version one of Function Probe, a multi-representational software that is built around the use of graphs, tables, equations and a calculator. As we have used and revised the curricular materials and software, we have found ourselves needing to alter significantly our understanding of functions and functional families. These changes have led us to reconceptualize our views of mathematics and how it is taught and learned. So, I would like to begin by discussing the theoretical view of mathematics that currently guides our research and design work, then discuss our design principles, and finally I will illustrate some of them using examples concerning functional relationships.

Theoretical Underpinnings in Piaget and Vygotsky

I have argued elsewhere that Piaget and Vygotsky's frameworks for intellectual development need to be integrated, but that the integration is not simply a matter of mixing and matching the two theories (Confrey, in press a; Confrey, 1993). In discussing one possible form of integrating the theories, I have begun by pointing out a fundamental characteristic that is shared by the two: genetic epistemology. Both theorists believed that the character of knowledge is comprehensible only by examining its genesis. This view is the central tenet of all constructivist epistemologies, and it is the strand which unravels the tenaciousness of absolutist or realist perspective of knowledge. Genetic epistemology leads us to abandon the view that knowledge can be detached from humanity and described as the accumulation of facts. Unless one believes in an immaculate conception of knowledge, one must recognize that the "true" (as eternal, universal and depersonalized) view of any knowledge no longer is viable. Genesis implies that knowledge develops in relation to humanity. It requires a knower and a known as an indissolvable pair. Thus, knowledge is necessarily embedded in a historical, cultural and environmental context.

Where Piaget and Vygotsky differ is in their choice of emphasis concerning the sources of knowledge. Piaget locates his primary source of constructive activity in the interactions between a person and his/her environment. He suggests that by examining the actions a person takes while engaging with challenging tasks, we can find the seed of his/her knowledge creation. Perturbations, disequilibrations or problematics (felt-needs to act) are the catalysts for constructive activity. These disequilibrations, experienced in relation to a person's current perspectives, then are either admitted or ignored, and if admitted, they either fit neatly and are assimilated to existing

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structures or they perturb the existing structures, forcing accommodation. Over time, the repetition in experiences lead to internalization through a process of reflective abstraction. This process of problematic, action and reflection can be described as the development of *schemes*, where a scheme becomes the means by which one anticipates, acts and mentally operates, and finally assesses the outcome (Confrey, in press a; Steffe, in press). Contrary to most charactures of constructivism, Piaget did recognize a significant role for others, but the role was one of socialization for the most part, where language served to capture and communicate the results of cognitive activity, rather than as guide to its construction.

In contrast, Vygotsky locates the major impetus for knowledge development in one's social interactions and cultural context. For Vygotsky, the higher cognitive ideas are first between people, interpersonal and then become internalized to be intrapersonal. ."It is necessary that everything internal in higher forms was external, that is, for others it was what it now is for oneself" (Vygotsky, cited in Wertsch, 1985, p. 62). Vygotsky's work has two primary intellectual roots. One is in the work of Marx and Engels, and is built around the idea that it is through labor, laboring on an object and with tools, that one witnesses fundamental transformations in the objects and discovers invariances. "Activities" which include labor, play, schooling, family living etc. then are the source of knowledge. For Vygotsky, his original focus on labor led him to assert that tools mediate knowledge. Vygotsky then extends the meaning of tool to view language as a form of a psychological tool.

The second influence on Vygotsky is Hegel's dialectics. When I first read Vygotsky, I treated the dialectic as a dichotomy, an "either-or"-- however, a dialectic involves the pulling apart of two components to create an opposition (as in a tension) which would collapse if either partner could not sustain its own integrity. And, a dialectic should invite one to examine the interplay in the space created by the opposition.. In Hegel, this allows one to work towards the creation of a "unity of opposites." For Vygotsky, knowledge evolves as the dialectic or interplay between thought and language which he argues have different roots. He creates his dialectic by locating the origins of thought in tool activity while locating the origins of language in social interchange. In the process of language development, he stresses the role of affect and the imitation of sound, rhythm and gesture. Language and thought begin to interact, according to Vygotsky, around age two and higher level development results from this interplay. Thought influences language development and language development influences thought.

I have argued elsewhere that Vygotsky's empirical work did not recognize two equal partners in this dialectic -- that in choosing "word meaning" as his unit of analysis, Vygotsky privileges the systematic, taxonomic forms of thought over what is learned in interactions with physical tools; however, theoretically, he argues for a genuine dialectic between thought and language in which tools mediate knowledge.

Applying the Thought and Language Dialectic to Mathematics

This same dialectic can be applied to the development of mathematical knowledge and the role of computers in this development. Clearly mathematics possesses the characteristics of both thought and language-- it 's origins can be traced to its usages as a tool and its communication through compact forms of language.

Consider the most common tools; they are grounded in human's forms of physical activity and they express, as it were, our common experience; they are constrained by our biological engineering. Think about what comes to mind when one envisions tools-- rakes, hammers, utensils, typewriters, screwdrivers, sewing machines and vehicles-- these are typically physical systems which allow us to accomplish some purpose, to do something, which usually involves physical activity. When we use tools, our attention is focused primarily on effecting an outcome-- succeeding in completing our task. And, with repeated tool use, we often discover that across "realizations"[3], the invariances in what we act upon become apparent. Emphasizing the role of action in the use of tools does not imply that master crafts people know only how to act directly with their tools. Clearly, they also know how to move systematically in a space (Millroy, 1990). But, the tool image, generates for us implicit connections to action-- and this is one of the issues I wish to stress in this paper. Action, and its ties to goals, operations, and reflections, are too often quickly neglected in our traditional presentations of mathematics-- and I would like to challenge the wisdom and necessity of this. Mathematics, I would claim, never loses its "grounding" in human activity-- even in the most complex forms of thought.

In contrast, viewing mathematics as like learning a language allows us to focus on the creation of an elegant structure and describe it in an symbol system with which we can construct logical propositions, forms of argument, legitimate rules of transformations and declare results via proofs. When we think of languages, we think of English, Spanish, Chinese, Afrikaans, symbol systems, traffic signs, code books, sign language and so on. Language, in Vygotsky, is not simply a matter of competent use of the syntax-- however, what avoids this is its interplay with thought. Language is not completely independent of activity; its meaning emerges from the interplay of language and activity. We have plenty of evidence of the ability of a computer to construct syntactically "correct" dialogue, but little or no evidence that such dialogue conveys any meaning. Vygotsky recognizes that language can be constructed syntactically by children through imitation of adults while lacking any real meaning, and argues for the importance of this kind of development as an essential stage in the learning process. His description of the "pseudoconcept" is aimed at acknowledging that children often use words before they have grounded its meaning in conceptual operations. Vygotsky suggests that this use of language that runs ahead of cognitive depth is an important part of learning-- and describes a key mechanism in how adults teach children to advance to higher levels of cognitive thought.

[3] The term "realizations" comes from its use in simulations where a "realization" is equivalent to a "run" of the simulation.

When children learn arithmetic, they draw on both characteristics of mathematics-- as a tool and a language. They may work with manipulatives, as tools to solve the problem, say to find the cost of a pen (13¢), a book (99¢) and an eraser (11¢), but they also quickly enter into a structural exploration as they use contrasting methods. One child may have figured that 99 + 11 was 110 by thinking of 11 as 100 + 10 while another figured the pen and eraser cost 24¢ averaging their costs and adding 12+ 12 and then adding 99+ 24 as 100 + 23. Sharing their results and convincing each other of their method's validity leads them into discussing the structural relations. That is, children regularly work with mathematics both as a tool, to accomplish the task, and as a language to move around systematically within a structure. In the past, we described the differences between the approaches of the two children as differences in "strategies"; what I am suggesting instead is to take notice of the children's competence in recognizing that additive relations form a whole structure. Asking students the question "why" can lead them to draw on both characteristics in justifying their answer.

Viewing mathematics as a dialectical interaction between a tool and a language opens up some interesting possibilities. For tools have action, doing, as their primary foundation, and language tends to place that action at a distance and to make systematicity and internal structure the most salient characteristics. Protecting the independent contribution of the two strands while exploring their interactions create a powerful orientation to mathematical insights and the role of technology. Historically these two properties have been cast as completing for the rightful claim of the discipline, pure vs. applied mathematics, but the distinctions grow increasing and appropriately murky with time. The computer has played a significant role in complicating this distinction, as it began as a number cruncher replacing the human calculator, evolved to assist in other manipulations but has since developed the capacity to present unique and flexible representational forms.

Mathematics as a Dialectic Between Grounded Activity and Systematic Inquiry

Vygotsky's dialectic of thought and language, and his recognition of the role of tools, both physical and communication-oriented, needs to be combined with Piaget's rich and varied examples of how children solve tasks to build conceptual operations to create a true dialectic. Piaget's work contributed to our understanding of how humans adapt to their environment. Furthermore, his work emphasizes that children's views do not necessarily mirror adult perspective. These contributors from Piaget balance Vygotsky's systematic empirical overemphasis on the value of language and hierarchical knowledge in comparison to action-based thought. This leads to the view that conceptual development entails an interplay of **grounded activity** and **systematic inquiry**. It recognizes that all of this is situated both in socio-cultural and biological/environmental constraints. And it implies that the flow of knowledge moves bidirectionally-- from others to oneself, from oneself to others. Tools accordingly can be viewed as assisting primarily in grounded activity or in systematic inquiry, although as is typical in the

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dialectic, those distinctions will interact as one examines the tool use in a broader context. Figure 1 is intended to convey this overview of knowledge development.

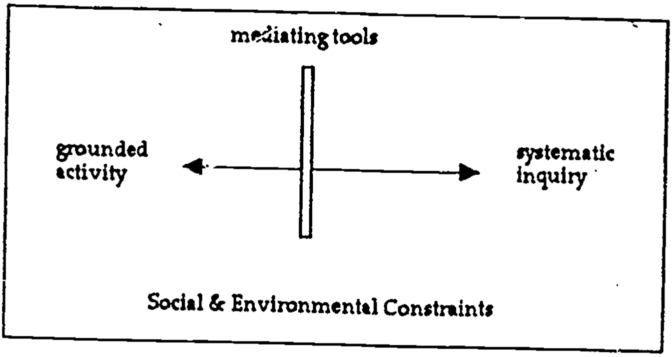

Figure 1: A Model of Conceptual Development

If one applies this figure to mathematics, it provides a fundamentally different view of the development of mathematical thought than the one in Figure 2. In Figure 2, one is led to view the development of mathematical thought as one in which a process creates an object which then creates process and an object and so on into higher and higher levels of abstraction. This description emphasizes increasing detachment from grounded activity. Abstraction involves removing that which is local and personal to produce the "higher" insight. The physical-- or bodily participation in the construction of knowledge is quickly neglected. Abstract, a word whose etymology is to extract away from, stresses a removal from context. It tends to imply not only generalization, but decontextualization. As a result, abstraction wipes away many distinctions in its pursuit of powerful generalizations. An alternative to this is to recognize the dialectic between abstraction and contextualization, and to view mathematics as emerging from the interplay between the two.

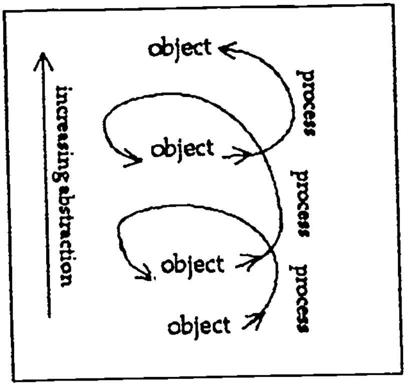

Figure 2: Conventional View of Mathematics

A simple example of the importance of viewing mathematics as both distinction and generalization can be illustrated with the quadratic function. "$y = ax^2 + bx + c$" is the generalized form of the quadratic. However, concentrating on that form hides the distinctions between the quadratic as a product of two linear equations and the quadratic as a sum of terms describing increasing velocity (and constant acceleration), constant velocity and an initial starting place. The second perspective leads us to view the quadratic as a description of falling bodies, with and without initial velocities, while the first allows us to recognize the value of the quadratic in describing maximum-minimum situations where two quantities are increasing/decreasing linearly and one is interested in their product. I am arguing for a recognition that the attraction of mathematics lies in moving flexibly among distinctions and contextualization, while appreciating generalizability.

Schemes are effective "units of analysis" for conducting research in that they allow us to keep our attention on both parts of the dialectic. They are investigable microcosms where one can examine both the grounded activities that spawn mathematics and the systematic language use that stabilizes and extends its use. They are anticipatory organizations of thought that allow us to recognize and/or cast a situation as requiring a certain type of mathematics and to act accordingly. And they represent the distinctions that our cultures have chosen to value and articulate.

Figure 1 could have displayed an interaction between the concrete and the abstract. I chose not to use this distinction for three reasons: 1) abstraction has become so identified with forms of higher thought that continuing its use would never allow an equal treatment of its partner. Thus, ceasing to use abstraction and replacing it with "systematic inquiry" might allow us to freshly

engage in reconceptualizing the territory; 2) abstraction has a confused use-- it may refer to a broad generalization, but as often it refers to difficulties in learning something: "it was too abstract;" and 3) the use of concrete has the unfortunate tendency to imply that mathematical ideas can be put into concrete materials-- like Cusenaire rods or Dienes blocks or computer programs. Also, the use of concrete tends to imply to people that there is no necessity of building up one's understanding, but that concrete objects are somehow immediately intuited. By using the term, grounded activity, I am signaling that mathematics evolves from the actions with the concrete materials.

Technology can be used to emphasize either or both aspects of mathematics. A compass, a straight edge, and protractor are technologies that emphasize grounded activity. A table of multiplication facts, a calculator, or a spreadsheet emphasize the systematic qualities of the mathematical enterprise. One can imagine sliding along the dialectic creating or identifying tools that highlight either the grounded activity or the systematic inquiry, while keeping in mind that it is in the interplay of the two from which powerful mathematics emerges. I wish to suggest this movement can create a useful way to imagine using and designing computer based tools. Computers are often called "semiotic tools" to indicate that they are used to communicate with symbols and signs, thus emphasizing more the structural, language-driven side of mathematics. However, in this talk, I wish to argue that computers should also be used with external devices[4], with laboratory experiments, and that they should incorporate dynamic forms of display to protect our connections with grounded activity.

The Role of the Interface

In Virtual Reality, a book by Howard Reingold, there is a discussion of using a computer for intelligence amplification (IA), an approach which its proponent, Frederick Brooks, contrasts with AI, artificial intelligence. In the IA, one seeks to use the computer to extend the capability of human beings to think, and in the second, the computer is viewed as a means to replace human intelligence. Intelligence amplification is based on the assumptions that humans can outperform computers in: pattern recognition, evaluations (rough determinations of a choice of action) and contextual interpretation. Computers can outperform humans in computational evaluation, data storage and simple memory. Designing software means considering how humans can most productively interact with the interface, which in turn drives and is driven by the program structure. Effective design seeks to make the fullest use of each of these capabilities.

The idea of an interface is essential in this description for the interface becomes the window through which a person is led to view to program's structure and capabilities. Interfaces, Reingold points out, are common. A doorknob is an interface on a door, a steering wheel on a car, a handle on a tool, an advertisement for a business, and the screen display on the computer. A poorly designed interface like a pot with burning hot handle, can make even the most delectable dish

[4] Hooking up external devices to machines is routine for data gathering and analysis in all kinds of science and engineering research.

inaccessible. I wish to argue that mathematics educator's need to pay close attention to the design of the interfaces, for it is here that representational forms can be designed and explored; it is here that the understanding of the child, student or novice presses up against the structure of the program and the student gains or loses entry. The interface is the place where we make our views of mathematics known by how we create the link to the program-- it creates the metaphors and forms of action we will come to identify with the ideas. Maximizing the connections of the user to their own forms of expertise and then providing resources to extend these into new learnings is the art of successful design. An interface should invite a user to find out what is here-- it should seek to engage as many of the sensory connections as possible to involve the student fully

We take visual displays as interfaces for granted, and yet a mere 20 years ago, using cards or paper tape with a computer was our only means of communication with the program's structure. As computer graphics became increasingly sophisticated, we have available whole new forms of interfaces-- they are visual, dynamic, non-linear and automated. They can display representations of objects (the desktop) and tools. Here too we, in mathematics, must seek to consider how we might design and use interfaces in light of a dialectic between grounded activity and systematic inquiry.

An Examination of Design

Given these preliminary remarks, as a context for our work, I would like to discuss the character of the design and research we conduct. The software has gone through many revisions which have been based on: 1) our research group's experiences in interviewing students; 2) a year-long project at the Apple Classroom of Tomorrow in Columbus, Ohio; 3) two years of software use in a large precalculus course at Cornell University; 4) two years of work at the Alternative Community School in Ithaca; and 5) three years of summer workshops with regional middle and secondary teachers.

Secondly, we propose a distinction between curriculum and software. Function Probe was designed for teaching a precalculus course. In that course, contextual problems were used to introduce families of functions. Those problems are available in a publication from Intellimation entitled Learning About Functions Through Problem-Solving. In the course, I stressed the role of action, visualization and the use of multiple forms of representations. Students learned to apply the tools of transformations to each of the family of functions. Function Probe was designed to assist in the course and to serve as a research tool. In designing it, we distinguished software from the curriculum. The software should allow a wide variety of choices of how to proceed; and, if one method is viewed as more efficient, then it should be encouraged through a curricular development.

Our experience with the design of computer technology suggests the value of the principles listed below. In the next section, a few of these principles will be illustrated using the function concept to demonstrate how the principles lead towards a richer view of mathematics learning with

technology. In the final sections, I will summarize what view of functions emerges and how it is connected with current initiatives in the field.

Design Principles

*Show profound respect for children's intellectual capacity and recognize and value the diversity of student methods.

* In designing new technologies, allow students' voices to guide us in reexamining and changing our understanding of the concepts we seek to teach.

* Create opportunities to use computers for communication, group interaction and/or student presentations. Software can be used to do this by allowing students to share methods, by requiring group cooperation, by encouraging the use of networks to collect, collate and interpret data, and by creating performance assessment resources.

* Pay careful attention to the role of the interface as the connection between a student's current intellectual viewpoint and the underlying structures of a program.

* Don't waste time mourning the changes in what is learned. Tools always mediate what is claimed as knowledge.

* Question the attachment of mathematicians to antiquated notation systems and mimic the irreverence of programmers who feel relatively unencumbered about creating new programming languages to respond to the demands of different tasks.

* Don't use computers as an excuse to neglect the importance of physical tool-based activity.

* Pay close attention to the development of operational schemes and how these are connected to physical tool-based activity and how they are embedded in language systems.

* Make exploration and experimentation easy and encouraged.

* Make effective use of activity structures that are inherently generative of imaginative activity. For instance, make use of design and play activities.

* Make significant use of multiple forms of representation and recognize that in every representation something is lost and something is gained. Coordination and contrast of representation is a powerful form of knowledge construction. No representation is transparent.

* Encourage students to build representations and other resources and then store them for future use in a personalized version of the software.

* Allow for student immersion in complex systems as a way for students to become proficient with the tool before understanding its composition.

* Build with an eye towards linking with other software.

Function Probe: A Software Tool for Teaching Functions Using Contextual Problems in a Multi-representational Environment

In the next section, I will take a four of these design principles will be discussed in light of our research work . The discussion will focus on a piece of software, Function Probe, that we have built and modified over the past six years. Function Probe is an analytic tool which can display four representations for functions, a graph, an equation, a data table and a calculator/keystroke environment. In the final section, I return to the question of how the concept of function has changed from this work and discuss what the broader implications of these changes might mean for widespread curricular reform.

* **Show profound respect for children's intellectual capacity, recognizing its non-conformity with adult perspective.**

The example chosen concerns the power of the table as a means of understanding functions. When we learned functions in secondary school, we relied primarily on the algebraic equations. In the introduction to algebra, we spent a year learning to manipulate algebraic expressions, factor, simplify expressions, combine like terms, etc. and finally we solved equations ad nauseam before we actually began to display these on graphs and work with the "umpteen" different forms for linear equations (point-slope, two point, slope-y intercept, x intercept-y intercept etc.). It was clearly the case that algebra dominated graphing and that tables were just a way of getting to the graph, an enabling resource. That teachers themselves promote and endorse such a view was encapsulated by one high school teacher who at the beginning of the time we worked together declared (approximately), "but if they can fill in a table. they must be doing everything that the algebraic equation captures only over and over again. Why not then just teach them to code it algebraically in the first place?"

Our examination of the use of tables with students has convinced us otherwise. Our interest in tables increased as we witnessed students frequently using the table as the primary means of entry to the problem. The process they went through of identifying the appropriate quantities to put in columns, and then quantifying those into a label and units of measure involved a dialectic process alternating between the data cells and the informal labels cells (See Figure 3). For instance on the cliff problem, where a person's house is 13 feet, 4 inches inches from a cliff that erodes at a rate of approximately 3 inches per year, they wanted to put some measure of time with some measure of distance. Time could be entered as the year, 1993, or it could be entered as years since 1993. Distance could be the total amount of erosion or the distance from the house to the cliff edge. Distance and rate of change could be described in feet or in inches. As the students worked out the informal labels (third row from top), specific data values were distinguished into categories which were transformed eventually into variables to be listed in the formal label cell of the table (second row down). (The icons in the top row are for sending the data values in two columns to the graph.) This process of quantifying the problem could be described as the construction of the variable-- and it allowed the students to become familiar with the idea of variation and its quantification into a rate of change-- a process which is often hidden once the assignment of a letter

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to the quantity is completed. This is a simple example of how an abstraction, naming the variable, leads to the loss of contextualization, the differences among the kinds and values of the numbers used.

Table

n	y	d	Δd	D	ΔD
number of years since 1993	date of year	distance from house to cliff (in)		distance from house to cliff (ft)	
0.00	1993.00	160.00	-3.00	13.33	-0.25
1.00	1994.00	157.00	-3.00	13.08	-0.25
2.00	1995.00	154.00	-3.00	12.83	-0.25
3.00	1996.00	151.00	-3.00	12.58	-0.25
4.00	1997.00	148.00	-3.00	12.33	-0.25
5.00	1998.00	145.00	-3.00	12.08	-0.25
6.00	1999.00	142.00	-3.00	11.83	-0.25
7.00	2000.00	139.00	-3 00	11.58	-0.25
8.00	2001.00	136.00	-3.00	11.33	-0.25

Figure 3: The Cliff Problem

One of the strategies we watched students use was to learn to choose a numeric value judiciously to allow them to gain insight into a phenomena. As reported in Confrey, 1992, students showed us how they used large numbers to distinguish a multiplicative coefficient from the additive one in a y= mx + b "Guess My Function" exercise (Confrey, 1991). The table window on Function Probe can encourage students to try negative, fractional or even irrational values, and each cell can be used for simple calculations and/or adjusted to display different numbers of decimal places. This exploration also reveals a limitation of software like Function Probe that transforms rational and irrational numbers into decimals masking the underlying actions that were used to construct the numbers.

A key issue in forming functional relationships has been adjusting the correspondence between the starting points of two columns. Frequently, we have seen students get an unintended outcome when the starting values are off by just one notch in either direction. This reveals more than a technical mistake-- as we have seen through interviews-- it often signals a confusion about the point of origin, whether it is meant to be thought of as a zero point in an additive world or as a initial unit of one, as in an exponential situation (Rizutti, 1991; Confrey, in press c).

One of the most important resources we provided was the Fill command (See Figure 4). Consider the traditional max/min problem, a rectangular field is located along a stream with fencing to be placed on only three sides. (The stream is the fourth side.) If there is 100 feet of fencing,

find the maximum area that can be enclosed in the rectangle. We can imagine the student here starts by trying the values of 50 for length and gets 25 for the width. She then tries 40 and gets 30 for the width. If she then tries 20 for the length and gets 35 for the width, she might realize that as the length goes down by 10 the width increases by 5, or possibly as the length goes down by 2, the width increases by 1. She might then complete a table of values by putting in one possible pair of length and width, perhaps even 100 (length) and 0 (for width), and then decreasing 100 by 2's to 0 and increasing the width from 0 to 50 by ones using "Fill" command. Because the students can create sets of columns that are related in their minds but are not algebraically dependent, we built in the resource of linking columns. Linked columns can be sorted, cut and pasted as if they were algebraically hooked, a resource students find useful. Also, we have found it useful to connect these kinds of coordinated "fills" or "co-fills" as we call them with the original problem context. Thus, one can imagine a 100 cm. string representing the 100 foot fence. To make the rectangular pen, giving one cm. to each width symmetrically will result in a loss of 2 cm. to the length. The covariation is reconnected with one interpretation of the physical action, that of symmetrically adding one to each of the two widths.

Table

l	w	A=lw
length (ft)	width (ft)	
50.00	25.00	1250.00
40.00	30.00	1200.00
30.00	35.00	1050.00
100.00	0.00	0.00
98.00	1.00	98.00
96.00	2.00	192.00
94.00	3.00	282.00

Figure 4 : Cofilling a Max-Min Problem

This sort of approach allows students to create a function by building one column, filling it and then building a second column and filling it. We have called such an approach to functions a "covariation approach" (Boyer, 1968; Confrey, Piliero, Rizzuti and Smith, 1990; Rizzuti, 1991) and contrasted it with the x to f(x) approach of an equation (which we call a "correspondence" approach.) We have argued that a covariation approach is intuitive for the students (Rizzuti, 1991) because it focuses student attention on the underlying action in the functional family (Confrey, 1988); it emphasizes rate of change; and it can allow the operation to be closely connected to the production of the coefficient in the equation.

Of course, such an approach will apply most easily to the linear and exponential function (where the rate of change is either a constant adder or multiplier). To apply this covariation

perspective to the quadratic, we built two resources, a difference command and an accumulation command. A quadratic produces a constant second difference (for equal units in x) and through a double accumulation and the appropriate use of constants, one can reproduce the original quadratic (See Figure 5). All higher powered polynomial functions can be explored likewise with repeated applications of the difference or accumulation command. To extend the use of the table as an introduction to calculus, we have built a version of a generic display and accumulator program. Built originally as a "bank account" program[5], it allows one to make an entry each time period and to see the resulting balance or visa versa. The simulation, or enactment, can be simultaneously displayed in the table in two graph windows as entry vs. time and as accumulated balance vs. time. In the generic version, the entries and balances can be anything from money, height, area, population etc.

Table

x	y=x^2	Δy	Δ(Δy)	Σ(Δ(Δy))+3	Σ(Σ(Δ(Δy))+3)+1
1.00	1.00	3.00		3.00	1.00
2.00	4.00	5.00	2.00	5.00	4.00
3.00	9.00	7.00	2.00	7.00	9.00
4.00	16.00	9.00	2.00	9.00	16.00
5.00	25.00	11.00	2.00	11.00	25.00
6.00	36.00	13.00	2.00	13.00	36.00
7.00	49.00	15.00	2.00	15.00	49.00

Figure 5: Showing the use of difference and accumulation commands for the quadratic

The exponential function provided us a special opportunity: to create a new notation. We created a symbol ® for the ratio between x_n and x_{n+1} parallel to the Δ for difference. Creating such a resource and conducting research on student understanding of the exponential has led us to assert that there are multiplicative units and a multiplicative rate that are poorly articulated within the mathematics literature and confused with the additive rates more commonly used (Confrey and Smith, in press b).

Once students have established a data set (x, y) (which can be done by entering data cell by cell, importing a data file, co-filling, or through an equation), the question of interpolation becomes salient. When two columns are related by an equation, interpolation is straight forward; however, with a co-fill interesting challenges are raised. For instance, with an exponential function, where they have filled x by ones getting 0, 1,2, 3... and y by multiplying times two

[5] We designed this software for Steve Monk as part of his NSF project on learning calculus.

getting 1,2,4,8,16. . . , they would have to reason out that the match in y for interpolated value of x= 1.5 would be obtained by multiplying 2 by $\sqrt{2}$. Much of our group's research has been focused on examining the historical development of this idea (Smith and Confrey, in press; Dennis, Confrey and Smith, 1993).

Support for the validity of the table window as an integral representation for functions has come from this reexamination of the historical record. During the period between Descartes and Leibniz, Wallis worked with tables with great inventiveness and creativity and used them to create an argument for the legitimization of rational exponents (see Dennis, Confrey and Smith, 1993). Liebniz also worked extensively with tables in order to develop what is now our familiar notation for calculus. He tried to make what was previously only accessible to mathematicians, calculus, into "mere child's play" to be done "in the blink of an eye." However, two years before his death, he realized that his success with notation could be detrimental to the development of other's mathematical minds, for the symbolism's apparent ease hides the underlying roots in tables. He lamented, "One of the noblest inventions of our time has been a new kind of mathematical analysis known as the differential calculus. But while its substance has been adequately explained, its source and original motivation have not been made public" (Euler, 1988). This type of historic work lends credence to the claim that many portrayals of the historical record are themselves distorted by the current emphasis on the importance of the f(x) form of equations. **Close listening** (Confrey, in press b) **to the student voice leads us to argue for a more equitable treatment of the different representational forms** (Confrey and Smith, in press a).

These kinds of student explorations have led us to envision the table as possessing a structure-- one which we might build on by combining our understanding of functions with some of the resources built into spreadsheets. Some of our current thinking along these lines is oriented towards creating a dynamic network for feeding into, and modifying and producing tables. The table window as designed for student use begins as a tool for solving the problem, grounded in capturing the physical actions of repeated addition or multiplication, but the table quickly becomes a more semiotic tool for systematic inquiry. The resources such as linked columns, fill, difference, ratio and accumulate commands create ways to examine the quantities and their numeric characteristics. Responding to student strategies required us to build new resources for the students and to coordinate these resources with the more traditional resources we had already envisioned. Thus what began as a format for listing data became a tool in its own right, because we allowed ourselves to respond to students' methods and build appropriate resources. By conducting historic research into the use of tables and drawing upon the dynamic resources of a computer environment we were able to build an inventive form of a table for student use.

*** In designing new technologies, allow students' voices to guide us in reexamining and changing our understanding of the concepts we seek to teach.**

Question the attachment of mathematicians to antiquated notation systems and their constraints and mimic the irreverence of programmers in creating new languages.

Although this point has been made in the previous example in that it argues for elevating the tabular approach to functions to an equal partner in the meaning of function, I will provide two additional examples in this section.

The calculator in Function Probe operates mostly as a traditional calculator that keeps a keystroke history. One of the distinctive features is that students can build buttons on it. This is used typically after a student has repeatedly carried out a series of operations and realizes she wants to automate the series. She chooses "add button" from the table menu and her line of keystroke code is listed in the display window and the ? key is added to the resources. By selecting the places in which she wants ? to replace a numeric value, and clicking on the ?, she gets the button's "coding" and then presses "OK." She gives the button a name and the button becomes available. (See Figure 6 where j1 is the button listed below that has already been built and is currently being applied to the value 5.)

For instance, on the cliff problem discussed earlier, if she wants to build a button that will take the number of years passed and produce the distance from the house to the edge of the cliff in inches, it would appear as 160 - ? * 3= . Such a notational record led us to recognize a key insight-- that in this kind of relation, the linear function in this context was more easily understood as [an initial amount] ± (x * increments of (±) unit change): that is, as $y = b + xm$. Such a form makes a closer connection with the exponential of $y = C\,a^{x}$ where it is then [an initial amount] */ (increments of (x) unit change)^{x}.

If the student wanted to output the distance to the cliff in feet, the button would look like this: 160 - ? * 3 = / 12 = (Figure 6). At a conference, I presented such an example and was quickly confronted by an irate university mathematician who took offense at a notation that appeared algebraic but as he put it "misused the equal sign." His comment was somewhat ironic to me, because in an initial design, we had used x instead of ? but a colleague felt it looked too algebraic so we decided that we'd use the symbol ? to signal the need for an "input" rather than a variable. I reiterated the fact that this was a keystroke notation (from an algebraic calculator in terms of the order of operations) but he was unwilling to see the equal sign used in this way as indicating the need to calculate a partial sum. The equal sign here is used as a substitute form of a parentheses.

Keystroke notation is neither globally better nor worse than traditional algebraic symbolism. It is an alternative that has proven useful in particular areas. It allows students to recognize the operational, procedural quality to algebraic expressions, and it has proven useful in teaching inverses of functions.

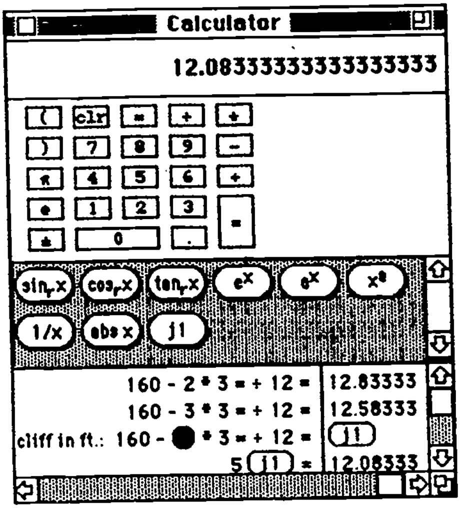

Figure 6: Using keystroke notation to build a button

The second example of how our calculator differs from standard calculators is that the log function is a binary function just as is + or *. That is, to use a log requires two inputs. The first input is the value one wishes to take the logarithm of, and the second is the base. Thus the key appears as $\log_a x$. Inputting 8, pressing the button $\log_a x$ and then entering 2 produces 3. We would argue that to have calculators that fail to do this is to limit the user unduly. Furthermore, the natural logarithm, as we have documented in our work on exponential functions, is often overused, when a base that is more closely connected to the description of the phenomena would suffice equally well. For example, in compound interest of 5%, 1.05 would be a much more direct and intuitive base with which to work.

Interestingly enough, this mild revision reverberated through the software design. In building log scales into the graph window, we were required to figure out how to create a scaling resource that allowed any base as the multiplicative unit measure on the axes. Our design, while not completely satisfactory, allows this to happen. If you wish to engage in an interesting intellectual challenge, decide how you might want the visual transformational tools to act on a log linear scale; for instance if one chooses a translation in the direction of the log scale, what

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algebraic result you would wish to see displayed?[6] There is a conflict between the visual action which looks like a translation and the multiplicative impact on the algebraic display.

In future design work, we have considered additional changes to standard notation. For instance, in the calculator window currently, we allow trigonometric functions to be entered using either degrees or radians, and the notation has been adjusted to reflect this: $\sin_d x$ and $\sin_r x$. However, these functions too could be considered as binary functions, where the second input represents the change per unit time represented in fractions of a circle. This might support a student who wants to divide up the unit circle into eighth turns and to use these as the "base" for the function; $\sin_{1/8} x$. Thus, $\sin_{1/8} 3$ would signal three one-eighth turns and produce $\sqrt{2}/2$. This approach would be an alternative to using a horizontal stretch on x before applying the function to accomplish this ($f(x) = \sin(\pi/4\,(n))$ where n = the number of one-eighth turns).

• Don't use computers as an excuse to neglect the importance of physical tool-based activity.

Function Probe as it exists relies on written problem statements to provide the contextual momentum. We recognize the limitations of such approaches and are working towards the development of alternative sources of problem or activity generation. Modest efforts have been implemented in our curriculum. We explored the use of simple apparatus for constructing curves as loci of points. For instance, one can take a straw, fold it in half, places it on a horizontal and put a thumbtack in the right half. By fixing the left endpoint, and sliding the right end of the straw until it reaches the left endpoint, the tack will trace out the shape of one quarter of an ellipse. This approach to functions is more in line with the approach of Descartes who used geometry to create curves as loci and then used algebra to describe them, a complete reversal of the mathematics now attributed to Descartes as we use his Cartesian plane (Smith, Dennis and Confrey, 1992). Function Probe can import data in a spreadsheet format such as from MBL systems (microcomputer based learning systems with sensors for movement, temperature, light, etc.) or from simulation environments such as "Interactive Physics." It can also import pict files that can be used to generate curves.

We are currently working with designing experiments for students to generate functions using physical apparatus. For example, imagine a wheel with a flashlight attached so that the flashlight always remains horizontal and roughly in the plane of the wheel itself as the wheel turns. If a sheet of paper was moved across a screen in front of the wheel at a constant speed as the wheel turned, the path of the flashlight could be traced to make a trigonometric curve. In a similar vein, Pat Thompson has experimented with having his middle school students trace a path around a circle with one hand while tracing their distance from the axis with the other. These kinds of

[6] These next two examples make reference to the dynamically mouse-driven transformational tools first available on Function Probe. These include stretching, translating, and reflecting graphs.

experiences are designed to try to provide students with experiential grounding for their mathematical ideas in mathematical actions. It is these kinds of experience that will eventually allow students to watch a spring move up and down such that its movement is "like" that of the Ferris wheel (and unlike it too). This will form powerful conceptions of trigonometric functions grounded in human activity.

The idea of the changing rate of change for the trigonometric functions, changing so that the rate of change is in effect a horizontal translation (and possibly a stretch) of the function itself is a remarkable characterization. In this respect, the trigonometric function shares some of the qualities of the exponential (both yield versions of themselves as their own derivatives). In this sense, so does the linear function. Note that this means that these three functions all embed a form of periodicity. How the exponential and the trigonometric functions can be viewed as related functions is a key issue that later resurfaces with complex numbers, but whose roots lie in just such explorations. Euler in his "Introduction" in 1748 constructed e and the radian measure of angles as a way to unify trigonometric and exponential functions within the complex numbers. I know of no one who experienced such a relationship as anything other than as a pure "abstraction." Returning to my earlier point, the grounded activity of examining rates of change and their ties to operations leads one to recognize this common periodicity, and the systematic extensions of these ideas without losing the basis in action forms the opportunities for deep and exciting mathematics.

We are currently engaged in building a number of more "experimental" activities. These include work with local teachers to design an approach we have labeled "experiment-simulation-analysis" (Noble, Flerlage and Confrey, 1993) for teaching integrated physics/trigonometry. Furthermore, over the summer, we are engaged in working with 9-11 year olds creating design activities that they can use with the software to explore the design of motorized Lego cars with gears, complex marble tracks and falling water balloons. In summer workshop, teachers recreated graphs into movements depicting position, velocity and time.

Also, I wish to mention two other research groups working along these lines that capture the sense of grounded activity and systematic inquiry. Andee Rubin, at TERC, began with a project called "Tape Measure" where she and other colleagues had students use videotape and software tools as a means of collecting data. In the demonstration tape, students tested various hypotheses such as that stride length is related to the speed of the runner. Under a new project, VIEW (Video for Exploring the World), she is extending her previous work further to allow students to explore such activities as rhythm and velocity position and time. In the research on students investigating rhythm, Rubin's students' examples include Double Dutch jump rope and dribbling basketballs. In projects on velocity, position, and time students enact performances and stories and test out their conjectures on videotape (Rubin, May 1993; 1993). In all these cases, we see the computer being used to build upon students' physical experiences in the world and to extend these experiences into

a computer medium to allow them to analyze the quantitative aspects and to create interesting representations of them.

Likewise, Ricardo Nemirovsky has been working with MBL, microcomputer based labs, with students where they explore such things as water flow, car movement and rate of change of surfaces. For example, he has built an interface between a graph sketching tool and two cars such that students can sketch a graph and then watch as the cars carry out that graphs' shape as movement along a track. Other work concerning position, velocity and time is being undertaken by Kaput who plans to implement "Math Cars" a visual simulation of these cars moving that explore relationships (Kaput, in press). Comparing the effectiveness of Nemirovsky's and Kaput's approaches will inform us better about the role of physical objects and their graphical displays.

We see this work as particularly important in that it can lead to the identification and articulation of schemes for anticipating and interpreting different functional relationships. Thus, we propose that students can learn to observe and participate by looking through functional lenses at their activities and that this might be well described by articulating the kinds of position-time or velocity time schemes that evolve from Rubin's and Nemirovsky's work. For example, we see the falling bodies scheme as an important way that students can recognize the quadratic, however, we see the idea of the product of two quantities each changing linearly as a different scheme. My claim is that these differences need to be connected with the differences in the grounding activities. And, that the differences need to be viewed within the larger framework of commonalty that underlies the family of functions we label as quadratic.

• In a multi-representational tool, no representation should dominate the others. And, in every representation there is both a loss and a gain. There is no such thing as transparency of representation. Protect the integrity of each.

Designing the transformational tool for stretch provides an interesting example of our perspective on the relative independence of representations. Our stretch icon has two original qualities. It includes a movable anchor line and it displays a visual view of stretch as a ratio relation. These characteristics exemplify the placement of the computer between a language and a tool, for: 1) we sought to create dynamic actions (implementing the first mouse-driven graph transformational tool) and in doing so, sought to create icons which drew upon the physical actions[7] -- while 2) creating a language for coding these actions and exploring their impact on other representational forms.

For us, the stretch is an action on one of two rubber sheets, (Borba, 1993, recognized in his study of students' understanding of transformations that there are two rubber sheets, one for the

[7] Understanding that the action of stretch creates a multiplicative operation on the function needs careful attention. We are not implying that the need for exploration of this is mitigated by the use of the computer.

plane with the axes and one for plane containing the curve). We chose to stretch the sheet with the curve and since one can imagining stretching a sheet, say vertically, so that some horizontal line is held constant (Imagine placing tacks across this line to hold it in place). Thus the anchor line on Function Probe is movable, because software tools which place it at the axes do so only for the algebraic convenience. Function Probe allows the student to place the anchor line on any horizontal or vertical line.

Once the anchor line is placed, the student touches the graph anywhere and a square "unit box" is produced. Its sides are f(x) - ⇣) long and wide. Holding down the mouse and stretching produces a stretched out spring and a change in the ratio of the height of the unit box to its width. This new ratio is the coefficient of stretch (See Figure 7).

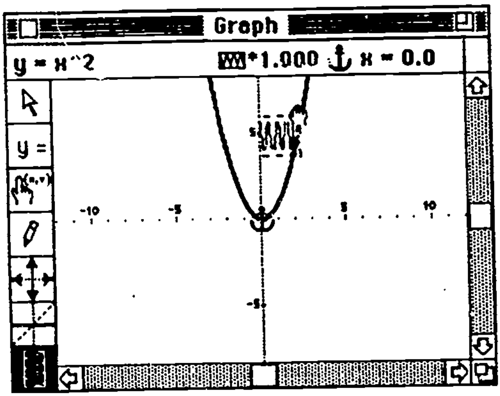

Figure 7

Students have not only grasped the use of the anchor line more easily than we have, but they have used it to great advantage. Trying to match two parabolas, they quickly align the vertices through translations, reflect the graph to the desired orientation[8] and place the anchor line there and stretch it (Afamasaga-Fuata'i, 1992 and Borba, 1993). And, in a lovely demonstration of the use of the anchor line at a teacher education workshop, a novice with the software used the anchor line cleverly. The problem was to lay out a stone path from a house to a bird feeder forty four feet away. There were to be fifteen circular stones all equally spaced with the last one placed against the bird feeder. The first stone could be at any distance from the house. This person imagined that all the stones could be bunched up against the feeder, going from 27 to 44, then a hook could be

8 Our reflection tool also allows one to place the axis of symmetry away from the x or y axis.

attached to the nearest stone and they would be stretched, keeping equal spacing, until that first stone reached the house. On Function Probe, she modeled the family of all possible solutions using a vertical stretch of her first solution, placing the anchor line at 44, and stretching until the first stone moved to the point (1, 1) (which would be up against the house if each stone were 1 ft. wide, see Figure 8).

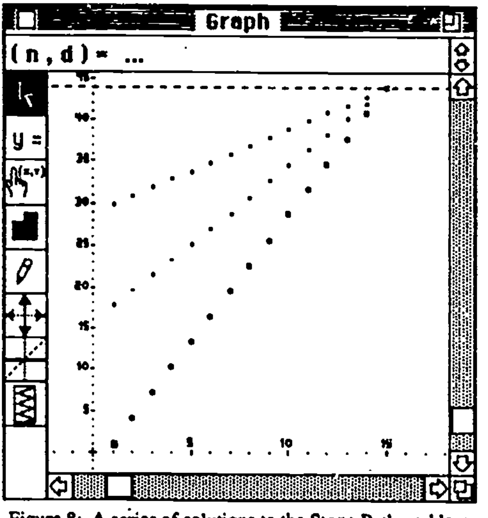

Figure 8: A series of solutions to the Stone Path problem

This example of allowing a movable anchor line demonstrates the importance of allowing the graph window to operate with as few algebraic constraints as possible. By protecting this independence of the windows, I proposed an alternative approach to functions through visualization, which later was explored and refined by Marcelo Borba in his dissertation (Borba and Confrey, 1992; Borba, 1993). In this approach, students are first encouraged to work visually with functions using the dynamic transformational tools. Then using a sampling procedure, they explore the impact of transformations on individual sets of discrete points by sending these to the table. A description of these transformations as algebraic equations follows. This is only one approach of four that we have worded with and we find it an interesting way to build students' powers of visualization.

To demonstrate the value of visualization, consider if a student is given the problem, you are riding on a Ferris wheel with a 12 foot radius. Each rotation takes 20 seconds. The lowest point of the Ferris wheel is 6 feet off the ground. Find a way to predict your height off the ground as a function of time. Now suppose a student wants to approach this visually. They will typically

begin with the idea that this is a trigonometric function (recognizing its periodicity). (This, of course, assumes some previous experience and understanding of the trigonometric function as converting constant circular motion into rectangular components, not a trivial problem.)

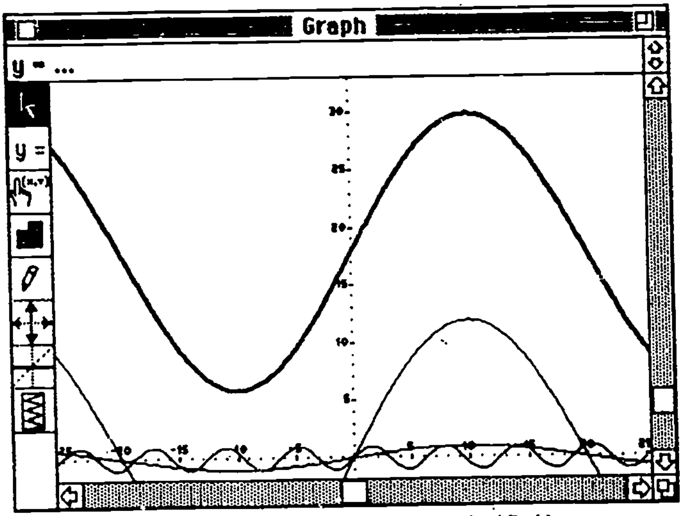

Figure 9: A visual approach to the Ferris Wheel Problem

Students using a visual approach to this problem will begin with the prototype y = sin(x). One way to proceed is as follows: stretch y = sin(x) horizontally to get its period to 20 seconds. Then, stretch the graph vertically to get the radius of the Ferris wheel to be 12 feet. Finally, a vertical translation of 18 feet will locate the function above the ground by 6 feet at the lowest point. This description appears relatively straight-forward, and its set of ordered transformations keeps the problem close to the context from which it came. However, graphical transformations also create a mathematical structure to be explored. For instance, the constraints on the order of operations in this problem raise interesting questions. (See Smith and Confrey, 1992 for a discussion of another problem which raises the issues of order of operations.)

The anchor line has become a very useful resource. Its independence from algebra was secured by our philosophy of protecting the integrity of each representational form, but its epistemological significance and usefulness was demonstrated by our students and teachers. We see the use of dynamical graphing resources as another way in which mathematics is kept close to its roots in actions. Appeals to such visual tools as anchor lines and springs that stretch encourage students to think of mathematics as a balance of systematic inquiry and grounded activity.

I described this as "an epistemology of multiple representations" as a way of describing our overall approach (Confrey, 1992). In such an epistemology, representations are considered useful in the ways that they complement and contrast with each other. We do not only focus on convergence-- because insight results in cases in which we find convergence between representations that appear to us to differ. This convergence within apparent divergence is what creates the sense in mathematics that a "discovery" must have been made. Seeing how the transformational actions apply to all families of functions is such an example of convergence across diverse cases. And, complementarily, the recognition of divergence within convergence adds mathematical pleasure to our explorations. This is illustrated by the way that even with the convergent application of transformations across families, different rules appear for each family. For example, we find that trigonometric and step functions do not allow horizontal stretches to be accomplished by vertical stretches, whereas linear, absolute and polynomial do with appropriate adjustments in the coefficients of stretch.

Redefining the Concept of Function in this Environment

When one works with Function Probe with students for a while, it becomes evident how impoverished our traditional approaches to functions are. In summarizing what we have learned about functions, we have:

* identified the key role of the table as a legitimate partner, in terms of usefulness and structural integrity, and with this, have argued that a covariational approach to functions needs equal attention with a correspondence approach;

* argued that functions have their roots in grounded human actions (prototypic actions) and that they evolve through our ways of generating operations and numbers from actions and displaying those in different representational systems;

* called for a much richer use of context, as a source of experimentation and data, as opportunities for play and design, and as a basis for simulation and modeling; and for careful attention to how those investigations can be led into examining the integrity of the mathematical system;

* illustrated the importance of rate of change as a key concept in the development of functions and are working towards equalizing its relationship with its sister concept of "accumulation;"

* demonstrated that visual transformational skills can lead one to a valuable understanding of the set of affine transformations as creating a boundary for a family of functions; and

* documented that functions involve more than a domain, a rule and an range, but that they involve the construction of a structured system, a selection procedure for a patterned subset of the system, and the production of a display of that system.

These insights are leading towards the development of new piece of software, Recursion Probe, which recognizes that: the issues of scaling are far more substantial than we originally conceived of them, building coordinate systems needs to be part of the children's mathematical

activities, and functions should be conceived of as a dance, a rhythm or a pattern displayed in relation to an organized space (Goldenberg, Harvey, Lewis, Umiker, West, and Zodhiates, 1988).

Implications for Equity and Diversity

Such an approach to functions signals the need for a significant change in the algebra curriculum-- not just as the precalculus level but all the way back into the elementary level. It requires a radical revision of what is conceived of as algebra. Algebra, as it is currently taught in schools, signals mostly the introduction of the variable, the equation and the manipulation of these forms. Later these manipulations of forms become the basis for the structural algebras of groups, fields, and for the discussion of number types, natural, integer, rational, irrational, real etc. However, a second history of algebra is the algebra of change (Klein, 1968). This is the development of ways to describe motion, growth and area and it is the basis for calculus, differential equations, etc. It is the algebra of quantity and geometry.

Algebra in schools needs to be revised to admit a broader category of studies and to draw upon both of these historic traditions. This, however, sets up a serious and important debate, for these changes are being advocated at the same time that there is a national call for "algebra for all" (Jetter, 1993). This movement is rooted in a justified critique that algebra has been misused as the sorting mechanism and has disenfranchised many students, especially those with less economic resources or orientation to schooling as a means of social advancement. This movement seeks to improve the retention and success rates of underrepresented groups and recognizes the place of mathematical training in allowing access to positions of power in our society. (For example see the work of "The Algebra Project" [Moses, 1992] in Jetter, 1993, The New York Times.) All of us committed to reform in mathematics education must make significant efforts to work cooperatively on changes to the curriculum. We must recognize also that the possibility of working at cross-purposes is especially likely if changes are dependent on technologies that are less available in schools with fewer financial resources.

I would therefore like to offer three recommendations: 1) We must make an aggressive commitment to produce versions of the curriculum and materials that can be used without expensive technological tools. For instance, tables can be produced off the computer and used for the same purposes described herein, but the use of them may be more tedious and time-consuming. However, with a calculator, and with students working cooperatively, the same ideas can be enacted. And along with this must come an aggressive commitment to getting solid technological innovations into all schools. 2) We must examine whether curricular changes that ground mathematics in common forms of human activity and experience will protect and encourage diversity. If we are to challenge effectively the elitism, racism and sexism practiced on a daily basis in mathematics from kindergarten to secondary school, to universities and colleges; and further into the offices, laboratories; and businesses, we must seek relief from mathematics that is portrayed as far harder and more inaccessible than it really is. We must remake the mathematics of

schools and escape from the echo chambers of increasing abstractions whose detachment from everyday experience is manufactured more than it is borne of logical necessity. 3) We must acknowledge the fact that new approaches will frequently originate in expensive technology, but accept our obligation to see those innovations enacted in less costly and more accessible forms.

Securing equitable access for unserved populations rightfully must be a driving influence, because without access, things will not change; and with diversity in the practices of mathematics, things will change more quickly than any of us can predict. At the same time, it would be unfortunate for our educational programs to lose touch with the new cultural tools. This is not to say that the tools themselves are beyond scrutiny, criticism and change. The history of technology is a history equally of invention, progress and improvement as it is a history of surveillance, imposition, control and violence. We, in mathematics education, must be very clear about the kinds of tools we see as promoting the kinds of learning we value. If our goal is emancipation and empowerment, we need to pay attention to how our design encourages or inhibits these developments. In conclusion, I have tried to argue in this paper that we must keep our tools grounded in forms of human activity while recognizing the power of giving all children access to systematic inquiry methods to investigate any complex system. Expertise lies in the combination of both types of pursuit: grounded activity and systematic inquiry, and with both, we can attempt to give mathematics a fresh start. We must recognize our obligation to encouraging and challenging all children to make technology serve them, and not to let our culture proceed the other way around.

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Computer-based Mathematical Modeling With In-service Teachers

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This paper discusses the process and products of K-12 teachers' work with mathematical modeling in a computer spreadsheet environment. The effort builds upon several prominent themes of mathematics education over the past decade--visualization, dynamically linked multiple representations, and new approaches to conceptualizing functions and variables--and it ties these themes together into mathematical modeling appropriate for primary grades through high school. It is hypothesized that features inherent in a well designed spreadsheet environment can help make the concept of "variable" more intuitively accessible. Introduction of spreadsheet modeling broadens the range of potential answers to current questions concerning thought processes involved in algebraic applications and problem solving.

In the past few years the professional literature has included an increasing number of reports advocating use of computer spreadsheets to support and enhance mathematics teaching and learning. See, for example, Bannard (1991), Cornell & Siegfried (1991), Fey (1989), Levin & Abramovich (1992), Masalski (1990), Maxim & Verhey (1988, 1991), McDonald (1988), and Pinter-Lucke (1992). Each of these reports suggests appealing applications for spreadsheets in the mathematics classroom, and each report makes persuasive arguments relating spreadsheet solution processes to current needs in curriculum reform. A spreadsheet environment is especially well suited to investigation of recursively defined functions, which sometimes can be more accessible for beginning algebra students than functions defined in closed form.

There is also growing attention to the importance of visualization in mathematics learning. Publications of the past decade are replete with references to the capabilities of computers to produce visual representations of mathematical ideas and allow manipulation of the visual representations. (See, for example, the variety of ideas in Zimmermann & Cunningham (1991).) Recent versions of computer spreadsheets (e.g., Microsoft Excel, Version 4.0 (1992)) include extensive graphic capabilities that facilitate dynamic manipulation of graphical representations of functions along with the analytic and tabular representations.

This paper discusses aspects of a project that involved 32 teachers of mathematics in grades K-12 in exploring spreadsheet problem-solving methods and developing models appropriate for classroom use. Recursively defined functions, dynamically linked multiple representations, and visualization of functional relationships were fundamental to the teachers' efforts.

Participants

The context of the work was the 1992 Summer Institute of the Northern California Mathematics Project. Thirty teachers of grades K-12 were selected from a pool of applicants. Selection criteria included participants' potential for providing professional leadership in their districts and the state. An additional seven teachers from past institutes were involved as mentor teachers and institute director, for a total of 37 teachers. Of the 37 teachers involved in the institute, 32 participated in the mathematical modeling activities discussed in this paper. The 32 participants' teaching assignments at the time spanned the range of K-12, with eight teachers of grades K-3, seven teachers of grades 4-5, seven teachers of grades 6-8, and ten teachers of grades 9-12. Nearly all of the teachers had some previous experience with a computer. Few had used a spreadsheet before, and none had used a spreadsheet for mathematical modeling or problem solving. The activities discussed in this paper were one facet of the month-long institute. The software used was Microsoft Excel 4.0 (1992) for Macintosh.

Spreadsheet Models

Two examples were used to familiarize the teachers with the spreadsheet's capabilities: the Fibonacci sequence and a square root extractor. Four other prepared examples were made available for teachers to explore on their own.

The square root extractor was adapted from a model suggested by Masalski (1990, pp. 8-9), based on the following familiar algorithm to compute the square root of the number n:

Step 1. Estimate the square root of n.

Step 2. Divide n by the estimate from Step 1.

Step 3. Compute the average (arithmetic mean) of the estimate from Step 1 and the quotient from Step 2.

Step 4. Replace the Step 1 estimate with the average from Step 3, and repeat the process until the desired precision is obtained.

Many of the teachers were already familiar with this algorithm, and it was easily implemented on the spreadsheet, as shown in Figure 1. The user chooses an "Input Number," n, and an initial estimate of the square root of n. Changing the estimate can reveal patterns in the computations. For example, it is readily apparent that the algorithm approximates the square root more quickly (i.e., with fewer iterations) when the initial estimate is more accurate.

Adding a graphical representation to the spreadsheet display facilitates exploration of the model. Features that are not as readily apparent from the table of values become salient in the graph. For example, while trying several estimates to explore the number of iterations necessary to approximate the square root adequately, one is likely to observe in the graph an apparent basic difference in the sequence

of successive approximations, depending on whether the initial estimate is too large or too small. As shown in Figures 2 and 3, when the initial estimate is *larger* that the square root, the algorithm produces a monotone decreasing series of approximations, each approximation smaller than the previous. However, when the initial estimate is *smaller* than the square root, the graph begins with a jump to a too large approximation, then continues with a monotone decreasing series. Depending on one's style, this can be simply a curious observation or a compelling conjecture that demands justification. Why wouldn't a too small estimate produce an increasing series that converges on the square root from below, as the too large estimate does from above? Or, given the beginning jump from too small to too large, why wouldn't the approximations be alternately too small and too large, with the graph showing a dampening zigzag as it approaches the square root?

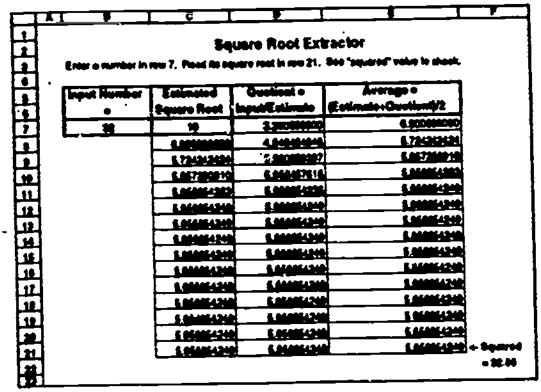

Figure 1. A square root extractor.

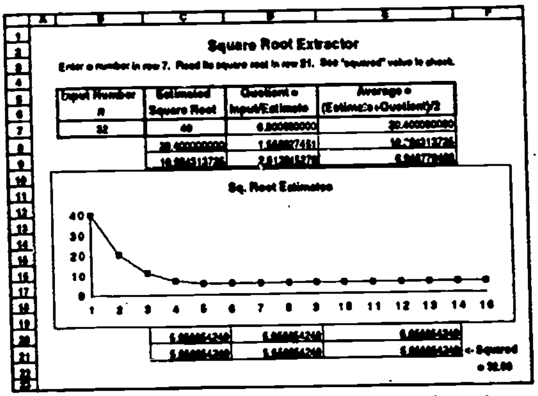

Figure 2. A too large estimate produces a monotone decreasing sequence.

Teachers' Exploration

Teachers were given the following four questions to explore:

1. What difference does it make whether the initial estimate is close to the square root or very far off?

2. What difference does it make whether the initial estimate is too big or too small?
3. What happens when you use a negative initial estimate?
4. What happens when you start with a negative Input Number, *n*?

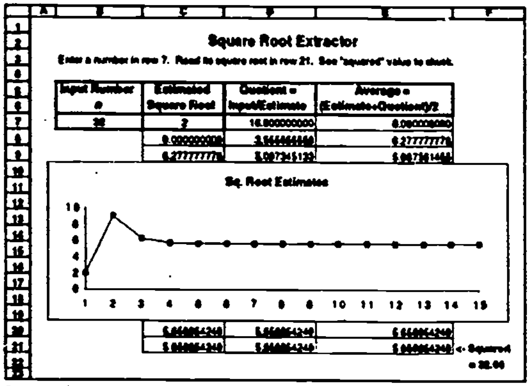

Figure 3. A too small estimate results in a jump to a too large estimate, followed by a monotone decreasing sequence.

Results of the first two questions have been discussed above. The suggestion to try a negative estimate (Question 3) aroused skepticism. Many participants predicted that it would not work, and from the discussion it was apparent that for some of the participants, this prediction was based on the general notion that when dealing with square roots, negative numbers "don't work." But as shown in Figure 4, entering a negative estimate simply produces a negative square root, and the graph of the approximations is a vertical reflection of that resulting from a positive estimate. The group as a whole accepted this result fairly easily, commenting about the difference between making a negative estimate and trying to take the square root of a negative number. This, of course, primed them for the fourth question.

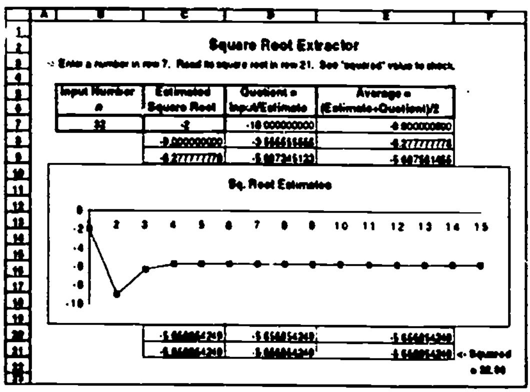

Figure 4. A negative estimate produces a negative square root.

Participants were more confident in their predictions for Question 4, that a negative Input Number, *n*, would be rejected or produce an error. Figure 5 shows a result of choosing a negative input. The apparently

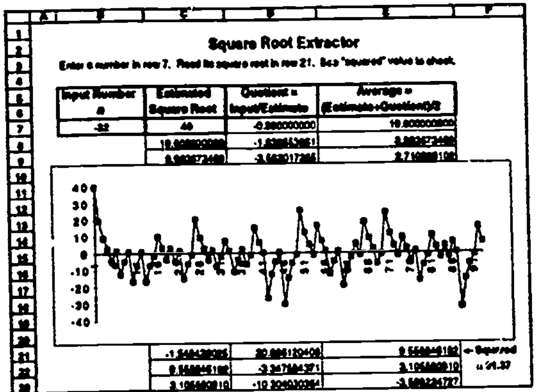

Figure 5. The result of a negative input.

chaotic behavior of the graph for a negative *n* was startling, and it led to much discussion, experimentation, and reexamination of the algorithm. Some participants extended their models to carry the algorithm through 100 or more iterations, looking for possible patterns. A few contemplated the difference between a mathematical concept (square root) and an algorithm (a means of extracting square roots under certain conditions). Because chaos had been a topic of earlier discussion, participants appreciated the unexpected result of entering a negative *n*, and they were enticed by the challenge to either find a pattern to the function or find adequate evidence to conclude that the function is truly chaotic.

Each of the observations above might have been accomplished from the spreadsheet's tables of values, without the graphs. However, it was the graphs that most often drew attention to possible patterns and that facilitated observation most effectively. A primary result of adding the visual representation appears to be to shorten the time to conjectures. The conjectures "jump right out at you." Then during the focused activity of gathering evidence to support or refute a conjecture, the visual representation continues to shortcut the data examination process. Patterns are simply easier to notice (and perhaps more compelling) in a graph.

<u>Teacher-Generated Models</u>

Following the introductory experiences outlined above, participating teachers were assigned the task of developing their own spreadsheet models. Teachers worked in pairs, with partners of their choice. The teachers produced 18 models, including "magic square" builders, maximization problems involving 2- and 3-dimensional objects, population models, and a host of investigations and applications incorporating Pascal's Triangle, fractals and chaos, and other mathematical ideas.

Observations and Discussion

One fundamental difference between the spreadsheet environment and the problem-solving methods already familiar to the participating teachers is the spreadsheet's facility to define variables for a function by simply clicking on cells containing the needed quantity. For example, to define a function

that sums the contents of cells C3 and D2. It is not necessary to identify the cells by row and column indexes and type "C3 + D2." Rather, the user can define a function by clicking the mouse on appropriate cells as needed. The user essentially says, "I want whatever is in this cell [*click*] plus [press '+'] whatever is in this other cell [*click*]," and the function has been created. This approach appears to have some of the same conceptual advantages as using open shapes (boxes, circles, etc.) instead of letters to denote variables in introducing algebraic ideas. Like an open shape, a spreadsheet cell can contain different values, and directly choosing the cell as part of a function may help to construct the difficult notion of "variable."

Some teachers were more ready than others to take this "naive" approach to creating functions, and a variety of tactics emerged. For example, a first grade teacher and a high school teacher working together chose to model the problem, "Would you rather receive a constant allowance of $10 per day or 1 cent the first day and double the preceding day's amount each succeeding day?" After beginning their project, the high school teacher missed a session, and the first grade teacher was left to continue on her own. It quickly became apparent that she was trying diligently to implement equations that her partner had worked out and written down for her. She was having considerable difficulty, and she was convinced that she could not complete the model without the help of her colleague. Despite this, she clung to the equations on the paper and she was reluctant to try another approach. After finally being persuaded to set the notes aside and take a fresh look at the problem on her own, she was able to construct two columns for daily allowances, one beginning with $10 and computing each day "the same as yesterday," and the other beginning with 1 cent and computing each day "two times the previous day." She then constructed a cumulative total column for each allowance scheme, with each day's total "yesterday's total [*click*] plus today's allowance [*click*]." This was clearly more intuitively accessible to her than her partner's formulas for linear and exponential growth with which she had been struggling.

Teachers' models targeted a wider grade level range than anticipated. The initial assumption was that applications appropriate for use in primary grade classrooms would not be found. In fact, several of the teachers' models could be viable in some primary grades. One model, a television viewing habits survey, was specifically designed with a second grade class in mind. Children enter information and draw conclusions about their daily and weekly use of time, as well as more long-term information such as how much TV they will have watched by the time they graduate from high school if they continue watching at their current rate.

Conclusion

This project involved experienced K-12 teachers in mathematical modeling, using variables and functions in a fundamental and intuitive context, and investigating the appropriateness of spreadsheet

modeling in mathematics across grade levels. Visualization played a vital role in the teachers' model building and exploration. In particular, having one or more graphs available on the same display with the rest of the model (and dynamically linked to change with every change in the model) greatly enhanced the process of making and testing conjectures.

A spreadsheet environment allows functions to be defined recursively and investigated dynamically. Once a model is set up, the initial values can be altered to investigate any number of similar problems, transforming a problem from a single instance into a rich generalization. Further, the spreadsheet allows for specifying variables for a function by simply clicking on the cells containing the needed quantity. This facility may prove beneficial in forming the often difficult concept of an algebraic "variable."

Wagner and Kieran (1989, p. 231) suggest several identifiable *dimensions* of algebraic thinking (including, among others, use of variables, understanding of functions, symbol facility/flexibility, generalizing, and ability to formalize arithmetic patterns). Two of the research questions posed by Wagner and Kieran in this context are:

1. What kinds of thought processes are involved in various algebraic topics?
2. What kinds of thinking processes are required to apply algebra to problem solving?

The potential answers to these two questions may well be broadened, and perhaps otherwise changed, by the style of algebraic problem solving, investigation, and visualization enabled through mathematical modeling in a well designed spreadsheet environment.

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OUT OF THE CUL-DE-SAC?

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Summary: *Students frequently construct and articulate mathematical relationships which are general within a microworld yet are interpretable and meaningful only by reference to the specific computational setting — they construct* situated *abstractions. Additionally, the computational setting can be exploited by the student to provide scaffolding onto which they can hang their developing ideas. In this paper we describe and illustrate these ideas and draw out some implications for research and development in mathematics education.*

For some years now we have been researching questions in relation to mathematical learning in the context of computational microworlds — largely but not exclusively exploratory Logo-based environments (see Hoyles & Noss, 1992 for a summary of our own and other research work in this field). During this time, a constant theme running through our work has been to try to unravel the relationship between ideas which the learner brings to a mathematical situation, and those which are expressed in the course of computational activities. In fact, we see this as a more general issue: to make sense of the ways in which the elements of a learner's mathematical knowledge coalesce into coherent mathematical ideas and concept. We see the computer as much a means of accessing these understandings and as a way of enhancing them.

We have established a framework within which to interpret children's mathematical behaviours, and it is this which we propose to outline and elaborate — as far as space will allow — in this paper. Our starting point was a recognition that a desirable feature of mathematical learning environments — and one which can be more readily operationalised in computational settings than elsewhere — is for pupils to be able to *use* (U) mathematical ideas before they 'understand' them; more precisely, to layer understandings in the course of use. In particular, we suggested that by using mathematical ideas in suitably-designed environments, it is possible to *discriminate* (D) and *generalise* (G) these ideas — and ultimately to *synthesise* (S) them with existing mathematical understandings (see Hoyles 1986, Noss & Hoyles, 1987).

Since we suggested this model, we have concentrated our energies on two specific aspects of the problem. First, we want to derive a mechanism by which the process U to D and U to S actually occurs; that is, while we have plenty of examples (and *in situ* explanations) of students using mathematical ideas to discriminate their essential components and to generalise from them, we would like to elaborate a theoretical basis for these transitions between mathematical states, in order to recognise (and perhaps, encourage) them in more general contexts. Second, we would like to better understand the states themselves: what exactly do we mean when we say someone has 'discriminated' the elements of a mathematical concept, or when we claim that they have made a 'generalisation'?

In our analysis of microworld interactions we have noticed two recurrent phenomena: first, students frequently construct and articulate mathematical relationships which are general within the microworld yet are interpretable and meaningful only by reference to the specific (computational) setting; and second, the computational setting can be exploited by the student to provide them with hooks onto which they can hang their developing ideas. In this paper we describe and illustrate these ideas and draw out some implications for research and development in mathematics education.

Situated abstractions

We have coined the phrase *situated abstraction* to describe the first phenomenon, in which students constructively generate mathematical ideas which are articulated in terms of the medium of construction. A situated abstraction is powerful enough to entirely encapsulate a mathematical relationship, but, unlike its expression in mathematical formalisation, it is bound into the setting, it is mediated by the technology[1] and its associated language. We will provide many examples below in our presentation from different settings.

It is clear that an abstraction such as this lacks universality; in particular it lacks precisely that element of (apparently) unmediated abstraction which is present in a mathematical (and only a mathematical) discourse. But what it gains is that, within the (computational) medium in which it is expressed, it articulates a general relationship which is structured by that environment. More importantly — from a pedagogical point of view — it is *constructed* by a learner who may have no access to the semantics and syntax of general mathematical language: it is the computational environment itself which affords the opportunity to build such abstractions.

[1] The situation does not have to involve the computer: a recent example involved Joe — aged eight — who, immersed at school in the use of pegs and pegboards, stated that 'a prime number is one which you can only make with a line of pegs').

Scaffolding

In observing children working in microworlds, we can argue loosely that 'the computer' 'offers' them the opportunity to express themselves mathematically; well and good. But how? Can we be more precise about the mechanism which is in operation? Some while ago, we proposed that the metaphor of *scaffolding* might be usefully applied to this kind of phenomenon (see Hoyles, 1991, Hoyles & Noss, 1987). Wood, Bruner and Ross (1979) refer to the idea of scaffolding as graduated assistance provided by an adult which offers just enough support (and no more) when needed so that a child can voyage into his/her *zone of proximal development*. More generally, cognitive psychologists working within this tradition have shown that learning can be facilitated by providing help in developing an appropriate notation and conceptual framework for a new or complex domain, allowing the learner to explore that domain extensively, then gradually *fading* that support. Learner participation is gradually increased — according to the needs and learning pace of the individual.

We would like to extend the scaffolding metaphor into a computational setting, and in doing so, it is important to clarify at least three areas in which it differs from the kind of process described by Bruner and his colleagues. First, the literature on scaffolding is mainly concerned with the support of skills, and we would like to apply it the *acquisition of concepts*. This is not, of course, straightforward, as learning to do mathematics is hardly comparable to say, basket weaving. Second, the traditional view of scaffolding is that it is universal; that is, that having voyaged onto the scaffolding erected in the zone of proximal development, the child is assumed to have 'developed' in a sense which is independent of the context in which the learning takes place, and the medium in which it is expressed. Our focus, conversely, focuses precisely on the setting, on the symbol system within which the ideas are expressed, and more particularly, the extent to which the scaffolding mechanism is domain contingent.

Most importantly, by extending the notion of scaffolding to a computational context, we are asking in what ways a computer might play the role normally ascribed to a human tutor. We need to clarify this point. What are the new possibilities for learning environments and new ways to conceptualise them, given the advent of sophisticated computer technology? Our concern is centred on mathematical expression:

• Can we describe how scaffolding operates within computational environments which provide a fluid and flexible support system by means of which a child can build situated abstractions?

• Is it possible for the computer environment to provide scaffolding for the construction of situated abstractions in ways which are *under the control of the learner* and which arise from distinctive features of the computer environment, rather than, say, by the provision of a computer-tutor embedded in the program?

Our experience leads us to believe that the scaffolding metaphor *taken together with* the notion of situated abstraction may have potential in developing a coherent theoretical framework for understanding children's mathematical conceptual development in computer-based environments. For example, the transition from seeing a mathematical relationship between specific cases to writing it in explicit general form is not straightforward. We have indications that the computer may allow pupils to "flag" what is varying and sketch out what might be the significant features of any relationships as mediated by the software — relationships that then can be constructed and tested out. We have also seen students using the computer as computational scaffolding in order to sketch out partial solutions to mathematical problems, to avoid having to shoulder the cognitive burden of solving the problem in its entirety and to 'add variables' at their convenience. We have observed students construct an approximate or visual solution, and then — aware of its limitations — use this as a means to generate a more analytic strategy. Most promisingly, the scaffolding metaphor may help in conceptualising child-computer interactions: in particular fading — again, under student control rather than as a decision (implicit or otherwise) of the tutor — becomes a means by which the student might be encouraged away from medium-specificity, and towards the abstraction of mathematical structures and relationships.

Rather than developing these arguments in the abstract, we now turn to a geometrical example which will serve to focus the discussion.

An Example: Interacting with Cabri Géomètre[2]

Cleo and Musha (age 14) were given the two flags shown in Figure 1.1 where one was the image of the other after a reflection. Their task was to find the 'mirror line' — the line of symmetry. They did not know the constructions necessary, and so, not unnaturally, they dragged the basic points[3], playing within Cabri to try to generate some clues. Slowly the activity became more focused and they started a little more systematically to drag the first flag about the screen, noticing the effects on the image. In using the medium in this way, they gradually developed a sense of *where the mirror must be* — until, after a short while, they could point to its position with certainty. But this intuitive certainty in itself gave no help as to *how* to construct the mirror: after all, to do so would require a language and a notation — not necessarily a mathematical one, but one which would allow them to express their intuition by more than simply pointing to the screen.

[2] Cabri Géomètre is a Euclidean geometry program within the genre of (but significantly different from) Geometry Inventor and Geometry Sketchpad.
[3] Dragging is a feature in Cabri in which any basic point in a construction can be 'picked up' by the mouse and moved about the screen.

Having developed this intuition, Cleo picked up the mouse, and started to drag the topmost and bottom-most points together (Figure 1.2). "That's it!" they both exclaimed, drawing the line joining these intersections with their fingers on the screen. They went further and dragged all the equivalent points together and in this way, actually 'constructed' the required line (Fig 1.3 illustrates the process). They were completely confident at this point that they had articulated (not verbally, but within the Cabri system) the mirror line to the two flags – that is they had constructed a situated abstraction:

The mirror line is what you see on the screen if you drag points and their images together

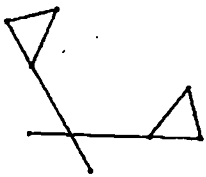


Figure 1.1

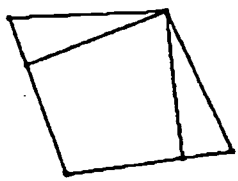
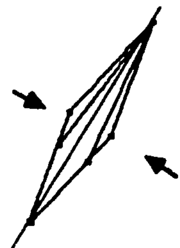


Figure 1.2



Line of symmetry (not drawn)

Figure 1.3

Now this is a completely correct strategy for finding a mirror line. The situated abstraction exactly expresses where it is, and the method is precisely generalisable to all reflections. On the other hand, it is self-evident that it is only meaningful and — perhaps most importantly — *constructable* using the scaffolding which operates within this particular computational environment. In this case (it is not always so easy) Cleo and Musha were able to dispense with this particular scaffold. Once they had 'seen' where the mirror line actually *must* be, they were able to surmise that they could 'fix it' by constructing the midpoints of the lines joining two pairs of equivalent points.

How far do they construe this as a general strategy which can be applied with, say, paper and pencil? Constructing midpoints might appear on the surface to be easily 'transferable': but things are not so simple. Paper and pencil construction would demand compasses (or, of course, measurement!); Cabri necessitates the opening of the appropriate menu item and clicking on two points. It is clear that these actions are only 'the same' from a very general mathematical perspective — one in which the role of the midpoint is

already understood. But the differences — and therefore, the difficulties of seeing the mathematical abstraction in the situated one — should not blind us to the fact that there are similarities as well. The Cabri midpoint is itself an abstraction. It wraps up much of the process of construction into a single action — we are using abstraction here (confusingly) in its standard mathematical/computer-science sense. The computer, for example, 'knows' what kinds of objects it 'needs' for a midpoint — a line segment or a pair of points. So some of the mathematical essence of the midpoint comes ready-made, it becomes both an object, and a tool with which to construct further objects. This duality is, of course, at the root of mathematical activity, and it is precisely this duality which many have considered to be so problematic (see, for example, the work of Douady, 1991).

Concluding Remarks

Now that we have begun to make sense of our own findings within the scaffolding/situated abstraction framework, we have to admit that we have started to notice them elsewhere. As an example, Confrey, Smith, Piliero and Rizzuti (1991) report that they found a keystroke representation of functions (as undertaken on a calculator) to be an efficient way for students to come to terms with the operational character of the function. One can imagine how students build up and represent functions using the keys of the calculator matched to feedback from the display as scaffolding for their situated abstraction. This approach is completely general and understandable — provided of course you are familiar with the meaning of the keystrokes!

What can we learn from this and all our own examples? In the first place, we would like to draw attention to the status of a situated abstraction, as lying somewhere on a dimension between intuition and fully-fledged mathematical abstraction. Consider the two students trying to find the mirror line with Cabri. We have seen how they appeared to be 'certain' of where the line should be, but unable to articulate how the position should be derived before they experiment with the software. So what *do* they know initially? Where the line should be, in the sense that it either 'looks right' or it doesn't. What in fact, is 'missing' is that they are unable to develop a mathematical model from their intuitive knowledge: they know when something is a reflection and when it is not, but they are unable to *construct* the mirror (or indeed if given the mirror and the object the reflection). Why? Because construction implies an explicit appreciation of the relationships that have to be respected within any situation, a mathematical model of the situation (how else do you know what to focus on, and what to ignore?).

It is here that we gain some insight into the scaffolding role of the computer. For without the computer, such a model can *only* consist of some formal mathematical material — 'knowledge' about perpendiculars, equal length constructions etc. In Cabri, that knowledge is wrapped up into a computational model — in this case, a model of

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Euclidean 'knowledge'. And so the level is notched up a rung or two: and abstractions can be made which are *situated* within the existing model, expressed through 'statements' (mouse clicks, pieces of programs etc.) which are already expressions of mathematical abstractions. Here, by the way, is a criterion for establishing the potential of mathematical software — the extent to which it manages to encapsulate or abstract a coherent and usable set of computationally-mediated tools.

Knowing roughly when a reflection looks right is a valuable intuition from which can be built a situated mathematical abstraction with the tools available. But this is not the end of the story. How do we interpret these student constructions? And how do we bridge the evident gap between these mediated articulations and those that somehow transcend the medium? This is not so straightforward as it seems, as it presupposes that we can really think of the issue as one in which 'the same' mathematics is expressed in two 'different' media? Such a view presupposes that there *is a* mathematics which encapsulates the essence of *both* situations. To consider this issue in detail would force into the uncharted waters of philosophy — we would need to consider just exactly what the relationship is between concrete experience and mathematical abstraction. But even from a psychological point of view, it should be clear that while we might want to consider a mathematical essence shared by, say, Cabri geometry and Euclidean geometry in terms of constructions and theorems, they are certainly not the same for the child.

We conclude with two further questions: first, if situated expr[illegible]sion of mathematical relationships and formal mathematical discourse are two sides of the same mathematical coin, we might call into question the gamut of research methodologies which insist on identifying transfer from one to the other[4]. There is (now) more than one way of appreciating and expressing mathematical relationships, even of constructing new mathematical objects. Perhaps we should acknowledge that mathematical abstractions are situated, all offer important insights, and to broaden our view of what counts as mathematical culture[5].

The second point is a partial converse of the first. Turning away from geometry, for example, can algebraic language be viewed itself as a medium to scaffold students' conceptual development? In our opinion there is no ontological reason why not — it is just that school algebra is usually not regarded as a constructive language! We contend that differences are not so much those of learning but are more to do with what is regarded as legitimate mathematics (this issue is explored in more depth in Noss (in press)). The challenge then becomes one of designing learning environments (involving computers or

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[4] See, for example, Jean Lave's eloquent critique of the transfer methodology: Lave J. (1988) *Cognition in Practice*. Cambridge, UK: CUP

[5] We would rather not be taken to be implying that traditional mathematical expression is *not* worthwhile — only that there may be more than one way to express!

not) in order that students can exploit them as scaffolding for the articulation of situated abstractions. At the same time, we need to work on the specificities of computational environments which offer ways out of the cul-de-sac within which a strictly situated view of learning traps us.

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TECHNOLOGY IN THE LEARNING OF FUNCTIONS: PROCESS TO OBJECT?

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This study of algebra learning in a computer environment aimed at exploring the nature of the cognitive shift that is involved in extending a numerical, process-oriented perspective of functions and their representations to a form-oriented, structural one. By means of a single-subject case-study of a 14-year-old who had already spent 20 hours learning to solve problems in the function-based, problem-solving, CARAPACE environment, but who had not yet been introduced to traditional algebraic manipulation, we followed the subject's evolving awareness of different families of functions and of the role of parameters of certain functions over the course of 12 sessions. During the study, there was no explicit teaching; rather the interventions took the form of a structured sequence of tasks accompanied by probing questions. We found that the subject's numerical approaches continued to evolve with respect to all representations. However, he did not become aware of the possible general forms of the notational representations that could be associated with the information he was able to derive from other representations of these functions.

There is a whole spectrum of possible ways to teach algebra; nevertheless, all of them can be roughly divided into two groups, according to the basic approach they promote:

a) the more traditional approach based on teaching algebraic techniques (such as manipulation of formulae, solving equations and inequalities, etc.) with no overall conceptual framework--a bottom-up approach in which the unifying concept of function comes late with no real connection to the techniques;
b) the functional approach: The unifying concept of function is introduced at the very beginning and all algebra is then built up from it.

On the face of it, the second approach is preferable, as it has the potential of making the learning much more meaningful. Indeed, many teachers, curriculum developers, and researchers seem to take the educational advantages of this approach for granted. However, new empirical findings (e.g., Linchevski & Sfard, 1991; Moschkovich, Schoenfeld, & Arcavi, in press) show that the functional approach, if adopted from the very beginning, may be quite difficult for the student. This finding gets its support and explanation from the freshly developed theory of process-object duality of mathematical thinking--or theory of reification (Sfard, 1991; Sfard & Linchevski, in press) which implies that new mathematical objects, such as functions, are not readily accessible to students and may not be well understood until the underlying mathematical processes have been learned and interiorized.

The functional approach to algebra may now become much more feasible, thanks to the increasing accessibility of computer-generated graphical representations. As has been pointed out by many researchers (e.g., Dreyfus, 1991) visual means may have beneficial effects on mathematical thinking. One possible explanation for this phenomenon is that visualization is considered to facilitate the acquisition of structural conceptions. Thanks to new technological means, graphical representations may now be used and analyzed even before algebraic symbolism has been introduced.

The aim of the study reported in this paper was to explore the feasibility of extending a functional, process-oriented approach to an object-oriented one when it is supported by the intensive use of graphical representations. By means of a single-subject case-study of a 14-year-old who had already spent 20 hours learning to solve problems in a function-based, problem-solving, computer environment (Garançon, Kieran, & Boileau, 1993), but who had not yet been introduced to traditional algebraic manipulation, we followed the subject's evolving awareness of different families of functions and of the role of parameters of certain functions. During the study, there was no explicit teaching; rather the interventions took the form of a structured sequence of tasks accompanied by probing questions. Our objective was thus not only to see how effective this sequence would be in helping our subject to develop object-oriented conceptions of functions, but also to uncover some of the cognitive processes involved in acquiring these conceptions.

Theoretical Framework

The historical-psychological analysis of different mathematical definitions and representations shows that abstract notions such as number and function can be conceived in two fundamentally different ways: **structurally (as objects) or operationally (as processes)**. The two approaches are contrasted in the following way:

> There is a deep ontological gap between operational and structural conceptions. ... Seeing a mathematical entity as an object means being capable of referring to it as if it was a real thing--a static structure, existing somewhere in space and time. It also means being able to recognize the idea "at a glance" and to manipulate it as a whole, without going into details. ... In contrast, interpreting a notion as a process implies regarding it as a potential rather than actual entity, which comes into existence upon request in a sequence of actions. Thus, whereas the structural conception is static, instantaneous, and integrative, the operational is dynamic, sequential, and detailed. (Sfard, 1991, p. 4)

The process-oriented approach, as we envisage it within the context of teaching algebra from a functions perspective, involves essentially using a point-wise lens to view graphs and their related representations. With this approach students work with *variables* and operate on numbers, replacing the variables by different numbers. Numerical calculations are an integral part of this perspective. This is in contrast with an object-oriented approach in which students work with *parameters* and operate on variables and their expressions (not necessarily using standard algebraic symbolism). Here, the form of the functional expressions is a key consideration. In a process-oriented approach, the focus is on *x-y* or input-output pairs which are used to generate points, and on the computational process involved in going, in general, from a non-graphical representation to a graphical one. Graphical interpretations tend to be local; even if global interpretations are requested, they tend to be justified in terms of process-based explanations.

The process-oriented approach is embedded in the CARAPACE computer environment (Boileau & Garançon, 1991; Boileau, Kieran, & Garançon, 1993). CARAPACE is a piece of software that was designed to foster a functional approach to problem solving (Kieran, Boileau, & Garançon, 1993). It involves student-constructed translation of problem situations into generalized statements (program algorithms) consisting of sequences of significantly-named variables and operations. The trial-and-error guesses subsequently generated by the student in attempting to solve a problem related to the functional situation are recorded in the form of input-output pairs in both tables and graphs. The resulting points of the Cartesian graph can be shifted dynamically by mouse-centered actions on the axes of the graph. With this approach, considerable emphasis is placed on the underlying process by which additional input-output pairs are generated and on the corresponding properties of the sets of points representing particular functional relationships, in both tabular and graphical representations. One of the pedagogical decisions of the CARAPACE team has taken the form of extending the process-oriented approach to include an object-oriented approach. Thus, the process-oriented

approach is expanded to integrate both the graphing of sets of points which can be made to appear continuous and also the links between their shape and the corresponding notational and tabular representations. This leads eventually to a study of the role of functional parameters and the various ways in which they can be represented and manipulated. In this shift to an object-oriented approach, a curve lens is used to view graphs and their related representations. The focus is on the global aspects of graphs and their interpretation. Students examine classes of functions and the key features that account for the variation among members of a class, in other words, the role of parameters on the shape and properties of the graph. The key issue here, and for this study, is the nature of the cognitive shift that is involved in extending a numerical, process-oriented perspective to a form-oriented, structural one.

The Study

This study is part of a long-term research program on the development and use of the computer environment, CARAPACE, as a tool for introducing certain conceptual aspects of algebra. Some of our past studies have focused on: a) the processes used in generating functional algorithmic representations (Kieran, Boileau, & Garançon, 1989); b) the kinds of strategies students adopt in reading and using tabular representations (Kieran, Garançon, Boileau, & Pelletier, 1988); c) the processes used in solving problems once the procedural representation has been generated (Kieran, Boileau, & Garançon, 1992); d) the beginnings of a transition to algebraic symbolism (Garançon, Kieran, & Boileau, 1990; Luckow, 1993); and e) the strategies students use in producing, interpreting, and modifying graphs of functional situations (Garançon, Kieran, & Boileau, 1993).

The study reported in this paper is one that involved a 14-year-old student, whom we shall call Kim, who was in the eighth grade. His marks in mathematics were above the class average. He was one of our subjects in the previous year's study during which time he was introduced to problem solving in the CARAPACE environment and to the use of Cartesian graphs as a tool for locating the solution interval for certain types of word problems. A brief overview of Kim's problem-solving approaches at the outset of this study is as follows. When Kim was presented with a problem (see Figure 1a), he usually carried out a numerical trial in order to help clarify the problem situation (see Figure 1b). From this, he generated a program algorithm (see Figure 1c) and then went to the table of values to both retry his paper-and-pencil value and enter a few additional ones. Then, he generally examined the graphical representation and often used this to find the problem solution (see Figure 1d). (Note that the headers of the CARAPACE screen shown in these figures, as well as the program excerpts, are in French because the study was carried out in that language; all problem questions and protocol excerpts have, however, been translated for this paper.)

We conducted 12 one-hour sessions with Kim during the winter term of the 1992-1993 school year, usually at a pace of two sessions per week. In each session, the interviewer presented Kim with functional situations--as opposed to problems to be solved--which he was to represent in the CARAPACE computer environment with three different representations: program algorithms, tables of values, and Cartesian graphs. All sessions were video-taped and transcribed; the computer work was recorded in dribble files.

The first five sessions, which included a pretest, focused on linear functions. The next seven sessions were devoted to quadratic functions (problem situations dealing with area, ticket sales, stopping distance of a car, throwing a ball, etc.) and the comparison of quadratic and linear functions in various representations, as well as a posttest. (Note that we had originally intended to include exponential functions in the study, but changed our minds when we realized that most of our exponential situations would involve the use of percentages, a conceptual area that, in an earlier study, was found to be a difficult one for many students of this grade level.)

A local basketball team is going to a tournament i... ris. A group of fans would like to go to cheer for their team. A travel agent arranges the following ... : For a group of 20 fans, the cost is $500 per fan; for each fan over 20, the cost for each person is reduced by $2. If the agent wants to take in at least $30000, how many fans must go?

Figure 1a

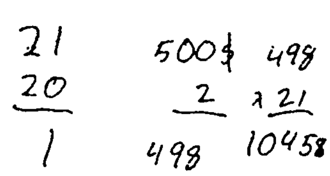


Figure 1b

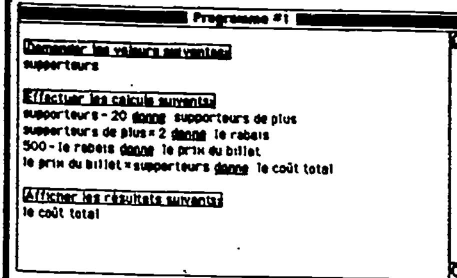
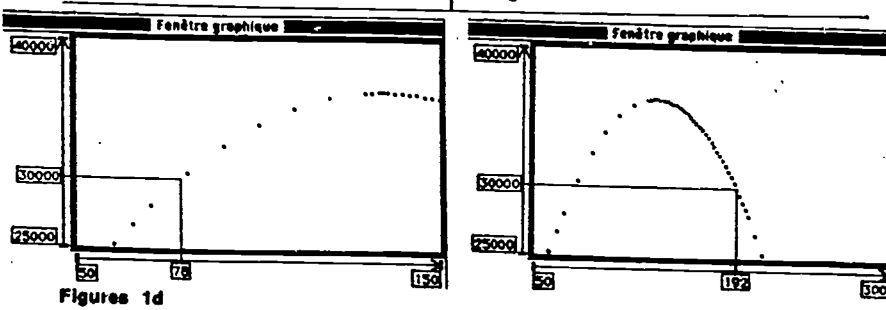


Figure 1c



Figures 1d

The flavor of the functional situations that we presented to Kim is illustrated by the following three examples. The first example comes from Session 5:

> There are two students who would like to earn some money during the summer by painting houses. Each decides to charge $10 an hour. They also have fixed costs which they charge even if no work is done. The first student charges $6 for his fixed costs and the second $8.
>
> 1. Write a program which will calculate what each of the students can earn.
> 2. Ask for a graph from 0 to 4 hours of work using a step size of 1.
> 3. Describe what you see in the table of values.
> 4. Suppose that a third student decides to compete with the other two. He wants to charge $10 an hour too, but his fixed costs are only $3. Predict what the graph of this student's possible wages will look like.
> 5. Suppose that a fourth student enters the picture. He also charges $10 an hour, but his fixed costs are $12. Predict what the graph of this student's possible wages will look like.

Figure 2 presents the last version of Kim's program and also the graph of this program for integral values of the input variable from 0 to 4.

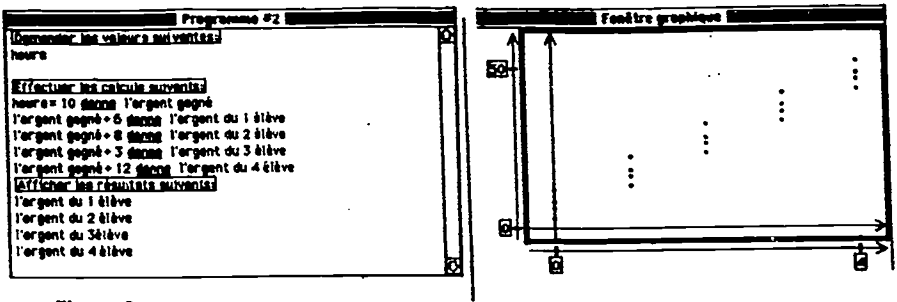


Figure 2

The second example concerns the following problem situation taken from Session 7: "The length of a rectangular field is eight meters more than the width. Write a program which can calculate the perimeter and the area of the field." Kim wrote the program shown in Figure 3.

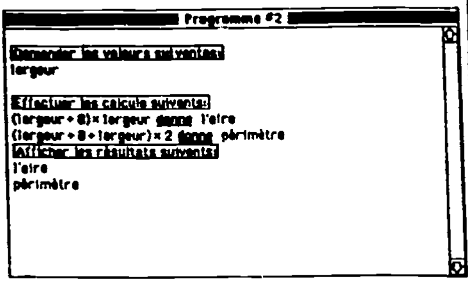


Figure 3

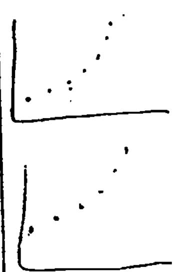


Figure 4

After Kim had written the program presented in Figure 3, Session 7 proceeded as follows (I is for the Interviewer; K is for Kim):

I: What do you expect the graph to look like for the two of them?
K: They will be a little curved and they will go up.
I: Would you do a little drawing to show me (see Figure 4).
. . .
I: OK. Now do the graph on the computer.

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K: OK (see Figure 5). ... The green is not the same curve as the red. (The reader will note that in the CARAPACE environment each function is plotted in a different color.)

I: What kind of curve is it?

K: It is not a curve; it's just a straight line.

I: Do you have any idea why?

K: I think because at 0 it would be 16, at 1 it would be 20, at 2 it would be 24. There is always a jump of 4.

I: Let's look at the table of values to see that a little better (see Figure 6). Now tell me about the perimeter.

K: <u>It goes up always by 4. That makes its regularity and makes it go straight instead of forming a curve, like the other one</u>.

I: What makes that one curve?

K: When the distance is not the same (pointing to successive points of the graph).

I: OK, and the table of values, what makes the difference?

K: Here from 9 to 20, there's 11. Then from 20 to 33, it's 13; then 15, then 17, and so on.

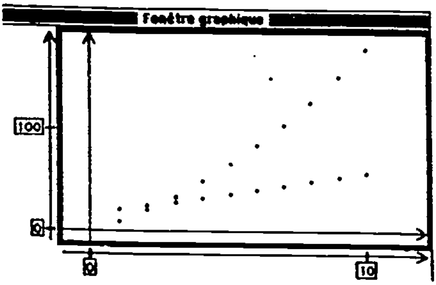
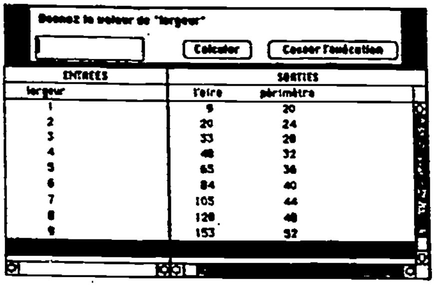


Figure 5



Donnez la valeur de "largeur"

Calculer | Cesser l'exécution

ENTRÉES	SORTIES	
largeur	l'aire	périmètre
1	9	20
2	20	24
3	33	28
4	48	32
5	65	36
6	84	40
7	105	44
8	128	48
9	153	52

Figure 6

A third example of the kinds of tasks used in the study is taken from Session 10:

Enter the following program: 3 * x <u>gives</u> a
x * x <u>gives</u> b

where x is an input and a and b are outputs. Predict the graphs. (After seeing the graphs), explain why the graphs are the way they are in terms of the program. And if we change the 3 in 3 * x to -3, how would the graph look? And so on.

Findings

The most salient finding of our study was the persistent development of Kim's numerical approaches for dealing with tasks involving various representations of functions. One characteristic of Kim's numerical resourcefulness was illustrated in the previously-seen excerpt from Session 7 when we noted how Kim used a table of values to explain why one of the graphs was linear and the other was not--in terms of a constant difference from one output value to the next versus a changing difference.

During successive sessions, Kim was asked a variety of questions that could have been answered by referring to the form of the program, but never were. His justifications were always numerical. For example in Session 10, he was asked to generate a program which had a line for its graph. He thought

of a simple context (a helium-filled balloon rising at a rate of 2 meters/second, which he remembered from the previous session) which he expressed with the program "m * 2 gives h." He was then asked to generate a program whose graph would not pass through the origin; this led to "m * 2 + 1 gives h." Thirdly, he was asked to generate one which climbed faster (to use his vocabulary), to which he produced "m * 5 + 1 gives $h1$." When asked to explain why this would climb faster, he replied: "Because here if you put 1 (in m * 2) it gives 2, and here if you put 1 (in m * 5) it gives 5; it's higher."

In the next question of Session 10, Kim was asked to write a program which would produce a curve for its graph. He thought of a context and mentally substituted a few values (0, 1, 2) to see if they resulted in a nonconstant difference for successive pairs of output values. He showed no sign of being aware of and being able to rely upon specific forms, such as x^2 (or x * x). When he finally generated by trial-and-error a program that yielded the curve shown in Figure 7, he was asked if he could produce one to go in the other direction. Again, he resorted to trial-and-error numerical verifications.

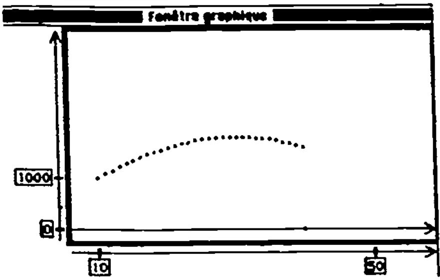


Figure 7

0	15
1	18
2	21
3	24

Figure 8

In Session 11, he was asked to enter the program "3(x + 5) gives k" and to try to create another program without parentheses that would give the same graph. He thought for a while and then began to write down a table of values (see Figure 8). He calculated the differences and thus was able to come up with the program "15 + x * 3 gives k." When asked by the interviewer if he could go directly from "3(x + 5)" to "15 + x * 3," he said he could not. This example suggests quite dramatically that students who have not yet learned to carry out some basic symbol manipulation might be hampered in their explorations of alternate forms of expressions and in their attempts to generate equivalent relations.

One of the questions of the last session was designed to disclose what resources Kim might fall back on if he were not permitted to rely upon his numerical approaches. It involved a set of 11 program excerpts (e.g., x * x -105 gives y) and for each one Kim was asked to state, "without doing any calculations in his head", which ones would have a graph that forms a line and which would have a graph that forms a curve. For all of those expressions having more than one occurrence of x, Kim responded that they would produce a curve. Thus, over the course of the study, he had not become aware of the crucial differences between, for example, x * x and x + x, with respect to their graphs. When asked to draw a rough sketch of the graphs of these programs, he graphed x + x just as he had graphed x * x; in addition, he stated that he had no idea how to do the graph of x * x - 105.

In a subsequent question asking him to write a program that would calculate the area of a square when the length of the side is supplied and to predict the graph that would result, he drew a half-parabola but

with the concavity inversed. After viewing the computer-generated graph of his function, he was asked to modify the program so that the graph would pass through the point (0, 10) instead of through the origin. He relied once more on numerical trial-and-error, not realizing that the simple addition of 10 to $x * x$ would do the job. Even though there had been several indirect attempts to focus his attention on the form of these second-degree programs, he had not abstracted the $ax^2 + c$ form nor the role of the two parameters a and c. (Note that there had been no intent on our part to include in this study any explicit focusing on the role of the parameter b in $y = ax^2 + bx + c$.)

A related issue is the question of whether or not for linear functions Kim had abstracted the $ax + b$ form or whether he was always relying on numerical testing. One of the posttest tasks, on the grouping of contextualized functional situations, indicated that Kim did not perceive all linear situations to be of the same type, that is, of the $ax + b$ type with only the a and b varied. He classified linearly decreasing functions as belonging to a different group from the linearly increasing functions.

Concluding Remarks

So what did Kim learn?

1. He learned how to read a table of values and to decide which ones did not represent linear situations due to the nonconstant differences. In the pretest, he had no global strategies for reading tables; he had to calculate input-output values in order to decide if a table of values matched or did not match a certain program.
2. He learned to read a graph and to decide if it corresponded globally to a program by looking at vertical distances to see if they were the same or changing. In the pretest, he had to use coordinates of points substituted into the program algorithm to see if they corresponded to a given program.
3. He learned to predict the shape of the graph from a program by carrying out mental calculations involving, usually, three values.
4. He learned to translate certain tables of values into linear programs.

What did Kim not learn?

1. He did not learn to abstract from a graphical representation the possible general forms of the programs that could have produced it.
2. He did not learn the role of the parameters for simple quadratic functions of the form $ax^2 + c$.

Kim clearly evolved in his global approaches to interpreting functional situations and their representations. However, he did not reach the levels of conceptualization that we had suspected he might attain. In terms of the process-object distinctions that were described briefly in the early part of this paper, Kim had, at the outset of this year's study, a kind of understanding of functional problem solving that could be characterized as process-oriented. This knowledge became more refined over the course of the study in that he learned to read the tabular and graphical representations in such a way as to extract information of a more global nature from them. Nevertheless, his awareness of the form of a notational algorithmic representation did not evolve to the extent that he could read its family or discuss the role of the parameters directly in non-numerical terms. One of the major issues here is the role of symbol manipulation. Kim had not yet learned to simplify algebraic expressions to a canonical form. The effects of such prior knowledge not only on the development of Kim's awareness of the notational forms of various families of functions but also on his initial approaches to representing functions in a procedural/algorithmic manner might have been such as to produce results profoundly different from those that took place.

In a recent study involving an object-oriented approach to the teaching of graphs with subjects who had all had some prior instruction in symbol manipulation, Moschkovich, Schoenfeld, and Arcavi (in press) found that "coming to grips with the object perspective takes time" (p. 25). In that regard, the study described in this paper was no exception. We found that certain basic facets of an object

conceptualization of functions, such as the forms of families of functions and the role of parameters, can take considerably longer to develop in their evolution from a process to an object conceptualization than one might expect. This study thus provides further evidence in support of Sfard's (1991) claim that an object conceptualization comes neither easily nor quickly.

Acknowledgments

The authors acknowledge the support of the Social Sciences and Humanities Research Council of Canada, grant #410-90-1041, and the Quebec Ministry of Education, FCAR grant #92-ER-1207. The authors also thank Michel Lalonde and Isabelle Gendron for their assistance.

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Modeling Using Geometry-Based Computer Technology

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Abstract

In the recent past it has been pointed out frequently that modeling is an important issue in the formation and appropriation of scientific knowledge. Here we will focus on the central role of basic and more advanced geometry knowledge for general scientific modeling.

We will show how software, based on new computer hardware and software technology, such as Cabri-geometré or Geometer's Sketchpad, can be used as a tool to practice modeling. The presentation will be illustrated with examples taken from maths (perspective drawing, space geometry, function graphing in polar and Cartesian coordinates, regression and least square approximation), mechanics (statics, composition of forces), optics (reflected images in ordinary and spherical mirrors, refraction, theory of the rainbow), physics (field theory, gravity, electricity), and astronomy (seasons, moon phases).

Research Papers

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Advanced Mathematical Thinking

APPROACHING INFINITY: A VIEW FROM COGNITIVE PSYCHOLOGY

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Abstract
In spite of the important role that infinity has played in almost every branch of human knowledge, cognitive psychology hasn't studied this concept deeply enough. We believe that different concepts of infinity, associated with different qualities and features, make active different cognitive processes. Some intuitive aspects related to infinity in plane geometry were studied by means of certain situations based on a theoretical classification. The population consisted of subjects aged 8, 10, 12 and 14 years (n=172). Results show that for some cases older subjects were better equipped to deal with, whereas for other cases (related to convergence) they had more problems than the younger subjects. Some differences concerning intellectual performance were found, mainly for the groups of 10 and 12 years.

The famous mathematician David Hilbert stated: "The infinite!, No other question has ever moved so profoundly the spirit of man; no other idea has so fruitfully stimulated his intellect; yet not other concept stands in greater need of clarification than that of the infinite..." (cited in Maor, 1987, p. vii). Nearly 50 years after Hilbert's death, what has been done in the disciplines that study knowledge, cognition and education to respond to this important challenge?

Throughout history the controversial and elusive concept of infinity has played an important role in almost every branch of human knowledge. In mathematics its study has presented many difficulties and in mathematics education, it is widely known that students face many problems when they study notions related to infinity. Such an important and particular concept of human mental activity should be an interesting subject for cognitive psychology. Paradoxically, none of the different theoretical approaches in cognitive psychology has studied this concept enough (Fischbein et al. 1979; Núñez, in press). In order to overcome this situation we suggest to consider the cognitive activity that infinity in mathematics requires as an independent scientific object, as many others are (e.g., ratio and proportion, functions, recursion and programming). This could help us to build a theoretical corpus that permits the creation of new concepts for a better discrimination and explanation of the phenomena that now aren't well identified. We believe that different qualities and features associated with endless iterations and with other concepts related to infinity, make active different cognitive processes, and because

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of this they should be studied systematically. According to this idea, we have conceived a classification of "different" infinities.

Classifying infinities from a cognitive viewpoint

The following classification supposes that, from a cognitive point of view, at the very base of the notion of infinity lies the operation of iteration. It has been conceived to study naive approaches and intuitive psycho-cognitive aspects of the concept of infinity as *pure* as possible (i.e., trying to isolate the mathematical training). By intuition we understand "direct, global, self-evident forms of knowledge" (Fischbein et al, 1979, p. 5). For this reason we have focused our study on the domain of plane geometry, in which it is possible to ask children about number and sizes of figures without recalling specific school-learned knowledge or technical notions. The cases of the classification are conceived as different types of transformations applied to plane figures (e.g., polygons). In order to simplify and to make operational our approach, we assumed that a plane figure can be transformed (its attributes being operated iteratively) in terms of number (cardinality (#)), and/or in space ((S); vertically and/or horizontally, being enlarged (D) or diminished (C)). Thus, this classification intends to differentiate the following aspects related to infinity.

The first aspect considered is the *Type* of infinity ("big" or "small" infinities). That is, whether it is conceived after a **divergent** ($T_{(D)}$) or **convergent** ($T_{(C)}$) operation of iteration. From a developmental point of view, important evidence shows that the idea of convergence is mastered and understood much later than that of divergence (Piaget & Inhelder, 1948; Langford, 1974; Taback, 1975). Moreover, the study of the history of the concept of infinity in mathematics reveals a similar tendency (Núñez, in press). The second aspect is the *Nature of the content* of the operation of iteration (infinite quantities or infinite big or small spaces). That is, whether the content is cardinal ($N_{(\#)}$) or spatial ($N_{(S)}$). We believe that a distinction between a cardinal content and a spatial content should be made, especially when talking about infinity. We share the opinion of D. Tall who says "cardinal infinity is ... only one of a choice of possible extensions of the number concept case. It is therefore inappropriate to judge the 'correctness' of intuitions of infinity within a cardinal framework alone, especially those intuitions which relate to measurement rather than one-one correspondence" (Tall, 1980, p. 271). The third aspect is the *Coordination* of type & nature ($T_{(D)}N_{(\#)}$; $T_{(D)}N_{(S)}$; $T_{(C)}N_{(S)}$). If we consider that there are two attributes iterated simultaneously, this is another aspect to be studied. In our opinion, what is interesting is not only to observe how many attributes should be coordinated in a given situation, but to observe what are the attributes of these elements.

These different qualities of the operations related to infinity could be represented in a 3-dimensional space (Núñez, 1991), a space of transformations (Figure 1), where axes are each of the most simple qualities that an attribute infinitely iterated may have

$(T_{(D)}N_{(\#)}; T_{(D)}N_{(S)}; T_{(C)}N_{(S)})$. Points in the space are each of the combinations of coordinations that hypothetically make active different processes from a cognitive point of view. We believe that such a model facilitates the conception and the operationalization of our hypotheses. For instance, among other hypotheses we could mention that, in general the closer one gets to the origin (no transformation), the easier to deal with the transformation; that situations on axis $T_{(C)}N_{(S)}$ are more difficult than those on axis $T_{(D)}N_{(S)}$ (due to the differences of difficulty levels between divergence and convergence mentioned above); that situations related to infinity represented by points on plane C are much more difficult to deal with, than those on A or B (because there is a coordination of qualities different in *type* and *nature*, which demands an activity of a superior level than that demanded in plane A or B, where at least qualities of the same order are present: type (divergence) and nature (space) respectively). Besides, these aspects may vary according to age, intellectual or academic performance and/or gender.

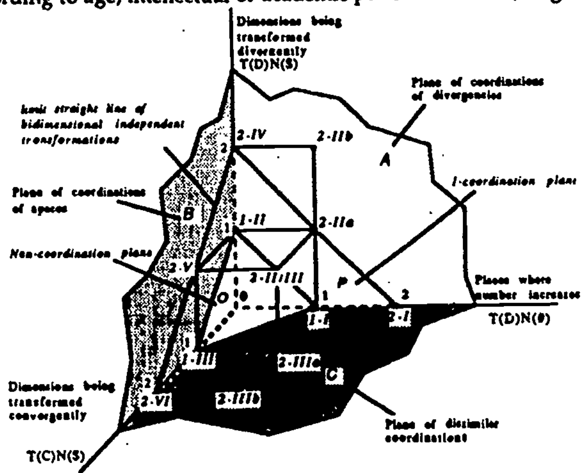


Figure 1

Method

Procedure

The study considered two parts. The goal of the first part was to explore with a quantitative approach the different cases of the classification, by means of questionnaires. In the second part we studied in depth, by means of individual interviews (approx. 50 minutes), four of the cases of our classification (cases 2-V, 2-II/III, 2-IIIa and 2-IIIb; see Figure 1), judged to be most interesting ones after the analyses of the results obtained in the first part. In this article, only some of the results of the first part will be analyzed.

Two variables related to performance were studied: Intellectual performance measured by the Raven test and academic performance (studied only in the second part) measured

by the marks obtained both in mathematics and in French (official language at school). In the first part the Raven test and the questionnaires were administrated during school time in the classroom. The individual interviews took place six months later and were videotaped.

Subjects

In the first part participated 172 students of two schools of the city of Fribourg, aged 8, 10, 12 and 14 years. Boys and girls were equally represented. Nearly 20% of this population participated in the second part (N=32; 8 subjects by age group, half high and half low intellectual-academic performers in each group).

Material

The cases of our classification were operationalized by means of geometrical figures that were transformed according to a certain rule (e.g., growing or diminishing in size, or number). The subjects were asked to explain what would happen with the "path around the figure" (perimeter) and the "amount of painting needed to color" the figure, as the transformation continues further and further "without stopping."

Results

The answers to what happen with the perimeter and the area were grouped in 4 categories (*Plus* or more, remains *Equal*, *Less* or diminishes, and *Missing* for answers difficult to classify or no answers). The goal was to analyze the general tendencies of the distributions of the answers by age and performance. At this stage we were interested in the distributions of these categories rather than in the correctness of the answer itself. The reasons that led the subjects to give the answers were to be studied in the second part of the research. According to age, results of the first part of the research reveal differences between the distributions of the answer categories for 5 cases of the classification (cases 1-II, 2-V, 2-II/III, 2-IIIa and 2-IIIb. See Figure 1). For certain cases the "correct" answer is more frequent as the age increases. The following is an example related to the perimeter:

Case 2-IIIb:

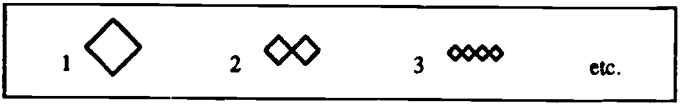


Figure 2 shows the distributions (percentage) of the four categories by age. The bold line represents the "correct" answer. A χ^2 test between the categories *Less*, *Equal* and *Plus* show that the differences are highly significant $\chi^2(6, N=159) = 34.92, p<.0001$.

For other cases, especially those asking about the surface and presenting a convergent iteration, the frequency of "correct" answers decreases with age, as it can be seen in the following example:

Case 2-II/III:

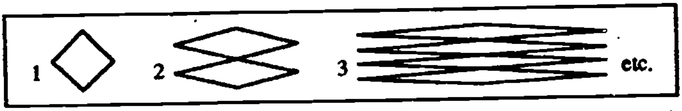


Figure 3 shows the distributions (percentage) of the four categories by age. The term "Conv. (F)" (for False Convergence) at the right of the graphic, indicates that 18% of the 14 years-old subjects who responded *Less* said that there will be no area, or that the figure becomes a line. A χ^2 test between the categories *Less*, *Equal* and *Plus* show that the differences are significant $\chi^2(6, N=155) = 14.76, p<.02$.

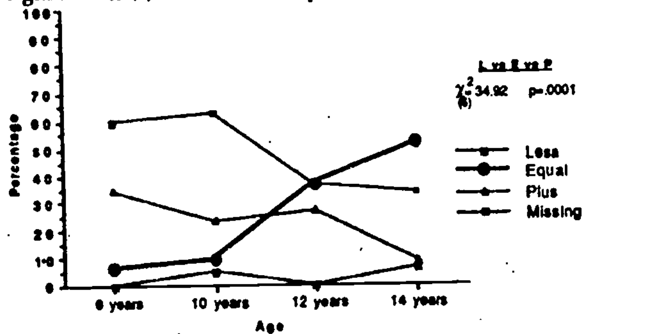


Figure 2: Percentage of subjects by answer and by age group (case 2-IIIb; perimeter). N=172.

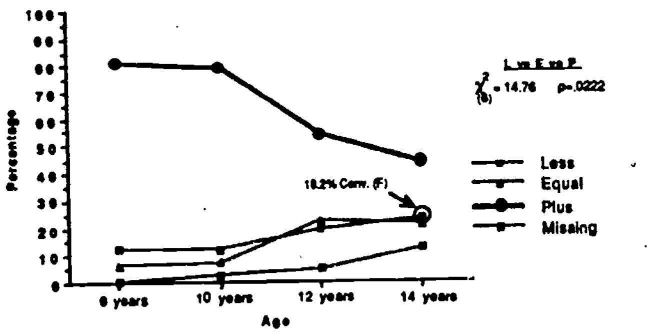
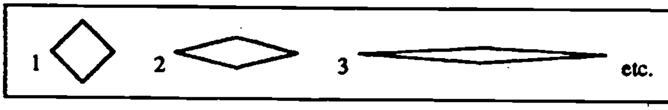


Figure 3: Percentage of subjects by answer and by age group (case 2-II/III; surface). N=172.

For other cases, although the "correct" answer is relatively stable according to age, differences were found concerning the frequency of "wrong" category. For example, the case 2-V (see next page), in which the area of the figure remains the same *ad infinitum*, the "wrong" answers *Less* and *Plus* are almost equally distributed at 8 years (ratio

Less/Plus nearly 1/1) but the latter decreases with age and the former increases with age (ratio *Less/Plus* nearly 16/1, see Figure 4).

Case 2-V:



Moreover, for those 14 year-old subjects who responded *Less*, 35.5% of them said that the figure will become a line without surface (Conv. (F)). A χ^2 test between the categories *Less, Equal* and *Plus* show that the differences are highly significant $\chi^2(6, N=157) = 25.47$, $p<.0003$.

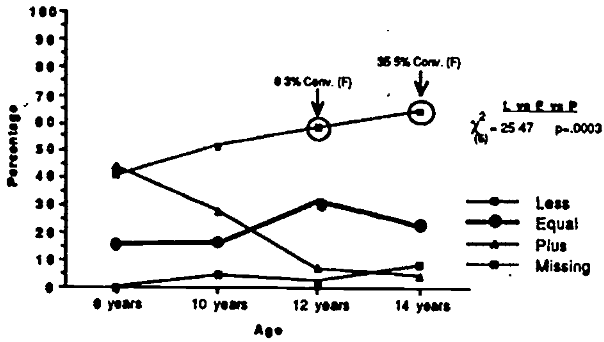


Figure 4: Percentage of subjects by answer and by age group (case 2-V; surface). N=172.

As far as intellectual performance[1] is concerned, differences between low and high performers were found only at age 10 and 12. At age 10 the "wrong" answer *Less* to the question about the surface of the case 2-V (Figure 4), was associated with high performance whereas the "wrong" answer *Plus* was associated with low intellectual performance ($\chi^2(1, N=33) = 3.88$, $p<.049$). For the same question at age 12, high performers answered more frequently the "correct" answer *Equal* than the low performers ($\chi^2(1, N=40) = 7.93$, $p<.0049$). A similar tendency concerning the "correct" answer was found at age 12 for the question about the perimeter of the case 2-IIIb (Figure 2) ($\chi^2(2, N=41) = 10.53$, $p<.0052$). Finally, no gender differences were found.

Discussion

The classification of the situations and the material we have used revealed that for certain combinations of features related to endless iterations interesting differences between age groups could be found. In those cases in which the complexity increases because of the *coordination* of different *type* and *nature* of the transformations (such as

[1] For these analyses the Raven scores were dichotomized by age group into high and low performers.

the case 2-IIIb on the plane C of Figure 1), age seems to help in dealing with these situations. Nevertheless, older subjects seem to loose their developmental advantage as soon as the effect of convergence appears in situations concerning surfaces in which the level of operability decreases. Thus, in a case like 2-II/III (Figure 3) in which measure and estimation become difficult, as age increases, the fact that the height of each figure decreases progressively is much more important than the increasing number of figures and the increasing width of the figures together. The growing effect of convergence as age increases is comparable to a "black hole" that attracts (absorbing and eliminating) any other effect that might affect the situation. Even in more simple cases like 2-V (on plane B of Figure 1), where a compensation of height and width is possible, convergence seems to play a much more important role as age increases. For certain of these cases, at age 10 and 12, maybe when a certain intuition of convergence begins to develop, intellectual performance shows similar tendencies as age does. Gender doesn't seem to play any role.

These results suggest that depending on the features of the attributes that are iterated to conceive infinity (convergence, divergence, cardinality, space), different cognitive processes may take place. The relation between cognitive development and the role of convergent iterations in the situations shown could be interpreted in the perspective of the concept of "epistemological difficulty" coined by the philosopher G. Bachelard, which says that it is in the act of knowing itself that appear, like a functional necessity, troubles and backwardness, and that these difficulties could be studied throughout history and educational practice (Bachelard, 1938). Anyhow, disciplines that are supposed to study cognition, knowledge and education should dedicate more efforts to clarify the question proposed by the mathematician D. Hilbert more than half a century ago.

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EPISTEMOLOGY AND COGNITION OF THE INTEGRAL:

ELEMENTS OF REPRODUCTIVENESS IN TEACHING-LEARNING SITUATIONS.

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We discuss in this study some results found in the teaching experience, of approximately one and a half year, with teachers and students, on the study of thinking of the advanced mathematics. In particular we refer to the concept of the integral within the frame of epistemological analysis, orientating the discussion toward teaching situations, where we explore links between construction patterns (accumulation and accumulated value) and visual arguments (slanted behavior).

SOME CONSIDERATIONS

Since the studies of Piaget, precisely on the notion of reflexive abstraction, an acceptance exists that characterizes the mathematical knowledge in terms of process and objects (Dreyfuss,1990).

Within this frame the mathematical knowledge is considered as a taking of processes or of objects related to the mathematical notions. That is, Mathematics deal, for example, with numbers, variables, and functions that can be considered as objects,related and connected, and are components of structures. While the processes are constituted by operations of these objects, they aretransformed by them. The structures can or cannot be preserved under these transformations.

Therefore, the complexity of the mathematical knowledge consists of two aspects: problem situation and the mathematical conception by the subject. Hence, taking the role of process or objects for many mathematical notions depends, on them.

In the learning of concepts, in this terms, we consider various phases, from concrete operations in the processes until their mental representation, which are transformed into new objects.

A mathematical concept that has been studied under this perspective of understanding the mathematical knowledge is the function. Its main problem consists in the difficulty of going farther by considering the function a rule of procedure and conceiving it as a mathematical object.

It is precisely in these terms that we want to discuss some pragmatic aspects oriented toward didactic reflections; teaching situations and curriculum considerations.

A DIAGNOSIS

Investigations on the understanding of the students facing the processes of differentiation and integration, report that the students have a marked tendency to reduce the mathematics of these themes to a collection of algebraic algorithms, avoiding graphs and geometric images. This leads the students to emphasize the aspects of the procedure over the aspects of the concept, generating response behaviors to the

problems. The recognition of patterns and the incorporation of meanings does not play an active part (in many occasions completely absent) in the argumentations of the students.

In a sample, on the conceptions and mathematical believes, of teachers of different higher education institutions in our country, we observed one symptom, the "need" and "possibility" in constructing a geometric model to interpret a given analytical situation or vice versa, does not exist in their arguments. This favors that in the teaching (teacher-student) the mathematical objects (for example, patterns with meaning) are seen as a meeting of procedures linked vaguely with symbols.

We describe a study on the concept of the integral, which consisted in identifying some basic elements in the meanings of the integral, when the student and the teacher have thought of Calculus under an ambience of continuous variation. (Cordero, 1991. 1992)

A STUDY

This study was performed in some engineering schools, where some aspects of thinking of the integral under specific situations of comparison and change were analyzed. Their explanations were characterized more from the situations than from the mathematical concept, finding a link with their conceptions and mathematical believes.

From these characteristics we captured arguments of the integral that are not found explicitly in the scholastic mathematical discourse. Such arguments are more in function of the cognitive structures than in function of the mathematical structures.

Later, the arguments were incorporated to a didactic exercise of Calculus - of one and various variables- in themes traditionally discussed in the first semesters of college, observing in one first stage, the impact of this approximation in the teaching- learning process from the discourse generated in the students, as well as studying, through their representations, their ways of processing numeric, parametric, and graphic information when facing the idea of continuous variation.

The manner of operating the mathematical content in this experience describes an unusual discourse in the teaching tradition, where we intend to understand and know the Mathematics in the phenomenon (Cantoral,1990), rather than treat Calculus with "applications" to Physics. In our case, we are dealing with phenomenon of change, that is we are dealing with a mathematical content in a fluid movement system against a mathematical-structured system. In this sense, the notions of prediction, accumulation, and distribution of a fluid were relevant.

Our objective is, then to identify the basic elements that coordinate the precedent notions to understand the concept of the integral. We discuss only three of these elements in this document: area, measure, and movement.

Aspects of the study. We worked three consecutive semesters, on Differential and Integer Calculus (in one and two variables) and ordinary differential equations, where we studied the meaning that students and teachers have given to the mathematical concepts taught.

The task consisted basically in formulating activities that led to knowledge, starting with the explanation of a known mathematical object and used daily in their school activity until dilucidating new situations for them. In the interviews we searched for natural notions and spontaneous representations.

The thematic content of Calculus can be described in the following scheme: relation of dependent and independent variables, variable and its variation, prediction and Taylor Series, accumulation and integration, movement system in its static state and movement system in its dynamic state.

We analyze movement in its most elementary forms, a simple displacement, that is, a change in position in addition to the traced paths. This form of analyzing movement configurates a mathematical structure, subject to situations of comparison, of change and of behavior. Such configuration traces a methodology that allows us to understand the phenomenon of variation through two elementary operations: sum and subtraction. And these, in turn, can be extended to other two elementary operations: the integration and the differentiation.

We name, in this experience, phenomenon of change (FC), the phenomenon of variation that includes a change in the initial quantity under a process of transformation and this other quantity is the last value in the process. The FC can be represented by two different situations: accumulation and accumulated value.

Then, the operations sum and subtraction, differentiation and integration in the context of accumulation and accumulated value situations derived strategies toward the solution of integration problems.In this manner, we found it interesting to discuss the interaction of the strategies used and the particular characteristics of the objects with which we operate.

We characterized the students by their attitudes toward the solution of problems. Then, we designed interviews, considering the most "common" words in their explanations, apparently not dependent on the given problem. The objective of the interview was to analyze the role these words play in the notion of the integral concept.

The interview was normed by presenting the same question in three different contexts. They were asked to calculate the area of a "rare" surface in three different situations: without reference axis, the AB segment as reference axis, and the cartesian coordinates X and Y as reference axis. From the "common words" they were asked what they understand under: length, area, and volume, and also under variable, variation, and accumulation.

Some of the typical verbal responses were:

"a variable is ....

a.) a representation of that what varies."

b.) a point in a plane that will have movement."

c.) used to know the change in a determined quantity when a variable varies."

"variation is ....

a.) the movement of something."

b.) that movement that allows us to reach a variable."

c.) the pattern of behavior that the variable follows."

"accumulation is...

a.) as if certain things stopped moving and gathered in a single place."

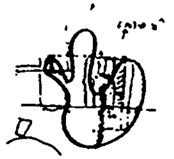
b.) the process of retention of a specific being."

c.) a quantity that varies and the new quantity obtained is added to the original one."

In the explanations of the word area, we found two types of answers:

"area is ...
a.) the product of two dimensions, length by width."
b.) a surface, that is a doubla integral, but it is also a pattern of basis by height."

When the students were asked to calculate the area of the region in the first box the strategy was similar in all students.



BOX 1

BOX 2

"I'll insert a known figure, such as a rectangle, the biggest possible that fits in the region. Then in the remaining region I'll insert small rectangles or triangles, depending on the form of the region to be covered, when I finish I will add all the areas of the known figures...".

In the calculation of the area of the second box, where the AB segment is considered as reference axis, the following strategies were found:

"If I can move the AB segment, I would move it up until it cuts the region, provoking a symmetry, then I would just calculate the area of one region, which I would do as in the previous problem..."

"If the AB segment is an axis such as X, then I can consider distances from the segment to the region and if I could only consider pieces of curves that delimit the region, then I could imagine a function for each piece of curve by inserting rectangles formed by the values of the function and the width of the function (the piece of the function belonging to the curve)..."

"If the AB segment is an axis such as X, I would also like to consider the Y axis..."

In the last question, where the Cartesian Axes were considered as reference axis, they recognized the use of Calculus but the strategies varied as follows:

"If we have axes the calculation can be done with the formula $\int f(x)dx$ and this would mean the same I have been explaining, because integrating is like adding all the f(x)dx, it is adding this dx as a constant and this f(x) as the variation of all the heights of our rectangles, then I imagine the area as the sum of all these rectangles from a to x..."

"If we have axes, I can now use Calculus, then the area A(x) can vary little by little so that the rectangle f(x)dx looks like what the area really is..."

DISCUSSION.

A static geometric region was considered in the analysis, that a priori does not refer to movement, therefore it does not have to refer to Calculus, but the strategies were of geometric nature reflecting important aspects in the conception ofthe integral:

- The area (length by width or basis by height) is considered as a pattern of measure adapted to the three situations. None yields an explanation of why this pattern is expressed by the product "basis by height".
- The strategy in the calculation of the area, without reference axis determines a situation of status and not of process, that is, the pattern of measure resembles the complete region in counter part to describing an approximation that reaches the region. Then the strategy is more focused on actions of completing than of reaching.
- In the calculation of the area, with one reference axis the functional relation of the pattern of measure is recognized, that is, the relation of basis and height as variables, without focusing attention in the variation. In this sense the situation of status is mantained.
- Considering the two axes as reference the functional relation of the pattern of measure can vary, but the status situation is maintained, whereas the taking of the differential element "f(x)dx" is of the same form and nature as the region to be calculated, besides "dx" is considered as something that does not change; therefore, the expression $\int f(x)dx$ is more related, in this context, to the action of completing than to the action of reaching.

Then, as a conclusion of this analysis we observe that the integral perceives a status situation that, on one hand, admits one representation without recognizing movement as is the case of adding the patterns of measure that recognize the region and, on the other, it admits a representation with movement as occurred with one student that had to imagine the area as a fluid (a river running under the curve) and calculate its accumulation.

Thus, considering the area under a curve, as a geometric model of the integral in an ambience of continuous variation demands moving the static. This category could explain why some students, independent on using the integral in their school activities, cannot explain that the expression $\int f(x)dx = F(b) - F(a)$ calculates the area under the curve $y = f(x)$. The other category, the relation of status maintained by the integral explains in the meaning plane the role that it plays on the phenomenon of continuous variation which expresses a change without describing its evolution. (Cordero, 1992).

IMPLICATIONS IN TEACHING

Considering the cognitive processes, facing the concept of the integral, the most relevant points that we have found in our study are:

- From the epistemological analysis, we could find a pattern of construction of the theory of integration configurated by the expression $\int f(x)dx = F(b) - F(a)$, where the difference $F(x+dx)' - F(x)$ and the conditions of a differential function F' play a significant role in the discussion. While in the frame of the "Riemann" arguments the fundamental aspect are the attributes of the continuity of the function f on the interval [a,b].
- A meaning to the integral was associated through the notion of accumulation. This meaning acquires relevance when two aspects of the phenomenon of continuous variation are shaded, through the sum and the subtraction, named accumulated value and accumulation.

$$\varphi(x) = \varphi(a) + \int \varphi'(x)dx$$
$$\text{and} \quad \varphi(b) - \varphi(a) = \int \varphi'(x)dx$$

Facing typical problems of the integral Calculus, which consist in calculating a last quantity in a process of increment (decrement) through two data: the initial

quantity Po and the ratio of change P'(t). We found that the answers are associated to the conceptions of the integral: accumulation and accumulated value.
- We managed to capture functional explications of the subjects, whose main characteristic consist in relating the meanings and objects in a local system, in contrast to an axiomatic structure.

On the other hand, with the "operations" of the graphs and from the analytical forms we managed to conceive the function not as a formal expression through which specific numbers are operated, but as an instruction that organizes behavior. The main part consists in varying the parameters associated to the functions and not the variables, this leads to estimate and compare an operation in two representations of the function: analytical and graphical (approximations in this sense, see for example [Confrey,1991] and [Farfan, 1991]).

In this sense, we find questions in the integration about behavior in two directions: integrating function and integral function.

$$\int f(x) = F(x) \qquad \alpha\int f(x),\ \int \alpha f(x),\ \int f(x) \text{ and } \alpha\ F(x), F(\alpha x).$$

Then the main part is not in the solution methods but in knowing patterns of primitives and derivatives.

These points have resulted to be important aspects providing new perspectives oriented to the didactic and reproductiveness in the educational system. For example, we identified different representations facing the knowledge of the integral and different arguments coming from the study on the subjects. The arguments make the notions of the concepts clear, that is, if the argument is associated to an idea of approximation the notion derived is in the function and its properties, as well as the limit. While if the argument is associate to the aim of behavioral, the notion is associate to the variation of the parameters that form the function, more than the variables. And then, when the arguments associates to the aims of comparission and prediction, the notions consist on looking at quantities that change, through a sum or subtraction.

On the other hand, there are different forms of representing the mathematical knowledge, we have here considered only two: graphical and symbolical. Looking at both representations through what we have called slanted behavior could suggest the link between the representations, even a representation not included in this document, the numeric representation, could be treated in the same manner.

Regarding the teaching situations and based on the explorations of thinking toward mathematical problems there are different approximations (Cantoral,1992); we, here. point out new arguments that could be translated into didactic sequences. The new arguments are normed by the notions of slanted behavior, a curricular consideration that favors this active role could be the use of calculators and software to make graphs (approximations in this sense can be found in [Tall, 1990]). The elements of reproductiveness can be found in these considerations, because an epistemology is made (Artigue,1992) that will serve a as conductive string to their reasoning and legitimate the reflexive work that will accept or refute in the study of some mathematical questions.

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Algebraic Thinking

NEGATIVE SOLUTIONS IN THE CONTEXT OF ALGEBRAIC WORD PROBLEMS

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ABSTRACT

Resistance to accepting negative solutions to equations and problems is found both in the history of the development of mathematics and in students who are beginning the study of symbolic algebra. In the latter case, avoidance of the negative solution is usually accompanied in practice by a separation of the operational manipulation of positive or negative numbers and the solution of algebraic equations. This article reports the results obtained by a clinical study carried out with secondary school students. The results reveal some of the conditions which propitiate the acceptance of negative solutions in algebraic word problems. Analysis in this study is guided by an earlier analysis concerning the acceptance and rejection of this type of solution in the history of mathematics.

INTRODUCTION

Studies such as those carried out by Bell (1982), Carraher (1990), Gallardo/Rojano (1990) and Resnik (1989), among others, indicate the difficulties students confront when they are faced with the conceptualization and operational manipulation of positive or negative numbers. The results of some of these studies show a separation of the development of operativity of whole numbers and their use in certain contexts. Thus, for example, in the case of the solution of algebraic equations, there are subjects that show resistance to the acceptance of a negative solution despite possessing fluid operativity with the numbers (see Gallardo, 1987). Teaching is one of the multiple causes to which this separation can be attributed, since it is quite common to find that school mathematics curricula treat the theme of whole numbers fundamentally as an extension of natural numbers, emphasizing the operative and paying little attention to the role these numbers play in the extension of the numerical domains of coefficients and solutions to algebraic equations.

In the search for other fundamental causes of the difficulties encountered in the conceptualization and operational development of positive and negative numbers, historical-epistemological, philosophical and psychological theoretical analyses have all been brought into play. Such is the case of the research carried out by Glaeser (1981), Schubring (1988) and Dreyfus/Thompson (1988). In the project entitled The status of negative numbers in the solution of equations[1], the research starts from a historical analysis of the acceptance and rejection

[1] Research project currently underway. Departamento de Matemática Educativa, CINVESTAV. Mexico.

of negative solutions in the context of the solution of algebraic equations. One of the most important conclusions of this analysis is that the acceptance of the first negative solutions presupposes a certain level of development of the language of symbolic algebra (advanced syncopation) as well as full operativity, advanced levels of interpretation of positive and negative numbers and the development of ad hoc methods in the case of negative solution (Gallardo, Rojano, Carrión, 1993). This type of conclusion has served as a basis for the formulation of hypotheses at an ontological level concerning the conditions in which it is feasible to go from primitive stages of conceptualization to stages of consolidation and formalization of the notion of relative number in the above mentioned context. This article reports on the results of the second part of the project in which some of these conjectures are put to the test by means of a clinical study with students 12 and 13 years of age, who were being introduced to the syntactic manipulation of algebra and the solution of algebraic word problems at the moment the study was carried out.

THE CLINICAL STUDY. Solution of word problems and negative solutions.

Data was gathered in two ways: a group of 25 students at 2• grade of secondary school in Mexico City were asked to answer questionnaires; individual clinical interviews were recorded by video and analyzed. The results of the questionnaire were used to select a group of students for clinical observation. The blocks of items presented in the interview dealt with the following themes:

1) Operativity in the domain of whole numbers.

2) Translation of situations expressed in words to symbolic language.

3) Use of pre-symbolic and symbolic language in the context of equations.

4) Solution of word problems.

This article gives the results of the clinical interview referring to the block "solution of problems". The dimensions of analysis in this part of the interview were:
method or strategy/problem solution/interpretation of the solution.

The methods used by the students to solve word problems were as follows.

PROBLEMS OF AGES. Luis is 22 years old and his father 40 years old, how many years must pass for his father to be twice the age of his son?

Method of Two. (Used by four students). The student finds the problem impossible because "the 2 is always there". This refers to the difference of 2 in the units of the data given in the problem as the ages of father and son advance. Example: One student established two lists of numbers, increasing the ages starting with the ages given in the problem: 22 years, 40 years. He writes: 23, 41; 24, 42; 25, 43; and so on. He notes that the difference in the figures of the units of each pair of numbers is always 2. He concludes that the problem does not have a solution.

Method of Duplication. (Used by three students). The student arrives at the correct solution, 18 and 36, the ages of son and father respectively, but he duplicates the ages and thinks that 36 and 72 is the true solution. There were also cases where the student thought that 36 x 2 = 72 and 72 x 2 = 144, is also a solution to the problem.

Method of the Difference. (Used by four students). The student finds the difference in ages, that is 40 - 22 = 18. He deduces from this that the son is 18 years old and consequently the father is 36.

Method of Altering the Difference. (Used by two students). The difference between the ages (18) is divided in half, 9, and this value is then added to the son's age, 22. The answer to the problem thus given is 31.

Ascending/Descending Method. (Used by four students). The student increases the ages of the father and the son and finds that the problem cannot be solved. He then decide to decrease the ages and arrives at the correct solution.

Algebraic Method. (Used by two students). Spontaneous formulation of the equation which solves the problem.

PROBLEM OF PURCHASING GOODS. A salesman has bought 15 pieces of cloth of two types and pays 160 coins. If one of the types costs 11 coins the piece and the other 13 coins the piece, how many pieces did he buy of each price?

Method of one equation. (Used by 15 students). The student looks for multiples of 11 and 13 that add up to 160. (This is equivalent to solving the equation 11x + 13y = 160. The existence of x + y = 15 is ignored). When the student does not find the multiples needed to solve the problem, that is 11 x 11 + 13 x 3 = 160, he uses an additional interpretation to explain his results, for example,

Student 1. He writes 66 + 91 = 157, and says, "he bought 6 pieces costing 11 coins and he had 3 coins left over".

Student 2. He writes 154 + 0 = 154 and explains, "he bought 14 pieces costing 11 coins each and none costing 13 coins".

Student 3. He writes 154 + 13 = 167 and says "he owed 7 coins".

Additive Method. (used by one student). The problem of the purchase of goods is modified such that the figures are smaller in order to facilitate solution. The equations which model the problem in this case are: x + y = 3; 2x + 3y = 40. The student assumes that each one of the pieces of cloth has a price different from that established in the statement of the problem in order to adjust the total price. He writes 1 x 2 + 1 x 3 = 5, thus, 40 - 5 = 35. He then says "the salesman bought 3 pieces: 1 costing 2 coins, another costing 3 coins and a third costing 35 coins".

Sharing out Method. This is also found in the modified version (x + y = 3; 2x + 3y = 40).A

student divides the total price, 40, by two. The result of the division, 20, is used with the other data of the problem 2, 3, and he formulates the sums: 18 + 2 = 20; 17 + 3 = 20. His answer is "he bought 18 pieces worth 2 coins each and 17 pieces worth 3 coins each".

It is important to point out that, contrary to what might be expected, the modified version of the statement (with small numbers) renders the problem impossible for many students. The conflict is accentuated since the solution is sought in the positive domain and the lack of adjustment between the data of the problem is more notorious than in the previous version ($x + y = 15$; $11x + 13y = 160$) where the magnitude of the numbers tends to hide the conflict. This obstacle disappears when it is suggested to the student that he use algebra to solve the problem.

Algebraic Method. (Used by two students). Spontaneous formulation of a system of equations to solve the problem.

Let us now look at the case of a student who, by using the process of substitution of the solution in a system of equations, managed to solve the problem which at first he had thought impossible. The student formulates the equations $11x + 13y = 160$; $x + y = 15$. He obtains the solution $x = 17.5$. The following dialogue then ensued:

Student: Totally impossible.

Interviewer: And now how are you going to find y?

Student: It can't be done, totally impossible.

Interviewer: Let's try anyway. (Student finds $y = 2.5$. Spontaneously he substitutes the values in the equations).

Student: It worked!

Interviewer: What happened then?

Student: Instead of buying, he gave 2 and a half pieces of cloth to the person he was going to sell them to, and bought 17.5 of the other. That is, the buyer gave the seller 2 and a half pieces of cloth and the seller gave him 17.5 pieces. It's like barter.

Interviewer: Why did you say it was impossible before?

Student: Because it's impossible with positive numbers.

Below there are two tables which summarize the dimensions of the analysis: method or strategy/problem solution/interpretation of the solution, for the two problems presented in this article.

PROBLEMS OF AGES

Method or Strategy	Solution	Interpretation of the solution
Method of Two	Does not exist	Impossible
Method of Duplication	Positive (one of various)	Occurs various times in the lives of father and son: 36, 72 : 72, 146.
Method of Difference	Positive Negative	Impossible 18, 36. 4 years ago. -4 years
Method Altering the Difference	Positive	4 years + 4 years
Method of Half	Positive	Luis is 31 years old
Ascending/Descending Method	Positive Negative	36, 18 4 years before. -4 years
Algebraic Method	Negative	$x = -4$

PROBLEMS OF PURCHASING GOODS

Method or Strategy	Solution	Interpretation of the solution
Method of one equation	Positive Debt Surplus	Impossible
Additive and Sharing Out Method	Positive	Change of problem data so as to adjust quantities involved
Algebraic Method	Negative	Interpretation of barter of goods.

CONCLUSIONS OF THE STUDY

From the analysis of the different problems exhibited in the study[2] the following was concluded:

1.- It is possible to solve the problem without expressing the solution in negative terms. The problem of ages (method of duplication, method of difference, ascending/descending method).

2.- The creation of specific methods from problem to problem occurs. Problem of ages (6 methods). Problem of purchasing goods (3 methods).

3.- The choice of the appropriate method requires the acceptance of a negative solution which is interpreted in the context of the problem. Problem of ages (method of differences, ascending/descending method, algebraic method). Problem of purchasing goods (algebraic method).

4.- When faced with problems with negative solutions the student turns to changes or adjustments in the data of the problem statement as well as the construction of sources of meaning which allow him to give plausible interpretations to the solution obtained. Problem of purchasing goods (method of one equation, additive method, sharing out method).

5.- A problem which can appear impossible to solve with arithmetical methods, is thought of as possible using algebra, once the negative solution is validated by being substituted in the corresponding equation or equations. Problem of ages (algebraic method). Problem of purchasing goods (algebraic method).

FINAL DISCUSSION

As we pointed out in the introduction to this article, historical-epistemological analysis carried out with respect to negative numbers in the solution of algebraic equations has guided this study at an ontological level and has allowed us to establish some of the conditions which propitiate the acceptance of the negative solution of word problems by secondary school students. We can thus affirm, at an ontological level, that for a student to accept a negative

[2] The study analyzed 10 problems of which 2 were selected for the purposes of this article.

solution, the following are necessary: the use of ad hoc methods, the construction of sources of meaning appropriate to the context of the problem, and, fluid operativity with positive or negative numbers. In the case of some problems of a commercial type, the use of algebraic language becomes indispensable for the possible arrival at the negative solution.

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THE NATURE OF UNDERSTANDING OF MATHEMATICAL MODELLING BY BEGINNING ALGEBRA STUDENTS ENGAGED IN A TECHNOLOGY-INTENSIVE CONCEPTUALLY BASED ALGEBRA COURSE

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This study examined the effects of a technology-intensive conceptually based algebra course (Computer-Intensive Algebra) on beginning algebra students' understandings of mathematical modelling ideas. The curriculum focussed on the use of functions to study real world situations, and students used computing technology (function graphers, table generators, curve fitters, and symbolic manipulators) to produce graphs and tables, to generate curves of best fit, and to manipulate symbolic expressions of function rules. Written exam questions, classroom and computer-lab observations, and task-based interviews provided data which revealed the nature of student understandings as they began to encounter mathematical modelling situations.

Mathematics classrooms are quickly embracing technology. Mathematics curricula that take full advantage of computing technology are rare, however, and there has been minimal opportunity to study how students learn mathematical concepts in such computer-intensive settings. In the few existing studies, there is consistent evidence that when mathematics curricula (calculus and algebra) focus student attention on concepts and applications and allocate the execution of routine procedures to computers and calculators, that students seem to learn the concepts and applications well and that their ability with routine skills does not seem to suffer (Heid, 1988; Heid, Sheets, Matras, and Menasian, 1988; Judson, 1989). There have, so far, been few analyses of the nature of student understanding of mathematical modelling ideas in such curricula (Zbiek, 1992). This study explores the beginning understandings of mathematical modelling by students who have experienced such curricula.

This study was designed to examine the understanding of mathematical modelling which beginning algebra students can acquire when their initial experience with algebra is a conceptually based course that centers attention on

families of functions, representations, and exploration of real world situations through related function rules.

METHOD

Sample

Two classes began the study of *Computer-Intensive Algebra* in January of their eighth grade year and continued through March of their ninth grade year. *Computer-Intensive Algebra* is a technology-intensive and conceptually based introduction to algebra which assumes student access to computing tools and which focuses on the development of the concepts of families of functions and the application of those families to real world situations. The five male students and three female students on whom the study focussed were chosen as a random and representative (with respect to course grade) sample from students in those two classes.

Data collection

The understandings of mathematical modelling attained by these eight students were examined through three data sources: student performance on selected items appearing on course examinations; student work on a mathematical modelling activity that included the data collection plan, the collection of data, the construction of suitable rules, and the use of those rules in reasoning about the situation; and hour-long task-based interviews that occurred at the end of the course (March of the ninth-grade year).

First, we analyzed student performance on end-of-chapter examinations. The examinations included items for which students had access to computing tools such as calculators, graphers, curve fitters, table generators, and aymbolic manipulators. Test items throughout the course were designed to assess how students evaluate given mathematical models, their use of various representations (symbolic, numerical, graphical, contextual) to explore a model, and the understandings students had of the implications of given or derived models.

This study mapped student progress throughout the course on the aforementioned aspects of mathematical modelling.

Second, we gathered data (transcripts of whole-class discussions and transcripts of student pairs working at computers) on how the *Computer-Intensive Algebra* students worked on several long-term mathematical modelling projects. The audiotapes recorded whole-class discussions and particular comments could be attributed to particular students through the supplementary field notes. The field notes of whole class discussions also recorded what was being written on the board and what appeared on the computer screen. In addition, several of the targeted students were observed in their collaborative work at the computer. One of the investigators constructed field notes recording the interaction between pairs of students, and these field notes were augmented by audiotapes of the discussion occurring among the students. The results for both types of data were detailed descriptions of teacher-student and student-student interactions in the course of work on mathematical modelling.

Third, the investigators conducted, transcribed, and analyzed hour-long task-based interviews with the targeted students. Interview tasks were designed to assess aspects of student understanding of the mathematical modelling process. Students had access to a variety of tools throughout the interview: computer with function graphers, curve fitters, table generators, and symbolic manipulators; paper and pencil; and calculator. Once again, audiotapes supplemented with interviewer notes resulted in the creation of verbatim transcripts of the interviews including detailed descriptions and screen dumps of the students' work with the computer.

Analysis of Data

We examined student responses to test questions and task-based interview transcripts for evidence of the nature of student understanding in each of the three following areas:

How students evaluated given mathematical models. Student ability to interpret a model. Role of the characteristics of families of functions in students' evaluating and reformulating models.

The use students make of various representations (symbolic, numerical, graphical, contextual) to explore a mathematical model. Use of the interaction among representations.

The understandings students had of the implications or limitations of given or derived models. The use of the mathematical model to reason about a situation. The confidence students have in particular models and their implications.

We examined class transcripts for explanations of the nature of student understanding in each of the targeted areas.

RESULTS

Written Examinations

These beginning algebra students were able to reason between and among representations of function rules. In particular, they were very able to answer questions about real world situations that fell into several major categories, including the following:

Given the value of an input variable they could find the value of an output variable (and given the value of an output variable they could find the value of an input variable), no matter which representation was given and even in cases when they had not previously encountered the form of the function rule.

Given a mathematical model for a situation, students could find the maximum or minimum value for the output variable.

Given a situation and a mathematical model, students could readily list factors that may have influenced the situation.

Some things remained difficult for the students even after they had experienced most of the *Computer-Intensive Algebra* curriculum. Some of the students still had difficulty recognizing and resolving contradictions among

representations; some of them had difficulty reasoning symbolically about the input/output relationship.

Classroom and computer-labwork observations

Classroom and computer-lab transcripts revealed the nature of students' insights and understandings about mathematical modelling as they progressed through the course. One set of transcripts provided information on students' understanding of mathematical modelling at the end of five months of work with the *Computer-Intensive Algebra* curriculum. This project represented the first modelling project that the students were asked to complete in its entirety. At that time, students were assigned a project that required them to determine the best price to charge for t-shirts that were to be sold by the Student Government for an upcoming "Powderpuff Football Game." Students gathered data so that they could generate a demand curve and used information from local stores to generate a cost function. They then used a computer curve-fitter to generate function rules and used the generated rules to make conclusions about the situation. Analysis of classroom and computer lab observations generated several interesting conjectures about the beginning of student understanding about mathematical modelling. Some of those conjectures are listed below:

- After students have spent a few months considering relationships between mathematical models and the real world, they can readily generate long lists of relevant independent variables. Their mathematical understanding lags behind this real world perception (see Zbiek, 1992) with the natural curricular approach concentrating initially on functions of one variable. Some teachers handle this discrepancy by suggesting that all of the additional variables are held constant but seldom referring to this restriction once the function rule has been established.
- After students have spent a few months considering relationships between mathematical models and the real world, they seem immediately to assess how well the function rule describes the situation. The nature of that assessment varies from student to student, with some students exhibiting a tendency to focus on trends in data and others having a tendency to focus on particular input-output pairs for the function rule. These

tendencies may also characterize the arguments particular students offer to support conclusions they draw about the situation.

When students begin to work on mathematical modelling problems in computer-intensive settings, they encounter and work on solutions for a number of different types of "problems."

Students differ in their approaches to generating function rules, some relying on goodness-of-fit measures and others relying on an initial commitment to a particular shape for the function's graph.

As students analyze problems that may involve several function rules, they sometimes give several meanings to the variablea involved.

As students work through the parts of the mathematical modelling process, they encounter and solve problems that draw significantly on their understanding of arithmetic concepts and procedures (e.g., the meaning of percent; closure of integers under addition, subtraction, and multiplication; and rounding).

Early in their mathematical modelling experiences students are able to reason about the feasibility of different one-variable models based on their graphical representations. For example, at this time students questioned whether a demand function had to be linear and whether a profit function necessarily had to have a break-even point.

When students encounter unusual output on a computer screen, they may make conjectures about its meaning but they do not necessarily check the validity of their conjectures with other available information. This tendency to make and use untested conjectures can also be seen in how students check computer input (e.g., several students concluded that a function rule was correctly input when they saw that it produced a table of values).

Task-based interviews

One focus of the analysis of the task-based interviews was to study how students made sense of the mathematical models they had created, of their answers to questions that were posed, and of contradictions and consistencies in their observations. Here are a few of our observations about sense-making by the targetted students.

- When they are given a mathematical model that is to describe a relationship in a situation, students vary in the ways they try to make

sense of the model. Some students immediately try to make sense of the rule by trying to generate the rule anew from the situation; some try to make sense of a function rule by looking at its component parts; some are reluctant partners in the sense-making business, answering questions about how a rule makes sense but never volunteering such an observation; and some make little connection between the function rule and the situation, treating the situation almost tangentially.

The techniques that students use to make sense of their answers include using favorite computing tools and representations to reproduce the answer, connecting their answers to other information in a somewhat local way (checking their results only against the most recently generated other results), and checking to see whether their answers fit a more global picture of the situation (e.g., checking answers against a perception that the function should be increasing or decreasing).

DISCUSSION

Eighth and ninth grade algebra students can begin to learn the process of mathematical modelling within the context of a curriculum that capitalizes on computing technology. In the process of this learning they draw on either local or global understandings of functions. For them, the process of learning about mathematical modelling is often the development of a balancing act between mathematical tools and representations and threads of reality.

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A THEORY OF THE PRODUCTION OF MATHEMATICAL SIGN SYSTEMS -THE CASE OF ALGEBRAIC REPRESENTATION OF BASIC GEOMETRICAL VARIATION NOTIONS-

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The object of this work is to present the results of a research concerning the appropriation of the notion of the straight line's slope, using the theoretical basis of the general explanatory semiotic approach [1] of the learning and teaching of algebraic processes. This theoretic perspective allows us, to advance the research hypotheses about of the cartesian representation of basic variation notions.

1. Introduction: A theory of the production of Mathematical Sign Systems (MSS).

This theory emerges from the necessity of accounting for the discourse that occurs when adult-expert and child interact conjointly in a mathematical problem solving-situation, side by side with the strict meaning of the mathematical text in question [2]. Or, we need the incorporation of some 'natural logic' that takes into in account the relationship between the meaning of the mathematical text and the meanings that are derived from the presuppositions, the immediate consequences or implications that occurs during tne previously mentioned communicative acts [3a]. In order to accomplish this, we propose consolidating in a theory of the production of MSS, the following notions: MSS [3b], sense [4b], and use of the mathematical signs culturally adequate or socially established [5a]. The last is adquired by means of cultural conventions [5b].

The role of a theory of the production of mathematical sign systems would be to explain the apprehension of school mathematical notions developed under a mathematical sign system and to describe the way to a competent and culturally adequate use of these notions.

Schematically all this will be done through the following steps: a) by the successive appropriation of mathematical sign systems with intrinsic characteristics. These characteristics derived from the mathematical signs and the particular way to operate with them, see fig. 1; b) the allocation, to the MSS, of the specific senses derived from the didactical model used during the school teaching; c) the recognition, ostentation, and the reproduction of the signs and their operation established before in an inferior MSS with respect to the level of the language used. All these mechanisms are possible because of a semiotic challenge for the user which consists of assigning new senses to signs and

operations used before in a culturally adequate way, in an MSS previously introduced through specific teaching models.

A theory with these characteristics complements the semiotic approach of the teaching and learning processes that had been developed previously.

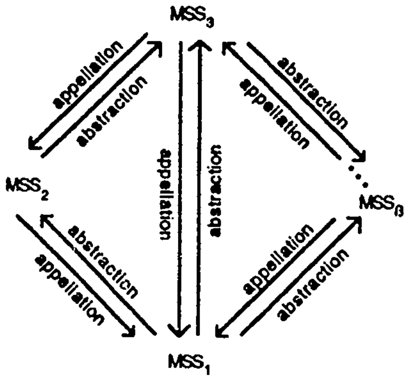


Fig. 1

From the research accomplished [6a, 6b, 6c], all with the same methodology for the experimental work, we have hoped to obtain on one hand the establishment of a series of important phenomena to be considered for the development of any mathematics didactic [4a, 4b], and on the other, a building block for the fundamental constructs of a theory used for the analysis of empirical observations, Local Theoretical Models.

In fact, this work was obtained from the analysis of empirical observations done in a previous work [6a] in the light of the three components of some Local Theoretical Models.

2. The aims and the methods of this empirical study.

This paper gives an account of aspects of a experimental study whose main aim is to underline the requirements necessary for the mastery of Thales' Theorem, the natural obstructions to its learning and the importance that these all bear in the use of a Mathematical Sign System where the rational numbers appearing are symbols whose referents are both the rational numbers appearing in the Arithmetical Sign System [6c] of fractions and those appearing in the Geometric Sign System of the ratios of continuous magnitudes. One can clearly state that, until we have a correct interpretation of the situations presupposed by the Similarity Theorem, this new mathematical Sign System stratum cannot have stable "objects" with which one could operate and establish order relations that would be the usual concrete referents of the operations, and order relations which will appear in Trigonometry.

The analysis centers in the point of a didactical cut represented by the learning of the Thales' Theorem [6a].

In order to observe it, there was established an experimental device where teaching was programmed over a period of two years, in the first and second years of Mexican Secondary School. There was control exerted over the teaching strategies to which the same two groups of 30 children were submitted over a period of two subsequent years.

The methodology used in this research was stratification of the population by means of a diagnostic test with three axes: syntactic skills, mastery of the solutions to equations and a third involving concepts and equation related to the understanding of the geometrical situation of the ramp of a straight staircase where the central and culminating point was the concept of slope of a straight line (the ramp in this case). This diagnosis was applied at the end of the first year of secondary school when the pupils were about 13 years old. The population under survey was divided into three classes which were derived from the grouping of the 27 classes produced from the three axes, each axis being an ordinal measuring scale. Two subjects were chosen for each class to undergo long duration video-taped clinical interviews. In these interviews the first object was to confirm the written diagnosis and observe the attack strategy employed in the solution of simple arithmetic equations.

Subsequently we observed the difficulties which each student experienced in learning the concept of slope of a straight line when the teaching approach was the same as the student would be involved in during the following school year. The clinical interview was, in itself, a didactic situation where the interviewer presented various aspects of the concrete situation, consisting of analysis of drawings of straight staircases, and tried to pin-point those ideas that the student had formulated spontaneously or through teaching strategies employed in the school year that was just ending.

The following year there were further interviews, with the most interesting subjects, in order to observe if the same kind of obstructions were still present or whether they had evolved. For these interviews a teaching strategy based on the previous year was devised. This however allowed the student not only to give a general explanation on the slope's constant value along the stair's ramp, but also to give an interpretation (in arithmetical language) of the facts present in the geometrical situation, which gave him the opportunity to explain statements made in our version of the Thales Theorem-the introduction of the concept of slope of a straight line.

Among the different cases observed, we can find those who can not exceed the difficulties stated by the arithmetical sign system of fractions and so could not assign an adequate social sense to the signs and operations of the Geometrical Sign System of continuous magnitudes, through the didactical model used.

But it is also true that there are cases in which it's possible to appreciate clearly: (A) how the Arithmetical Sign System of fractions constitutes a language stratum indispensable for the sense assignment [4b] to the new signs and operations of the Geometrical Sign System of continuous magnitudes; and (B) why the new notions -in this case we are talking of straight line slope notion and correlated concepts- which emerges in the new MSS which consolidates the ASS of fractions and the GSS of continuous magnitudes will not be stable until they can be formulated and explained through the referents established in a previous language stratum, that is, in the ASS of fractions.

In this paper, there will only by the report of the analysis of one of these cases. Later on some other results will be published. This is part of a bigger Research Project that deals with the Algebraic and Cartesian Representation of Basic Variation Notions.

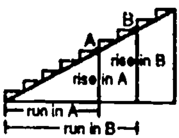
3. The case study: toward the algebraic representation of the basic geometrical variation notions.

This case, one 13 years old subject from the high-medium stratum, according to the diagnostic classification, shows stratification of the language of MSS of the rational numbers toward the conformation of algebraic and cartesian representations of the basic variation notions.

The given geometric problem, which initially any child of the observed population (and constitutes the Didactical Cut) could not solve in any socially established manner, is about the slope of a straight stair ramp:

A (adult-expert): Compare the fraction of rise and run in A with the fraction of rise and run in B. ¿What is the relation between this fractions:

$$\frac{\text{rise in A}}{\text{run in A}} \qquad \frac{\text{rise in B}}{\text{run in B}}$$



¿Are they equal or is one greater than the other?

The whole population answered that the ratio in B was greater than the ratio in A. They gave a "natural" erroneous geometrical reading of this problematic situation.

For structuring the questions in the interaction between the adult-expert and the child, knowledge of the Didactical Model is very important.

The parameters that are used to define the situation consist of the utilization of the Didactical Model of the straight staircase and the structuring of the series of the problems presented to the child.

The progression of the interview was based on the previously detected sense by means of the analysis of the required formal competencies (of the ideal user) and his translation by the Didactical Model in question (both methodological principles required by the Local Theoretical Model for the empirical investigation of the processes).

This progression can be analyzed in terms of a strategic sequence of steps toward the solution of the given problem.

The strategic sequence of steps identified is the following:

1. Comparison of equivalent fractions with specific numbers obtained from the concrete context presented;
2. Comparison of equivalent fractions where small numbers intervene;
3. Comparison of non-equivalent fractions in the Didactical Model presented;
4. Comparison of non-equivalent fractions with small numbers in the numerator and the denominator;
5. Variation of geometrical ratios (continuous magnitudes) in the Didactical Model presented, confirming their equivalence for any pair of points on the ramp.

In the case that we are reporting the subject eventually manifests and adequate cultural use of the equivalence operation of arithmetical fractions (to rectifie the initial natural erroneous geometrical reading of the problematic situation):

> T (girl): These are equivalent. (T writes and equals sign between the fractions 180/360 and 90/180 obtained by measuring the diagram of the staircase).
> A (adult-expert): And why did you tell me that you had a greater slope...?.
> T: I thought... (signaling to the staircase, reflects, looking at the given diagram) No... they have... (she smiles) "I made a mistake"
> A: What mistake did you make?
> T: That the same slope... some lines don't have the same slope... But this anywhere is the same.

A: Or, if I calculate in another point, let's see, in B', how much was it? How much do you calculate here?
T: (Writes) "15*10/ 30*10"
A: What is that? is the slope en B, isn't it?
T: This is equivalent. (T writes) "150/300 = 15/30 = 1/2" "These are equivalent"
A: Or, also in B the slope is the same.T: In all, in all it's the same.

And also the child manifests a socially established use of the notion of the slope of a straight line:

A: ... You told me that the slope was grater because one rises and advances more...
T: I told you that it wasn't the same slope... but... it's the same slope. The slope is the same on any part of the staircase because the staircase is a straight staircase (T makes a hand motion in the air, to draw the line that would pass through all the steps of the given staircase).

...

A: Or, what you told me in the beginning was wrong? You said that if one was bigger or if both were bigger, anyone would know that the fraction was grater. It doesn't always happen.
T: Well, to me... (T takes away the page that shows the completed arithmetic calculations and signals on the diagram of the staircase two distinct points), the one I have here and this one here, is the same slope (T "draws" again, in the air, the straight line in the diagram of the staircase).

Some results.

The study indicates that the proportional variation with discrete magnitudes is a necessary antecedent to recognition of the continuous linear variation represented by a linear equation and requires an interpretation of the straight line's slope in arithmetical theorems, due to the instability that we observed one year later in one of the series of three clinical interviews that took place at the end of the subject's second year of junior high school. The instability occurred in spite of the fact that during the second year of junior high school, the student has continued working with the Didactical Model of the staircase. This indicates that it is necessary to formulate and explain the notions that emerge in the process of abstraction by basing them on referents established in a previously acquired MSS.

Also, was confirmed that the main obstacles for the appropriation of the notion of the straight line's slope, were determined by the sense that the student had assigned both the rational numbers appearing in the Arithmetical Sign System of fractions [5c] and those appearing in the Geometric Sign System of de ratios of continues magnitudes. The possibility of representing lines through the cartesian plane, with algebraic expressions (linear equations), requires mastery of the continuous linear variation. This is essential in

order to understand the possibility of having two related continuous variables, such as the ones represented in the equations such as $y = kx$ which vary in a direct way and can be represented as straight lines through the translation of analytical geometry. Furthermore, we have also observed that the possibility of solving arithmetical problems on the variation of continuous means, like that which appears in the problems of mixtures, requires the handling of the concept of linear function with a continuous variable, but already as an object to be operated with objects belonging to the same semantic field (other linear equations), in order to generate other relations [8].

Finally, in agreement with the theoretic perspective of the production of MSS, it is probable that the principle obstructions that the pupil confronts in his journey toward competence in the use of the cartesian representation of basic variation notions, are located in the senses that he or she has developed by the previous teaching sequences that concern the variation of a point throughout a rectilinear trajectory; the proportionality in the different contexts of use; the geometrical ratios and the notion of fraction.

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FAMILY OF ARITHMETICAL & ALGEBRAIC WORD PROBLEMS, AND THE TENSIONS BETWEEN THE DIFFERENT USES OF ALGEBRAIC EXPRESSIONS

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This paper is based on the results obtained from experimental classroom studies and clinical interviews with students between ages 15 and 16. The studies focused on the students' solution of arithmetical and algebraic word problems. Some previous results can be found in Filloy, E. and Rubio, G. ([2] and [3]). The studies have to do with the application of three Didactic Models for the solution of arithmetical/algebraic problems.

THREE DIDACTIC MODELS

The three Didactics Models are:

1) The Method of Successive Analytical Inferences (MSAI).- The statement of the problem is conceived as a description of "a real situation" or "a possible state of the world"; the text is transformed by means of analytic sentences, i.e., using "facts" which are valid in "any possible world". Logical transformations are made which act as a description of the transformation of the "possible situation", until one such inferences is recognized as the solution to the problem. MSAI is the classic analytical method for solving this kind of problems using arithmetic only. 2) The Analytical Method of Successive Explorations (AMSE).- AMSE is a method of solution which uses numerical explorations in order to begin an analysis of the problem, thereby reaching a solution. 3) The Cartesian Method (CM).- Here, some of the unknown elements in the text are represented by algebraic expressions. Then, the text of the problem is translated into a series of relationships expressed in algebraic language, leading to one or several equations whose solution, via a return to the translation, brings about the solution to the problem. This approach to problem solving is the usual one in current algebra textbooks.

AIMS OF THE PROJECT

The project aims to describe the kinds of difficulties, hindrances, and conveniences produced by the use of any of the three methods when solutions to word problems appearing in algebra textbooks are attempted. We are interested in discovering the kinds

of skills generated by the use of AMSE that can make the user competent in the use of CM. And also, which skills generated by MSAI are necessary for a competent use of CM.

Our research project contemplates the relationship between the solution of problems and a competent syntactic use of algebraic expressions, in such a way that we can: (a) detect and describe the knowledge skills needed to set the analysis and solution of certain types of problems in motion; (b) find out if the transference of algebraic operativity is propitiated and/or reinforced, in the context of the problems, via the application of each one of the three methods for problem solution; (c) Show whether a proposal for teaching which uses MSAI and AMSE among its didactic methods might not allow the student to use language strata which are close to algebra, yet more concrete than algebra itself, in the solution of problems; (d) analyze how the use of the elements of the Sign-System of Arithmetic, via MSAI, could lead the user to give meaning to the symbols that are found in the algebraic expressions we call equations; (e) make it possible that algebraic expressions are given meanings external to mathematics; and that with the development of these skills, the foundations could be laid for the student to, (f) confer meanings, starting from the algebraic language itself, to the symbolic formations of algebra; and, (g) study the relationships between the ability to carry out the logical analysis of an arithmetical-algebraic word problem in MSAI, and the ability to analyze and solve a problem via the other two methods, AMSE and CM.

THE RESEARCH PROJECT

The following phases will characterize this experimental project:

PHASE 1. Exploratory study.

a) A theoretical analysis in order to formulate a Local Theoretical Model (see [1]) which would allow the empirical observations to be analyzed in terms of three components: 1) Formal Competence, 2) Teaching Models, 3) Cognition. A brief description of these components can be found in [2].

b) An experimental process: the classroom activity of student groups was monitored (1990-1992). The groups were classified according to three key areas of competence, and, essentially, as to the predominance of the following prerequisites for problem-solving: I) Arithmetic, II) Pre-algebra and syntax, and III) Semantics.

In the teaching component of this phase, only MSAI was used in some groups, and only AMSE in others.

[3] describes the theses that led to this phase of the work, which also constitutes the framework used for designing the next phase.

PHASE 2 (1991-1992). In this phase, five exploratory questionnaires were used to classify the population which was to be monitored in class. The results were used to select cases for clinical study, in order to observe in detail the theses presented in [3] as well as those which arose as the observation progressed throughout the school year. Essentially, MSAI and AMSE were used together in the teaching process so that, at the end, the students' skills with CM could be analyzed. This paper reports the clinical study.

SOME PRELIMINARY RESULTS

I) Through the use of certain strata of the Mathematical Sign-System (MSS) belonging to the CM, users generate intermediate senses linked only to these strata. And this allows them to simplify the solving of certain Problem Families. Once these senses are mastered, the use of this new sign-system, with only these strata, brings about the simplification of certain problems (e.g., Krutetskii's [4] chickens and rabbits). Thus, teaching models such as AMSE propose the development of ad hoc intermediate strata that can be identified among the more concrete, as well as in the realm of the abstract ones, necessitated by the CM, in order to simplify problem-analysis. At the same time, the aim is to progressively generate senses for such algebraic representations as are going to be implemented by the use of the CM. Each Problem-Family determines the representation levels required for its solution.

II) In order to solve a problem with MSAI, as for instance, the problem of chickens and rabbits, the required competence can reach the almost expert-level. That is why the natural tendency is, for instance, to use trial and error, thus trying to circumvent the series of successive analytical inferences required by the arithmetical logical analysis of the situation. These inferences call for representations which permit an analysis, and this, in turn, demands more advanced uses of such sign systems. In other words, Arithmetic and the natural language are being intertwined and set to work; and beside, a competence is needed to make a logical-semiotic sketch which makes the solving strategy meaningful. What makes such an analysis and logical-semiotic sketch

complicated is the fact that, for some problems, an intensive use of the work-memory is required, and this implies a training which only expert problem-solvers have attained.

III) When employing MSIA, what is generated is not a unique representation of a certain style; rather, this changes according to each Problem Family. With the use of Algebra and the CM, on the other hand, certain characters (the algebraic ones) are always used and by means of which one will write, arrange, work; in other words, the representation is arrived at through "canonical forms", and this constitutes a part of the sense in the use of such sign-systems for these problem situations. When using MSAI, on the other hand, representations must be invented for each problem, and this will require a certain use-capacity of the work-memory in order to keep representing the actions proposed in the solving logical-semiotic sketch, while leaving new marks and signs (or new chunks in the memory) by means of which the previous results can be clustered and not left "hanging". AMSE, as a solving method for word arithmetical-algebraic problems, requires that the students learn how to leave marks which progressively liberate memory units, thus allowing the user to use such units in the progressive unchaining of the analysis and the solution of the problem. When using AMSE, some students do not leave enough marks; therefore, in his or her system of representation, only some of the equations proposed are valid ones.

These intermediate representations arrange information in chunks of more complex organization, even if one cannot distinguish this in the signs produced by the user. During the interviews, some students arrive at a representation of the problem where, most likely, they made calculations (as, for instance, by means of a calculator) and, in the end, they just write the numerical solution to the problem. Then, they establish an equality and recover the operations (as dictated by one of the AMSE phases), so as to subsequently produce an equation.

Neither is it sufficient to propose a more abstract representation to solve such problems; likewise, it is not sufficient either --if arithmetical methods are being used, for instance-- that the student maintains in the work-memory everything that he is producing.

For instance, in the case of the problem where it is asked when and where the two hands of a clock come together, some solvers try to progressively approach the point which would yield a result; yet they fail, because through such an approach they will never find an equality of the terms in the two columns where they codify the positions corresponding to each hand, when adhering to the idea of more and more

approximations. Since they use a representation through decimal numbers, and the sought-after position must be expressed in elevenths, they will never be able to have in one column the number appearing in the other one; nevertheless, this is the usual solving strategy. Here, the problem does not have to do with a faulty recording of calculi; the difficulty lies in that an equation is not being sought starting from the logical-semiotic sketch (as a representation of what is happening in the problem). In order to advance, what is going to be required is a "more abstract" representation, or at least a better articulated representation, of the problem solving process, such as the one proposed by the AMSE, wherein the aim is to have access to a process where a series of phases are being implemented permitting to clarify which are the specific relationships among data, in such a way that some of these relationships can be later on identified as related one to the other(s). The objective, in this case, is not to find a numerical solution to the problem, but to establish the linear equation modelling it.

It can be seen, however, in the corpus of diagnoses, as well as in the clinical interviews, that AMSE phases can be known and yet these may be meaningless for systematic use. As it seems, the main concern among these users is to solve the problem, and not to represent it algebraically, something which, sometimes, they can do only after they solved the problem. They always rescue operations from an equality, notwithstanding the fact that the teaching model shows, in PHASE 3 of AMSE, that this can be done after any of the comparisons resulting from the numerical explorations.

IV) What has been pointed out above is a cognitive tendency which is observed in a great part of the population. It consists in a difficulty to attribute a meaning to the algebraic representation when students are immersed in a dynamics of numerical resolution. This is seen, for instance, when the systematic use of AMSE is proposed, after having showed how convenient it is for solving certain problems, through a trial and error procedure. What is observed here is that some students realize that they have performed such and such operations, only after they have reached a solution to the problem; that is, they realize there exist relationships among the data when they establish an equality (as in PHASE 3 of AMSE). This tendency to obtain an equation in order to follow the teaching requirements, induces the student to represent some unknown quantity or magnitude in the problem merely as a label, and not as a generalized number. In fact, when it is used, a solution to the problem has already been obtained and, in the best of cases, the value found by other means is associated to the letter appearing in the equation. In these cases, letters used in the equation the student has obtained possess a name status, but these letters are not associated with the assumed numerical values which were used to find the representation of the problem solution.

V) Eventually, trying to solve more and more complex Problem Families, MSAI requires AMSE-type representations. Once this stage is reached, representations are established which already show a separation from the use of MSS as used in the MSAI, for in that case the representation of something unknown is being implemented to operate with it, whereas in MSAI the unknown is merely going to be represented, and inferences are going to be made which speak of such a representation. Yet, the student will never operate with it. This is one of the greatest differences between such a use of the unknown in MSAI, and the use which is attempted to be given in the intermediate MSS as employed with the AMSE, not to speak of the CM.

The latter is related to the empirical-type results reported in [2] and [3], which tell us that some subjects do not admit the possibility of making inferences about something which is unknown. Other cases, not much different from the one just mentioned, are those where it is simply avoided to operate with the unknown, i.e., there is a resistance on the part of certain users to implement operations with something unknown, even when a use is being made of AMSE in teaching.

VI) Once MSAI has been used, the student, at that moment, becomes unable to generate problems within one and the same Family. Formerly, some users solved such problems by trial and error. When, however, such problems are solved by means of MSAI, this permits the users to recognize that there exists a logical structure for the problem, i.e., that there is a logical relationship between the quantities involved, and this becomes a hindrance to generate the Family. In this natural tendency, users come to realize that in such problems there are certain logical components which they had not perceived and which they do not master. And this is going to inhibit them from creating other similar problems.

This a clear clue that when solving a problem a logical semiotic sketch is first performed.

VII) From the results so far obtained it can be said that not until certain types of problems are solved will the natural tendency to approach them by means of pure MSAI be modified. Subsequently, making the unknown vary will be implemented, until, at a later stage, it is possible to solve Problem Families which will require that the user makes use of other representations, where unknown quantities must be represented, and then used to make inferences. Finally, it will be necessary to operate with the representation of the unknown.

Thus, representation needs generate new senses which bring on the possibility to make more abstract uses of the MSS's employed to establish the representation of the problem, from the logico-semiotic sketch. The essential difference between the introduction of algebra and these previous approaches lies in that in the latter, when solving problems, the unknown is represented, although it is not operated. Inferences are made with a reference to the representation of the unknown; but if operated, this is always done by means of the data: if a mention is made of unknowns, this is only in terms of the results of operations which are being done with the data.

As more complex Problem Families are solved, the sign systems used become more abstract. These (generalization and abstraction) processes operate on Problem Families, either when the students find common elements (generalization), or when they perform negations about some of the Family elements (in this case we would be speaking of an abstraction process).

Thus, a mechanism explaining why mixture problems are harder than those of other Families would be based in observing the need to break with the practice of just inferring from the representation of something unknown, in order to find a use for the representation where such unknown parts vary. This happens, for instance, when, from just using a directly proportional relationship, one turns to use several directly proportional relationships, jointly, within the same problem. This use of the representation of the unknown is the most complex one to be found in this study. It corresponds to the use of algebraic representations related to direct variations and to inversely proportional variations.

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Assessment and Evaluation

TEACHERS' USE OF ALTERNATE ASSESSMENT METHODS

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Summary:

It is becoming increasingly clear that teachers' evaluation practices influence the nature of children's mathematical experiences. Assessment practices send a powerful message to students about the mathematical thinking, experiences, and content that is valued. This paper reports on two projects involving teachers' use of and attitudes toward alternate assessment. One project involves implementation of alternate assessment on a county wide scale; the other deals with five individual teachers' efforts to use alternate assessment. Both projects address teachers' uses of alternate assessment, factors that facilitate and impede their use, and the interaction of that use with both their beliefs about mathematics and their role as teachers of mathematics.

There is a growing consensus among mathematics educators about the value of teaching mathematics from a constructivist perspective. This movement is consistent with various reform minded documents (e.g., NCTM, 1989;1991) that convey a process orientation toward mathematics teaching and learning. The problems of realizing process-oriented classrooms are manifold, including the importance of what and how mathematical understanding is assessed. Typically, teachers and students will, at some point, focus on questions such as, "Will this be on the test?" and imperatives like "This is important. Make sure you know it for the next test." Such negotiation is fundamental to determining what gets valued and what gets learned. Crooks (1988) emphasizes that classroom evaluation guides students' thinking in terms of what they should value and what standards they are expected to meet and hence "appears to be one of the most potent forces influencing education" (p. 467).

It is well documented that many, teachers hold or communicate a limited view of mathematics (Brown, Cooney, & Jones, 1990) and that students often think of mathematics as a set of a priori algorithms to be executed in well defined circumstances (Borasi, 1990). Further, teachers' evaluation practices sometimes reflect this limited view as they conflate the notion of an item being difficult with the notion of an item assesing a deep and thorough understanding (Cooney, 1992; Cooney, Badger, & Wilson, 1993).

Two projects involving teachers' use of alternate assessment items (e.g., items that require solutions

other than a pre-determined number) and alternate assessment techniques are the focus of this report. In the first project (hereafter referred to as the Individual Teacher Project (ITP)) five teachers incorporated alternate assessment items and techniques into their teaching. The teachers were given items that they could use in their teaching and were provided suggestions for generating their own items. Data from this project consists of two interviews with each teacher, three observations of their classes, information shared during three project meetings, and copies of tests and materials that the teachers sent us. In the second project, hereafter referred to as the Large County Project (LCP), 122 fifth grade teachers were asked to incorporate assessment materials into their teaching in an effort to have a continuous achievement program. The assessment materials were designed to compliment the curriculum materials the county had designed in an effort to move teachers toward a more "hands on" approach to teaching mathematics. Data from this project consisted of a survey conducted in the fall of 1992 on the teachers' use of the assessment materials and telephone interviews conducted in March, 1993, with the 12 teachers who were willing to be interviewed. Data from the ITP and the LCP were used to address the following three questions: In what ways and how do teachers use alternate assessment items in their teaching? What factors seem to facilitate or impede teachers' use of alternate assessment methods? and In what ways and to what extent does an emphasis on alternate assessment methods interact with teachers' beliefs about mathematics and about their role as teachers of mathematics? We consider each of these questions in turn.

In what ways and how do teachers use alternate assessment items in their teaching?

In LCP the most common use of alternate assessment items cited by the fifth-grade teachers who were interviewed was the identification of objectives and skills that need to be re-taught; the most common scenario was that the teachers retaught a skill if many of the children in the class performed poorly on an assessment item. Although the assessment items in LCP were designed for use with small groups, half the interviewed teachers mentioned using them in a whole class format. The teachers indicated that the assessment materials were also used as worksheets, as teaching tools, as classwork to be sent home to parents, as problems for the whole class to discuss, and as sample work for portfolios. The results were used to determine individual students' strong or weak points and to structure lessons. Five of the twelve teachers interviewed stated that they did not use the results of the alternate assessment items in determining grades. The other seven teachers said that the results contributed to the students' grades.

In the ITP Carol, a middle school teacher in an urban school, used open-ended questions that required

students' explanations such as "Explain why 4.25+7.3 cannot equal 12.28," or "Identify the mistake Terry made when he said that 1.26+3.4 equals .150?" She also had her students write about their feelings toward mathematics and the kinds of difficulties they experienced in solving mathematical problems.

Katie, a secondary teacher in a university town, provided her own analysis regarding the changes she underwent (see table 1). Observations of her target class revealed that her use of alternate assessment items

Table 1: Katie's analysis of her changes in assessing students' understanding

Traditional Questions	Alternate Test Questions
1. Find the GCF of 12 and 15?	1. Why can't 12 be the GCF of 6 and 18?
2. Round each of the following to the nearest tenth: a. 48.128 b. 1399.92435 c. 4.2378	2. Create a number that would round to each of the following: a. 54.32 b. 853.8 c. 91.998
3. Categorize each of the following numbers as prime or composite: 81, 2, 53, and 111.	3. Buddy thinks that 91 and 93 are prime because they end in an odd digit. Pete says that Buddy is wrong. Who is correct and why?

had also influenced her teaching of mathematics as she prodded students with questions like, "Why is that true?" or "How did you get that answer?"

For Dave, a secondary teacher in a private school, it was not unusual for 25% of his test items to consist of open-ended items. His teaching mirrored this emphasis as well. The following items are typical of items Dave created: "Is it possible for an isosceles triangle to have a right angle? If so, give an example. If not, state why," or "Peter thinks he can find an equation for a parabola that has no real x-intercepts (roots), a negative y-intercept and a vertex in the second quadrant. Lucy disagrees. Who is correct and why?" Dave feels that open-ended questions give more information about a student's understanding than purely computational ones since he "could more easily define a student's strengths and weaknesses." Interviews with Dave's students indicate that they sense his method of teaching is different from other mathematics teachers they have had. Comments like, "My other mathematics teachers didn't ask us to explain things." were typical.

Lisa, a colleague of Dave's in the same school, used alternate assessment in all of her classes. Approximately 29% of her test items were open-ended. A typical question is, "Does $x^{12}/(x^3 \cdot x^9) = 1$ for all values of x? Why or why not?" Lisa also has an interest in portfolio assessment. She conveys this interest to her students, "I talk about portfolio assessment a lot, I say I'm looking at your whole, everything you do, every piece of work you give me, everything you say to me, every action." In keeping with this, Lisa has her students create portfolios in which they react to each topic and discuss problems or insights they had while studying the topic.

Elizabeth, a geometry teacher in an urban setting, used alternate assessment items primarily as bonus

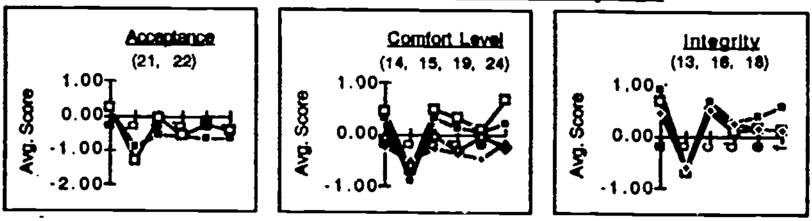
or warm up activities. However, she gradually tended to use items such as the following on her tests.

Ian said that the missing side of this right triangle is 17 because it contains a Pythagorean triple. Is he correct and why?

What factors seem to facilitate or impede teachers' use of alternate assessment methods?

In the LCP survey the teachers' reactions to the close-ended questions were coded on a Likert scale: -2, -1, 0, +1, +2. Table 2 shows the positive responses toward the assessment materials. In general, the teachers felt the materials were clear and concise and generally felt comfortable using the materials in small groups albeit this usage was rather uneven across the fifth grade teachers, indicating significant interaction while gouped by years of experience (a-f, see figure 1).

Figure 1: Conceptual Clusters of LCP Teachers' Reactions vs. Years of Experience



The teachers felt that they had reasonable support although they indicated that their best support was from their peers. They also expressed concern that the assessment materials were in a notebook separate from the curriculum materials--thus making the integration more difficult. Although the teachers felt that the assessment materials did a good job of testing the curriculum's objective, in reality not all of the teachers bought into the curriculum's objectives as some relied heavily on texts and minimized a "hands on" Table 2: Teachers' Positive Reactions Toward the Assessment Materials (n=122)

NO.	Question	Avg.	(p <)
13	The procedures for use are clear and concise	+ .46	.0001
14	I feel confident in using the materials	+ .1	.2
15	I feel comfortable assessing students in small groups	+ .31	.001
16	The assessment materials are helpful in determining students' understanding of mathematics	+ .19	.05
17	Information gained from the assessment materials has caused me to alter my classroom instruction	+ .14	.1
18	The assessment items do a good job of testing the curriculum objectives	+ .21	.05
27	The parents of my students will be supportive	+ .15	.05

approach to teaching mathematics. Teachers who made extensive use of the assessment materials reported that the students liked working with the materials. In general, the teachers felt that the implementation process should have been more gradual rather than being imposed on them in a single year.

Table 3 shows the negative responses toward the assessment materials that are regarded as impeding factors toward its implementation. Several conclusions seem warranted. First, the teachers felt that using

Table 3: Teachers' Negative Reactions Toward the Assessment Materials (n=122)

No.	Question	Avg.	(p <)
19	It is difficult to choose samples of students' work for the portfolios that illustrate the students' achievement level	-.32	.0001
20	The materials limit my creativity in classroom instruction	-.63	.0001
21	My previous evaluation methods gave me better insight into my students' understanding of math	-.56	.0001
22	The time and effort invested in using the materials is worth it, given the quality of information I obtain about my students' understanding of mathematics	-.4	.0001
23	My students seem to be frustrated in responding to the assessment tasks	-.31	.001
24	In using the materials, I have difficulty grading all students fairly	-.18	.05
25	In using the materials, I am concerned that my assessment of students' responses will not be consistent with that of my fellow teachers	-.12	.15
26	I am concerned that the materials will not prepare my students for the newly revised form of ITBS	-.41	.0001

the materials limited their creativity in teaching--signaling that the teachers saw the assessment as being separated from instruction. Too, they were not confident that they could select appropriate samples of students' work to put in the portfolios.

The ITP teachers universally cited time as a limiting factor in using alternate assessment. Also mentioned was their comfort level, both with mathematics and, particularly, in creating their own items. Thus, they noted that items created for them by the project staff as being important facilitating factor.

In what ways and to what extent does an emphasis on alternate forms of assessment interact with teachers' beliefs of mathematics and their role as teachers of mathematics?

In interviews with the LCP teachers, the comments focused on procedures involving the assessment materials rather than its substance. Teachers' time was a primary concern to many although some teachers minimized this concern. Positive comments included that it allows progress according to student ability, forces students to think, provides uniformity across the county, and promotes student interactions. Some high correlations were found (see table 4) while LCP teachers' reactions toward single items were grouped conceptually, indicating that teachers' attitudes toward the assessment materials are related to their beliefs.

Table 4: Pearson Correlation Coefficients among concepts regarding the assessment materials

Concepts (Questions in Parenthesis)	Support	Comfort	Integrity	Acceptance	Inf. on Instruction (17, 20)
ITBS (3, 26)	.39	.53	.51	.44	.46
Support (12a, 12b, 12c, 27)		.55	.58	.42	.50
Comfort Level (14, 15, 19, 24)				.54	.53
Integrity (13, 16, 18)					[illegible]
Acceptance (21, 22)					[illegible]

With respect to the ITP, it is difficult to conclude that how the teachers saw mathematics was related to whether the teachers used alternate assessment items and techniques. It is clear, however, that how they saw mathematics was related to the types of items they gave their students. For example, Carol strove to make her students feel successful in the classroom and to develop a positive attitude toward doing mathematics; hence, she frequently used hands on materials. Yet her tests had a strong computational orientation.

Carol, Katie, and Elizabeth struggled in their attempts to create open-ended items much more than Dave and Lisa. It is not clear whether this circumstance relates to their beliefs about mathematics (Dave and Lisa communicate a broad view of mathematics) or to the fact that Dave and Lisa teach together in a relatively risk free environment in which they are free, indeed, encouraged, to experiment. At the beginning of the project Dave and Lisa were already familiar with the NCTM Standards and hence "had a jump" on the other teachers. While it is striking to note that the public school teachers (Carol, Katie, and Elizabeth) were more likely to emphasize computational outcomes as they struggled to create and use alternate assessment items, it is also interesting to note that Carol and Katie exhibited more change in their teaching than did the other teachers and demonstrated considerable professional growth in doing so.

Conclusion

Green (1971), in analyzing teachers' beliefs systems, allows for the possibility of beliefs existing in clusters, isolated from one another. We see evidence of this in this study, particularly with the LCP. For many of these teachers, assessment is seen as something quite different than teaching. For the teachers who saw the assessment materials as an encroachment on their time, they asked the question of why they also had to do assessment.. But for other LCP teachers, time was not so much of an issue or at least this concern was minimized because of the perceived importance in using the assessment materials. These teachers saw considerable harmony between the assessment materials and their normal way of teaching mathematics. Neither did this isolation or non-isolation relate to a school's economic standing. Indeed, some of the more vocal concerns were from teachers who taught in high profile schools, whose students generated high test scores on the ITBS, and whose parents were very involved in school affairs--or at least "watched over" the school.

The ITP provides a somewhat different twist on the notion that beliefs about mathematics influences instruction (Thompson, 1984; 1992). Here the evidence points to the fact that a teacher's beliefs about

mathematics are primary (Green, 1971) and the manifestation of that in terms of test and class questions is derived from the primary beliefs. Perhaps this should be no surprise. But it is also the case that where teachers demonstrated considerable professional growth (in terms of moving toward the Standards), the entry point for this change came from a concern about assessment. While this may be a chicken and egg question, it certainly raises the question of whether assessment must always be relegated to the caboose of reform or whether in fact in could serve as the engine for driving reform.

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Epistemology and Cognitive Processes

MATH TALK IN A HETEROGENEOUS GROUP

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Abstract

Changes in the mathematical discourse of four third-grade students in a guided learning group, and the teacher behaviors accompanying these changes, are described. Less capable students who possessed relatively weaker mathematical competencies than their more able peers constructively influenced the discourse of their peers under particular conditions of adult guidance. The positive discourses was sustained over time by the group. The less capable students in this study monitored and regulated the activity and provoked "sense-making" exchanges among all students. These outcomes suggest that mathematical discourse can be enhanced by adult-led heterogeneous ability learning groups.

There is mounting pressure to change instructional practices for teaching mathematics (National Council of Teachers of Mathematics, 1989). Recommendations for change suggest moving away from an emphasis on memorizing procedures toward an emphasis on mathematical thinking. The social atmosphere believed to be most conducive to thinking generally, and mathematical thinking in particular, is one in which students have an opportunity to talk, reflect, and work together under particular forms of guidance from their teacher (Brown, Campione, Reeve, Ferrara, & Palincsar, 1991; Lampert, 1990; O'Connor, 1991; Stevenson & Stigler, 1992). The goal is for mathematics classes to be places in which all students participate in sustained discussion and collaborative activities--places in which opportunities for students to communicate mathematically are maximized (Pimm, 1987).

The press for more interactively oriented learning environments of necessity means that children of widely varying competencies will interact with each other. Yet relatively little is known, either theoretically or empirically, about the cognitive effects of interactions between more and less capable learners in these increasingly heterogeneous settings, nor how teachers can maximize mathematical talking and thinking in these environments. The purpose of the

work reported here is to: (1) examine mathematical talk and opportunities for mathematical thinking that emerge in interactions between more and less capable learners in a small guided learning group, and (2) investigate the conditions that support these interactions.

It is generally believed that there are cognitive benefits for less capable students in heterogeneous groups. This belief is consistent with Vygotskian theory that predicts that learning is enhanced when a zone of proximal development is created by the presence of a more capable individual (Vygotsky, 1987). The focus of most studies based on Vygotskian theory has been on describing the cognitive development of the less capable learner and on identifying critical guiding behaviors of the more capable that are believed to facilitate this development (LCHC, 1983; Rogoff & Gardner, 1984). Although it has been suggested that interactions in heterogeneous groups may be mutually beneficial for both more and less capable learners (Palinscar, 1992), the ways in which less capable learners create zones of proximal development among their more capable peers have not been specified.

Methods

The data for the proposed project were collected as part of a larger study ($N = 58$) designed to enhance mathematical understanding in elementary school children using collaborative and guided practice procedures (Reeve, Gordon, Brown, & Campione, 1990). This two-year study took place in two inner-city parochial schools on the south side of Chicago, attended primarily by minority students. All the students in the participating classes were assigned to one of four performance levels based on their Total Mathematics score on the California Achievement Tests (Forms E & F). Groups were then created by randomly assigning one student from each performance quartile to each group. The resulting groups, balanced by gender, were thus composed of four students, one from each of four distinct performance levels. Half of the groups were taught using collaborative and guided practice procedures. A major goal of the project was to examine ways in which students appropriate mathematical constructs, as reflected in their talk. The role of the adult in these groups was to move children out of an "other-directed" framework into a "self-directed" framework--a kind of cognitive socialization in which the teacher was the socializing agent. Three "reflection

boards" were used as a medium for externalizing thinking throughout the problem-solving activity. The remaining groups participated in one of two control activities: unguided cooperative learning, or paper-and-pencil practice with routine teacher comments. All students worked to solve three term "combine" problems in which the first, second, or third addend was the unknown. We used this problem type because it proved sufficiently difficult to induce students to reflect, for example, on part-whole relationships:

> Bill is counting the cars he passes on his way to school. He counts 81 cars altogether. He counts 28 red cars, 34 blue cars, and some black cars. How many black cars does he count on his way to school?

One of the 7 intervention groups was selected for the current study. The group met 21 times over the course of 7 weeks. The data source was protocols and written work from these sessions.

Subjects

For the purpose of the current analysis, the two students drawn from the lower two quartiles of their class, Tyron^{L} and Linda^{L}, were considered "less capable" (denoted hereafter by the superscript L), while the students drawn from the higher two quartiles, Rochel^{M} and Clark^{M}, were considered "more capable" (denoted by the superscript M). All the students in this group were familiar with one another and had been classmates for at least one full school year. They all had the same instructional background in mathematics.

Tyron^{L} was mildly neurologically impaired. His speech was sometimes difficult to understand, his physical movements became noticeably spastic when he was ill at ease, and his handwriting was large and poorly controlled. His classmates openly referred to him as "stupid" and "weird". Tyron^{L}'s total battery score on the California Achievement Tests, administered to his class in March of the previous school year, was in the 7th percentile nationally. His weakest performance was on the mathematics subtests (2nd percentile) and his best was on the Language subtests (15th percentile).

Linda^{L} was an engaging, willful, and active child. Her performance in class was consistent with that of students described as learning disabled. Although seemingly quite

bright, Linda[L] frequently appeared distracted, had difficulty carrying out tasks that required written work, performed poorly on paper-and-pencil tests, and was a terrible speller. Linda[L]'s total battery score on the California Achievement Tests was in the 5th percentile nationally. Her weakest performance was on the Language subtests, where she obtained a minimum score, and her best was on the Reading and Mathematics subtests (11th percentile nationally).

Rochel[M] and Clark[M] performed markedly better in school and on standardized measures of mathematics than did either Tyron[L] or Linda[L]. Rochel[M], for example, was an academically proficient, cooperative and popular student with a mild speech dysfluency. She sometimes stuttered when directly questioned, but communicated carefully and concisely when initiating conversation, or volunteering to respond to a question in class. Rochel[M]'s total battery score on the California Achievement Tests was in the 49th percentile nationally. Her weakest performance was on the Mathematics subtests (26th percentile) and her best was on the Language subtests (78th percentile). While clearly one of the more successful students in her class, Rochel[M] maintained a low profile--she was a little remote and not easy to "see".

As soon as a question was asked, and sometimes before the question was completed, Clark[M] had his hand up. His impulsive behavior and insistence on being recognized, irritated many of his peers. His classmates, however, did believe that Clark[M] knew what he was talking about--they said Clark[M] was "smart". Clark[M]'s total battery score on the California Achievement Tests was in the 45th percentile nationally. His weakest performance was on the Mathematics subtests (20th percentile) and his best was on the Language subtests (73rd percentile).

Results and Conclusions

Findings indicated that learners were exposed to a variety of opportunities to think and learn as a result of discourse with their less capable peers. Less capable students stimulated mathematical discussions in their more competent peers in the following ways:

(1) The less capable students monitored and regulated the activity of the more capable. They noticed omissions, inconsistencies and novelty, and insisted on corrections. They made task components more explicit by requiring that peers break down the problem-solving

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activity into steps. Their participation generally slowed down the pace of the activity, which, rather than being a cost, resulted in more reflection on the part of high performers.

(2) The less capable students provoked "sense-making" exchanges. They offered contrasting (correct or incorrect) answers that played the role of naturally occurring counter-suggestions. On the 13th day of the intervention, for example, the group struggled with how to represent and solve a relatively new type of problem. A comment by Linda^L initiated an exchange in which Clark^M, after an initial protest, thought about an alternative approach:

Clark^M:	For the [pause] I draw a circle. Draw a circle.
Linda^L:	Can I say something?
Rochel^M:	Okay, what.
Linda^L:	*We can't draw no circle.*
Rochel^M:	Yes we can.
Clark^M:	Yes we can.
Rochel^M:	We did that before.
Clark^M:	*Maybe you're thinking about those boxes.*
Linda^L:	Yeah.

While Rochel^M appeared to hold to the view that what worked before should work again, Clark^M thought twice, literally, instead of rejecting Linda^L's comment out of hand. Furthermore, now that two approaches were put on the table, a comparison was possible. On other occasions, Tyron^L and Linda^L questioned rote and abbreviated performance by Rochel^M and Clark^M, which resulted in discussions in which the rationales behind performance decisions were made explicit.

(3) Changes in the less capable students prompted reflection. Progressive changes in the abilities of the less capable students on the target task were visible to their high performing peers. For example, on the 5th day of the intervention, the following exchange took place when Linda^L answered for Tyron^L:

Teacher:	You know, answering for him is not going to help him.
Linda^L:	He needs it!

Teacher: You have your idea, Clark has his idea, and I want to hear Tyron[L]'s idea.

Linda[L]: *He doesn't have one.*

But four days later, when a similar event occurred, Linda[L] said, "You don't have to tell him. He know how to count!"

The more capable became better able (and more willing) to reveal their own strengths and weaknesses, to display uncertainty, and to consider alternative solutions:

Clark[M]: *What's the information?* Um, there was, um, 20, there were 20 costumes. Um, *wait a minute!* . . . . Just put 'there were 8 vampires,' *or* you can put the whole thing.

The particular kinds of interactions among students that were observed in this study appeared to result from conditions intentionally established by the teacher. The teacher: (a) *established a constructive collaborative social climate* by modeling, and requiring practice in, respectful responses to all student comments; (b) *promoted student-student discourse* by explicitly inviting comments from all group members, limiting her own participation, and turning questions back to members of the group; (c) *engaged the group in metacognitive thinking* by asking reflective questions, and consistently using student ideas, particularly those of less capable participants, as opportunities to initiate talk about mathematical procedures and ideas.

These results illustrated that students of all proficiency levels can engage in mathematical discourse under specific conditions of guidance, and suggested that less capable students may be an important and overlooked classroom resource. How the interactions reported here compare to interactions among more capable students in homogeneous groups, or in heterogeneous groups with more widely varied ability levels, for example, remains to be studied.

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EPISTEMOLOGICAL CHANGES IN COMMUNITY COLLEGE MATHEMATICS STUDENTS

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ABSTRACT

The study examined community college students' beliefs concerning the nature of mathematics. 182 students were administered a Likert survey designed by the authors. With rare exceptions, students manifested a limited view of mathematics, e.g., as numbers and their manipulation in well-structured problems. We found no significant differences between students from remedial and relatively advanced classes. A semester-long constructivist intervention in a remedial arithmetic course failed to result in significant differences concerning their belief about the content of mathematics, a result we argue may be due to the students' interpretation of the course activities as outside the boundaries of mathematics.

There is a growing literature examining the nature of students' epistemological beliefs and the effects of these beliefs on students' learning. For example, Songer & Linn (1991) argue that students who hold a more dynamic view of science will be better able to integrate their newly acquired knowledge. Schoenfeld's (1989) finding that most high school students believed if they had not resolved a math problem within five minutes, it was time to give up, reveals an impoverished model of the nature of mathematical problems and activity. More generally, Resnick (1991) argues that a major focus in the classroom should be an "enculturation" into mathematics as a way of

thinking and valuing.

Our study extends this line of inquiry to community college students. The literature of students' epistemology has largely ignored community college students, a population of older students who tend to have a weak academic background with many experiences of failure.

The purpose of the study is two-fold. First we aim to examine the beliefs these students hold about the nature and structure of mathematics. Second, we aim to develop instructional interventions that can foster the development of community college students' epistemological perspective, especially, their conceptualizations of the defining boundaries of the domain of mathematics, its structure, and the nature of mathematical activity. For the intervention phase of the work, we choose the most extreme population within the community college math classes, adult students who had not yet mastered the rudiments of arithmetic. We assume that, in addition to repeated experiences of failure, these students have been exposed to a very limited view of the nature of mathematics and mathematics activity.

METHODOLOGY

Subjects

The subjects of this study were drawn from a community college in Los Angeles County. The population of the college was predominantly minority and lower S.E.S. All students in six courses participated in the study: four sections of a basic arithmetic course (with no prerequisite), a geometry course (with a prerequisite of elementary algebra), and a general math course for non-math majors designed to transfer to four-year colleges (with a prerequisite of intermediate algebra).

Research Design

Students in each of the six classes were administered the survey instrument on the first day of their respective semester-long course. Two of the basic arithmetic classes were also administered the instrument a second time at the end of the course. One of these sections functioned as the control group. The other section, taught by the first author, functioned as the experimental group for the instructional intervention.

Survey Instrument

A Likert scale was designed to identity students' epistemological beliefs. The scale consisted of 33 statements. A set of statements corresponded with each of the aspects of our conceptual framework: the nature of mathematics, the structure of mathematics and its relation to other domains, as well as their beliefs about learning mathematics. The statements from these sets were presented in randomized order.

Within each set, statements were included from naive and sophisticated perspectives. For example, concerning the nature of mathematics, students evaluated statements such as "Mathematics is a precise science that always has a logical and definite answer," "Thinking mathematically almost always means reasoning with numbers," and "The thinking of mathematicians often involves creativity." Each epistemological aspect is scored separately in terms of a mean score derived from the 1 to 5 Likert ratings (after adjusting for directionality of the statements). This paper focuses on the analysis of beliefs concerning the nature of mathematics.

Instructional Intervention

The instructional intervention was implemented in one of the basic arithmetic sections. The intervention consisted of the entire semester of classes, held twice a week over a 16-week period. Students were informed that if they stayed in the section, they would be part of a research project, which would involve video- and audio-taping (no

students opted to change sections).

The intervention sought to modify the students' beliefs by transforming the arithmetic curriculum into ill-structured problems, situated in semantically rich contexts of interest to the students. Problems were collaboratively defined by students and instructor and pursued by small groups or the class as a whole. Plans for data collection and analysis were scaffolded by the instructor. Concepts were introduced opportunistically as the students saw a need within their process of problem resolution. Procedures were both invented by the students and negotiated between students and instructor. Examples of the kinds of problems that the students undertook include investigations about people in their community, such as the typical duration of a marriage, and the design of a park. Across the semester, the instructor avoided any explicit discussion of the mathematical enterprise in order to not bias the results of the second survey. The control consisted of a traditional arithmetic course. This group learned the requisite subject matter by a combination of text, direct instruction, and practice.

RESULTS AND DISCUSSION

Two way analysis of the variance by class and gender reveal insignificant differences. Basic and relatively advanced classes all manifested low-level beliefs about the nature of mathematics. This finding is not surprising, given that the students' epistemology literature has found naive perspectives even on the part of university students (Hammer, in press).

Analysis of pre- and post-survey scores indicated that the gain of the experimental group over the control group did not reach significance (Table 1). These

initial results suggest that the students' epistemologies may have been robust, too robust to change in a single semester.

	Experimental	Control	
Male	$n = 11$ $\bar{X} = 0.20$ $s = 0.46$	$n = 13$ $\bar{X} = -0.03$ $s = 0.60$	$n = 24$ $\bar{X} = 0.08$ $s = 0.53$
Female	$n = 11$ $\bar{X} = 0.15$ $s = 0.30$	$n = 11$ $\bar{X} = 0.06$ $s = 0.28$	$n = 22$ $\bar{X} = 0.11$ $s = 0.29$
	$n = 22$ $\bar{X} = 0.17$ $s = 0.38$	$n = 24$ $\bar{X} = 0.01$ $s = 0.47$	

Table 1: Sample size, means, and standard deviations of changes in average score for study samples

Examination of student verbal feedback during the course and in written form at the end of the semester led us to conjecture an aspect needed in a more successful intervention. Our purpose was to expand students' conceptions of the nature of mathematics. We aimed to work toward this goal by framing the course across a rich range of problem formulation, problem solving, and mathematical reasoning. However, many students failed to reconceptualize the nature or boundaries of mathematics accordingly, but rather were concerned that the course was not focusing on mathematics, as manifested in such comments as, "When are we going to start doing real math?" Despite extensive involvement in mathematical sense making that did not involve numbers, student responses at the end of the course indicated only 23% disagreed with the statement "Thinking mathematically almost always means reasoning with numbers," and only 9% disagreed with the statement "Mathematics is mainly the

study of numbers and how to apply them to problems."

We question whether participation in a classroom framed in terms of the culture of mathematics may be sufficient to change students' epistemological perspective. Students may simply interpret the course content as arbitrary selections on the part of the instructor and the nature of the activities as solely teaching techniques. We suggest future studies examine the effectiveness of combining an active engagement in authentic mathematics with explicit meta-analysis of the bounds and nature of the mathematical enterprise.

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Functions and Graphs

INTERNAL AND EXTERNAL REPRESENTATIONS RELATED TO THE FUNCTION CONCEPT

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Summary

From our intuitive experience, it is known that reasoning is related to mental images. In particular, if a subject posses a mathematical concept, it should be represented in some way. The subject has at that moment an internal representation of the concept. In this work we study internal representations of the concept of function teachers have, through the external representation they present in a given task. Fourteen questionnaires were designed, different tasks related to the concept were asked to teachers, using different external representations of tha concept of function.

Introduction and objectives

Studies involving questionnaires which concerned with the concept of function have been carried out with mexican mathematics teachers at pre-university level since 1988. On the grounds of these findings and those from Vinner (1983), Markovitz et al (1986), Vinner and Dreyfus (1989), Dubinsky and Harel (1992), the next stage in our research was to explore the problems teachers havo to understand the concept of function. We take as a guide Johnson-Laird's idea (1983, p. x) that "human beings construct mental models of the world, and that do so by employing tacit mental processes".

In what follows, we will refer to mental representations as internal representations, and to any kind of symbolic or graphical representations as external representations. These latter are materialized outside the subjects' minds. Generally, mental representations (that is, internal representations) students and teachers hava are not completely integrated. Whenever teachers are posed a problem using an external representation, for example of a given function, they appeal to their internal representations to answer. What kind of internal representations are favored by teachers to answer in a given task?, which of them yield errors?.

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Our study is intended to approach these questions by taking into account different performances teachers showed on paper when they were given a specific task.

We are focusing on Marr's idea (1985) related to systems of internal representations, when he states that:

> Modern representational theories conceive of the mind as having access to systems of internal representations; mental states are characterized by asserting what the internal representations are obtained and how they interact. (p. 104)

Methodology

According to our aim, fourteen questionnaires were designed. The study was carried out with 30 mathematics teachers (pre-university level). Two questionnaires per week were given to the teachers during 7 weeks. The questionnaires and tasks were related to: Identification of functions (C1), Identification of points of Domain, Image and Image set (C2), Tabulation and graphic (C3), Identification of functions and the writing down of a definition (C4), The calculations of the function at given points (C5), Translation from the algebraic to the graphic representation (C6), Translation from the graphic to the algebraic representation (C7), Construction of functions with some properties (C8), Identification of functions which are equal (C9), Articulation between representations: pictoric, symbolic-algebraic and physical context (C10), Articulation between representations: physical context, symbolic-algebraic and pictoric (C11), Operation with functions (C12), Proof or construction of non-examples (C13), Decision about the truth or falsity of definitions of function (C14).

It was found that the percentage of correct responses increased along the answering of the questionnaires, but the intuitive ideas induced by strong internal

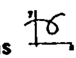
representations turned up and some of them yield errors. For the purposes of this manuscript we analyze a few questions of the questionnaires.

Analysis of answers

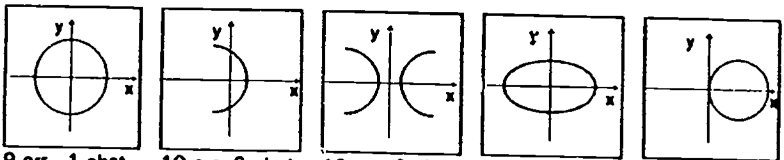
In C4 we posed the open question: What is the definition of function?. 18 teachers answered using the "Rule of correspondence", 10 teachers answered in terms of "Sets", and one gave a wrong definition (two images belonging to one point). In questionnaire 14 the teachers were given definitions of function which can be found in the mathematics textbooks they used: In terms of variables, in terms of sets (ordered pairs), in terms of a rule of correspondence, in terms of INPUT-OUTPUT.

Informants were asked to decide which definitions were true and which were false and, which of them they used for teaching purposes. 14 teachers said they used the definition in terms of "Rule of correspondence"; 13 in terms of "Set of ordered pairs"; 3 in terms of "variables". Also, 11 teachers said that the definition of function in terms of "variables" was false and 11 teachers said that the definition of function in terms of "INPUT-OUTPUT" was false.

Now, if teachers prefer a definition of function, can they correctly decide if a proposed graph represents a function?.

In C1, we presented 26 graphs and the informants were asked to decide which of them represented a function and why. The second item was  , 29 teachers said that this curve did not represent the graph of a function (there was only one error), the arguments of the teachers were distributed as follows: two teachers used a definition of ordered pairs, 10 teachers wrote that there were more than one image in certain points, six teachers explicitly used a vertical line cutting the curve in more than one point, one teacher said that it was not the graph of a function (without

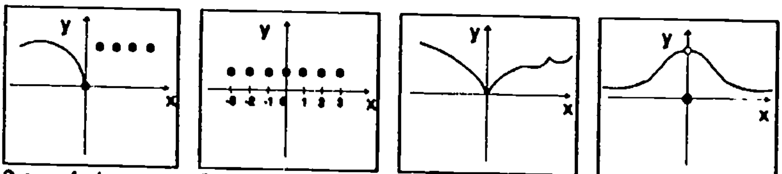
justification). When teachers were given conic curves like those showed in Figure 1, the six teachers who used a vertical line followed the same strategy, answering correctly. Are errors related to conic curves due to the existence of an analytical expression? It seems that the answer is affirmative. That is, it seems that the existence of an analytical expression is part of the internal representations of the concept of function teachers have. Moreover, it seems that that belief is stronger in some teachers than the formal definition of function they have.



9 err., 1 abst. 10 err. 2 abst. 12 err. 0 abst. 9 err., 0 abst. 8 err., 1 abst.

Fig. 1. Errors and abstentions linked to conic curves

When presenting a conic curve to one third of the population, the existence of an equation related to the curve, made an appeal to an internal system representation connected to an algebraic expression (formula). In other words, the teachers did a false recognition of a representation of a function, by means of the analysis of the shape of the curve. The teachers who used an argumentation with a vertical line answered correctly the questions related to the conics. These teachers have integrated the idea of the process of using vertical line into their internal system representation.



3 err., 4 abst. 3 err., 3 abst. 4 err., 1 abst. 1 err., 5 abst.

Fig. 2. Errors and abstentions linked to different functions

It is known from the history of mathematics that the idea of analytical expression was immersed in the definition of function given by Bernoulli, Euler and Lagrange. Moreover, it seems that in their definition, the idea of continuous function was involved in their mental representation. We tried to verify if that is case in the internal representations teachers have. Questionnaires C8 and C9 concerned with this issue.

The questions in C9 (equality of functions) were designed to compare teachers' performances in simple direct tasks, and in complex tasks in the questions of C8 (construction of functions). Then, for example, item 2 in C9 asked if $f(x) = 2$ was equal to $g(x) = \sqrt{4}$, for every $x \in R$. There were 26 correct answers. In contrast, in problems designed to compare those performances in C8, most of the teachers failed. In question 12 of C8, subjects were asked to construct three functions of real variable satisfying $|f_1(x)| = |f_2(x)| = |f_3(x)| = 2$. The results were as follows: four teachers gave no answer; one teacher gave three erroneous functions; six teachers constructed the three functions correctly; 10 teachers constructed the two first functions correctly and for the last one they just repeated one of the functions (for example, $f_1(x) = 2$, $f_2(x) = -2$ y $f_3(x) = \sqrt{4}$ or $f_3(x) = \sqrt[3]{8}$); nine teachers presented correctly the first function and they tried to construct two more different functions. Formally, these nine answers might be considered as correct. For example, these teachers wrote $f_2(x) = (6x+2)/(3x+1)$ y $f_3(x) = (4x^2-4)/(2x^2-2)$. Then, since f_2 is not defined in $x = -1/3$ and f_3 is not defined in $x = \pm 1$, the three functions ara different. But, if we compare their answers with those produced in question 18 (see below), it is evident that in a complex task, they really believe that a different writing in an algebraic expression produce a different function.

Now, we analyze the answers given to quastion 18, which asked for two functions of real variable such that $(f^\circ f)(x) = f(f(x)) = 1$. No one teacher could give a correct answer constructing two different functions. There were 21 answers to the first function ($f_1(x) = 1$). But, for the second one, the teachers gave the same function, written differently. For example, 12 teachers wrote $f_2(x) = sen^2x + cos^2x$.

Summarizing, teachers' favorite definitione of function were those given in terms of "Rule of correspondence" and that of "ordered pairs". It is surprising they rejected the definition in terms of relation between variables, which it seems is close to their mental representation of function.

These results show, firstly, that most teachere did not construct a function using mcre then one rule, and secondly, that most teachers did not consider discontinuous functions.

We can see that teachers "could not breke the function". That is, they could have broken tha function if they have constructed a continuous or discontinuous function using two algebraic expressions. It seems that in the case of the history of the concept of function, mathematicians like Bernoulli or Euler had a mental representetion of the function concept related to continuous functions and to analytic expressions (or formula). It seems thet our teachers heve en internel system representetion, like Bernoulli end Euler hed, linked to continuous functions exclusively constructed by one enelytic expression.

It seeme thet when fecing e given complex tesk, there is en orgenizetion of knowledge end possibly the intuitive ideas or obstecles emerge et thet very moment producing e correct or incorrect enswer. Teechers fevored not en isoleted internel representetion but one kind of system of internel representetion.

Are these teachers inducing the students to construct e similar intarnal representation?

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PERSISTENCE AND TRANSFORMATION OF A STUDENT'S ALTERNATIVE STRATEGY IN THE DOMAIN OF LINEAR FUNCTIONS

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Summary

The goals of this case study are to describe the alternative strategy a student used for determining the y-intercept of a line or an equation, show how this strategy persisted when the student was confronted with the standard strategy, and show how the strategy was revised rather than erased, as learning proceeded. This student, supported by tutoring and a computer environment for graphing equations, generated a "counting horizontally" strategy for finding the y-intercept of a linear function. We examine the evolution of this strategy and show how it co-existed with the standard "counting vertically" strategy.

Introduction

Research on student conceptions and strategies in mathematics has documented particular student ideas and described how they are at variance with expert ideas. However, it has not resolved crucial questions regarding the nature and transformation of student conceptions or strategies (Smith, diSessa, & Roschelle, in press). Analyses of student conceptions which describe errors and misconceptions have focused largely on the "mis-" aspect of student ideas and have supported the perspective, either explicitly or implicitly, that learning occurs as these misconceptions are erased and replaced with "correct" or "expert" knowledge (Moschkovich, 1992a; Smith, 1992). If we take the constructivist perspective seriously, however, some aspects of student conceptions and strategies must be reasonable, applicable in some contexts, or refineable. Some alternative strategies which at first glance, seem to be simple mistakes or the result of misconceptions may, in effect, be useful or have the potential to be revised.

This study provides evidence supporting two claims about the nature and transformation of student-generated strategies and conceptions. The first claim is that students generate conceptions and strategies because these are useful and applicable even if only in limited contexts (Smith, diSessa, & Roschelle, in press). The second claim is that students do not necessarily discard these strategies and replace them with the correct ones as learning occurs (Moschkovich, 1992b; Smith, 1992). This study shows an instance of a strategy that is applicable and useful in some problem contexts but not others, presents an example of the co-existence of alternative and standard strategies, and describes how one student revised an alternative strategy.

*This paper was a collaborative effort. Authors appear in alphabetical order.

In this discussion, we define a strategy as the set of actions taken to solve a particula: problem and a conception as the set of connected and coherent ideas underlying a specific strategy (or several related strategies). A description of a strategy includes the problem context in which a student applies the set of actions. We assume that strategies arise from underlying conceptions, that conceptions underlie strategies, and that both conceptions and strategies are integrally related to and dependent on the problem context and the social context in which a student is working.

As an illustration of a strategy, consider the problem "Given the graph of the line y = x find the graph of the line y = x + 4." In order to solve this problem, one could use the standard "counting vertically" strategy which entails: a) starting at the origin; b) counting 4 units up on the y-axis; c) identifying the final point (0,4) as the place where the graph of the second equation will cross the y-axis. On the other hand, since in this case the slope is 1, one could also use a "counting horizontally," rather than vertically, strategy as follows: a) selecting a starting point on the line y = x; b) counting 4 units horizontally to point P; c) drawing a line through point P parallel to y = x.

Strategies, like other student actions and activities, are context dependent and reflect underlying conceptions. For example, the values of the slope of each line in the problem presented above are part of the problem context. Likewise, the presence or absence of a grid (or the line y = x) is also part of the problem context, since a student may use a strategy that relies on the grid (or the line y = x). In this case, this student's use of a "counting horizontally" strategy reflects the conception that the graph of y = x + b was a *horizontal* translation of *b* units of the graph of y = x.

Methods

In this study, we examined six hours of videotape of an eighth grade student (AK) and a tutor (CK) working through a curriculum dealing with linear functions and their graphs, over a period of six weeks. This student/tutor pair was the first of four pairs in a larger study of tutoring and learning conducted by the Functions Group in its lab at U. C. Berkeley. The curriculum was designed to familiarize students with the domain of linear functions as represented by equations, tables, and graphs.* The tasks were framed by an investigation of slope and y-intercept in the y = mx + b form of a linear equation. AK had continual access to in-house software called GRAPHER (Schoenfeld, 1990), which allowed him to manipulate algebraic and graphical representations of functions.

*This curriculum was developed by A. Arcavi, M. Gamoran, J. Moschkovich, and C. Yang and it drew on materials designed by various people in the Functions Group at U C Berkeley including S. Magidson.

An entire unit of the five unit curriculum was designed to support the discovery of the standard "counting vertically" strategy for locating y-intercepts. However, the curriculum was informal and flexible enough to allow students to make conjectures and construct alternate strategies. The tutor did little didactic teaching; instead, she probed the student's understanding, occasionally intervening with leading questions aimed at directing the student's attention to contradictions in his work (Schoenfeld, Gamoran, Kessel, Leonard, Orbach & Arcavi, in press). In our analysis, we focused on the segments in which y-intercept played a prominent role, tracing the change in AK's strategies through selected y-intercept episodes. This data was analyzed microgenetically in the spirit of Schoenfeld, Smith, and Arcavi (in press) and Moschkovich, Arcavi, and Schoenfeld (in press).

Results and discussion

In the following sections we describe the alternative strategy for determining the y-intercept developed by AK. Using examples from the videotape transcripts we show how this student-generated strategy arose and persisted. We believe the alternative strategy persisted despite its limited applicability because it was useful and made sense to the student. We also describe how the student refined and revised his alternative strategy. He refined the "counting horizontally" strategy by specifying the contexts in which it was or was not applicable and he revised it by comparing it to another strategy. In particular, AK did not abandon his alternative strategy simply as a result of the curriculum's or the tutor's emphasis on the standard strategy. Thus, the alternative strategy was not merely replaced by the standard strategy; instead these different ways of determining and using the y-intercept co-existed.

The dominant strategy developed by AK involved "counting horizontally." This is in contrast to the standard "counting vertically" strategy. While the standard strategy was the one presented and emphasized in the curriculum, AK persisted in using his alternative "counting horizontally" strategy, even though it was applicable only in limited contexts when $m = 1$. AK used several variations of the "counting horizontally" strategy described above. He started counting: a) from the end point of the segment of a line of the form $y = mx$ that appeared on the grid, b) from another point on that segment in Quadrant I, or c) from the origin. We will refer to all of these variations as one strategy under the label "counting horizontally."

Emergence of the "counting horizontally" strategy

The student developed a "counting horizontally" strategy to determine the y-intercepts of linear functions near the end of the second tutoring session. Asked to graph several equations of the form of $y = x + b$, including $y =$

x and $y = x + 4$, AK first drew the line $y = x$. Then, beginning at (10,10), he counted horizontally four units to the left of the line $y = x$. He then drew a line through (4,8) which was parallel to the line $y = x$, thus correctly graphing $y = x + 4$. In later problems he modified this by beginning to count from a point on the line $y = x$ rather than from the corner of the grid.

Several factors such as the tutor's style, the sequence and focus of the curriculum, and the presence of a grid on the page contributed to AK's construction of this strategy. The pedagogical stance of the tutor provided AK the freedom to generate this strategy. Instead of presenting the standard "counting vertically" strategy through explicit instruction, she allowed AK to construct a solution on his own. Moreover, the tutor did not reject or correct AK's initial "counting horizontally" strategy. The curriculum's sequence of tasks also supported the construction of this alternative strategy through the frequent use of the reference line $y = x$ and through problems focusing on the effect of changing y-intercepts. As a result, AK typically described lines as variations of the reference line $y = x$.

Finally, the available material resources such as the reference line $y = x$, the 10 x 10 grid, and the grid points facilitated AK's construction of this strategy (Stevens, 1991 and 1992). When using the 10 x 10 grid, AK described the effect of changes in y-intercepts as moving the lines diagonally from "the upper left to the lower right" and focused on the edges of the grid, rather than the x- and y-axes, as the reference objects for describing line movement. In one instance during this problem, AK started at (10,10) and counted the grid points along the edge of the grid to determine the horizontal distance between $y = x$ and $y = x + 4$, "it's going to be 4 away." Since the horizontal distance between lines with slope 1 equals the vertical distance, this method works for generating the correct y-intercepts whenever $m = 1$. The tutor's pedagogy, the curriculum, and the material resources all contributed to AK's construction of this successful alternative strategy for finding y-intercepts.

"Counting vertically" or standard strategy

This was the first task in which AK was asked to give the equation of a line with a slope different from 1 and a y-intercept different from 0. AK was asked to find the equation of a graphed line ($y = 2x + 6$) given both its graph, and the graph and equation of $y = 2x$. AK generated the equation y equals eight x after noting that the graph of $y = 2x + 6$ passes through (1,8). The tutor then asked him if that equation worked for other points on the line, and he said "no."

Returning to his horizontal strategy, AK then drew the reference line $y = x$ and counted eight dots near the top of the grid to announce "It's y equals eight plus x . . ." The tutor again reinforced her prior assertion that the

solution must "work for all of these points." The tutor then focused AK's attention on the vertical distance between the lines. AK counted six dots and said "y equals six times x?" When the tutor told him to consider the graph of y = 6x, AK switched to "y equals x plus six?" AK then graphed the equation y = x + 6 on the computer and noticed that the "slant" of the line was wrong. Finally, AK changed the "slant" to 2 and specified the effect of the number 2 as corresponding to the "slant" or "angle" of the line and the effect of the number 6 as making the line go "up and down." In summary, during this problem AK, with guidance from the tutor and feedback from the computer, articulated the standard "counting vertically" strategy for finding y-intercepts.

Persistence and transformation

AK returned to and elaborated on the "counting horizontally" strategy during a later problem in the fourth tutoring session. He extended the problem contexts in which he used this strategy and explicitly considered the contexts in which it was applicable. AK had initially used the "counting horizontally" strategy by starting to count from the the endpoint of the line segment of y = x drawn on the grid, the point (10,10). During this problem he extended the strategy to include counting horizontally from other points on the line segment y = x besides the end point. AK was working on generating several lines from given equations. After successfully employing the "counting horizontally" strategy to graph y = x + 3, he then also graphed the equation y = x + 7 by counting seven units to the left of the line y = x. He then noted that the horizontal distance between the line y = 3x and the line y = x was the same at several points along the two lines, including the distance between the origin and the x-intercept of the second line.

AK then considered the application conditions of the "counting horizontally" strategy. Recognizing the importance of the line's slope, he said, "Since I know it's on the same slant, if it were on a different slant, it [the "counting horizontally" strategy] would be harder," beginning an examination that he would continue in the next problem. Finally, the two strategies co-existed as he also used the standard strategy, saying, "I've got this other way to figure it out" and counting three dots up from the origin to attain the same solution. Thus, AK's "counting horizontally" strategy evolved as he came to recognize both the scope and the limitations of the contexts in which it was applicable. At the same time, the alternative strategy was not erased by the introduction and use of the standard strategy; instead it continued to co-exist along with its vertical counterpart.

Comparison to the standard strategy

In the next problem AK compared the "counting horizontally" and "counting vertically" strategies, evaluating how easy each was to use, how each made sense, and the contexts in which each was applicable. To find the equation of the graph y = x + 5 he first successfully applied the "counting horizontally" strategy and then also used the "counting vertically" strategy. He then noted that the initial horizontal strategy was "a little bit harder way to count," an assertion he reiterated two times. In addition, he considered how each strategy made sense. Although he initially argued that the "counting horizontally" strategy is "an easier way to think about it," he concluded that the "counting vertically" strategy made sense as well since "they're five apart, too."

Even after the standard strategy had been introduced by the tutor, AK found a problem context in which his strategy was more useful than the standard strategy. AK compared the two strategies for finding the equation of a line he traced on the computer screen, approximately y = x + 15. AK noticed that counting up the y-axis didn't work well on the 10 x 10 grid, since (0,15) was off the graph, but he could easily apply his "counting horizontally" strategy: "You can't do that [use the "counting vertically" strategy] if it's way up here." Thus, AK compared the two strategies in terms of the contexts in which they are useful and found an instance in which the "counting horizontally" strategy was more useful than the standard one. He also argued that the "counting vertically" strategy was easier to use but that it was not useful or applicable in all situations.

Refinement and revision

In the following problem AK refined and transformed the "counting horizontally" strategy by specifying the contexts in which it was or was not applicable. In particular, AK did not abandon his alternative strategy simply as a result of the curriculum's emphasis on the standard strategy. Even when he encountered limitations to the strategy, AK responded by refining his "counting horizontally" strategy rather than replacing it. For example in one problem where m = 2, AK graphed the equation y = 2x + 7 by counting 3 1/2 units to the left of y = 2x, moving horizontally in half-unit increments. This corresponds to viewing y = 2x + 7 as y = 2(x + 3.5) and generating the graph of y = 2x + 7 by shifting the graph of y = 2x horizontally 3.5 units instead of vertically 7 units.

After graphing several lines on the computer, AK noticed that "anything [any linear function whose y-intercept is 0] will always go through the origin . . . like 2x, 5x, 10, anything without plus or minus." AK then related this information to his horizontal strategy, "you know I can't do it that way ["counting horizontally" strategy] because it's a different slant." He elaborated, "it doesn't work this way ["counting horizontally" strategy] because it only goes

a half each time [comparing y = 2x and y = 2x + 7]." He also said that his "counting vertically" strategy always worked even in when the slope is not 1, "you can always get it that way [moves pen up and down, "counting vertically" strategy]." Although this episode suggests that AK might have abandoned his "counting horizontally" strategy by this point in the tutoring sessions, this was not in fact the case. On the contrary, AK continued to use both strategies in several subsequent problems and while playing a computer game at the the end of the five curriculum units.

Conclusions and Implications

This study contributes to current research and theory on misconceptions and conceptual change in several ways. It provides data to support the claim that students generate strategies and conceptions because they are useful. We also show how student-generated strategies can be revised rather than replaced. The examples presented illustrate the power and usefulness of alternative strategies and further our understanding of the nature and transformation of student strategies and conceptions.

There are two implications of this study for classroom instruction. First, it is important for teachers to recognize that standard or textbook strategies are not the only correct ones and that the student-generated strategies can also be mathematically interesting and productive. These alternative strategies need not be treated as "misconceptions" to be rooted out, but rather as sensible constructions to be explored. Secondly, student-generated strategies need to be respected and taken into account during classroom instruction. This study suggests that alternative strategies will not fade simply because students are shown the standard strategies. Students need to be motivated to change their own strategies. The traditional instructional goal has often been to replace student generated strategies with standard ones. In contrast, this study suggest that instruction should assist students in identifying both the power and limitations of their own strategies and comparing their strategies with the standard ones in terms of criteria like generalizability.

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Body Motion and Children's Understanding of Graphs

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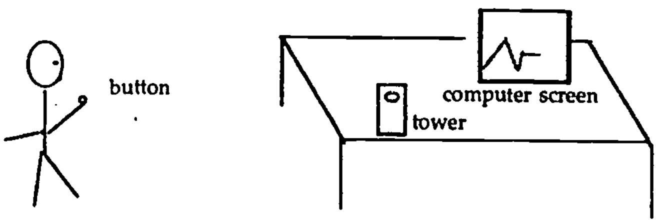
This paper is a partial report of a study on children's understanding of graphical representations for a continuous variation. Three children were interviewed through several sessions working with a motion detector that generates computer graphs in real time. We worked with graphs of position vs. time and velocity vs. time. This paper analyzes the initial encounter of the children with the motion detector set on the distance vs. time graph— what they did, the tasks that they posed to themselves, their language, and the significance of their learning.

As part of the Students' Conceptions of the Mathematics of Change project[1] we are conducting a study on children's understanding of graphical representations for continuous variations. In this particular set of interviews with elementary school students, we are looking at the domain of motion. In this paper, we summarize the activity of three children in their initial work with a motion detector and then discuss three aspects of these interviews that particularly drew our attention. To exemplify these aspects we will use excerpts from an interview with one of the children, Eleanor. In a longer paper, we will include episodes from interviews with other children.

The motion detector we are using consists of one small object—the "button" —whose position is measured, a sensor or "tower", and a computer. The children hold the button or place it on a moving object such as an electric train. For the interviews reported in this paper we have set the interface so that the computer monitor displays in real time a graph of the changing distance between the button and the tower. Beyond a distance of five meters the detector does not measure and the computer graph flattens. The motion detector measures only the distance to the tower.



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The Interviews

The interviews consist primarily of conversations between the interviewer and the child as the child uses the motion detector, during which the interviewer is trying to understand the child's learning. In planning these sessions, we do not know ahead of time what will be rich; they are deliberately open-ended. Although we plan a possible sequence of tasks for the interviews, each interview is a unique event. Thus, it is not "replicable" research. Our goal is to understand how the child is making sense of the situation. We are not limiting our interest to what she knows; we also look at what interests her, how she goes about finding out more, and how firmly she holds her beliefs. We ask the children to talk aloud about what they are thinking. The interviewer responds naturally with expressions of interest and also intervenes at times with restatements, questions, or to focus the child's attention on aspects of the situation or on the child's own statements or actions. The interviewer cannot learn what the child thinks without affecting how she is thinking.

We videotape the interviews. We meet as a group to view them together. The questions we pursue when we analyze the videotapes arise out of the data we observe. The question for this paper is simply: *What is the nature of what children do when first working with a motion detector?*

Summaries of Three Interviews

Joe, age 8, Dina, age 9, and Eleanor, age 10 first worked with the motion detector in the second of a series of hour-long individual motion interviews with the interviewer, Tracey. In the first interview they had drawn representations to instruct Tracey to move a hand-held toy truck and an electric train in ways each one established. All three children came to include instructions whether to go fast or slow, where to start or stop, and when to change direction.

In their second sessions described here, the children began working with the motion detector. The interviewer posed no particular tasks to Eleanor and Dina, encouraging them to take charge of the investigation. Because Joe had become engrossed primarily in controlling the motion of the train, she posed a task to direct his attention back to the motion detector .

All three children began by investigating what motions elicited a response from the motion detector. Each of them soon concluded that the line on the screen gets higher when you move away from the tower and lower when you move toward it. They then made use of their findings to explore something of interest to them.

Joe begins his investigation by moving his arm with the button near the computer, up and down, around in a circle, left and right. *I see. If I move it backwards or farther away, it goes higher*. He moves way back, and considers the effect of getting out of range and of turning the button away from the tower. Then he asks if he can put the button on the electric train. *It would be pretty cool to have the button on the train*. With the train he investigates how to make a

horizontal line as near as possible to the top of the computer screen, but the button goes out of range. *The computer knows that it is going to go too high and go out of the picture.* He solves this by shortening the train track and then marking the place on the track where the button on the train gets out of range. *Once it gets to here it just goes straight.* (Once the train gets to here, the graph goes horizontal.)

Dina spends little time figuring out how the motion detector works. From the beginning she moves mostly away from and toward the sensor, making high and low lines on the screen. She began the interviews with the idea that high lines on a graph show faster speed. She had developed this idea during her first session making graphs for Tracey to interpret. In working with the motion detector, she puts together her assumption that high on the graph means "fast" with her experience that further away makes a higher line. *That would kind of be at a low speed. ... because I think that low is slow.* When Tracey asks what could you do to make it even "faster", she responds, *Get further.*

From the beginning and throughout this process, Dina evaluates the overall graphs aesthetically. In comparing her first two graphs she says *the other one is more attracting because of all the bumps..* When one of the designs looks to her like an "N" she begins to see the designs as pictures. She decides to draw different pictures and letters. Through planning how to draw a box, she recognizes the constraint *Can you get this to go back?* (to the left)... *I don't think you can.* She takes account of that constraint in planning other drawings. She recognizes that she can make an "N" and an "M" but not an "a" *(it would only be a bump)* nor a "T" *(it would be like an upside down L).*

Eleanor approaches the motion detector systematically, posing questions about certain patterns on the screen that result from her motions. She works almost completely independently of Tracey, watching the computer screen as she investigates moving with the button in various ways. Once she has established the parameters of the motion detector, she begins to explore. *I'm going to try and make a pattern.* She envisions an image that she tries to draw and in the process asks and answers more questions about the way the motion detector works. She finds that she cannot make the graph go backwards (right to left) or draw a vertical line.

Perspective-Taking

Eleanor's exploration with the motion detector can be characterized as a process of perspective-taking. Piaget and Inhelder (1967) initiated a series of studies about children's understanding of how "others" perceive an object from a different viewpoint than their own. Their work was further developed and broadened by many researchers. Flavell (1990) distinguished between two "levels" in perspective-taking. In the first one the child is aware

that the "other" sees the object or the landscape differently from how she does, but she is not able to describe or articulate what is the other seeing. In the second level the child is able to specify how the object looks from the viewpoint of the other. In our study, Eleanor is not dealing with an object but with her own body actions within a certain space (walking, running, moving her arms, etc.) and the "other" is the motion detector. The motion detector "sees" certain aspects of the body actions and not others. It shows what it sees through a graphical representation that is responsive only to selected features. By exploring the graphical responsiveness of the motion detector, Eleanor discriminated what aspects of her motions the motion detector senses, among all that are possible to "see". Eleanor's learning embedded a transition from the first to the second level of perspective-taking, from awareness of differences to their articulation. She constructed a perspective from which she could anticipate the motion detector's way of representing her motion.

As with any other process of perspective construction, Eleanor not only discriminated what aspects of her body motion were noticed by the motion detector, but also how they were expressed in the computer graphs as part of a whole system with its own consistency and inner constraints. We identified three different threads woven into her perspective construction: (1) Interplay between seeing the computer graph as a response to her motions and seeing the computer graph as a pattern, (2) Incorporating the inner constraints of the tool, and (3) Using the qualities of one domain (graph or body motion) to describe the other one.

The Computer Graph as a Response to Actions and as a Pattern

Eleanor's use of the motion detector involved frequent shifts of focus. One focus was on the relationship of her body actions and spatial relationships in the room to the graph (*So if you get closer to it, it goes down low*); another focus was on the graph in relationship to a pattern that she envisioned (for example a regular wave). These shifts marked a change of attitude; from trying to figure out how the tool responded to her movements, to doing something with it and analyzing the graph's congruence to the pattern she had imagined.

Initially Eleanor explored the relationship between her actions and the computer graph:

> Eleanor begins her investigation with the motion detector by setting a sequence of tasks for herself. *Let me move it farther away.* She walks backward away from the tower. The line goes up and then flattens. *Maybe this is the farthest it can go. What if I move it up higher?* She reaches up high and the line remains at the same height. She walks forward toward the tower.. The *closer to the tower it gets, the lower, I think.* She walks backward away from the tower. *And I think it gets... It gets higher until this line.* She walks in and points to the screen where the line has flattened. *I don't think it makes any difference if you go like that*. Standing near the tower, she holds the button way up in the air and then way down to the floor. She asks to move the tower because perhaps the tower cannot see the

button through the table: *Because it might not be able to see it if I put it down here.. Maybe it can't see the button under the table.*

Next Eleanor shifted to analyzing the graph as a pattern—how parts of it differ from each other and how the whole differs from the pattern that she had intended:

> *OK, I'm going to try and make a pattern.* She alternately walks forward and reaches out her hand toward the tower and then walks back and pulls her hand away from the tower. After a few in and out motions she doesn't back up as far each time. The graph is a series of sharp zigzags diminishing in height as it moves from left to right. As she moves back and forth, Eleanor watches the graph forming. When she sees that the zigzags are diminishing, she comments: *Actually this is not exactly the same pattern.* Eleanor goes a shorter and shorter distance away from the tower. When the graph finishes, Eleanor and Tracey comment on the wavy pattern.... *(T: And the whole thing has kind of a shape, too, doesn't it?). Yeah, it's all like zig-zags through that side, but I mean they're all...* She does a zigzag motion with her finger. *They look like, kind of like mountains or something. At first I was going to have it stay on this line,* (tracing her finger on a horizontal line from the top of the highest zigzags) *but they got... hey kept on getting smaller.*

Tracey then posed a question that moved Eleanor to shift the focus back to the graph as it related to her body actions:

> *(Why did they get smaller?) Because I... I didn't walk as far. Maybe I can mark where I walked to, and then like... (T: Sure) E: Maybe I can like just put this like right here.* She places a little note pad on the floor. To make the next graph Eleanor walks six times at a regular pace up to the tower and back to step on the pad on the floor. The seventh time she steps back beyond the pad.

Again, Eleanor analyzed the graph for its consistency within itself and compared it with her plan:

> *That one* (peak) *went up a little too high, but. . .* (Touching each of the peaks on the graph)*That one* (the overall zigzag pattern)*was kind of more the same, but a little bit different.*

Incorporating the Inner Constraints of the Tool

Some aspects of Eleanor's learning were not about the relationships of the graphs to her actions or to the patterns that she envisioned, but about discovering and taking into account constraints of the motion detector's methods of displaying information. Eleanor uncovered constraints of the motion detector when she thought of a pattern that it could not make. She reflected both on differences among the possible and impossible patterns and on the relationships between the actions and the patterns. She imagined what patterns she would

like to make, compared them to what she was able to make, and then went back to analyzing the actions.

Eleanor wondered about the impossibility of the computer graph showing a vertical line:

> *Let's see* (running her fingers straight up and down on the screen). *I wonder if you could get it to go straight up? Not like diagonal* (tracing her finger over a slanted line on the graph). *Probably you couldn't because if it would go straight up it would have to just be the same time, because it's moving along* (running her fingers from left to right across the screen) *no matter what you do. (T: Moving along in time?) So you'd have to kind of stop the time and go like that* (tracing straight up). *And go like this* (moving as if to rush away from the tower). *Because, because it's moving along this way* (to the right) *the same time it's going that way* (straight up). *(T: Do you think you can make a steeper line than this?) Maybe, maybe if you do it faster. (T: OK, shall we try that?)* Eleanor starts by running a short distance toward the tower and back and then stands still moving her arm quickly forward and back. *That's almost straight up.... Even though you can't make it go like straight up, you can get pretty close if you do it faster.*

Using the Qualities of One Domain to Refer to the Other One

As Eleanor began to anticipate and explain the computer graphs, her talk and gestures combined in complex ways the qualities of the graph with her body actions. At times she described each with the qualities of the other.

This language manifests her fusion of the action with the graph:

> *(T: What does this pointy thing mean?)Well, the point is where I like went the other way. Because I went towards it going down* (tracing a finger along a downward sloping line on a zigzag) *and then away from it going up* (tracing a finger along an upward sloping line on a zigzag).

She ascribes the graph's motion ("go down low") to the person:

> *At the bottom, I think if you go down too low, you will like—it gets a flat.*

She ascribes a person's motion ("forwards") to a motion of the line on the graph:

> Tracey points to one of the upward slanting lines. (T: So the line up was when you were walking how?) *When I was walking backwards. And the line forwards was that way* (pointing at a line on the screen with downward slope).

As the session progressed Eleanor's talk reflected more and more meanings for certain words. In the rich conversation with Tracey they both used context to sort out the ambiguities.

Discussion

We understood Eleanor's initial interaction with the motion detector as a process of perspective construction; she tried out different kinds of body motions and compared the tool's response to her expectations. As a result, she developed ways to plan and interpret graphs that were consistent with the ones produced by the motion detector. Within this process of perspective construction we identified three aspects that she enacted: constant shifts between seeing the computer graph as a response to actions and seeing the computer graph as a pattern, incorporating the inner constraints of the tool, and fusing the qualities of the graph with the qualities of the body actions.

Understanding the nature of what children do when first working with a motion detector has ramifications for how we think about how children make sense of graphs. Eleanor, as well as the other children that we have interviewed, showed us how graph learning can be a richer experience when explored on the basis of dynamic responsiveness to body actions. Given the complexity inherent in body motion, it is remarkable that the motion detector enabled children to develop a consistent focus on particular elements of their body motion. This was especially true because they held the button in their hands and got feedback in real time. This tool also created an opportunity for children to take a qualitative approach to graphing as opposed to more traditional point-to-point process. By focusing on the general shape of the whole graph or on particular elements within the graph, children became invested in issues of scaling. Their sense of scale grew out of exploring what "out of range" meant in the graph and the room, as well as using landmarks connecting positions in the room with locations on the graph. Beyond the particularities of using the motion detector we believe that this work informs the place of qualitative graphing in the elementary school curriculum.

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Language and Mathematics

MAKING SENSE OF GRAPHS AND EQUATIONS DURING PEER DISCUSSIONS: STUDENTS' DESCRIPTIVE LANGUAGE USE

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This study examined students' use of descriptive language during exploration of the domain of linear equations and their graphs. The analysis of the videotaped peer discussions focused on the differences between the everyday and the mathematics registers, the negotiation of meaning for descriptive terms, and the transformation of students' language use. This paper summarizes selected results from the study describing how three pairs of students used relational terms and phrases to describe the translation and rotation of lines.

Introduction

This study explored the process of learning mathematics through peer discussions by focusing on students' use of descriptive language. Through their discussions and computer exploration of graphs and equations the students negotiated the meaning for terms such as "steep," "steeper," "less steep," "moves up," and "moves down" and the choice of reference objects for describing line movement. While their use of descriptive language was initially ambiguous and reflected the everyday meaning of terms, five of the six students came to use these terms in a manner more consistent with the mathematics register.

The construction of mathematical knowledge was examined in light of two themes in current theory and research-learning through social interaction and the relationship between language and learning mathematics. The study began with the assumption that knowledge is socially constructed through interactions with other people and that this construction is mediated by language (Vygotsky, 1978, 1987). Following Solomon (1989), I view competence as knowing how to act in specific situations involving lines and their equations, including knowing how to use language. While Vygotskian theory and research have emphasized adult guidance and the mechanism of internalization in the teaching/learning process (Vygotsky, 1978,1981; Wertsch, 1981, 1984, 1985), investigators have stressed the need to include an analysis of peer work as well (Forman and Cazden, 1985; Forman and McPhail, 1989).

While interaction with peers can be considered one possible context for supporting learning, the details of how peer discussions function as a context for developing conceptual knowledge remain largely unspecified. Researchers have begun to address how conversations between peers might support conceptual learning in mathematics (Forman, in press) and to explore the relationship between language and learning mathematics (Cocking and Mestre, 1988; Durkin and Shire, 1991; Pimm, 1987; Richards, 1991). Many of these studies have

focused on one aspect of mathematical discourse, the mathematics register[1]. Durkin and Shire (1991), Pimm (1987), and Walkerdine and Sinha (1978) have explored how the vernacular and mathematical registers differ and how discontinuities between these two registers sometimes present difficulties for students. O'Connor (n.d.) has described how students negotiate and define the use of particular terms in the mathematics classrcom. In light of this work, the study focused on the differences between the everyday and mathematical registers for this domain and the negotiation of meaning for relational terms.

Subjects and Methods

The subjects for this study were three pairs of ninth and tenth grade students from an exemplary pilot first-year algebra course. These students participated in videotaped discussion sessions with a peer of their choice. Before volunteering for the discussion sessions these students had participated in two curriculum units, each six weeks long, focusing on linear and quadratic functions. The two chapters included modeling of real world situations, use of graphing calculators and computer software, and student group work with some whole-class discussions.

The discussion sessions were conducted after school in a classroom and lasted from two to six hours over a period of two to four days. In these sessions the students explored slope and intercept using Superplot, graphing software which allows students to graph equations. To structure discussion of different conjectures and predictions, the students followed an instructional sequence similar to the classrooms method for classroom discussions in science (Hatano; 1988; Inagaki, 1981; Inagaki and Hatano, 1977). The problems were designed on the basis of student conceptions suggested in previous research (Moschkovich, 1989; Schoenfeld, Smith, and Arcavi, in press) and in classroom observations (Moschkovich, 1990). The introduction to the discussion sessions included an explanation of terms used in the worksheet (steep, steeper, less steep, origin, move up or down on the y-axis, etc.) using examples. Transcripts of the videotaped discussion sessions for the three case studies were analyzed by coding instances where a student described a line, described its movement, or compared two lines.

Students' Descriptive Language Use

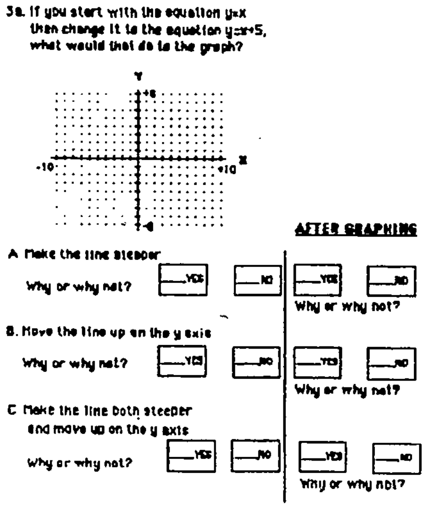
Learning to participate in mathematical discourse and learning to use the mathematical register for a domain is not simply a matter of learning vocabulary definitions. Instead, it involves learning how to use language

[1]A register is "a set of meanings that is appropriate to a particular function of language, together with the words and structures which express these meanings " (Halliday, 1978) A register is a set of words and expressions that have a particular meaning within a particular activity. The mathematics register is the set of meanings, words, and structures appropriate to the practice of mathematics. Some words may overlap between the everyday and the mathematics register, but they have a different meaning

appropriately while solving and discussing problems in different contexts. In the case of linear equations and their graphs, the mathematics register includes more than technical terms such as "slope" and "intercept". Even though neither the discussion problems nor the teachers used these technical terms, the students still negotiated the meaning of the less technical versions used in the problems. The students did not simply learn to use the technical terms slope and y-intercept. Instead, they learned how to describe lines and their movement in a manner more consonant with the mathematics register by refining the everyday meaning of terms, by choosing reference objects to describe the movement of lines, by focusing on vertical translation for the form y=mx+b, and by separating rotation and translation as independent properties of lines.

The language used to describe lines mathematically differs from the everyday language used to describe spatial objects and the relationships between them. As students are encouraged to discuss this mathematical topic and allowed to use their own language, the differences between the two registers become more obvious. For instance, the term "steeper" has a different meaning in everyday language than it does in the mathematics register in that the prototype (Lakoff, 1987) of a steeper hill seems to be a hill which is also higher. Steepness and height do, in effect, co-vary in everyday life; that is, steeper hills will tend to be (or seem to be) higher (or viceversa). This is similar to the way we interchangeably use the words "older" and "bigger" in reference to children, so that these words have to be "renegotiated" in reference to a huge five year old or a small six year old. While in everyday use one is neither careful nor concerned with the separation of the two properties of lines, steepness (m) and location along the y-axis (b), in mathematics discourse these two properties of lines are seen as inherently and crucially independent.

The excerpt below is taken from the discussion of a pair of students who had difficulties with the meaning of the term "steeper" and used it to include translation. This meaning for the term "steeper" was first evident while these students were discussing Problem 3a. In this problem students were asked to predict whether changing the equation y=x to y=x+5 would make the new line steeper or not, and then to decide whether their prediction was right after graphing the two lines on the computer.



The discussion of the meaning of the term "steeper" is most striking after they had graphed the equation y=x+5 on the screen:

HE: (*Reading*) After graphing, is it steeper?
FR: Isn't it steeper? No.
HE: It's not steeper, is it? (*Moves hand to the screen.*)
FR: Are we talking about the same thing?
HE: Yeah . . .
FR: I think it's steeper right here (*points to the y-intercept of the line y=x+5.*) Cause look at it . . . 1,2 and 1,2 (*counting up to 5 on the y-axis and then to 5 on the x-axis, the axes are labeled with a slash every two units.*) This is the same.
HE: (*Mumbles, then places a pen where the line y=x would be.*) It's going up.
FR: So it's no, right? The answer is no . . .
HE: I guess so cause it just moved up (*moving the pen up and parallel to where the line y=x would be.*)
FR: Can you make it deeper....steeper?
HE: (*Demonstrating with the pen.*) Steeper it might go like that (*rotating the pen counterclockwise from the line y=x*) or like that (*moving the pen up to (0,5) and also rotating the pen counterclockwise.*)
FR: Steeper like that (*moves pen below the x-axis so that it takes the place of aline which is steeper than y=x and also has a negative y-intercept.*) This way right?
HE: Steeper is like this (*the pen is in the position of a line steeper than the line y=x*) but more like this (*moves the pen up and down the y-axis*) . . . It's no? Do you agree? Yes, no?
FR: Yeah . . . Are you sure? Do you agree? So this is "YES" also . . .
HE: So we're going up on the y axis . . . and "make the line both steeper and move on the y axis" (*referring to Question C*). . . You don't want it steeper you just want it to move up on the y axis . . . so . . . yes or no? (*Looks at FR.*)
FR: Mhm . . . no.

At first HE said the line on the screen was steeper than the line y=x, while FR was certain that the line was not steeper. When they explicitly began to disuss the meaning of the word "steeper," FR seemed to think that a line is steeper at the y-intercept. When FR asked HE whether one can make a line steeper, HE demonstrated that a "steeper" line can be either a line which is only rotated counterclockwise from the line y=x or a line which is first translated up the y-axis and then rotated. FR explicitly used a line that was translated down to the y-axis to exemplify "steeper," and HE represented a "steeper" line as first a line steeper than y=x but which was also translated up or down. Even while looking at the line y=x+5 on the screen (and apparently knowing where the line y=x would have been located) these two students alternated between using "steeper" to refer to both translation and rotation, and using "steeper" to refer only to rotation. The gestures which accompany their dialogue are especially striking on the videotape, since they leave no room for doubt that they are referring to translation as well as rotation

Students in all three pairs used the term "steeper" to include translation at some point in their discussions and had to negotiate the meaning of this term. One of the students who used the term 'steeper" to include translation explicitly said that she had understood the term to mean "higher." In sum, the data from the peer discussions for these three pairs show that there were discontinuities between the everyday and mathematics registers for terms and phrases such as "steeper," "less steep," "move up," and "move down." The instances where students explicitly discussed and negotiated word meaning involved this set of phrases, which were all used in the discussion problems, as well as terms and phrases generated by the students such as "moves left," "moves right," "moves to the sides," and "through the origin."

Students negotiated two aspects of their descriptions, the meaning of individual terms (as seen in the excerpt above) and the choice of reference objects for describing movement. They negotiated the starting and the ending reference objects for describing where lines started and ended: whether their descriptions used specific reference objects such as the axes, the origin, the line y=x, and salient points on the lines (such as the x- and y-intercepts); or more general references such as up, down, right, left, and side (or sides), which can be ambiguous. Some students used the x- and the y-axis as a reference for describing the steepness of lines. For example, when MA and GI initially disagreed about whether the steepness of a line they had graphed had changed, MA used the x-axis as a reference object to argue that this was the reason why the line was less steep. MA described the line y=x as being "between the x and the y(axes)" and defined the term "less steep" as meaning "closer to the x than

to the y (axes)". While this choice of reference objects to define and compare steepness might seem natural, other students did not arrive at such a useful choice of unequivocal reference objects or terms to describe either rotation or translation. For example, one of the students in the third pair (MJ and DI) initially introduced the terms "left" and "right" to refer to rotation. This choice of words made the discussion problematic and inconclusive, since they also used these terms to refer to translation.

MJ and DI were working on a problem structured like Problem 3a except that in this case the two equations were y=x and y=3x. Although MJ moved her hand clockwise and counterclockwise, she described the movement of the line y=3x as "to the left or to the right" of the line y=x, using the same terms DI would later use to refer to translation, "right or left." DI initially accepted MJ's definition of rotation as "to the left or to the right" but then added that it moved "clockwise" (the line in effect moved counterclockwise). MJ also described the effect of a change in the slope as "it moves more to the left and gets steeper." This is an ambiguous use of the word "left" since it is not clear whether MJ was using left to refer to translation or rotation. Moreover, if she was using "left" to refer to rotation, then she was describing the line as being both translated and rotated[2]. In the subsequent dialogue, DI continued to use "left" to refer to translation while MJ insisted that the line y=3x had moved "left." The ambiguity in these two students use of reference and relational terms continued to plague them throughout their discussion session.

The descriptive language used by these six students during the last few problems reflects increasing conceptual knowledge and is more consistent with the mathematics register: it reflected an increased coordination between the two representations, a separation of the effects of the parameters m and b, and an increased focus on vertical, as opposed to horizontal, translation. Each of these students' use of descriptive terms moved closer to the mathematics register in at least some ways. Of the six students, only MJ continued to use ambiguous terms or reference objects.

Initially the students mixed references to the two representations and their descriptions, reflecting the lack of a coordination between the two representations. They later came to limit their descriptions of lines to graphical terms only, showing an increasing coordination of the way the two representations are connected, and

[2] There are other important aspects of this dialogue which are discussed in more detail in a longer version of this paper. First, while DI used the connection between a change in an equation and a change in a line as a part of her explanations, MJ neither initiated this sort of explanation nor was she convinced by DI's use of this connection between the two representations Second, MJ refered to the y-intercept to describe rotation. Lastly, DI initially focused on horizontal, rather than vertical, translation.

began using phrases such as "if you add, the line goes up" and "if you multiply, the line gets steeper". Five of the six students also coordinated the two representations by coming to drop descriptions not easily connected to the form y=mx+b. These students had initially described lines as moving "left" or "right" as well as "up" and "down". They later focused on the result of changing the b as specifically moving lines up or down. Their later descriptions also omitted mention of movement along the x-axis, reflecting an increasing coordination not only between the algebraic and graphical representations, but also a focus on the specific form of the algebraic representation used in the problems, y=mx+b. Lastly, while students initially conflated rotation and translation in their descriptions, they later used these two movements independently of each other, thus differentiating between and separating rotation and translation as independent properties.

Conclusions

The data collected in the discussion sessions show that there were evident differences between the everyday and the mathematics registers for the meaning and use of relational terms and phrases. Students frequently used the term "steeper" to refer to translation as well as rotation of lines; they also used other terms ambiguously to refer to both translation and rotation. Students negotiated the meaning of individual terms and the choice of reference objects for describing line movement. Through their discussions and computer exploration of lines and their equations they came to describe line movement and compare lines in a manner more consistent with the mathematics register.

There were several aspects of the peer discussions which may have supported the transformation of students' language. Students used their own terms (or the terms used in a problem but with their own meaning) to describe lines, they asked their partner to clarify the meaning of different terms and phrases, and most of the time they negotiated a consensual meaning for terms. One of the benefits of peer discussions might be the occurrence of such negotiations of meaning. The negotiation and transformation of the students' descriptive language was an important aspect of making sense of lines and their equations. The negotiation of meaning that students engaged in, the fact that most of the students arrived at a consensual use of these descriptive terms, and that students' language use moved closer to the mathematics register all show that peer discussions can be a productive context for transforming students' language use.

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Modeling

MATHEMATICAL MODELING: A CASE STUDY IN ECOLOGY

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To better understand the role that modeling and simulation might play in mathematics education, we must first develop an understanding of how modeling and simulation are used by practitioners in the scientific and engineering community. Two initial questions guided this particular investigation: what are the different kinds of models that scientists use and how are these models used. Using a case study methodology, three scientific research teams at Cornell University were interviewed and observed during the fall of 1992. In this paper, we will describe our work with the ecological modeling group and discuss some of the implications of that work for mathematics education.

Introduction

Modeling is a diverse activity with practitioners in all areas of science, engineering and mathematics. Within and across these disciplines, many substantially different kinds of models are used. Two initial questions guided this particular investigation: *what* are the different kinds of models that scientists use and *how* are these models used to describe, predict and interpret phenomena. This case study involved in-depth observation of three groups of modeling practitioners at Cornell University: one in operations research, one in bio-mechanics and the third in ecological modeling. In this paper, we will review the findings for the modeling activities of the theoretical ecology research project in order to understand the goals of the modeling effort, the underlying mathematics, the use of data in the model, the use of visualization, and the conceptions of time and space. We will discuss some of the implications of that work for mathematics education.

Background

This ecological modeling research project is studying the changes in the dynamics of relatively simple systems as the size of the system changes. The research team is interested in theoretical questions of ecology: what is the correct scale to be looking at the problem?

How big do you need your system to be in order to capture the variability in the system? For example, you would need a smaller study area for a grassland than for a forest. But no one really knows just how much smaller. The current study is looking at: What happens in these systems when you increase their size? How do the dynamics change? Prof. Richard Durrett is a mathematician with expertise in the theory of interacting particle systems. Linda Buttel is a research support specialist in ecology with extensive experience in computer modeling. This collaboration is an effort to apply some of the mathematical results from interacting particle systems to biological models of ecological systems.

Methodology and Data Sources

Over the course of two months in the fall of 1992, I conducted a case study of two members of the research team, Buttel and Durrett. As an observer-participant, I met with the research team during several working sessions in which the details of the models were discussed and actually implemented. During several meetings with Buttel, we discussed how their particular model fit into the broader scheme of ecological modeling. All the sessions were audio-taped. Background readings, interviews, computer programs, various graphics output and video were analyzed and a final case study report was written. The transcripts of the audio-tape, the computer programs and the video of the model comprise the primary data sources for the analysis. The quotes in this paper are from the transcripts.

Results

Kinds of Models. Buttel defines two broad approaches to modeling: in the first approach, the researcher models the behavior of groups of individuals. Hence, one would find, for example, average germination rates and average dispersal rates. This type of model is generally defined by systems of differential equations based on the notion that the population at time t+1 is given by the population at time t plus the birth rate times the population at time t minus the death rate times the population at time t plus the number that immigrate in minus the number that emigrate out. This approach is widely used by biologists; most predator-prey models fall into this schema.

In the second approach, however, the researcher looks at the behavior of each individual entity within the system. Buttel describes the difference between the approach

of tracking the individual versus following the groups of individuals as follows:

> One approach is to stand on the bridge and look at all the leaves going underneath the bridge. So stay at one point in time and let everything go by. And the other thing is to get on the leaf and follow the one leaf and see where it goes and what it does. And those are the two different approaches. And to some degree it is driven by the biology and by what you can parameterize and how good your parameterizations are. There is so much variability say in annual plants; what you would do then is pull out a distribution. Sometimes it is better to do things... to average things out because we really don't have enough information to do it any other way.

Buttel suggests several factors that would influence the choice of approach: the way the problem is conceptualized, the ease of computation, the kind of data that is available, and the kind of entities that are being studied. For example, annual plants in a landscape are very similar and therefore it would make sense to use an average distribution. For trees in a forest model, it would make more sense to study the individual trees and their interactions with their neighbors. It is this last notion of studying the interactions with the nearest neighbors that is the key distinction between the two modeling approaches. The models which track the individual can incorporate spatial explicitness in the model. That is to say, where you are located in space does matter. What happens to you as the system evolves depends on where you are located. The differential equation models based on groups of individuals do not incorporate this spatial explicitness.

How the Models are Used. Two particular models were investigated over the course of our meetings. The first was a predator-prey model (Dewdney, 1984) and the second was an epidemic model. Four salient features characterize both these models: the use of data, the incorporation of spatial explicitness, the use of visualization, and the notion of continuous time.

Biological data had almost no role whatsoever. The closest use of any data was with somewhat vague and almost arbitrary use of facts like 'measles lasts 14 days'. This would appear to be a direct consequent of the fact that the goal of the modeling effort is to study the dynamics of the system itself. Buttel describes this as follows:

> Rick's models are much more abstract. They really don't have much biological reality to them. Biological reality can be imposed from above. In other words you can say, this particular kind of mathematical interacting particle system behaves like a forest fire. Or behaves like a competition model between a predator and a prey. But in a very abstract sense. But the nice thing about that is that you don't get bogged down in all the biological detail and you can see very clearly the effects of the spatial interactions.... So one of the things we are addressing is what happens in these simple systems when you increase the size of the grid. Do the dynamics change? We're looking at the behavior without the data.

By taking this approach, they can study fairly complex behavior with a very simple model. As Buttel comments, 'you don't need a complicated model to get at complicated behavior.' This is, in fact, borne out by the implementation of the epidemic model. The code for this model is only several hundred lines long and is elegant in both its structure and its simplicity. The essence of the model is contained in about thirty lines of code. The simple rules of this model actually generate complex behavior.

For both the predator-prey model and the epidemic model, space is considered to be a discrete grid wrapped on a torus. For the epidemic model, each grid space (or site) is thought of as being in one of three states: infected, susceptible or recovered. One can think of a person occupying each site and being in exactly one of these three states. The relationship between these three states is cyclical. A site (or person) can change from being susceptible (to a disease) to being infected to being recovered to being susceptible again. Whether or not a particular site becomes infected depends upon whether or not its neighboring sites are infected. Thus, what happens to an individual depends upon where that individual is located in space and upon the transition rate between the states.

The two aspects of the dynamics of the system that were of greatest interest were its initial starting configuration and the critical values of the transition rates for which the system reached an equilibrium in which the epidemic was sustained. There were three configurations used to start the system: a random distribution of infecteds, a small square of infecteds at the center of the grid, and a rectangle of recovereds with a thin layer of

infecteds on top. For each of these initial configurations, Durrett was interested in the transition rate from recovered to susceptible, the regrowth rate:

> From a random start, what is the smallest regrowth rate so that the epidemic sustains itself for a given size system? The regrowth rate scales like 1 over the log of the size of the system...There is a critical value (for which the epidemic is sustained) of delta (the regrowth rate) that does depend on the details of the model. For example, if I change from four neighbors to eight neighbors then delta is going to go down by a lot.

Durrett's interest is in the equilibrium state of the system in which the epidemic persists. By varying the parameters of the system, you can see the infection running through the system and either dying out or sustaining itself as a second wave of infection runs through. Durrett comments: 'Presumably there is only one interesting equilibrium state which you get into whenever the infection doesn't die out. And that's sort of the point of investigating these things on the computer is that it's very hard to analyze mathematically, but we sort of like to look at pictures to see what happens.' This seems to suggest that for Durrett the 'pictures' of the model are some sort of vehicle for generating or getting at the hard core of the analytical, and hence more real, mathematics.

A final aspect of both models is the treatment of time. Time for these models is treat continuously. To understand how the notion of continuous time is both used and implemented in these models, let us first consider the use of discrete time for the predator-prey model. According to Durrett and Levin, discrete time models are used almost universally in the biology literature (1992, p.1). Let us begin the model with sharks and fish distributed at random over a spatial grid. Let's assume in a discrete time model, that th model is at some time t and we wish to update it to time t+1. To do this, we would start at tl corner of the n by n grid and then proceed systematically through the grid. Suppose, for example, that in the first cell there is a shark and then shark looks to one of his neighboring cells and finds a fish there. You need to save that information in a separate array so that when you proceed to the next time step, the shark can move to that new location and eat the fish. But there is another problem that has to be considered. Suppose the next cell you look at also has a shark in it. Further suppose that that shark looks at the

same neighboring cell as the previous shark did and finds the fish there. Now you have to define a set a rules that governs this kind of collisions. Do you choose the shark who got there first? Then what do you do for the second shark? This can also introduce an artifact of directionality. You have to choose some way to go through your grid. But this has to be done in such a way (along with your collision rules) so as not to introduce any bias of time or direction into the model.

Finally, when you have gone through all of the grid sites, you then update your information and this becomes the state of the system at time t+1. You continue this process through as many time steps as you wish. The implementation of this discrete time model requires extremely careful programming so as to correctly deal with the timing problems (to correctly synchronize events) and with directionality problems. These timing problems can, in fact, become fairly complex. The implementation of this discrete time model generally requires doubling your arrays, so as to keep track of both the state at time t and at time t+1, and in some cases other bookkeeping information as well.

The continuous time model is actually much simpler to analyze as well as to implement. Suppose the n by n grid of your model is at some time t and you wish to update it. You pick a site at random from the n squared sites, and you update that site. So, for example, if you found a shark at that site and that shark looked to one of its nearest neighbors and found a fish there, the shark would immediately move to that site and eat the fish. So when you choose your next site at random, if it happened to be near the site you had just chosen, the second site would find the previous sites already updated in place. After you have chosen n squared sites at random on the grid (most likely some will have been chosen more than once and others not at all), this can be used to define one unit of time. Durrett and Levin compare these two approaches as follows:

> The main difference then is that we update one site at a time rather than all sites at once. In the terminology of the theory of cellular automata we use asynchronous rather than synchronous updating. Asynchronous updating makes it easier to prove theorems since the state of the system changes gradually rather than abruptly. From a modelling point of view asynchronous updating is simpler since we do not need

"collision rules" to decide what should happen when several events try to influence a site at once. (1992, p.16)

Both the predator-prey model and the epidemic model were done using this approach of continuous time.

The four most salient features of this modeling effort seem to be the minimal role of data, the incorporation of spatial explicitness, the use of visualization to gain insight, and the treatment of time as a continuous variable. These characteristics distinguish this work from much of the traditional biological modeling that is currently being done. These particular models are geared toward answering ecological research questions that are both theoretical and abstract.

Concluding Remarks

The mathematics of this model presents itself in two very distinct ways: the elegant simplicity of the algorithms embodied in the code of the epidemic model and the visual presentations of the patterns that were being examined. The computer algorithm had no formal algebraic representation during any of the discussion or background readings. It can better be characterized as a set of rules which governed the interactions between elements in the system. The behavior of the system is then seen in visual patterns which are consequences of the actions of the researchers in choosing parameters and starting configurations for the system. Most of the actual working sessions alternated between changes to the parameters of the model and changes to the code and reflection on the visual display that was generated. The pictures were integral to the modeling process.

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FROM SCHOOL MATHEMATICS TO APPLIED MATHEMATICS : LEARNING TO BUILD MODELS WITH A WORD PROBLEM MIND[1]

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If mathematics is to be integrated with science teaching, modeling is likely to become an important part of the curriculum. The present paper examines the kind of positive and negative transfer which junior high- and high school students spontaneously do from school mathematics to modeling situations. The results show that, when dealing with a situation taken from economics, they make large use of proportional and functional reasoning. Their performances in collecting information and in making predictions were however strongly influenced and sometimes weakened by a tendency to assimilate these tasks to problem solving situations.

A growing number of scientists and educators propose to integrate the teaching of mathematics more closely into science. In such a perspective, teaching students how to build models will become an important common objective for mathematics and science teachers. Learning to correctly perform these tasks such as collecting data, representing them in tables or graphs, using the available informations to predict the values for non explored areas of a phenonenon and eventually to construct equations, should then receive a larger place in the mathematics curriculum than the one it has now. Many researchers have already shown that pupils graphing skills are currently too weak in order to become useful tools to support their reasoning (see Leinhardt, Zaslawsky, & Stein, 1990, for a review). Transforming tables or graphs into equations also remains a difficult task as late as grade 12 (Schw_tz, Dreyfus, & Bruckheimer,

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1990). Much less is known, however, about how pupils spontaneously organize and perform in the collection of information or in making predictions using partial information. A possible reason could be that these tasks are generally done by the teacher or under his/her control. In math classes, when studying functions, data are often provided together with the problem; in science classes, teachers tend to specify on which basis data should be collected ("Write down the temperature of the liquid every minute", "measure the height of the plant every day", etc....).

The present study tries to investigate how junior high and high school students spontaneously construe and perform these tasks of collecting data and drawing predictions (di Sessa, Hammer, Sherin, & Kolpakowski, 1991). What mathematical knowledge will they transfer from current school mathematics to applied mathematics? (Larkin, 1989).

Procedure and subjects

The observations which we will use to discuss these issues come from a laboratory situation in which subjects dealt with a fictitious ground transportation company simulated on a computer. Subjects were asked to explore the company's fare policy in order to make themselves able to later predict the fare for any given distance. Information about the fares were obtained through a public call box, which, for any distance proposed by the subjects, indicated the correponding price. The fares were made of a base rate and two additional rates, one for short distance travel and one for longer distances. The number of authorized calls was deliberately limited in order to prevent testing one mile after the other. The range of attainable distances was suggested by pretenting that the entire simulation took place on an island which was comparable in size to that of Western Switzerland (the region where all the subjects came from). No indication was provided concerning how students should organize and

conduct the collection of data. No direct reference was made to math or science. Graph and plain paper, pencils, rulers, and a pocket calculator were placed next to the computer. Subjects worked in pairs thereby confronting their ideas and negotiating their requests in case of disagreement. They were 40 ninth graders and 36 eleventh graders; they represented all the pupils from four classes, two at each level. These levels were selected because, in Switzerland, they corresponded to year 1 and year 3 in algebra. All subjects had some experience with computers, but none of them had ever worked on simulations nor been taught modeling before.

Results

Not surprisingly, some of the flaws generally noted in problem solving also appeared in the way students approached the information collection task. Insufficient planning and poor use of available informations were frequently observed, especially among the 9th graders. For instance, only 11% of the 9th graders refered to the previously indicated size of Western Switzerland when choosing their probes. In contrast, 53% of the 11th graders made this reference. Despite these weaknesses, 9th graders did rather well in identifying the important parameters of the system.

Table 1: Proportions of pairs of pupils who correctly identified various parameters of the system at each grade level.

Grade level	Base rate	Short distance rate	Long distance rate
9 *(N=20)*	70	90	25
11 *(N=18)*	50	72	22

As one can see in Table 1, ninth graders found the base rate and the short distance rate more often than their older counterparts. They identified the long

distance rate as many times as the 11th graders. Finding these values was obviously the goal which most of the ninth graders and part of the eleventh graders had set to themselves for that task. Some other observations confirm that these subjects tended to construe the task of collecting information as a problem solving situation. They also spent considerable efforts and probes in order to isolate the point at which the rate changes, the point "that contains the enigma" as one subject said. Expecting to end the task with a single answer, as in word problems, they showed enormous embarrassment when they sometimes realized that one could give various answers to a same question. The following quote is an illustration of this embarrassment:

Exp: Does it work or not?
Sandie: No, it doesn't; one gets .4 with some numbers and .25 for some others; it is the computer that has a problem.

To capture their attitude at a global level, we call these subjects the "problem solvers". Some students however did not approach the task in the same way. They tended to distribute their probes more evenly along the scope of what they considered as plausible distances. Few of them even did so in a very systematic way, dividing the size of the island by the number of authorized calls. They generally did not loose trials asking for the fares of two consecutive kilometers. They also did not check the price for fractions of kilometers. We call these subjects the "information samplers". The majority of them are 11th graders.

What are the consequences of such different views of the information collection task? More specifically, is there a difference in the accuracy of the predictions made by "problem solvers'" and by "information samplers"? The answer is yes but the way the predictions differ depends on what it is made

upon. When asked to predict the fare for short distances, or to predict what distance could be covered with less money than the base rate, those who dedicated most of their probes to try and find answers about the base rate and the price per mile made better predictions than the "information samplers" ($t(27)= -3.215$, $p=.003$). "Problem solvers", however, were further off with their predictions in the long distance range ($t(33)=2.89$, $p=.007$). This apparent contradiction becomes easily understandable when one examines upon which basis the predictions were made. The "problem solvers" relied essentially on computation; they tended to derive arithmetically their predictions from a base rate and a price per mile rate. The "samplers", on the contrary, used much more approximations and post-hoc adjustments of what they had computed, in order to take into account a larger number of available data. As a result, the problem solvers are more accurate when, as they say, "the normal rate" applies; the "samplers" surpass them when it does not.

Discussion and conclusion

This research clearly shows that there seems to be no risk that introducing modeling in the math curriculum at the junior or at the high school level will turn students away from doing mathematics. Although we deliberetaly avoided explicit mention of mathematics while presenting the tasks to the students, their performances and their verbalizations are filled with mathematical reasoning and arguments. However, 9th graders and, to a lesser extend 11th graders, seem to have a restricted image of what doing applied mathematics means. Students' practice of solving word problems leads them to construe such tasks as collecting data or making prediction as problem solving situations. As a result, they tend to search for immediate answers and make scarced use of heuristics to make good predictions. This observation is perfectly in line with Schoenfeld's (1985) descriptions of non-experts' problem

solving strategies. It also confirms Larkin's (1989) observation that problem solving strategies, such as to try and set subgoals, clearly belong to the kinds of knowledge that tend to transfer rather easily from domain to domain.

But the present results also should draw our attention to an important issue for any attempt to teach modeling, particularly at the junior high school level. At this age, collecting data and making predictions are not seen as interdependent tasks. Because they construe the collection of data as a problem solving situation in itself, most 9th graders and part of the 11th graders tend to focus their attention on specific parts of the phenomenon while ignoring large parts of it. This proves that they do not see the collection of information as a means to ensure good prediction over the whole range of the observed phenomenon, but only as a goal on its own. Without particular efforts to clearly underline the interdependance of all the tasks included in building models, the whole process of modeling might remain morcellated out in the students' minds. It is our hope that studies like this one could make aware of this risk all the program designers which try to integrate mathematics in the teaching of science.

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PROSPECTIVE SECONDARY MATHEMATICS TEACHERS AND MATHEMATICAL MODELS

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This paper reports the mathematical modelling portion of a larger study (Zbiek, 1992). It discusses the understandings of mathematical modelling displayed by 13 prospective secondary school mathematics teachers, all of whom completed at least four university mathematics courses beyond calculus. Its findings include three grounded hypotheses regarding how the subjects developed, evaluated, and used functions as mathematical models in the presence of curve fitters, graphers, symbolic manipulators, and other computing tools in a four-month mathematics-education mathematics course.

How do prospective secondary school mathematics teachers with substantial background in formal mathematics view the world mathematically? In particular, how do they develop, evaluate, and utilize mathematical models? These questions arise naturally if one expects prospective secondary mathematics teachers to construct their own real-world applications of mathematics rather than to memorize and recall previously seen applications.

Background of the Study

Prospective teachers most likely have prior experiences involving links between their perceptions of the world with their knowledge of formal mathematics. These experiences are often direct applications of mathematical algorithms to compute one or two values given well-defined tasks within limited real-world settings, as in traditional word problems. More open-ended mathematical modelling tasks increase the uncertainty of prospective teachers as they mathematize situations. Computing tools (e.g., curve fitters) alleviate computational burden and allow less contrived real-world problems. However, key components of the modelling process, including selecting variables, choosing the mathematical form of a model, and evaluating the model's relevance, remain crucial issues. As researchers and educators exploring and facilitating these learners' sense-making processes, we ask, What are the struggles that prospective secondary school mathematics teachers encounter when they attempt to construct, validate, and use mathematical models in the presence of computing tools?

Studies in the spirit of the Students-and-Professors problem (e.g., Clement, 1982) suggest that college students, including students with mathematics-intensive majors, have difficulty developing simple mathematical models from verbal descriptions of relationships

found in everyday experiences. The more extensive mathematics backgrounds of prospective secondary mathematics teachers may (or may not) enhance their ability to develop models.

One study directly considered open-ended mathematical modelling activities with pre-service mathematics teachers (Trelinski, 1983). Trelinski analyzed the written work of 215 subjects to assess the degree to which graduate mathematics students in Poland preparing to be teachers were ready to introduce mathematical applications to students. She concluded that each subject typically tried only one major solution path and many subjects seemed to follow no mathematically consistent scheme. No subjects presented complete solutions and several omitted relevant variables. They seemingly made assumptions, perhaps unconsciously, about the whole process but failed to use these assumptions in consistent ways. Few constructed formal (symbolic) models. According to Trelinski, at least some subjects knew their models were flawed. She concluded that these prospective teachers did not demonstrate a natural transfer of (supposed) abstract mathematical knowledge to modelling situations.

Others approached issues involving mathematical modelling and secondary school teachers in less direct ways. For example, Binns, Burkhardt, Gillespie, and Swan (1989) constructed modelling modules for teachers to use with their students. The purpose of the construction and trial-based refinement of these modules was to provide inservice teachers with resources and not to study how inservice or preservice teachers themselves cope with mathematical modelling activities. The focus of their work and other studies involving mathematical modelling and teachers was on empowering teachers with the seeming intent to produce changes in their classrooms. The existence of such works alone indicates that there is at least a perceived lack of transfer from formal mathematics not only to the teachers' personal mathematical modelling endeavors but also to their classroom modelling activities.

Prior research thus indicates that college students and teacher certification candidates do not regularly apply relevant mathematical ideas that were supposedly part of their coursework. The evidence however is insufficient in important ways. There was little attempt to determine whether prospective teachers *could* connect their existing mathematical

understandings with modelling tasks. Some of these subjects may actually possess adequate understandings of necessary mathematical ideas but not be able to access them well in open-ended modelling situations–like Wollman's (1983) subjects who did not construct appropriate equations to represent relationships although they could work with similar equations in the abstract and could answer questions about real-world settings of the Students-and-Professors variety. A second missing piece is the examination of what prospective teachers do when faced with more open-ended situations involving familiar real-world happenings. For example, a potential barrier to Trelinski's subjects success was their lack of familiarity with the biological setting of their task. A third issue not addressed in previous studies is the impact of computing tools. Trelinski's subjects may have attacked the problem very differently given access to computational tools for devising functional relationships from hypothetical data.

Data Collection and Analysis

Subjects. The current qualitative study explored the ways in which 13 prospective secondary school mathematics teachers intuitively viewed the world mathematically and the nature of developments in their mathematization processes over a four-month period. Ten subjects reported no prior exposure to mathematical modelling, with three of them confusing it with observational learning. The other three subjects encountered mathematical modelling briefly in a methods course in which they used TI-81 calculators to generate models that represented various quantities as functions of time. Only two of these three subjects defined mathematical modelling, both claiming it was using mathematics to predict future events.

Data sources. The data collection occurred in the context of a university mathematics-education mathematics course for prospective secondary school mathematics teachers in which mathematical modelling was one component and during which the subjects had continual access to computing tools. The subjects engaged in various tasks as part of the course that required them to devise mathematical models in open-ended situations, evaluate given models, and use models to describe real-world situations. Data sources include audio tapes, transcripts, and written work obtained through three individual interviews with each

subject, classroom and computer lab observations by the investigator/instructor, and subjects' paper-and-pencil questionnaires responses, written reflections, computer lab reports, and individual projects and papers.

Tasks. The two other foci of the mathematics course were function and proof. Given this context and the propensity of computing tools for function representation, the subjects almost always used real-valued functions of one or more variables to model the real-world situations. However, the tasks considered within any one of the various data sources differed in several key ways. Occasionally, and early in the course, the investigator gave subjects potential models to use and to evaluate. During the last four weeks of the course subjects were totally responsible for the construction of the models. Tasks also differed in the extent to which the form of the model might be suggested through the subjects' knowledge of physical and social sciences. Models of some situations could be generated through the application of a known formula, such as the perimeter of a rectangle. The form of other models might be verified or predicted by the subjects' knowledge of physical and social science, as in the quadratic nature of trajectories. Other tasks involved situations for which no obvious formula or theory seemed applicable, as in the Transportation Data Scenario from the final interview. For this task, subjects saw data for 50 states and the District of Columbia for each of the eight categories; the eight variables and the data for the state of Alabama appear in Figure 1.

	Variables	*Alabama*
1	The age at which a person may obtain a driver's license	16 years
2	Number of licensed drivers in the state	21 million
3	Number of registered autos, buses, and trucks in state	41 million
4	That state's tax on a gallon of gasoline	13 cents
5	Number of miles traveled annually by vehicles from that state	39,684 million
6	Amount of motor fuel consumed annually for highway travel in that state	2,605 million
7	Amount of motor fuel consumed annually for non-highway travel in state	59 million
8	Information about that state's safety-belt-use law	No law

Figure 1. Transportation Data Scenario variables and sample data (Hoffman, 1991, p 170).

Data analysis. The analysis of data followed the guidelines for development of grounded hypotheses outlined by Glasser and Strauss (1967) and was consistent with criteria

for data analysis and collection suggested by Merriman (1988). Analysis began with coding of the interview data, noting any reference to mathematical modelling activity, followed by the identification of patterns therein. The investigator then returned to the interview transcripts, looking for plausible alternative explanations as well as supporting evidence of these patterns. Treating other data sources similarly, the researcher identified and compared any response or passage that addressed mathematical modelling against the working hypotheses.

Findings and Discussion

In the pages that follow, at least one illustration of subjects' modelling work within the Transportation Data Scenario exemplifies each of three grounded hypotheses that emerged.

> Hypothesis 1: *Prospective secondary school mathematics teachers view the world in complex ways; they verbally describe real-world relationships that involve multiple variables and incorporate composite functional structures. However, the models they actually construct lack the corresponding mathematical sophistication.*

One common form of mathematical sophistication introduced by the subjects as they verbally described interrelationships among several variables was function composition. Mark was one of eight subjects who used function composition in the Transportation Data Scenario:

M: Because [age] would determine how many drivers there are and then [the number of drivers] would determine how many miles you traveled, then [the miles you traveled] would determine the amount of motor fuel you used. ... I mean indirectly because it affects other things that [eventually affect] the fuel consumption. (Mark, Final Interview)

Composition was sometimes mentioned in combination with the introduction of additional variables, the most common feature discussed by all of the subjects. For example, 11 of the 13 subjects noted additional factors such as total amount of fuel consumed in the state or population of the state in the Transportation Data Scenario.

However, the mathematical models subjects actually constructed and used were usually devoid of intervening variables. Mark, for example, then created a linear model of the number of registered vehicles as a function of the number of licensed drivers: F(D) = 1.11D + "10 to 20 percent." Other common traits of the constructed models were dependency on computational relationships (8 of 13 subjects in the Transportation Data Scenario) such as computing numbers

of miles per gallon, and frequent use of cause-effect relationships (7 of 13), especially a belief that increasing the number of licensed drivers increases the number of registered vehicles.

Hypothesis 2: *The processes by which individual prospective secondary mathematics teachers develop and evaluate mathematical models fall into general categories. Key features of the process used by any one of these prospective teachers include the ways in which the subject uses computing tools, the criteria by which the prospective teacher selects among proposed models, and the extent to which the prospective teacher personalizes the situation.*

The most striking characteristic of the subjects' uses of computing tools in both the construction and validation of models was their extremes of dependence on the tools. Seven subjects voluntarily used curve fitters to construct models in the Transportation Data Scenario. Some of these subjects claimed that the highest tool-generated goodness-of-fit measure undoubtedly determines the best of the potential models but the others analyzed how well potential models matched their personal realities. Six subjects however never used any tool capacity beyond simple arithmetic computations. At this extreme was Dorothy who devised a model based on a miles-per-gallon notion, using only the calculator to perform the division. Her written work appears in Figure 2. She argued that her model was adequate because it worked well for the three data points. In contrast, Phyllis rejected all of her models saying there would always other variables to include and so none of her simple models could suffice.

Total Consumption	Results of Dividing by 10		Results of Dividing by 15
2664	M/10	3968.4	M/15 = 39684/15 = 2645.6
total = H + N 271	m/10	384.1	M/15 = 3814/15 = 266
2009	M/10	3.424.7	M/15 = 34247/15 = 2283

Figure 2. Dorothy's written computations leading to $f(m) = \frac{m}{15}$ beginning with total fuel consumed for highway and non-highway travel [headings added].

Hypothesis 3: *Prospective secondary school mathematics teachers answer in various ways questions about phenomena for which they receive or develop models. Many subjects use personal experience or selected data points rather than models in answering the questions. Others use models in some way, noting that the quality of their answers depends on the appropriateness of the models.*

All subjects were skeptical of models they generated, including those that they thought had the greatest potential. For example, each subject answered questions of these three types for whatever relationship that subject chose to model in the Transportation Data Scenario:

Trend: What happens to input values as output values increase?
Interpolation: What is the output value for an input value within range of input data?
Extrapolation: What is the output value for an input value beyond range of input data?

Nine of 13 subjects qualified their answers as being valid only if their models were valid. All subjects successfully devised function models. Yet, although they were confident in their work in producing the models, they seemed to have very little confidence in the relevance of their final products to the real-world situations they supposedly modeled.

Conclusion and Implications

The current study identified several characteristics of the mathematical modelling behavior of prospective secondary school mathematics teachers who had studied formal mathematics at the university level. These future teachers could use tools but did not use them automatically. They could use models to answer questions but doubted their answers. They saw sophisticated relationships in the world but constructed simplistic mathematical models. Mathematical modelling insights, skills, and confidence arose neither naturally nor quickly. In fact, subjects' understandings of the process developed slowly and in widely different ways regardless of similarities and difference in prior mathematical experiences or achievement.

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Number and Proportion

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THE LANGUAGE OF FRACTIONS AS AN ACTIVE VEHICLE FOR CONCEPTS

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We have developed an ample qualitative study centered on syntactic components, the basic meanings and the signification processes that meet in the "language of fractions". This report is referred to one of the instruments used in said research: an exploratory questionnaire applied to 37 pupils of fourth grade. Our purpose has been to recognize how children developed in the several planes forming said language and in the continuos transit from the latter to the verbal language and vice versa. In our follow up on the students' individual performance, special attention was given to elementary situations of the recognition of the numeral, in response to exercises that contained representations of a different nature (among others, pictorial representations). The analysis model adopted to interpret the results did favor the syntactic, semantic and "transit or passage" planes from one language to the other. In addition, we recognized in the semantic plane a preliminary expression space for the concepts built by the children.

During the last few years, the research bound to the *semantics of fractions* and the *rational numbers* has shown a noticeable enrichment. With the designations of *meanings, interpretations* and *constructs* there has been identified several suitable contents to be constructed in connection with such numbers (among the various existing sources we want to highlight those of Kieren, 1988).

In other fields of the arithmetical research, with regard to the *addition and substraction of integers*, studies have been conducted at the several levels forming the *arithmetical language* which were subject to an exhaustive investigation. In this regard, Nesher's (1982) experimental work has allowed to establish a set of variables attached to the *semantic, syntactic* and *logical* language planes, that bear upon the children actual *performance* in the framework of *word problems*. Sastre (1984) with a different approach, was able to recognize certain *connections* and *difficulties* that can be detected among diverse language expression environments introduced by the teaching (that is, among the *verbal language*, the *arithmetical symbols* and the *graphics representations*) that are also connected with the addition and substraction of integers.

On the basis of the fundamental aspects of the above mentioned

research, our work is intended to explore the *components* and *processes* that join in the "language of fractions" previously identified. The character we assign to their development is that of an *active construction* by each user (in which we agree with the approach set forth by Laborde, 1990, with reference to *mathematical language* in general).

We have selected a portion of said exploratory questionnaire to prepare this communication and concentrated in certain elementary written activities that demand the children the *recognition of the fraction*. In order to design such exercises, we considered some of the phenomena we were able to isolate during our previous investigation (Figueras, Filloy y Valdemoros, 1986, 1987). In the follow up of the results, we did give preference the semantic contents detected through the *actual performance* of the children, since at that level of active language development, the concepts set forth by them can be detected; however, this has not implied neglect of the concomitant syntactic constructions nor the "transit or passage" processes from verbal to arithmetical language and vice versa.

METHOD

The said "Exploratory Initial Questionnaire" was applied to pupils of *fourth grade* in a *public grade school* (that shows an average performance level in the national education system). The group was formed by 37 boys and girls with ages between 8 and 11.

The block of exercises oriented to the identification of the fraction was supported on *different meanings of fraction* (specifically the *whole-part relationship, quotient, measure* and *ratio*). In the text of each exercise, were included representations of different nature (particularly, pictorial representations and graphic signs involved in the *verbal and symbolic-arithmetical languages*). The task consisted of the direct and indirect identification of the fraction by the student (that is, through the writing of numerals or the use of "pictorial representations"). The questionnaire, before being applied was pilot-tested with students of different schools and ages; also, it was presented to some specialists in the field of arithmetical research.

The results were submitted to a *qualitative analysis* in which we considered different levels of language consolidation, according to our original proposals (Valdemoros, 1990, 1992):

-In the semantic plane.

-In the syntactic plane.

-In the "passage" processes from the verbal language to the arithmetical language.

SOME RESULTS AND THEIR INTERPRETATIONS

We have selected some typical responses that were frequent in this group of students. Through them, it is possible to recognize some relevant obstacles faced by those children, in the construction of the respective language.

1. IN THE SEMANTIC PLANE

We were able to identify different *semantic contents* unrelated to fractions, that were attributed to those numbers by the students (some of which we had recognized in Figueras, Filloy y Valdemoros, 1986, 1987). For this paper, we have selected some answers that lack such background in our previous studies. We refer to the *rejection of important semantic restrictions* in the context of the sharing tasks.

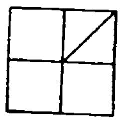
The most general semantic restriction involved in those situations, belongs to the self nature of the arithmetical model attached to any concrete case of sharing; it is not a non-differentiated distribution fashion of certain type of objects among a number of persons, but it *necessarily* requires an *equitable sharing*. This demand introduces an important *restriction of sense* in what is commonly called "daily experience", in order that said *arithmetical model* can be suitably carried out. Frequently, students are familiar with other realsahring experiences; that is one of the reasons why many children do not take into consideration said *general semantic restriction*. This is shown in Figure 1.

With regard to other situations, in which the character of the respective *semantic restriction* is markedly local and specific, we have selected the case of the *pictorial representation* of the sharing of liquids. A sizable number of students made the corresponding subdivision, ignoring the represented substance and by tracing perpendiculars to the base of the drawn containers, as if it were a partition of an area. In this context, the corresponding *semantic restriction* is closely related to the use of the chosen

pictorial representation and will lead to reject subdivision manners such as those shown in Figure 2.

Exercise XII

Five friends intend painting, jointly, a wall like this.



How could they fairly distribute the work to be performed? Show it in the above drawing. Thus, each friend will have to paint 1/5 of the wall.

Figure 1

Exercise XIV

Eight boys take part in a celebration. Show how they can divide these beverages of different flavors so that each receives the same amount.

Write the name of each boy by the part being assigned him. Thus each boy receives one half and 1/8 of the mixed beverages in the party.

Figure 2

II. IN THE SYNTACTIC PLANE.

The aspects of interest in this regard are those belonging to the *concrete ways of articulating the several mathematical signs* and to the *concrete use of the syntactic rules* giving them support. In a less-frequent manner (given the elementary organization of the tasks herein considered), we approach the *combinations of arithmetical signs* involved in a *compound answer*. The fact that we are working on *fraction recognition exercises*, clarifies the reason why we did not detect a wide repertoire of relevant syntactic constructions in this group of exercises.

The answer chosen for this section (shown in Figure 3), refers to the *violation of a tacit rule that regulates the counting and the subsequent pairing of numerator and denominator* in contexts joined to the *part-whole* relationship. This rule states two different and autonomous counting times, linked to the recognition of both symbolic constituents of the fraction. Said problem consists in that the student merges both processes into one, with which the objects that form the part are counted twice in a same sequence.

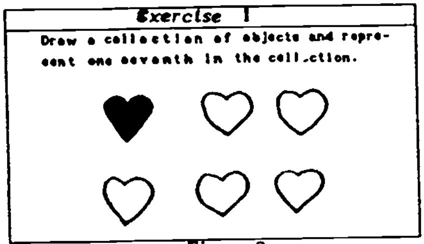


Figure 3

The implicit thought in the just-exampIified answer, is of the type *"There are six objects plus one more, shaded"*, that opens way through its false correspondence with the denominator of the fraction presented in the respective statement.

Contrary to many other anwers recorded in the Exploratory Initial Questionarie (in which the pairing of the numerator and the denominator shows the absence of inclusion of the part within the whole); what is affectec in this one is the manner of such inclusion, which appears by means of an inadequate development of the count.

IN THE PLANE OF "PASSAGE" FROM ONE LANGUAGE TO OTHER.

We selected a case showing the *combined used of different* languages in a single answers, a circunstance that generates a certain degree of ambiguity (as shown in Figure 4).

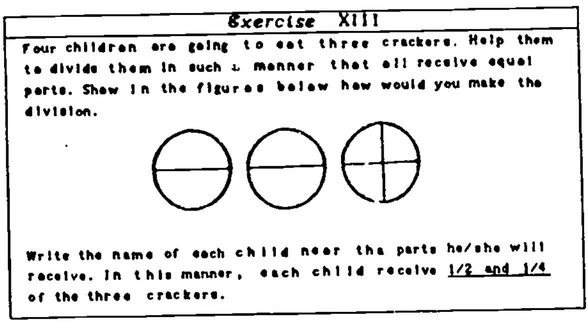


Figure 4

In what is apparent, this answer would have been formulated as an indirect reference to the sum of both fractions. However, the evidences given by another block of arithmetic tasks included in the same questionaire[1] (where many students established expressions similar to those identified in Figure 4 and relating them indiscriminately with the addition and substraction of fractions), make us believe that the approach of the students show in that figure, corresponds to a *juxtaposition of heterocilte signs* to which the students did not yet assign the meaning of the addition of fractions. In these cases, the students do not seem to notice nor clarify such a connection between what they themselves recognize and the involved operation in the established situation.

CONCLUSIONS

The examples with which the recognized problems have been illustrated, do show some of the numerous and subtle difficulties experienced by these students in the incipient constructions of the "language of fractions". Even when the aforesaid fits into a *selection of typical phenomena of a different nature*, they also allow to isolate a few components and specific processes of the semantic and syntactic planes, as well as of the "transit" from the linguistic expresions towards the arithmetical language.

Due to the elementary charscter of the works considered here, the detected syntactic components did not show within this framework a great diversity and remained centered in the required links by the pairing of the numerator and the denominator, in the context of the part-whole relationship. On the semantic ground and connected with certain sharing tasks, we recognized some restrictions of sense that were frecuently omitted by the students, since they were not compared against the "daily and familiar experiences". Lastly, the mixed used of linguistic expressions and arithmetical notations by many students, put us closed to the assumption that such answers did show trends toward the juxtsposition of signs of different origin and that were very far from channeling explicit additive combinations.

1 We refer, specifically to the problems prepared by the children themselves in connection with work instructions of this nature: "Invent a problem that contains 1/4 + 1/2".

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GROUP CASE STUDIES OF SECOND GRADERS INVENTING MULTIDIGIT ADDITION PROCEDURES FOR BASE-TEN BLOCKS AND WRITTEN MARKS

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Small groups of second graders were asked to add horizontally presented 3- and 4-digit numbers using base-ten blocks and written marks recordings of the block procedures. Most children displayed increased conceptual understanding of place value and multidigit addition and demonstrated better written addition competence at the end of the five- to eight-day learning situation. The six groups displayed individual patterns of invention and learning that were dependent upon the personalities and mathematical understandings of the group members. Children easily added with the blocks, devising accurate strategies for multiunit sums of ten or more (e.g., twelve tens, sixteen ones, eleven hundreds). Many children did not spontaneously link the block addition to marks addition, instead operating in two separate worlds. When blocks addition was linked to marks addition, the blocks were a powerful support for conceptual understanding of marks addition. Blocks words were in some cases a more powerful support than were English words, and complete verbalization of trading seemed to be very helpful in facilitating understanding.

This article reports how, through complex intertwinings of personalities and children's different mathematical understandings in each of six groups, unique patterns of group interaction and paths of learning occurred. Six small groups of four or five second graders participated in this study, three during each of two data-gathering sessions. These children were in the second-grade math class that was the top of three in their school. They were assigned to groups that were balanced by gender and homogeneous with respect to conceptual and procedural competence in place value and multidigit addition and subtraction as assessed by pretests. An adult experimenter videotaped and took live notes of each group's meetings and guided their initial experience with base-ten blocks. During problem-solving, experimenters intervened to curtail rowdy behavior or to redirect sustained incorrect mathematical thinking. Children used digit cards during the first data-gathering session, a large paper pad ("magic pad") during the second session, and individual papers during both sessions to show their marks addition. See Fuson, Fraivillig, and Burghardt (1992) for further details of the methodology and early learning in the groups.

The focus of this report is on group case studies of the addition portion of the study. The analysis of children's mathematical interactions relies on the theory of multiunit understanding in Fuson (1990). Personality factors combined with the mathematical strength of individual children to create different group learning paths and different addition procedures with the blocks and the marks. Over half the children had had a first-grade teacher who used the blocks to teach place value but not addition, so the children were quite heterogeneous with respect to initial knowledge of the blocks. Children ranged on the pretest from solving no 2- to 4-digit addition problem correctly (6 children) to solving all problems correctly (4 children); they showed a similar range in place-value knowledge and conceptual explanations for 2-digit and 4-digit trading and alignment of uneven problems. On the posttest and/or in the videotaped data most of the children demonstrated increased conceptual knowledge concerning place value and

multidigit addition and in the ability to do written multidigit addition (four children were at ceiling on the pretest written marks problems but showed increased understanding in some other task).

Official leader and checker roles rotated daily among children in a group. The intent of assigning these roles was to increase equality of participation among the children (Cohen, 1984). Children did respond to the "leader" roles by participating more actively in the groups' problem solving on their "leader" day, but "natural" leaders who led the group on most days also emerged in all groups.

Group 1: First session, high initial knowledge. These children, two girls and two boys, made few errors in adding numbers in written form on the pretest; they did their most interesting work with the blocks in subtraction (to be reported in another paper). During the study, they worked backwards from the written marks to the blocks and took two days to work out all of the details of relating their written procedure to addition with blocks, verbalizing their blocks addition, and showing the written marks procedure with the digit cards. On the first two days they made three vertically-aligned rows of blocks, one for each of the addends and one for the sum. They physically traded in ten of one kind of block for one of the next larger block, which they then put above the blocks of that kind (just as the 1 is written above the next left column in the standard U.S. procedure). The experimenter tried to get them to think of another way to add with the blocks or in written form, but they could not -- at this point they seemed to be too constrained by the standard written procedure. On the third day, however, they solved a problem by adding with the blocks from left to right and did the trading correctly. They set up and solved a 4-digit plus 3-digit problem correctly with the blocks but aligned the 3-digit number on the left with the digit cards and on individual papers. But because they solved the problem with the blocks and recorded the written solution from the blocks, their answer was correct. On their fourth day they were asked to use only the digit cards and to just talk about the blocks. They left-aligned a 4-digit plus 3-digit problem and got an answer that they recognized was too large. They figured out that the 3-digit number only had hundreds and therefore was aligned incorrectly. In response to urging from the experimenter, one girl invented a new digit card procedure in which she used the digit cards as named-value numerals (all numbers were made with extra zeroes to show their value: 2678 was made as 2000600708). This procedure was demonstrated and discussed on the final day of addition. The children agreed that this answer was too large (i.e., these are not standard written marks). The children worked together fairly well in this group with the exception of one boy who was quite disruptive and negative and repeatedly involved the other boy in physical and verbal disruption and picking on the girls. This was probably exacerbated by the fact that this addition work was too easy for these children; their behavior improved in subtraction especially with zeroes in the minuend.

Group 2: Second session, high initial knowledge. This group of two girls and two boys also made few pretest errors in written addition. From the beginning, these children vertically aligned the blocks. They disagreed about whether they should use separate blocks to show the sum or just push the addend blocks together. On the first day, they used extra blocks to show the sums of the ones and tens but not of the hundreds and thousands. They just counted the blocks in both addends on subsequent problems. This group began by adding the blocks from the right

(as in the standard U.S. written procedure) and continued this for all problems. One child on the second day started adding blocks from the left, but he was stopped by the other members. For the first three days they did not physically trade the blocks when the sum exceeded nine, but recorded the trade in the written procedure and talked about how they could not write two digits and so had to trade ten to the next column. everyone agreed about the blocks and written procedures, but the explanations were not very full. The experimenter continued to say that they should do everything with the blocks that they did with the marks, but they did not seem to see the necessity of trading the blocks physically even though their explanations sometimes used block words and described block trades. On the fourth day the experimenter asked children to make explanations of their written marks procedures. The children spontaneously used block words and fully described the required block trades (e.g., saying "I took ten flatheads and put them together to make another fatty" to explain the 1 written above the thousands place). On the fifth day finally, on the last problem, the children spontaneously traded the actual blocks. They then exclaimed that they understood what the experimenter had meant when she asked them to do with the blocks what they had done with the written marks. That day, the group also progressed from aligning the first 3-digit and 4-digit problem on the left to solving another such problem without aligning digits at all, but adding the correct multiunits both with the blocks and digit cards. They aligned all subsequent uneven problems correctly. This group worked fairly well together though there was some antagonism between the mathematically strongest boy and girl. They were often distracted and silly, again perhaps because the problems were not very challenging to them. In general they continued to attend to their mathematical tasks at the same time as they carried on irrelevant discussions. They also went on to show their best work and thinking in subtraction.

Group 3: First session, medium initial knowledge. This group of two girls and two boys had two members O and M who worked hard at understanding addition with blocks and the digit cards, one boy D who had strong conceptual understanding but frequently dropped out of problem solving unless prompted by the experimenter, and one girl U who sometimes disrupted the group activities, except when she was the "leader." U gradually withdrew from group participation, with moments of engagement occurring late in the session (see burghardt, 1993, for a case study of this child). The group began by setting up the addends with blocks (second addend above the first) and adding the blocks mentally from the left to get the answer: three thousand twelve hundred sixty two (the ones column sum was twelve and was mentally added to the tens sum fifty). One child then showed the hundreds to thousands trade with the digit cards and described it in block words, saying you couldn't have twelve hundreds. Thus began five days during which the group quickly figured out how to add the blocks, trading correctly moving either from the left or from the right, but floundered with the digit cards, inventing several wrong marks procedures as well as frequently using the correct standard procedure of writing a 1 above the next left column. (See Fuson & Burghardt, 1993, for a report of these incorrect procedures.) During this time they did not link the blocks addition closely to the digit card procedures, and they discussed the digit card procedures only in terms of digits or English words, rarely in block words. The experimenter on the sixth addition day forced the children to link the blocks and

the digit cards at each addition step and to describe digit card moves using block words, and the group agreed on a correct procedure. On the next day O again showed confusion when adding the tens but corrected himself when the experimenter asked him to think about the blocks. O then suggested a new marks procedure in which the top addend is increased by one (e.g., a digit card 5 is replaced by a 6) if a block needs to be traded to that column because that is how the children did it with the blocks: they put the new block in with the top addend blocks (this is actually the written procedure that was learned by second graders in Fuson, 1986). Discussion continued that whole day comparing O's new written solution and the standard method of writing the traded 1 above the addend. O was confused on one more problem about the trading of too many tens and convinced D using the English words "tens and ones," but M used block words (with the experimenter's support to withstand the boys) to establish the correct trade to the hundreds. The mathematical work would have progressed more smoothly if D had been more dominant and if U had not been so disruptive; U began with good written competence but learned little during this study because she was physically or attentionally absent from so much of the group activity.

Group 4: Second session, medium initial knowledge. This group consisted of three girls with one or no errors on the pretest written addition problems, and two boys who did no written problem correct on the pretest. Overall the group was enthusiastic and worked well together. They spent much of the second day and some of the third struggling to write a block number they had made with twenty nine teeth (unit cubes). They recognized that writing two digits for the units would make the wrong number and suggested many different nonstandard notations to show this number (e.g., 3 4 29 or 3 | 4 | 29). The experimenter finally asked them if they could make any exchanges with the teeth and the licorice. This group then invented a written procedure in which they added each kind of multiunit (ones, tens, hundreds, thousands), wrote the sum in two digits if necessary, and then fixed this answer to be in standard notation using only one digit per multiunit. This procedure evolved from their use of blocks: they first added each kind of multiunit, recorded their sum with marks (e.g., 3 12 5 12), traded ten of any blocks that had ten or more for one of the next larger block, and recorded the successive fixed sums (e.g., 5 12 5 15 became 6 2 5 15 and then 6 2 6 5). Some children continued to write problems in horizontal form throughout, while others wrote problems vertically aligned. At this point, all children understood addition and their written marks fixing procedure conceptually when supported by blocks, but none were able to "fix" sums without the support of blocks. On the following day, however, the group talked themselves through the trades using marks only and successfully fixed an answer. Some children continued to work on devising and understanding a fixing method for their written marks procedure during the final two days, describing what they were doing with block words and using the blocks when necessary. The fixing usually proceeded from left to right. Others devised a general method of fixing that did not depend on talking through the fixing with block words: for the 2-digit sums they crossed out the 1 and wrote a 1 above the next left digit. When the group had to move on to subtraction, all but one child could carry out their invented add-first-fix the sum method with written marks only and could explain this procedure in terms of trading multiunits. This group worked well together partly because the two most dominant members (one boy and one girl)

were exemplary "good" rather than "bossy" leaders and had the strongest mathematical knowledge.

Group 5: First session, low initial knowledge. Of this group of two girls and two boys, one girl, M, was the dominant group member. M had little initial conceptual and procedural knowledge, while the others showed moderate to perfect pretest performance on written addition solutions. Through the first three days of addition the children, led by D and X, worked toward a blocks and digit-card procedure in which the blocks were aligned vertically and sums over nine had ten of that block traded for one block in the next left column. There was disagreement about the order of making the addends and whether to add from the left or from the right (each was done on different problems). Descriptions and explanations sometimes centered on the number of blocks and omitted the kind of block, leading to errors and prolonged discussion, and full verbalizations of the block trading were not given (they focused either on the new one ten or hundred or the old ten ones or ten tens but did not verbally describe the ten tinies traded for the one rectangle). Over the next four days M invented and imposed a new procedure in which the goal was to leave only nine in a given column (because "you can't have more than 9 in a column"); the excess over nine (or sometimes over ten) was taken away. This excess was often put above the next left column, but was sometimes dropped (M's procedure led to answers like 6999 or 4999). This 9's procedure competed with the ten-for-one trading procedure over four days, with children frequently using the 9's procedure with the blocks and the digit cards, and the standard algorithm on their individual worksheets. All four children changed their views repeatedly within and over days, frequently expressing confusion. During this confused period, children talked about how many they had to take away from column sums over nine to make that sum small enough. On the third such day the experimenter encouraged the children to keep the blocks and the written marks connected and reviewed the ten-for-one trades with blocks. Over the final two days of addition, the experimenter continued to support linking the block and written marks procedures and queried the children about the size of the blocks. The children eliminated their 9's procedure in favor of their written trading procedure. At the end, all of these children were able to verbalize some understanding of the correct ten-for-one trading, although their explanations were still incomplete.

Group 6: Second session, low initial knowledge. This group of three girls and two boys ranged on the pretest written addition tasks from making only one fact error to getting all the sums wrong. During the first few days, the boys and girls argued about how to write and solve problems, but soon the girls became established as the most actively engaged members of the group, and the boys deferred to the girls. The group presented the first problem horizontally with the blocks and then added the blocks beginning with the thousands. When they got a sum of twelve breads (hundreds), one child said that there couldn't be two numbers in the sum so "you put the two down and add the one to the top of the other side." This verbal description arose from procedural knowledge of the standard written algorithm, but did not specify sufficiently where the "one" should be written. Because they were moving from left to right some children wanted to write the 1 above the next column, i.e., at the top of the tens column. Confusion over where to write the 1 persisted over the next four problems. "Regrouping" was referred to as a written

method unrelated to the blocks; it had to do with writing the 1 somewhere. Children wrote the next problems vertically, some asserting that you can't regroup with horizontal problems. For the next two days, this group continued to add blocks from the left and write the 1 above the column to the right, in the mirror image of the trades that they had been previously taught. Soon, however, the girls said that the 1's were wrong because they were adding in the wrong direction ("You move to the left, the opposite of writing"), but the boys continued to insist on adding from the left. One child focussed on the size of the blocks representing each trade, so the blocks were traded to the next larger column. On the fourth day of addition, addition began from the right and, on the fifth day, the experimenter asked the children to add one problem both from the left and the right. This produced two different solutions, a correctly traded answer from the right and an incorrect solution from the left due to a mirror-image trade. Although the children had previously traded correctly when adding from the left, the mirror-image trade occurred when a child allowed her written addition to dictate the blocks trade. A heated discussion followed and, from then on, the children added from either the left or the right flexibly, using blocks, trading correctly, and recording correctly. The girls by the end all had given conceptual explanations for various blocks trades but, when helping the boys, more often gave procedural explanations to them. On the eighth day, the experimenter asked the children to do a problem on the magic pad and explain it by talking about the blocks. The girls could all do so, but the boys required help. One of the boys was very shy throughout, and the other boy frequently withdrew from active participation.

Discussion

A striking aspect of all of the group work was the relative ease with which children invented accurate quantitatively-based multiunit addition with the blocks compared to the many inaccurate invented multidigit written marks procedures (see Fuson and Burghardt, 1993). Children never added different block multiunits but did add written digits for different multiunits (e.g., hundreds and thousands). The block quantities also suggested what to do when children had too many in the sum of a given multiunit (e.g., twelve tens) and provided language to convey the quantities involved in these solutions. The written digits instead elicited nonquantitative procedural language ("Write the 1 up there") even when the digits were being used to describe block moves. Using block words (e.g., tiny, long legs) to describe written digit procedures was sometimes more helpful than using English words (one, ten) because the block words require a child to be clear about both the kind of multiunit and how many multiunits there are. These can get confused in English: a child would say "ten" to mean either "ten ones" or "one ten," but had to say "a tiny" or "ten tinies" or "one long legs" when using block words. The ambiguities in English led to confused communication among children and allowed erroneous written procedures; blocks and block words clarified these confusions. Many children did not spontaneously link blocks addition and written marks addition, resulting in erroneous written marks procedures. When experimenters forced children to link the blocks and written marks for each multiunit (e.g., children had to write the hundreds marks as soon as they added breads), the quantities in the blocks enabled children to correct their written marks procedures. Verbalizing what had been done with the blocks,

especially with blocks words, also proved to be helpful to some groups in correcting written procedures.

Personalities and mathematical knowledge both contributed to the quality of the mathematical work of a group. When dominant members had good mathematical knowledge and were good rather than bossy leaders, the groups made better mathematical progress. Most groups were not very good at identifying group members with inadequate understanding, and some such members hid their lack of knowledge fairly successfully. More focus on such helping, a longer time on addition for some groups, and more time to do backwards linking with everyone discussing the marks procedures in blocks words would have helped the weakest children. The strongest children could have handled more difficult questions such as "What are differences between adding from the right and from the left?" Second graders can do interesting mathematical work in this environment, but they do need some help from a teacher to maximize their use of group work, to relate the block quantities to written digit procedures, and to verbalize their solutions conceptually. We are presently analyzing data from low- and middle-achieving children to see how these results generalize.

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Arithmetic from a Problem-Solving Perspective: An Urban Implementation with Reports on Students, Teachers, and Classroom Interactions

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This study reports the results of an urban implementation of Cognitively Guided Instruction with information on student achievement, staff development, teacher observations, and questions on about performance with minority populations. Twelve first-grade teachers participated in a staff development effort designed to focus instruction on the process that students used in their solutions rather than the production of written answers to exercises. Students in experimental classes performed significantly better in solving word problems as well as completing number facts than students in comparison classes. Systematic observatons revealed differences between CGI and non-CGI teachers in their teaching of arithmetic.

After the recent positive findings of constructivist mathematics instruction, often focused in middle class settings, the authors sought to replicate this work in an urban setting. Brophy and Good, in *The Third Handbook of Research on Teaching*, state, "Interactions between process-product findings and student SES or achievement level indicate that low-SES-low-achieving students need more control and structure from their teachers: more active instruction and feedback, more redundancy, and smaller steps with higher success rates. This will mean more

review, drill, and practice and thus more lower level questions."(p. 365) The authors found that CGI was effective in such settings.

Twelve first-grade teachers participated in a staff development effort designed to provide them with opportunities to examine ways to teach first-grade mathematics curriculum from a problem-solving perspective using a Cognitively Guided Instruction framework. These teachers were one of two teachers identified by a principal in each building to participate. Using a quasi-experimental design, a group of 12 first-grade teachers and their classrooms, from schools that matched the treatment schools' population characteristics, comprised a comparison group. The percentage of minority students in these 24 urban classrooms ranged from 57% to 99%.

Experimental teachers participated in staff development that followed the Fennema and Carpenter (1989) model. The program involved a 19-hour July workshop and three two-hour sessions in September, October, and December. Comparison teachers participated in staff development that focused on problem solving in elementary school mathematics, but that did not include CGI principles or research. This program included two 1.5-hour workshops in October and January.

Teachers in the experimental group taught arithmetic through the use of word problems, and their students spent considerably less time on skill worksheet drills. Instruction in experimental classes focused on the process that students used in their solutions rather than on the production fo written answers to exercises.

A 14-item arithmetic word problem test from Carpenter et. al. (1989) study was used as a pretest (early October) and posttest (late February-early March) in each of the 24 classrooms. For the purpose of analyzing student performance data. 6 boys and 6 girls were randomly chosen from each of these classrooms for a tolal of 144 subjects in the CGI group and 144 in the non-CGI group. These students were interviewed individually to assess the processes they used in solving a variety of word-problem types and the strategies they used in completing number facts. The student interview measure from the Carpenter study (1989) consisted of two parts: (a) six word problems solved with access to counters and (b) a number facts test without access to counters. Each word problem was read to the child. and after each response, the interviewer coded the response on a coding sheet and recorded the student's explanation of the solution process on a student response sheet. For the facts part of the interview, each fact. printed on a 4 x 6 card, was shown to the child. The child was asked to read the number statement adn complete the fact. If unable to complete the fact using recall or a derived fact, the child was encouraged to use fingers and direct modeling.

Students in experimental classes performed significantly better in solving word problems as well as completing number facts. There was a significant difference on the pretest favoring the CGI group ($t(22)=2.98$, $p.01$). Adjusted for the pretest scores were computed. Table 1 shows significant differences with CGI students showing gains, both from pre-test to post-test and in comparison with non-CGI students on a written problem-solving test. word problem interviews. number

facts interviews.

Table 1

Classroom Means, SD's, Adjusted Means, f-values for

Arithmetic Word Problem Tests (N=24)

		CGI		Non-CGI		
Achieve measure	Max	Mean (SD)	Adj	Mean (SD)	Adj	f-value
Written ps-pretest	14	2.26 (1.04)		1.25 (.56)		
Written ps-posttest	14	9.67 (1.76)	9.41	2.92 (.94)	3.18	87.60**
Interview word prob	6	5.54 (.35)	5.44	2.83 (.61)	2.93	114.45**
Interview no. facts	5	4.76 (.47)	4.68	2.92 (.56)	3.00	46.47**

In addition to the analysis of correct responses, an examination of student use of advanced strategies (counting on, derived fact, or recall) was conducted on the students interviewed. The mean number of times that students used an advanced strategy in solving a word problem or completing a fact item is reported in Table 2.

Table 2

Means, SD's, and t-values for the

Use of Advanced Strategies for Word Problems and Fact Interviews

		CGI	Non-CGI	
Interview Measure	Max	Mean(SD)	Mean(SD)	t-value
Word Problems	6	3.00 (1.75)	0.67 (.69)	4.29*
Facts	5	3.65 (1.58)	0.90 (.76)	5.45

A total of 82 classroom observations were conducted of both experimental and comparison classrooms. The goal of the observations was to gather information on the mathematics content, instructional behaviors, and in noting specific categories of teacher behavior, such as focusing on process and questioning students. The study presents information from these systematic classroom observations that indicate that CGI teachers consistently read word problems to students and asked students to explain how they arrived at their solutions. They did not teach specific strategies. The CGI teachers in the study conducted instruction in a variety of formats with a mix of whole group, medium group, and small group settings. CGI-trained teachers using strategies with which they were comfortable to allow students to solve problems and communicate their results on a regular basis to peers and the teacher. Communication was common place in the CGI classrooms, with teachers consistently asking students to explain how they arrived at a result Moreover,children were not observed spending time working alone on skills worksheets.

In contrast, all but one of the non-CGI teachers followed a very predictable routine. The lesson would begin with a brief explanation of what was going to be learned that day followed by a demonstration of a procedure to add or subtract two numbers or solve a word problem. This was followed by an example or two and then an explanation on how to complete fact exercises on worksheets. Children in the non-CGI classrooms were asked to complete addition or subtraction facts and to give answers to word problems, but they were seldom asked to explain how they

solved a problem or completed a fact. Children in these classes were observed working alone on skills worksheets for anywhere from 50% to 75% of the time dedicated to mathematics.

This study was consistent with the findings that children need not master computational and other lower-order skills before they can develop their problem-solving skills (Carpenter et al., 1989). Although non-CGI students spent more time completing worksheets on number-fact drills, the treatment students showed significantly greater achievement on completion of number facts, recall of number facts, and recall and use of advanced strategies in completing number facts.

Although all teachers taught essentially the same content, CGI teachers spent considerably more time on word problems, not only in teaching addition and subtraction, but in teaching other topics as well. They taught mathematics by placing children in a problem-solving situation using word problems that were as relevant to the children as possible. Non-CGI teachers, on the other hand, were observed following the textbook very precisely. These teachers spent more time teaching specific procedures for completing number facts, writing number exercises, and dealing with worksheets. Their students spent more time alone on worksheets and producing single number answers with very little opportunity to explain their thinking.

There is a widely-held belief, attributed to many who represent minority or disadvantaged populations, that students, especially those who are minorities or disadvantaged, must learn the basics, often in a rote manner, before moving on to

problem-solving and process-oriented mathematics. This study challenges that belief. While care must be taken in generalizing the results of this study, they do suggest the applicability of CGI principles, i.e., constructivist principles, in urban classroom settings. Further research studies with this focus are necessary to the mathematics reform movement in order to meet the challenges raised by the process-product research findings and the beliefs held by many teachers of disadvantaged minorities that process-oriented learning only works for the middle class majority student.

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ALGORITHMS SUPPLANT UNDERSTANDING: CASE STUDIES OF PRIMARY STUDENTS' STRATEGIES FOR DOUBLE-DIGIT ADDITION AND SUBTRACTION

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A year-long study of first, second, and third-grade students indicates that conceptual knowledge may be extinguished through an emphasis on procedural knowledge. The students' prior understandings of place value in double-digit addition and subtraction became subordinate to and subverted by teacher-taught algorithms which the children accorded higher epistemological status than their own successful, invented strategies.

Framework

The importance of connecting procedural knowledge and conceptual knowledge in mathematics instruction has been acknowledged by researchers (Carpenter, et al., 1990; Hiebert and Lefevre, 1986; Kamii, 1988; Burns, 1992; Simon, 1993) and by NCTM (1989). Hiebert and Lefevre (1986) describe conceptual knowledge as rich in relationships and part of a larger web of ideas. It is constructed actively by the learner, and represents the "understanding" piece of our knowledge. Procedural knowledge embodies the step by step instruction on how to complete tasks. It consists of rules and algorithms, and requires a familiarity with symbols and language.

Contrary to the recommendations of the Curriculum and Evaluation Standards for School Mathematics (1989), most elementary teachers permit and support rote learning of procedures in their classrooms either because it is easier for them instructionally (Burns, 1992) or because it results from the limited extent of their own knowledge (Simon, 1993). Not surprisingly, elementary textbooks pander to the practice.

Prior to and outside the school experience, children demonstrate remarkable inventiveness which illustrates conceptual understanding of arithmetic and is more powerful than their knowledge of algorthims (Carraher & Schlieman. 1985; Groen & Resnick, 1977; Carpenter, et al., 1990).

Methodology

To identify addition and subtraction strategies of children before and after instruction, we interviewed and videotaped 19 children; ten first-graders (5 boys and 5 girls), nine second grade students (4 boys and 5 girls), and ten third grade students (8 boys and 2 girls). All children come from the same rural elementary school and represented a wide range of mathematical abilities and socio-economic backgrounds. The first graders were interviewed once at the end of the school year. The second grade students were interviewed three times: in November before instruction on the arithmetic algorithms, in February/March, and again in May. The third grade students were interviewed three times as well, mainly to ascertain if age and repeated practice had any significant effect on students' conceptual understanding.

All students were asked to solve double-digit addition and subtraction problems embedded in simple word problems and in familiar contexts [involving terms like "stones" and "marbles"]. They were asked to solve it first using base 10 blocks and then mentally or with paper and pencil as they chose. The students were also questioned whether they knew of any alternative ways to solve the problem.

Results

Before Algorithm Instruction

Almost all of the students interviewed before instruction of the addition and subtraction algorithms took place demonstrated invented strategies which used non-traditional, front-end approaches (not the usual left-to-right order). They were almost all successful with addition, though much less so with subtraction. Many of their errors consisted of counting while keeping track during counting up and counting back.

Of the ten first grade students, nine students attempted a non-traditional approach for addition, and one used the traditional algorithm. Eight students were accurate. In subtraction all students attempted a non-traditional approach and two students were accurate.

Similar results were observed from the second graders. Before receiving instruction in the traditional double digit addition algorithm all nine second grade students solved the problems with a front end approach --- adding the tens and then

counting on to add the ones. Seven out of nine students correctly solved the problems using base ten pieces and six of the nine correctly aolved the problems mentally.

The students were less successful with subtraction, but no less inventive. Only three students attempted subtraction mentally, and all three were successful with non-traditional approaches. Six students solved the problems using base ten pieces; three were successful. Five students solved a subtraction problem in writing; four had strategies that were unclear and inaccurate, and one used a non-traditional approach that was successful.

After Algorithm Instruction

At the time of the winter interviews, the second grade students had several weeks of instruction in the addition algorithm and had only just been introduced to the subtraction algorithm. Student performance on the addition problems indicated a gradual movement awar from their own non-traditional methods towards a committed use of the addition algorithm. Six of the nine students switched to the traditional method when working on problems mentally. In their written work; seven students correctly used the traditional algorithm, one student vacillated between both methods, and one student simply added all of the numbers. While using base 10 pieces, eight of the nine students used the traditional algorithm successfully. The ninth student was not observed.

By the spring, all nine students exclusively used the traditional algorithms for addition, and all but one were successful. The results for subtraction were similar, though delayed due to the timing of instruction. Again, by the spring, all of the second grade students used the traditional subtraction algorithm, but only three were successful.

On The Possibility Of Alternative Strategies

All 19 second and third grade students were asked at the end of the year if they could think of alternative solutions other than the traditional methods they used after instruction. Specifically, for the problem 35 + 27,the students were asked if they thought it was possible to add the tens first. They all thought that the traditional methods were the best, and most believed it impossible to solve the problems anv other way. Twelve students thought it was impossible to solve addition problems in any way other than the traditional method. The other seven thought it was possible, but they could not explain how. Fourteen of the students thought it was impossible to solve subtraction problems with any but the traditional method. The remaining five thought it may be possible, but they couldn't do so themselves. All 19 second and third grade students were confident that the traditional method for subtracting [41 - 18] was the best and only method for them.

We repeatedly observed students shed their prior understandings in favor of the authoritative knowledge they learned in school. One example follows:

Jamie's Case:

In November, Jamie (a second grader) added 19 and 26 mentally. She verbalized, "I know I have 30 because I have a group of ten and two more tens. Then if I take 1 from the 6 and give it to the 9, I'll have another group of 10. That leaves five left, so the answer is 45."

In February, Jamie attempted to add 34 and 99 by beginning to group the 9 tens and 3 tens, then stops and says, "Oh, I have to add the ones first." She then grouped the units, and traded for a ten to solve the problem.

In May, in response to a question as to the possibility of solving the problem by adding the tens first, Jamie emphatically stated, "No you never add the tens first." Instead she suggested that another way to solve the problem might be to know the answer from memory. Finally, she was confronted with her own invented strategy as a strategy "someone used" to add 49 + 19 (think of 50 + 19 and then subtract one to get 68). When asked if she thought this method would work, she replied, "If you know that way it's okay, but it's much, much better to just add the ones first."

Misconceptions After Instruction

Finally, we identified numerous misconceptions regarding place-value among the second and third graders. However, the most remarkable observation was the students' willingness to disregard their doubts about unreasonable answers if they believed that their teacher or their book thought such answers correct. For example [27 + 35], an offer of an alternative solution:

Paul (high achieving second grader): You can add the tens and put 5 down and add the ones and get 12 so you have 512.

Interviewer: Can you make up a story for that?

Paul: I had 35 rocks and 27 more, so that makes 512.

Conclusion

We believe that by encouraging students to use only one method (algorithmic) to solve problems they lose some of their capacity for flexible and creative thought. They become less willing to attempt problems in alternative ways, and they become afraid to take risks. Furthermore, there is a high probability that the students will lose conceptual knowledge in the process of gaining procedural knowledge.

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STUDENTS' REASONING ABOUT RATIO AND PERCENT

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This work comes from a larger project of several investigations of students' conceptual and procedural knowledge of percent and their strategies with ratios involving 100. This paper describes students' knowledge of percent and ratio problems with 100 just prior to instruction in percent as compared to those near the conclusion of instruction. This study tested 124 students on written tasks, and 30 students were individually interviewed. Most students exhibited some intuitive conceptual knowledge of percent as a part/whole relationship, but their understanding was limited by particular contexts and numbers. Students demonstrated some intuitive proportional strategies and some incorrect additive strategies. After instruction students used various strategies with these problems, but understanding of percent continues to be limited and some incorrect strategies persist.

Area of Concern

The ability to understand and solve problems involving percent is important for students in many fields of study, and the use of percent continues throughout adult life in managing one's personal finances and in understanding economic issues. Percent has been a school mathematics topic for many years, and the NCTM Standards (1989) continues to recommend its inclusion for today's curriculum. Numerous studies indicate that students' understanding of percent is extremely deficient (Brown, Carpenter, Kouba, Lindquist, Silver & Swafford, 1988; Hart, 1988), despite its common use in our society and inclusion in the school curriculum. This study addresses the concern that there is very little current research on students' intuitive reasoning or informal conceptual or procedural knowledge of percent.

While there are several different methods for solving percent problems, several mathematics educators have proposed the most meaningful development of percent is through the use of ratios and proportional reasoning (Glatzer, 1984), which is a method of solving percent problems that closely models the underlying mathematical relationship of a ratio with one hundred compared to another ratio. These recommendations appear to be based on personal opinion and experiences rather than on research. A concern of this study is to seek evidence of student's use of ratios or proportional reasoning to suggest extending these strategies to percent.

Theoretical Framework

The theoretical basis of this work is consistent with the constructivists' theory of learning, which considers learners as active participants in building personal knowledge, not as passive recipients of definitions and procedures. Numerous studies have reported how children enter school with considerable personal knowledge of numbers and of

problem solving, prior to any formal instruction (Gelman & Gallistel, 1978; Carpenter & Moser, 1984). There are a number of successful teaching studies which base instruction on the child's thinking and which build upon or extend the child's correct intuitions (Mack, 1990; Carpenter, Fennema, Peterson, Chaing, & Loef, 1989). Investigations with older children and adults have demonstrated that the construction of knowledge, including self-invented strategies, continues to take place even after formal schooling has begun (Carraher, Carraher, & Schliemann, 1985; Lave, Murtaugh, & de la Rocha, 1984). This study assumes that children establish meaning for mathematics symbols by connecting the symbols to a referent, which is an event or series of operations with which the child has experiences and which provide meaning for the symbols (Van Engen, 1949). It has been proposed that the connecting of symbols with a referent is the first process toward symbol competence (Hiebert, 1988). All of these works support the basic assumptions of this paper; namely, that an understanding of the student's mental or physical referents and their reasoning about ratios and percent should prove helpful in planning an improved curriculum for instruction.

Previous Works in Proportional Reasoning and Percent

Studies measuring children's performance in proportional reasoning have reported it to be a difficult concept in that most children from 13-15 years of age make very little progress in working with formal proportional relationships Hart (1988). Numerous works report the use of incorrect additive strategies to solve problems involving proportions (Hart, 1988; Karplus, Pulos, & Stage, 1983). In contrast to these data, a number of studies report that individuals develop informal, context specific proportional strategies through occupational use (Carraher, Carraher, & Schiemann, 1985; Lave, Murtaugh & de la Rocha, 1884). Teaching studies with young children that emphasize experiences in familiar contexts report student success with proportional relationships (Streefland, 1985; Tournlaire, 1986). These studies suggest that the reported poor performance on standardized tests may not accurately reflect students' informal knowledge or their capability to understand ratios and percent. There is little current research in percent or research about children's reasoning with percent.

Methodology

A theoretical and cognitive analysis of percent was developed and used to design written tasks to reveal the student's reasoning about ratios and percent. The 37 tasks concentrate on situations in familiar contexts but include some non-contextual, symbolic examples. The tasks were administered during a regular mathematics class at two public schools in a mid-size city in the Northeast. Frequencies of the students' responses to the written tasks were recorded according to correctness, the actual answer, and the solution strategy where applicable by the researcher and verified by an interrater with a 93% agreement. Individual student interviews of about 20 minutes in length were administered to fifteen students selected at random from each group to gain further insight into their

reasoning. The sample consists of 61 students before instruction and 63 students near the conclusion of percent instruction. Descriptive detail and statistical methods of significant differences on particular tasks or questions and groups of tasks were used to compare the students' reasoning and strategies.

Discussion of Results

Student Intuitions About Percent

The interviewer asked each student how he/she would explain the idea of percent to a friend who did not understand it and the replies were recorded for context and for the specific percents used. Of the 15 younger students, 6 used shopping examples and 6 used school grades, while none of the older students mentioned school grades and only three used shopping examples. Several older students used hypothetical examples with money, and four supported their ideas entirely without context, suggesting that some defining characteristics of percent have been extracted from specific instances or contexts.

The younger students spoke primarily about 100% and 50% as equivalent to a "whole" and a "half," and approximately half of this group gave a real example, i.e., "if you had 20 problems, a 50% grade would mean you got 10 correct." The students were asked to explain 30% to indicate the generalization of their understanding of percent. Only two of the younger students and about half of the older students could extend their knowledge to explain 30%. These replies suggest that 100% and 50% serve as a referent or prototype for percent and that the conceptual knowledge of percent is somewhat number dependent. Success in solving familiar situational problems with specific numbers is significantly different between the two groups but follows a similar pattern within each group, namely, success with 100% and 50% is superior to success with other percent numbers (see Table 1 below).

Table 1: Performance on Three Problems with Specific Percent Numbers

Percent Number	Younger: Mean	Older: Mean	p-value
100%	1.46	1.98	.0016
50%	1.34	2.08	.0007
10%	0.84	1.54	.0000
25%	0.30	1.49	.0001
75%	0.25	1 08	.0001
30%	0.21	0.86	.0001

The higher means of the two groups with the 100% and 50% problems is further evidence that these numbers serve as a prototype or referent for percent.

Considering the various data from both the interview and the written tasks, it appears that the majority of students in each group have a limited interpretation of percent as a part/whole relationship. For example, when given a choice of representations, the majority of students from both groups selected a part/whole model in a diagram or verbal form rather

than a ratio expressed as "parts per 100," or "parts for every 100," which no students selected. Part/whole explanations were commonly given by students in each group, such as "with 4 blocks, 1 would be 25%," or "here is a pie, cut it in half to show 50%."

In summary, the younger students have some intuitive knowledge of percent; however, it appears to be limited conceptually to a part/whole model, to prototypes of 50% and 100%, and to school grades and shopping situations. Most older students conceive of percent apart from specific numbers or situations, but they also exhibit primarily a part/whole interpretation.

Success in Solving Problems Involving Ratio and Percent

While the interview data on the younger students serve as an indication of intuitive knowledge of percent, these students' successes on written tasks are additional indications of knowledge of percent and of ratios with 100. Some of the tasks required only an estimation or number sense and were intended to reflect conceptual knowledge, others probably required some problem solving strategies or solution procedures, and some required decimal and fraction representations of percent. The success rates of the two groups on the various sections of these tasks are given in Table 2 below.

Table 2: Average Percent Correct On Written Tasks

Problem group	Younger group	Older group
10 items: Estimate %, in familiar contexts	43%	58%
6 items: Ratios with 100, in familiar contexts	31%	51%
6 items: Percent, in familiar contexts	22%	44%
6 items: Percent exercise, no context	7%	35%
9 items: Represent percent as fraction or decimal, no context	12%	40%

These data suggest that the younger students have some number sense of percent and can solve some problems in familiar contexts involving ratios and percents; however, their success drops sharply when there is no context and they are working with only symbolic representations. It appears that procedural knowledge to solve problems with only percent symbols is not developed informally or that percent symbols are not associated with the percent or ratio concepts prior to instruction. It seems that the "ratio with 100" problems are more manageable than percent problems by students before instruction, indicating that this reasoning might provide a natural bridge to percent problems. The older students had considerably more success in solving symbolic percent problems apart from a context and in translating among symbolic representations. Their success may be a result of practice in symbolic manipulations or a result of improved understanding; however, it should be noted that their overall success on these tasks was quite low.

Strategies with Ratio and Percent Situations

The students' thinking in terms of solution strategies was investigated by 12 problems posed in familiar situations on the written survey. Half were explicit percent problems, stated using the term or symbol for percent; and half were implicit percent problems, stated as a ratio "for every 100." Table 3 shows the sums of the two most frequently used strategies by each group on these problems:

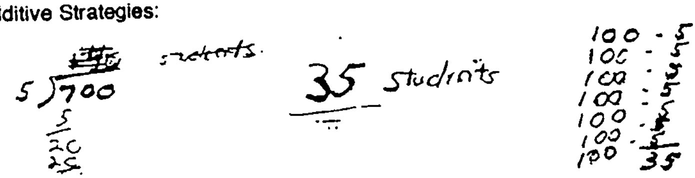
Table 3: Most Frequently Used Strategies on Percent and Ratio Problems

Younger Students		Older Students	
not shown	254	not shown	317
incorrect add/subtract	211	ratio pattern	152
ratio pattern	48	(50%, halves)	40
(50%, halves)	45	incorrect add/subtract	72
incorrect mult/div	20	incorrect mult/div	36
random calculations	9		

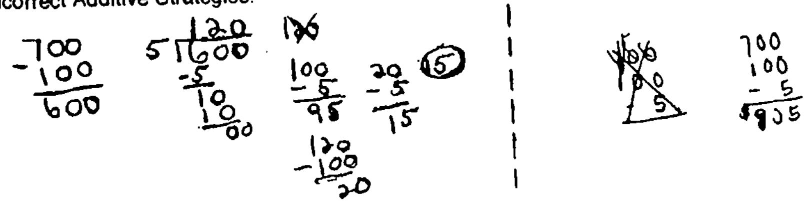
These data reveal the most frequently used strategy by the younger students was the incorrect addition/subtraction strategy, where a sum or difference is used incorrectly to represent a ratio or percent situation. This is not to be confused with addition used as a substitute for multiplication. Of note is the second most frequent strategy used by the younger students, the "ratio pattern" strategy. This category includes informal procedures such as the building-up of number patterns or equivalent fractions. While incorrect additive strategies are also commonly used by older students, their most common strategy was a ratio pattern. Examples of students strategies to a problem are as follows:

> A school reported that usually 5 out of every 100 students needs special help in reading. If this school has 700 students, how many would you expect to need the special help?

Correct Additive Strategies:



Incorrect Additive Strategies:



Correct Ratio Strategies.

$\frac{5^{\times 7}}{100_{\times 7}} = \frac{35}{700}$, 35 students

stud. 5 : 100 students
10 : 200
15 : 300
20 : 400
25 : 500
30 : 600
35 : 700

35 would need extra help.

The younger students frequently tried to directly model the situation by diagrams or charts, and they frequently resorted to random operations on the numbers contained in the problem. Additional strategies used by older students include use of a formal proportion with an unknown, multiplying by a decimal equivalent , and dividing to obtain the desired percent. Both groups used some unique and successful, personal strategies.

The Z-test yielded significant differences between the two groups in the proportion of students who used a particular strategy in 5 cases out of the 19 tested on the 12 problems (the Z-test requires that at least 5 have the attribute of interest and at least 5 do not; therefore, not all frequencies could be compared by the Z-test). The instances of significant differences include the correct ratio strategy used more by the older group on 2 problems and the incorrect additive strategy used more by the younger students on 3 problems. The strategies that did not yield significant differences in 14 of the 19 cases were incorrect mult/div, incorrect addition, and the correct ratio pattern. It would seem that additive strategies with ratio and percent are intuitive and somewhat resistant to change over time and instruction. While the two groups frequently used similar strategies, the younger students can be characterized as using primarily additive strategies and the older students as using primarily multiplicative strategies.

Conclusions

The common use of percent in our society has resulted in some intuitive knowledge of percent before instruction. While the concept of percent is limited in terms of context, specific numbers, and to a part/whole interpretation, it is suggested that an improved curriculum would help students extend these intuitions to additional contexts, numbers and conceptions of percent. An improved curriculum should help students use intuitive problem solving strategies with ratios to develop meaningful and effective procedures for solving percent problems. In addition, an improved curriculum should recognize the intuitive and persistent incorrect additive strategies and plan instruction to intentionally confront these misconceptions.

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A COMPARISON OF CONCEPTUALLY BASED AND TEXTBOOK BASED INSTRUCTION IN GRADES 1-3: A THREE YEAR STUDY

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Two groups of students were followed over a three year period in Grades 1-3. One group of students received instruction that focused on developing meaning for place value concepts and multidigit addition and subtraction. The other group received textbook based instruction. The group receiving the alternative instruction were more successful in solving place value, noninstructed computation, and story problems.

The reform efforts in mathematics education (e.g., National Council of Teachers of Mathematics, 1989, 1991) call for instruction that emphasizes conceptual understanding. Theoretical arguments in favor of teaching mathematics for understanding have a long and rich tradition (e.g., Brownell, 1935; Davis, 1984; Fehr, 1955; VanEngen, 1949). It is assumed that learning with understanding has both short- and long-term benefits such as flexibility, transfer, and increased learning over time. This paper will report the results of a study which examined the effects of implementing conceptually-based instruction on multidigit addition and subtraction during the first three grades.

Method

Sample

The sample consisted of two groups of students attending a large suburban elementary school with a wide range of ability levels. One group of students (N = 29) received alternative instruction during a portion of the school year in all of Grades 1-3. A second group of students (N = 24) received conventional textbook based instruction throughout this same time period.

We began with 153 students in Grade 1. During Grades 2 and 3, the school assigned students to different classrooms and each year the alternative instruction was provided in some of the classrooms. The groups of 29 and 24 students represent those who received the same kind of instruction on addition and subtraction all three years, the alternative or textbook based instruction.

Instruction

The instruction was designed to emphasize an understanding of place value and grouping-by-ten ideas and to encourage students to develop procedures for adding and subtracting multidigit numbers based on their understanding of place value. The underlying rationale was that students are more likely to understand if they are given opportunities to construct relationships, relationships between their current knowledge and new information, relationships between different forms of representation, and relationships between alternative solution procedures (Hiebert & Carpenter, 1992).

We hypothesized that understanding place value involves building connections between the key ideas of place value, such as quantifying sets of objects by grouping by 10 and treating the groups as units (Fuson, 1988; Steffe & Cobb, 1988), and using the structure of the written notation to capture this information about groupings. Different forms of representation for quantities, such as physical materials and written symbols, highlight different aspects of the grouping structure, and building connections between these yields a more coherent understanding of place value. These principles were operationalized by presenting students with contextualized problem situations, encouraging students to represent quantities with both physical materials (base- blocks) and written numbers, developing solution strategies with both representations, and sharing and discussing their solution strategies with the class. The sharing of strategies led to the students in these classes solving fewer problems each day than the students in the textbook based classrooms.

The special instruction took place when the students reached the relevant chapters in their textbooks. This involved approximately 6 weeks in the first grade, and 12 in each of second and third grades. The length of the instruction period on these topics was the same in the conventional and the alternative classrooms.

Assessments

The students were given group tests at the beginning and end of each school year and also after the place value units in grades 1 and 2, about mid-year. During the first year, approximately one-half of the students were randomly selected to be interviewed. Only the results of the group tests will be presented in this paper.

The group tests consisted of items focusing on place value, multidigit addition and subtraction, and story problems. The place value questions assessed the students' use of groups of 10 and 100 to quantify sets and their understanding of the positional nature of the number system. The multidigit addition and subtraction items included both instructed and noninstructed items. The noninstructed items involved either more regroupings than had been discussed in class and/or involved larger numbers than had been introduced in instruction. The story problems all involved combining familiar numbers with no regrouping necessary to solve them.

Results

The results of the initial test in Grade 1 indicated that the two groups of students did not differ significantly on any of the categories of items, place value or computation, at the beginning of the study. After the first test in Grade 1, the group receiving the alternative instruction outperformed the textbook group on all subgroups of items (place value, instructed computation, noninstructed computation, story problems). The differences were not always significant, but the rank order of their positions did not change on any of the subsets of items over the three year period.

Performance levels over the three year period on Place Value, Noninstructed Computation, and Story Problems are shown in Figures 1-3. It is important to note that the same items did not appear on all of the tests. For example, the place value items on Test 3 of Grade 1 were not the same items as on Test 1 of Grade 2. This means it is inappropriate to compare performance of a single group of students over time but it is appropriate to compare the performance of the two groups of students over time.

Figure 1 illustrates the results on the place value items. The students receiving the alternative instruction exhibited a significantly greater ($p<.01$) understanding of place value concepts than did the other group of students. This pattern continued over all three years and, in fact, the differences remained fairly constant over time.

Figure 2 illustrates the results on the noninstructed computation items; only the results of the first test are included for Grade 3 as all computation items became instructed computation by the end of the school year. The second assessment in Grade 1 was given after a place value unit. The students in the alternative instruction classrooms apparently used their new understanding of place value to a significantly greater extent ($p<.05$) to increase their performance on the computation items. For example, on an item such as 27 + 30, the students may have thought about what happens when three tens are added to a given quantity. This difference ($p<.01$) continued on the third assessment of the first grade. At this time, the noninstructed items were those that required regrouping, a topic not discussed in any of the classrooms. Once again, the students in the alternative instruction classrooms apparently were using their place value understandings to adapt their instructed procedures to accommodate this new type of problem. For example, one-third of these students correctly solved the problem 38 + 24 as opposed to 12% of the students in the textbook based classes. The most frequent response (39%) of the students in the textbook based classes to this problem was 512; only 2 of the 153 students in the alternative instruction classes gave this response. Apparently the students in the alternative instruction classes recognized that a 3-digit response was inappropriate when combining 38 and 24, a possible result of their greater place value understandings. A similar difference ($p<.05$) occurred on the second assessment in Grade 2; this assessment also occurred after the students had completed a place value unit.

The results on the Story Problem items are illustrated on Figure 3. As described earlier, the story situations all involved numbers the students had

been using in class and the computation did not require any regroupings. The students in the alternative classrooms responded correctly significantly ($p<.01$) more often to these problems at the end of Grade 2 and throughout Grade 3 than did the students in the textbook based classes.

Discussion

The instruction described in this study is one of many approaches to assist students in understanding mathematics. The results indicate that instruction that supports students' efforts in making connections between different forms of representations, and the sharing of their solution strategies, can prove to be beneficial. Although the students in these classrooms solved fewer problems than those in the textbook based classrooms, competency on the instructed computation did not suffer and, in fact, they frequently performed at a significantly higher level. One might speculate that the additional time spent on each problem had led students to reflect on their actions, a reflection that assisted the students in solving noninstructed problems.

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PLACE VALUE

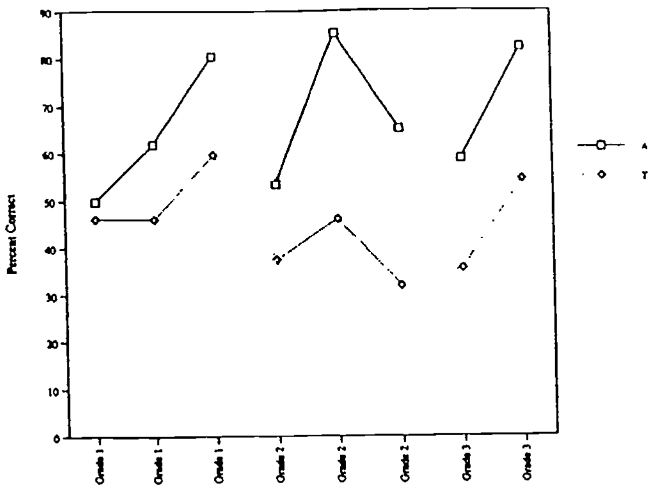

Figure 1

NONINSTRUCTED COMPUTATION

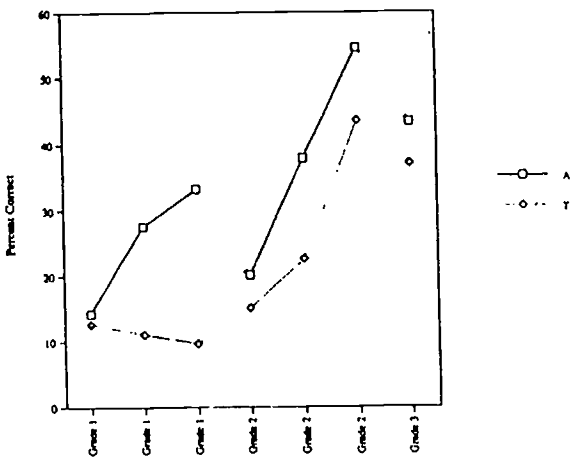

Figure 2

STORY PROBLEMS

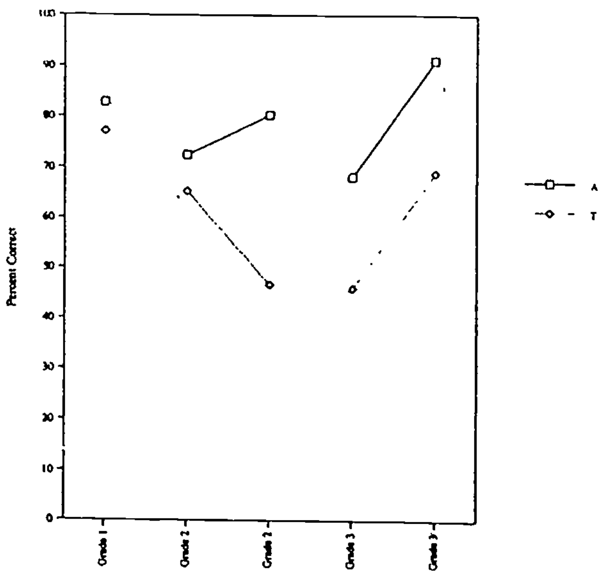

Figure 3

What is the meaning of 12.34five? Preservice teachers' interpretations.

Rina Zazkis, Simon Fraser University

This study investigates preservice teachers' concepts of place value, multidigit number structures and decimal fractions using representation of numbers in other-than-ten bases. Based on the analysis of twenty clinical interviewes, we describe students' construction of non-decimal number and speculate on their understanding of the decimal number system.

The general concept this study investigates is the concept of place value, emphasizing the place value in a fractional part of a rational number. More specific concept is a non-integer number represented in bases other than ten. These numbers, such as 12.34_{five} or 46.23_{seven}, are referred to as "non-decimals". Various researchers ([e.g. [9],[12]) have pointed out that a correct algorithmic performance in a given domain doesn't necessarily indicate students' conceptual understanding. Therefore, assuming teacher's ability to manipulate correctly decimal rational numbers, we asked them to make sense of "non-decimals" by performing addition and subtraction with these numbers and also by converting them to base ten representation. These non-standard tasks helped to eliminate the possibility of students' rote learned patterns and assisted in focusing on place value. From students' efforts to interpret non-decimals we learned about their concepts of rational numbers and multidigit structure in base ten.

The conventional way of converting 12.35_{five} to base ten is to use the following expanded notation: $12.34_{five} = 1 \times 5 + 2 \times 1 + 3 \times 1/5 + 4 \times 1/25 = 7\frac{19}{25} = 7.76_{ten}$.

Zazkis & Khoury ([13]) provided a detailed analysis of the domain of non-decimals. They also categorized student's problem solving strategies and possible errors or misconceptions in interpreting non-decimals. The current study is based on the analysis of clinical interviews and it attempts to provide a conceptual framework for student's mental constructions in this domain.

Theory

Action-process-object framework

Our theoretical perspective is a constructivist approach, based on the ideas of Piaget. The particular interpretation of constructivism used in this study is the adaptation of a constructivist perspective to the action-process-object developmental framework ([5]) Dubinsky discusses the action-process-object theoretical perspective as an adaptation of ideas of Piaget to the studies of *advanced* mathematical thinking. Previously it was used in the studies of undergraduate mathematics topics like calculus and abstract algebra. (e.g. [2], [3], [6]) We suggest that this theoretical perspective is appropriate for the discussion of mathematical knowledge development in general, not necessarily of what is considered to be "advanced". One of the goals of this study was to examine this claim.

The essence of the theoretical perspective developed by Dubinsky is that an individual, disequilibrated by a perceived problem situation in a particular social context, will attempt to reequilibrate by assimilating the situation to existing schemas available to her or him, or, if necessary, to reconstruct those schemas at a higher level of sophistication. The constructions which may intervene are mainly of three kinds -- actions, processes, and objects.

An *action* is any repeatable physical or mental manipulation that transforms objects in some

way. When the total action can take place entirely in the mind of an individual, or just be imagined as taking place, without necessarily running through all of the specific steps, we say that the action has been *interiorized* to become a *process.* New processes can also be constructed by inverting or coordinating existing processes. When it becomes possible for a process to be transformed by some action, then we say that it has been *encapsulated* to become an *object.*

As an example of a not-so-advanced mathematical concept, we consider the number 5 and describe the development of this concept using action-process-object framework. In the beginning of number acquisition the number 5 is identified with the counting of five attributes ([10]). A young learner has to touch the things counted to build a one-to-one correspondence between the number of attributes and the counting sequence. At this level the number 5 is an action for the learner. Later, when the learner can count a group of five in her or his mind, or recognize a group of five , we would say that the number 5 has been interiorized to become a process. When a learner is able apply an action to the number 5, such as adding it to another number, we say that number 5 has been encapsulated to become an obje .

In many mathematical situations it is essential to be able to shift from an object back to a process. One of the tenets of the theory is that this can only be done by *de-encapsulating* the object, that is, to go back to the process which was encapsulated in order to construct the object in the first place. In our example de-encapsulating of the object of number 5 may take place when the learner is asked to compare number five with number six. In this case the learner may go back to the process of counting to discover that number 5 is reached in the counting sequence before the number 6, therefore it is smaller. The concept of de-encapsulation plays a major role in the analysis of our data.

Method

Subjects. Twenty pre-service elementary school teachers in the last year of their teacher preparation program participated in the study. The topic of number representations in different bases was revisited during their "methods" class. However, the idea of extending other-than-ten base representation to the fractional part of the number was not introduced prior to the administration of the interviews.

Instrument. In a clinical interview setting students were presented with two different types of problems. In the first type of problems the students were asked to perform addition and subtraction with non-decimals. In the second type of problems the students were asked to convert non-decimals to base ten. During each interview, that lasted about one hour, participants were prompted where appropriate for understanding, that might not have been apparent from their initial response. Clarification and probing questions were asked, additional tasks were offered where necessary, to establish or to confirm the strategy used by a student. The interviews were transcribed and categorized in terms of different questions, their difficulty, and repeating error patterns used by students. The action-process-object theoretical perspective was used to analyze the interviews for ways in which the students appeared to think about the specific topics.

Results

All the participants passed a screening assessment, that is, they were able to perform correctly on tasks of conversion from various bases to base ten and addition/subtraction with integers, represented in bases other than ten. With non-integer numbers the situation was different. Sixteen out of twenty

participants were able to perform addition/subtraction correctly using different reasoning strategies. Four other students implemented consistently "error patterns", some of which were consistent with some of the examples described by Ashlock and VanLehn ([1], [12]). Only 10 students out of 20 converted correctly the given non-decimals to base ten representation. The remaining ten were consistent in using self- invented incorrect methods through different examples. In what follows we give some examples of student's responses to the proposed conversion tasks. The computational tasks of addition and subtraction aren't discussed here due to space allotation.

Conversion Tasks

In order to perform conversion Ann used the expanded notation method, similar to the above introduction of non-decimals. This method was used by 9 out of 10 correctly performing students.

Linda's conversion of 1.23 in base four below was unique in this sample.

> *Linda:* *Well what I mean is that um, I'm just using this decimal point to divide each number to the right of the decimal point over um, like 4, 4^2 each, uh, yeah, 2 groups of, hmm, well 23 over 100 over 100th, okay, but that stands, that's not 23, that is 2,3 [read: two-three] in base 4 which is 2 groups of 4 plus 3 units, okay? Does that make sense?*

For Linda, the fractional part of the number .23 in base four means 23/100, where both 23 and 100 are base four numbers. Therefore $23_{four}/100_{four}$ is interpreted as a fraction 11/16. Unlike Ann and many others, who assign place value to every digit and work with expanded notation of the number, Linda treats the fractional part as one conceptual entity.

Moving the decimal point error pattern

In a new situation, where no formal algorithm for symbolic manipulation was provided, Joan tried to use familiar problem solving strategies. One such strategy was applied when performing division by decimal: the decimal point "is moved" in order to work with integers.

> *Joan:* *Well I'm multiplying. Like, if I'm going to move a decimal point that would be mult... like if I have 3.2 and I want to move this decimal point to make it a whole number, then I would multiply it by 10's and my decimal point would go, so if I'm moving this decimal point two places I multiply by 100, but I'm not familiar enough with this to know that if I once have an answer I can divide by 100 and adjust the decimal points. (pause) Whaa! not a nice question.*

In order to convert 3.25_{six} to base ten, Joan moved the "decimal" point two places to the right, then she converted the number 325_{six} to base ten, she got 125 and "adjusted" the decimal point moving it two places to the left to get the result of 1.25. She wasn't sure in what she did, but it was "the only way (she) could think of".

Reading bug error pattern

Michelle applies a strategy that was referred to by Zazkis & Khoury ([13]) as a "reading bug". Rather than assigning place value to each digit in the fractional part of the number, she treats the fractional part as one entity and assigns to it the "place value" of the smallest unit involved. That's how

0.12_{four} is interpreted as 12/16 (in base ten) . The name "reading bug" refers to the reading rules for decimals: saying "twenty-three hundredths" brings to mind 23/100, rather than 2/10 and 3/100. The equality between the two expressions was misinterpreted when applied to other-than-ten bases.

Interviewer: *How about converting 3.12 base 4 into base 10?*
Michelle: *Umm, yeah sort of, the fractions kind of stump me. Okay, so this is basically 12/100ths, so oh it isn't 12/100ths, it's 12/16ths, it's 3 units, so it's 3 ones, it's 12/16ths*
Interviewer: *How about something like 4.34 in base 5 convert that to base 10.*
Michelle: *See, I think that I could see this better if I actually um, okay this is groups, this is 34/25, no, that can't be right, because it's more than a whole. Yeah, you can't have that, that's wrong.*

Michelle later adjusted her strategy to take care of cases where numerator appeared to be bigger than denomenator.

Common fraction substitution error pattern

Marina used a pattern identified in the previous research as "common fraction substitution" . She interpreted the place values in the fractional part of non-decimal as 1/4 and 1/40 or 1/6 and 1/60 for bases four and six respectively.

Interviewer: *Alright, where did you get the numbers 3/4ths and 2/40's from?*
Marina: *Well because the 4 is the reverse of this one, because the place value goes like this, the place value goes 1, like in 10's, it goes 1's, 10's, 100's or 10 squared and then here on the other side of the place value it would go, um 1/10th, 1/100th, so that's why I was thinking of that but I get confused when I try and convert it into 4ths.*
Interviewer: *And if you had a number .452 in base 6? What is your first reaction to that?*
Marina: *That would be, that would be also the reverse, that would be 4/6, 5/60ths and 2/600ths.*

Conversion - what comes after assigning place values?

It seemed to us that what was left in the conversion task after assigning place values to digits, was just a matter of simple arithmetic. Marina and Jill convinced us that the issue isn't that simple. Not that they failed in "simple arithmetic", they just had difficulty to decide what arithmetic had to take place. Marina felt stuck after assigning place values to the digits of number 1.32_{four} and didn't know how to proceed with the task. Jill assigned correctly place values, but didn't look at the sum of the products. Instead, she tried to estimate the number:

Interpretations and Discussion

According to our theoretical perspective, students' ability to perform addition/subtraction with non-decimals indicated that they treated these numbers as *objects*, since they were able to perform *actions* on them. The conversion task requires students to move from an object back to a *process*, which is done, according to the theory, by *de-encapsulating* the object. The objects (rational numbers) that had to be de-encapsulated by the interviewees were constructed long time ago and probably over a long period of time. How difficult or how natural is the de-encapsulation in this case? Dubinsky ([5])

conjectured that "an epistemological obstacle occurs when an action, process or object conception of some mathematical topic existed for some time and was very useful in dealing with many problem situations, but now is presented with problems that it cannot handle." Our findings defir· ·ly support this conjecture and provide an example of partial or incorrect place value conceptions th ..rved our subjects faithfully throughout their college years, but caused an obstacle when used to interpret non-decimals. Our results confirm that de-encapsulation takes the student back to the process which was encapsulated in order to construct the object in the first place. This process may not be sufficient to deal with problems on higher level of sophistication and needs to be reconstructed. This reconstruction appeared to be very difficult for some students and was a major source of error performance.

Constructing non-decimals

Ann and Linda demonstrate two different but correct mathematically interpretations of non-decimal. In both cases the object of non-decimal is de-encapsulated to a process. For Ann the object of 0.23 is de-encapsulated to 2/10 + 3/100 For Linda the object 0.23 is de-encapsulated to 23/100. Using the base ten analogy, both Ann and Linda coordinate this process with the process of representing the *fundamental sequence* for a given base. In what follows we discuss the concept of the "fundamental sequence".

In base ten representation the fundamental sequence is a geometric sequence of powers of ten. The construction of this sequence starts as early as children learn to assign to the digits of multidigit numbers values of "tens and ones". This sequence is reconstructed and expanded later to become "hundreds, tens and ones" and "thousands, hundreds, tens and ones" and so on. When the learners are introduced to the idea of decimal fractions, this fundamental sequence is reconstructed to include "tenths, hundredths, thousandths, etc." It is later understood that this sequence is infinite in both directions, that is,

..., 1000, 100, 10, 1, 1/10, 1/100, 1/1000 .... or ... $10^3, 10^2, 10^1, 10^0, 10^{-1}, 10^{-2}, 10^{-3}$... .

Assigning place values to the digits of multidigit mixed decimal numbers may be no more than matching the digits to the elements of this sequence. In order to create a fundamental sequence of another base, the decimal sequence has to be de-encapsulated to a process. In this process the learner has to be able not just to name the elements of the sequence, but to create the next element in line and explain the relationship between the elements. If this process means to the learner "taking consecutive powers of the base of ten" or "multiplying by ten to create the element to the left and dividing by ten to create the element to the right of the given one", it can be easily reconstructed and generalized to the process of fundamental sequence in any given base. In an example of base five, the fundamental sequence is ... 125, 25, 5, 1, 1/5, 1/25... or ...$5^3, 5^2, 5^1, 5^0, 5^{-1}, 5^{-2}, 5^{-3}$... .

Coordinating Ann's process with the fundamental sequence process leads to expressing 0.23_{five} as a sum 2/5 + 3/25. Coordinating Linda's process leads to expressing 0.23_{five} as $23_{five}/100_{five}$, which is 13/25. We summarize those constructions in Figure 1.

Constructing non-decimals - What can go wrong?

Majority of the error patterns that were observed in this study can be explained as a defect in one of the transitions described in Fig. 1. The reading bug pattern for example is an inadequate coordination of Linda's process with a fundamental sequence. Instead of interpreting 0.23_{five} as $23_{five}/100_{five}$, students using the reading bug interpreted this number as $23_{ten}/100_{five}$. The common fraction

substitution error pattern results from construction the fundamental sequence of place values in base five as 1/5, 1/50, 1/500 etc.. This happens when a process of 1/10, 1/100, 1/1000 is constructed by "adding zeros", as Marina stated, rather than dividing by ten or multiplying the denominator by 10. This also may happen when the learner acknowledges multiplication by ten, but doesn't focus on the essence of ten as a base and treats it as a constant value to generate sequences for other bases.

The understanding of a multidigit structure requires more than the understanding of place value ([1]). It requires an understanding that the multidigit number is a sum of the products of its digits, so called "face values", by place values. Even though assigning place values is the essence of the conversion task, it is insufficient to solve the problem. It may be the case that for Marina, for example, the object of 0.23 is de-encapsulated as "two tenths and three hundredths", where "and" means concatenation, writing values sided by side, but not addition. This may explain why Marina couldn't proceed in the task of conversion beyond assigning place values.

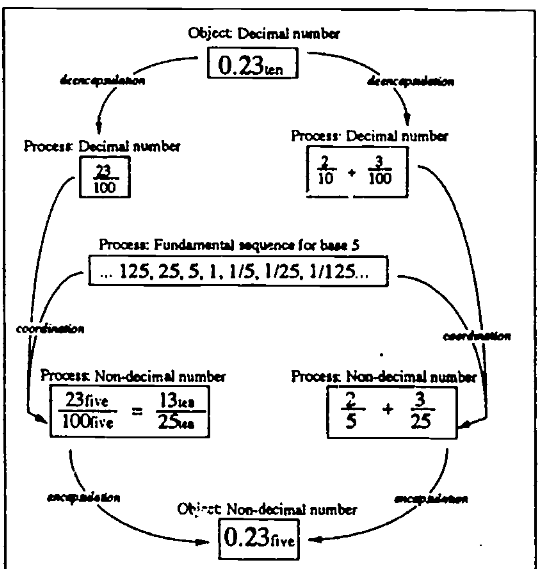

Figure 1
Construction of non-decimal number

Conclusion.

Evidently, from the students' constructions of the fundamental sequences in other bases, as well as from their de-encapsulation of fractional decimal numbers, we learn about the deficiencies in and the development of their understanding of base ten system. The results suggest that preservice teacher's understanding of place value number system is fragile and incomplete. This partial understanding doesn't interfere with the teachers' correct algorithmic performance, but may result in incomplete constructions in the minds of young learners. We believe with Steffe ([11]) and many others that the

improvement of mathematics education in our schools starts with the improvement of mathematical knowledge of teachers. Place value is a basis of positional system arithmetic. We believe that the analysis of teacher's knowledge in this domain in this study highlights possible pitfalls and conceptual difficulties.

According to Ashlock ([1]), the children have difficulties with computational algorithms because they do not have an adequate understanding of multidigit numerals at the time they are introduced to the algorithms. When such children become teachers, many of them still hold adequate understanding of multidigit numerals. Dienes ([4]) advocated the use of multibase arithmetic to reinforce the understanding of base ten positional system. We find renewed interest in this topic at the college level, for education of preservice mathematics teachers . Hungerford ([8]) suggests and strongly encourages a new instructional approach for teaching a "new arithmetic", since it "seem to improve students' understanding of the mathematics involved". On the other hand, Freudentall ([7]), arguing with "innovators", who "like to do structures on other bases", claimed that "if compared with mathematics resulting from pondering more profoundly the subject matter and its relations to reality, unorthodox positional systems are a mere joke". Even though, Freudental didn't exclude other-than-ten bases for "remedial use", and actually stated that "it is a good didactics to motivate pupils by jokes, and an unorthodox positional system may even be a good joke".

Regardless of how one may call the learning/teaching experiment with other-than-ten bases -- be it a powerful tool or a "good joke" -- we appreciate its value for acquiring deeper understanding of place value numeration and multidigit structures. We suggest that non-decimals become an integral part of this topic.

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Probability and Statistics

THE DEVELOPMENT OF ELEMENTARY TEACHERS' STATISTICAL CONCEPTS IN RELATION TO GRAPHICAL REPRESENTATIONS

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The objectives of this research were to determine if there were patterns to elementary teachers' development of statistical ideas. Center of the data and typical of the data were the two concepts studied. Comparisons of teachers' responses before and after instruction were made to determine areas of fixation and ideas about measures of center. Before instruction teachers tended to fixate on large graphical features. After instruction teachers focused on measures of center, particularly the median, to explain their ideas of center, rather than graphical features. More teachers focused on data intervals after instruction to explain typical in the histogram, but these ideas were not stable over the two graphs. We conjecture that fixations and stability are two factors in determining the statistical conceptual development of elementary teachers.

BACKGROUND

Principles of constructivism that reflect the theories of Piaget, von Glasersfeld, and Vygotsky formed the theoretical framework of this research (Davis, Maher & Noddings, 1990). These perspectives assume that knowledge is constructed by the individual. During construction, previous constructions are accessed to restructure schema based on the need to adapt to the environment. The stability of the constructions across different contexts, such as different graphical representations, is another important factor in cognitive development (Clough & Driver, 1986; Garcia, 1991).

Graphical representations of data are central to statistical investigations, and therefore, play an important role in the development of statistics concepts using different contexts. Fixations or centrations were described as errors caused by centering predominantly on larger elements in graphs or other images (Clement, 1989; Bell, Brekke, & Swann, 1987; Inhelder & Piaget, 1964). Berenson, Friel, and Bright (1993) determined that before instruction, many elementary teachers tended to fixate on gross features of graphs to explain their ideas about center, typical, and prediction. These fixations produced a variety of alternative conceptions that were not always stable across contexts.

Russell et al. (1990) found that elementary and middle grades teachers tended to define average or typical as the arithmetic mean. There have been few other investigations of elementary teachers' and students' concepts of other measures of center (Bright & Hoeffner, 1993; Shaughnessy, 1992).

We used the theories of constructivism, including the notion of stability of concepts over different contexts, as the platform for this study. The purpose was to investigate areas of fixations, and the ideas about measures of center that elementary teachers used when explaining center and typical of the data. The stability of ideas across different contexts was considered before and after instruction to determine if there were patterns in the development of these ideas.

METHODOLOGY

The state-wide sample consisted of 55 elementary teachers selected to participate in an extended professional development project. Teachers were given an open-ended, paper and pencil instrument (Berenson & Bright, 1992) two months before a three-week, residential workshop, and at the end of that workshop. The workshop presented many different data analysis activities, involving the teachers in posing questions, collecting data, analyzing data, and interpreting data (Friel, 1993). Center and typical were concepts embedded in workshop activities associated with analyzing data.

These data were analyzed to understand the fixations, the alternative conceptions, and the stability of the concepts when interpreting a line plot and a histogram. Answers and explanations were grouped into 13 categories (See Tables 1 and 2). The first eight categories related to areas of fixation on large elements of the graphs such as: range, horizontal scale, mode(s), vertical scale, data intervals or concentrations, absence of data, number of columns, and sample size. The next three categories were the mean, median, and combinations of mean, median, and mode. Two additional categories were added for missing and other/partial responses. Other responses included a combination of two fixations and/or concepts, and ideas that could not be categorized. Partial responses were those where a number was given, but no explanation as to how that number was obtained. Posttest results are compared to pretest results in Tables 1 and 2.

RESULTS

<u>Center or Middle of the Data</u>

The line plot represented the number of raisins found in 30 boxes; the a horizontal scale was 26 - 40, and the range was 28 - 40. The modes were 28 and 35, the mean was 32.3, and the median was 31.

After instruction, 44 teachers found the median to determine the center of the data in the line plot. Several strategies were used to find the median: halving the sample size (the 15th data point), working from both sides of the data to balance or create two equal groups of data points (15 data points on either

side), and "betweeness" (between the 15th and 16th data points). Two teachers used combinations of measures of center. On the pretest, areas of fixation were: range, horizontal scale, modes, data intervals absence of data, and number of columns. On the posttest a few teachers remained focused on the scale, range, or the absence data. Results are shown in Table 1.

The histogram was a bimodal, normal distribution of lengths of 24 cats with a horizontal scale in inches from 10 - 40, and the range from 1[illegible] - 36. The vertical scale of 0 - 5 was labeled "count." The mean was 29.8, the median was 30, and the modes were 30 and 32.

After instruction, there were 31 teachers who identified the center of the data as the median in the histogram with similar strategies used in the line plot. Two teachers identified the mean as the center of the data and 7 teachers used combinations of mean, median, and mode to define "center." Fixation on the range, modes, or a variety of data intervals were noted in the histogram for the remaining teachers. On the pretest teachers fixated on the horizontal or vertical scales, range, modes, data intervals, absence of data, and sample size.

Table 1. Changes in Teachers' Fixations and Concepts of Center or Middle of the Data

Alternative Conceptions and Fixations	Line Plot		Histogram	
	Pretest	Posttest	Pretest	Posttest
Fixations.				
• Range	12	1	10	6
• Horizontal Scale	9	4	7	0
• Modes	2	0	3	1
• Vertical Scale (most frequent frequency)	0	0	3	0
• Data Intervals (concentration of data)	3	0	4	5
• Absence of Data	3	2	0	0
• Number of Columns	1	0	0	0
• Sample Size (eg. n=24)	n/a	n/a	3	0
Statistics.				
• Mean	1	1	1	2
• Median	13	44	12	31
• Combinations: Median, and/or Mean and/or Modes	0	2	1	7
Incompletions.				
• No Response	6	0	9	0
• Other/Partial	5	1	2	3

n = 55

In summary, after instruction it appeared that fixation on large graphical features in both the line plot and the histogram decreased while the instances of statistical concepts increased. Also, there was a

decrease and less variability in the number of alternative conceptions after the workshop. A few teachers extended their conceptions to include combinations of measures of center. The number of strategies to find the "median" is of interest.

Typical

"Typical" was a term used by Russell et al. (1990) in conjunction with average. Friel and Joyner (1993) further described the search for a replacement of the word "average" to use with elementary teachers and students. An open-ended term was needed to capture ideas of center and the term, "typical" was sufficiently interesting and ambiguous, and controversial for this investigation.

The line plot contained two questions concerning typical:

1. *Some teachers thought that the typical number of raisins in a box was the middle of the data. Do you agree? Explain.*

2. *Other teachers thought that the most frequent number was the typical number of raisins in a box. Do you agree? Explain.*

Teachers were asked to simply identify the typical cat in the histogram and to explain their answer.

On the posttest, responses to the above question 1 were: 15 teachers selected the mean, 4 selected the median as typical, and 1 teachers determined that typical was either the mean or the median in the line plot. The remaining teachers fixated on modes, a wide variety of data intervals, and the absence of data points in question 1. See Table 2.

After examining the answers on the posttest to question 2, it was noted that after instruction some teachers remained fixated on the two modes (n = 10) or the absence of concentrated data points (n = 7) to explain why they disagreed that the modes were typical (refer to question 2). These ideas are represented separately in Table 2, together with those who thought that the modes (n = 8) or data intervals (n = 11) were typical. Nine teachers selected either the mean, median, or combinations of statistics to describe typical in question 2.

In the histogram 19 teachers selected either the median or mean or combinations of mean, median, and mode as typical on the posttest. Five teachers used the modes as typical values and 29 teachers selected different intervals of data to describe typical. Many of these teachers stated incorrectly that the majority of cats fell within their intervals, or did not balance the intervals around 30. Fixations on the pretest were range, horizontal scale, modes, vertical scale, data intervals, and sample size.

In summary, after instruction many teachers did not fixate on large graphical features with the exception of data intervals to determine typical. The wide range of intervals selected and the apparent disconnectedness with measures of center for many of the intervals deserves note.

Table 2. Changes in Teachers' Fixations and Concepts of Typical

Alternative Conceptions and Fixations	Line Plot				Histogram	
	Quest. #1		Quest. #2			
	Pre	Post	Pre	Post	Pre	Post
Fixations.						
• Range	2	0	0	0	2	0
• Horizontal Scale	0	0	0	0	1	1
• Modes	8	7	20	8	19	5
• Two Modes not Typical	1	1	10	10	0	0
• Vertical Scale	2	0	2	1	2	0
(most frequent frequency)						
• Data Intervals	3	11	0	11	7	29
(concentration of data)						
• Absence of Data Intervals	0	0	0	7	0	0
(therefore not typical)						
• Absence of Data	1	7	0	0	0	0
• Number of Columns	0	1	0	0	0	0
• Sample Size (eg. n=24)	n/a	n/a	n/a	n/a	2	0
Statistics.						
• Mean	11	15	4	6	4	7
• Median	5	4	1	1	5	10
• Combinations: Median, and/or Mean and/or Modes	0	1	0	2	0	2
Incompletions.						
• No Response	10	1	8	1	5	0
• Other/Partial	12	7	10	8	8	1

n = 55

Stability

Pretest fixations and conceptions were examined for stability over the two graphical representations for center and typical. The majority of responses on the pretest indicated fixations on large graphical features to explain ideas of center and typical. However, these fixations were not stable over the two graphical representations on the pretest. We propose that teachers may have been in the process of defixation or decentration to focus on different parts of the field of different contexts to reduce their errors.

> *There is not only centration, there is decentration. By centering on different parts of the field successively, the subject creates a situation in which these parts affect each other reciprocally, thus reducing the errors cause by centering predominantly on the larger elements* (Gruber & Voneche, 1977, p. 648).

The stability of responses on the posttest were somewhat different for the two concepts. Recall that the median statistic was used by more teachers to explain center, and data interval fixations were used by

more teachers to explain typical. More stability was noted for the concept of center, and less stability for the concept typical. However, the following combinations were noted for both concepts on the posttest:

Statistic a & statistic a - the same statistic was associated with the concept in both representations.

Statistic a & statistic b - different statistics were associated with the the same concept.

Fixation a & fixation a - the same fixation was associated with the same concept.

Fixation a & statistics a - a fixation and a statistic were associated with the same concept.

Fixation a & fixation b - two fixations were associated with the same concept.

More research is needed to fully understand the role of stability in concept formation. The number of different contexts presented for investigation may be a consideration. Additionally, interviews will provide more detailed information for understanding the complexities of development in teachers' statistical thinking.

CONCLUSIONS

In the coming years, there will be a greater need to teach statistical concepts and graphical representations of data to preservice and inservice teachers. The variability of fixations and conceptions have implications for the content and pedagogy of professional development. Further research can ask questions concerning how to link the fixations and conceptions to build stable constructions. What activities, problems, graphical representations, and sequence of curriculum affect construction? Are there additional alternative conceptions created by these interventions that may block more complex construction? Do children's constructions parallel those of their teachers?

Data and its analyses are pervasive in this age of uncertain information. Mathematics educators recognize that elementary school is not too early to begin construction of statistical concepts. The results of this study indicate that strong professional development programs are needed to assist elementary teachers in the construction of stable and viable statistical schema.

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STATISTICS KNOWLEDGE OF ELEMENTARY TEACHERS

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This study examined 56 elementary school teachers' knowledge of statistics before and after a three-week statistics workshop. Content knowledge was assessed through a paper-and-pencil instrument of twelve, open-ended statistics questions; responses were scored holistically. Understanding of relationships among four critical statistics concepts (selected from the state curriculum) was assessed through analysis of concept maps. Performance on eleven of the statistics questions improved significantly. Teachers' written responses improved in completeness and clarity. Teachers included more of the four concepts in their second concept maps and showed more relationships among the four concepts. There were not dramatic increases, however; concepts were stilled viewed mainly as independent. Future instruction for teachers needs to deal more explicitly with relationships among statistics concepts.

Since 1989 statistics has been one of the seven strands of the North Carolina Standard Course of Study for elementary school mathematics. However, the statewide, end-of-grade tests that have been given have included so few items on statistics that teachers have not felt the need to devote significant amounts of instructional time to teaching the statistics strand. New tests are now in place (beginning in 1992-93) that include roughly equal representation from each of the strands. Teachers will have to begin to teach statistics if their students are to score well on these new tests. Unfortunately, many elementary school teachers have little background in statistics, so in preparation for the new tests, significant inservice is needed. These inservice programs should be informed by understanding of the knowledge of statistics that teachers bring to the inservice programs. This research was a beginning attempt to develop understanding of teachers' thinking about statistics, both before and after an inservice intervention.

Research on understanding of statistics concepts (as opposed to probability) has almost exclusively dealt with students older than those in elementary school (e.g., Bright & Hoeffner, 1993). The results suggest that inappropriate use of statistics concepts and processes in problem solving persists well into adulthood. Very little attention has been paid to the knowledge of teachers of any grade level (Shaughnessy, 1992), though Mokros and Russell (1991) did begin to identify specific strategies that elementary school teachers use to solve statistics problems related to an understanding of the concept of mean. All of the research that we have found reports understanding of individual concepts (e.g., mean) or strategies (e.g., balancing data around the mean). None of the research focuses on teachers' general ideas about statistics content or their understandings of relationships among statistics concepts.

Understanding participants' notions of relationships was critical, since the content of statistics for this project involves statistics as a process of investigation: pose a question, collect data, analyze data, and interpret results. The parts of this process are intended to be viewed as an interrelated whole, rather than as separate steps. It was important, therefore, to know if participants shared this view of statistics.

Procedures

Subjects

This study is part of a larger project (TEACH-STAT, funded by the National Science Foundation) to assist elementary teachers develop deeper understanding of statistics concepts and learn instructional strategies for teaching those concepts to children. Planning for the inservice program was carried out during 1991-92, and the first group of teachers participated in a residential, three-week workshop during July, 1992. The teachers (N=56) came from across the state and had been selected from each of nine regions because of their enthusiasm for teaching and their desire to learn about how to teach statistics more effectively. Thus, they cannot be assumed to be representative of the "typical" teacher. However, there is no evidence in the background information that we collected that would suggest that these teachers have any more mathematics knowledge (or statistics knowledge) than other teachers.

Instrumentation

Two data gathering techniques were used. One was a Content Survey developed by the TEACH-STAT central project staff, with input from the faculty site coordinators. The Content Survey included twelve open-ended items:

1-3. Read and interpret already represented data in a histogram, boxplots, and a scatterplot

4. Make a representation of provided data. The representation involved categorizing the data.

5. Describe how they would explain measures of center (mean, median, mode) to fifth graders

6-9. Answer questions about the concept of mean

10. Answer questions about what is "typical" about a given set of data

11. Distinguish between sample and population

12. Answer questions about the concept of normal curve

The second technique was a concept map of the concept, statistics. The purpose of obtaining concept maps was to understand how participants related various aspects of statistics.

Procedures

Teachers at each site met with the site coordinator (local university mathematics educator, science educator, or statistician) for one afternoon during late spring 1992. Part of the time at this meeting was devoted to completion of the untimed Content Survey; response times ranged from 20 to 45 minutes. The Content Survey was administered a second time in summer 1992 (at the end of the three-week workshop on statistics). These data were available on all 56 subjects.

On the morning of the first day of the summer workshop, concept maps were explained and instructions for making concept maps were provided. Simple illustrations of concept maps were given to illustrate both the hierarchical nature of a concept map and the interconnections that could be drawn among concepts across the various levels in any particular map. Correct and incorrect ways of drawing components of concept maps were identified (e.g., all connecting lines would be labeled in a correctly drawn concept map), though it was repeatedly emphasized that a concept map as a whole was never viewed as right or wrong, only as reflective of that individual's thinking about the concepts. Instructions for drawing a concept map concluded with the advice that a list of concepts might first be organized

according to groups of ideas that seemed to fit together, after which the concept map could be drawn. Participants were asked to think silently about the concepts that came to mind when the word, statistics, was mentioned and to make a list of those related concepts. Participants were asked to organize their list of concepts into related groups. Then they were asked to write statistics in an oval at the top of a blank page and to begin work on drawing the concept map. About 35 minutes was allotted for construction of concept maps, though everyone completed their maps in about 30 minutes.

Concept maps were gathered again in February 1993 at a meeting of all participants and site coordinators. The instructions for preparing a concept map were repeated before participants were asked to draw their maps. Because of low attendance at this meeting, only 32 pairs of concept maps were available for analysis.

Analysis

The items on the Content Survey were scored on a holistic basis on a scale of 0-4; 0 indicated that there was no response; 1, that the response was irrelevant; 2, that partial understanding displayed; 3, that almost complete understanding was displayed; and 4, that the response was fully acceptable. The scores for individual items were compared across the two administrations with t-tests.

The concept maps were analyzed in terms of four critical concepts in the state curriculum for grades K-6 as identified by the researchers: graphing, collection of data, prediction, and measures of center. Maps were first coded according to whether these concepts were included in the maps. Then relationships among the four concepts were analyzed by categorizing pairs of concepts as dependent or independent. Concepts were classified as dependent if they appeared on the same branch of the map; otherwise, they were classified as independent. For each pair of concepts, tallies were made of the number of times one concept was subordinate to the other and the number of times the concepts were independent.

Results

Content Survey

On the first administration of the Content Survey, there was a wide range of responses to each of the items. For questions related to reading histograms, approximately one-third of the responses reflected good understanding of the content knowledge being assessed. Teachers' knowledge with regard to interpreting boxplots and scatter plots was quite limited. Responses to questions on measures of center were similarly weak; justifications of answers were often confusing, incomplete, or inaccurate. Analysis of responses provides insight as to what they do and don't know.

On the second administration, the variety of responses was not so great. More importantly, however, there were noticeable changes in the clarity of explanations offered by teachers, in part evidenced by greater use of statistics vocabulary.

Comparison of the scores across administrations indicates that performance improved significantly on all items; performance was significantly better at the .001 level for six items, at the .01 level for three items, and at the .05 level for two items. Only one item (#9) did not show significant improvement.

Concept Maps

Inclusion of critical concepts. The concept, graphing, was subdivided into three parts: graphs/graphing only, specification of specific school graphs (i.e., bar, pie, circle, line, picture, Venn diagram), and specification of specific statistical graphs (i.e., histogram, line plot, stem and leaf, box plot, scatterplot). Inclusion of "graphs only" in concept maps decreased from 41% to 22% of the maps, while inclusion of "school graphs" and "statistical graphs" increased, respectively, from 47% to 72% and from 9% to 78% of the maps. Clearly, there was a shift away from consideration of graphs in general to consideration of specific graphical representations.

The concept, collection of data, was also subdivided into three parts: surveying (i.e., surveys, polls, interviews, rating scales, questionnaires), counting (i.e., tally/count, observation, experiment, measurement, investigation), and sampling (i.e., sampling, population). Inclusion of the each of the three parts in concept maps increased. "Surveying" increased from 41% to 50%, "counting" increased from 38% to 41%, and "sampling" increased from 22% to 41%.

Prediction and measures of center (i.e., mean, median, mode, average) were tallied as specific concepts. Inclusion of "prediction" in concept maps increased from 16% to 28% of the maps, and inclusion of "measures of center," from 66% to 78%.

Counts were also made of the number of maps that included the four concepts. For the first/second set of maps, 3%/0% included none of these concepts, 16%/3% included one concept, 34%/25% included two concepts, 41%/50% included three concepts, and 6%/22% included all four concepts.

Relationships among concepts. In the first set of maps, for graphing and collection of data (which appeared together in 18 maps), graphing was a subordinate concept in 9 maps, and the concepts were independent in 9 maps. No one suggested that collection of data should be subordinate to graphing. For graphing and measures of center (which appeared together in 20 maps), measures of center was subordinate in 1 map, and the concepts were independent in 19 maps. For collection of data and measures of center (which appeared together in 13 maps), measures of center was subordinate in 2 maps, and the concepts were independent in 11 maps. For graphing and prediction (which appeared together in 5 maps), the concepts were independent in all 5 maps. For collection of data and prediction (which appeared together in 3 maps) prediction was subordinate in 1 map, and the concepts were independent in 2 maps. For prediction and measures of center (which appeared together in 3 maps).the concepts were independent in all 3 maps.

In the second set of maps, for graphing and collection of data (which appeared together in 27 maps), graphing was a subordinate concept in 10 maps, collection of data was a subordinate concept in 1 map, and the concepts were independent in 16 maps. For graphing and measures of center (which appeared together in 24 maps), measures of center was subordinate in 5 maps, and the concepts were independent in 19 maps. For collection of data and measures of center (which appeared together in 21 maps), measures of center was subordinate in 8 maps, and the concepts were independent in 13 maps. For graphing and prediction (which appeared together in 9 maps), prediction was subordinate in 1 map, and the concepts were independent in 8 maps. For collection of data and prediction (which appeared together in 8 maps)

prediction was subordinate in 4 maps, and the concepts were independent in 4 maps. For prediction and measures of center (which appeared together in 8 maps) the concepts were independent in all 8 maps.

Finally, for each set of maps, those that contained more than one of concepts (26 maps from the first set and 31 maps from the second set) were coded according to the number of possible relationships that were actually drawn. For example, if two concepts were included, there would be only one possible subordinate/superordinate relationship that could be drawn in the map, while if four concepts were included, there would be six possible subordinate/superordinate relationships possible. Since the number of possible relationships was dependent on the number of concepts included in a map, data were collapsed according to whether there the number of relationships drawn was zero or one or more (Table 1).

Table 1. Percentages of Maps Illustrating Possible Relationships

Number of concepts in map	Number of relationships 0	1 or more
	Map One	
2	81.8	18.2
3	46.2	53.8
4	50.0	
	Map Two	
2	87.5	12.5
3	37.5	62.5
4	12.5	87.5

Discussion

Understanding of content. The workshop seemed to have a measurable effect on teachers' performance in answering statistics questions. First, their performance improved significantly on almost all items on the Content Survey, suggesting that their content knowledge generally improved. However, there continued to be numerous gaps and holes in their understanding. Analysis of these gaps allowed the TEACH-STAT faculty to redesign the workshop so as to attempt to fill those gaps. Second, all four critical statistics concepts appeared more often in the second set of concept maps than in the first set, and individual teachers included more of these concepts in their second maps than in their first maps. This suggests that the teachers' views of statistics had broadened, perhaps indicating richer cognitive understanding of the content.

Relationships among concepts. First, for the first set of maps, less than half of the teachers included three or more of the four concepts in their concept maps. Graphing was almost universally seen as important to understanding of statistics, while measures of center and collecting data are viewed as important by half to two-thirds of participants. This suggests not only that many teachers may not have been very familiar with the objectives of the statistics strand of the state curriculum but also that teachers individually apparently did not think of these four concepts together when they consider what might be

included in the domain of mathematics labeled "statistics." For the second set of maps, almost three-fourths of the teachers included three or more of the four concepts in their concept maps. Graphing, collection of data, and measures of center were individually included in more than three-fourths of the maps. Prediction remained a concept infrequently considered by teachers as part of statistics. Teachers views of statistics in their second maps, however, certainly reflected more attention to concepts that are important in the state curriculum.

Second, in the initial maps, understanding of subordinate/superordinate relationships was not common among teachers, but when such relationships existed among pairs of concepts, teachers always saw those relationships in the same direction. There were no differences of opinion among teachers about which concept in a pair was subordinate to the other. This is somewhat surprising, given that there are typically many different views of appropriate sequencing of mathematics topics. In the second set of maps, the number of subordinate/superordinate relationships illustrated increased, but it was still common for teachers to view the four concepts as independent. Almost half (45%) of the teachers showed no relationships among these concepts at all. With one exception, however, when relationships were shown they were again shown consistently across maps. Since the summer workshop did not deal with hierarchies of relationships, this result raises the possibility that there may be some "intuitive" or "natural" development of relationships among these four concepts. Understanding this intuition might prove fruitful for helping teachers understand students' thinking so that instructional decisions can be made to take advantage of students' thinking.

Third, in both sets of maps, there was a positive relationship between the number of concepts included in a map and the percentage of possible relationships that were actually drawn (Table 1). Inclusion of more concepts in a map might reflect a richer understanding of statistics, and one might expect that as the richness of the conception of statistics increases, the more likely that relationships will be seen among those concepts. This finding might suggest that one goal of statistics instruction for teachers is to improve the richness of understanding of statistics so that those teachers see more relationships among concepts. Increasing personal understanding of relationships might be a necessary first step toward helping teachers improve in their ability to teach students about the relationships among concepts.

General comments. Teachers thought somewhat differently about statistics before and after the workshop. They recognized more of the "bits and pieces" of statistics that mathematicians and mathematics educators might want categorized as "statistics content." However, teachers seemed to have poorly developed relationships among these bits and pieces. Rather, teachers seem to view concepts as isolated from each other. For example, although more teachers included "prediction" in their second concept maps, none of them suggested in either map that prediction might be developed from knowing measures of center. That is, none of them showed a subordinate relationship for prediction to measures of center. If teachers are to understand these kinds of relationships, workshop leaders are probably going to be quite explicit about discussing or modeling them.

Teachers' lack of relational understanding would seem to inhibit their effectiveness in helping elementary school students begin to develop relationships among the concepts. For example, teachers may

not be able to help children discuss clearly the relationships between pairs of concepts. Too, teachers are perhaps unlikely to connect pairs of concepts during instruction, so students are unlikely to develop connections between the pairs of concepts. Further, this lack of relational understanding would seem to inhibit teachers' skill at structuring worthwhile statistics problems for elementary school students and to assist those students overcome difficulties that inevitably arise while solving such problems.

There is still much we need to know about how to help teachers develop an adequate understanding of statistics so that their instruction is effective. Yet, if the statistics standards (e.g., NCTM, 1989) are to be implemented, teachers must develop that understanding. It is possible, especially based on the data from both the Content Survey, to make recommendations about the kinds of professional development experiences that are appropriate in helping teachers "round out" their content knowledge of statistics in order to be better prepared to address the learning needs of the students they teach. Such information, particularly when tied to the observations of the concept maps, may be useful to mathematics educators who are concerned with what teachers' know about statistics content.

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UNDERSTANDING THE LAW OF LARGE NUMBERS

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Twelve students answered questions involving the distribution of sample means both before and after an instructional intervention. Correct performance improved on these problems but dropped on problems having to do with the distribution of samples.

Perhaps the most fundamental concept in inferential statistics is the sampling distribution. The most basic property of sampling distributions is the law of large numbers (LLN) — that statistics of larger samples (e.g., means) more closely approximate the corresponding parameters of populations and are thus less variable than those of smaller samples. Mastery of this law can carry a student a long way in statistics. Early research by Kahneman and Tversky (1972) suggested that people had fairly poor intuitions about the LLN. Below is a problem we adapted from them.

> Post-Office Problem (Tails version). When they turn 18, American males must register at a local post office. In addition to other information, the height of each male is obtained. The national average height for 18-year-old males is 69 inches (5 ft. 9 in.). Every day for one year, 5 men registered at a small post office and 50 men registered at a big post office.
>
> At the end of each day, a clerk at each post office computed and recorded the average height of the men who registered there that day.
>
> I would predict that the number of days on which the average height was more than 71 inches (5 ft. 11 in.) would be
>
> a. greater for the small post office than for the big post office.
> b. greater for the big post office than for the small post office.
> c. There is no basis for predicting for which one, if either, it would be greater.

Only 15-20% of the undergraduates in a study by Well, Pollatsek, and Boyce (1990) answered this problem correctly (option *a*). Roughly 65% answered *c*, suggesting that they believed sample size has no effect on the variability of the mean. Kahneman and Tversky (1972) concluded that this answer is indicative of the "representativeness heuristic," according to which the likelihood of drawing a particular sample is judged by considering how similar the sample is to the parent population. In this problem the two samples differ only in size, but because sample size is not a feature of a population, it is not used as a criterion to judge the similarity between the samples and the population. Accordingly, people tend to think even small samples will adequately represent the critical features of the population.

We have been designing software that demonstrates various features of the LLN and testing it in tutoring interviews with undergraduates. The software permits drawing samples of various sizes from a population of elements and plotting the means of those samples in a

sampling distribution. By comparing the empirically constructed distributions of different sample sizes, one can observe that the means of larger samples tend to cluster more closely around the population mean than do means of smaller samples. In an earlier study we conducted, students received instruction on sampling distributions of the mean. Computer simulations were used to construct sampling distributions based on different sized samples. This intervention increased correct responses to questions concerning the variability of means as a function of sample size by a factor of four. However, when we asked questions about the distribution of elements within a sample, many subjects over generalized what they learned about statistics to the distribution of the samples themselves (Lohmeier, 1992; Well, Pollatsek, & Boyce, 1990). In this study, we look more closely at the nature of students' responses to various problems before and after similar computer-enriched instruction.

Method

Tutoring interviews were conducted with 12 undergraduates recruited from psychology courses at the University of Massachusetts. The individual sessions lasted about 75 minutes and were videotaped. Before introducing the simulation software, students solved two problems. The first problem "Post-office tail" was identical to the one given above. The second problem, "Post-office lure," is given in abbreviated form below:

> Post-office lure. When they turn 18 . . . 50 men registered at a big post office [same paragraph as in the Post-office tail problem].
>
> At the end of one day, a clerk at each post office computed the percentage of men who registered that day who were less than 66 inches tall.
>
> I would expect that the percentage of men less than 66 inches would be [same three options as in the Post-office tail problem].

This problem was constructed to test the degree of understanding of the LLN after instruction. Note that the lure problem does not ask about the distribution of sample *means* from the two post offices over the period of a *year*, but about the distribution of the *individual men's heights* at the post offices on a particular *day*. Because the expected value for the requested percentage in both samples equals the corresponding percentage in the population, the correct answer to the lure problem is *c*.

For each problem, students were instructed to first read the problem aloud and then to solve it while "thinking aloud." The interviewer (Lohmeier) encouraged students to vocalize their thoughts and also probed them when responses were unclear. After selecting and justifying an answer, the page on which the problem was written was turned over, at which time the student was asked to "restate the problem in your own words."

Following the instructional intervention, which lasted about 40 minutes, students again solved the Post-office tail and Post-office lure problems under the same instructions. These

were followed by three transfer problems. The first of these, "Post-office center," was similar to the Post office tail problem. The only difference was that the former asked about "the number of days on which the average height was within 2 inches of the national average (69 plus or minus 2 inches)" rather than about the frequency of means in the tail of the sampling distribution. Thus, the correct answer in this case is *b*, that the larger post office will likely record more such days. The other two transfer problems used different cover stories but were structurally compatible with the Post-office tail problem ("Geology tail") and Post-office lure problem ("Treasury lure").

Results

We discuss three aspects of students responses: a) their answers, b) rationales for their answers, and c) the accuracy of their restatements of the problem.

Answers to Problems

Answers (along with students' restatements of the problem) are summarized in Tables 1 and 2. Table 1 shows performance on the tail and center problems, Table 2 on the lure problems.

Table 1. Summary of performance on tail and center problems

Before				After											
Post Office Tail				Post Office Tail				Geology Tail				Post Office Center			
S#	ans	qty	object	S#	ans	qty	object	S#	ans	qty	object	S#	ans	qty	object
8	S	#	mean	8	S	#	mean	9	S	#	mean	7	L	#	mean
1	E	#	mean	1	S	#	mean	1	S	%	mean	8	L	#	mean
11	E	%	mean	7	S	#	mean	2	S	%	mean	10	L	P	mean
3	E	?	mean	2	S	P	mean	7	S	%	mean	3	L	?	mean
4	E	?	mean	10	S	P	mean	4	S	P	mean	11	L	%	elem
5	E	?	mean	9	S	%	mean	12	S	?	mean	9	L	%	?
9	E	?	?	6	S	?	mean	8	S	?	?	5	L	%	?
7	E	#	elem	12	S	#	?	6	S	%	elem	6	L	P	?
2	L	#	elem	3	S	?	?	11	E	%	mean	1	L	•*	•
12	L	#	elem	11	E	#	mean	5	E	P	mean	4	L	•	•
6	L	P	elem	5	L	#	mean	10	E	?	mean	2	S	#	mean
10	L	P	?	4	L	#	elem	3	E	%	?	12	S	?	mean

* Missing values indicate that the student was not asked to restate the problem.

We look first at answers, coded under columns labeled "ans." On the tail problems, the correct response is that S (the smaller sample) is more likely; for the center problem, L (the larger sample) is correct. Correct answers are grouped at the top of a column and enclosed in a

box. The number of correct answers on the Post-office tail problem increased from 1 to 9 over instruction, and remained reasonably high (8 and 10) on the 2 transfer problems. Moreover, for the most part, students who responded correctly had not simply learned to answer "smaller," since they gave the correct answer "larger" when the question concerned the frequency of values in the center as opposed to the tails of the sampling distribution. Judging from these results alone, one would conclude that the instruction had been quite successful.

Table 2. Summary of performance on lure problems

Before				After							
Post Office Lure				Post Office Lure				Treasury Lure			
S#	ans	qty	object	S#	ans	qty	object	S#	ans	qty	object
1	E	%	elem	3	E	%	elem	9	E	%	elem
3	E	%	elem	9	E	%	elem	1	S	%	elem
4	E	%	elem	11	E	%	elem	2	S	%	elem
5	E	%	elem	12	E	#	elem	6	S	%	elem
11	E	%	elem	5	S	%	elem	8	S	%	elem
9	E	%	?	8	S	%	elem	10	S	%	elem
7	E	•	•	1	S	#	mean	12	S	?	elem
6	L	%	elem	2	S	?	mean	4	S	?	mean
8	L	%	elem	10	S	?	mean	3	L	%	elem
10	L	%	elem	4	L	%	elem	5	L	%	elem
12	L	%	elem	6	L	%	elem	7	L	%	elem
2	L	%	?	7	L	%	elem	11	L	#	elem

Performance on the lure problems appears to show a negative effect over instruction. Before instruction, 7 students correctly answered the Post-office lure; after instruction, only 4 correctly answered the same problem, and only 1 correctly answered the Treasury lure. These results replicated findings of an earlier study (Well, Pollatsek, & Boyce, in preparation), suggesting that after instruction students have some understanding of the LLN, but are not yet able to fully discriminate between situations when it is and is not applicable.

Restatements of Problem

After students had given an answer, they were asked to restate the problem in their own words. These statements gave an indication of how students encoded the problem, providing a context for further evaluating their answers. We were interested, in particular, in the degree to which incorrect answers resulted from encoding errors. If, for example, students misinterpreted a tails problem as asking for the distribution of elements in a sample (i.e., as a lure question), the answer E (equal) would be correct given their interpretation. Entries in Tables 1 and 2 under the columns headed "qty" (quantity) and "object" were obtained by

coding students' restatements. The object column shows whether the student said the problem was asking about a) means of the two samples or b) sample elements (elem). The qty column codes a student's restatement according to the quantity sought: number (#), percent (%), or probability (P). For example, below is the restatement by S6 of the Post-office tail problem before instruction, which was coded as "P, elem" based on the underlined phrases:

> What the odds would be that someone 5'11" would be, would it be greater that they'd go to the smaller post office — well, that you'd get someone that height at the smaller post office or the larger post office....

Overall, the accuracy of encoding the tail problems improved over instruction. Whereas before instruction only 3 students correctly restated the tail problem, after instruction 8 gave correct restatements on the Post-office tail problem and 7 on the Geology problem. (Though "number" was the specified quantity for the tails and center problems, "percent" and "probability" were also considered correct encodings.) Moreover, although before instruction there were 5 clearly incorrect (as opposed to incomplete) responses on the Post-office tail problem, in each of the post-instruction problems only 1 student's restatement was incorrect.

Nine of the restatements of the lure problem were correct before instruction, and none were incorrect. ("Number" was considered an incorrect encoding of the quantity on lure problems.) While the number of correct restatements remained about the same after instruction (8 and 9), the number of incorrect responses grew to 4 and 2, with half of these reinterpretations involving means rather than elements. This is further indication that performance on questions involving distributions of the mean improved, to some extent, at the expense of performance on questions involving distributions of samples.

Our major interest in coding restatements was to determine to what extent incorrect answers resulted from related encoding errors. People may respond "equal" (E) to the problems about distributions of means because they interpret them as questions about the percentage of elements in the sample distribution above or below some value. We obtained little evidence for this hypothesis. Of the 12 E answers given over the 4 sampling-distribution problems, in only 1 case was the problem restated in terms of elements.

Students who responded L to the tail problems may also have been interpreting the problems as asking about sample elements. This answer would be reasonable if they misinterpreted the question as asking not only about sample elements (rather than means), but also about the number (rather than percent) of elements. The responses of S2 and S12 on the pretest were consistent with this interpretation. However, they gave the same answer to the lure problem before instruction, and yet correctly interpreted the questions as concerning percents. We say more about this below.

Rationales

We can make only a few comments about rationales students gave. First, after instruction students who correctly answered the sampling-distributions problems gave rationales that were, for the most part, consistent with the LLN. Summing over the 3 after-instruction problems, 22 of the 27 correct answers (81%) were accompanied by rationales that were consistent with the LLN, as exemplified by the rationale of S9 to the Post-office tail problem after instruction.

> I don't think there is any basis again . . . no, no, no. These are means. It would be greater for the small post office. This is a shift that I didn't make from talking about a population distribution to the distribution of sample means. This is the same question I read before, isn't it? And they're talking about two different distributions. I did not make that shift . . . . When the sample is smaller in a distribution of sample means, the standard deviation will be greater from the mean, and for any given number away from the mean, there will be a greater number of points.

This excerpt suggests this student learned a new distinction as a result of instruction. This is the only student who, after instruction, answer all problems correctly.

Students who incorrectly answered E to the sampling-distribution problems for the most part argued as did S3: "I just don't think there is enough information for me to answer that." This same argument was also given by most students who correctly answered E to the lure problems. The vagueness of this argument suggests that most correct responses to the lure problem before instruction were not based on the understanding that, on average, the two samples would have the same percentage of elements in the specified area. It also suggests that incorrect responses of E on the tail problems were not based so much on the representativeness heuristic, but on the belief that there is not enough information given to make a prediction — a variety, perhaps, of an equal-ignorance argument.

Finally, the majority of students (78%) who gave incorrect answers of L to the lure and tail problems justified these with what we refer to as the "more-is-more" argument. This is the belief that the larger sample will have larger x, where x can be scores, means, percentages, variability, etc. For example, below is the justification given by S2 to the Post-office tail problem before instruction:

> There would be more people going to the bigger post office. So the average number of days would be greater for the big post office . . . . If they have more people going to the big post office, then the average is going to come out higher than the small one.

For a few students, this belief may consist primarily of a failure to understand the difference between number and percent. Some made no apparent distinction between the two; others distinguished number and percent but vacillated between them during the interview;

still others apparently recognized some difference but, reasoning from the more-is-more notion, thought it unimportant, as illustrated, again, by S2:

> I guess it would be greater for the big post office. When you're talking about percentage, you're talking about a number of people .... You'd have less people going to the small post office and more at the big post office ... more of a chance at the big post office.

The more-is-more rationale may explain L answers on lure problems by students who nevertheless correctly encoded the problems as dealing with percents rather than numbers.

Conclusions

We should emphasize that our objective in these tutoring studies is not to design a one-hour intervention for teaching the LLN. It may even be overly ambitious to expect much understanding after a semester of statistics instruction. Rather, we are interested in exploring related intuitions that exist prior to instruction and the nature of difficulties that arise as various concepts are introduced as part of instruction. Our research suggests that students' difficulties learning the LLN are not based on incompatible intuitions, such as the representativeness heuristic, and that, indeed, by using the computer to demonstrate the construction of sampling distributions, most students quickly develop some understanding of why means of larger samples are less variable than those of smaller samples. However, poor understanding of percentages, and whatever other misunderstandings support the more-is-more rationale, present a barrier for many students. Finally, we used lure problems in this study to test the degree of understanding of the LLN. We are not satisfied that students understand the LLN as long as they continue applying it to distributions of sample elements. While it was not part of instruction in this study, we would expect that it would facilitate understanding if during instruction students were given problems which deal both with distributions of means and with distributions of samples, addressing explicitly why the LLN applies in one case but not in the other.

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YOUNG CHILDREN'S INTERPRETATIONS OF CHANCE SITUATIONS

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SUMMARY

Research literatures offer discrepant views concerning what understanding of chance entails, its relation to thinking probabilistically, and the nature of alternative interpretations. This study capitalized on the technology of videotapes to closely examine children's interpretations within tasks involving randomness and a qualitative level of differential probabilities. The author worked individually with 12 kindergartners and 12 third graders in their exploration of four different spinners. Two independent coders analyzed all of the tapes for manifestations of interpretation. Codings across 258 data points resulted in a 82% inter-rater reliability. Understanding of chance followed understanding of probability. Chance apart from probability was rare. However, probabilistic reasoning without chance was common, in the sense that children held strong or absolute expectations about outcomes based on relative distribution of colors in the immediate spinner.

There are increasing calls for educators to incorporate chance and probability in the mathematics curriculum beginning at the primary grade years (*e.g.* from the National Council of Teachers of Mathematics and the National Sciences Board of the National Academy of Sciences). This recommendation comes at a time where there is little agreement concerning the psychology of chance. Analysis of intellectual history and the research literatures of children's and adults' cognition offer a miriad of conceptualizations of what chance entails and how hard it is to grasp.

Within the adult probabilistic reasoning literature, Tversky and Kahneman's research together with the numerous studies their work has evoked have been characterized by an intense debate concerning the adequacy of adults' understanding of chance and probability. On the one hand, Nisbett, Krantz, Jepson & Kunda (1983) argue that adults frequently use ideas of chance and probability in everyday reasoning and that these ideas and the heuristics based upon them are "part of people's intuitive equipment."

However, Tversky and Kahnemann (Tversky & Kahneman, 1974, Kahneman, 1991) argue that adults fail to recognize the extent to which chance contributes to what they experience of the world and this failure (even on the part of the statistically sophisticated) leads to consistent "cognitive illusions."

Within the children's cognition literature, Piaget's (1975) seminal work on the subject posits that prior to about age seven, children fail to grasp chance due to their "overriding sense of order." Around seven, children come to conceptualize indeterminacy, *albeit* initially without a sense of the patterns that can emerge across chance events. Piaget argued that probability emerges in adolesence, first understood as the relation between variability and constancy within these patterns. Again a literature in response has manifested numerous apparent inconsistencies with the original research. For example, Kuzmak and Gelman (1986) conclude that by five years of age children understand randomness in that they differentiate between determined and indetermined events.

Different literatures also present contradictory views of the genesis of probability and its relation to chance. Falk, Falk and Leven (1980) assume Piaget's stance that probability is built upon the underlying idea of chance as randomness, but come to radically different conclusions concerning the time at which the child can understand it. They conclude that even the first grader can understand probability, at least on some intuitive level. From the perspective of historical analysis, Gigerenzer *et al.* (1989) argue that probability does not necessarily entail an understanding of chance. Probability may first emerge in conjunction with a purely deterministic sense of expectation (as in the thinking of Pascal and Pierre Simon de la Place).

From the current state of these literatures, it is ambiguous what adult or child understands of chance, its relation to probability, and the nature of alternative interpretations. This research project capitalizes on the rich range of conceptualizations of chance raised in the various literatures and the power of videotape technology to re-examine the nature of children's understandings. Kindergartens and third graders individually participated in an instructional experiment across a sequence of tasks involving aspects of chance including randomness, a qualititative level of probability, and sampling. This first paper examines children's chance, probabilistic and alternative interpretations in a spinners task.

METHODOLOGY

Subjects consisted of 12 kindergartners and 12 third graders, evenly divided by gender, who volunteered from a multi-ethnic school of middle and lower middle S.E.S.. Each subject worked individually with the experimenter/author across the five task series, in three or four sessions of approximately 30 minutes in length. All sessions were videotaped.

The study took the form of an instructional intervention, structured to foster reflective analysis upon data the child collected across a range of situations. Each activity was designed to both diagnose and foster the children's understanding of some aspect or aspects of chance. This paper focuses on the second activity, a series of spinner games[1]. Spinners were selected since the simultaneous visual display of the entire sample space would seem to support a probabilistic interpretation. With spinners students are also able to generate their own data in a relatively efficient manner.

Materials in the spinner activities set included a game board, colored chips and a variety of spinners. The game board was in the form of a large bar graph with two columns of eight squares. A red chip and a yellow chip were used in conjunction with three spinners of varying color distribution: 50% red / 50 % yellow, 75% red / 25% yellow and 90% red / 10 yellow %. A yellow chip and a green/red chip (two chips glued together) were used in conjunction with a four-color spinner, with an even distribution of red, green, blue and yellow.

Each game began by the child making a spinner choice or chip choice, with the goal of trying to reach the top of the game board first. Players advanced when it was their turn and the spinner landed on their color. (To avoid the potential control of gentle spins, the spinner had to spin around at least twice.) Prior to each game, the child made predictions concerning who would win and where the anticipated loser would end up on the board. The child and experimenter played many games with the different spinners, as well as repeats of games with the same spinner.

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[1] The spinner task was preceded by an elaboration of Piaget's (1975) marble tilt box problem, involving predictions and explanations of progressive randomization of marbles, initially arranged by color.

Table 1: Abbreviated Version of Coding Schema

Category	Coding Criteria
Personal Aesthetics	• S makes selection of chip or spinner on the basis of aesthetics, *cf.* color preference
Personal Control	• S assumes one can or should be able to control spinner outcome by means of nuances in how one acts on the spinner or general competence with spinners
Rejection of Probability	• S explicitly denies differential expectations for different outcomes that have in fact unequal probabilities
Data Driven	• Interpretations based on patterns in what has already happened, devoid of any sense of spinner function as affecting these patterns
Probability without Chance	• Deterministic probabilities (O or 1) • No aspect of chance • Focus on given outcome, not distribution of events
Chance without Probability	• Conceptualization of spinner as random generator • No anticipated patterns across the random event; no greater likelihood of some outcome(s) than others
Chance with Probability	• Conceptualization of random generator • Outcome(s) cannot be predicted with certainty, but some outcomes more likely than others

FRAMEWORK OF INTERPRETATIONS

A coding schema was designed to differentiate the subjects' interpretations within the spinner games. (See Table 1.) A primary focus of the framework was the relation between chance and probability. In the most primitive category, *Rejection of Probability*, the student explicitly denies differential expectations for multiple outcomes that in fact have different probabilities. The form of probability in the category *Probability without Chance* is limited to a deterministic prediction of who will win the game or where the spinner will land next. This interpretation accords with the *Outcome Approach* as defined by Konold, wherein the "the primary goal in situations involving uncertainty is not to arrive at a probability of occurrence but to successfully *predict* the outcome of a *single* trial. . . Probability values are evaluated in terms of their proximity to anchor values of 100%, 0%, and 50%." (Konold, 1991, p

146). In *Chance without Probability,* children conceptualize the indeterminancy of spinner and game outcomes, but fail to understand the different probabilities associated with the various outcomes. In *Chance with Probability,* children have a sense both of the indeterminancy of spinner and game outcomes and the differential probabilities associated with the different colors (at least on a qualitative level of greater or lesser chances). The framework also included two alternative interpretations: *Personal Control*, wherein children thought of the spinner function as within their control, and *Data-Driven*, where the children reasoned on the basis of correlations or patterns in the situation without conceptualization of any connections of these data to spinner function.

Coders assessed interpretation across a range of situations: before a spinner game begins (*e.g.* choice of spinner and rationale, anticipated effect of playing with a different spinner, projections concerning game outcome, and projections as to where on the bar graph the anticipated loser will be when the game ends), during the game (*e.g.* anticipation or interpretation of spinner outcome, interpretation of the game's progression, projections and changes in projections about game outcome, and implicit theories about spinner function) and after the game (*e.g.* accounting for the results and their relation to the anticipated results, and anticipations about what would happen if the game was repeated or a different spinner was substituted). Two coders independently coded the entire data set.

RESULTS AND DISCUSSION

The assignment of one of nine categories (the seven above plus "Other" or "Uncodeable") to any manifestation of interpretation resulted in 258 data points and an inter-rater reliability of 82%. All disagreements were resolved by consensus, upon joint reviewing of the videotape. The findings revealed a complex of developments from the kindergarten to third grade level. (See Table 2.)

Table 2: Cross-Age Comparison of Interpretations

Interpretation	# of K's applying	Total K instances	# of 3rd's applying	Total 3rd instances
Personal Aesthetics	(8) 66%	(16) 10%	(0) 0%	(0) 0%
Personal Control	(6) 50%	(18) 11%	(3) 25%	(4) 4%
Rejection of Probability	(6) 50%	(26) 16%	(2) 17%	(3) 3%
Data Driven	(9) 75%	(29) 18%	(5) 42%	(10) 11%
Probability without chance	(10) 83%	(38) 23%	(5) 42%	(27) 28%
Chance without Probability	(3) 25%	(5) 3%	(1) 8%	(2) 2%
Chance with Probability	(4) 33%	(11) 7%	(7) 58%	(41) 43%
Other	(3) 25%	(3) 2%	(1) 8%	(1) 1%
Uncodeable	(4) 33%	(10) 16%	(2) 17%	(2) 2%
To code/ not to code disagreement Final decision not to code	(6) 50%	(7) 4%	(1) 8%	(1) 1%
Total coded episodes		163		95

Although the majority of kindergartners acted on the basis of *Personal Aesthetics* at some point in their protocol, no third graders ever manifested this irrelevant interpretation. The other two spurious interpretations, *Personal Control* and *Rejection of Probability*, were rarely used by any third grader but together accounted for 27% of the kindergartners interpretations. *Chance without Probability* was rare at both grade levels, accounting for just 3% of kindergartners' interpretations and 2% of the third graders Whereas the two grade levels exhibited *Probability without Chance* at about the same relative frequency (23% among the kindergardners and 28% among the third graders), *Chance with Probability* was much more common among the third graders (7% in comparison with 43%).

Two issues of particular theoretical interest emerge in these findings. In violation with Piaget's model of the genesis of chance and probability, chance in the sense of random generator emerges after probability and chance without probability was rarely seen. In line with historical analyses of the development of these ideas, probability was first manifested as deterministic expectations. More generally, the manifestation of the alternative interpretations of Personal Control and the ubiquitous

Probability without Chance reveal the underlieing challenge of conceptualizing the source of the variability, the bounds of the predictable, and the bounds of control.

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Research Papers (Continued)

Problem Solving

TRACING MILIN'S BUILDING OF PROOF BY MATHEMATICAL INDUCTION: A CASE STUDY[1]

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Milin, a fourth grade boy, was observed in a videotaped small group assessment discussing a mathematical activity with three other students. His contribution to the solution of the problem posed to the group was based on an argument by mathematical induction. This paper traces the development of his construction of this method of justification.

Children working in social settings (small group or whole class) frequently have opportunities to observe and listen to the solutions and justifications of others (Davis & Maher, 1990; Davis, Maher & Alston, 1991). Individuals, however, often begin by building initial solutions of a problem which are their own construction, then consider the ideas of others, comparing them to their own for a more refined representation (Maher & Martino, 1991).

In a longitudinal study of children's thinking, Milin has been videotaped during classroom problem-solving sessions and in follow up task based interviews since first grade (Davis, Maher, & Martino, 1992). He has been in mathematics classes where children are given opportunities to explore mathematical ideas in open problem-solving situations (Maher and Alston, 1991). In grade three his class was presented a counting problem asking pairs of students to build all possible towers four cubes tall, given unifix cubes of two colors. The following day the students shared their strategies and results in whole class discussion. Questions that the students were unable to resolve at this time were left open for future consideration. In grade four, Milin's class worked on a similar task which called for building all possible towers five cubes tall, given unifix cubes of two colors, red and yellow (Maher & Martino, 1992). This class activity was followed by individual task based interviews with a number of the children and a small group assessment discussion with a group of four (Maher, Martino & Davis, submitted for publication).

Episodes from the videotaped data showing evidence of pivotal moments in Milin's construction and reconstruction of his mathematical argument were identified and analyzed with respect to the strategies and language used to build and justify his solutions. In particular, the following behaviors were considered: (1) questions and reflections that Milin posed to himself; (2) his interactions with classmates; and (3) his response to teacher questioning.

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Documentation reported in this paper was taken from the videotapes and Milin's written work from five mathematical activities, during two months of his fourth grade year (See Table 1).

Table 1 Mathematical Activities

ACTIVITY	CONTENT	DATE	LENGTH
Classroom Lesson	Building towers of 5; justification of solution	February 6, 1992	80 minutes
First Interview	Reconstruction of solution using cases; consideration of simpler problems	February 7, 1992	54 minutes
Second Interview	Extension to towers with 3 colors; building from towers of 1 to towers 2 cubes tall	February 21, 1992	33 minutes
Third Interview	Organization in "families" for towers of 1-5 cubes tall	March 6, 1992	30 minutes
Group Assessment Discussion	Solution and discussion of justifications for towers 3 tall with Stephanie, Jeff and Michelle	March 10, 1992	45 minutes

In two subsequent written assessments, in May and October of 1992, the durability of Milin's method of justification was also documented[2]. Four months later, in a different problem situation extending the tower problems to ideas about conditional probability, we again observed Milin persuading his classmates of the validity of his solution based on mathematical induction.

Episodes documenting Milin's building of his solution and justification are presented chronologically beginning with an excerpt from the fourth grade classroom session and concluding with episodes from the Group Assessment Discussion.

Episode 1: Beginning of Reorganization from Pairs of Opposites to Cases

During the class session, Milin and Michael, his partner, generated pairs of towers by building duplicates with the color of the cube in each position in the first tower replaced by the other color in its "opposite". The only indication of recognition of an exhaustive "case" was when Milin, in the 4th grade session, explained the two single color towers:

Teacher: Is there anything else that helps you to make new towers?
Milin: We already know that we did 5 (cubes) of these (all red) and 5 of these (all yellow) - so, we won't do that anymore.

[2] For a detailed analysis of Milin's written assessments refer to *Four case studies of the stability and durability of children's methods of proof* by Maher and Martino in this volume.

The boys tentatively settled on 32 as a solution, noting agreement with several other groups. Milin defended their solution when questioned:

> Milin: It gets harder (to find more) - If you go about 4 minutes without finding one, you're probably done.
> Teacher: Can you know?
> Milin: No - Try for 10 minutes! - for one hundred hours!

In sharing solutions with the class, some children grouped particular towers according to certain characteristics. First noted were the five towers with one red cube and four yellows arranged in a staircase pattern, along with the opposite staircase of one yellow and four reds. The two single color towers were also noted. A second staircase, with two adjacent red cubes, generated 4 more towers. Another set of 4 towers that were opposite in color to this set, was added. Members of the class offered similar staircases for the two red cubes separated by one, two and then three yellow cubes. Milin volunteered to explain.

> Milin: For 1's (towers with the 2 reds separated by 1 yellow) there's only 3. For 2's (towers with the 2 reds separated by 2 yellows) there's only 2 and for 3's (towers with the 2 reds separated by 3 yellows) there's only 1.

Stephanie added that there would be 10 altogether with exactly 2 red cubes, noting first the four with 2 adjacent red cubes and then the 6 that Milin had just described. Another child volunteered that there would be 10 more towers like these but opposite in color.

Episode 2: Milin's Explanation based on Cases

In the first interview, Milin said that the class sharing session had shown him how to be "sure about the towers". He quickly described and built the towers for cases A, E, B, F, C and G (See Figure 1.)

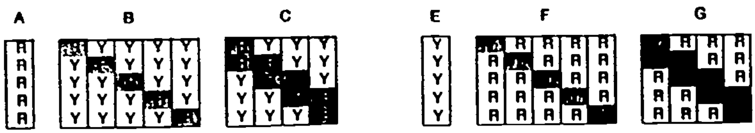

Figure 1 Milin's Partial Construction of Towers 5 High

By using his original strategy of opposites, he came up with 12 more towers for a total of 32 but was unable to provide a convincing argument.

> Teacher: You really think you have them all?
> Milin: Uh huh.
> Teacher: You know you have all of these (Groups C and G) ... and you know you have all of these (Groups B and F).

Milin: But when we came to the separating part we kept on checking them ... and we took about 10 minutes and still didn't find any.

Episode 3: Considering Simpler Problems

Later in the first interview, Milin considered towers 4 high and 3 high. He approached these problems by looking at the various cases for towers 5 tall, subtracting the appropriate number of towers from cases B, F, C and G and then estimating a total of 24 for towers 4 tall and 18 for towers 3 tall. When asked about towers 2 tall, he responded:

Milin: I could do this right now.
Teacher: Sure.
Milin: One of these (one red and one yellow), switch that around like this. .. about 4
Teacher: You think?
Milin: On one's (towers one cube tall) there would be two.

Episode 4: Taller Towers Built from Smaller Ones

At home Milin built a set of 16 towers 4 tall and recorded his results for towers 1, 2, 3 and 4 cubes tall with two colors. He returned to the second interview with this data.

Teacher: If you were going to go from towers of ... one and make them into towers of two? How would you do that?
Milin: You put one tower on top of the other.

In conversation, Milin said that he remembered a similar problem in 3rd grade, but thought that the task had included three colors..

Teacher: Instead of with two? would that make it different?
Milin: Uh huh ... if there's 3 colors then you could make three (towers 2 tall) of these (holds up one of the cubes) ....

Milin began to build from the 3 towers one cube high that he had identified (white, yellow and red). However, his first three towers were each a single color, which he failed to consider later when looking for the third member of a "pair".

Milin: (Talking to himself) .... Put a red on top of that ..by this there will be pairs of three.
Teacher: What do you mean?
Milin: See ... because ...(Indicating his towers of 2 with white and yellow and white and red) .. by this there'd have to be a pair of three somehow.

Milin moved a yellow and red tower into the group to make his "pair of three".

Episode 5: Organizing by Families

Milin began the third interview by building 4 towers of height two after choosing two towers, each one cube tall, one a black and the other a blue cube.

Teacher:	Why did that happen?
Milin:	This happened because you had to put something else on top of these.
Teacher:	What could you put?
Milin:	If you had blue, you could put another blue ... or a black on it.

When asked about towers 3 tall, Milin first looked at the four 2 tall towers that he had built.

Milin: OK ... You could have these four .... Another four.

Milin generated the set of 8 towers 3 tall by taking each of the four towers of height 2, building two duplicates of it and placing a blue cube on one and a black on the other. He explained as he positioned the first set alongside its corresponding tower two cubes tall.

Milin: See .... That would go into this family.

Milin used the term "family" throughout the rest of the interview to explain the relationship from shorter to taller towers. After building towers of height three, Milin explained that this strategy would also work for towers four and five cubes tall.

10 minutes into the interview:

Milin:	.. and that would work with all these too (Points to the row of towers 3 cubes tall).
Teacher:	OK - so how many are there going to be?
Milin:	16 - 2 for this, 2 for this, 2 for this, 2 for this, 2 for this, 2 for this, 2 for this, and 2 for this (Pointing to each of the 8 towers).
Teacher:	Yes.
Milin:	... and once you get to 16 ... 32 .... You get two for all of them and you get 32.
Teacher:	Oh, really?
Milin:	But it doesn't work on 6, towers of 6.
Teacher:	Why?
Milin:	Cause it's different .... 'cause I got 50. I made staircases .....

Milin then organized particular "families", by building towers first from 3 to 4 cubes tall, then from 4 to 5 cubes tall and finally from 5 to 6 cubes high. As he did this he made several conjectures about why the "family" strategy would not work for towers 6 cubes tall.

19 minutes into interview:

Teacher:	You're saying you couldn't do that for all of them going from 5 to 6?
Milin:	Uh huh, because there's going to be less.
Teacher:	Why?
Milin:	(Laughing) Because some of the "families" can't afford them!

21 minutes into the interview, referring back to his written records :

Teacher:	Does this strategy then just not work after a while?
Milin:	Uh huh, after 5, maybe cause 10 is an even number and you can divide by 5 or something like that.

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Milin continued building towers of height 6 from the towers 5 cubes tall, asking, as he worked, whether anyone else had been thinking about these problems. As the interviewer turned to talk with colleagues about setting up a group assessment, Milin broke into their conversation:

Milin: but this ......(Referring to the recorded solutions on his paper) ...this might not do it, but it might.
Teacher: It might?
Milin: I might be wrong or something.

The interviewer then concluded the interview by asking Milin to summarize what he had found out about towers with 3 colors.

Teacher: You've done a really good job of keeping some of these records. I'm going to ...
Milin : (Interrupting) But, I think I did something wrong on ... ummm, from 32 to ... going to 6 -... I think I did something wrong.
Teacher: Why?
Milin: I don't thing that rule would break down like .....
Teacher: If it didn't break down, how many should there be?
Milin: 64.

Episode 6: Argument Based on Mathematical Induction: Milin's Explanations in the Group Assessment Discussion

Throughout the discussion the other three children based their arguments on number patterns and instances of cases. Milin made a drawing and, on 8 different occasions, offered explanations that used an inductive argument (See Figure 2 for Milin's drawing).

Figure 2 Milin's Drawings of Towers Two and Three Cubes High

His explanation in each instance was based on generating from a shorter tower exactly two towers that were one cube taller. For example:

Teacher: Can you explain for me why, from 2, you could get 4.
Milin: I know -
Teacher: OK - Milin.

Milin: (Pointing to his towers that were one cube high) Because - for each one of them, you could add one - No - two more - because there's a black, I mean a blue, and a red --- See for that you just put one more - for red you put a black on top and a red on top - I mean a blue on top instead of black. And blue - you put a blue on top and a red on top - and you keep on doing that -.

During this discussion, he expressed certainty that this pattern would continue.

Milin: ... and for each one you keep on doing that and for 6 you'd get 64.
Teacher: Does that make any sense?
Milin: We followed the pattern to 5 - Why can't it follow the pattern to 6?

Conclusions

Several observations can be made about this development. The first is that the development of a stable and durable argument took time and occurred gradually through a number of varied experiences. Each successive activity provided Milin opportunities to build on his earlier constructions. He was encouraged to think more deeply about the ideas and to continue this at home. Milin's building of a stable justification was not done in isolation. His first strategy was to build a justification by cases that were triggered by classmates' observations. When he pursued the heuristic of thinking of a simpler problem he was able to construct an argument that made sense to him. Milin's construction was facilitated by his classroom community, through strategies that were shared and the knowledge that other children were engaged in similar exploration.

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BLOCKING METACOGNITION DURING PROBLEM SOLVING

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Abstract: The paper presents a case study (Barclay) in which the subject consciously chose not to be aware of his metacognitive activity and concurrently seemed to remain relatively constant in his problem-solving success and to shrink in his confidence and willingness to try problems. Data included journal entries, group and written problem solutions, four videotape sessions of talking aloud while solving a problem, a pre- and post-Attitude Inventory, and prompted retrospection.

Metacognition has been widely discussed and accepted as an important factor for success in problem solving. Yet, the path of its development, and the mechanisms that spur someone along that path, as well as the effect of that development have received much less attention (Schoenfeld, 1992). Earlier work established that self-report data can be used to study the development of metacognition (DeGuire, 1987, 1991). The present paper is a companion piece to DeGuire, 1993. The purpose of the 1993 paper was to present a case study (Jackie) that chronicled Jackie's positive development in metacognition, which in turn produced significant growth in problem solving success and several characteristics related to such success. Mechanisms that spurred development in Jackie all had to do with externalizing thought processes in some way, e.g., writing "metacognitions" in a separate column in the written solution of problems. The purpose of the present paper is to present a case study (Barclay) in which the subject consciously chose not to be aware of his metacognitive activity and concurrently seemed to remain relatively constant in his problem-solving success and to shrink in his confidence and willingness to try problems. Though there is no universally accepted meaning of the word "metacognition," all usual definitions include the monitoring and regulation of one's cognitive processes; this is the aspect emphasized here.

Context of the Case Study

The subject of the case study (Barclay) was a student in a course on Problem Solving for the Elementary Classroom, taught by the researcher. The data for the case study were gathered throughout the semester-long course. The course began with four sessions (two weeks) devoted to an introduction to problem solving, several problem-solving strategies (e.g., make a chart, look for a pattern, work backwards), and metacognition. After the introduction, the course progressed from fairly easy problem-solving experiences to quite complex and rich problem-solving experiences. Throughout the course, students discussed and engaged in reflection and metacognition. Often, especially early in the course, students solved problems in pairs, with one student serving as the "thinker" and the other as the "doer" (Schultz & Hart, 1989). Most other in-class problem solving took place in groups of 3 to 5 students.

During the course, the students were given 8 problem sets to be solved and written up outside of class in order to be evaluated. The written report was to be an in-progress record of all work on the problem, including blind alleys, and was to include a separate column for "metacognitions." All solutions were evaluated so as to give more weight to the cognitive and metacognitive processes in the solution than to the final solution of the problem. Students in the course also completed an Attitude Inventory (Charles, Lester, & O'Daffer, 1987, p.27) twice, once early in the course and once at the end of the course, and wrote 10 journal entries. The topics of the journal entries were chosen to encourage reflection upon their own problem-solving processes and their own development of confidence, strategies, and metacognition during problem solving.

Barclay was one of two students who volunteered to be videotaped while thinking aloud during problem solving. There were four videotape sessions, spaced approximately equally apart . The problems used in the videotape sessions are representative of the problems used throughout the course. In abbreviated form, they were the following:

Videotape Session 1: (Number Problem) Three whole numbers multiply to 36. Five more than the sum of the numbers is a perfect square. What is the sum of the numbers?

Videotape Session 2: (Fence Problem) You have 10 boards, each 1 unit by 1 unit by 2 units. (The red rods from sets of Cuisenaire rods were used as a model for this problem.) You want to build a fence 10 units long by 1 unit wide by 2 units high. It is possible to build the fence in a variety of ways. (Side views of 2 ways were pictured.) With how many different arrangements of the boards could you build the fence?

Videotape Session 3: (Locker Problem) There are 1000 lockers, numbered from 1 through 1000, and 1000 students. Each student walks by the lockers one at a time. The first student opened all of the lockers. The second student then closed every other locker, that is, the one with the even numbers on them. The third student changed the status of every third locker, that is, opened it if it was closed and closed it if it was open. The fourth student then changed the status of every fourth locker. One by one, the students changed the status of the appropriate lockers. At the end, which lockers were left open and which were left closed?

Videotape Session 4: (Extension of Checkerboard Problem) How many different rectangles are on an 8-by-8 checkerboard? Note, rectangles are considered different if they are different in position or size. So, a 2-by-1 rectangle is considered different than a 1-by-2 rectangle.

The following case study is the story of the Barclay's development. Throughout, direct quotes are from his written problem solutions, journal entries, or videotape transcripts.

The Case Study of Barclay

Barclay was older (mid-20's) than the typical college student, working full-time and taking courses at night. He had taken mathematics through Calculus I, earning A's throughout. He had had no teaching or classroom experience. As a mathematics learner, he described himself as "a sponge." He considered himself a "whiz at mathematical problems. . . . If you count puzzles, that's one of my main hobbies." However, he had had no training in problem solving or in problem-solving strategies.

After the introductory phase of the course

In a journal entry shortly after the introductory phase of the course, Barclay felt his approach to problem solving and his confidence in his problem solving abilities had been reinforced in the course.

He reported that his most commonly used strategy was "work on simpler problems. . .and put them together to make the whole thing work." This strategy, as he seemed to interpret it (i.e., break the problem into parts and work on each part separately) remained his preferred and dominant strategy throughout the course. At this point, he was consciously working on being aware of his cognitive and metacognitive processes.

At the first videotape session, Barclay solved the problem rapidly and called it "simple." He had very little difficulty reporting his cognitive processes but reported few metacognitive processes. He did monitor his local procedures in solving the problem but not whether or not he had considered all possibilities. When this fact was pointed out to him, he checked further for other combinations.

A couple days later, in his journal, he wrote about his conception of metacognition and his own level of awareness of his cognitive and metacognitive processes.

> Metacognition is the thought process that goes on that a person may or may not be aware of that guides him through a problem. It's what decides on a strategy and gives ideas on how to go or what works and what doesn't. It's as an overseer. I think I was fairly aware of my thought processes [before this course]. I would put myself at a 5 on a scale of 1 to 10. . . . This class has started me thinking about this more and I would say that I am <u>much</u> more aware of my thought processes now. I pay more attention to how I think. I would put myself at a 7 now.

So, Barclay understood metacognition primarily as monitoring, whether consciously or not, and he was definitely trying to be aware of his "thought processes."

<u>During the second part of the course</u>

Barclay's written problem solutions during the second part of the course showed increasing metacognitive activity. In one solution, he began with his favorite strategy and soon excitedly exclaimed that he saw a pattern, though he had not consciously decided to look for a pattern. In another solution during this time, Barclay expressed genuine amazement at the simplicity of a solution done in class in which a pattern was found and used to find the number of ways a task could be done. In his solution of this problem, he had laboriously (and correctly!) generated all 252 ways!

It was during this time that Barclay began to express some frustration with metacognition. Shortly before the second videotape session, he wrote in his journal,

> I am sometimes very aware of my cognitive processes, but most of the time they go on without me noticing. . . . I guess, for me, sometimes I just need to let the thoughts flow. If I think about it too much, it's a hindrance. I've gotten into these loops where I'm thinking about my thoughts about thinking about thoughts. . .you get the idea.

In our discussion at the beginning of the second videotape session, when asked whether his awareness or his cognitive or metacognitive processes had changed in any way, he clearly indicated that he was making choices about whether or not to attend to his metacognitive activity.

> Well, it depends. If I think about it, I'm seeing more than I ever saw. But. . .I don't always do too much thinking about what I'm thinking. . . . But at times I am more aware when I try to make myself more aware. Especially when we do the written problems with the metacognitive column.

In his solution of the Fence Problem at the second videotape session, Barclay began counting possible arrangements but with many lapses in his verbal reports. Soon he expressed some frustration. "This is getting complicated. There's got to be an easier way to do it." Clearly some kind of monitoring was taking place. He then began using internal patterns to verify his counts, again indicating monitoring of local procedures. An overall pattern eluded him. There were many 5-10 second pauses during this struggle, with facial and bodily expressions (e.g., tapping on desk while staring) suggesting unreported mental activity. When prompted to try to think out loud, Barclay grimaced and commented, "It messes me up." However, he complied. Several minutes later, he again commented "I know this has got to be the worst way to get the answer." During the next several minutes, he seesawed among pursuing his counting pattern and commenting on his sense of being overwhelmed and his approach being too complicated, with several 5-10 second lapses in his verbal report and occasional lapses as long as 30 seconds. Throughout this time, there were occasional prompts from the researcher to think out loud, each greeted by Barclay with varying degrees of a facial grimace. At one point, he decided, "Maybe I could just go brute force, count up all the 4 H's and just.... If I write it all down, maybe I'll be able to see it." He shifted his point of view but only to count it in a slightly different way. This time, however, he was successful. "There you go. (laughter) I don't know why I didn't see that before...[proceeding to count, with lots of uninterpretable mumbling].... Add them all up, I get.. .89. That's the number of combinations.... 89 ways. That seems like an odd number. [long pause]" He was "not 100% confident" of his solution. After several seconds of interchange, with the researcher probing Barclay to explicitly consider the in-class solution approach to the similar problem he had recalled earlier in this solution, he made the connection. "A simpler problem? . . .AHHHH!!! doing a simpler problem!!! Light went on! (laughter)" He then proceeded to use this approach and solve the Fence Problem rapidly in less than 2 minutes. Then he felt "100% confident" of his solution.

Of course, Barclay preferred the second solution path to the first. The researcher pointed out that, during the solution, he had picked up on the signal of being overwhelmed, that he had several times noted that there had to be an easier way to solve it, and that he had even thought of a similar problem, yet he had never stepped back from the problem long enough to let these pieces fall together to suggest the second solution path. Time did not permit further probing on these points.

During the third part of the course

After videotape session 2, Barclay's written solutions changed in that what he reported in his "metacognitions" column were mostly cognitions (i.e., the processes he used) with little or no attention to monitoring and looking back. In a journal entry he reported that "I'm still uncertain as to how to define 'metacognition." Somehow, his earlier, clear conception of metacognition had become clouded.

In the discussion at the beginning of the third videotape session, Barclay gave the following status report on his view of himself as a problem solver and his awareness of his metacognitive and cognitive activity.

> I think I have more tools to solve problems now than I had before. . . . I think that it's actually my cognitive awareness that's increased. However, I think it's something you have to consciously work at. I don't think I've been working at it as much as I could. . . I feel it hindered me. It slowed me down. . . . I think that a lot of these tools. . .I use them but it's more on the metacognitive level, so I'm not aware that I'm using them. . . . [They're] probably automatic to some extent. . . . On a scale of 1 to 10 with 1 being low awareness, I would say I'm average, a 5. . . . There was a point in this course when I put it higher. I think it's higher than when I started this course.

After an initial reading of the Locker Problem for the fourth videotape session, Barclay began looking at the first 10 lockers and recording status changes in a grid. He began to focus on how many lockers were changed each time. Later, he changed his focus to which ones each student changed. He was mumbling a lot with several lapses into silence, some short and some fairly long (15 to 30 seconds). It was not clear what pattern he was attempting to follow. He continued in this manner for about 20 minutes, grimacing when reminded to say what he was thinking, occasionally complaining that this was getting too complicated. At one point, when reminded to say what he was thinking, he said,

> (grimacing) I'm tired of making a lot of mistakes. . . . Here's what's really going on in my mind. Because I know I'm doing all this busy work, my mind's busy working subconsciously. But I'm not really aware of what's going on. . . . Eventually I'll come up with a better idea working it this way.

He returned to counting and looking for a pattern. After several more minutes, he glanced at the original problem. Since time was growing short, the researcher prompted him to reread the problem. He realized he needed to look for <u>which</u> lockers would be left open, not how many. After prompting by the researcher to see whether he knew for sure any lockers that would be open, he paused (11 seconds), then checked his chart and again paused (10 seconds), and finally replied, "1, 4, 9. . .squares. . .The next one's going to be 16. . . . Now I feel a little stupid." He went on to complete the solution but could not explain why the lockers worked that way, even after considerable probing by the researcher. Later, in his journal, he completed the explanation of the solution. He wrote, "I liked this problem. . . . But this problem really frustrated me! It wasn't really the problem's fault. . . . I wrote one the numbers in RED ink. This prevented me from seeing the pattern of perfect squares. My mind disregarded the red ink." He seemed unable to relate his difficulties to not ever stepping back from the problem to assess his overall progress and to reread the problem.

<u>During the last part of the course</u>

During the last part of the course, Barclay's performance on his written problem solutions was quite uneven. In his "metacognitions" columns, he continued to report primarily cognitions, rarely monitoring beyond the level of local procedures. Toward the end of this time, Barclay summarized his concept of metacognition and his view of his own levels of metacognition and awareness.

> Metacognition is the background thinking that goes on behind problem solving. . .the "little voice" inside your head that tells you when you're on the right track or when you need to try something new. Very often this "voice" will give "hints." This is metacognitive thinking. . . . I feel that when I first learned about metacognition, I was confused about it. As I looked more and more at my background thinking I was able to get a better understanding of metacognition. Through all this I have become more aware of my own thought processes. . . . Now, my

> awareness has become less, but not to the point where I started. I think that practicing being aware is helpful, but the mental effort of this awareness is a hindrance. I would rate myself at a 4 on awareness at the beginning of the course. I think at one point I would have rated myself at an 8 or 9, but now I think I've settled into the area of a 6.

Barclay's solution of the Extension of the Checkerboard Problem during the fourth videotape session was amazing in that he actually solved it by brute force and counted all 1296 rectangles! Throughout the solution, he did internal checks of local procedures. He struggled for quite some time with generalizing while he continued to count. There were frequent pauses during this time. When asked what he was thinking, he replied, "I'm not thinking. I'm just letting my brain come up with an idea. It's not coming up with one." He was gradually filling in his chart. Finally, "all I've got to do is just add up all these. . . . I'm pretty sure that will be the answer although I'm not sure how I check that right off." He continued to struggle for some time with finding

> a way of adding these up without doing this. There's got to be a way. . . . Meanwhile I'm trying to think of a way subconsciously. I'm going to go ahead and keep figuring because that keeps my mind off of what I'm thinking about, which is my problem. Now I'm thinking about what I'm thinking about what I'm thinking about and that gets even worse. So. . .it's a trap. That happens to me a lot. I'm thinking about what I'm thinking about. Then it just turns into a big loop. Anyway, back to the problem.

Soon, he came up with the sum. Unfortunately, this approach left him with a specific solution that was, at best, difficult to generalize. Only with extensive prompting and guiding from the researcher was he able to generalize it to an n-by-n checkerboard.

At the end of the course, Barclay had experienced some changes in his attitudes, as measured by the Attitude Inventory. In the "willingness" category, 3 of his 6 responses had changed, to indicate he was less willing to try problems. In the "perseverance" category, 4 of his 6 responses had changed, to indicate he was more persevering in solving problems. In the "self-confidence" category, 4 of his 8 responses had changed, to indicate he was less confident of his problem-solving abilities.

Discussion

Barclay seemed to have mixed feelings about trying to be aware of and monitor his cognitive and metacognitive processes, insisting that such awareness hindered him. Thus, he consciously chose at times to block his metacognitive processes from his consciousness, such as in the third and fourth videotape sessions and possibly in the second. During these sessions, he agonized through difficult and complex counting procedures, only reaching confidence in his solutions with help from the researcher. Thus, he was frustrated in these sessions. His ability to persist in these counting strategies was quite amazing! Yet his frustration took a toll on his willingness to focus on his metacognition. One can only hypothesize how smoothly and quickly these sessions might have gone if he had allowed himself to step back from his solution processes long enough to see other ways of solving the problems.

The contrast with Jackie's development (DeGuire, 1993) during the same course gives some in-sight into what might have been for Barclay. Jackie began the course with much more limited

problem-solving experience than Barclay. She eagerly embraced metacognition and her problem-solving abilities seemed to develop continuously with no indication of leveling off. She struggled with the problems in the first 2 videotape sessions. By the third videotape session, she correctly solved the Locker Problem and went on to explain why the status changes of the lockers fit the pattern. In the fourth videotape, she quickly solved the Extension of the Checkerboard Problem and went on to generalize it to an n-by-n checkerboard. By the end of the course, changes in her Attitude inventory showed a clear increase in her persistence, her enjoyment, and her confidence in solving problems.

Barclay often attributed his lack of success of seeing a pattern or solving a problem to his notation, his carelessness, his lack of "tools," or to other factors. Yet, he had creative and useful notation, he was meticulously careful enough to actually succeed in counting large numbers of possibilities, and he clearly had the "tools" when strategies were pointed out to him. These factors seem less plausible as an explanation than to attribute his lack of success to his reluctance to step back from his solution process long enough to assess and monitor his progress and to consider other solution approaches. He preferred to let his subconscious somehow feed him ideas and solution approaches rather than to take charge and consciously search his available tools. Such reluctance to interfere with approaches in which he had been successful and which had become rather automatic for him is understandable. Yet, the temporary interference caused by focusing on his metacognitive processes would likely have produced substantial gains in ease in his problem-solving success and perhaps prevented the decline in his confidence and willingness to try problems. Either Barclay did not believe metacognition would produce such results or he was unwilling to take the risk. Jackie's development (and other research results about the role of metacognition in problem solving) make it clear that the risk would have paid off well for him.

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THE ARITHMETIC-ALGEBRA TRANSITION IN PROBLEM SOLVING: CONTINUITIES AND DISCONTINUITIES

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Abstract

Considering the arithmetic experience acquired by students in problem solving over many years, the introduction of algebraic reasoning in problem solving gives place to conflicts and structuring of thought. The fundamental changes that mark the passage from arithmetic to algebra will be examined from the point of view of problems traditionally presented in algebra, and from the point of view of the procedures used by 132 students of secondary I (12-13 years old) in these problems, before any introduction of algebra. These spontaneous reasonings will then be confronted in individual interviews with what is normally expected in algebra.

Introduction

. If we consider the children's previous experience during their primary schooling and at the beginning of secondary school (6 to 13 year olds), problem solving is not new to them. They have developed over many years in these problems a certain number of strategies, a certain number of concepts. An arithmetic culture is already in place, that manifests itself in a particular body of problems, and by different procedures, different ways of approaching these problems. These procedures are carried over to certain knowledges concerning numbers and operations, and to certain relations of mathematical symbolism. (Kuchemann, 1981; Booth, 1984...).

When a student begins problem solving in algebra, he is faced with a new mathematical "culture," with reference to new problems, to a new way of approaching and reasoning these problems requiring a readjustment of his previous knowledge concerning number and operations and a different relation to symbolism. This process of acculturation leaves open possible conflicts and structuring of thought in the passage from one mode of solution to another. Problem solving seems therefore to be an interesting area in which to examine, from a didactic perspective, the possible continuities and discontinuities in the passage from one mode of treatment to another. The present research focuses on the study of these continuities and discontinuities in the passage from arithmetic to algebra: on what reasoning, on what arithmetic experience can this learning of algebra be built? On what reasoning, and on what experiences must it work against?

These questions raise the issue of the reasoning used by students, before any introduction to

algebra, in problems traditionally presented in this domain. particularly the types of stable arithmetic reasoning used by them. Moreover the question of the arithmetic-algebra transition requires us to reflect on the nature of the problems given in the two domains, and on the pedagogical strategy used to assure this transition.

Approaches to algebra by problem solving: the arithmetic algebra transition.

For several centuries the teaching of algebra was based on a certain body of problems, for which arithmetic and algebra proposed different methods of attack (Chevallard, 1989). In this approach, algebra was presented as a **new and more efficient tool** for solving problems which had previously been solved by arithmetic, as an indispensable tool that allowed one to attack problems that arithmetic could only treat locally. It also appeared as a **privileged means of expressing general solutions** to a whole class of problems. In this regard we find the following definition in an old textbook (Arithmétique des écoles, 1927) "algebra is a science that simplifies problem solving and generalizes the solutions by establishing the formulas to solve problems of the same type." (p. 406). These two functions of algebra guide the choice the ordering of problems used for its introduction (see table 1), and orient the pedagogical approaches elaborated (see table 2).

Table 1

Types of problems used in teaching, during the introduction to algebra.

(1) Simplify problem solving, deal with problems that arithmetic can only treat locally.
We find here (Arithmétique des écoles, 1927) problems of the type "unequal sharing", the relations involved in this sharing being increasingly complex.
ex. *We divide a sum between three persons A, B, C. A receives 800$ less than B, who receives 200$ less than C. What does each person receive, if the total to divide is 3000$.*

(2) Generalize solutions by establishing formulas: we find here, for example, problems without numeric givens explicitly envisioning a generalization.
ex. *We divide a sum of money between two children, giving to one child twice as much as to the other. How can we establish the amount given to each.?*
or another example:
Two cyclists start together and go in the same direction, one going a certain number of kilometers more than the other. What distance separates them after a certain number of hours?

We attempt in this way to gradually show the pertinence of this new science that we call algebra (see Clairant, Éléments d'algèbre, 1760, p. 88-95):

Table 2

Pedagogical strategy to assure the passage to algebra.

- Starting with a particular problem, similar to those the first algebraists presupposed (problem of unequal sharing)
- we give the solution of this problem the same as we could find it without algebra
- then we retake the algebraic method to express the solution to the problem (rhetorical solution, that is then expressed in symbolic form at each step).
- Variations around the same problem will then be proposed to show the power of the solution developed (increase in the number of relations, number of parts, increase in the complexity of the relations, new problems of the same type in another context).

The evolution of the teaching of algebra shows that this junction between the two domains (arithmetic and algebra) that is found here in problem solving was subsequently eclipsed. In this way algebra would not appear as a tool for solving complex problems, since the problems posed to students at the start could easily be solved by arithmetic, nor as giving access for the solving of a large class of situations.

Under these conditions how does the passage from arithmetic to algebra take place for the student? Given the importance of problem solving in arithmetic and algebra, and the difficulties students have when they attempt an algebraic solution, some reflection on the profound nature of the changes that mark the passage from arithmetic to algebra is imperative.

Method

Different types of problems chosen on the basis of a reference framework developed by the team were given in written form to four groups of students in secondary I (132 students, 12-13 years old), before any introductory algebra had been presented.

Individual interviews were then conducted with some of the students who were chosen on the basis of different profiles in the arithmetic procedures used to solve these problems. These interviews aim essentially to determine students' reactions to an algebraic treatment of problems.

Reference framework

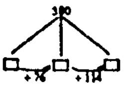
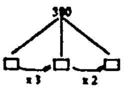
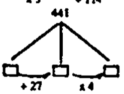
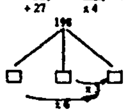
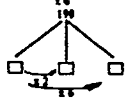
The characteristics we have previously given of the introduction scenario to algebra that has been established over several centuries, allowed us to undertake the analysis of the problems traditionally presented in algebra, and their complexity, from another point of view than the one actually considered in teaching, that is, the equation. We have seen previously that the gradation of the proposed problems were conceived in function to the relations implicated in the structure of the problem. This is the perspective that was retained by our research group. Our framework, developed through a systematic examination of different types of problems found in the arithmetic and algebra sections of textbooks at various grade levels and not restricted to current textbooks, is based on the "relational calculus" (Vergnaud, 1976) involved in the representation of such problems (the nature of the relationships between the givens, knowns and unknowns, the ordering of these relationships...). The research team studied various types of problems (problems of the type of unequal division, problems involving transformations of quantities, problems involving links between non homogeneous quantities and a rate...)[1]. This framework allows us to bring out, in this a priori analysis of problems, the cognitive complexity of the task's demands on students in terms of relational calculus.

Results

In what follows we will restrict our discussion to a well defined class of problems, whose "type of unequal dividing" are generally given in introductory algebra. Table 3 illustrates the difficulties the students have in some of the problems used in the research, and confirm the influence of elements of complexity that were identified previously in the problem analysis stage: the influence of the type of links (composition of two additive versus multiplicative relationships, non-homogeneous composition of two relationships; of the sequencing of links (linear versus non-linear, direct or indirect)).

1 For an analysis of such problems see Bednarz, Janvier, Mary, Lepage (1992).

Table 3

The problems	*Problem structure*	*Success rate*
a) 380 students are registered in sports activities for the season? Basketball has 76 more students than skating and swimming has 114 more than basketball. How many students are there in each of the activities?		5.21%
b) 380 students... Basketball has 3 times as many students as skating and swimming has twice as many as basketball. How many students are there in each of the activities?		15.16%
c) 380 students... Basketball has 3 times as many students as skating and swimming has 114 more students than basketball. How many students are there in each of the activities?		2.78%
d) 441 students... Basketball has 27 more students than skating and swimming has 4 times as many students as basketball. How many students are there in each of the activities?		11.82%
e) Three children are playing marbles. Together they have 198 marbles. Pierre has 6 times as many marbles as Denis et 3 times as many as Georges. How many marbles does each child have?		5.56%
f) Three children... Pierre has 6 times as many marbles as Denis et Georges has 2 times as many as Denis. How many marbles does each child have?		31.59%
g) Three children... Georges has 2 times as many marbles as Denis et Pierre has 3 times as many as Georges. How many marbles does each child have?		19.44%

Furthermore, an analysis of procedures used by the students in the set of problems they were given allows us to point out different arithmetic profiles of reasoning, illustrating different ways of managing the quantities and relationships involved (see table 4).

Table 4

	For example
[1] The known in the problem is taken as a starting point. They treat it as an initial state and operate on it to generate the various unknown quantities (198 is treated here as a state that allows the application each of the relationships).	In the problem... e (Table 3) They do 198 + 6 = 33 198 + 3 = 66
[2] They give themselves a starting state by using a fictional number. Here we find procedures like numeric trials. (They fix a number for which everything can be reconstructed.)	In the problem... b 10 20 25 + 30 + 60 + 75 60 120 150 100 200 250
[3] They create a known state by using a strategy of equitable sharing (dividing the whole by the number of categories). The number obtained serves as a generator to find the various unknown quantities (the division of the whole gives one of the desired quantities. This number serves to generate the other quantities).	In the problem... e 198 + 3 = 66 and 66 + 6 = 11 66 + 3 = 22
[4] They take into account globally the different relationships involved. (they seem to see 1 share, 3 shares and 6 shares)	For example, in the problem... b, They do 380 + 10 = 38 then 38 x 3 = 114 and 114 x 2 = 228

The results of the interviews of the subjects representing different arithmetic solution patterns point out some fundamental changes in the passage from arithmetic to algebra. Among what appears to be the major difficulties are: the ability to think in terms of an unknown, the difficulty to carry out a substitution of one unknown quantity by another, and the composition of different unknown quantities that allows one to keep track of the relationships present and the successive states. The three first profiles of reasoning (table 4) enables us to anticipate this last difficulty by a sequential arithmetic treatment resting on intermediate states that characterize this reasoning.

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A COMPARISON OF PAIR AND SMALL GROUP PROBLEM SOLVING IN MIDDLE SCHOOL MATHEMATICS

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SUMMARY

This study carefully examined how two different cooperative learning configurations affected the quality and quantity of student interactions and the accompanying teacher responses. Middle school students using a mathematics curriculum designed for group work were observed during two academic quarters. It was discovered that students who worked in groups of two gave more frequent and detailed explanations of their thinking than did students working in groups of four. Futhermore, students working in pairs seemed less withdrawn and engaged in less off-task behavior than did students in groups of four.

INTRODUCTION

Although there is a growing body of research which addresses the issue of cooperative learning, there remains a gap in the understanding of how this complex instructional approach is enacted by groups of students and teachers. Few studies have systematically examined the interaction process that occurs in small groups; most studies have attempted to predict achievement from a few characteristics of the individual, group, or classroom. Without data on students' actual experiences in these groups, research presents an incomplete picture of the influence of group work on individual learning.

In part, the success of cooperative learning and its subsequent popularity has been attributed to the emphasis on student discussion and elaboration. Research in cognitive psychology has found that if information is to be retained in memory and related to information already in memory, the learner must engage in cognitive restructuring, or elaboration, of the material (Wittrock, 1978). One of the most effective means of elaboration is explaining the material to someone else. Noreen Webb (1985) found that the students who gain the most

from cooperative activities are those who provide elaborated explanations to others. While those who receive explanations learn more than students who work alone, it is clearly the explainers who benefit the most. Furthermore, receiving only answers without explanations or receiving no response at all to a question was found to be detrimental to students' achievement. Finally, students gained more from giving explanations to individuals perceived as not understanding an idea, such as other students, rather than giving explanations to those who are likely to know how to solve a problem, such as teachers (Webb, 1985).

METHODOLOGY

The purpose of this study was to examine the dialogue of cooperative learning and to describe the quality of interactions between students and between students and teachers. In this paper, cooperative learning was defined as: students working in a group contributing their expertise and knowledge while seeking a solution to a problem. All group members must understand the material and contribute to the group process.

The authors of this paper selected the following criteria as indicators of the effectiveness of cooperative groups:

1.) Students are observed giving explanations to other individuals or to the entire group.

2.) All students participate in the group in a positive way that contributes to the problem solving task.

3.) Students show understanding of the problem and the process and attempt to find a solution. Students who do not understand the problem make their needs known to the rest of the group.

4.) Students respond to other students' questions.

5.) Students show interdependence and work together rather than

relying on the teacher.

These general characteristics helped to provide a basis for comparison of two types of student groupings: students working in pairs and students working in groups of four. The following exchanges were recorded during the observations: **EXPLANATIONS**, descriptions of thought processes longer than five seconds; **QUESTIONS** among group members including questions to and from the teacher; **ANSWERS**, solutions without accompanying explanations; **INFORMATION**, defined as any observation, illustration, or model contributed by a group member or the teacher which was not sufficiently detailed to constitute an explanation; **OFF-TASK BEHAVIOR**, comments or other actions which distracted or deterred the group from solving the problem.

Middle school students using the curriculum Visual Mathematics (1991) were observed for this study; this program incorporates cooperative groups in daily problem solving. Initially, a preliminary pilot study of students working in groups of four was conducted. During this study, behavioral characteristics of groups emerged which suggested that a comparison study of students working in pairs would be informative. In researching the groups of four students, twenty 45-minute observations were conducted, (five observations in each of four teachers' classrooms in grades six, seven, and eight), during winter term. Subsequently, one teacher agreed to use only pair problem solving during the spring quarter, and five weeks later, eleven observations of this classroom were conducted.

RESULTS AND DISCUSSION

Results of the study are as follows:

* Students working in pairs spent more time giving explanations than did students working in groups of four. In groups of two, the mean length of time students spent giving detailed explanations during

each class period was 2 minutes compared to 49 seconds in groups of four.

* The mean number of explanations given by students also increased in groups of two. Students in groups of two gave an average of 6 explanations each class period, while groups of four gave only 2.

* There were fewer instances of no explanations offered by students working in pairs. In groups of two, during 9% of the classes observed, no explanations were given by the students. In groups of four, during 31% of the classes observed no explanations were recorded.

* The role of the teacher in classrooms using pairs and groups of four was different. Teachers interacted much more with groups of four than with groups of two, but much of this interaction was for management and discipline. This was largely due to the observation that 48% of the groups of four were judged to be dysfunctional due to excessive off-task behavior, while only 9% of the pairs could be similarly classified.

* Related to the above observation, students working in groups of four made 21% of their explanations to the teacher, while students in pairs tended to give explanations to each other, offering only 6% of their explanations to the teacher.

* Perhaps the most striking difference between the groups is their relative degree of involvement in the problem solving process. As measured by the number of verbal exchanges by each student, we observed that only 9% of the pairs had a non-engaged member compared to 48% of the groups of four. In both types of groups there tended to be a dominant individual who made more contributions. However, the reluctant member(s) seemed more likely to make a contribution in the smaller group. In 36% of the student pairs, each partner contributed

equally in the dialogue, asking questions and giving explanations. This was never the case in groups of four.

It was noted that although only one teacher from the study of groups of four volunteered to try pair problem solving spring quarter, by the end of the term two of the three remaining teachers had also adopted this configuration in their classrooms. When asked about this, they cited management and total group involvement as reasons for the change. One teacher reported that frequently students in groups of four tended to split into two pairs which worked independently of each other. The second teacher, concerned about lack of total participation among all group members, had unsuccessfully tried seating groups of four at smaller tables, hoping closer proximity would encourage collaboration.

It was also observed that the role of the teacher changed noticeably when students worked in pairs; the teacher acted as a facilitator between groups, asked probing questions, and encouraged students to explain their thinking. In contrast, when students worked in groups of four, the teacher intervened more often for management/discipline purposes, and tended to ask more leading questions in order to involve reluctant students.

Based on the observations made during this research, grouping students in pairs appears to be more effective than grouping in fours. Students engage in more problem solving related dialogue in pairs and off-task behavior decreases markedly. Pair problem solving seems to provide maximum opportunities for students to explain their thinking, defend their procedures and conclusions, and question other students about their methodology.

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FOUR CASE STUDIES OF THE STABILITY AND DURABILITY OF CHILDREN'S METHODS OF PROOF [1]

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This research is a component of a longitudinal study, now in its fifth year, of the development of mathematical ideas in children. Four children were presented with a counting problem in grade three and again in grade four. In constructing justifications for their solution the children invented methods of proof. This report describes their reasoning and traces the durability of their construction in subsequent follow-up assessments to grade five.

There has been considerable interest over the last few years (NCTM, 1989) in empowering students to explain their thinking about mathematical situations and provide justifications for their solutions to problems. In our study of the development of mathematical ideas in children, we have found that children can invent, in a natural way, mathematical proof. This paper traces a segment of the development of proof for each of four fifth-grade children: Jeff, Michelle, Milin[2] and Stephanie. Analyses of the videotaped data and of the children's written work have made possible assessments of what students can do as they are engaged in problem-solving situations in which mathematical tools are available (Davis, Maher & Martino, 1992; Martino, 1992; Maher & Martino, 1992a). The study takes place in an elementary school in New Jersey which has been a site for mathematics teacher development since 1983. This long-term partnership with teachers has resulted in cooperative research of children's development of mathematical knowledge (Maher, Martino & Davis, submitted for publication; Maher & Martino, 1992b). In tracing the stability of student 's invented methods of proof, three modes of assessment were used: (1) the children's construction of justification to a problem (March 10, 1992) arising out of a four member group discussion; (2) the children's written productions of justification arising out of a paired-problem solving format, three months later (June 15, 1992); (3) the children's productions arising out of their individual written justification (October 5, 1992) four months later. [3]

The Tasks - Three variants of a combinatorial task consisting of building towers of a particular height were presented to students over a two-year period. Students were asked to build as many different towers as possible four cubes tall (grade 3, October 1990), and five cubes tall (grade 4, February 1992) when Unifix Cubes in two colors were available. Also, they were asked to determine whether they had built all possible towers without omitting or duplicating any, and to provide a convincing argument that all possible arrangements had been found. The design of the lesson allowed students to work in pairs and later share their strategies and results during a whole class discussion led by the teacher on the following day.

Questions that students were unable to resolve were left open in anticipation of a later administration of variations of this task.

A variant of the task (towers three cubes tall) was given to four children in grade 4 (March 10, 1992). The children, Jeff, Michelle, Milin and Stephanie, were asked to determine the number of towers of height three that could be built when there were two colors available, and to convince each other that they had accounted for all possibilities. Out of this discussion, three methods of proof emerged: *proof by cases, indirect method of proof, and proof by mathematical induction* (Maher, Martino & Davis, submitted for publication; Davis, 1992).

To test the durability of the children's mathematical behavior demonstrated on the March 10th assessment, two other assessments followed. The first (June 15, 1992) was administered to children working in pairs (Stephanie and Milin; Jeff and Michelle) and the second (October 5, 1992) was given individually. These two subsequent written assessments were used to trace and monitor the justifications presented by students during the first assessment session (March 10th).

The following questions guided the study:

1. How do students represent their problem solutions? What strategies are employed? What notations are invented? What are the justifications and arguments presented?

2. Are the justifications used by the students stable or do they change over time? For example, do students refine or modify their own methods? Are the ideas of others incorporated into their own methods?

The Data - The data for this study come from several sources: (1) transcripts and analyses of videotapes of children working in pairs or groups of four during the problem-solving sessions; (2) the written work which students produced during these taped sessions; (3) researcher observations recorded on-site, and (4) two individual written assessments for building towers of height three (June 15 and October 5). The sessions include: students working in pairs building all possible towers four cubes tall in grade 3 and all possible towers five cubes tall in grade 4; the March 10th assessment tape of the discussion of four students in grade 4; and students working in pairs to produce a written justification for their solution to the tower problem (in this case, three cubes tall) in June of grade 4.

Results - In building a justification for their solutions on March 10th, three methods of proof emerged: proof by cases (two versions), indirect method of proof, and proof by mathematical induction (Maher, Martino & Davis, submitted for publication; Davis, 1992; Maher & Martino, 1992a). Stephanie presented a *proof by cases* in which she classified her eight towers of height three when selecting from two colors (red and blue) into the following organization: all towers with no blue cubes, all towers with exactly one blue cube (three towers with exactly one blue cube in the top, middle and bottom positions forming a

staircase pattern), all towers with exactly two blue cubes "stuck together" or adjacent, all towers with three blue cubes and all towers with two blue cubes "took apart" or separated by a red cube (see Figure 1).

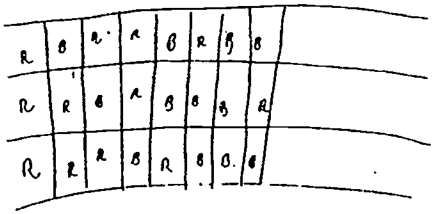

Figure 1. A written production of Stephanie's March 10th version of proof by cases.

The videotape of the March 10th session shows that Milin, Jeff and Michelle were listening to their classmate Stephanie's version of *proof by cases*. They proposed a modification of her organization of proof by cases by suggesting that she place the towers with exactly "two blue cubes separated" and the towers with "exactly two blue cubes adjacent" into the same category, "exactly two blue cubes".

For the case of towers with exactly one blue cube, the children introduced an *indirect method of proof* to justify that all towers with one blue cube were accounted for. They proposed that once a blue cube was placed in each of the three tower positions (the first with blue in the top position, the second with blue in the middle position and the third with blue in the bottom position) to place the blue cube in a different position (lower or higher) would violate an initial condition (towers of height three and exactly one blue).

Milin, during the same March 10th session, explained to his classmates that for the four possible towers that were two cubes tall when selecting from red and blue cubes there were only two blocks (red or blue) that could be placed on top of each introducing a form of *proof by mathematical induction* (see Alston & Maher, this volume).

Two Written Assessments (June 15, 1992; October 5, 1992) - Two written assessments were administered following the March 10th session to monitor the durability of students' productions, their invented justifications, and the influence of the ideas of others over time.

Analysis of the children's work indicated that their original justifications produced during the March 10th session were stable. Also, individual arguments presented appeared to influence the ideas of others. This was manifested by the appearance of more than one method of proof (Michelle and Stephanie) and a more elegant variation of original methods (Jeff and Stephanie). These are described as follows:

Jeff and Michelle [4] - On the June 15th written assessment, Jeff worked with Micbelle to produce an organization of three groups of towers when selecting from green and black cubes (see Figure 2).

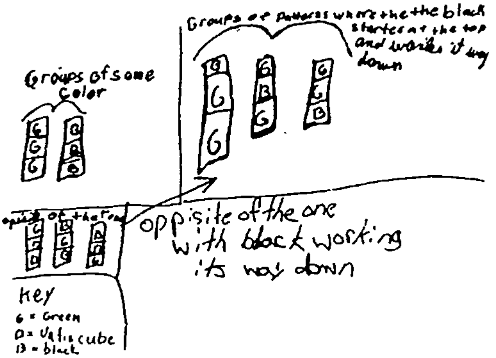

Figure 2. Jeff's June 15th justification.

He wrote: "same color" (all towers with zero cubes of one color and three cubes of the opposite color), "patterns where the black starts at the top and works its way down" (all towers with one black cube and two green cubes) and "opposite of the one (group) with black working its way down" (all towers with one green cube and two black cubes). This recursive organization satisfied Jeff's accountability for all towers with exactly two black cubes. He reasoned that consideration of the case of exactly two black cubes was more easily monitored by considering the case of exactly one green cube.

On Jeff's October 5th written assessment, he quickly replicated his *recursive method*, calling it the "step method". Thus Jeff had produced a method of proof which was specific to the case of towers three cubes tall (see National Research Council, 1993).

Michelle, on her October 5th assessment, wrote 3 x 2 = 6 + 2 = 8, and wrote that 3 x 2 represented "the height of the towers" (three cubes tall) multiplied by the "number of colors" (which was two, black and white). She then considered the two towers which were a single color and wrote "2", possibly to account for the "all black" and "all white" towers (Figure 3). Perhaps Michelle was representing Jeff's June 15th "step method" for three towers with exactly one black cube and three towers with exactly one white cube ("3 x 2") with the two solid-colored towers added on ("+2").

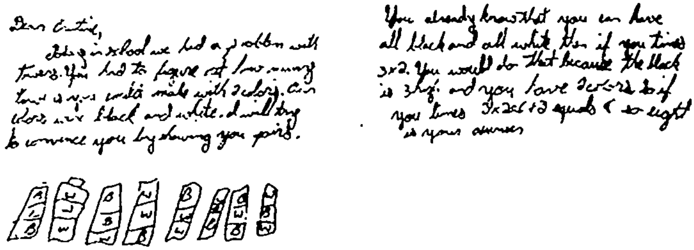

Figure 3. Michelle's October 5th justification.

In a subsequent interview in which Michelle was shown her solution to the October 5th assessment she was unable to recall what she had written earlier to represent her solution Instead she presented a proof by mathematical induction to justify her solution and built a "tower tree" with the plastic cubes. She also considered the case of building towers when selecting from three and four colors and represented her solution with a "tree" made with the cubes. [5]

Stephanie and Milin - Since Stephanie and Milin each attempted to present a complete proof on March 10th, the evolution of each child's method of proof will be discussed separately.

Stephanie Recall that on March 10th, Stephanie developed a version of proof by cases which distinguished between "two blue cubes adjacent" and "two blue cubes separated". On June 15th, using green and black cubes, Stephanie first considered the two towers of a single color. She then considered all possible towers with two black cubes and all possible towers with two green cubes. Her organization within these two cases indicated a refinement of her March 10th construction of proof. She incorporated the tower with two black cubes separated by a green cube and the two towers with two black cubes adjacent to each other into one group which she called "two black". She did the same thing for the group she now called "two green". Perhaps she had considered the comments of Jeff, Milin and Michelle during the March 10th discussion when each had suggested that all towers with exactly two cubes of a color be grouped together resulting in reorganizing 'ier cases into a more concise method of proof. Stephanie also referred to the "doubling pattern" (attributing it to Milin) and used it as a way to monitor her construction of towers as well as to predict the number of towers of any height produced when selecting from two colors of cubes.

On her October 5, 1992 assessment, Stephanie used a more elegant form of her March 10th version of proof by cases as her primary argument for convincing others that there were eight and only eight towers of height three when she selected from red and white cubes (see Figure 4). Cases were organized according to number of red cubes in which there were no longer two separate categories for towers of

exactly two of a color. This example illustrates that although Stephanie's own invented method of proof was durable, she was able to consider the ideas of others and incorporate them into a more elegant version of her proof.

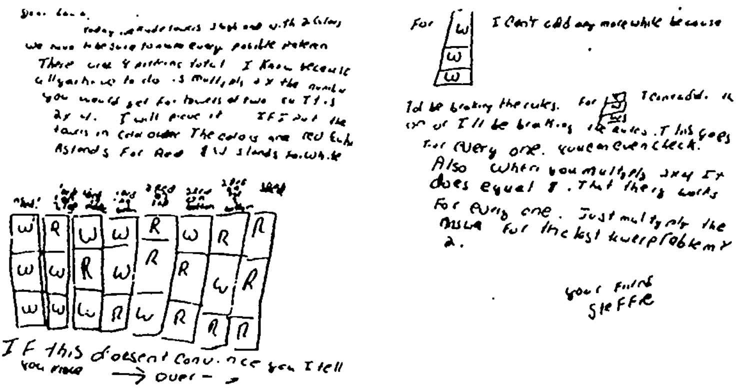

Figure 4. Stephanie's October 5th justification.

On this assessment we also observe Stephanie incorporating the ideas presented in the March 10th session and the first assessment in which she worked with Milin. She again referred to a "doubling pattern" (there are 4 towers two cubes tall, 8 towers three cubes tall, etc...) as yet another method to support the results of her proof by cases.

Milin Milin's method of proof by mathematical induction was also resilient. On both his first assessment (with Stephanie) and his second written assessment, he used his "building up" or "doubling" argument to obtain the total number of towers. In the October 5th assessment Milin presented a procedure for generating numbers of towers of different heights. Milin's notation to describe his method was also interesting (1=2, 2=4, 3=8, 4=16, etc...). Without the benefit of the earlier videotape to substantiate his reasoning, a reader may have difficulty making sense of his written explanation. However, one might reasonably infer that his argument given in the written assessments was consistent with earlier observations. However, it was presented in a much more abbreviated format (see National Research Council, 1993).

Conclusions

All of these instances indicated that once Jeff, Michelle, Milin and Stephanie had developed their own method of proof, each in turn was ready to listen to and consider the ideas of the others. Generally, the methods of proof constructed by each child were durable. Refinements could be traced from earlier

conversations. Although no student completely discarded his argument in favor of the argument of another student, each student further refined an earlier method, thus producing more elegant justifications.

Endnotes

1. This work was supported in part by grant number MDR 9053597 from the National Science Foundation. The opinions expressed are not necessarily those of the sponsoring agency, and no endorsement should be inferred.

2. For a detailed account of Milin's development of a method of proof see (Alston & Maher, this volume).

3. The written assessments of two of the children in this report, Jeff and Milin, appear in Measuring Up (1993) published by the National Research Council.

4 Recall that on March 10th, Jeff and Michelle each offered a modification of classmate Stephanie's version of *proof by cases* into an organization which focused on one attribute, the number of blue cubes in each tower. Each also recognized the usefulness of the additive "doubling pattern" used by Milin in his proof by mathematical induction. In fact, when Jeff was interviewed one month prior to the March 10th assessment he noted the "doubling pattern" and used it to calculate the number of towers for heights four, five, six and eight when selecting from two colors. Michelle had also discovered this pattern in an interview which took place two weeks prior to March 10th.

5. In a subsequent class session (February 28, 1993) Michelle and Milin worked together on an activity which extended the ideas developed on March 10, 1992. Both students constructed a "tree" with towers built from plastic cubes to justify the total number of towers to height six when selecting from two colors.

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PROBLEM SOLVING IN AND OUT OF SCHOOL

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Building upon previous research in mathematics practice in and out of school, problem solving in several everyday situations is contrasted with how problems from these contexts are solved in school situations. The use and form of problem solving were examined in the everyday contexts of carpet laying, interior design, and retailing. This out-of-school problem solving was then compared to the problem solving of students in school situations as they solved problems from these contexts. Textbook exercises set in these contexts were also examined.

Introduction

Prior to the last decade, conventional wisdom was that mathematics was culture-free knowledge. Now it is generally accepted that mathematics has a cultural history and that mathematics learning occurs during participation in cultural practices as well as in school. However, researchers that have examined mathematics practice in school and mathematics practice in out-of-school situations have noted the gap between these two (e.g., Carraher, Carraher & Schliemann, 1985). Knowledge gained in out-of-school situations often develops out of activities that: (a) occur in a familiar setting, (b) are dilemma driven, (c) are goal directed, (d) use the learner's own natural language, and (e) often occur in an apprenticeship situation allowing for observation of the skill and thinking involved in expert performance (Lester, 1989). Knowledge acquired in school all too often grows out of a transmission paradigm of instruction and is largely devoid of meaning (lack of context, relevance, specific goal).

Research Design

This paper is based on several studies examining the mathematics concepts and processes used in everyday work situations. One study explored the mathematics practice of carpet layers (Masingila, 1992b), another examined how interior designers use mathematics in solving problems that arise on the job, and a third explored the mathematics used in retailing. In each of these studies the problem solving done in the out-of-school work situation was compared to the problem solving done by students in school situations. Textbook problems were also examined in each of these studies. Various conceptual, theoretical, and methodological frameworks guided the conceptualization, design, and conduct of this study: a cultural

framework of ethnomathematics, an epistemological framework of constructivism, a cognitive framework of activity theory, and a methodological framework of ethnography. (For a more detailed discussion of these frameworks see Masingila, 1992a, 1993.)

Methods and Data Sources

In the first phase of the studies I examined the mathematics practice in an everyday work situation. In each of the everyday contexts I observed and informally questioned persons working in these contexts. These studies used four methods of data collection for the field work: participant observation, ethnographic interviewing, artifact examination, and researcher introspection. The data were analyzed using activity theory as a framework. This allowed for the data to be interpreted within the context in which they were collected and thus be as meaningful as possible. Following the example of Scribner (1984), I used occupations, work tasks, and conditions to represent the three levels of analysis—activities, goal-directed actions, and operations. I analyzed my field data through a process of inductive data analysis using two subprocesses that Lincoln and Guba (1985) called unitizing and categorizing.

Phase two of the studies involved observing and questioning pairs of students working on problems that occurred in the work contexts. In the carpet laying study, the students were ninth graders who had been through approximately the same school mathematics as the carpet layers. In the interior design and retailing studies, the students were college students preparing to enter these two careers. Data were collected for this phase by observation, informal interviewing, and researcher introspection. I analyzed the data by examining what methods each pair used to solve the problems, and then compared these with how the persons in the work context had solved the same problems. I looked for how the students understood the concepts in the problems, and how they went about solving the problems. I also analyzed comments made by the students that might give me insight into why they did what they did.

Comparing In-School and Out-of-School Problem Solving

There appear to be two common threads running through the research literature on mathematics practice in everyday situations. First, the fact that problems are embedded in real contexts that are meaningful to the problem solver motivates and sustains problem-solving

activity. Secondly, in solving problems that arise or are formulated in everyday situations, problem solvers often use "mathematical procedures and thinking processes that are quite different from those learned in school. Furthermore, people's everyday mathematics often reflects a higher level of thinking than is typically expected or accomplished in school" (Lester, 1989, p. 33).

Lave (1988) has found evidence that mathematics practice in everyday settings differs from school mathematics practice in a variety of ways. In everyday settings: (a) people look efficacious as they deal with complex tasks, (b) mathematics practice is structured in relation to ongoing activity and setting, (c) people have more than sufficient mathematical knowledge to deal with problems, (d) mathematics practice is nearly always correct, (e) problems can be changed, transformed, abandoned and/or solved since the problem has been generated by the problem solver, and (f) procedures are invented on the spot as needed.

I found several differences between the problem solving as done by persons in the everyday work contexts and by students in school situations. In everyday situations: (a) a mathematical concept is sometimes understood and used differently than the way it is taught in school; (b) people are flexible in dealing with constraints inherent in a problem; and (c) problem solvers develop a "feel" for their work and trust that sense.

Different Conceptual Understanding

Percentage of change is a common concept in retailing and in school mathematics. In school, percentage of change is understood to be the amount of change from the original amount. A typical textbook exercise involving this concept might be the following:

Find the percent of change for a video game system that cost $29 in 1980 and $99 in 1990. (Davison, Landau, McCracken & Thompson, 1992, p. 262).

A student finding the answer to this exercise would subtract $29 from $99 to get a $70 increase, then divide $70 by $29 to get an increase of approximately 241%. Percentage of change in retailing, however, is understood to be the amount of change from the retail price. Thus, for the situation in the textbook exercise above, a retailer would divide $70 by $99 to get an increase of approximately 71%. Since the final result in retailing is sales, all percentages of

change are based on retail prices. In this case, the problem solving done in the everyday context is different because of the different conceptual understanding of percentage of change.

Flexibility in Dealing with Constraints

Problems that occurred in each of the everyday situations I examined were filled with constraints. Constraints I observed in the carpet laying context include that: (a) floor covering materials come in specified sizes (e.g., most carpet is 12' wide, base (vinyl pieces glued around the perimeter of a room) is 4' long, most tile is 1' x 1'), (b) carpet pieces are rectangular, (c) carpet in a room (and usually throughout a building) must have the nap (the dense, fuzzy surface on carpet formed by fibers from the underlying material) running in the same direction, (d) consideration of seam placement is very important because of traffic patterns and the type of carpet being installed, (e) some carpets have patterns that must match at the seams, (f) carpet seam placement for commercial jobs is sometimes determined by how the amount of carpet ordered is divided among the rolls sent from the manufacturer, (g) tile and wood must be laid to be lengthwise and widthwise symmetrical about the center of the room, and (h) fill pieces for both tile and base must be six inches or more to stay glued in place. Some particular situations have more constraints, such as a post in the middle of a room that is being carpeted or a pipe sticking out from a wall where base is being installed.

Interior design work has the same type of constraints as carpet laying—size of materials, patterns and seam considerations, physical structures in a building, symmetry of ceiling grids—while in retailing one has to consider factors such as markups, markdowns, shrinkage (negative difference between the final book inventory and the actual verified physical inventory), how the time of year affects sales, and operating expenses.

The students who worked problems from the carpet laying context often had difficulty dealing with the constraints involved in the problems. For example, in a problem involving the installation of tile, the students struggled to figure out a way to install the tile so that the constraints about lengthwise and widthwise symmetry and fill pieces being at least six inches wide were fulfilled (see Masingila, 1992b for more discussion about the students' problem-

solving work). Similarly, the students training to be interior designers had difficulty trying to determine the amount of wallpaper needed while considering the repeat pattern.

The students were also not as flexible as the experienced workers in seeing more than one way to solve a problem. In a pentagonal-shaped room that needed carpeting, the ninth-grade students were able to see only one way (without guidance from me) to install carpet. The estimator, on the other hand, was able to visualize how the carpet would be laid if it were installed with the nap running in the direction of the maximum length of the room and with the nap running in the direction of the maximum width of the room. By having more than one solution, he was able to weigh cost efficiency against seam placement and make a decision while considering these constraints (Masingila, 1992c).

One of the problems I gave the interior design students to work was as follows:

You need to purchase some materials for upholstering some chairs. If you buy the whole bolt, which has 60 yards of material, it will cost $5.00 per yard. If you purchase less than a full bolt, it will cost an additional $1.50 per yard. At what point does it become more economical to purchase the whole bolt of material?

One pair of students solved this problem by first finding that the whole bolt would cost $300. They then guessed that 50 yards at $6.50 might be close to $300, and then tried 40 yards, 45 yards, and 46 yards. They decided that if they needed to purchase anything more than 46 yards, they would buy the whole bolt. The interior designer, who was faced with this problem, also found that the whole bolt cost $300 and then divided by $6.50 to find that 46 yards and 5 inches is approximately the amount at which it becomes more economical to buy the whole bolt. However, she decided that if she needed an amount close to 46 yards, "like if I needed 44 yards, I would buy the whole thing because I'm spending less than 5% over what I need and I can most likely use the material for something." For the students, this problem had only one answer; for the interior designer, the answer depended upon the situation.

My interpretation of this difference in flexibility on the part of the students and the carpet layers, interior designers, and retailers is that the students, for the most, have not been exposed to problems with real-life constraints that must be considered and addressed in order to find

solutions. Although there are many exercises in school mathematics textbooks that are set in a floor covering, interior design, or retailing context, the exercises are typically devoid of real-life constraints and, as a result, do not require students to engage in the type of problem solving required in the everyday contexts (Masingila & Lester, 1992).

Developing a "Feel"

During my conversations with floor covering installers, retailers, and interior designers, they all mentioned that after awhile they developed a "feel" for their work. In the carpet laying context, developing a feel for the work includes being able to: (a) know when the carpet is stretched tight enough, (b) make difficult 45° angle cuts to match carpet seams, and (c) look at a room and know what the obstacles are. Developing a feel also involves number sense; for example, knowing if the square yardage of carpet you calculated makes sense given the dimensions of the room. The following conversation illustrates the use of number sense in the retailing context.

Joanna: Since the markup is based on the retail price, how do you figure out what to have as the retail price when you buy something?

Bob: Well, if the markup is 50%, then you just double the cost.

Joanna: How about if the markup is 40%?

Bob: After awhile you develop a feel for it. Suppose you bought something for $2 and you want to mark it up 40%. So the retail price should be around $3.40. I've done this for so long that I just sort of know what the price should be.

When I posed this same situation to the retailing students, none of them had a ready answer. They each punched things out on a calculator by guessing a certain number, trying it, and then picking a better number based on that calculation. Eventually, they arrived at suitable values for the retail price, but they did not display the same number sense that Bob demonstrated.

Closing the Gap

There is growing evidence that very little knowledge and very few skills can be "transported directly from school to out-of-school use . . . To be truly skillful outside school,

people must develop situation-specific forms of competence" (Resnick, 1987, p. 15). If we hope to close the gap between doing mathematics in school situations and doing mathematics in out-of-school situations, then we need to learn about and from mathematics practice in everyday situations and engage students in school in *doing* mathematics--developing and using processes that will enable individuals to have the resources to become competent in specific situations when needed.

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Social and Cultural Factors Affecting Learning

HOUSEHOLD VISITS AND TEACHERS' STUDY GROUPS: INTEGRATING MATHEMATICS TO A SOCIO-CULTURAL APPROACH TO INSTRUCTION

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This paper presents a collaborative project between university and school teachers. Our goal is to develop classroom teaching that builds on the students' and their families' knowledge and experiences (funds of knowledge). Via household visits and study groups, these funds of knowledge are uncovered and used to develop learning modules. This paper reports on two aspects of the project -- the study groups, where we have an ongoing dialogue about what mathematics is and how to tap into the mathematical funds of knowledge, and an example of a classroom implementation, where we were especially interested in integrating mathematics into the learning module.

Bishop and Abreu (1991) in writing about the gap between out-of-school and in-school mathematics comment that there is not enough research on how to actually bridge such gap in school teaching. Nunes (1992) writes "how can teachers identify and capitalize on mathematics learned outside school?" (p. 557). This paper addresses these issues and responds to the challenge presented by Wilson and Mosquera (1991), namely "we are challenging all researchers to consider a culture inclusive approach to mathematics education" (p. 26).

Background

The work reported here is part of a research project that has as a primary goal to gather information on the knowledge and resources existing in the students' homes and community to then develop classroom learning modules based on this information. The teachers in this project teach in schools with a large Mexican-American and/or Yaqui Indian student population. Our position is a rejection of the deficit model for minority education and a belief that students, their families and other members in their community share a rich body of knowledge and skills that can become valuable resources for school instruction. A key concept in this research is that of Funds of Knowledge, which are "the essential bodies of knowledge and information that households use to survive, to get ahead, or to thrive" (Moll, Vélez-Ibáñez, & Greenberg, et al.,1990, p. 2). There are three components in this project--household visits, study groups, and classroom implementation. The teachers receive instruction in ethnographic interviewing. They then visit the homes of some of their students to learn about the funds of knowledge in these households. Questionnaires on the family structure, parental attitudes towards child-rearing, labor history, household activities are used to provide some structure to these home visits.

The study groups reflect the collaborative nature of this project. Elementary school teachers and university faculty (in anthropology, language, reading and culture, and mathematics education) come together to share their ideas and knowledge about classrooms, teaching, learning, and the findings from the households. These sessions are instrumental to the development of learning modules that build on students' experiences and that promote their active participation in the learning process.

Originally, the focus of the project was to develop learning communities for reading and writing in bilingual classrooms. The expertise of most members in the project is in the area of bilingual literacy. Some of the guiding principles of instruction for their classrooms are: use of academically challenging activities; the goal of reading and writing is the making of meaning; build learning on the students' and their families' funds of knowledge. These principles are also applicable to the teaching of mathematics. Thus, the project has now expanded its scope to encompass mathematics instruction. This paper presents some of our efforts in this area.

Conceptual Framework

Two related bodies of research provide the conceptual framework for the work reported here: research on the gap between inside and outside school mathematics (Bishop & Abreu, 1991; Lave, 1988; Saxe, 1991), and research on the development of classroom communities where mathematics is socially constructed (Cobb, 1991; Lampert, 1986; Schoenfeld, 1991). Quite often minority students end up receiving a mathematical education that stresses basic skills and rote learning (Porter, 1990). Instead, this project encourages a mathematics teaching that stresses students' construction of meaning and connections to their outside-school world (Charbonneau & John-Steiner, 1988). Maier (1980) argues that school mathematics should become closer to folk mathematics--that is "the way people handle the mathematics-related problems arising in everyday life" (p. 21). The differences and apparent lack of connection between in-school and outside-school mathematics are well documented (Bishop & Abreu, 1991; Carraher, Carraher, & Schliemann, 1985; Lave, 1988; Saxe, 1991; Schoenfeld, 1991). Several of these studies show that people are very competent in dealing with mathematical tasks that they view as relevant to themselves. This idea of relevance is key to the project teachers as they develop the learning modules.

My primary role and interest in joining the project is to work on developing classroom mathematics communities that reflect a two-way dialogue between the school and the students' households. In these communities I envision students working in small groups using and communicating to others their "informal" knowledge of mathematics; mathematics activities being contextualized on the knowledge, skills, and experiences present in the students' households; parents (and other household members) participating in this learning community.

I turn now to two components of our work--the study groups and an example of a classroom implementation.

About the Study Groups

The teachers in the project consider these regular two-hour after school meetings as one of the most important aspects and one of the main reasons why they wanted to participate in the project.

> The study group is the point of clarification. During the study groups, I can reflect with trust. I feel no fear of being judged. I am free to think in a flow of thought. Many times that flow of thought gains clarity as other people respond or share their reflections. The study group does provide a set for exploration. [first/third grade teacher]

A variety of issues are tackled in any of these study group sessions: discussion of the findings from the households, brainstorming on possible learning modules based on those findings, classroom management, impact of the project on the students.

Since I joined the project, we have been having an ongoing dialogue on what mathematics is and on how to develop mathematical activities from students' experiences and background. In the rest of this section I focus on two teachers' reflections about mathematics, in particular in relation to our work on integrating mathematics in the project. These teachers reported their being less familiar with thinking about mathematics than about literacy. The first/third grade teacher expressed her need to have a teacher model different approaches to teaching mathematics in order for her to feel that she can do it. She said that she knew how to let her students play with language, but that she did not know how to let them play with mathematics. She expressed her need to have a theory about mathematics teaching, not just a series of activities to do. The fifth grade teacher said that she wanted "to get out of the rut of doing math with paper and pencil and textbook." She viewed my having joined the project as an opportunity for her to work on how to get her students to talk about mathematics in cooperative groups. In her view, the biggest obstacle to blending the teaching of mathematics with the spirit of the project was the fact that her

students were resistant to change and seemed to have "the idea that real work isn't getting done if it's not oriented to a page in the book and a worksheet filled out and checked off."

Both teachers commented that, as they started to think about mathematics in the project, they noticed how much the spirit of the project could carry over how they teach mathematics:

> The philosophies are really parallel, learning occurs when it's authentic, when it has something to do in the child's life at that point in time, and I think it's very important in both the literacy and the mathematics, but I had more training on how to do this in literacy, and I have not had the training on how to do that in mathematics... When we [the two teachers] met informally to discuss it [the learning module], again most of the discussion was on the literacy. We both felt we had more expertise in that area; on the mathematics; it was, well, let me ask Marta... and you had to be there for us to even think about these issues. [First/third grade teac' -:]

The seeds for promoting the development of mathematics classroom communities are there, yet there are also some obstacles as I have illustrated in this section. The study groups have served as a forum to start a dialogue about teaching and learning mathematics. But in order to really integrate mathematics in the project, I think that a more determined effort needs to be made. The teachers need to experience mathematics as learners themselves, they need to "see it happen." The study groups by themselves are not enough. More focused sessions working on developing a specific learning module, and looking at the mathematics potential in the module is one step further, as this next section will show.

In the Classroom

In this section I dc -ribe an example of a learning module around the theme of construction in a second grade class. This theme emerged as the teacher noticed that the families she had visited were very knowledgeable about construction (framing a house, bricklaying, ...). In the summer prior to the classroom implementation, the teacher, two other project members and myself met regularly to plan the module. This teacher uses a thematic approach in her teaching. Thus, the final learning module integrated children's literature, writing, reading, mathematics, and social studies. In mathematics, the teacher wanted to develop her students' awareness for different shapes as well as work on measurement (including the concepts of perimeter, area and volume), estimation, and patterning.

But a key aspect of this project is to build learning on students' experiences and knowledge. This teacher believes very strongly in following her students' agenda. This belief shaped our planning sessions and made me realize an issue that I have not yet resolved. It is the issue of preserving the

"purity of funds of knowledge" (students' experiences driving the learning) while at the same time pushing for "the math potential of certain situations." Our goal is teacher initiated change. The teachers make the ultimate decisions as to what to try in their classroom, and for this teacher giving her students a voice in the planning of the module is a must. Thus, the module was developed and modified as she and her students were working on it (as Henderson & Landesman (1992) note, in this approach, time becomes a key issue as it is hard to extract the mathematics potential, plan, and implement at the same time). As a starting point to a theme, she has the students make a web to find out what they know about the theme and what they would like to learn about this topic. The theme develops largely out of her students' questions. In the construction module, some of these questions were: "how do you put a 2 x 4?", "how do you frame a house?", How do you put a cooler?" At home, the parents contributed their expertise by helping the children on these questions and commenting on their uses of mathematics in the construction tasks.

The students worked on a series of mathematical activities, such as building with manipulative materials, measuring bricks, following a recipe to make adobe bricks, creating patterns, estimating the number of bricks in a wall, making a blueprint for their haunted house for Halloween, and as a final project, building their community. We videotaped some activities, audiotaped the pre and post module interview of six students (three pairs), collected students' writing, and I took field notes of classroom observations. There was clear growth at the literacy level (in their written expression and in their overall acquired knowledge about construction). But I find it harder to convey what happened in mathematics. Many of the ingredients that I envision for a classroom mathematics community are present in this second grade class. The students are used to working with each other in a constructive way, and are eager to engage in conversations about their work. The atmosphere in the class is such that students are encouraged to contribute their ideas and to use their "informal" methods in mathematics. I was impressed by their persistence and their willingness to pursue a problem. The students took on any question I would bring up, such as two girls trying to figure out how many cans of paint would be needed to paint a house they had made on the geoboard; or a group of three students working on patterns took on the challenge to figure out a Fibonacci pattern I made with pebbles. They all appeared to be engaged and enjoying what they were doing. The teacher commented that now that she was teaching

using a thematic approach and involving her students in the decision making, she kept hearing her students saying "teacher, you do the neatest things" and she had never seen them bored.

Mathematics is integrated throughout their work and this reflects the teacher's view of what doing mathematics is:

> I think that's what the whole project is about, making them realize that there is a lot that they know about mathematics already, everything you do is mathematics, math is not this worksheet they give you, ... I think in the past we thought of mathematics as being formulas and you know, lots of numbers in a written form, and now, my concept of mathematics is totally different, it's meaning, to me, it's finding meaning to why some things work and it doesn't necessarily have to be in numbers, it can be in written language, in literature, (pause) I don't know, just finding out why something works, that's how I see mathematics; and I think that's where they were at, I think that if we start looking for that, if we start looking to see if they are making meaning out of what they're doing, building, making, that's mathematics.
>
> I think that their understanding became more profound, their understanding of what they knew, like M. [a student] when she measured that brick, and said, "oh, I know the other side is going to be the same", she knew that that brick was square and that the other side would be the same and I think that for her it was like "oh, I already knew that"; she realized how much she knew about a square, and to me that's mathematics; to me that was a real breakthrough.

Conclusion

These teachers feel comfortable with the household visits and with the process of developing learning modules based on the funds of knowledge. They have a firm belief in grounding instruction on students' experiences. When asked about the effects they have noticed in their classroom, they all report a dramatic change in their perceptions of their students as learners. They attribute this change to their having experienced the wealth of resources in their students' households. They also remark on a high parental involvement, now that they have redefined for themselves what parental involvement means. The second grade teacher, referred to the motivation and excitement that her students have about learning. The first/third grade teacher talked about the development of her students' awareness of the fact that they have a lot they can contribute to their own learning.

Our current interest and effort is on the pedagogical implications of this work. We are focusing on how to document and analyze what takes place in the classroom. What can we do beyond providing a rich description of the classroom atmosphere? The students appear to be using mathematics to make sense of their environment. What is the nature of their mathematical discourse? What kinds of connections are they making? Since this project emphasizes contextualized learning, I am particularly interested in pursuing the issue of transfer of learning across situations.

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ANALYSIS OF INTERACTIONS BETWEEN AFFECT AND COGNITION
IN ELEMENTARY SCHOOL CHILDREN DURING PROBLEM SOLVING

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As researchers we seek to infer particular emotional states in students from our observation of their mathematical behavior. We discuss the analysis techniques used in identifying interactions between affect and cognition in a structured, task-based interview. This is part of an exploratory longitudinal study looking at elementary school children's mathematical development over three years. We consider an excerpt from the interview of one fifth-grade student solving a "jelly bean" problem, to illustrate an interaction between affect and cognition, and to conjecture affective pathways constructed by the child.

Introduction

The role of affect in mathematical problem solving is of great interest to the mathematics education research community (Goldin, 1988; McLeod and Adams, 1989), and affect has also become of broad interest in the study of learning (Renge and Dalle, 1993). As researchers, we face the crucial question: How does one measure affect? To be more specific, how can we reliably infer particular emotions from our observations of a subject? What are valid methods for eliciting affect, in the context of task-based problem-solving interviews?

In our research, we have been considering two levels of affect -- "local affect" and "global affect". Goldin (1988) defines "local affect" to be the "changing states of feeling during problem solving", and treats it as an internal *system of representation* for problem solving, on a par with imagistic representation, formal notational representation, verbal representation, and a system of planning and executive control. Global affect, in contrast, refers to general feelings and attitudes, reinforced by belief structures, that solvers may bring to a problem situation but that are not so readily modified. This paper, through exploratory observation and analysis, begins to describe possible methods of studying both global and local affect during problem solving.

Our analysis includes a number of techniques: (1) analyzing student responses cognitively, based on videotapes of clinical interviews; (2) considering tone of voice; (3) looking for "meta-affect" -- affective responses referring to oneself solving problems; and (4) using the Maximally Discriminative

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Facial Movement Coding System (MAX). MAX was initially developed by C.E. Izard in 1979 for measuring the emotion signals in the facial behaviors of infants and young children (Izard, 1983). Using these four tools, we hope to construct an affect profile for each subject.

The MAX system codes the movement of three facial regions; 1) the brow region, 2) the eye/nose/cheek region and 3) the mouth region. The basic assumption underlying the system for identifying emotion expressions is that emotion activation results in organized patterns of facial movements or appearance changes that represent or signal the emotions of human experience (Izard, 1983). The coding of each region is done separately according to categories designed to distinguish among the types of movement observed. A MAX number is assigned to each second of taping to identify which category of activity (if any) has taken place. Observers are trained until an eighty percent reliability in coding is achieved. After all regions have been coded, a correspondence is drawn between combinations of the three codes, and the states of feeling that are to be inferred: states such as interest, surprise, enjoyment, sadness, anger, disgust, discomfort/pain or fear, as well as various blends of these. For instance the combination of raised eyebrows, an enlarged, roundish appearance of the eye region, and corners of the mouth pulled back and slightly up corresponds to a facial expression of "enjoyment" in the MAX system (these descriptions are highly abridged).

The overall purpose of the techniques we are using is to be able to identify *interactions between affect and cognition* (DeBellis and Goldin, 1991), as well as to propose *affective pathways constructed by the solver* (Goldin, 1988). This type of analysis is further motivated by the need to find ways of observing solvers that permit inferences about deeper mathematical understandings (Lesh and Lamon, 1992). The aim of our analysis is to independently infer internal cognitive and affective representational states for each subject. Interactions are identified when these are compared at corresponding times.

Subject and Method

The initial group of subjects in a study currently in progress (Goldin, et al., 1993) were twenty-two elementary school children from a cross-section of New Jersey communities: two urban schools (5 third-graders and 4 fourth-graders); one school in a predominantly blue-collar, "working class" community (7 fourth-graders); and one in a suburban, "upper middle-class" district (6 third-graders). These students are being followed in a three-year longitudinal study which looks at how children's internal systems of cognitive representation develop over time. As part of that project, six task-based interviews are being conducted with each subject. For this report, we have selected one subject and will describe a single episode that occurred this year during the third (of the six) task-based interview. Our subject is Jerome; he is ten years old (two weeks before his eleventh birthday), and presently (at the time of the interview) in the fifth grade. He attends school in the "working class" community.

Three task-based interviews were conducted at each of the above schools during 1992 and the first

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half of 1993. Normally, two video cameras were set up to record each interview session. But in the third interview, to look more specifically at affect, three video cameras were used. One camera focuses on the problem solver's work (since the interviews elicit the use of external materials); the second camera focuses on both the problem solver and the clinician; and the third camera, though at a distance from the child, zooms in on his or her facial expressions.

The interviews are conducted by members of a team of clinicians (a professor together with graduate students working toward advanced degrees at Rutgers University). All of the clinicians have teaching experience. The clinician follows a carefully-structured script. In the interview we describe, two problems are successively introduced after a short discussion to elicit aspects of the subjects' global affect. The children are encouraged to "think aloud" while solving the problems, and are sometimes asked to describe the way they are feeling.

The question "Could you think back to the first time you remember doing mathematics?" is followed by several further questions to engage the child in describing as vividly as possible a recollection of an early mathematical experience. After a description is elicited, questions such as "How did you feel about that?", "How did you feel when that happened?", "Did you enjoy it?", "Was there anything you didn't like about it?" are asked. Finally, the question "Do you think this experience has anything to do with how you feel about mathematics now?" is asked in an attempt to connect the early experience described to the child's global affect in relation to mathematics today. Similar questions regarding mathematics in school, and mathematics at home or with friends, are also asked. Several other questions address global affect about problem solving: "Do you think you're good at solving problems?", "Why do you think that?"; "Do you like solving problems?"; "What do you think makes someone a good problem solver?"; "Who do you think solves problems best in your class?", "Why do you think (name) is a good problem solver?".

After these questions, the first problem is presented (orally). "Which would be easier ... to cut a birthday cake into three equal pieces or four equal pieces?" On the table in front of the subject are a ruler; three styrofoam "cakes" -- one with round, one with triangular, and one with rectangular cross-sections; a ball of thin white string; thin markers of several colors; two pencils; a pair of scissors; a pad of 1/4 inch graph paper; a pad of white paper; and two jelly beans -- one orange and one green. Several problem variations are then explored focusing on whether the shape of the cake matters, and the task is varied by introducing icing on the top and on the sides of the cake. The goal of each question is to elicit a coherent description and external representation from the child.

The second problem is also presented orally; simultaneously the clinician places two small glass jars of jelly beans on the table: "The next problem is about jelly beans. This jar has 100 green jelly beans" (clinician points to the jar containing the green beans) "and this jar has 100 orange jelly beans"

(clinician points to the jar containing the orange beans). "Suppose you take ten green jelly beans from the green jar, and put them into the orange jar" (clinician points to the orange jar), "and mix them up." (clinician pretends to transfer the jelly beans but does not do it) "Then suppose you take ten jelly beans from this mixture and put them back into the green jar." (clinician pretends to transfer the jelly beans but does not do it) "Which jar would have more of the other color jelly beans in it? Would there be more green jelly beans in the orange jar, or would there be more orange jelly beans in the green jar?" The child is left free to solve the problem, at first without hints or suggestions. Then s/he is encouraged to explain the problem, to justify any answer or conjecture, and to elaborate through questions such as, "Why?" "Will it always come out that way?" "Can you explain to me using the jelly beans?" If no answer is given, the child is encouraged directly to experiment with the jelly beans.

Affect and Cognition During One Episode of the "Jelly Bean" Problem

When Jerome was solving the "jelly bean" problem, his initial response to the question, "Which jar would have more of the other color jelly beans in it?" was (incorrectly) that the orange jar would have more green jelly beans in it. He stated, "I think the orange jar, because if you mix it up the chances of you taking ten jelly beans out of here with some already ... I just think it's a better chance." As the interview continued, he changed his answer to be (correctly) that the number of the other color in each jar would be the same, after testing an example which transferred nine orange jelly beans and one green jelly bean back into the green jar. When the clinician asked, "Will it always be equal?" he responded, "If you're using ten ... I think." The clinician then asked, "What if you're not using ten?" Jerome concluded that if you're transferring an odd number (in your handful) then the numbers of the other color in each jar will *not* be equal, but if you're transferring an even number of jelly beans from jar to jar, then the numbers of the other color in each jar will always come out the same.

He maintained this position and decided to run two experiments: one experiment which transferred ten jelly beans (his "even" choice) and a second experiment which transferred eleven jelly beans (his "odd" choice). He ran his first experiment (ten jelly beans), and correctly concluded there were an equal number of the other color in each of the jars. He ran his second experiment (eleven jelly beans), and found that there were seven orange jelly beans in the green jar and seven green jelly beans in the orange jar. This contradicted his conception that with odd numbers the amounts would be different. He spontaneously said that he didn't think that could happen, and justified the outcome by saying there must have been "another one" (jelly bean) in the jar from the last problem. At this point, the clinician asked Jerome if he would like to start the "odd" experiment over. He said, "I don't care" and began to set up the experiment for another trial. Prior to starting the "odd" experiment for a second time, the clinician asked Jerome, "What do you think is going to happen?". He answered, "Well, they're gonna be two different numbers." Then he did the experiment, transferring eleven jelly beans for a

second time. Below is a section of our transcription of what happened next ("..." implies a short pause in speech):

Jerome:	How many went over? (whispers to self) One, two, three, four, five, six, seven, eight, nine. Two greens went with this one ... nine ... nine oranges ... that's eleven. Now, one, two.
Clinician:	You can dump them out.
Jerome:	(begins to count jelly beans in other jar -- silence for 57 seconds) Umm ... two, four, six, eight, nine. Well there won't be ... they're both ... they keep on equaling the same amount even if they're odd! But? Two ... two green went over ... and ... two green with that and nine orange and then nine green are left. Well they still stayed the same. The same amount in each one.
Clinician:	How can that be?
Jerome:	I don't know. (silence for 15 seconds)
Clinician:	What do you think is going on?
Jerome:	I dunno. (silence for 3 seconds)
Clinician:	What are you thinking about?
Jerome:	UHHH ... I'm just trying to figure out how did this happen. (silence for 17 seconds)
Clinician:	Can you figure it out?
Jerome:	(shakes head NO after 6 seconds and stops actively solving the problem)

Observations and Interpretations

Analysis of this episode begins by analyzing Jerome's internal strategic/heuristic/executive representation of the problem. We conclude he has prior to this episode *established a subgoal* (of testing or demonstrating his "even/odd" conjecture. He is *executing a plan* for reaching this subgoal by conducting an experiment (one case each). He *believes* his conjecture to be true, and is carefully *monitoring* a second trial of the case that was at variance with it. Transcription of the tape is labeled by the time that has passed during the interview: every videotape displays the hour:minute:second:tenth of second on the screen. Independently, the facial videotape has been coded using the MAX system; the MAX coding is also labeled by the time (to the second) that has passed during the interview.

The MAX system provides the information that Jerome was enjoying himself as he was attempting to solve this problem. When the clinician asked him "How can that be?" and he responds "I don't know", the MAX coding indicates a surprise/enjoyment blend, based on Jerome's facial expression the instant after the question was asked. This seems to confirm our inference that Jerome was surprised to find the numbers of the opposite color jelly beans to be the same. During the 17 seconds of silence in the above episode, the MAX coding suggests that Jerome's facial activity indicated an anger/enjoyment

blend. This is particularly interesting, since it was the first suggestion of "anger" during the interview. Soon after that, Jerome shook his head "no" and stopped solving the problem. It seems that for Jerome, the moment that anger became a part of his problem solving experience, he chose not to continue to solve the problem. Jerome's tone of voice seems to indicate he is interested in the problem, consistent with the "enjoyment" state. We are unable to identify "anger" in his tone of voice.

Finally, meta-affective questions were asked in reference to Jerome's experience of solving the jelly bean problem. The clinician asked him when he felt the best. Jerome responded, "When I used ten and both had the same ... same amount". Then the clinician asked, "When did you feel the worst?" Jerome replied, "When the eleven had the same amount ... cause I didn't want that to happen. I thought I was right." The clinician then asked, "How do you feel now?" Jerome said, "I still don't know what's going on ... I just can't picture it happening." Again, we find further evidence to support our inference that Jerome was feeling badly at this time.

The responses about Jerome's "first experience" doing mathematics indicate that he has positive global affect toward mathematics. He fondly remembered an early experience doing mathematics with his family, describing how his father tried to teach him to add when he was about four or five years old. He remembered having "no clue" about what his father was talking about. He said that math is "pretty fun", but frustrating when he cannot do something. He also reported that he doesn't like "subtracting that much". When asked "Why?" he just said that it "annoys" him. Jerome reported that he has also done math at home with his brothers, who are "a lot older". One brother is a junior in high school, and the other brother is a senior in high school. He said that his brothers told him that he was "too stubborn", and have stopped doing math with him. Jerome also said he likes to solve problems. When asked why, he replied, "cause if you get into the problem and it's *interesting* then I like it but if it's just like *boring* then it has *no meaning* to it ... like I'll still try it out but I wouldn't have the same *intensity* like if I was interested. Like definitely try and figure out ... like I *know* I want to." Here he talked about his own feelings (meta-affect) in his description of why he likes to solve problems. He identified the difference in his level of "intensity" to solve problems when they are of interest to him or mean something to him. When the clinician asked Jerome who solves problems the best in his class, he responded, "Matt, because he refuses to stop until he's done." For Jerome, the "best" problem solver is the one who perseveres. Yet Jerome himself did not persevere when his experiment led to an unexpected outcome.

Jerome's Affective Pathway

The preceding discussion permits us to conjecture, in a very preliminary way, a developing pathway relating Jerome's internal affective and executive representations (Goldin, 1988). We inferred affective states involving enjoyment, enjoyment/surprise, and enjoyment/anger. The enjoyment state allows Jerome to establish subgoals, develop and/or access plans for reaching them, and carry out these

plans. Taking these actions maintains the enjoyment state. The affect of surprise occurs in response to an outcome at variance with Jerome's prior expectations. It evokes an effort to "figure out how did this happen." But after a brief, unsuccessful attempt to do that, he seems to feel anger. This state apparently *impedes* Jerome's further access to productive strategies, and he stops solving the problem – the quality of perseverance that he himself articulated as central to being a good problem solver is lost.

Conclusion

Our descriptive analysis permits some detailed inferences and conjectures concerning affect during problem solving, and its interactions with other internal representational systems in children. But much more remains to be done before we can have complete confidence in such inferences.

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MEASURING COGNITIVE AND AFFECTIVE BEHAVIORS IN THE MATHEMATICS CLASSROOM: A MULTIDIMENSIONAL APPROACH

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In this paper we describe the cognitive and affective behaviors of one small group of grade 7 students assigned an open-ended non-routine mathematical task: determining the feasibility of a new school canteen. We relied on self-report measures, and observable and inferred indicators of behavior. Data were analysed at different levels. From these, integrated descriptions of students' affective and cognitive engagement were developed. Operationalizing affective, observed and inferred behaviors were important objectives. Gender differences in these behaviors were also of interest.

Introduction

The *Handbook of Research on Mathematics Teaching and Learning* (Grouws, 1992) reveals concern among researchers about the role and measurement of affective factors in student learning of mathematics. Schoenfeld (1992) claimed that the domain of beliefs and effects was underconceptualized and in need of new methodologies and explanatory frameworks. McLeod (1992) argued that little impact on mathematics education had been made by the vast research literature on affective factors. Work concerned with gender differences in mathematics learning formed a noteworthy exception.

Good, Mulryan and McCaslin (1992) provided an extensive overview of research issues associated with small group learning in mathematics. Included among the many factors reported as contributing to student behaviors, interactions, and achievement in these learning settings were gender, group composition, students' achievement levels, and task. Good et al. (1992) noted a "paucity of process data to indicate what happens during small-group instruction" (p.167) and claimed that more information was needed about "how groups can facilitate certain student attitudes and problem-solving abilities" (p.193).

It has long been recognized that attitudes are not simple to define. Despite recognition of the complexity of the attitude concept, more limited operational definitions have often been employed to meet measurement needs. Attitudes to mathematics, for example, are typically measured using a variety of pen-and-paper self-report instruments, the most common of which were described by Leder (1987). In studying attitudes related to mathematics learning, omissions of the context of learning and the evaluation of overt behaviors are the

most frequent.

Objectives

Attempting an integrated description of students' cognitive and affective behaviors in a mathematics classroom setting was an important aim of this study. To do so required the operationalization of the affective behaviors observed and inferred. Gender differences were of interest as were comparisons between individual students' beliefs, reflected in their responses to the self-report instruments, and their behaviors when engaged in learning mathematics in a small group.

Theoretical framework

Variables included in models concerned with gender differences in mathematics learning, summarized by Leder (1992), served as a useful context for our work. Persistence, confidence, expectations for success, attributional style, social expectations, contextual effects, beliefs about mathematics, perceived usefulness and enjoyment of mathematics, stereotyping of and sex-role congruency with mathematics, willingness to work independently, mastery orientation and learned helplessness were among the variables of interest. These variables guided the choice of students' affective and cognitive behaviors to be described functionally. The observational scheme for social behaviors and information-processing components presented by Clements and Nastasi (1988) is similar to the approach adopted in this study for operationalizing and categorising these behaviors.

Methodology

The design and methodology of the study were shaped by the following:

- the setting was to be the regular grade 7 mathematics classroom
- the task set by the teacher should be realistic and challenging
- the observed group needed to be mixed so that behavior patterns could be examined for gender effects
- affective beliefs were to be described through self-report data, and behaviors through observational measures
- an extended period of observation and detailed, easily re-examinable, records of behaviors and interactions were needed if high as well as low inferences were to be drawn

Decisions about the research techniques adopted arose from the nature of the research problem. Observation methods, a descriptive system, videotaped records, interviews, and self-report data were used. The observational and self-report data were analysed qualitatively. The

latter were also quantitatively examined; these findings have been reported elsewhere (Leder & Forgasz, 1991).

The sample, setting and procedures

The sample comprised three female and two male students working on a non-routine open-ended mathematical problem in a small group. The teacher had organized the class of 28 students into six groups of four or five students and nominated the sample group as one worth observing. Each of the five students had been rated by the teacher as either very good or excellent at mathematics.

The small group was observed for seven of the eight consecutive (45 minute) lessons in which they worked on the task. The lessons were videotaped and the tapes carefully transcribed. Field notes supplemented the lesson transcripts and students' reactions to each lesson were monitored. Self-report measures of students' attitudes and beliefs about mathematics and themselves as learners of mathematics were gathered prior to the onset of the small group task and again some time later. At the end of each lesson students indicated their feelings about and understanding of the work just completed. Interviews took place towards the end of the school year.

Operational definitions of the variables of interest were developed and excerpts from the lesson transcripts were used to clarify them. One example of an operational definition is shown on Table 1.

Table 1: The operational definition of 'persistence' used in the study

Variable	Definition	Example
persistence	continues with constructive work at task or with verbalized idea when encounters difficulty, rebuff or failure without coaxing or encouragement.	B: This isn't what we need M: How about if you give everybody one survey.. B: This isn't what we need, M M: ...so they can have a look at them (grabbing sheets from B) C: Yes, it is (C takes a sheet from B). It's 10 out of 13 wanted a new canteen out of the teachers and 3 out of 13... B: This isn't what we need. It's that profit stuff, (indistinguishable word) know how much people... J: This'll help too. B, this'll help too, all right. So calm down

NB: C, Ch and J are female students, B and M are males.

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Results

The cognitive (cog) and affective (aff) behaviors were each classified at two levels: lower (LI) and higher inference (HI). Cognitive behaviors, including oral contributions to mathematical discussions, were rated as high (HL) or low level (LL). Task engagement was categorized as central (C) (to the mathematical task) or peripheral (P).

High inference analyses of cognitive and affective behaviors were preceded by an outline of the sequence of lessons observed and detailed descriptions of the content of each lesson. These, together with information about the students' perceptions of the lessons and their attitudes and beliefs about themselves and mathematics, provided an awareness of the context in which the fine-grained analyses of event episodes occurred. The different sets of data were informative in their own right and were essential prerequisites for the integrated analyses.

Specific examples illustrate the approach used. As shown on Table 1, B's persistence was categorized as aff, HI, and was associated with mathematical engagement which was cog, LL, HI, C. This excerpt illustrates a pattern of interpersonal behaviors discerned over the entire sequence of lessons which appeared contextually bound to the composition of the group: the marginalisation of C, the dominance and personal confidence of J, and B's capacity for independent thinking. Also highlighted was an exception to a more generalized pattern. During the eight lesson sequence, M and B had worked very effectively and were mutually supportive of each other's efforts. Yet here, M appeared to have ignored B's appeal for backing.

At times, the fine-grained analyses revealed conflicts with the profiles of individual student's attitudes and beliefs compiled from the self-report data. Different interpersonal dynamics were occasionally noted when the teacher (DM) was engaged with the students. DM was also found to respond differently to members of the group. The correspondingly different patterns of affective and cognitive student behaviors were put down to the contextual effect of his presence. During the lesson from which the following excerpt is taken, the verbal exchange took place during the third, and longest (4½ out of a total of 7 minutes), of DM's four visits to the group.

DM: Well, how do they make the profit? How is the profit made?
J: (M begins to speak simultaneously but stops) By people buying things and people said they'd buy...

DM: And, by buying things at a cost greater than what they pay for it, right?
J: Mmm
DM: Now, how do you increase your profit? You either increase the cost, increase your price or...
B: or increase the number of people that come in
DM: (nodding) or increase the number of people, good, increase the number of buyers in the...

DM's questioning style in this excerpt appears 'leading'. Generally, however, he seemed to allow students to explore their own ideas. Often exchanges with students would conclude without closure having been reached.

The contextual effect noted in the excerpt accompanied B's independent thinking (cog, HL, HI, C). B received positive affirmation (aff, HI) for his response in the discussion. J, however, was prevented from completing her contribution and, uncharacteristically, did not persist with her seemingly valuable contribution. Within the global context of the sequence of lessons, analyses revealed that J commanded much more of DM's time than any other member of the group. Aware of J's propensity to dominate, perhaps DM, by ignoring J in this instance, may have helped encourage others to volunteer their ideas.

Overview of cognitive behaviors

Generally, the group 'worked' well with minimal prompting from the teacher. The task proved challenging and the level of mathematics in which some members of the group were engaged might not be so enthusiastically embraced by grade 7 students in a more traditional setting. Grappling with percentage increases with inconvenient numbers was one example. The task set by the teacher was meaningful to the students within the school context as they had a personal stake in the eventual outcome. This appeared to provide some motivation for a genuine attempt at the task. The teacher had also outlined the basis on which the group report submitted would be assessed. To some extent the requirements for assessment appeared also to have influenced the directions undertaken and the effort expended by some group members on various aspects of the task. Presentation was a notable example, particularly for the girls.

During the eight lessons, the five students were often engaged in different activities. As the sequence of lessons progressed, task demarcation, along gender-stereotypic lines, became more evident. The two boys were more often engaged in mathematical tasks, the girls in more peripheral activities related to presentation. This might account for the girls having spent more time than the boys in off task

conversation. There was, for example, little to stimulate mathematical discourse when coloring in bar charts already plotted by the boys.

Overview of affective behaviors

Self report data after each lesson indicated that the girls were more confident than the boys that they had understood the work associated with the task. Analyses of group interactions also revealed a clear pattern of peer hierarchy to which all members of the group appeared to adhere. That is, the contextual effect of group composition appeared to influence strongly the exchanges and task engagement of students. The interviews revealed that the boys more so than the girls had found the task enjoyable, challenging and interesting. The teacher was also found to have interacted differently with students. For example, only some received positive feedback from him.

Integrated analysis

The subtle responses and reactions discernible from the fine-grained analyses enhanced the development of the integrated descriptions of students' cognitive and affective behaviors. When affective behaviors were compared with the self-report data, there appeared to be greater consistency in beliefs and observed behavior for the boys than for the girls. The tasks in which the students were engaged might partially account for the belief-behavior mismatch for the girls. During the lessons, there was little indication that the girls were dissatisfied with their predominantly non-mathematical role. The peer hierarchy, however, seemed to exercise a controlling influence on the directions of the project, on the behavior and language used by group members, and on ultimate satisfaction with the project. At interview, C who had been virtually excluded from meaningful contributions to the task, and Ch whose initial leadership status had been challenged, articulated that they had not really enjoyed the activity, and that there had been little to do. J, M and B who had been the more active participants in tackling the task had found it more interesting and challenging.

Conclusions

An important aim of the study was to attempt an integrated description of students' cognitive and affective behaviors in a mathematics classroom setting. The multi-layered approach adopted provided a more 'unified perspective' on students' cognitive and affective behaviors. The nature of the tasks undertaken, the quality and level of students' cognitive engagement and their affective involvement could be reported in more specific detail and more globally. The rich data base and

varied analyses revealed differences in task engagement and more subtle differences in students' learning experiences. Affective measures gathered from self-report data and from observed classroom behaviors did not always match. The relationship of affective behaviors to cognitive task engagement suggests the possibility of longer term implications for students' mathematical learning outcomes.

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THE MATHEMATICS IN AN EPISTEMOLOGY OF ETHNO-MUSIC

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Summary
In this paper we examine the mathematics grounded in the scientific practice of an Ethnomusicologist as he systematically attempts to bridge a Javanese music system and Western music. In support of our prior theoretical work, we provide evidence for splitting and a covariational approach to exponential and logarithmic functions. We also contrast the mathematics arising in this scientist's practice with his experience with mathematics in the school curriculum.

This study is part of a larger investigation of the uses of mathematics, particularly exponential and logarithmic relationships, in the practice of scientists. Our goal is to begin to understand the scientists' mathematics from their perspective within the context and culture of their practice. In a sense, we are applying an Ethnomathematics approach to our own university community, based on the interview methods we have used in our previous work with students. We examine the ways scientists use mathematical models to describe, predict, and interpret phenomenon; that is, the ways in which mathematics plays the role of an "epistemological object" in their practice. Through this we will extend our theoretical work on splitting and exponential functions, as well as gain insight into the relationship between the formal mathematics of the school curriculum and the math as it is developed and used in diverse practices.

The Ethnomusicologist

In this paper we discuss our work with a Cornell music professor, Dr. Martin Hatch, who works in the field of Ethnomusicology and is currently studying the music of Central Java, an Indonesian island. Our data include a number of discussions with Hatch, including a ninety minute videotaped interview, and observation of his introduction to a new computer tool for representing musical sound (Vaughn, 1992). We also used papers by Hatch and others in the field. Finally, he reviewed and suggested revisions to an earlier draft of this paper.

Ethnomusicologists, as Hatch explained, seek to document varieties of musical experiences around the world for reasons of describing the "richness of humanity and the vitality of culture", as well as contributing to an understanding of the cultural evolution of music. His current work is focused on the nature of "tuning" within a Javanese musical system which is considerably different from Western tuning. Hence he is faced with the cross-cultural dilemma of trying to hear "*what the Javanese hear*", while beginning, as he must, from his own Western perspective. Hatch described this as "*building a bridge*" between the Western and Javanese music systems. In the passage below, he explains this dilemma and how mathematics or "numbers" function as a tool in bridging the musical systems.

> We are trying to discuss ... how there are different ways of using something that we all have available to us: that is numbers. We're talking about using them in the Western academic setting, but now I'm adding a new dimension: confirming that the rules of tuning that I find through this analysis are actually operative for Javanese singers.

> I am trying to devise a universal way of understanding tuning in musical systems. I'm not sure that I have the correct combination of tools, but at least *I'm convinced that using numbers will help me get closer to that goal.*
>
> Now the issue of the relationship of universals to local cultural activities comes into play to the extent that academic work -- the system of research and rational observation and analysis -- can produce a scientific language that will be able to account for human activities all over the world. But we have found that this language is not enough. We have to broaden the application of words that we use in the West to accommodate different conditions for musical systems around the world. ... Somewhere in my mind's justification for the work with numerical analysis of tuning is the idea that what I'm doing here has some validity for what goes on in Java and that my work will contribute to an understanding of why and how all cultures and all human beings make music (italics added).

Hatch's challenge, then, is to develop an epistemology of cross cultural music; a coordination of Javanese and Western representations of musical sound using Ethnomusicological analytical techniques and drawing, intuitively, upon mathematics.

In studying Javanese tuning Hatch is investigating both *a cappella* songs and gamelan. The gamelan is the primary ensemble of instruments in traditional Javanese musical performance and is made up largely of fixed-pitch percussion instruments, including drums, gongs, metallophones, cymbals, bells, and rattles (Rossing, Shepard, 1982) and a few stringed instruments. Singers may also accompany the gamelan.

The Representation of Musical Sound

In Western music, an "octave" implies a multiplicative relationship between pitch frequencies. Raising a note one octave doubles the frequency of the pitch. Notes in the Western system are also "equal tempered," that is, the ratio of the frequencies between any two successive notes in a chromatic scale is constant.[1] Although the Javanese do have an octave that represents approximately a doubling of frequency, the notes in between are not equally tempered. The ratio between successive pitches in the Javanese scale varies significantly. To analyze this variance, Hatch measures the pitch frequencies of Javanese *a cappella* songs and gamelans. The measurements, sampled and displayed on a computer screen, provide a verification of what he "hears" in the songs.

While frequency gives Hatch a quantification of musical pitch, he is often interested in the *"intervals"* or *musical distance* between notes. Working with frequency presents a problem -- a problem which Hatch resolves with a mathematical transformation. The frequency of each successive octave in the Western system is related to the previous octave by a 2:1 ratio. Therefore, as Hatch described it, if A below middle C is 220 cycles per second, then the next A is 440 cycles per second and the next 880. Though the *additive intervals between* 220 and 440 and between 440 and 880 are different in terms of the number of cycles per second (Hertz or Hz.), *musically,* Hatch said, *"there is something the same"* about them. Musically, it is the same distance -- one octave (See table 1.). Hence, Hatch explained,

[1] A chromatic scale includes all 12 half steps of a Western equal tempered scale and is generated, for example, by hitting 12 successive keys, white and black, on a piano.

musicologists use a logarithmic conversion of the cycles per second to get the same number of units for each interval. The conversion creates 1200 units called "cents" for every octave. So in the Western system of 12 half steps per octave, there are 100 cents in a half step interval.

The unit "cents" was developed by Alexander John Ellis (Ellis, 1884) for the "measurement of exotic music systems". Hans Peter Reinecke (1970) later published a logarithmic table providing a value in "absolute cents" corresponding to each frequency value in the audible range (as distinguished from "relative cents" below). Since there are 1200 cents per octave and an octave represents a doubling of frequency, the conversion from frequency to cents is done by taking the logarithm of frequency (F) using a base of 1200th root of 2. Reinecke provides the following equations.

$$C_{abs}\ \text{(absolute cents)} = \log_{2^{1/1200}} (F\ \text{Hz.}) \quad = j1(F)\ \text{in table 1.} \qquad (1)$$
$$10{,}538\ C_{abs} = \log_{2^{1/1200}} (440\ \text{Hz.})$$

Where C_{abs} is absolute cents such that 1 C_{abs}.corresponds to 0 Hz. It is also common for Ethnomusicologists to use "relative cents" to describe the distance or interval between musical pitches. This is done either by subtracting absolute cents or converting the ratio between the pitch frequencies (F2/F1) to cents.

$$\Delta C_{rel}\ \text{(relative cents)} = \log_{2^{1/1200}} (F2/F1) \quad = j2(C)\ \text{in table 2.} \qquad (2)$$
$$1200\ C_{rel} = \log_{2^{1/1200}} (880\ \text{Hz.} / 440\ \text{Hz.}) \quad \text{using equation (2)}$$
$$1200\ C_{rel} = \log_{2^{1/1200}} (880\ \text{Hz}) - \log_{2^{1/1200}} (440\ \text{Hz}) \quad \text{using equation (1)}$$

Now the distance or the difference between musical pitch can be described using the additive quantity "cents". Thus the conversion to cents is logical because it allows Hatch and other Ethnomusicologists to work with additive rather than multiplicative distances -- he can represent musical distance by subtracting pitch values represented in absolute cents. Using the additive unit, one can see 100 cents for every equal tempered half step and 1200 cents per octave in the table 1 below.

In the Javanese "slendro" scale being studied by Hatch there are five notes per octave, labeled 1, 2 3, 5, 6 ('4' is skipped in order to reconcile slendro with the alternative seven note "pelog" system). The scale differs from the Western scale above in that the notes are not equal tempered: the frequency ratio or the number of cents between successive notes is variable rather than constant. Furthermore, the musical spacing between notes also depends on the "mode" a piece is played in. Not only may the interval between 1 and 2 be different from the interval between 2 and 3, but the spacing between 1 and 2 alone may not be consistent from piece to piece or even from octave to octave[2]. Hatch describes this variance in the following passage from the transcript.

[2]A gamelan may be tuned to different modes in different octaves. Hatch called this a "shift in commitment".

Well if you take 1200 cents in an octave, then you can divide that octave into any number of different interval sizes by pitches. [In] slendro there are five pitches in the fixed pitch tuning, so if you take 5 into 1200 you get around 240 cents. In the slendro system, 240 is more or less an average of the different tunings one finds in Javanese gamelans. ...But...in no actual gamelan do you find 240 cent intervals... On all gamelans two of the intervals are wider than the other three. Two of the intervals have parameters that are between 260 and 320 cents, and then the other three are generally between 190 and 230 cents.

Modal practice, then, involves the notion of a "gapped scale", where the mode in which a piece is performed or a fixed pitch instrument is tuned determines which intervals are wide and which are narrow in the scale. Hatch called this the "signature interval" of a mode: "Signature intervals are the intervals that give the character to a particular mode." For example, the mode *Manyura* has wider intervals between 6 and 5 and between 3 and 2. *Nem* has wider intervals between 3 and 2 and between 1 and 6. These modes, two of the three principle modes of Javanese slendro tuning, are shown in table 2.

	F	●F	C=J1(F)	d=ΔC	c=Σd	r=F/440	
Equal Tempered Notes	Frequency (Hertz)	Freq. Ratios between each note	Absolute Cents	Relative cents between each note	Relative cents from A (Additive distance)	Freq Ratio from A (Multiplicative distance)	Musical Interval from A
A	440	1.06	10538	100	0	1.00	
Bb	466	1.06	10638	100	100	1.06	
B	494	1.06	10738	100	200	1.12	Second
C	523	1.06	10838	100	300	1.19	
C#	554	1.06	10938	100	400	1.26	Third
D	587	1.06	11038	100	500	1.33	Fourth
Eb	622	1.06	11138	100	600	1.41	
E	659	1.06	11238	100	700	1.50	Fifth
F	698	1.06	11338	100	800	1.59	
F#	740	1.06	11438	100	900	1.68	
G	784	1.06	11538	100	1000	1.78	
G#	831	1.06	11638	100	1100	1.89	
A	880		11738		1200	2.00	Octave

Table 1. All the Western half steps between A 440 Hz. and A 880 Hz. are shown in the Notes column.
Frequency: The corresponding frequencies.
Frequency Ratio: The approximate ratio between each half step, or successive frequencies.
Absolute Cents: The conversion of each frequency to absolute cents using equation (1).
Differences: The difference between half steps in cents or relative cents.
Additive Interval: The interval between A, 10,538 abs. cents, and each successive half step in cents.
Multiplicative Interval: The ratio between A, 440 Hz., and each successive half step.
From the last two columns, one can see that the multiplicative interval (musical distance) for an octave is 880:440 = 2:1, and the additive interval is 1200 cents.

	M	ΣM	R=j2(M)			N	ΣN	r=j2(N)
Javanese Slendro Scale	Manyura Intervals in Cents	Additive Difference from 6 in Cents	Manyura Intervals in Freq Ratios		Javanese Slendro Scale	Nem Intervals in Cents	Additive Difference from 6 in Cents	Nem Intervals in Freq Ratios
6		0			6		0	
5	220	220	1.14		5	310	310	1.20
3	190	410	1.12		3	210	520	1.13
2	290	700	1.18		2	260	780	1.16
1	200	900	1.12		1	190	970	1.12
6	300	1200	1.19		6	230	1200	1.14

Table 2. The Javanese slendro gapped scale in *Manyura* and *Nem* modes, portraying musical intervals in relative cents (additive difference) and frequency ratios (multiplicative difference).

Ethnomusic Analysis and a Covariational Approach to the Logarithmic Function

Hatch is investigating "tuning" in Javanese music, specifically, how it is that singers and instrumentalist tune to one another. His unit of analysis is a "musical phrase". He looks at the musical distances between the notes at the end of the phrase to determine the prominent pitches in a particular melodic configuration. This indicates the mode of a piece, as interpreted by the singers and performed on the fixed pitch gamelan, which is crucial in determining the "background system for tuning". Hatch uses the mathematical conversion of frequency to cents in his analysis of these intervals. He describes this use of "numbers" in the passage below.

> I'm very interested in knowing how it is that singers tune, what they use as their background system for tuning. But I have to take *a cappella* songs, unfortunately because the machines we have now are only good enough to trace the fundamental of a single voice. And I must commit myself to saying that when I hear the fundamental *I can represent what I hear by a number* that is the same value as what an acoustician would apply to that in hertz or cycles per second value. And that number has a musical interpretation... And if I can apply those numbers to equal tempered systems... then we can say that the distance between C and A is the distance between 523.3 cycles per second and 440... and that can be translated into a certain number of cents. *Between C and A is a minor third so that would be 300 cents.* Now if I can lay over that grid of what lies behind at least the primary level of theoretical construction of Western tuning...if I can lay over against that grid the background system that I perceive in monophonic song in Java, I've gone some distance toward comparison between tunings in Javanese non-fixed pitch realizations and tunings that lie behind Western ideas of tuning (italics added).

In this and other passages from our interview, we see evidence of a *covaritional* approach to the coordination of the *multiplicative* relationships of frequency ratios and the *additive* relationship of cents in describing musical intervals (Confrey and Smith, 1992). For example, Hatch relates the ratio expressed as

a "minor third" directly to 300 cents. He coordinates the relative musical distance represented by the frequency ratio with the additive difference represented in cents. The logarithm or the rule that maps the domain to the range is less important, in his analysis, than the covariance of the domain and range. Hatch illustrates this approach again in describing "modes" in Javanese slendro tuning.

> If you take the slendro parameter from 190 to 230 cents, that's around a whole step. Then the parameter from 260 to 320 cents is around a Western minor third. So in Javanese slendro tuning all the gamelans have three intervals that are around a major second and two intervals that are around a minor third.

Hatch's research into mode and tuning within the Javanese slendro system has led him to the tentative hypothesis that Javanese singers provide their own basis for tuning, rather than tuning to a background system based on the fixed pitch gamelan. Note that this idea is counterintuitive in a Western culture where singers are thought to tune in relation to an equal tempered tuning system developed around instrumental music. Hence, in Java the gamelan provides a *"shimmering background against which a ...singer can lay his or her interpretation of the mode"*[3].

The Ethnomusicologist's Mathematics

Hatch relies on mathematical actions, constructs, and representations in coordinating multiple representations of musical sound in his Ethnomusicology practice. In addition to the discussion of functions, ratios, and intervals above, mathematical notions of unit, scale, variation, approximation, parameterization, and distribution are applicable to his music research. Yet while *we are convinced* that Hatch is engaged in significant and interesting applications of mathematics in his practice, Hatch himself does not see it that way. In fact he was concerned about our expectations in that sense and began the interview with the following "disclaimer".

> I don't know much about math applications other than what I learned in high school. I don't, for example, do calculus. I remember logarithms, some, but I don't use them in any of this process. I just use the results of the number tables and simple multiplication, division, addition, and subtraction. It's kind of sad in a way, because I never was in a position to learn math for my profession, in a way that would open up to me the possibility of doing research along those lines; that is research using math applications to apply to musical formulations... That's a disclaimer.

Hatch does not view what he is doing, beyond multiplication, division, addition, and subtraction, as mathematics . His perception of mathematics is based on his experience with math in the school curriculum -- abstract, decontextualized, formal mathematics. As a result, the intuitive mathematics of his practice is not viewed by him as mathematics at all.

Rather than formalized concepts learned in math class, Hatch's mathematics is practical. It has arisen naturally in the context of problems and resolutions within his practice, in a cycle of problematic - action - representation - reflection (Confrey, 1991). For example, Hatch quantifies pitch and interval in order to describe the Javanese music apart, in some sense, from a Western framework. However

[3]This argument is developed further in a longer version of this paper which is available upon request.

quantification in terms of frequency creates a *problem* for analysis of musical intervals. Using some of Hatch's data, one can see that it is often much easier to estimate musical distance additively than multiplicatively. For example the interval from 10,094 cents to 9985 cents is just over 100 cents, hence a halfstep. Estimating this relationship from the frequency ratio 340 Hz. / 320 Hz. would be more difficult. Hence, the mathematical *action*, the conversion from cycles per second to cents, is the result of the problematic associated with estimating intervals using Hertz. *Representation* of the interval as cents or a half step, or a minor second follows.

Conclusion

We see this professor's practice as an authentic and valid use of mathematical intuition in an epistemology of musical relationships. He has developed mathematical insight in response to problematic situations in the context of his research. These insights are largely separate from his experience with formal mathematics curriculum. We also believe his practice has validity as a context for teaching mathematical relationships to students, where ideas such as exponential functions arise naturally out of the need to coordinate representations of data in the counting and splitting domains.

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GENDER DIFFERENCES IN STUDENT-TEACHER INTERACTIONS IN SOME GRADE SEVEN MATHEMATICS AND LANGUAGE ARTS CLASSROOMS IN CANADA AND CUBA: A PILOT STUDY

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In most of the industrialized Western World, women are underrepresented in mathematics and mathematics-related fields. However, this is not the case in many socialist countries. This study set out to identify some ways in which the school, as a social institution, may contribute to such differences. Specifically, it tried to determine whether there are any significant relations between the sex of students and the quantity and quality of their interactions with teachers in mathematics and in language arts in Cuba and Canada, two countries in which there is a wide variation in women's participation in mathematics-related occupations.

In the last twenty years women's lack of participation and under-achievement in mathematics has been the subject of a considerable amount of research. This research has encompassed three broad areas dealing with differences in the biological, psychological and socialization factors with respect to men and women. Researchers focussing on the biological considerations have suggested that gender differences in mathematics performance result from innate differences between boys and girls. Those arguing from a psychological point of view have examined a number of different areas including students' confidence in learning mathematics, their ability to work independently, the attribution of success or failure in the subject, and perception of mathematics as a useful field of study. Researchers who concentrate on socialization factors focus on the influence of widely-held cultural stereotypes which portray mathematics as a male domain. The processes by which such stereotypes are passed to students via their parents, teachers, and society in general, have been extensively investigated. In emphasising the socialization factors in the learning of mathematics, this paper presents a number of conclusions that call into question the relative importance of both psychological and biological factors in the learning of mathematics.

In schools, the values and mores of the dominant culture are passed to the next generation. While learning mathematics in schools, students also learn the values, attitudes and expectations attached to that subject by the society of which they are a part. The teaching and learning of mathematics is not value-free. It is culturally bound. If this is indeed the case, the question arises as to whether the underrepresentation of women in mathematics-

related fields in most of the industrialized western world is a function of the cultures of such societies rather than any biological or psychological attribute of the women themselves

Within this broad conceptual framework of socialization, this study set out to identify some ways in which the school, as a social institution, may contribute to the differences in participation and achievement in mathematics between boys and girls. It focusses on the contribution of the classroom teacher, an important bearer and transmitter of culture. Specifically, the study tried to determine whether there are any significant relationships between the sex of students and the quantity and quality of their interactions with teachers in mathematics classes. Recognizing the already-stated importance of cultural differences, it is based upon observations of mathematics classrooms in two countries with different political, economic and social systems as well as wide differences in women's participation in mathematics-related occupations. The first country is Canada, an English-speaking, industrialized capitalist society in which there is relatively low participation of women in mathematics-related fields. The second country is socialist, Spanish-speaking Cuba in which there is high participation of women in mathematics-related occupations. The investigation was expanded to include language arts classes as a control.

The Canadian sample was drawn from the school district of Burnaby (population approximately 150, 000), a suburb of Vancouver on the west coast of Canada. The Cuban sample was collected from a number of towns across the main island of Cuba, located in the Caribbean. Grade seven classes were chosen for observation because the literature suggests that significant differences in mathematics performance do not reliably emerge before that grade. The total sample size was limited by the accessibility of classrooms in Cuba. Collecting data in Cuba proved to be very difficult. In a twelve week visit, it was only possible to observe eight grade seven classes in which mathematics and language arts were taught by the same teacher. Therefore, the same number was applied in Canada. The data collected, then, consists of thirty two hours of classroom observations. A modified version of the Brophy-Good Dyadic Interaction Observation System was used for data collection. This system, which is the one most commonly used in interaction studies is designed to quantify systematically the types of interactions in which teachers are involved with individual students.

The comparison of the Canadian and Cuban data on student-teacher interactions in mathematics and language arts classes, indicates some interesting differences between the classes observed in the two countries. The main one is that, in Canada, boys interact more with teachers in mathematics than they do in language arts. The reverse situation was observed in Cuba.

When the interactions were divided into teacher-initiated and student-initiated, further differences were found. In Canada, teachers initiated more process questions with boys and had extended interactions more with boys than with girls in both types of classes. In Cuban mathematics classes, although teachers did not initiate more process questions with girls, they were involved in more extended interactions with them. The opposite was true in language arts classes. In Canada, boys were disciplined more frequently than girls in both subjects, whereas in Cuba, in general, few students were disciplined. Of those who were, the difference was slightly in favour of boys.

Student-initiated interactions also showed gender-related differences. In Canada, boys initiated more product questions in language arts whereas girls initiated more in mathematics. No differences were found with respect to process questions. In Cuba, girls initiated more product questions in mathematics and more process questions in both subject areas.

Unsolicited student call-outs were also found to vary between the two countries. In general, boys called out more frequently than girls in Canada than in Cuba. In addition, there were differences in the quality of the call-outs between the two countries. In Canada, students, most frequently boys, called out usually as a result of, or in reaction to, a disciplinary action from the teacher. All of the call-outs observed in Cuba were related to academic matters. For example, when a teacher made an error while working at the blackboard, one or more students called out to correct it.

In both types of classes in both countries a small group of students dominated classroom communications. In Canada, this group was consistent across subject area and was composed primarily of male students. In Cuba, however, the dominant group varied with the subject. A small number of girls dominated interactions in the mathematics classrooms whereas a small number of boys dominated interactions in the language arts classrooms.

A striking differences between the Canadian and Cuban data related to the quantity of total teacher-student interactions. The total interactions in Cuban mathematics classes was approximately half of that in Canadian mathematics classes (53%) while the total interactions in Cuban language arts classes was approximately 70% of that in Canadian language arts classes.

These results are more surprising when the different teaching methods of the two countries are taken into account. In both mathematics and language arts classes in Canada, the teacher generally lectured the class as a whole for less than 15 minutes and students subsequently worked by themselves with individual help from teachers. During the rest of the hour, the whole class was occasionally brought back together for a few minutes while the

teacher explained a question or a common problem to them. Questions not finished by the students in class time were generally assigned as homework.

In the sixteen Cuban lessons, on the other hand, students were never observed to work independently. The teacher worked at the blackboard with each class for the full hour and then assigned extra book questions for homework, usually after the end of the class. Therefore, although Cuban mathematics teachers worked with the class as a whole for the full hour, there were fewer interactions per class compared to Canadian classes where teachers instructed the whole group for fifteen minutes or less. The reasons for this difference may include the different teaching styles in the two countries.

A number of differences in classroom teaching styles were observed. For example, a technique commonly employed by teachers in Canada is to ask a rapid series of short-answer questions of a large number of students, particularly at the begininning of a lesson as a warm-up, or as a review at the end of a lesson. In Cuba, on the other hand, a question was asked of an individual student and then the teacher and the class waited while the student concerned thought about the answer. This delay may result from the fact that, in Cuba, a student was generally required to accompany an answer with a justification, particularly in mathematics classes. For example, when a student was asked: "What is x if 2x + 4 = 10?", he stood at attention at the side of his desk and thought, for what, to a Canadian observer, seemed an embarrassing amount of time. Eventually the student responded: "If 2x + 4 = 10 then x must be 3." After each correct response, the teacher always replied, "Correct. Now Explain." The student continued,

> "In order to check my answer, (and this was stated for every example) if x equals 3 then the left hand side of the equation would become 2 x 3 + 4 or 10 and this is the same as the right hand side. Therefore I am correct."

For every question the reverse operation was stated as part of the answer. This system of asking students to rationalize an answer in mathematics was observed in classes from grades two to twelve.

Another difference in teaching styles in the two countries was observed when a student gave a wrong answer. In Canada, teachers were frequently observed to redirect the question to another student or, on occasion, another student would call out the correct answer to the class. Neither situation was observed in Cuba. If a student gave the wrong answer or the wrong reasoning for the answer, the teacher stayed with that particular student until the difficulty was clarified, despite a few pleas of "Profe, yo" (Me, teacher) and the raised arms of other students.

These extended interactions often involved long periods of silences while students were allowed to think about the error that had occurred. Teachers in Cuba were involved in twice as many of these types of interactions with girls than with boys in mathematics classes.

As in mathematics classes, lengthy answers were also given by students in language arts classes. In this case, a distinct difference was observed between boys and girls. Boys exhibited a great deal more confidence than girls when talking in front of the class. When asked to explain a line of poetry or the meaning of a passage in a story, boys stood at attention at the sides of their desks, thought for a few moments and then started to talk. Responses fifteen to twenty sentences long were not unusual. The student was not rushed nor encouraged to be brief. When finished, boy students were frequently cautioned by teachers in the use of "um" or "y" (and) or the dropping of final consonants, a common phenomena in Cuban speech. The main focus of the teacher seemed to be on fluidity and clarity in speech. Content was commented on less frequently.

This was not the case with girls who were much more shy when talking in front of the class. Many of them stood at the sides of their desks, but not at attention, and spoke very quietly whilst moving their bodies from side to side and fingering the hems of their skirts in what appeared to a Canadian observer, as a stereotypical shy pose. They were not encouraged to speak up and were never observed being told to speak more clearly or to avoid using hesitating speech effects. In addition, girls' answers were considerably shorter than those of boys. Teachers often followed up on answers from boys, thereby producing other lengthy responses but rarely did so with girls. Furthermore, the majority of the follow-up questions which did involve girls occurred during the grammar parts of the language arts lessons and were, therefore, of shorter duration.

Such differences in teaching styles are possible explanations for the striking differences between the numbers of total student-teacher interactions in Cuban compared to Canadian classes. Thus, despite the shorter direct teaching time, Canadian teachers were involved in a great many more interactions with individual students than were Cuban teachers.

Finally, the sample teachers in both countries were asked informally if there were any differences between boys' and girls' abilities in mathematics and language arts. All of the Canadian teachers agreed that there were no differences in abilities in language arts. However, almost half of them (3 out of the 8) suggested that, although girls were neater and worked harder, boys had superior skills in mathematics. On the other hand, although all 8 of the

Cuban teachers claimed that there were no differences between boys and girls with respect to language arts, 6 out of the 8 responded that girls were superior at mathematics.

To conclude, it should be re-emphasized that this was a pilot study of teacher-student interactions in grade seven mathematics and language arts classes in Canada and Cuba. Its intent was not to produce generalizations about the totality of mathematics education in Canada or in Cuba. Rather, the objective was to answer the specific questions raised with respect to teacher-student interactions within the sample populations.

The purpose of this study was to examine whether gender-differences existed in student-teacher interactions in grade seven classrooms; whether interactions were different in mathematics compared to language arts classes; and whether there were differences in interactions between the sample classrooms studied in the two countries, Canada and Cuba. Sixteen classes were observed in each country - eight each of language arts and mathematics - yielding a total of approximately 2 000 interactions. The results of the study confirm the findings of many other researchers that boys and girls are treated differently in classroom interactions. The results of the Canadian sample reflect those of most interaction studies conducted in North America, Europe and Australia in that boys consistently received more teacher attention than girls in mathematics classrooms. The Cuban results were extremely different from those in Canada with respect to cross-subject interaction patterns. In other words, in Cuba, girls were more dominant in classroom interactions in mathematics classes while boys were more dominant in language arts classes. In both sets of data, a small group of students were seen to dominate classroom communications: A group of boys in both Canadian mathematics and language arts classes; a group of girls in Cuban mathematics classes and a group of boys in Cuban language arts classes.

It is being inferred here that there is a relationship between classroom processes and student achievement. However, a consideration of achievement data for the students in the sample classrooms in each country has not been included primarily because of the difficulty of collecting this type of information in Cuba. In a fuller study, the link between classroom interactions and student achievement would have to be investigated more thoroughly.

Schools are microcosms of the societies in which they exist. Inequalities which exist in the larger society are manifested in the schools. The results of this study suggest that adolescent girls can and do achieve in mathematics if they live in a society in which this is the accepted norm. In other words, social values and attitudes towards achievement in a subject can decide how well students perform in it. This study suggests that student achievement in mathematics education is at least partially pedagogically and hence culturally determined. The

question of why women are underrepresentated in mathematics-related fields in the industrialized world is complex but this study indicates that it is related both to the way in which different forms of knowledge are gendered and to women's place in society.

As noted in the limitations of this study, there were a number of difficulties in obtaining large, comparable samples in both countries. Nevertheless, the results at least hint at differences between the two countries in this area which merit further research. Firstly, would the same results be sustained in a more scientifically-rigorous, larger-scale study? Secondly, do the findings in grade seven classrooms in Cuba apply at all grade levels there? Thirdly, are mathematics-related occupations held in the same esteem and similarly rewarded in the two countries? Finally, are the differences in teachers' attitudes, as well as the differences observed in teaching styles in mathematics and language arts in the two countries truly representative?

This pilot study suggests that women's difficulties with respect to mathematics participation and achievement are, at least in part, culturally induced. It also suggests that, given a different set of values or intentions, or political will, the degree of womens' success in mathematics in any society could be increased significantly.

Students' Beliefs and Attitudes

UNDERSTANDING THE IMPLICATIONS OF CONTEXT ON YOUNG CHILDREN'S BELIEFS ABOUT MATHEMATICS

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First-grade children in different classroom environments were interviewed to determine their perceptions of what it means to do mathematics. Although children's responses were similar, their rationale and their interpretation of the conditions under which they held such beliefs were different. The contextual nature of the questions played a key role in determining children's perceptions. All children were able to relate to questions situated within the content of addition and subtraction. However, familiarity with different classroom situations limited some children's ability to respond to questions. In order to effectively determine children's perceptions of mathematics, it is critical consider children's classroom experiences.

Introduction

Within mathematics education, researchers have successfully attempted to understand and document the knowledge that students' bring to their learning of mathematics (Carpenter & Moser, 1983; Kouba, 1989). The accumulating evidence suggests that understanding children's knowledge of the content and how it develops enhances our understanding of mathematical learning and thus. how to structure the learning environment. Yet, these investigations focus only on knowledge of mathematical thinking within a particular content domain. Little is known about other aspects of knowledge that children bring to the learning situation. Potentially, different aspects of students' knowledge interact with any given situation to influence the students' learning (Greeno, 1989). One aspect of knowledge that may play a role in the learning of mathematics is the students' perceptions of what it means to do mathematics. Understanding the relationship between students' perceptions of what it means to do mathematics and the situations within which students interact can build on the work documenting the mathematical thinking children bring to a situation, and lead to a more complete understanding of student learning. In this paper we present a method for documenting children's perceptions of what it means to do mathematics and discuss the potential for using this method with students within different classroom environments.

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Students' Perceptions of Mathematics

Educational reformers currently suggest that for meaningful mathematics to occur students should be making conjectures, abstracting mathematical properties, explaining their reasoning, validating their assertions and discussing and questioning their own thinking and the thinking of others. These ways of doing mathematics suggest that "mathematics is fallible, changing and like any other body of knowledge, the product of human inventiveness" (Ernest, 1991, pxi). However, researchers report that children typically view mathematics as a set of rules and procedures, in which problems are solved by applying explicitly taught computational algorithms that have been presented by the teacher. Students expect these algorithms to be fairly routine tasks, which require little reflection and yield correct answers (Garofalo, 1989). Thus, these children view mathematics as a static body of knowledge which is not created but replicated. There is little more than this documenting children's perceptions of mathematics, and very little documenting young children's perceptions.

Documenting Students' Perceptions of Mathematics

Few studies have investigated students' perceptions of mathematics in a manner consistent with the work on children's thinking about the content. Not surprisingly, much of the earlier work examining students' perceptions of mathematics only reports children's responses to yes/no questions, or likert scales items. Except for anecdotal reports, few studies provide an in-depth examination of student perceptions of what it means to do mathematics. We were interested in ascertaining students' perceptions of mathematics where we could understand the rationale and conditions under which students felt their perceptions of mathematics applied. Thus, we developed a structured interview where questions were situated within the contexts of particular mathematical content and classroom events.

Interview questions were designed to address several themes about the teaching and learning of mathematics, the students' views about what it means to do mathematics, and the role of the teacher and the student. The questions were phrased as situations for the children to respond to, such as "Suppose you were working with a partner in your class to solve a math problem and you and your partner get different answers. How do you know which one is right?" The interview questions were designed to elicit what children thought was important to communicate about mathematics and problem solving,

how they resolve conflict when students' answers were different, their acceptance of alternative ways of solving problems, their perceptions of authority in the classroom, and the identification of the locus of control for determining how problems were solved, including the selection and use of manipulative material, and the correctness of answers. (The complete interview can be found in Carey & Franke, 1993.) The interview was used to collect information about students' perceptions of mathematics in two different learning environments.

Problem solving classroom environments

We elicited first grade children's perceptions of doing mathematics in classrooms where problem solving was the focus of mathematics instruction (Carey & Franke, 1993). Specifically, these were Cognitively Guided Instruction (CGI) classrooms. In these classrooms children are provided the opportunity to engage in problem solving, discuss their solution strategies, and build on their own informal strategies for solving problems, ideas consistent with the current reform movement.

The children' responses to the interviews indicated that they perceived mathematics as an open-ended problem solving endeavor, where communicating mathematical thinking and use of materials to solve problems was an integral part of the task. The children recognized and accepted a variety of solutions and assumed a shared responsibility with the teacher for their learning. The children varied in their perceptions of what it meant to succeed in mathematics, however, success was not determined only on the basis of speed and accuracy.

The children' abilities to respond to the questions asked exceeded our expectations. The children were articulate and explicit. The open-ended questions allowed the children the opportunity to talk in-depth about what was relevant or important to them about mathematics, rather than simply agreeing or disagreeing with a given statement. When given the opportunity the children were eager to talk about what they thought it meant to do mathematics in their classroom, their responsibilities as learners of mathematics, and the role of the teacher and other students in their own learning of mathematics. The fact the children in these problem solving classrooms were able not only to respond to the questions asked but also to provide detailed information about their perceptions of mathematics leads us to believe that this way of gathering information of students' perceptions of mathematics is a fruitful one.

Traditional classroom environments

We used the same set of interview questions to determine children's perceptions of what it means to do mathematics in traditional, textbook-based classrooms. The mathematics instruction in these classrooms was based on the sequence and content of the mathematics textbook. The children in these traditional classroom environments responded to questions in general terms. They defined the content of mathematics as an independent, paper and pencil activity. Their reference to specific content focused on the operations of addition and subtraction. Most of the children described what it meant to do mathematics in terms of procedures. For example:

I: Some of the kindergartners are wondering what it would be like to do math in the first grade. What would you tell them about the kinds of things you do in math?
S: You have to listen.
I: Okay. What else?
S: You have to pay attention to the teacher.
I: Okay. What else would you tell them about what you do in math?
S: You have to listen very well. And follow directions.
I: Would you tell them anything about the math that you do?
S: You have to do subtracting, adding.

The children recognized and accepted a variety of solution strategies, similar to children in problem solving environments. When children were asked to evaluate strategies, they selected the one they felt most comfortable with rather than the most efficient strategy. Most of the children did not identify recall of a number fact as the best way to solve a problem. As one child indicated, "The better one is, you have to count with the cubes...it would be better. So when the teacher asks you, where is your answer, you can count on the cubes."

Although the children recognized and successfully used direct modeling to solve problems, they did not discuss solution strategies when asked questions within the context of classroom situations, nor did they acknowledge problem solving or the use of manipulatives when responding to these questions. Even though they possessed a variety of strategies, they did not consider a discussion of strategies as part of what it meant to engage in mathematics.

The Role of Context

The contextual nature of the questions seemed to play an important role in ascertaining children's perceptions about mathematics. By context, we mean the content of addition and subtraction and familiar classroom situations.

Mathematical content

By situating questions within the content of addition and subtraction, the children were able to draw on their own mathematical thinking in responding to questions. Their own experience solving addition and subtraction was a part of the context to which they could relate. For example, the children were given descriptions of a range of solution strategies that children used to solve the number fact 3 + 4, which included direct modeling, counting, and the use of a derived fact. All of the children, except one, stated that all those solutions were good ways to solve the problem "because they used their fingers, and they used counters. And some of them knew it was seven."

Familiar situations

Children's familiarity with different learning contexts was reflected in their responses. Their lack of familiarity with specific situations limited our ability to determine their perceptions of what it meant to do mathematics. For example, all of the children indicated that they had worked with a partner. However, the children in traditional classrooms had no sense of what it meant to negotiate some shared meaning of the solution to a problem. They interpreted working with a partner differently from what our question was trying to document. What we considered as working with a partner was not what these children had experienced.

I: Suppose you were working with a partner in your class to solve a math problem, and you and your partner get different answers. How do you know which one is right?
S: Me and my partner. My partner, I think, is wrong or right, or me, I'm right...And I count with my fingers, I cover [my] answer. I give it to Ms. S, my teacher, and Ms. S sees it. I think I'm right or wrong.
I: Okay. You're working with a partner trying to solve a math problem. You and your partner get different answers. How do you know which one is right?
S: Mine.

In most cases the children stated that either one partner or the other would just be right. In some cases they said they just did not know which one would be right. This question was designed to elicit children's ideas of authority in the classroom. We were interested in whether the children would resolve the difference themselves or defer to the teacher for the answer. However, this question was not effective with children in traditional classrooms because the nature of the experience of working with partners was qualitatively different than the experience of children in problem solving classroom environments. For children in traditional classrooms, negotiating a solution to a problem

was not an issue. As a result, another question that more closely matched their classroom experience was need in order to determine the issue of authority.

Because of their lack of familiarity with the context, the children were often unable to elaborate on their responses. They were able to provide an initial response to questions but had difficulty providing a rationale. Unlike children in problem solving classroom environments, they were not able to articulate their strategies. Although they successfully used modeling and counting strategies, they only communicated the answer to a problem and not the strategy they used to solve the problem. The following conversations suggest children's limited experiences with the situation.

I: Daniel has 8 stickers. How many more stickers does he need to collect to have 15 stickers altogether?
S: [Directly models the solution.] Seven.
I: Ms. C wants to know how you did in solving that problem. What would you tell her?
S: I did great.
I: Okay. Do you think that's what Ms. C would want to know?
I: Yeah.
S: Why?
I: She knows I passed all of my math facts, and I know my problems, and I do problems for homework, too.

Another student who successfully solved the word problem seemed to be unsure about how to respond, beyond finding the solution to a problem:

I: Ms. B wants to know how you did in solving that problem. What would you tell her?
S: I would tell her that-um-I don't know what I'd tell her.
I: What do you think that Ms. B would ask you?
S: She would ask me to do – like sometimes she asks us to do it again if we get the answer wrong or something.

In determining the context of the questions asked it is imperative that the contexts are ones that the children have experienced and can interpret in ways that are consistent with the goals of the questions. The familiarity of the questions does make a difference in what we learn about children's notions about doing mathematics.

Summary

Asking children questions set in familiar situations about mathematical concepts they have worked with allowed us to gather in-depth information about the children's views on what it means to do mathematics. We found that the children's initial responses to

a question often told only part of the story. By probing the children's responses we learned how their initial response fit. We were able to understand some differences in the children's perceptions about doing mathematics, as well as some unexpected similarities. However, we also learned that providing questions situated in contexts more familiar to the children in the traditional classrooms may have provided us more complete information about both the children in the traditional classrooms and the children in the problem solving classrooms.

The information gathered demonstrates that even young children have some definite ideas about what it means to do mathematics. The children in the traditional classrooms possessed notions about what it means to do mathematics that were consistent with the current mathematical reform movement, they were able to talk about different strategies and recognize their appropriateness. This should not be surprising given that we know that these children come to school with informal notions about solving problems that are consistent with these ideas. However, it does show that these notions need to be fostered for the children to continue to see how they fit with the doing of mathematics. Thus, understanding children's perceptions about doing mathematics can help us both understand children's learning and develop learning environments that build on notions of doing mathematics that are consistent with the work on children's mathematical thinking.

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MINORITY STUDENTS' BELIEFS ABOUT MATHEMATICS

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The purpose of this study was to investigate minority remedial college-level students' beliefs about mathematics. Seventy-five students from four levels of developmental mathematics courses participated in four focus group interviews. The interview sessions allowed students the opportunity to discuss their beliefs and attitudes about mathematics and the learning of mathematics. This paper reports the results of three key features of those sessions: students' views of mathematicians, students views of mathematics in relationship to themselves and students' beliefs about their own mathematical skills and understanding. The results indicate that students believe that they are not part of the mathematical community and that for many, learning mathematics is fruitless.

Alan Bishop, in his book, Mathematical Enculturation: A Cultural Perspective on Mathematics Education, discusses the social aspects of mathematics education: cultural, societal, institutional, pedagogical and individual. Just as the educational institution is influenced by the society and culture which it represents, individuals are also influenced by their cultural inheritance. Most institutions of higher education were formed and governed by the culture that represented the American society for most of its history. Today, however, the American society is changing and many of these institutions no longer reflect those changes in the society.

The results of the Fourth Mathematics Assessment of the National Assessment of Educational Progress (1989) indicate that black and Hispanic students performed consistently below white students on all items except one: whole number computation. The results for females were not much better: males continue to outscore females, especially in higher levels of mathematics. Furthermore, women and minorities take fewer mathematics courses and are greatly underrepresented in science and technological careers. At the same time, women and minorities are enrolling in colleges and universities in larger numbers than ever before. Many enter college with quite limited mathematical skills. A 1985 Department of Education report

indicates that over 25% of incoming freshmen enroll in developmental mathematics courses (Hall, 1985). Recent reports indicate that the situation will not improve over the next decade and may even get worse (National Research Council, 1989). If these incoming students are to be successful in college level mathematics it is necessary to understand how students develop a sense of community and ways in which students are dependent upon their beliefs and attitudes in the learning of mathematics.

As individuals explore and internalize their world, they do so in a unique relationship to their own culture and society. Green (1978) and MacCorquodale (1988) found that because of undesirable cultural characteristics associated with mathematics, native American and Hispanic women were not encouraged in mathematics and math related careers. Charbonneau and John-Steiner (1988) cite that making the linkages between the mathematics taught in school and "their own childhood understanding" may be difficult for minority children. Ginsburg (1977) points out the necessity of helping students relate school mathematics to their own real world. How can we do that unless we know what that world is and when students' real worlds are so diverse?

Methodology

Seventy-five students from a summer bridge program for minority students participated in this study. These students were enrolled in one of four levels (arithmetic, elementary algebra, intermediate algebra: part I or intermediate algebra: part II) of developmental mathematics courses offered in the General College of the University of Minnesota. The students participated in four focus group interviews, where they were asked to react to questions about mathematics, about ways of learning mathematics, and about various solutions to mathematical problems. Each of the focus group sessions was led by one of the two investigators. There was a prescribed script for each of the focus group sessions, but investigators were free to follow a line of discussion when they felt that the discussion was appropriate to the topic. The focus group interviews were audio taped.

Results

This paper will reflect only on several of the key features that were gleaned from this study. The three focal points of this study which are presented here are: students' views of mathematicians and mathematics teachers, students' beliefs about mathematics in relationship to themselves as learners of mathematics and students' beliefs about their own mathematical skills and understanding.

Student Views of a Mathematician

During the focus group interviews, students were asked to describe a mathematician. In doing so, many students described their previous mathematics teachers. Overall, student responses portrayed mathematicians as male, highly intelligent, and anti-social: "They have beards. I don't know why. . . . it's just one of those things."; "Usually a man."; "I find mathematicians very intelligent people. You either understand mathematics or you don't. . . To me mathematicians command a lot of respect."; "I'd say they keep to themselves. It's not your everyday Joe out on the street. Someone who's just a loner type in isolation, where it's quiet."; "I'd say they don't know how to relate to people."; "Most mathematicians, you know, can't bring their knowledge down to someone who doesn't know their level. You know, they're always above you when they're explaining something."; (referring to the previous comment) "I had a teacher that way. 'Are you stupid? Any stupid person could get this.'".

The students' views that mathematicians are mostly male, intelligent, and anti-social have implications for the way in which students themselves approach the learning of mathematics. According to Borasi, "students need to see themselves as mathematicians in order to fully achieve the goals of thinking mathematically, valuing mathematics, and being confident and ready to use their mathematical expertise whenever appropriate" (Borasi, 1992). Students may view themselves in ways that are contrary to their views of mathematicians. In particular, the view that mathematicians are male may be a barrier to a female envisioning herself as a

mathematician. No student commented on the ethnicity of a mathematician, but that may be due to the sensitivity the students had towards the 'white' leaders of the focus groups.

These views of a what a mathematician is and does as presented by the students in this research study may keep the students, especially minority and female students, as 'outsiders' among the mathematical community. Mathematics, to these students, is not personalized. As a result, students may only try to mimic what a mathematician (or a teacher) shows them, not believing that they, too, have the power to think about and create mathematical ideas, to be a mathematician.

Student Views of Mathematics in Relationship to Themselves

Also during the focus group interviews, the students were asked to complete the analogy, "Mathematics is to me as ________ is to ________.". Many students had difficulty working with the analogy and instead responded to the open ended statement: "Mathematics is to me . . .". Student responses to "Mathematics is to me" were categorized into six basic beliefs about mathematics: 1. rules to memorize; 2. always forgotten; 3. in the way, forced, or irritating; 4. something useful in the future; 5. something needing insight; and 6. something that gives a challenge.

Although the last three categories seem positive in nature, reflecting usefulness, insightfulness, and challenges, the student responses indicate a lack of personal need for learning mathematics: "[Mathematics] must be a key to understanding, and until I have the key I am at a disadvantage. . . I'm actually glad that I'm forced to take math now, because . . . if I had the choice, I probably would have never taken math.".

Furthermore, the first three categories of student responses reflect negative attitudes, extrinsic motivations, and a lack of control when learning mathematics. Some student responses are: "[Math] is like a French language I'll never understand."; "[Math is] like a revolving door. You always go back to what you should have learned the day before."; "[Math is] like a car in a traffic jam. You can't get around it, but you've got to do it."; "Mathematics is to me as vegetables are to children. And I say

that because I really don't like math that much. I have to take it, and if I have to do it, then I will."

When students do not believe in good reasons to learn mathematics, learning based on understanding breaks down and is replaced by learning based on respect and obedience (Skemp, 1987). Students need to see connections between mathematics and their real world or between mathematics and their own curiosities and ways of thinking. As demonstrated by the majority of student responses, learning mathematics is not motivated by intrinsic means, but instead is something that is, as one student responded "just set in front of you . . . you don't sit there and ponder, you just do them, do the steps, . . . get it done. And there isn't much to talk about." There is a need to foster growth and development in positive personal meanings about learning mathematics in the classroom.

Students' beliefs about their own mathematical skills and understanding

At several points during the focus group interviews, students were asked to talk about their own mathematical skills and understanding. Responses did not vary within the different levels of mathematics. Most students felt inadequate about their mathematical knowledge: "I think I'm math illiterate."; "I think I know more. I'm not being conceited or anything but, since I repeated it 2 or 3 years, I should know more."; "I just wish I could remember what I learned."; "To me it's embarrassing to fail, and failing tests, every single test you take you fail. It's like, you think you did so good, and like I'm trying to pull myself up. They won't let me get out of my math class, and I tried so hard, and they won't let me out. It's painful because no matter how hard I try."; "It makes you feel hurt."; "I hate math. I hate it so much. I don't get it at all." "You can get the wrong answer, and you think that it's right. Like on a test."; "I think the reason why I don't like math, is that I like things that I'm doing to have a purpose, and with math, like most of the problems and things, you are wondering why you would ever have to know this stuff. . . I can't see the purpose. "

Students may attribute their negative views about their mathematical abilities to previous failures in mathematics, or perceived failures, and a lack of purpose for taking and learning mathematics. Developmental mathematics courses are designed to help students overcome earlier failures and deficiencies. Unfortunately, it appears that students beliefs about their own mathematical ability may a barrier to seeing this as a new beginning, thus perpetuating the cycle of failure.

Although these responses about the students' mathematical abilities are not surprising. However, the view was so pervasive that it begs consideration. Students believe that taking and learning mathematics is important, but believed, for them, it is useless. According to McLeod (1992) beliefs and attitudes play significant roles in the learning and teaching of mathematics. Bishop (1988) stresses the importance of students becoming part of the culture of the mathematical discipline. For this group of minority students, mathematical enculturation appears non-existent.

Conclusion

When considering students views about their own mathematical abilities, their belief that mathematics courses lack purpose for them and their views of mathematicians and themselves in relationship to mathematics, the future looks bleak for minority students whose high school mathematics courses did not prepare them for college level mathematics.

In mathematics education, it is necessary to understand how students come to know mathematics, especially for those students who have been traditionally underrepresented in mathematics (females and minorities). As the college population becomes more culturally diverse, it is imperative that we begin to understand the perspectives of all students in the development of curriculum and instruction. This is especially important in the area of mathematics.

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Teacher Education

ASSESSMENT OF FRACTION UNDERSTANDING: STUDENT PERFORMANCE AND TEACHER REACTIONS

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The development of fraction understanding by middle-school students depends upon the quality of the instructional experiences they receive in the classroom. A well-informed teacher is the key to the development of students' understanding, and according to Harel (in press), a well-informed teacher knows content, epistemology, and pédagogy. In this study, five middle school teachers received the results of a conceptually-based fraction test that they had administered to their students early in the year. The teachers spent three hours discussing the results with fellow teachers and researchers. The discussions were taped and transcribed and teachers' comments and questions were analyzed to reveal 1) teachers' surprise at the types and percentages of student conceptual errors, and 2) the difficulties teachers had in conceiving instructional experiences that would promote conceptual understanding.

The purpose of this investigation was twofold: to assess the fraction knowledge of fifth- and sixth-grade students (in this case, students of teachers with whom we work), and to assess teachers' understanding of their students' performance and the effect of this understanding on future instruction. The investigation is part of a long-term study focusing on two fundamental questions, each of which has important implications for mathematics learning in the middle school. (1) How does teachers' understanding of rational number and the multiplication operator, and of quantitative reasoning and proportional reasoning, influence the manner in which they teach? (2) As teachers' understanding of these topics develops and as teachers become more aware of how students learn these concepts, how is student learning enhanced? The conceptual framework for this work is discussed in Sowder (1992).

Teacher knowledge has been the focus of much discussion in recent years. Harel (1991) has proposed that there are three components of a teacher's knowledge base: mathematics content, epistemology, and pedagogy, where mathematics content refers to the depth and breadth of the mathematics knowledge, epistemology to the teacher's understanding of how students learn mathematics, and pedagogy to the ability to teach in accordance with the nature of how students learn mathematics.

In this project we work to link and extend these three components. We have two fifth-grade and three sixth-grade teachers participating in the study. The teachers meet in seminars approximately every two weeks. The seminars during the fall term focused on rational number understanding. Towards the beginning of the term, a test was designed and administered to assess the fraction knowledge of students.

The test was conceptually rather than procedurally based, and items included comparing, ordering, and recognizing constructs of fractions; determining the size of the unit; and understanding operations on fractions including estimating results. (The test was also given in classrooms of some of the teachers who were in this project last year.) After the results had been tabulated, the teachers were provided a summary of their students' test results, the individual tests of each of their students, and an overall ranking of the problems from most to least missed. A three-hour seminar was devoted to discussing these results.

There were two general themes that arose during this seminar, both of which will be described in this manuscript. The first involved the teachers' surprise at their students' performances. This surprise was related to the teachers' difficulties understanding how their students were thinking. The second general theme revolved around how the teachers might use the information about their students' thinking when making instructional decisions. In particular, we will focus the issue of the teachers' identification and understanding of the mathematical ideas to be taught to their students.

The teachers' difficulties understanding their students' thinking

Tables 1 - 3 provide the cumulative results of three of the seventeen items from the student assessment test analyzed by grade level. (Note that the teachers were not provided this general information. They were only provided data on their own students.) Throughout the seminar discussion, the teachers expressed their surprise at the performances of their students. Ken commented on the fact that less than one third of the students in his two sixth grade mathematics classes correctly answered #1h see Table i), and Sean, a fifth grade teacher, joined the discussion. (Teachers are denoted by their first names, researchers are denoted by R_i, with i representing the ith researcher):

Ken: I cannot believe 1h. I always tell my kids, "Sound it out. What is this number? Seven tenths; one-seventh. They don't sound the same, folks."

R_1: One-third of the students got it right.

R_2: It's not only that they missed it, but the number that thought they are equal.

Ken: I couldn't understand how two things that sound totally different....

R_3: Equivalent fractions sound totally different.

R_2: Yes, it's not that they sound different.

Sean: I can see they could confuse that.

Ken: That they're equal?

Sean: Yeah.

R_1: How do you compare these?

Sean: Well, we know they are seven tenths and one seventh, but the kids.....

R_1: Wait a minute. So you look at 0.7 and say seven tenths. How do you know seven tenths is bigger than one seventh?

Sean: Okay, 1/7 is less than a half; 0 7 is bigger than a half. But the kids aren't mature enough to do that.

R_1:	So first they must see zero point seven as seven tenths.
Sean:	Yes, which I feel they didn't.
R_1:	Then they need some benchmarks and to know what 1/7 is....
Sean:	Yes. There's a lot involved in this.

Table 1: Percentages of students in each response category for Item 1 (Bold type indicates the correct response.).

1. Look at each pair of numbers. If one is <u>larger</u> than the other, circle it.
   If the numbers are the same, put = between them.

			First number larger		Second number larger		Equal		No Answer		Other	
		Grade*	5	6	5	6	5	6	5	6	5	6
1a.	$\frac{5}{4}$	1	**43**	**47**	53	45	4	4	0	2	0	3
1b.	$\frac{2}{7}$	$\frac{3}{7}$	32	15	**66**	**78**	2	5	0	1	0	1
1c.	$\frac{5}{7}$	$\frac{5}{9}$	**51**	**45**	45	48	2	3	0	2	2	1
1d.	$\frac{1}{2}$	$1\frac{1}{2}$	6	4	**89**	**89**	2	4	2	1	0	2
1e.	$\frac{2}{3}$	$\frac{3}{4}$	47	16	**32**	**63**	19	17	2	3	0	1
1f.	$\frac{3}{4}$	$\frac{6}{8}$	47	22	32	45	**21**	**29**	0	3	0	1
1g.	$\frac{3}{8}$	$\frac{8}{13}$	45	29	**38**	**55**	17	13	0	3	0	1
1h.	0.7	$\frac{1}{7}$	**45**	**29**	26	22	21	46	9	2	0	1
1i.	0.5	$\frac{6}{12}$	40	18	30	50	**21**	**27**	9	4	0	1

*Grade 5 (n=47). Grade 6(n=143) except on items 1a, 1d, and 1h, for which (n = 113).

Ken's response was typical of those given by the teachers and illustrates their surprise that their students had not thought about particular problems in ways that the teachers had expected the students to think about the problems. Ken could not comprehend how his students could miss this problem; after all, if the students had simply sounded the two numbers out ("seven-tenths" and "one-seventh"), it would be obvious (to Ken) that these two numbers could not represent the same value. Sean eventually jumped into the conversation, stating that he could see how students might have confused those two numbers, but then he began to gloss over the fact that in order for his students to make sense of that problem the way he was thinking about it, they would have to be able to use benchmarks with the same flexibility he had, in particular, that one-seventh is less than 1/2, whereas zero point seven is the same as seven-tenths, is greater than five-tenths or one-half. This protocol becomes even more telling when considering that it was preceded by an in-depth discussion of problems 1g, during which time we discussed the usefulness of finding benchmarks.

The teachers experienced difficulty trying to understand how their students were conceptualizing these mathematical problems. They tended to assume their students were thinking about all of the problems conceptually, when in all likelihood the students often were not. Consider the discussion of problem #14 (Table 2), which was missed by 3/4 of the students. (Ken is a teacher, R_1 and R_2 are researchers):

Ken: What I can't understand is why they would pick 5.5.

R_2: They ignore the one-half.

Ken: But it sticks out like a sore thumb. You can't say it's zero. There's got to be something.

R_1: But you're already thinking conceptually.

Table 2: Percentages of students in each response category for Item 14. (Bold type indicates the correct response.).

14. Circle the correct answer for: $5 + \frac{1}{2} + 0.5$

a. It can't be done.		b. 5		c. 5.5		d. 6		e. 10		No Answer	
Gr 5*	Gr 6*	Gr 5	Gr 6	Gr 5	Gr 6	Gr 5	Gr 6	Gr 5	Gr 6	Gr 5	Gr 6
30	26	2	3	26	24	**24**	**28**	13	14	4	6

*Grade 5 (n=46). Grade 6(n=143).

Ken could not understand how a student might select 5.5 as an answer, when clearly (to Ken) 5 + 0.5 was 5.5, and therefore adding 1/2 to this must have resulted in an answer that was greater than 5.5. However, this line of reasoning only holds if the students realize that fractions and decimal numbers belong to the same number system and are thinking about the problem conceptually . In fact, when this problem was given to students in interviews during another research study (Markovits and Sowder, 1991) students frequently claimed that "fractions and decimals are different kinds of numbers and can't be added together." This reasoning accounts for both the a and c answers. Another reason could be that students may have invoked buggy algorithms, or incorrect rules, void of the important underlying conceptual ideas. For example, some of the students who incorrectly thought that 6/8 was greater than 3/4 in 1f correctly concluded that 8/13 was greater than 3/8 in 1g. We discussed the possibility that the students were applying a rule that stated the larger numbers in the "top" and "bottom" results in the larger fraction. Some students missed 1b, stating that 2/7 was larger than 3/7. Here the bug that we decided upon was that some students, while working with unit fractions such as 1/5 and 1/7, had concluded that when working with fractions, things are just the opposite of what one might expect. That is, even through 7 is larger than 5, 1/7 is smaller than 1/5. Using similar reasoning, one might conclude that 2/7, which "looks" smaller than 3/7, must instead be larger. Another rule emerged during our discussion of 1f. It was hypothesized that students might have thought that 3/4 was larger than 6/8 because the difference between 3 and 4 is smaller than the difference between 6 and 8. All these possibilities were discussed with the teachers.

The last example we will provide here of the teachers' difficulty understanding their students' thinking occurred during a discussion of #6 (Table 3), a problem that appeared on a NAEP assessment. (Sylvia and Vaneta are teachers, whereas R_1 is a researcher):

R_1: Over all 22% answered correctly.

Sylvia: One third of mine got it right.

R_1: Vaneta, the most popular answer in your class was 42.

Vaneta: I know. If you multiply, you still don't get 42.

R_1: Do you know where they got 42?

Vaneta: Where?

Sylvia: Did they add them all together? They added them all together.

Vaneta: That's bizarre. Why did they do that?

R_1: That's what 2/3 of your class did.

Vaneta: I know. That's why I'm wondering what logic is behind adding them all together? I don't know why they did that.

R_1: Vaneta, you asked why they would add all those numbers. My question would be, "Why not?"

Vaneta: I guess if you have no idea.

Sylvia: You have a plus sign there.

R_1: What does this show that they might not understand?

Vaneta: A lot of things. They don't understand that they need a common denominator.

R_1: Do you need a CD?

Vaneta: Well, no, not really. You could draw pictures and add things and estimate. I don't know if they got "about"--that that meant estimate. We've had no experience with fractions in the (sixth-grade) classroom prior to giving this test.

Sylvia: That's what I would say. They didn't even know that you were talking about fractions. Or else that they've been exposed to fractions, but not really picturing what a fraction is.

Vaneta: I don't think my kids have a clue what a fraction is. That's not true. They have some intuitive knowledge. That's why I like that paper folding thing. I think they could get into that. They could bring in the intuitive knowledge that they know and the background that they have from their experiences, but in the formal sense, I don't think that they've been taught fractions at a manipulative concrete level meaning.

Table 3: Percentages of students in each response category for Item 6. (Bold type indicates the correct response.).

6. About how much is $\frac{9}{10} + \frac{11}{12}$? Circle the closest answer: 1, 2, 20, 22, 42

1		2		20		22		42		No Answer/ Other	
Gr 5*	Gr 6*	Gr 5	Gr 6	Gr 5	Gr 6	Gr 5	Gr 6	Gr 5	Gr 6	Gr 5	Gr 6
11	5	**30**	**19**	19	26	17	20	19	24	4	7

*Grade 5 (n=47). Grade 6(n=143).

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There are three points we would like to make about this transcription. First, during the first portion of this transcription, Vaneta could not imagine why her students might have added all of the individual numerators and denominators together. She found this "bizarre," providing yet another example of a teacher's difficulty understanding her students' thinking. The second point is that at one moment, R_1 asked the teachers, "Why shouldn't students add all of the numbers?" This question was an example of the researchers' continual attempts to help the teachers understand the thinking of their students. We often tried to pose questions designed to orient the teachers toward ways of looking at problems that were very different from ways they'd considered, but were very possibly ways considered by some of their students. The last point we would like to make will lead to our next section. In answer to the question, "What does this show that they might not understand," Vaneta responded, "A lot of things. They don't understand that they need a common denominator." There is no need to find common denominators in this problem, and although Vaneta came to understand that point, her initial reaction was that common denominators would be necessary. This leads to the second general theme that arose during the seminar: the teachers' understanding of the important mathematical ideas.

<u>How might the teachers use the information about students' thinking when making instructional decisions?</u>

In order for the teachers to be in a position to use information about their students' thinking when planning lessons, the teachers must themselves possess a conceptual understanding of the mathematics. In the previous transcripts, Vaneta provided an example in which she either did not understand or overlooked the essential mathematical idea that estimation does not require first finding common denominators. Another example of this arose while discussing problem 1c, involving comparing 5/7 and 5/9:

R_1: What would you like for your students to be able to think when they look at 5/7 and 5/9?

Sean: I would like for them to say, "Oh, 63 is a common denominator." (Laughs)

Ken: I don't think I would like the common denominator.

Carolyn: I wouldn't want to see a common denominator.

R_1: Would you, Sean? Would you like that?

Sean: Yeah, I would. Ultimately I would. I would like them to know. But ultimately I would like for them to be able to find the common denominator.

Ken: You know what I would like to see on this little problem is some kid say, "Oh yeah, 4/7 is a little bit larger than half; 5/9 is a little bit larger than half; but 5/7 is a little bit more than more than a half. In other words, the increment kind of deal. If they could explain. Or if they could visualize it: 5/7 of a week is 5 days and then--maybe they could do something like that. For them to say 4/7 is a little bit more than a half and that's about equal to 5/9 and then one more increment of a seventh--that makes it a whole lot more than 5/9 kind of deal.

Carolyn: I would expect them to see maybe a pizza with seven pieces and a pizza with nine pieces. You would know in a pizza with nine pieces, the pieces have to be smaller. And if they're the same number (of pieces)--that's what I would expect.

Ken: Oh yeah. Numerators.

R_2: The same number of pieces?

Carolyn: Um hmm. Because you have the same numerator, and that's what I would expect them to see.

Sean, when asked to describe the kind of reasoning he would ultimately have liked to see his students develop, responded with a procedure that sidestepped the important underlying mathematical idea. Jim's response, although conceptual, also sidestepped the underlying idea that Carolyn noted. It is noteworthy that this session occurred after the teachers had attended several seminars on students' understanding of fractions from "experts," including Tom Kieren, Nancy Mack, Judy Sowder, Barbara Armstrong, and Nadine Bezuk. In fact, the researchers working with the teachers during this particular session were struck by the difficulty the teachers experienced while trying to plan how to facilitate the development of their students' thinking. In much the same way that the teachers struggled because of incorrect assumptions they had made about the thinking of their students, we, the researchers struggled because of incorrect assumptions we had made about the thinking of the teachers.

Final comment

In subsequent seminars, we often returned to the discussion of students' views of what fractions mean and what teachers can do to help students develop strong conceptual understanding of fractions and fraction operations. The teachers themselves frequently returned to the two notions of unit fractions and benchmarks as essential for helping students understand fraction relationships and fraction operations. In their instructional planning activities, these two notions played central roles. Although these (and other) notions were brought up in the previous seminars, it seemed that it was only after the teachers reviewed and discussed their own students' responses to the test items that they convinced themselves that understanding unit fractions and being able to use benchmarks were central to fraction understanding.

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MATHEMATICAL JUSTIFICATION : A CLASSROOM TEACHING EXPERIMENT WITH PROSPECTIVE TEACHERS

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This paper analyzes a group discussion in a mathematics course for prospective elementary teachers. The data presented show intensive aspects of negotiation of classroom norms and standards for mathematical validation (justification) between a teacher with a vision of mathematics and classroom activity based on current mathematics education reforms and students whose expectations are rooted in their experience of traditional school mathematics. Four levels of justification are observed as this negotiation transpires.

Current mathematics education reform efforts promote a vision of classroom mathematics in which students engage in exploration of mathematical situations, communication of ideas, and verification, modification, and validation of those ideas. The vision of students creating mathematics, evaluating the mathematics created by members of the classroom mathematics community, and negotiating shared approaches to and standards for these activities contrasts sharply with traditional mathematics classes, in which the teacher and textbook serve as the source of mathematics and the evaluators of mathematical validity.

The *Professional Standards for Teaching Mathematics* (NCTM, 1991) argues that one of the five "major shifts" that must take place in mathematics classrooms is "toward logic and mathematical evidence as verification - away from the teacher as the sole authority for right answers" (p. 3). The shift of authority for verification and validation of mathematical ideas from teacher and textbook to the mathematical community (the class as a whole) is important because it can enable students to see mathematics as created by communities of people, based on the community's goals and its accepted forms of practice; it can give students rich opportunities for understanding mathematics which result from involvement in the creation and validation of ideas; and it can result in the students' sense that they are capable of creating mathematics and determining its validity.

This paper documents the negotiation of new classroom norms for determining mathematical validity in a mathematics course for prospective elementary teachers. It provides a picture of a class in which the teacher attempted to promote a shift in authority (from teacher to students) with respect to mathematical validity.

Understanding this development is potentially important for three reasons. First, if competence in generating mathematical justification is an important goal of mathematics education, mathematics educators must understand how such competence develops and become aware of obstacles to that development. Second, mathematics education reform proponents must understand the challenge for teachers of attempting to implement reform principles in the context of classrooms populated by students who hold traditional views of school mathematics. Third, in order to prepare teachers to carry out the mathematics education reform, mathematics teacher educators must understand the process of fostering change in prospective teachers' conceptions of mathematics and, in particular, their notions of how

mathematical validity is established. The context of a mathematics course for prospective elementary teachers allowed us to pursue knowledge in these three areas.

This study was part of the Construction of Elementary Mathematics (CEM) Project[1], a three-year study of the mathematical and pedagogical development of prospective elementary teachers. The project involved an experimental teacher preparation program designed to increase preservice teachers' mathematical knowledge and to foster their development of views of mathematics, learning, and teaching that were consistent with the views espoused in recent reform documents. Subjects for the study were 26 prospective elementary teachers. Data collection proceeded through a mathematics course (from which the data in this paper derive), a ten-week course on mathematics learning and teaching, a five-week pre-student-teaching practicum, and a fifteen-week student-teaching practicum.

The project's research and instruction were grounded in a social constructivist view of learning (Cobb, Yackel, & Wood, in press; Ernest, 1991). The research on the mathematics course and the course on mathematics learning and teaching employed a constructivist teaching experiment methodology (Cobb and Steffe, 1983). We adapted that methodology to research on classroom mathematics (in the manner of Cobb, Yackel, and Wood, 1988). The first author taught all classes. Classes were videotaped and project researchers (including the second author) took field notes. Following each class the first and second authors met to discuss what they inferred to be the conceptualizations of the students at that point and to plan for the next instructional intervention. These analyses and planning sessions led to the development of models of students' understanding.

The two units in the semester-long mathematics course focused on developing understanding of multiplicative structures. The first involved the development of understanding of evaluation of the area of a rectangle as a multiplicative relationship between the sides (Simon & Blume, in press a). The second focused on mathematical modeling of real world situations, from which students were to learn to model situations with ratio, distinguishing between situations which could be modeled additively and those that could be modeled multiplicatively (Simon & Blume, in press b). The course goals with respect to mathematical justification included:

1. Acceptance by the classroom mathematics community of the authority to judge mathematical validity;
2. Renegotiation of standards for justification.
3. Appreciation of the need for and power of deductive justifications; and
4. Engagement with "proofs that explain."

This paper examines dialogue from the initial mathematics experience in the first CEM course. Excerpts of the class discussion provide a view of the negotiation of classroom norms and standards for mathematical validation (justification).[2]

The "Why multiply?" Episode

The students' first mathematical task was to determine, without overlapping rectangles and without rectangles overlapping the edge of the table, how many rectangles of a particular size could be placed on the table at which they were seated. Students solved the prohlem in small groups and reported their group's approach(es) to the whole class . Each approach involved using the group's rectangle to

measure the length and width of the table and then multiplying these two measures. The teacher asked, "Why did you choose to measure this edge and that edge and multiply?" What follows is an intensive negotiation between the students and the teacher. At issue are what constitutes appropriate mathematical behavior in this class, and what counts as mathematical justification.

Bobbie: It seemed like the easiest way to come up with an answer... alls [sic] we would have to do is multiply two numbers and come up with one answer.

Bobbie's response described her motivation for multiplying but lacked an attempt to persuade others in the class that multiplication worked, that it was an appropriate operation to perform. The teacher then attempts to focus the class on his initial request for justification and to describe what would constitute justification. (In the dialogue that follows, "Simon" refers to the teacher.)

Simon: Why is it - You said it was an easy way to get an answer. Is it, is it an easy way to get a correct answer? Now why, why measure along an edge and another edge? How is doing that and multiplying two numbers related to covering this whole table with rectangles? You seemed to all think that was a good way to go about the problem. Why do you think that was?

Deb then appeals to authority to justify the use of multiplication.

Deb: Because, um, in previous, previous math classes you learned the formula for area is length times width. So probably everybody has the idea.

Using a humorous reference to an earlier discussion, the teacher attempts to shift authority for judging validity to the class.

Simon: All those evil math teachers you were talking about before, and now you are taking their word for it? How do we know if they are right?

Several students then respond with descriptions of how their teachers demonstrated to them that multiplying length by width produces the same result as counting squares.

Molly: The teachers. They showed us how it worked.
Simon: Ah, what did you think? Jonnie?
Jonnie: They showed, um, it, for example if you put down all the square, all the rectangles. And then you times like the width and the length together, and you got an answer and then you, you added all of them together, and the answer was the same. Both ways. So you knew that that method, the length times the width, would work because they had us add them up after we timesed them.

The teacher then focuses the discussion on whether the procedure always works, attempting to create a need for a deductive argument.

Simon: So it worked. And it worked on the ones you had in school before. Does that alwaya work? [pause] All the time, most of the time, some of the time?
Molly: In rectangles or squares.
Simon: It always works for rectangles or squares?
Molly: From what I've seen.
Simon: From what you've seen. There might have been some rectangles you haven't met yet. Is that it?

Molly's response suggests that she was continuing to base her justification on inductive, empirical evidence. The teacher continues his attempts to create a need for a deductive argument.

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Simon: Will it always work - if we count the rectangles along here and the rectangles along there and multiply, that we'll get the same answer as we would if we counted them all up? [pause] How many believe that will always work? Raise your hand. How many are sure it won't always work? How many are not sure?

Molly responds by articulating the connection between counting the rectangles and the use of multiplication.

Molly: Well, it would work because, um, multiplying and adding are related in that multiplying is, is like adding groups, and so it would always work, because you add them up to see how many is in the square and to multiply the, the groups that go like that, that'll always work. You would get the same number, I'm saying if you added them or if you multiplied that side times that side. Because you're adding, I mean, you're multiplying the number of groups by the number in the groups, which is the same as adding them all up.

Molly has produced a deductive argument for "Why multiply?" Although, this is consistent with what the teacher was attempting to promote, he gives no indication of it. He is interested in what sense the other students are making of Molly's argument. His goal is for the students to evaluate the argument and appreciate the power of deductive arguments to validate and explain beyond that of inductive arguments.

Simon: You multiply the number of groups times the number in the group. [writes "# groups x # in groups" on board] Does anybody understand what she means by that? [pause] Eve?
Eve: Well, you take, if you had like five circles and inside each circle you had five dots. The five dots would be the number in a group, and the five separate circles would be the number of groups. [laughs] And if you multiplied the five times five then you would get how, wait, how many numbers of all the dots.
Simon: How many dots in all?
Eve: [Yes.]

After a bit more discussion, the teacher attempts to determine whether students understand the relationship of the term "group" to the rows and columns in a figure. This continues the teacher's attempts to establish that a rigorous justification is one in which all parts are connected.

Simon: Can somebody paraphrase what Molly has given us without using the word "group?" Georgia?
Georgia: Okay, basically what she's saying, without using the word "group," is that she took three squares. Okay. Lengthwise and, you know, up and down, and she took those three squares and she multiplied them by the other three squares. That works for the width wise. And she got, there's three squares in each, portion and then there's three square in each width. And she got them. That would give you your answer.
Simon: [pause] I lost track of the "why multiply?"
Georgia: Why multiply? I didn't say "why multiply."
Simon: Okay. That, that was - maybe my question wasn't clear.
Georgia: Why multiply?
Simon: Molly was giving us a reason why we multiply. Right?

In response to this question, Georgia returns the discussion to where it started by providing a justification that presents a motivational argument and does not engage the issue of <u>why</u> multiplication is a valid operation in this situation. When pressed by the teacher to tell why, she responds with a resort to authority.

Georgia: We want to find the area, but you're, that's, that's what you do when you find the area. You multiply.
Simon: Why?
Georgia: Cause that's the way we've been taught. And it's an easy way to do it.
Simon: I'm one of those people who doesn't believe anything unless he is persuaded.
Georgia: If we had to sit here and measure, go across, and, and do every one of these, we would miss our next class. [class laughs]

Jonnie then provides another appeal to authority without directly addressing Molly's justification. In Jonnie's justification, she introduces A, B, and C to represent squares in the figure. Our interpretation of this use of "variables" is that it is an appeal to the language of authority.

Simon: I'm getting clear that this is a time saver and that you'll be able to go to your next class that way for those of you who are hoping to. Um, but, why does it work? [pause] Jonnie?
Jonnie: I was always taught in school that it's a mathematical law that if you take, like, if you say, um, if I told A plus B plus C times A plus B plus C that's, you're gonna get, you know, how many, how many squares there are total in that area of the rectangle.
Simon: What do the A, B, and C refer to?
Jonnie: Well the group. Instead of saying groups, I'm saying A plus B plus C cause there's, on that right there. Like one, uh, rectangle would be A, the other rectangle would be B, and then the next one would be C. Okay, if you add those together you get 3. And then you do the same thing with the, um, length. One is A, one, the other one is B, the other one is C. You add those together you get 3 then you times them together, you get 9.
Simon: And that's just a law?
Jonnie: That's what I was told and I always asked why and they're like well, well it's -
Simon: Can we repeal it? Change it? (pause) Lois?

For many of the students, Molly's explanation was not particularly powerful. However, Lois offers a justification that seems to be connected to Molly's.

Lois: Like going, like going down, each one of these [squares on the left side] represents one of these [rows of three squares]. So when you multiply, you're saying, this is really worth three not just one. So when you're going down, like you're saying like, this is three, this is three, this is three. But, an easier way to do it, is just count each of these as three but only show it as one. So you are saying like one of these rows can be represented just going down the side instead of going one two three, one two three, one two three. You just go one two three. Because like this is three across...

Lois has provided a justification for multiplying by constituting composite units of 3. However, later analysis of student journals provided evidence that only a few students recognized the deductive arguments as being different from the other justifications offered.

Summary and Conclusions

This episode demonstrates the teacher's attempts to engage the students in mathematical justification and to challenge them to respond to the demand for justification in more sophisticated ways. "Sophisticated" was an emerging concept for the teacher. Although it was informed by his goal of rigorous deductive justification and his knowledge of previous empirical and theoretical work with respect to proof, he continually had to infer the nature of the students' thinking and hypothesize potential growth based on his inferences. The tension of responding to the students' thinking and working towards the teacher's learning goal is discussed in depth in (Simon, in press).

Analysis of students' responses led to a characterization of levels of justification as follows:

Level 0 - Responses identifying motivations that do not address justification, ("Why?" is interpreted as a question of motivation rather than validity);

Level 1 - Appeals to external authority;

Level 2 - Empirical demonstrations;

Level 3 - Deductive justification that is expressed in terms of a particular instance (This level is not seen in the episode presented.); and

Level 4 - Deductive justification that is independent of particular instances.

These levels represent a progression from reliance on externally determined validity (level 1) to an engagement with the mathematical relationships that provide the connections between results and known initial conditions and that are based on a group's taken-as-shared knowledge (levels 3 and 4). These levels are consistent with the classification of justifications previously done by Balacheff (1987) and Van Dormolen (1977).

The episode presented is not one in which the students spontaneously seek to validate an idea. As a result of their examination of the inductive evidence (and perhaps their past encounters with authoritative sources), the CEM students were persuaded that the multiplication of length times width was an appropriate method for counting the total number of rectangles and were unlikely to search for another method of validation. The teacher, aware of the power of deductive argument and its importance in other mathematical communities, attempted to establish deductive justification as a part of the classroom culture. (This is an example of what Cobb, Yackel, and Wood (1988) refer to as "mathematical acculturation.") The teacher's primary strategy for negotiating with the class what would count as a justification was to focus the discussion on why multiplication was appropriate and whether they could create compelling arguments that the pattern always holds. The question of why multiplication is appropriate was an attempt to promote discussion beyond what is the pattern? to why does this pattern exist? This further level of discussion involves students in understanding how the method works, the relational aspects of the mathematics in question (Hanna's (1990) "proofs that explain"). Thus, the move from inductive to deductive justification involves not only reconstituting what it means to justify (in the context of this mathematics community), but also understanding the particular mathematics content and negotiating what it means to understand.

Our analysis of data supports the idea that mathematical justification is a cognitive and a social process, *the process of working within socially constituted and accepted modes of establishing validity to collectively determine what is cognitively compelling*. Establishing a mathematical justification involves developing an argument[3] which builds from the community's taken-as-shared knowledge. Initially these modes of establishing validity must be established. Although the teacher plays a key role in this process, he cannot impose these norms. In episode 1, we see that even after Molly provides her deductive justification, other students continue to respond to requests for justification based on their experience with mathematical validation in other settings. The mutual constitution of standards for mathematical justification is an active and ongoing process of all those involved. The shift from old to

new mathematics education paradigms is not a one time event. Students continue to operate sporadically out of the old paradigm even as they are productive in the new.

Footnotes

[1]This material is based upon work supported by the National Science Foundation under Grant No. TPE-9050032. Any opinions, findings, and conclusions or recommendations expressed in this material are those of the authors and do not necessarily reflect the views of the National Science Foundation.

[2]A longer paper, which analyzes three episodes and raises a broader range of issues, is available from the authors.

[3]Argument is used broadly here. It is not meant to exclude justification by presentation of empirical evidence

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"More than Repeated Addition" - Assessing and Enhancing Pedagogical Content Knowledge about Multiplicative Structures

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This study examined prospective elementary teachers' pedagogical content knowledge of multiplicative structures. The teachers were assessed on a variety of tasks to measure their initial ability to recognize and differentiate multiplicative structures and how those abilities changed as a result of the experiences in the methods course. The findings suggest that the teachers experienced reasonable success in categorizing multiplication problems. However, they had more difficulty constructing multiplication problems that conformed to a particular structure and their success at categorization was fragile when confronted with more open-ended tasks.

Conceptual Framework

Shulman's definition of pedagogical content knowledge as "the knowledge which goes beyond knowledge of subject-matter per se to the dimension of subject matter knowledge for teaching" (Shulman, 1986, p.9) served as a working definition for this study. That is, for mathematics, pedagogical content knowledge includes "the most useful representation of those ideas, the most powerful analogies, illustrations, examples, explanations, representations, and demonstrations - in a word the ways of representing and formulating the subject that make it comprehensible to others" (Shulman, 1986, p.10). We also utilized a model of the transformation of subject-matter knowledge into pedagogical content knowledge which emphasizes the influence of beliefs about the learner and the content on pedagogical content knowledge. Thus, this study examined subject-matter knowledge with respect to a specific content area and how that subject-matter knowledge changed as a result of experiences in the methods class.

Multiplication was chosen as the content domain for investigation because multiplicative structures are a major component of the middle grades curriculum and these structures are often poorly understood by students, including college students (Graeber & Tirosh, 1988). Taber's (1993) analysis of students' strategies for solving various kinds of multiplication problems documents the strong associations many students construct between the operation of division and some kinds of multiplicative situations. If teachers are to help students recognize multiplicative structures, then their pedagogical content knowledge should include the ability to recognize various kinds of multiplicative structures. We not only examined prospective teachers' understanding of multiplicative structures, but gave assignments that encouraged them to extend their content knowledge and pedagogical content knowledge.

Methodology

Subjects

The subjects were 39 prospective elementary teachers enrolled in a Mathematics Methods course at a major state university. All of the subjects participated in all of the tasks except for the final task, the card sort. Seven subjects were chosen for the card sort task on the basis of their pretest performance - 2 high, 2 middle, and 3 low. The Elementary Methods course provided an opportunity to assess subject-matter knowledge and the initial process of transforming subject-matter knowledge into pedagogical content knowledge. During the course, examples of various multiplicative structures (equal grouping, part-whole, compare, array, and Cartesian product) were discussed. Writing

narrative problems for specific numerical examples was a typical class activity and homework assignment.

Tasks

The teachers were given four tasks designed to assess pedagogical content knowledge of multiplicative structures. The first was a pre- and posttest of 12 narrative problems comprised of 4 part-whole, 4 compare, and 4 distractors. The teachers were directed to write an expression that would solve the problem. The posttest consisted of the same 12 problems given 14 weeks later. Subjects did not see their pretest at any time during the semester and the specific problems were not discussed.

The second task involved writing a narrative example of a multiplication problem for each of the following symbolic problems:

a) 5/8 * 56 b) 143 * 27 c) .65 * 42 d) 2/3 * 4 1/5

e) .53 * .8

Though students were encouraged to write a narrative example of each of the multiplicative structures described in class (equal grouping, part-whole, compare, array, and Cartesian product), many found this to be extremely difficult, given the numerical constraints of the problems, and most wrote equal grouping or part-whole problems.

The third task involved analyzing narrative multiplication problems in terms of the multiplicative structures discussed in class. The teachers were directed to ask 4 of their friends to write narrative problems for each of the 5 numerical multiplication problems. The teachers were then to categorize

and analyze the multiplicative structures of their problems and their friends' problems (25 in all).

The fourth task involved 7 teachers who were given 16 cards with narrative multiplication problems (7 compare, 4 part-whole, 3 equal grouping, 1 array, and 1 Cartesian product) and directed to categorize the problems in any way that seemed reasonable to them.

Results and Discussion

The pretest, posttest, and the card-sort were analyzed for changes in the teachers' ability to recognize multiplicative structures and discern structural similarities in multiplication problems with different kinds of numbers. The analysis task provided a measure of their ability to write problems that conformed to a particular structure and to analyze the structure of the multiplication problems written by others.

The pretest measured recognition of the multiplicative structure of 4 part-whole and 4 compare problems; the mean number of correct responses was 5.33 with a standard deviation of 1.90. Teachers had the greatest success on two of the compare problems (97% and 100%) and the most difficulty on two part-whole problems (36% on both). Examples of these are provided below:

> Compare A small bottle of perfume contains 7/8 ounce. A larger bottle contains 3 times that amount. How much perfume is in the larger bottle?
>
> Part-whole A bicyclist sets out on a trip of 487 miles. He stops for a rest after going .05 of the distance. How far has he traveled?

On the most difficult problems 50 % of the teachers wrote division rather than multiplication expressions (49% and 54%). It was not merely the presence of a rational number in the

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problem that affected their success, but the nature of the number that was the multiplier. Teachers were much more successful on problems with whole number multipliers than on problems in which the multiplier was a rational number less than one.

On the posttest the teachers improved by one approximately 1 problem with 17/39 choosing correct operations for all of the problems. The mean was 6.51 with a standard deviation of 1.54. The same problems as on the pretest were easiest and most difficult; however, fewer teachers chose division as the operation for those difficult problems (13% and 28%). Clearly, the teachers improved in their ability to recognize multiplicative structures in these problems.

On the **writing and analysis task (II)** the teachers were successful at recognizing multiplicative structures but experienced difficulty writing problems that conformed to a particular structure. For the problem 143 * 27, most wrote equal grouping problems and on problems with one or more rational number factors, most attempted to write part-whole problems. When there were two rational number factors, 24% of the attempts to write narrative problems were unsuccessful. Teachers made few errors in identifying the structures of the problems they wrote for the three examples with at least one whole number factor, but 11 of the 39 not name the correct structure for the problems they wrote for 2/3 * 4 1/5 and 15 could not for .53 * .8. The teachers experienced little difficulty in correctly identifying the multiplicative structures of the problems written by their friends, which were mostly part-whole or equal grouping.

The chief source of difficulty in their analyses was in discriminating between equal grouping and part-whole problems on the one hand and unsuccessful attempts to write such problems on the other hand.

Their comments in their analyses of these problems reflect their changing beliefs about the nature of multiplication. Class discussions prior to this assignment revealed that their initial concept of multiplication was that of repeated addition. Through writing and analyzing these problems, the teachers gained an awareness of the difficulties involved in constructing multiplicative situations to fit problems like 2/3 * 4 1/5 and .53 * .8 and of the variety of situations that can be represented by the multiplication operation. The comments of many of the teachers are represented by this excerpt from Allison's analysis.

> *Although, I must take into consideration that my friends are quite intelligent and were taking this experiment quite seriously, all four of them struggled with the problems for extended periods of time.*

The goal of the card-sort task (III) was to determine whether the prospective teachers could discern similarities in multiplicative structures among problems with different kinds of numbers and to further illuminate their understanding of multiplicative structures. Though those who participated were chosen to represent a range of understanding based on the pretest, the card-sort task revealed that for 5 of the 7 subjects the most salient feature was the nature of the multiplier (whole number or rational number less than one). Their classification strategies ranged from an integrated understanding of multiplicative structures in which both problem action and the type of multiplier were used to determine problem categories to

schemes in which categories were determined by contextual features. Several grouped all problems with whole number multipliers except the Cartesian product into a category that they called, variously, "repeated addition", "ratios or proportions", or "simple multiplication". Similarly, others grouped all problems with fraction or decimal multipliers together in group called "part-whole" or "percent of a whole." No one categorized all of the problems according to the problem action.

Conclusion

The results provide convergent evidence that many prospective teachers were able to recognize that part of a quantity can be found through multiplication by a number less than one, though some persisted in their understanding of this situation as a division operation. Results of the problem analysis and card-sort task indicate that the teachers were more successful on directed than on open-ended tasks, suggesting some teachers' knowledge of multiplicative structures was fragile and dependent on the context of the task. Thus, while the experiences provided in the methods class facilitated the transformation of subject-matter knowledge into pedagogical content knowledge, they also indicate the necessity of strong subject-matter knowledge.

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CHARACTERISTICS OF MODELS FOR THE DEVELOPMENT CF MATHEMATICS TEACHING

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To guide efforts to help teachers create mathematics practice which is consistent with the current reform movement, we think that it is important to begin developing models of the process of teacher change. We propose a set of four components to consider in formulating models of the process of teacher change: (1) qualitative reorganizations of understanding; (2) orderly progression of changes; (3) the contexts and mechanisms by which transitions are effected; and (4) individual motivational and dispositional factors.

The current mathematics reform movement has recognized that new forms of mathematics teaching will be needed to support the proposed curricular reforms. If students are to learn mathematics for understanding, they are going to have to be taught in a way that encourages them to experience mathematics as a subject area that can, in fact, be understood. To this end, conventional instructional strategies will need to be supplanted by new forms of teaching.

Because these new forms represent a radical departure from traditional instruction, the motivation for helping teachers to reform their mathematics practice is very high. Yet the means by which teachers develop their practice are as yet little understood. It is critical that we develop some models for the growth of teaching practice if we are to succeed in stimulating such change on a wide scale.

We propose drawing upon cognitive-developmental theories to stimulate the process of model building. While different theories of cognitive development offer different accounts for the processes of intellectual growth, they all address several core issues: (1) qualitative reorganizations of understanding; (2) progression of changes; and (3) transition mechanisms. To these three components we propose to add a fourth, (4) individual motivational and dispositional factors. Models for transforming mathematics teaching which are based on these four components will have the advantage of being grounded in principles of intellectual development which are heuristically powerful. Below we consider how researchers might begin to think about these four components of change with respect to the development of mathematics teaching.

Qualitative Change. Theories of cognitive development begin with descriptions of the state of individuals' understanding at different points in time: often these different states are assumed to represent qualitatively different mental organizations which are (re)constructed over time by the learner. In developing models of changing mathematics practice, the field

will need to determine whether it is possible to describe qualitatively different stages, or levels, of teaching.

Current research and intervention projects are beginning to provide some descriptions of changing teaching practices (for example, Cobb, Wood, & Yackel, 1990; Fennema, Carpenter & Loef, 1990; Hart et al., 1992; Lampert, 1991). There have also been two recent efforts to begin framing descriptions of teacher transformation with respect to levels of teaching practice (Schifter & Simon, 1992; Thompson, 1991). Both of these formulations seek to characterize the development from traditional forms of mathematics instruction to teaching which is grounded in a constructivist epistemology, and propose similar kinds of levels of practice. Both also maintain that developmental models of mathematics teaching will need to account for qualitative change in three "strands" of the craft: (1) epistemology as it shapes classroom practice; (2) personal understanding of mathematics as it shapes classroom practice; and (3) instructional strategies.

The levels of teaching in both frameworks describe movement from beliefs about the "transmissibility" of knowledge toward beliefs about its "constructivity." Teachers' relationships with their students and with mathematics itself also begin to change as they become less intent on helping students to acquire facts and procedures and more involved in building on what (and how) their students understand. At the final level, Schifter and Simon and Thompson each describe a kind of teaching which is qualitatively different from the current norm.

Orderly change. A second characteristic of developmental theories is that the progression of stages is assumed to be sequential and invariant (Case, 1985; Fischer, 1980; Piaget, 1970; Werner, 1948). Although not considered to be teleological, cognitive development is viewed as growth toward increasingly more complex, differentiated, and mature forms of knowing. This growth proceeds according to a single, fixed order, whereby the challenges to be faced and resolved in the present stage become the base from which the next stage's challenges are created.

The view of stage progression as unitary and invariant holds at some levels of analysis and not at others. When taking a broad view, viewing cognitive development as the growth of the logical thinking, it is possible to maintain that development is characterized by inexorable forward movement according to a single pathway. When the focus is on the development of understanding in more particular content domains, however, the strong form of the claim is not necessarily substantiated.

Formulating specific models for the development of mathematics teaching should include some treatment of this general issue of order. Rather than searching for strictly ordered change, however, we would propose that models of mathematics teaching investigate its order*liness*. By this we mean that there are aspects of the process that are predictable, given an understanding of a teacher's current "profile" with respect to the three strands of

epistemology, mathematical understanding, and practice, but not that there is a single, fixed sequence through which teachers change. For example, some teachers may find that they need to develop a deeper personal mathematical understanding before they teach as effectively as they would wish. Others may find their greatest challenge lies in understanding their students' thinking.

We imagine some relatively small number of different pathways that teachers take as they seek to change their practice. Among other factors, these pathways would depend on how developed each of the three contributing strands are when teachers decide to make substantial changes; on the kind of external support and encouragement they receive for making changes; and on their own motivational and dispositional makeup. Researchers might choose to direct their attention to exploring whether or not this claim is supported empirically and, if so, they might begin to map out some of the alternative pathways.

Mechanisms of change. A third characteristic of developmental theories is that they propose mechanisms to account for transitions from one stage of understanding to another. As such, they are a key theoretical element for those seeking to facilitate change rather than to describe it, for it is thought that finding ways to activate these mechanisms will stimulate development. Current theories of cognitive development emphasize both psychological and sociocultural mechanisms of change.

At the psychological level, the emphasis has been on those mechanisms that promote individuals' reorganization of their own knowledge and understanding. Traditional Piagetian theory emphasizes processes of assimilation and accommodation, by which individuals seek to reduce the disequilibrium between their current knowledge and those experiences in the world which challenge that understanding (Piaget, 1970). It has recently been proposed that there are other transition mechanisms which do not involve the experience of cognitive conflict. These involve processes by which individuals can consolidate and reflect on their understanding through mechanisms of "proceduralization," by which behaviors become increasingly routine and automatic, and "explicitation," by which individuals come to represent knowledge that was previously available only through action (Karmiloff-Smith, 1992). Still other theorists have proposed that cognitive development proceeds by processes of the differentiation and integration of cognitive schemas (Werner, 1948), or the increasingly abstract representation and coordination of skills, schemas, and executive control structures (Case, 1985; Fischer, 1980). What all of these mechanisms share in common is the emphasis on internal, psychological processes of change.

Other theorists take a more socially-embedded view of cognitive development, arguing that the construction of knowing is not simply a matter of individual, solitary construction of understanding, but a dialectical process firmly grounded in a system of social relations (Broughton, 198; Cole, 1985; Lave, 1988; Vygotsky, 1978). These theorists emphasize that *all* bodies of knowledge are social constructionsnot only individuals' personal understanding, but

also the very disciplines that we each seek to know and the social organizations in which we study, work, and play.

It is Vygotsky's writings about culture, thought, and development that have received the most attention within the American psychological community. Within this perspective, qualitative restructurings of thought are related to the acquisition and use of powerful new tools and signs for mediating thought (Vygotsky, 1978; Wertsch, 1985). These tools and signs are cultural creations, and help to shape the structure and organization of individual thought by emphasizing particular socially valued relationships and reasoning processes.

The mechanism for intellectual change here would lie in the individual's acquisition and exercise of the socially constructed tools and signs. These are acquired, in part, through socially mediated interventions within the individual's *zone of proximal development*--that "space" between the individual's current level of intellectual functioning and his or her potential for understanding as assessed by success in problem solving situations that are guided by adults or more cognitively sophisticated peers (Vygotsky, 1978; Wertsch, 1985).

Little work in mathematics education currently focuses specifically on either the psychological or sociocultural mechanisms of teacher change, although there are a number of intervention projects currently exploring different models for inservice professional development which may yield information about these mechanisms (Cobb, Wood, & Yackel, in press; Fennema, Carpenter, & Peterson, 1989; Hart et al., 1992; Schifter & Fosnot, 1993). We would encourage systematic study of both psychological and sociocultural mechanisms in investigations of the process by which teachers develop new forms of mathematics practice.

Missing Elements: Motivation and Disposition. Modern theories of development have paid relatively little attention to the influence of motivational factors on learning. While there have been some recent efforts to integrate emotional factors into discussions of intellectual development (Case, 1988; Fischer & Lamborn, 1989) there has yet to be a well-developed account of what leads people to undertake the often difficult and frustrating task of reconstructing their understanding. A complete model for the development of mathematics teaching, however, should include considerations of the effect of motivational processes and individual dispositional factors on the course of teachers' developing practice.

Perhaps the simplest reason is that people are motivated to undertake significant change when they feel the need to do things differently. In the case of mathematics teachers, this "felt need" for change can originate in many places--a desire to "be current" pedagogically, a sense that standard teaching methods are not adequately serving some students, a recognition that their students have a considerable amount of intuitive understanding that is not being invoked in traditional curricula or activities, a strong mandate from administrators. This initial motivation is extremely important, and those who seek to provide professional development for teachers need to know how to encourage them to find a compelling reason to undertake the task of transforming their practice. Beyond this, however, is the importance of

understanding more about what *keeps* teachers motivated to work on their teaching during those times when it may feel terrifying, seemingly unproductive, frustrating, boring, and even "business-worse-than-usual." Schifter and Fosnot's (1993) case studies of mathematics teachers in transition suggest that courage is an important ingredient in the motivation equation.

In addition to understanding more about motivational factors which help to impel and sustain teachers' efforts to reconstruct their mathematics practice, we propose that certain individual, dispositional characteristics are likely to influence teachers' actual course of development. These dispositional factors--teachers' particular interpersonal and intellectual orientations to the world--will influence the ways in which they approach and resolve certain kinds of issues about students, classroom structure and functioning, use of curriculum materials, relations with parents, colleagues, and administrators, and so on. All teachers will encounter these issues in some form or another, but the nature of their experiences with them may vary, in part as a function of their own internal psychologies. What may provide a welcome opportunity for reflection and development for some teachers may become serious stumbling blocks for others, and perhaps non-issues entirely for other teachers.

General Directions for Future Research. By identifying that which is orderly about the development of mathematics teaching researchers will be able to create a kind of road-map of the process. This road-map, together with the understanding of the mechanisms of change, and the ways that these interact with motivational and dispositional factors, can help to guide the design and planning of intervention programs which aim to help teachers work toward developing new forms of mathematics practice.

Currently there is little systematic information about how teachers in the process of changing their mathematics practice encounter and resolve the kinds of issues described above. At present, therefore, we can neither characterize the process of teacher change in any general way, nor can we hope to facilitate the process by being able to recognize certain familiar dilemmas, crises, or choice points and understanding something about the typical range of ways through them. If we are to do so, we will need to supplement the current case study approach with longitudinal studies of larger cohorts of teachers engaged in changing their practice. These will be particularly useful for providing information about the range and variation present in the process of constructing new forms of practice.

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PRESERVICE ELEMENTARY TEACHERS' EXPLANATIONS OF
ACHIEVEMENT IN MATHEMATICS

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Preservice elementary teachers from Québec and Ontario were compared as to the causal attributions they used to explain mathematical achievement. Subjects were asked to explain their own former achievement and the achievement of a high- and a low-achieving student. The Ontario group attributed achievement more to "ability" and less to "easy subject" when explaining both their own and a high-achieving student compared to the Québec group. Moreover, the attributions used by the two groups suggest they had different perceptions about the role of the teacher in explaining achievement.

The relevancy of teachers' perceptions for student learning has been well documented (see e.g. Clark, 1988; Clark & Peterson, 1986; Cooper & Good, 1983). Motivation theory based on attributions has been widely used as a framework to explain how teachers' perceptions influence student achievement. Some important results are summarized below:

1) Teachers' attributions used to explain students' achievement are related to the expectation teachers hold for students' achievement (Cooper & Lowe, 1977; Cooper & Good, 1983 p. 97 ff.).
2) Teachers use different feedback strategies, depending on how they attribute students' achievement (Cooper & Good, 1983, p. 101 ff.).
3) The causal attributions students use to explain their own achievement are similar to those attributions teachers use to explain these students' achievement (Darom & Bar-Tal, 1981).

Closure of this line of reasoning is reached by the fact that students' achievement and their causal attributions are interrelated by a process model of the achievement motive (Heckhausen, 1974; Heckhausen, Schmalt &

Schneider, 1985 p. 129 ff.; Kuendiger, 1990, p. 11 ff.).

The importance of teachers' achievement-related perceptions raises a number of questions. For example, what perceptions are held by those individuals who plan to become teachers and do they differ in the way they attribute similar achievement? To partially answer the latter question, the achievement-related perceptions held by preservice elementary teachers from two Canadian provinces were compared. The attributions they used to explain their own former achievement as well as students' achievement in mathematics were investigated. The data provided below were gathered as part of a larger research project (Kuendiger, Gaulin & Kellenberger, 1992).

Subjects were 286 preservice elementary teachers enrolled in their first year of a three-year program at Laval University, Québec City, Québec, and 167 preservice elementary teachers enrolled in a one-year program at the University of Windsor, Ontario. The following information was gathered via a questionnaire:

1) Preservice teachers' perceptions about themselves as former learners of mathematics
   - perceived former achievement in mathematics:
     Subjects were asked to indicate their achievement level during their own schooling on a 5-point scale.
   - attributions of the above achievement:
     Five attributions were provided which are commonly associated with high achievement (ability, effort, interest, easy subject and good teaching) together with the corresponding attributions commonly associated with low achievement. Preservice teachers were asked to indicate which of these reasons they considered to be "most applicable" and "somewhat applicable" to explain their achievement.
2) Their perceptions about the mathematical achievement of two hypothetical students
   - attributions of a high- and a low-achieving student:
     Questions were structured in the same manner as the question related

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to the attribution of their own achievement. For a high-achieving student, the five attributions related to high achievement were provided together with the additional attribution "advanced cognitive development". For a low-achieving student the corresponding reasons were provided.

Likelihood ratio X^2-tests were used to compare the responses of the two groups of preservice teachers.

Preservice teachers from Québec and Ontario did not differ significantly in their perceived former achievement ($X^2(1,N=453)=2.030$, $p>0.01$). The median of each group was between "average" and "above average". Since most preservice teachers judged their performance as above average, it was not surprising that attributions commonly associated with high achievement were generally mentioned more often than those related to low achievement (see Figure 1). Although the two groups used similar attributions to explain both their own achievement as well as students' achievement (see Figures 1 and 2) some significant differences were found.

Of particular importance were those attributions for which the two groups differed significantly when explaining both their own and students' achievement. The Ontario group consistently used "ability" more often and "easy subject" less often for their own and a high-achieving student. The same group used "poor teaching" more often to explain their own and a low-achieving student. Typically, attribution of one's own achievement to "ability", an internal, stable factor, rather than external factors such as "easy subject" is viewed as an indication of a positive self-esteem. However, these two attributions also share another aspect: A teacher cannot influence either of these reasons. When this aspect is linked to the above finding that the Ontario group also used poor teaching more often, the two groups appeared to differ in their perceptions about the role a teacher plays in learning mathematics. This interpretation was supported by the fact that the Ontario group also used "good teaching" less often to explain their own achievement compared to the Québec group.

Figure 1: Comparison Between Québec and Ontario

Attribution of Former Achievement in Mathematics

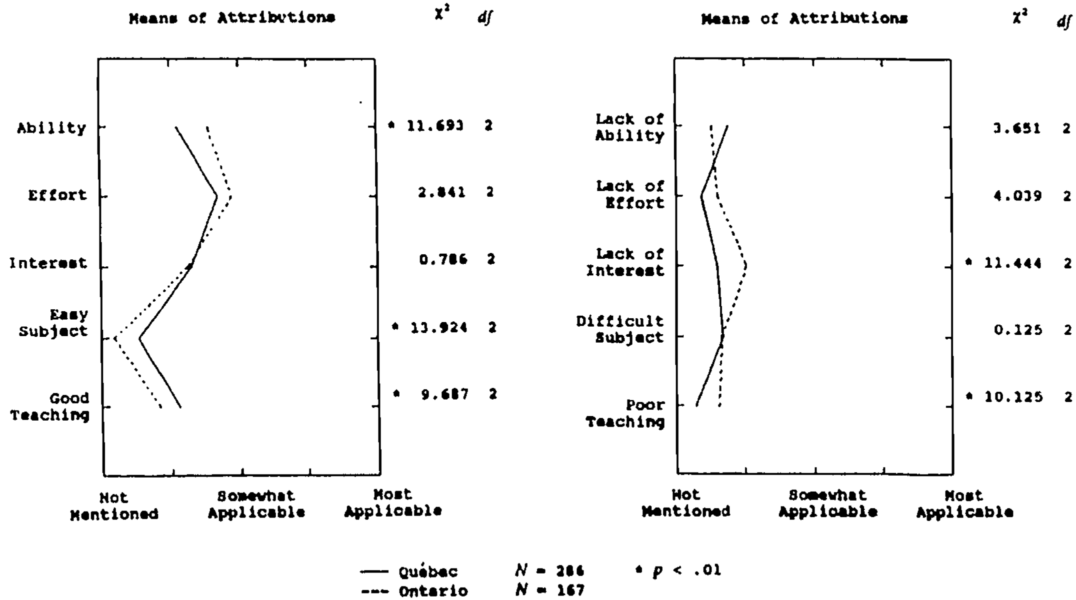

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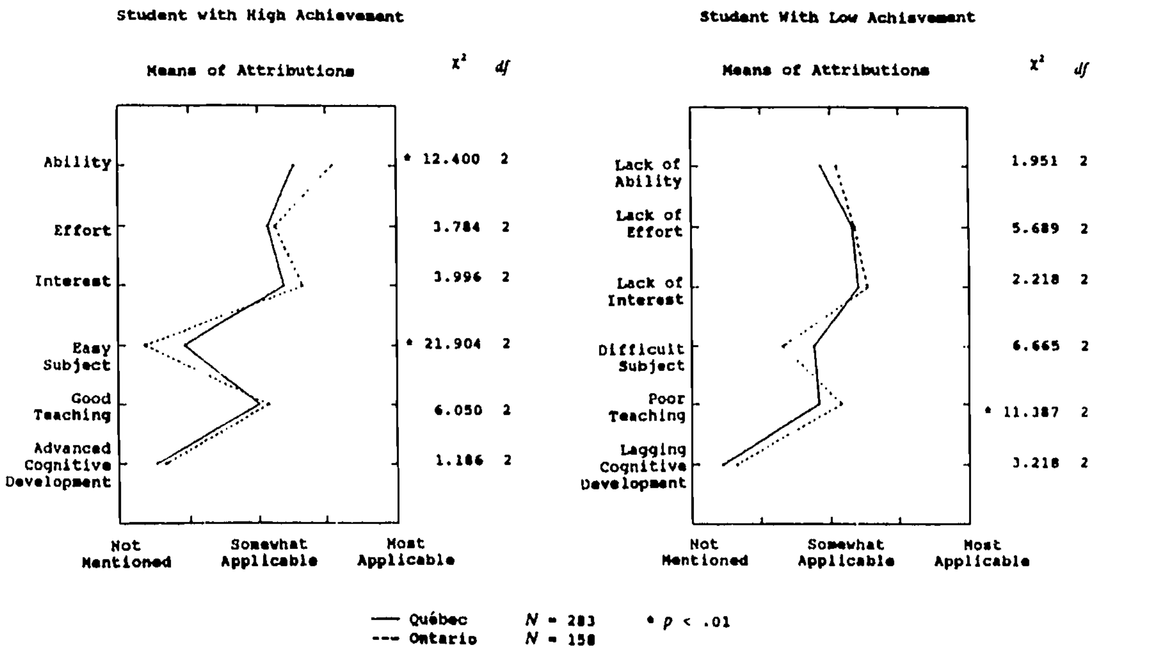
Figure 2: Comparison Between Québec and Ontario

Attribution of Student Achievement in Mathematics



In summary, Québec and Ontario preservice teachers differed in some of their attributions when explaining mathematics achievement. In line with the above literature review, these differences suggest that Québec and Ontario preservice teachers may use a different set of attributions to explain their future students' achievement. The specific impact this may have on students' motivation and achievement has not yet been investigated. However, former achievement together with the casual attributions have been used to describe a motivational framework called "learning history" (Kuendiger, 1990), which has been related to presarvice teachers' sense of efficacy as future teachers (Kellenberger & Kuendiger, 1993). The question arises whether or not attributional differences between Québec and Ontario preservice teachers are due to differences in their learning history which consequently influence preservice teachers' sense of future efficacy.

Although data from only one university in each province were collected, the above interpretation is based on the assumption that the results are representative of the two provinces. If this assumption is valid, then the explanations of mathematical achievement may depend upon the cultural environment. Support for this interpretation comes from an ongoing research study in which the attributions used by preservice elementary teachers from Windsor and two German universities are being compared (Kuendiger, Schmidt & Kellenberger, 1993).

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CARICATURES IN INNOVATION: THREE "MIDDLE-SCHOOL TEACHERS" TRY TEACHING AN INNOVATIVE MATHEMATICS CURRICULUM

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This paper discusses teachers' adjustment to use of an innovative middle school mathematics curriculum. As data on content knowledge, beliefs, and teacher practices were collected throughout the first full year of the curricular innovation, three recognizable categories of teacher seemed to emerge. These are presented through three caricatures: "The Frustrated Methodologist," "The Teacher on the Grow," and "The Standards Bearer." Struggles with giving up control of the mathematics, with allowing students to be confused on their way to understanding, and with stressing pencil-and-paper computation characterize the Frustrated Methodologist. The Teacher on the Grow is faced with teaching unfamiliar mathematical concepts but seems to learn a lot in the process. The Standards Bearer is not only confident in her knowledge of mathematics but also quite successful in facilitating student-centered use of the investigations provided by the curriculum.

The Connected Mathematics Project (CMP) of Michigan State University, with funding from the National Science Foundation, is developing a complete mathematics curriculum for grades 6, 7, and 8. Writing and pilot-testing of the CMP materials began in 1991-92 and will continue for a total of five years. Teachers and students using the CMP materials are engaged in teaching and learning quite different from the traditional transmission/reception mode because CMP is committed to developing materials that engage students in learning mathematics through contextualized "investigations"—activities that involve groups of students with mathematical concepts and applications and in reflective writing and discourse about these same ideas.

Evaluation data for CMP are being collected from classrooms in San Diego, CA; Portland, OR; Mt. Pleasant, MI; Pittsburgh, PA; Portland, MI; Queens, NY; Flint, MI; and Chapel Hill, NC. Thus, the teachers and students using the CMP materials represent quite varied geographic locations as well as diverse academic abilities (ranging from learning disabled to gifted), socioeconomic levels (ranging from upper middle-class to poor), and ethnic backgrounds (including large representations of African Americans, Latinos, Asians, and non-native English speakers at some sites). The entire 6th grade CMP curriculum was first tried during the 1992-93 school year. In all, feedback was obtained from more than 30 teachers who used the materials, and a recorder was hired at each site to provide more in-depth documentation of curriculum use by two target teachers. Recorders (most of whom were either experienced teachers or graduate students in education) attended a 2-day training session in the summer 1992, observed weekly in the target teachers' classrooms; administered project-developed questionnaires to target teachers and students three times during the school year; and conducted periodic interviews with target teachers and a subset of their students. The recorders also collected archival data such as teacher plans and student work from target classrooms, and were asked to write up profiles of target teachers and students, as well as vignettes describing interesting or important classroom events. Although the recorders collected a considerable amount of information on students as well as teachers, this paper focuses on the reactions of the target teachers during their first year of using the 6th grade CMP curriculum.

Our earliest plan for this paper involved presenting case studies of several of the target teachers involved in the CMP trials. However, as we collected data throughout the first full year of CMP classroom use, we began to observe commonalities among teachers that could not be adequately described by writing about individuals. Three main classifications of teachers seem to be emerging from our analyses of the data. As a result, we have adopted the method suggested by Noss and Hoyles (1993) of describing our findings via caricature. Each of our three caricatures is written as if it describes an individual teacher, but is actually a composite description that attempts to embody the salient characteristics of the group of teachers it represents. The first caricature (The Frustrated Methodologist) represents a group of teachers who have relatively strong math backgrounds but who seem quite resistant to the changes in their teaching methods required by the CMP curriculum. The second caricature (The Teacher on the Grow) comes from a set of teachers whose weak backgrounds in mathematics have confronted them with difficulties in teaching certain of the CMP investigations, but who have readily adopted new methods of teaching and who seem to be developing gradually, but steadily, into confident and effective CMP teachers. The third, and most successful, group we have caricatured as The *Standards* Bearer. These teachers, with strong backgrounds both in content and pedagogy, seem to have found a ready match between their own educational philosophies and that of the CMP curriculum, and, as a result, have embraced it whole-heartedly and successfully. Our caricatures are still evolving since we are continuing to gather data on CMP teachers even as we write this paper. As more teachers are included in our evaluation of the CMP project (in 1993-94, when both 6th and 7th grade materials will be used and in 1994-95, when 8th grade materials will be tested as well), we hope to note additional characteristics of successful and problematic adaptation to use of the CMP curriculum, and we will remain open to the possibility of developing additional caricatures.

The Frustrated Methodologist

Billie has been teaching for nine years. She majored in mathematics education in college and has been teaching mathematics at the junior high level for the past seven years. During this time, she has developed classroom routines that are comfortable for her and that, at the same time, have generally won the acceptance of students and administrators. Since her mathematical background is broad, very little of the mathematics encountered in the 6th grade CMP materials has been new to her. Billie believes that a good mathematics teacher should be clear, enthusiastic, and strong in interpersonal skills. Because one of her main concerns is that each child be successful, Billie worries about making the class too frustrating. She is quick to alleviate any evidence of student confusion, and uncomfortable about leaving a topic until she feels certain that the majority of students have mastered it.

When asked about her teaching style, Billie claims that she uses "the discovery method." However, classroom observations indicate otherwise. Most of Billie's classes are actually very teacher directed, as illustrated by the following discourse which occurred after Billie wrote 2/4 = 1/2 on the overhead.

Billie: What is the name of the top number of the fraction? [Billie pointed to the fraction 1/2.]
Morgan: One.
Billie: Please raise your hand to answer.

Julio:	Numerator.
Billie:	Mark, what is the name for the bottom number?
Mark:	Denuminator.
Billie:	What did you say? Try it again.
Mark:	Denominator
Billie:	Now if 2/4 is equal to 1/2, what could we say about the numerator of the one fraction in relation to the numerator of the other fraction?
Tiffany:	It's a pattern. The bottom number is a 2 and the top number is a 1. You went up one on the numerator and two on the denominator.
Billie:	Natosha?
Natosha:	You can multiply both by the same number and get the second.
Billie:	Yes. Banks, what did we multiply 1 by to get 2 in the numerator? . . .

This brief excerpt of classroom conversation illustrates several characteristics of Billie's teaching style: she prefers to maintain tight control over classroom discussions, she systematically discourages or ignores most statements (such as Tiffany's) that may be incorrect or that could lead to "confusion," and she often seems to be more concerned with procedural facility than with conceptual understanding. Billie demonstrates many of the methodological characteristics of "Jeanne" in Thompson's (1984) case study, who "believed that it was her responsibility to direct and control all classroom activities . . . avoiding digressions to discuss students' difficulties and ideas" (p. 120).

Billie worries that there is not enough review in the CMP materials, that she is not always able to stop at a place in the CMP activities where she can assign homework, and that her students' computational skills are deteriorating due to lack of practice. Because of these concerns, she sometimes saves time (and avoids student confusion) by limiting student experience with manipulatives, generally preferring to demonstrate with manipulatives on the overhead rather than to have students handle materials at their seats. Billie allows her students to work in pairs, but rarely in groups of three or four even though much of the CMP curriculum is built around small-group investigations. It seems that Billie does not really trust students who are working together to be on task. Billie has replaced the "partner" quizzes provided as part of the CMP curriculum with her own teacher-made individual evaluations. Billie supplements the CMP curriculum with computational practice and occasionally gives a quiz or test without the use of calculators, even though calculators are always available for non-graded activities.

Billie is generally reluctant to change her tried-and-true teaching methods and—as a result—is often frustrated by the inevitable incongruence of trying to teaching a curriculum that does not match her teaching style. However, we have observed some aspects of Billie's teaching that are changing. In the second half of the year, Billie began letting go of some of the routines she had been using for years (like using the first 10-15 minutes of each class for computational or problem-solving "warm-ups"). And she recently has seemed more willing to tolerate confusion in her classroom; her wait time when questioning students is longer now than earlier in the year.

Billie is not a bad teacher. On the contrary, she does many things in her classroom that casual observers would classify as good teaching. However, we have observed a constant tension between the intentions of the innovative curriculum and her routine practice. Billie may be at what Hord, Rutherford, Huling-Austin, and Hall (1987) term the mechanical level of use of innovation—a level characterized by

disjoint and often superficial use of curriculum materials. This disjointness of old and new practices leads to the frustrations inherent in being torn between two philosophies of teaching.

The Teacher on the Grow

Darcy has been teaching for 10 years. She has a degree in elementary education, but for the last four years has been teaching middle school mathematics. Darcy characterizes the first seven years of her teaching as "very traditional," but says that during the past three years she has begun to attend workshops and meet people that "opened doors for me." Unlike Billie, Darcy's background in mathematics is not strong; she had only nine hours of mathematics in college and openly admits this weakness, claiming "if I had a better background in mathematics I would be a better teacher." Darcy believes that a deeper understanding of mathematics would enable her to ask more appropriate probing questions—questions that, in turn, would help her students make their own mathematical discoveries. As Fennema and Franke (1992) point out, lack of mathematical knowledge can be quite influential in a teacher's instructional decision-making.

Although we realize that Darcy needs help in understanding the mathematics of the 6th grade curriculum at times, we also see her as willing to learn and to take risks in exploring what to her is uncharted territory. Her approach to teaching is one that allows students the opportunity to explore and discuss ideas. These explorations occasionally lead to unanticipated dilemmas, but more often provide situations for learning to take place. The following is a recorder's description of the classroom interactions of a group of boys (two African Americans, a Latino, and an Indian) in Darcy's class.

> They are not afraid or embarrassed to ask each other or an adult a question.... If they come up with different solutions, each person will argue his case. If any one person in the group doesn't understand a question or how to get a solution, someone else in the group will explain it. . . . they explain concepts to each other until the person understands.

The most promising aspect of Darcy's teaching is that she is committed to being a learner. One demonstration of this came after the unit on area and perimeter when she wrote, "This is the first time I really understood how to find the actual area of a circle." Darcy is also learning new methods of teaching. Toward the end of the year, the recorder observed:

> Darcy now tries to create a classroom environment that promotes and encourages student involvement in class activities. Instead of presenting a math concept first and illustrating that idea by working several problems to practice on, she now investigates a series of problems with interactions between her and the students to develop mathematical concepts. Darcy wants her students to be able to articulate some of the mathematical concepts that were introduced to them.

Not all of Darcy's teaching episodes turn out as the writers of the CMP curriculum had envisioned. Sometimes the difficulties are due to Darcy's limited mathematical background—for example, she once confused the class for an entire lesson by mistakenly telling them that adjacent angles were consecutive angles in a polygon. On other occasions Darcy sometimes seems to get so involved in the creative aspects of a student activity that she overlooks students' mathematical errors. For example, an activity in the CMP

unit on rational numbers asked students to lay out a garden with specified fractional parts designated for each vegetable. Darcy had students illustrate their selections by gluing seeds to miniature gardens drawn on pieces of cardboard and some of the gardens were quite creative. However, in grading these colorful projects, the fact that some of the gardens showed 12/10 or 15/10 of the garden planted escaped Darcy's attention entirely.

As we can see from the preceding discussion, Darcy is not a outstanding teacher. On the other hand, we have noted growth in her teaching from the beginning of last school year to the present. She herself has been learning about data analysis, area, and probability. She has been willing to try journal writing as a means of gaining access to students' thoughts. And, although at first reluctant, she now enthusiastically uses partner quizzes as a means of aligning her assessment with her instructional methods. Being open to change, and anxious to learn, has allowed Darcy to grow both in content knowledge and in pedagogical expertise.

The *Standards* Bearer

Natalie has been teaching for 13 years, with experience in both self-contained and departmentalized middle grades classes. While her degree is in elementary education, she has picked up a number of additional mathematics courses over the years. Her mathematics background, in terms of hours, would certainly be the equivalent of a minor in mathematics, although not enough for a major. She describes herself as having been "very good at teaching algorithms" during her early years of teaching. " I used concrete materials, broke down the skills, and felt quite successful." After attending an NSF workshop, she began to reconsider her thinking about teaching and "went looking for inservice experiences." She has in the past few years attended many workshops on mathematics teaching and methodology and is currently involved in a project on alternative assessment. Natalie wants her students to be able to "talk, explain, probe, prove what they state, write, tackle a problem, never stop, problem solve, action plan, apply, and transfer concepts later in the year." Her most common teaching approach is to pose a problem, have groups of students work on it, share their ideas with the whole class, explore related problems, and then return to the original problem.

Natalie has had no difficulty adapting to the methods and mathematics of the CMP materials because her personal philosophy of teaching already aligned quite well with the philosophy of the program. She very quickly moved past the personal concerns that many teachers must work through in implementing an innovative curriculum, and focused her concern on student adaptation to the innovation. Natalie is particularly interested in her students' view of assessment in this new curriculum. She writes, "Many are really hung up on assessment. They are concerned with the lack of grades in my gradebook. A few dislike the 'unfairness' of less able students doing well because they are working with a student who understands."

Our analysis of Natalie's teaching is that she has a comfortable command both of the mathematics she teaches and of appropriate techniques for facilitating student growth. Her content knowledge is strong enough that she is not threatened by students' questions that extend beyond the content in the textbook. Nor is she concerned when students' conjectures are erroneous. Natalie is knowledgeable enough to

know where to look for counter-examples, as demonstrated in the following vignette. (Students had previously found the area of polygonal figures by superimposing clear centimeter grids and counting squares.)

Natalie: Now, how do you measure the area of irregular shapes? What do you do with the pieces and parts? Pieces and parts seem to pose a problem. [Natalie draws a "bean" on an overhead grid.] Who can tell me the area of this shape?

Sam: I put an x in all of the squares. If it's almost a square, it gets an x. Then I take a calculator and add. [Student is putting x's in all squares and "almost squares." When he reaches half a square he claims:] This would be .5, this one over here, .25.

Natalie: Other kids may be thinking of those as fractional parts of the square, but Sam doesn't want to add fractions, so he put it in decimal form. Renaldo, do you want to share your strategy?

[Silence, then Renaldo nods his head. He comes forward, carefully places a piece of string around the perimeter of the "bean" and then stretches the measured length of string into a square. Natalie demonstrates to the class with hand gestures.]

OK. What did you get?

Renaldo: 200.

Natalie: [Referring to an earlier lesson.] Remember the polygon we found with area of 14 and perimeter of 30? We changed it so that the perimeter stayed the same, but the area changed?

Carla: The perimeter is the same, but the area is different, that's confusing!

[Natalie gives the students some time to try out "Renaldo's Theory" of finding the area of irregular shapes by comparing the results obtained (a) by tracing around their open hands and figuring an approximate area by counting squares versus (b) by wrapping a piece of string around the figure, making the measured string into a square, and figuring its area. Then the class comes back together.]

Thomas: My square is going off the paper. Renaldo, your theory is wrong!

Natalie: Renaldo, can you tell us your theory?

Renaldo: You take the perimeter and you make a regular polygon, then you measure the square.

Janelle: Your theory is wrong!

Natalie: [To Janelle] Why?

Janelle: I don't know, it's just wrong.

Natalie: How many found that the square had a greater area? [Many hands] . . . I just wanted to say that this is what mathematics is, coming up with a theory and testing it out. Renaldo is really acting like a mathematician...

This vignette captures the essence of The *Standards* Bearer. Natalie is able to entertain students' notions in the classroom discourse, whether they are right or wrong. This session could have stopped with the first student's suggestion of how to find the area and Renaldo would have continued to wonder about his approach to finding area. Also, the other students would have missed out on the opportunity to test a theory and show by counter-example its weakness. This classroom episode was made possible becsuse Natalie is confident enough of her mathematical background to allow students to make mathematical conjectures. Furthermore, she is willing to risk letting students assume some control over what happens in the classroom, realizing that student may experience momentary confusion when they share ideas, but that such discussions often lead to greater opportunities for student learning.

Discussion

From our caricatures, it is interesting to speculate how Billie and D cy might have reacted to the situation described from Natalie's class. Billie, in her concern to remove all potential confusion, would likely have ignored or quickly corrected Renaldo on his theory. Darcy, due to her limited mathematics

background, might have praised Renaldo for his creativity and accepted his method as appropriate (especially if his estimate for the bean-shaped figure had proved at all reasonable). In either case, a valuable learning opportunity would have been lost. In Billie's case, a chance for the class to really "do mathematics" would have been squandered. In Darcy's case, a misconception might have been perpetuated—both for her and her students.

As a result of our data collection this year, we feel confident that level of mathematics training is one of the important variables in successful implementation of the innovative CMP curriculum. Another variable of interest seems to be degree of prior agreement with the methodological philosophy of the curriculum. Interestingly, we have noted some characteristics of teachers from which no pattern is developing. One of these is the number of years of teaching experience: there are teachers in all three caricatures with double-digit years of experience. Another is amount of inservice education and workshops attended: most of the CMP teachers have had numerous such experiences, but only some seem to have been significantly impacted by them.

Our study of the teachers involved in curricular innovation is still very much in its formative stages. As our evaluation of the CMP curriculum continues over the next several years, we will continue to make assertions and to look for evidence to confirm or disconfirm our claims. In the coming year, we intend to further connect our research with the literature on teacher change as well as with that on teacher content knowledge and beliefs about mathematics. The caricatures that we have developed so far have already raised an interesting question. Which is the more difficult adaptation for teachers to make—using and appreciating the power of unfamiliar methodologies (as Billie must do), or recovering from lack of mathematical background when unfamiliar mathematics content is encountered in student materials (as Darcy must do)? We hope that further study of the teachers who together comprised our caricatures of Billie, Darcy, and Natalie—along with additional information from teachers who will be involved in the piloting of the seventh and eighth grade CMP materials will shed some light on this, and other, important questions.

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LIMITS OF SEQUENCES AND SERIES:
PROSPECTIVE SECONDARY MATHEMATICS TEACHERS' UNDERSTANDING

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Abstract

This article is about the investigation of teachers' subject matter knowledge and its interrelations with pedagogical content knowledge and curriculum knowledge in the context of teaching the concept of limit. Thirty-eight prospective secondary mathematics teachers completed a nonschool-like task in an open-ended questionnaire regarding to their knowledge about limits. The analysis shows that many subjects possess basic understanding of the conception of limit. Understanding of the other categories of the concept of the limit was missing, and very few could differentiate between sequences and series. Evidences have indicated that lack of the conception of limit influenced the subjects' pedagogical content knowledge and curriculum knowledge the context of teaching of the concept of the limit. Therefore, when describing limits for students, many used their own basic understanding about the limit concept to explain what a limit is, and were unable to come up with different ways of explanation.

The focus of research on what teachers need to know in order to teach has shifted from quantitatively examining teachers' standardized tests scores to emphasizing knowledge and understanding of facts, concepts, and principles and the ways in which they are organized, as well as knowledge about the disciplines (Ball, 1991; Even, 1993; Lee,1992; Leinhardt & Smith, 1985; Shulman, 1986; Wilson et al., 1987). Although researchers argue that teachers' subject matter knowledge is interrelated with pedagogical content knowledge and curriculum knowledge, there is little research evidence to support and illustrate the relationships (Even, 1993). Lee (1992) provides a framework for understanding the subject matter knowledge for teaching the concept of limit. This framework consists of five understanding categories; such as basic understanding, computational understanding, transitional understanding, rigorous understanding, and abstract understanding. The general aim was to investigate pre-service teachers' subject matter knowledge, pedagogical content knowledge and curriculum knowledge in the context of teaching the concept of limit. In particular, we examined prospective teachers' understanding about the limits of sequences and series.

* This paper is based on part of the author's doctoral dissertation, completed at Michigan State University in 1992. The author gratefully acknowledges Glenda Lappen, William Fitzgerald, Bruce Mitchell, Perry Lanier, and Thomas McCoy for their help.

Method

Subjects: The subjects in this study were 38 prospective secondary mathematics teachers in the last stage of their professional education. This group was selected because their knowledge reflected the knowledge prospective teachers have gained during their college education, but before they started teaching. They came from six universities in the United States of America.

Instrumentation: Subjects were asked to write down the responses as if they were teaching. The task was a non-school problem with two parts and was given in Figure 1. The task provided several pile up fraction bars, and unit fractions were shaded successively in descending order. In part one, subjects were asked to write down a sequence based on the given geometrical figures and to provide the limit. In part two they were asked to write down another sequence formed by the partial sums of the sequence they wrote (i.e. the infinite series) and to provide the limit.

1.Figure (A) below illustrates a fraction wall formed by fraction bars. Consider the infinite sequence formed by the individual shaded fraction bars in figure (B) below:

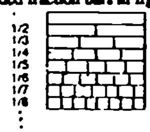


Figure A

a) Write down the infinite sequence formed by the individual shaded fraction bars in figure (B), and what is its limit?
b) Write down the infinite sequence formed by the partial sums of the sequence in (a), and what is its limit?

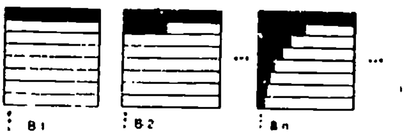


Figure B

Figure 1

Procedure: Subjects were given the questionnaire in a paper-pencil format. The administration of the questionnaire took place in the regular methods class by the instructor in the year of 1991. The task was examined and scored by three individuals consisting of one

mathematician, one high school teacher and the researcher herself based on a pre-described scoring system. The subjects' responses were scored 2, 1, and 0 points based on the correct responses, partially correct responses, and incorrect responses or no responses, respectively. The scoring system of the prospective teachers' responses in our sample was shown in Table 1.

Table 1--Scoring System

2 pt.-- Providing the correct sequence with correct matching limit. Examples: If on (a) a subject states the harmonic sequence $\{a_n=1/n\}$ and says its limit is 0, or If on (a) a subject states the harmonic series $\{a_n=\sum_{k=1}^{k=n}\frac{1}{k}\}$ and says this sequence diverges or its limit is infinity. If on (b) a subject gives the sequence is $\{a_n=\sum_{k=1}^{k=n}\frac{1}{k}\}$ and the sequence is divergent and has no limit, or If on (b) a subject gives the sequence in the numerical representation as 1, 1+1/2, 1+1/2+1/3, 1+1/2+1/3+1/4,..., 1+1/2+1/3+...+1/n,... and gives the limit is positive infinity.
1 pt.-- Providing the correct sequence with incorrect limit or with no limit number given. Examples: If on (a) a subject states that the sequence is $\{a_n=\frac{1}{n}\}$, but the limit is 2, or some other finite number rather than the true limit which is 0; or If on (b) a subject gives the sequence $\{a_n=\sum_{k=1}^{k=n}\frac{1}{k}\}$ and gives 2 for the limit or other finite numbers.
0 pt.-- No response or incorrect response. Examples: If on (a) a subject responds that the sequence is $\{B-1/n\}$ and the limit is B, or If on (b) a subject gives $a_n=\sum_{k=1}^{k=n}\frac{1}{2k}$ and gives 2 for the limit, or If on (b) a subject gives $a_n=\frac{1}{n-1}$ and gives the expression $\lim_{n->20}\frac{1}{n-1}$ for the limit.

Results

The results consisted of two sections. The first section was the distribution of the raw scores and the other section was the categories of the responses. Based on the scoring system, the correct and the incorrect responses were scored. In part (a) of the task , less than half (45%) of the responses were correct and were scored two points; 29% of the responses were scored one point; and 26% of the responses were scored 0 points. In part (b) of the task, only one sixth (16%) of the responses were correct and scored two points; 26% of the responses were scored one point; and more than half (58%) of the responses were scored 0 points.

The responses of the task were then grouped based on the following categories: (1) correct sequence with correct limit; (2) correct sequence with incorrect limit; (3) incorrect sequence

with correct limit; (4) incorrect sequence with incorrect limit; and (5) no response. The responses in each category indicated numbers of correct response, types of errors, kinds of mistakes, and how many of them. For example, one subject responded " the sequence was $\{a_n = \frac{1}{n-1}\}$ and gave the expression $\lim_{n->20} \frac{1}{n-1}$ for the limit." This response grouped in category (4) which was incorrect sequence with incorrect limit.

Discussion

The purpose of this study was to elicit the prospective secondary mathematics teachers' subject matter knowledge of understanding about the concept of the limit. The knowledge of understanding about the concept of the limit were categorized as 1) basic understanding, 2) computational understanding, 3) transitional understanding, 4) rigorous understanding, and 5) abstract understanding (Lee, 1992). In order to find the limit of a given sequence, one needed first to identify five different representations of sequences: listing the first few terms, one dimensional graph, two dimensional graph, algebraic expression, and geometrical figures. (Lee, 1992). Then based on the knowledge of understanding about the concept of the limits, one would be able to find the limit of a given sequence. The task presented here was a geometrical representation of sequences. The subjects' task was first to transfer (Putnam, 1987) the geometrical figures into listing the first few terms and/or stated the general term (an algebraic representation). The subjects were then demonstrated their knowledge and understanding about the concept of the limit by showing how to teach the limit of this given sequence in written formats The discussion consisted of two sections. First, we used part (b) of the task to demonstrate prospective secondary mathematics teachers' understanding about the concept of the limit of this geometrical representation Then, we showed three pervasive mis-understanding of the concept of the limit.

Five Categories of Understanding of the Concept of the Limit

The five categories of understanding of the concept of the limit in terms of the geometrical figures in the task were concluded as follows:

Basic understanding. By looking at the either the geometrical representation, or the listed first few terms and/or the algebraic expression, subjects intuitively stated that the limit is infinity or the limit did not exist.

Computational understanding: Subjects stated that

$$\lim_{n\to\infty}\sum_{k=1}^{k=n}\frac{1}{k}=1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\frac{1}{8}+\frac{1}{9}+\frac{1}{10}+\frac{1}{11}+\frac{1}{12}+\frac{1}{13}+\dots+\frac{1}{2^n}+\dots$$

and $$S_{2^n}=1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\frac{1}{8}+\frac{1}{9}+\frac{1}{10}+\frac{1}{11}+\frac{1}{12}+\frac{1}{13}+\frac{1}{14}+\frac{1}{15}+\dots+\frac{1}{2^n}$$

$$=1+\frac{1}{2}+\left(\frac{1}{3}+\frac{1}{4}\right)+\left(\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\frac{1}{8}\right)+\left(\frac{1}{9}+\frac{1}{10}+\dots+\frac{1}{16}\right)+\dots+\left(\frac{1}{2^{n-1}+1}+\dots+\frac{1}{2^n}\right)$$

$$>1+\frac{1}{2}+\left(\frac{1}{4}+\frac{1}{4}\right)+\left(\frac{1}{8}+\frac{1}{8}+\frac{1}{8}+\frac{1}{8}\right)+\left(\frac{1}{16}+\dots+\frac{1}{16}\right)+\left(\frac{1}{32}+\dots+\frac{1}{32}\right)+\dots+\left(\frac{1}{2^{n-1}+1}+\dots+\frac{1}{2^n}\right)$$

$$=1+\frac{1}{2}+2\left(\frac{1}{4}\right)+4\left(\frac{1}{8}\right)+\dots+2^{n-1}\left(\frac{1}{2^n}\right)$$

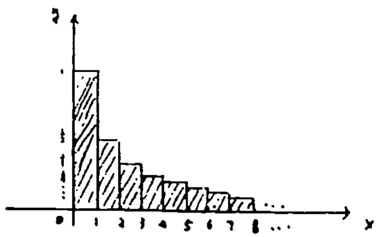
$$=1+\frac{1}{2}+\frac{1}{2}+\dots+\frac{1}{2} \qquad (n \text{ terms of } \tfrac{1}{2})$$

$$=1+\frac{n}{2}$$

S_{2^n} —> infinity, whenever n—>infinity

Thus, $\lim_{n\to\infty}\sum_{k=1}^{k=n}\frac{1}{k} > \lim_{n\to\infty} S_{2^n}$ is also divergent.

Transitional understanding: Subjects transferred the geometrical representation into vertical format, such as,



and knew that the sum of the shaded area was actually the sum of the harmonic series $\sum_{k=1}^{k=n}\frac{1}{k}$, however, they might not know how to prove it.

Rigorous understanding: Subjects proved the sum of the harmonic series was actually equals to infinity by showing the connection to the definite integral.

$$\lim_{n\to\infty}\sum_{k=1}^{k=n}\frac{1}{k}=\int_1^{\infty}\frac{1}{x}\,dx=\ln x\Big|_0^{\infty}=\ln\infty=\infty$$

Abstract understanding: Subjects applied and transferred the above knowledge into tasks in mathematics and/or in other field.

However, the results indicated this group of prospective secondary mathematics teachers' knowledge of understanding about the concept of the limit was restricted to the basic understanding regarding to this geometrical representation.

Three Misunderstanding of the Concept of the Limit

In addition, the results also indicated three misconceptions subjects possessed. These were finite point of view of the concept of the limit, mis-interpretation of the geometrical representation, and confusion between sequences and series.

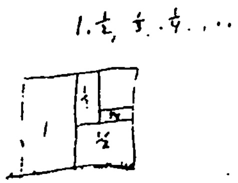
Finite point of view of the concept of the limit: Recall the definition of a sequence was defined as "A sequence is a function whose domain is the set of positive integers." The responses of some of the subjects were stated otherwise. For example,

i) The sequence is $\{B - \frac{1}{n}\}$ and the limit is B.

ii) $An = \frac{1}{n} + \frac{1}{n+1} + \frac{1}{n+2}$ and the limit is 0.

iii) $\lim_{n \to 20} \frac{1}{n+1}$

Mis-interpretation of the geometrical representation: The response given by one subject was given in the following figure:



This subject obviously showed no understanding of the notion of fraction and thus misinterpreted the geometrical representation of the limit of the harmonic series with the geometrical series $\{a_n = \frac{1}{2^n}\}$. Because of the confusion between these two given sequences $\{a_n = 1/n\}$ and $\{a_n = 1/2^n\}$, 11% of the subjects came to the incorrect conclusion.

Confusion between sequences and series: Students in calculus class often confuse infinite sequences with infinite series (Davis, 1982). Several subjects in this study also had a hard

time to differentiate between these two. Two of them claimed that they did not know what the term "partial sums" means. Half of the subjects provided no responses in part (b) indicating the lack of the ability to form a new sequence from the series.

Conclusion

A powerful content-specific pedagogical preparation based on meaningful and comprehensive subject matter knowledge with the correctly connected curriculum knowledge would enable teachers to teach in the spirit envisioned in the Professional Standards for Teaching Mathematics (NCTM, 1991). However, this group of prospective secondary mathematics teachers' knowledge and understanding about the concept of the limit was restricted to the basic understanding. The ability of transferring the geometrical fraction bars into listing the first few terms of a sequence and/or the general term seemingly was not there. This was an evidence of the lack of curriculum knowledge; especially how to connect the notion of fraction to the concept of limit. This also indicated the lack of pedagogical strategy for teaching the concept of limit. How to properly integrate the intuitive notion of limit earlier into the lower mathematics curriculum and pedagogical preparation is a question needing further research.

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FROM LEARNING MATHEMATICS TO TEACHING MATHEMATICS: A CASE STUDY OF A PROSPECTIVE TEACHER IN A REFORM-ORIENTED PROGRAM

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This study examines the experience of a prospective elementary teacher in the context of a reform-oriented mathematics teacher education program and in her subsequent student teaching in elementary classrooms. It highlights the shift in Toni's relationship to mathematics resulting from her mathematical experiences in the program and her struggles to provide her students with opportunities to learn as she had. Toni instituted many of the teaching practices that she had benefited from as a student. However, her use of these strategies was not supported by an ability to identify and understand the "big ideas" involved in what she was teaching nor by an elaborated personal theory of how children learn mathematics.

Mathematics teacher preparation programs may be conceived based on a sound foundation of research and theory. However, it is the prospective teachers' understandings of their experiences in such programs, not the conceptions of the researcher / teacher educator that determine the effectiveness of the program: This study was an attempt to look at the sense-making of one prospective elementary teacher in the context of a reform oriented instructional program and subsequent field experience.

Toni was one of 26 prospective elementary teachers who volunteered to participate in the mathematics education courses of the Construction of Elementary Mathematics (CEM) project.[1] The two semester-long courses, created specifically for the research project, were based on constructivist theory and current mathematics education reform ideas (NCTM 1989, 1991). The first course was a mathematics course, which dealt primarily with understanding multiplicative structures, and the second was a course on mathematics learning and teaching. The courses were the focus of research using a constructivist teaching experiment design (Simon & Blume, in press; Cobb & Steffe, 1983), and both were taught by the first author. Following her participation in the courses, Toni took part in a 5-week pre-practicum and a 15-week student teaching practicum in elementary school classrooms.

The case study of Toni[2] was based on data from interviews conducted with her before, during, and after the program; videotapes of the project classes; her reflective journal; her written work; and videotapes of her teaching during the pre-practicum and practicum. The purpose of the case study was to better understand the development of prospective elementary teachers as they progress from narrow views of mathematics and traditional views of mathematics learning and teaching towards views more consonant with those of the current mathematics education reform effort. Toni's story chronicles her growth as a learner of mathematics and her struggles to foster similar growth in her students and focuses on the sense that Toni made of an innovative mathematics learning experience and its relationship to her subsequent teaching.

Case Study Data

Prior to CEM

Toni entered the program with an acceptable academic record (2.9 grade average) including two college mathematics courses. Nonetheless, she reported that she had learned very little mathematics in the past and that mathematics caused her anxiety. Her anxiety and the negative experiences that she heard described by other students regarding the university's mathematics for elementary teachers course

caused Toni to postpone taking this required course and to sign up for the CEM course (in lieu of the mathematics for elementary teachers course) when the opportunity arose. She hoped that the CEM course would be better, more applicable to her career as an elementary school teacher, and less stressful. Despite her anxiety about doing mathematics, Toni was eager to teach mathematics. She was committed to creating a better experience for her students "to prevent other people from feeling the way I feel about math."

Toni's mathematical knowledge consisted of some rudimentary understandings and a host of mathematical algorithms. These algorithms were "black boxes" whose workings she never questioned. She seemed to have no expectation that she could understand these algorithms. She was able to use her algorithms in the context of the problems for which they had been taught, she generally remembered how and when to use them. In addition, Toni monitored her work by judging the magnitude of her answers to quantitative problems and by using unit analysis to check the appropriateness of her computation.

Toni believed that her mathematical preparation was inadequate due to the limitations of traditional mathematics teaching. However, a role-play prior to the start of the project courses, in which she played the teacher working with one student, revealed that she had not as yet constructed an alternative approach to mathematics teaching. Her response to a student who did not understand regrouping in subtraction was procedural in focus and consisted exclusively of teacher explanation.

Participation in Course One: Mathematics

Although Toni approached the mathematics course with some anxiety, it did not keep her from participating immediately. The nature of her contributions were consistent with what we had observed in pre-program interviews - procedural explanations and attention to when an algorithm should be used. Her initial journal reflections were brief and answer oriented. However, soon Toni began to explore ideas independently and to focus on what it would mean to understand an idea. Toni's journals in the latter part of the course provided evidence that her work between classes involved wrestling with mathematical questions and evaluating her understanding of the ideas involved. She began to consider her ability to verbalize an idea as an indicator of her level of understanding. However, in contrast to her view of journal writing as a chance to verbalize an idea to develop and assess her understanding, we find no evidence that she viewed collaborative group work in this way.

Whereas prior to the course, a ratio was a black-box for Toni, in the latter half of the class she asked thoughtful questions about the meaning of the ratios being used in an attempt to understand ratio as a measure of an attribute of a physical situation (see Simon & Blume, in press). Over the course of the semester, Toni increasingly engaged in empirical exploration in order to evaluate the validity of a particular mathematical formulation. She used a bathroom scale to investigate the relationship between weight and pressure. She mixed and tasted different concentrations of ice tea in order to choose between additive and multiplicative explanations for concentration, both of which she found compelling in class. Toni expressed enthusiasm about being able to "experiment" in these ways. Her view of mathematics

seemed to be expanded by including empirical investigation. However, we noted that she was unable to reason about proposed arguments in the absence of the empirical results.

Toni's growing enthusiasm for experimenting led her to conduct an independent investigation in response to a question posed by a fellow student. She was puzzled by why an n% inc ... followed by a n% decrease did not return an employee to her original salary level. Toni reflected, "Our 100% theory seemed very logical, because it seems to me that if you [add] 20% of a number [to the original number], the original number divided by the new number will equal 80%."[3] Toni calculated the results of the increase and decrease and recorded the results in a table to look for a pattern. She wrote,

> I am having a difficult time determining what is wrong with our theory. I did find that the smaller the percentage increase, the closer the original number is to a percentage that will bring you to one hundred percent. [Toni expected that the percent increase plus the quotient (original salary divided by the new salary) would equal 100%.] I'm not sure what this means, but I think it is a significant factor in determining what is wrong with our theory.

This excerpt shows not only Toni's engagement in exploring patterns, but also her sense that the pattern is a partial step towards the understanding that she is seeking. Whereas earlier in the course, she might have focused on her inability to get "the answer," she now seemed to view her sense making as an ongoing process.

Following her participation in the program, Toni described herself as often feeling frustrated in this first semester. "I just felt sometimes I just wanted to say, 'Just give me the answer. I don't want to explore it anymore." She identified her concern for her grade in the course as compounding her tension and frustration. She felt a need to earn a good grade in the course, yet felt a lack of control at times to do so. Unlike previous mathematics courses in which the teacher indicated behavioral criteria that would constitute success, this course did not aim at competence with a particular set of behaviors.

Toni showed growth in her commitment to understand, her sense of what it means to understand aspects of mathematics, and her confidence in her ability to understand. However, at times Toni's original answer orientation was in evidence. A change seemed to be occurring in Toni's relationship to mathematics; yet, her previous relationship had been developed over many years and remained a significant influence. While Toni came to value exploration and reflection, she became frustrated periodically with her lack of progress and focused on hearing from others ideas of how to solve the problem. It seems important to recognize that although a shift had developed in her relationship to mathematics, it had only moderate impact on her mathematical concepts over the course of one semester. Many of Toni's underdeveloped mathematical concepts such as her difficulty in distinguishing between multiplicative and additive relationships remained problematic.

Participation in Course Two: Learning and Teaching Mathematics

In the second course, Toni continued to explore mathematics and to increase her confidence in her ability to understand and to solve problems. In addition, the alternative to traditional mathematics teaching that she was seeking was taking shape. In her writings and her class participation, she stressed focusing on and promoting children's thinking, understanding, and communication of mathematics. Her experience impressed upon her the power of using manipulatives. Analyzing mathematical knowledge to articulate what it might mean to understand a concept remained difficult for her. She seemed to make

little progress on questions such as "What might understanding division be beyond counting out cubes to solve a division word problem?"

Although Toni's experiences in the CEM mathematics contributed to her evolving image of mathematics teaching, she made distinctions between teaching prospective teachers and teaching children. She asserted that the CEM mathematics lessons were for students who "had learned the basics of mathematics." And in a post-program interview, she indicated that the frequent lack of closure regarding a correct answer or procedure would be less appropriate for elementary students.

Student Teaching

Toni did her student teaching in classrooms that permitted a significant amount of freedom to teach mathematics as she saw fit. The content that she was expected to teach derived from the school district's mathematics curriculum guide. From her initial lessons, it was clear that Toni was committed to creating a form of practice which departed from traditional mathematics teaching and which incorporated what she had learned in her teacher education program. Her teaching made frequent use of small collaborative groups and manipulative materials. Rather than lecture and demonstration, Toni tended to begin her lessons by posing questions or problems to the students.

Toni's approach to teaching mathematics often involved posing questions to lead students through a series of responses that were designed to build to the level of response which she had identified as her goal for the lesson. (In post-teaching interviews she often used the language, "lead them to understand.") The responses were what she considered to be indicators of understanding given that the children had not been provided with a model to imitate. When her goal was to teach students to name and write simple fractions using area models (our language) she followed a four-part plan: (1) ask the students to say the number of parts, (2) ask them to say how many of those parts are indicated, (3) ask them for the fraction name, (4) ask them for the written form. From Toni's perspective, she had designed a plan to develop understanding. It began with something that the students knew, how to count the number of parts, and developed sequentially with each piece connected to the one before. Toni's instruction was predicated on the expectation that someone would be able to answer the question appropriately. There did not seem to be an attempt to inquire into children's understanding. As a result, there was no explicit attempt to base the lesson on students' current concepts. During the lesson, her attention was not on what concepts students were constructing, but whether they were producing the behaviors that she considered to be evidence of understanding. She did not focus on creating a problematic situation that would result in accommodation.

Toni was limited by her knowledge of mathematics and her ability to identify key mathematical ideas. In addition, her views of mathematics learning were relatively undeveloped. She seemed to believe that learning involves active participation as opposed to passive taking in of information. She believed that understanding must be built on prior understanding. Her lessons, however, suggest that this building process is the result of sequencing of a particular set of behaviors.

Our sense is that, prior to a lesson, Toni did not analyze the mathematical understandings nor did she question whether she was teaching concepts or skills. Rather Toni's question seemed to be, "How

can I teach this topic (found in the curriculum guide) in a way that will foster more understanding than just teaching a rote algorithm?" "Understanding" in the context of thinking about teaching rather than about learning evokes an association with a set of teaching strategies which Toni thinks of as contributing to understanding. Thus, Toni focused on using manipulatives, asking questions, and having students work together. This focus on general teaching strategies does not involve Toni in important issues of the mathematics and of children's thinking.

Summary and Conclusions

The mathematical experiences in which Toni engaged during the CEM classes gave her a feel for what mathematics learning might be. Although she struggled at times with the lack of closure on mathematical ideas and with the exam process, Toni felt herself growing in understanding and in her confidence to engage in mathematical experiences and discussions. She approached her student teaching with a commitment to adapt to the elementary classroom that which had been powerful for her, including opportunities for students to develop their own ideas and strategies, to work with manipulatives and diagrams, to encounter a variety of approaches to problems, to articulate their ideas, and to see the connections between mathematical ideas. These commitments cover some of the key aspects of recent mathematics education reform documents (cf. NCTM, 1989 and 1991).

For Toni, what were the key determinants of her CEM mathematics learning experiences? Her reflections and subsequent teaching suggest that Toni focused on the structure of the class and the behaviors of the teacher, including the use of cooperative groups, manipulatives, whole class discussion, and teacher questioning of students. Notable, as well, were the teaching strategies not in evidence such as lecture/ demonstration and telling students whether their answers were correct.

However, as Toni attempted to apply these strategies in the classroom, they became problematic. Students did not necessarily learn what Toni intended for them to learn. How could she help them make progress without telling them what think? What manipulatives should be used, at what point? What problems should the cooperative groups be engaged in? What issues should be pursued now and which ones addressed later? Often, when Toni viewed videotape of her lessons, she was able to recognize when she was not doing the type of teaching that she intended to do. However, knowing how to promote the learning of particular concepts was not clear to her.

Whereas Toni had adopted a potentially powerful set of teacher behaviors, she was relatively unaware of what went on inside the teacher's head. She was unaware of the knowledge and thinking which informed the teacher's decision making. (Simon, in press, provides an empirical and theoretical exploration of the teacher's decision making in the CEM mathematics class.) A teacher's mental activity can be discussed focusing on three broad areas: mathematics, mathematics learning, and mathematics teaching. These three interdependent categories allow us to organize our analysis of Toni's limitations as a teacher of mathematics.

Mathematics: Toni created her lessons based on the topics listed in the curriculum guide. She did not seem to inquire into the "big ideas" that were implicated in these topics, nor did she attempt to specify the web of concepts that might be related to the principal understanding at hand. We

hypothesize two reasons for this. First, Toni's own understanding of the mathematics was often weak. Second, Toni did not see the teacher's analysis of the mathematics as foundational to the kind of mathematics learning opportunities that she had come to value.

Mathematics Learning: Toni had not developed an explicit theory of mathematics learning. Although many of her views seem compatible with a constructivist orientation, careful examination suggests that her ideas are about teaching and not learning, that is, they are about how to support learning and not the mechanism by which learning takes place.[4] For example, Toni believes that when students have the opportunity to generate their own ideas rather than listen to the teacher's ideas, they develop greater understanding. Contrast that idea with a constructivist notion that each person constructs their own understandings in all situations (including lectures), and that learning is a process of construction which is triggered by a problematic situation in which current schemes are inadequate to accomplish the goal of the individual. Toni's belief implies a particular type of classroom activity. It does not address the knowledge of the individual and how that knowledge is transformed. As a result of her attention to teaching rather than learning, Toni does not focus on understanding students' current conceptions. Without an explicit personal theory of learning, she is unable to analyze teaching beyond sorting teaching behaviors into those which are more and less effective.

Mathematics Teaching: Toni's teaching and her reflections on her teaching suggest that she repeatedly returns to a set of teaching behaviors which she believes result in learning. Using manipulatives, posing real-world problems, organizing cooperative groups and whole class discussions, and questioning students rather than telling them information are the tools that she brings to the classroom. However, her lack of grounding in the mathematics and in how students learn, leaves her with a limited set of strategies for helping students to progress.

Toni's teaching consists of leading students through a set of behaviors towards behaviors which she considers to be demonstrative of the knowledge she is trying to promote. She attempts to start the sequence at a point where she can count on appropriate responses. This novice approach to teaching-without-telling may appear to be successful as long as there is one or more students who can answer her questions. However, when Toni becomes aware that students are not following what is transpiring, her approach does not help her know how to proceed.

While Toni is leading her students towards her goal, she frequently engages in behaviors that are likely to be beneficial. She asks students why they did what they did and to represent their ideas with concrete representations. However, the potential of such strategies is limited by the mathematical value of the activity and the appropriateness of the activity to the students' current mathematical knowledge. In addition, her lack of discrimination between logico-mathematical knowledge and social knowledge prevents her from making an informed choice among a wide range of strategies - even telling. (Of course, Toni does engage in telling, but it seems to creep in when she is intending not to.)

For Toni, the two-semester CEM instructional program was significant. It gave her a chance to change her relationship to mathematics and to have an experience of mathematics learning with understanding in a collaborative classroom community. Toni also acquired a set of potentially powerful

teaching strategies. Her reflections in her journals and interviews suggest that much of what she experienced and read are useful to her as she thinks about learning and teaching experiences in which she is involved.

As important as these changes are, Toni's story also points at the inadequacy of her preparation to teach mathematics. Observations of her teaching and subsequent analysis of her reflections on her teaching suggest that the following are key questions:

1. What is the extent and nature of instructional intervention that would provide her with a strong conceptual background for teaching mathematics in the elementary grades?

2. How might Toni be assisted in identifying "big ideas" in the elementary mathematics curriculum?

3. What experiences would lead Toni to develop a useful personal theory of how children learn mathematics and the commitment and ability to pursue children's understandings and thinking?

4. How might Toni learn the "invisible" aspects of teaching, the mathematical conceptual analysis, the building of models of students' thinking, the hypothesizing about potential learning (discussed in Simon, in press)?

Clearly the answers involve a program of greater duration and the opportunity for extensive teaching experience in reform-oriented classrooms supervised by educators who have a deep understanding of these issues. Regular opportunities to conduct mathematics interviews with children and to analyze the content and process of those interviews in a group of colleagues also seems indicated. However, many of the answers to these questions are less easily identified.

Notes

[1]The CEM project is supported by the National Science Foundation under Grant No. TPE-9050032. Any opinions, findings, and conclusions or recommendations expressed in this material are those of the author and do not necessarily reflect the views of the National Science Foundation.

[2]A much longer paper describing the case study is available from the first author. In this 7 page version, transcript data were deleted.

[3]Note that the 20% refers to 20% of the original number while the 80% refers to 80% of the new (larger) number. The students were not making this distinction.

[4]Our language here is overly dichotomous. We recognize that having a personal theory of learning is not all-or-nothing. Certainly Toni has some rudimentary notions about learning. For example, she seems to believe that learning is sequential. However, we believe that an important point is made by contrasting her orientation towards teaching strategies with a perspective grounded in how students learn.

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A PRESERVICE TEACHER LEARNING TO TEACH MATHEMATICS IN A COGNITIVELY GUIDED MANNER

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This paper reports the results of a study designed to answer three questions: 1) To what extent can a preservice elementary school teacher adopt cognitively guided beliefs with respect to the teaching of mathematics? 2) How does an exemplary student teacher transfer into practice her beliefs about teaching mathematics in a cognitively guided manner? 3) What are some of the issues /factors that influence the degree to which a student teacher can teach mathematics in a cognitively guided manner? Results indicate that a student teacher can adopt cognitively guided beliefs and experience some success at implementing these beliefs during instruction. However, this student teacher experienced many difficulties. The reasons for these difficulties are discussed and comparisons are drawn between the beliefs and practices of the student teacher and those of her master teacher. Implications for teacher preparation are discussed.

There is a desperate need to change the way mathematics is currently taught, and researchers have begun to study the beliefs and knowledge teachers must possess in order to be in a position to make these changes. This research has been supported by the theoretical framework proposed by Shulman (1986) who introduced the term *pedagogical content knowledge* in an attempt to link content knowledge and curricular knowledge into an area devoted to "the ways of representing and formulating the subject that make it comprehensible to others" (p. 9). An important question related to teacher education and teacher change is the extent to which teachers use pedagogical content knowledge. A research program conducted at the University of Wisconsin and referred to as Cognitively Guided Instruction (CGI) suggests that primary school teachers who receive detailed research-based knowledge about addition and subtraction problem types and solution strategies use this increased understanding of how children think to learn more about their own students' understanding, resulting in better mathematics learning and increased confidence by their students (Carpenter, Fennema, Peterson, Chiang, & Loef, 1989; Peterson, Fennema, Carpenter, & Loef, 1989). These results, however, have not been extended to preservice teacher education. That is, we do not know whether people who are in the process of becoming teachers stand to benefit by receiving detailed research-based knowledge about children's solutions strategies.

The current study investigated this issue by studying one student teacher to determine the degree to which she developed CGI beliefs and the extent to which these beliefs resulted in CGI instruction in a first grade classroom. We used case study methodology to investigate this student teacher during her second semester of a two-semester fifth-year elementary school credential program. We chose an exemplary student teacher because we were seeking an existence proof: that is, can a preservice teacher

utilize research about how children think in such a way that it influences her practice, and if so, what are the issues and factors that influence the degree to which this occurs?

METHOD

The subject for this study, Miss T, was selected because her background and circumstances distinguished her from other elementary school preservice teachers. First, her mathematics background was stronger. Second, she had completed a graduate course in mathematics education prior to the start of her fifth year in which she was exposed to CGI material. Third, during the year she worked on her credential, she took a mathematics methods course that included CGI material. Fourth, during her second semester student teaching, she was working with a first grade teacher who had herself been involved with a CGI inservice project during the past year.

Data were collected for a period of one-and-a-half years, from the time Miss T was enrolled in her graduate course until her completion of student teaching, with the majority of data collection taking place during her second semester of student teaching in a first grade inner-city classroom. Data included scores on belief scales taken before and after the graduate course, interviews conducted before she began and after she completed her student teaching assignments, 12 classroom observations of her student teaching, and a planning session with her cooperating teacher. We also observed the master teacher on six separate occasions, and interviewed her at the end of the student teaching semester. All of the observations and post-observation discussions of Miss T were audiotaped and selectively transcribed, and we videotaped one lesson. All interviews were transcribed in their entirety.

RESULTS AND DISCUSSION

Miss T's Orientation Prior to Student Teaching. Miss T's took the pretest belief scale at the beginning of her student teaching semester. These scores, along with her preinterview, indicated that she possessed an orientation toward the following beliefs: (a) children construct their own knowledge, (b) skills should be taught in relationship to understanding and problem solving, (c) mathematics instruction should facilitate children's construction of knowledge, and (d) children, if provided appropriate materials and a chance to discuss their thinking, can create mathematical meaning for themselves. She was critical of the type of mathematics instruction generally found in schools involving demonstrations, guided practice, and homework from the textbook, and she viewed the textbook as a resource to be used but not to be obeyed. She intended for problem solving to form the basis for much of what she did in class, with students encouraged to share their thinking. She believed that conceptual understanding ought to precede the development of procedures.

Translating Beliefs Into Practice. The following were common characteristics of all the lessons we observed Miss T teach. Textbooks were never used. Students were consistently asked to explain how they got their answers. Word problems were often either used as the basis of the lesson or students were asked to compose their own. Miss T was never satisfied with just one correct answer, and the students seemed comfortable with being asked for additional strategies.

The hallmark of CGI is the teacher's use of content-specific research knowledge about students' mathematical thinking. One example of this occurred early in the teaching of Miss T's geometry unit,

when she wanted her students to develop vocabulary while at the same time work on their understanding of shapes by describing the difference between a square and a rectangle. One student commented that "This (referring to the rectangle) is weird and this (referring to the square) is not" and another said, "This is long and this is like a box." Miss T responded by asking the student, "Are sides of the square different from sides of the rectangle?" One boy stated, "one is longer and the other is shorter," a girl said, "one is smaller and the other is taller," and another girl said, "one is skinny and one is fat." Miss T then commented that, "I am trying to get to one thing." At this point in the lesson, Miss T passed out scissors and the students worked on cutting out some shapes.

During the discussion after class, Miss T explained that she had wanted her students to state that the square is made of up four equal sides whereas the rectangle has two pairs of equal sides. While discussing the difficulty her students were having with analyzing various properties of shapes, Mrs. A suggested that Miss T read a paper that had just been sent to her about CGI and geometry (Lehrer, Fennema, Carpenter, & Osana, 1992), which included a thorough discussion of van Hiele's levels of geometric thought. During a discussion with Miss T three days later, she was asked what she thought of the article. She commented:

Miss T: I liked it. It was difficult for me to ... I read Van de Walle (1989) for this unit. I had thought that I had understood the levels. The research with regards to those levels really helped me ...

I: Did that article affect anything you were thinking about with regards to how you will teach? Let me put it this way, did re-understanding level 0 (of the Van Hiele Levels) change the way you will teach?

Miss T: Yes.

I: How?

Miss T: I am not going to focus on specific features or specific language. I'll bring it up, because some kids might get the fact that a square has it and a rectangle doesn't, but I am not going to sit there like I did the other day and try to get the entire classroom to get that.

And she didn't. Miss T spent the duration of the geometry unit letting her students work at a number of different centers she had created so that her students would be able to engage in activities which might be said to address van Hiele level 0 (visualization), instead of van Hiele level 1 (analysis). These activities were well-received by her students and seemed appropriate. This was an example of Miss T's use of research-based knowledge of students' thinking when making instructional decisions.

The fact that Miss T was able to utilize detailed knowledge of students' thinking when making instructional decisions is encouraging. However, although there were times when she was able to do this, there were many occasions when her knowledge of students' thinking did not translate into instructional decisions. During one discussion about how she uses knowledge of individual students when making curricular decisions, she stated:

> Maybe I will take home their book and their portfolios, so I can begin to tell what they know. That wouldn't even drive me that much, because if I found out that three-fourths (of the students in the class) were direct modelers and one-fourth (of the students) were counters (using counting

strategies), with some use of derived facts. I don't think that would help me that much to help me focus on what I am doing. I think the problem is the subject matter is so limited.

Miss T knew that many of her students were not yet able to count-on, but she still had trouble designing lessons which she felt would address her students' levels. The next session will describe some possible reasons for this difficulty.

What are some of the issues/factors that influenced Miss T's implementation of CGI?

There are a number of reasons why a teacher's knowledge of students' mathematical understanding might not be accessed when making instructional decisions. We cite five:

1) The teacher's knowledge of mathematics might not be sufficiently structured to allow her to fit observations of her students' understanding into a coherent mathematical picture. That is, without possessing a rich mathematical road map, the teacher may not see the possibilities related to structuring the curriculum (McDiarmid, Ball, & Anderson, 1989).

2) The teacher might view the curriculum as given, and she may not believe she has any freedom to vary from that curriculum. For example, many teachers view the textbook as the curriculum to be closely followed (Romberg & Carpenter, 1986).

3) Teachers may not possess a map of the curriculum. Although this is related to reasons #1 and #2, it merits separate mention. Whereas a teacher may posses mathematical knowledge rich in relationships (#1) and may not believe that the mathematics curriculum is defined by the textbook (#2), she still may not possess a sense of what important mathematical ideas should be included at a particular grade level.

4) The knowledge of students' thinking may not fit into a larger framework. In her case studies of English teachers, Grossman (1990) noted that teachers who do not possess a framework with which to make sense of their students' understanding may find it difficult to learn from experience. An example of this can be found in this study. Miss T understood that her students were having difficulty explaining their geometric reasoning, and even though she had read about the Van Hiele Levels, it was not until she went back and thought again about the levels that she was able to make sense of her students' thinking in terms of their development of geometric reasoning. As a result, her renewed understanding of the Van Hiele Levels provided the larger framework into which she was able to place her knowledge of her students' thinking, resulting in her ability to make appropriate instructional changes.

5) Teachers may be constrained by their view of how children develop understanding. This is especially true among novice teachers, who often underestimate the time required for students to develop conceptual understanding. The many forces acting on elementary school teachers help explain why this view continues to predominate many teachers' thinking. Behavioral objectives, for example, which are still extolled as important for beginning teachers to understand to help them focus their lessons (Zumwalt, 1989), carry with them the unintended (or perhaps intended) message that anything can be taught to anyone in forty-five minutes.

In describing Miss T's difficulties, we will discuss how her difficulties compared with the difficulties experienced by her master teacher, Mrs. A. Two consistent issues plaguing Miss T were

classroom management and her students' difficulty articulating their thinking. Although both of these issues also bothered her master teacher, classroom management did not seem to trouble Mrs. A nearly as much as it troubled Miss T.

The five reasons stated above did not equally account for the difficulty Miss T experienced when she attempted to use her knowledge about students' thinking when making instructional decisions. The first two reasons did not seem to play a major role - her knowledge of the mathematical content was deep and she did not view the textbook as being overly important. The fourth reason also did not seem to apply because not only is the framework which describes the addition and subtraction problem types and solution strategies well formulated, but Miss T possessed an excellent understanding of this framework, perhaps even better than Mrs. A's. For example, note the following dialogue between Miss T and one of the researchers:

I: Have you talked to Mrs. A about what she would like them to be able to do by the end of the first grade?

Miss T: Not that specific question. But I asked her, "Before we get to geometry, what do you want me to cover?" She said she wanted them to be able to do start unknown problems, which surprised me.

I: Why did that surprise you?

Miss. P.: Because it is so specific. Some of them may not be able to get there yet. It seems to me a developmental issue. There are some of these kids that won't be able to get that.... It would be like forcing someone to have that cognitive flexibility before they have that cognitive flexibility. Forcing them to think the words in the problem before they really understand what it means or how they can change the numbers or understand the situation so they can do it themselves.

We believe that reasons 3 and 5 accounted for much of Miss T's difficulties. Having only student taught for half of one semester at the 5th-6th grade level, Miss T had virtually no opportunities to consider what are the big mathematical ideas toward which first grade children ought to be oriented. That is, she had not created a mental map of the curriculum to which she might turn when making decisions about the class. We are not in a position to comment as to whether or not Mrs. A possessed such a map of the curriculum, but it is noteworthy that Miss T thought her master teacher possessed one: "I think she has a master plan in her brain that she has mentioned several times that she may not be able to verbalize." She also said, "I asked her what she wanted me to cover and I don't know if she is trying to torture me by not telling me specifics." The other issue which seemed pertinent to Miss T's ability to utilize information about her students' thinking was #5, involving her inexperience at knowing how long one might reasonably expect students to take to develop addition and subtraction problem solving abilities and associated processes of reasoning. This was an issue that did not seem to concern Mrs. A, who often mentioned that her chronologically younger students required the passage of time before they might come into their own.

Perhaps the most interesting difference between Miss T and Mrs. A was their view of themselves as CGI teachers. One of the questions they responded to during separate interviews conducted at the end of the semester required each of them to place herself on a CGI continuum in terms of where they thought they were and where they would like to be. Figure 1 and 2 display their respective responses.

Non-CGI _______X____________________________O CGI
Teacher Teacher

FIGURE 1: Mrs. A's self evaluation of where she is as a CGI teacher (X) and where she would like to be (O).

Non-CGI ___________________________________X__O CGI
Teacher Teacher

FIGURE 2. Miss T's self evaluation of where she is as a CGI teacher (X) and where she would like to be (O).

In describing why she placed herself where she did, Mrs. A explained that she still had a long way to go in implementing CGI. Miss T, on the other hand, explained that although she was not completely implementing CGI, she felt that she had adopted CGI beliefs and it was only because of other issues, such as classroom management and her students' inability to express themselves, that she was not able to further implement CGI.

The difference in where Miss T, the student teacher, and Mrs. A, the master teacher, placed themselves on the continuum reflects their different views of that which constitutes CGI. Mrs. A, like the other inservice teachers who have placed themselves on this continuum, saw herself in terms of what she was able to implement in the classroom. Miss T, however, viewed herself in terms of her own beliefs, separated from the many factors that impeded her ability to implement CGI as she would have liked.

The three authors, independently using an instrument designed to locate one's development of CGI based on one's practice (Franke & Fennema, 1992), indicated that both Mrs A and Miss T held the rudiments of CGI philosophy and were beginning to translate that philosophy into instruction. We placed Mrs. A a little higher than Miss T on the scale.

We hypothesize that the differences in where the master teacher and the student teacher placed themselves on the scale may highlight an important difference between how inservice teachers and student teachers come to understand new learning theories. While both inservice teachers and preservice teachers must filter new theories through their current understanding, inservice teachers may constrain the new information in such a way as to take in only that which they are able to assimilate into their current schema of teaching. Student teachers, on the other hand, who do not possess a great deal of knowledge about the practice of teaching from the perspective of the teacher, may be more apt to take in

the theory as a whole without necessarily knowing how or even if they will be in a position to implement that theory during instruction. In the current study, Mrs. A placed herself low on the CGI Teacher scale because she was not implementing CGI as much as she hoped to. Miss T placed herself high on the CGI Teacher scale in spite of her perception that she was not able to implement CGI to the extent to which she would like, because she thought she possessed a good understanding of CGI principles.

FINAL COMMENTS

This study, which provides an existence proof that a preservice teacher *can* utilize pedagogical content knowledge about how children think in such a way that it influences her practice, carries important implications for teacher preparation. Miss T developed strong cognitively guided beliefs that affected her classroom practice when student teaching with a CGI master teacher. Furthermore, these beliefs were not shaken even in an environment where implementation was difficult. The student teacher consistently based lessons around problems, encouraged students to provide a variety of solution strategies, and assessed her students. However, she experienced much difficulty consistently applying her knowledge of her students' understanding when making curricular decisions. Our analysis of this difficulty suggests areas to be addressed during teacher preparation. In particular, the possibility that the process by which inservice teachers assimilate novel pedagogical information may be qualitatively different from the process by which student teachers assimilate such pedagogical information may have important implications for both teacher preparation and teacher inservicing.

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TEACHERS ENTER THE CONVERSATION

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With the emergence of a new vision of mathematics instruction, the mathematics education community finds itself in need of specific, concrete images in order to 1) communicate basic theoretical principles that underlie that vision and 2) identify for discussion issues intrinsic to the practice of the "new mathematics pedagogy." A project was designed to support teachers as they wrote reflective first-person narratives about mathematics learning process in their own classrooms. The paper describes the project and its products and discusses responses to the papers produced. Finally, it considers what teachers learned in the very process of writing such papers.

Out of the convergence of changing social needs with two decades of research in cognitive psychology, a new vision of mathematics instruction has emerged. Though codified in the NCTM Standards and embraced by influential segments of the education policy community, it has yet to be shown what this vision will look like when translated into the day-to-day life of the mathematics classroom. Constructing the practice that will realize the principles animating the Standards has only just begun and it follows from the very nature of those principles that classroom teachers must be the primary agents of that process.

With increased authority--and responsibility--for shaping mathematics instruction in their respective classrooms, teachers will also need to invent new forms of collegiality. They must demonstrate to their peers, as well as to the rest of us, what, concretely, the new mathematics classroom will look like. But once acknowledging this need, the absence of teacher voices from the professional conversation becomes deafening (Cochran-Smith and Lytle, 1990). This has provoked some teacher educators, myself included, to wonder, What are the forms and the forums through which teachers might share what they are learning as they begin to transform their practice along the lines mandated by the Standards?

In recent years, a consensus has been growing that stories are more helpful to people who need to learn to think in new ways about complex, context dependent domains like teaching than are theoretical expositions (Barnett, 1991; Carter, 1993; Shulman, J., 1992; Shulman, L., 1986). Only through the telling of stories about their classrooms can teachers convey the richness and interconnectedness of what they have come to understand--about their students, say, and their schools and communities; about subject matter; about established classroom structures as well as experimental practices--as they face the stream of challenges that constitutes everyday life in the classroom. The current mathematics education literature provides examples upon which such teacher narratives might be modeled: case studies of classroom teachers written by researchers (e.g. Schifter & Fosnot, 1993; Wilcox et al., 1992); case studies conducted by university faculty who also teach K-12 mathematics and make their own teaching the object of their research (Ball, in press-a in press-b; Borasi, 1992; Lampert, 1988, 1989); and cases written by full-time

classroom teachers (Barnett, 1991; Countryman, 1992). Such studies can provide rich accounts of classroom process, illustrating the kinds of dilemmas that arise in daily instruction and how teachers think about and resolve such dilemmas.

In this paper I describe an experimental project which borrows elements from each of these models. Designed to support teachers writing about their own mathematics instruction, the Mathematics Process Writing Project (MPWP) was conducted by SummerMath for Teachers, a K-12 in-service mathematics program. Since 1983, SummerMath for Teachers, which is based at Mount Holyoke College, has offered summer institutes and courses based on constructivist perspectives on learning (Schifter, 1993-a; Schifter & Fosnot, 1993; Simon & Schifter, 1991).

The Project

Although project participants did conduct "research" (defined as systematic and intentional inquiry [Cochran-Smith & Lytle, 1990]), the MPWP emphasized "the teacher as writer" rather than "the teacher as researcher." Its goal was to produce rich, reflective narratives of classroom process which explored teachers' goals and decision making.

The course comprised two major activities: reading assigned materials and writing. The readings were by teachers writing about their own teaching--for example, articles by Ball (in press-a, in press-b) and Lampert (1988, 1989) as well as articles coming out of the writing movement (Atwell, 1985; Hillocks, 1990). In addition to such works, the second and third groups of teacher-writers read papers written by their predecessors. All readings were critically examined for both content and writing style.

The writing component of the course was fashioned after the process-writing model that many of the elementary teachers already used in their own classes. Consistent with the mathematics pedagogy we endorse, process writing involves students working cooperatively to analyze and edit writing projects. For the first several weeks, specific assignments were given so that teachers could begin to explore pedagogical issues and experiment with writing styles (e.g., transcribe a classroom dialogue and then write a narrative, based on that dialogue, about what happened; describe a student who has revealed to you that he/she has learned something that you are trying to teach; write about a student who expresses a mathematical idea that surprises you.) Eventually teachers determined the direction of their own writing and worked on final projects--15- to 40-page reflective narratives on topics of their choosing. Throughout the course, teachers met in both small and large groups to share their work and solicit feedback. All of their work was turned in to me, the project director and instructor, and I responded in writing. Upon request, I met with teachers either in class, in my office, or over the telephone.

The Products

The 49 papers produced by project participants are quite varied. Some of these explore particular grade-specific mathematical topics--third graders working on graphs, sixth graders using Logo to discover properties of triangles, high school students constructing meaning for the concept of variable. In contrast to traditional classroom presentations of mathematics activities, these papers position activities and

problems in the flow of a particular classroom. The reader learns about the teachers' goals for the lesson, about what was happening before a particular problem was posed, what happened afterwards, about the questions students asked, the ideas they proposed, how they interacted with one another and with the teacher, what students learned, what the teacher learned, etc.

Other papers address issues that classroom teachers are likely to face as they engage the new mathematics pedagogy: How does one teach students to listen to one another, work collaboratively, and participate in mathematical inquiry? How does one meet the needs of all students in a mixed-ability classroom? How might published materials be adapted to meet the particular needs of one's class?

In still other papers, teachers write about their own process of change--conquering math phobia, living with and working through conflicting paradigms of learning and teaching, experimenting with new teaching strategies, and engaging in inquiry about students' cognitive constructions.

See Schifter (1993-b) for excerpts of teachers' papers. A subset of these papers will appear in an anthology (Schifter, in preparation) to be published by Teachers College Press.

Responses to the Papers

Since the first set of papers was produced 24 months ago, I have distributed them to teachers and teacher educators along with a questionnaire about them. In addition, my colleagues and I have used the papers in our own in-service courses.

Teacher educators who have used the papers have found that the vivid descriptions of classroom process provide their students with grounding for theoretical principles where contexts for interpreting these abstractions are lacking. For example, one colleague who assigned to her pre-service students a paper about third graders' explorations of multiplication reported that, although her students had been talking about "discovery" and "discourse" all semester long, it was only after reading that paper that they had an image of what those words might mean, concretely, for an elementary classroom. From then on, these students continually referred back to the paper as they discussed instructional principles and possibilities.

In contrast to decontextualized presentations of the theoretical basis of the new mathematics pedagogy, the classroom situations described in the teachers' papers are instantly recognizable to their peers, helping them to interpret those principles in terms of their own teaching. In "Making Graphs Is a Fun Thing to Do," Val Penniman describes her confusion when she realized that students who had followed her from second grade into third could not answer simple questions about graphs even though they had done a hands-on unit together the year before. As she planned her graphing unit for third grade, she decided to make the task more, rather than less, complex--it had to involve a problem which challenged her students to construct new understandings. Her paper, which follows her class through the unit, allowed some teachers to translate a theoretical principle--that the construction of new understandings is stimulated when established structures of interpretation are defeated by novel experiences--to their own teaching situations:

[The paper] changed the way I always thought about helping children to understand a concept. When the children in Val's class . . . encountered difficulties in graphing, she . . . increased the difficulty of the situation. Prior to reading her paper and seeing the response that her students had, I would naturally have simplified the task. I learned that by increasing the difficulty of some tasks, you can encourage better understanding.

Another conception new to most teachers and dramatized in the papers is that mathematics is not a finished body of discrete facts and computational routines, but a dynamic field which involves posing questions, making and proving conjectures, exploring puzzles, solving problems, and debating ideas. For example, in "Decimal Multiplication: Down the Rabbit Hole," Rita Horn leads the reader along the route traced by her students: "We multiplied and got a smaller answer! How can that be?" "I think when you multiply by a decimal, you get an answer smaller than the number you started with." "Not always! Look at when you multiply 3 by 2.7. The answer gets bigger than 3." "So why does it work some of the time, but not all of the time?" And in "Of-ing Fractions," Joanne Moynahan's class of sixth graders is trying to figure out which operation--addition, subtraction, multiplication, or division--fits the following problem: The Davis family attended a picnic. Their family made up 1/3 of the fifteen people there. How many Davises were at the picnic? As readers follow the children's discussion, they are confronted with questions about the meanings of multiplication and division, about relationships among operations, and about the match between a mathematical model and the situation it represents. For some teachers, papers like these provide a first opportunity to think through such conceptual issues for themselves.

I [was] very interested in the math of Rita's paper. I had never thought about decimal multiplication that way and I was intrigued.

Others find that the papers yield new insights into their students' ways of thinking:

"Of-ing Fractions" was helpful to me mathematically. I was at a confused state about how and why multiplying fractions was so confusing to kids and hard to teach and that paper really provided the "momentary stay against confusion" I needed at a certain time.

And still others realize that perhaps their own students could engage in the kind of mathematics portrayed in the papers:

I learned that within most children's conversations about mathematics there are some very important ideas, and if I listened in my own class, I would begin hearing them.

Clearly, it is the very concrete and specific nature of the papers that induced teachers to examine--indeed, for many, to formulate for the first time--their own beliefs about such deep and important matters.

Precisely because the papers were narrated from the teacher's perspective, teacher-readers could see themselves in their colleagues' shoes and begin to imagine themselves in similar situations.

Conclusion

As a social consensus has crystallized around the instructional paradigm perhaps best articulated in the NCTM Standards, teachers of mathematics in the United States are being asked to transform their practice in unprecedented ways. But at this stage in the reform process, the principles that underlie the Standards are largely abstractions: the construction of the day-to-day practice that will concretely realize them has only just begun. The animating premise of the writing project reported on here is that it is the teachers themselves who must, collectively, invent the new mathematics pedagogy and, in so doing, discover what those principles really mean.

However, even as I argue that teachers must assume primary responsibility for producing and disseminating the knowledge necessary for transforming mathematics instruction, I also believe they cannot assume that responsibility until--and unless--they themselves undertake to transform their own teaching. But teachers whose practice enacts traditional paradigms of learning and teaching cannot begin this process on their own. Through in-service programs, teachers must be provided with opportunities--for example, in explorations of mathematical ideas--to experience challenges to their often unarticulated beliefs about learning, teaching, and the nature of disciplinary content.

Writing project participants had previously taken at least one in-service mathematics education course with me or one of my colleagues at SummerMath for Teachers: some had been working with us for as long as seven years; others had entered the program the previous summer and were just now beginning to work through what it means to enact a practice based on a constructivist view of learning. All were reflective about their teaching and all had demonstrated the ability to write fluent prose.

These teachers viewed the writing project as an opportunity to further develop their teaching. It provided yet another context for examining--in a very profound way--the assumptions that form the basis of their practice. In the words of one participant:

> I found the writing process forced a scrutiny of what goes on in my classroom that I have never experienced before--not from having observers in my room, not from being evaluated, and not from writing in a journal. When I first began, it was a painful experience to read what I had written. There were so many incidents and situations that looked different when I read about them that I began to question my teaching skill. Now that I have had a chance to think about the experience for a while, I realize that writing and then reading about what happened puts you and your observations some distance from the situation written about. It allows an objectivity in a more leisurely setting which helps to clarify thinking.

No less important was the collective reflection that took place each week as teachers shared their writing and participants recognized the power of the work they were doing together to construct the new pedagogy. As one teacher-writer put it:

> Meeting with a group of teachers each week helped me not only by giving me feedback on my writing but on the math that was happening in my classroom as well. It also gave me a chance to read and hear about what was going on in other classrooms. Developing this habit of reflecting and sharing has been a pivotal part of my change. It has struck me several times that these are pieces that are often missing from teacher education programs and from our daily professional lives, and for me, these were pieces that were essential.

At the beginning of each MPWP course, participants were anxious, intimidated by the task set for them. These teachers did not think themselves capable of creating a significant piece of work, one that addressed a complex pedagogical issue and honestly represented their teaching. And they were afraid that others would scorn--or, at best, be indifferent to--their work. Thus, they began with short assignments and as they received lots of feedback that pointed out successful writing and identified important ideas, their confidence grew. Writing week after week--for 14 weeks--with encouragement and suggestions from me and from their peers, the teachers slowly developed drafts of their final papers. They then had an additional ten weeks to complete their projects, calling on me and on one another for further support as needed. Clearly, such writing is time consuming, exposing, and difficult. And in the absence of serious, well conceived programs designed to encourage it, very few teachers will undertake it.

Finally, education policy in general, and teacher education in particular, have traditionally shared in what might be called the "culture of professional expertise"--with its mechanisms of legitimation and its hierarchical models of knowledge production and dissemination--that are so pervasive a feature of modern society. Teachers will have to fight for the idea that it is they themselves who must invent the new mathematics pedagogy and that a literature of teacher narratives can give voice to that process. As for teacher educators, we will have to get used to the idea that our role is to help teachers challenge the very culture of expertise that has authorized us to be part of this exciting process.

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Teachers' Beliefs and Attitudes

CHANGING VIEWS OF MATHEMATICS LEARNING: A CASE STUDY OF A PROSPECTIVE ELEMENTARY TEACHER

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Recent mathematics education reform efforts have focused attention on the need to reconceptualize mathematics teacher education. Such a reconceptualization requires an empirical base with respect to teacher development in the direction of mathematics reform principles. This case study examines the experience of one prospective elementary teacher in the context of a two-semester reform-oriented instructional sequence. Our reconstruction recounts Georgia's struggle with a shift in mathematics education paradigms.

This brief report is part of a longer case study of Georgia, which includes analysis of her student teaching experiences. The analysis of her teaching provides additional insight into the nature of her beliefs and understandings.[1] The study was part of the Construction of Elementary Mathematics Project[2], a three-year study of prospective elementary teacher's development which involved two semester-long classroom teaching experiments, the first a mathematics course and the second a course on mathematics learning and teaching. All classes were videotaped, participants were interviewed periodically, and journals and written work were collected. In addition, a group of prospective teachers targeted for more extensive data collection were videotaped (student) teaching in elementary school classrooms, and post-teaching interviews with them were recorded.

Georgia's story is told in chronological "slices," reflecting a natural segmentation. The first slice portrays Georgia before the classes began. The subsequent slices included here detail the beginning of the first semester, and the remainder of the two-semester instructional program.

Prior to Instruction

Georgia arrived fall semester on the main campus of a large university, coming from a small college where her father was an instructor. She had some difficulty adjusting; her trips back home each weekend to visit her family and boyfriend provided emotional support for the transition.

In the first interview, Georgia began to establish herself as one of the most outspoken members of the group. Unlike most of the prospective teachers, she asked a number of questions at the end of the interview: "Why is this so important?" "Why are we doing this interview?" "We can be honest with you - it won't affect our grade?" Her blunt, outspoken manner was to be evident all through the project.

Georgia confessed in the first interview that "I have a difficulty with math. It's not one of my stronger subjects so I thought I'd try [the project course]." Georgia's performance on a project pretest and a series of mathematics problems administered during interviews supported her claims of being weak in mathematics. "I know that when I was in elementary school it was not taught enough, that's one reason I don't like it, and I wish to change that." "I never really had teachers who really took time to really see how I was ...how the progress was coming. It was kinda just hurry up and get this over with so we can move on to something else." For Georgia the poor teaching she had received was the cause of her current lack of confidence and competence in mathematics. "I mean the experience I have had with math teachers is that they're so cotton pickin' cold and hard."

As Georgia discussed her expectations of the pilot course, it became clear that she found fault with the teachers that she had encountered, not with traditional approaches to mathematics instruction. She indicated that a good course would involve large numbers of practice problems, an ongoing feeling of success(frustration was to be avoided), regular praise by the teacher, and significant individual attention. It was important to Georgia to know whether her answers and procedures were correct. As she was working problems during an interview, she looked to the interviewer for confirmation, "I'm not getting any feedback. [pause] Are you an effective teacher? [nervous laughter]"

Georgia was asked to explain how mathematics should be taught so that students could understand it. She said that it would be important to put students in groups so the bright students could help the slower students and that after she had her students work in groups, she would stop the students and have people who got the right answers put them on the board. "If they did it the wrong way or the way I didn't want it done, possibly I could show the way I want it done and the proper way to do it."

We used a mock teaching situation to compare prospective teachers' comments in interviews with their "teaching behaviors." Georgia role-played a teacher who was helping a sixth-grader who was incorrectly regrouping while computing 300 minus 124. The interviewer offered to provide any teaching materials requested. Georgia worked with the "student" by repeating the rules for subtraction, and asking, "Now, do you understand?" When asked if she would do anything differently if she were to do the role play again, she said that she would give him "an example problem ... like 200 minus like, 123 ... because he had trouble with his zeros, so I would use zeros." This statement fits with an idea of competence in mathematics as being able to apply procedures you have learned to a series of closely related problems.

Beginning of the Mathematics Course

Beginning with the first mathematical experience, the project mathematics class ran counter to Georgia's expectations. Georgia expressed her dissatisfaction non-verbally, verbally, and in her journal. She objected to the lack of positive reinforcement and the amount of time spent on individual problems. How could she learn anything if the teacher never told her what was right? Although mathematics classes had not worked well for her in the past, Georgia clung to traditional expectations of a mathematics class. "We should be getting a worksheet every day with at least five problems on, so that we have practice and when we enter a test, we don't have total blank knowledge of what is going to be on there." Speaking up in class #7 she asserted, "I know you won't do this, but I kind of wish you would just like tell us. ... right now I am dumbfounded about [the class process]. ...I just don't see us going forward....I am so frustrated. I just wish...that it would be normal."

This was her most threatening mathematics experience yet. Not only was she thrown again into the frightening sea of mathematics, but this time she was left without many of her traditional life preservers; there was no book to study, no examples for extra practice, and no sample solutions that she could follow in solving similar problems. "I'm getting to the point where I'm getting a little ticked off," Georgia reported with obvious understatement. The videotape of class #9 showed Georgia as not only uninvolved but resistant. She did not participate in the first part of the class- leaning on her elbow and looking bored. When she was assigned a partner and asked to work on some problems with him, she

shook her head and indicated that she did not want her small group participation to be recorded. Her partner tried to interest her in the assignment, but Georgia merely yawned and glared at the instructor.

Another way Georgia resisted the direction of the class was by aligning herself with the other easily identified resistor, Sherry. Sherry contested the instructors efforts by insisting that her rights were being violated and threatening to refer the matter to higher authority. Georgia and Sherry spent time together between classes and supported each other verbally during challenges to the class process.

Georgia seemed convinced that this traumatic experience was the work of an insensitive teacher. "I don't like that night exams.[sic] He doesn't realize that at night you just want to kind of crash. You know, and you are tired. You know, you have a full day. But he doesn't care because it is at his convenience, you know, and he doesn't want to take away from class time which is totally ridiculous."

In her journal after class 9, she wrote, "Lastly I am still very frustrated and feel awkward about saying the frustration is still there. I feel in NO way ready to take the exam nor do I feel that I will be able to do well on it. The reason for this I believe is because nothing in the class has been determined as right yet." The teacher had not respected traditional rules about preparing students for an exam.

Georgia took her exam and the results were more devastating than she had anticipated - no points earned for any of her answers. Crushed and crying, Georgia headed home. Her dad would come to her defense. They talked until late into the night, her father trying to support her but yet suggesting that perhaps the professor might be trying some reasonable approaches to mathematics. During spring break, looking for allies, Georgia described the class to a former mathematics instructor and showed her the exam. She received the confirmation that she sought. The former instructor agreed that Georgia's professor's approach was crazy and that the exam was cruel. Georgia was ready to drop the course.

However, as she sat with her decision to quit, she became less sure that it w s appropriate for her. She would not feel good about herself if she gave up. It would put her a semester behind. Most of her classmates were making it. Georgia returned to campus and confronted the instructor, expressing her anger and hurt and describing her indecision as to whether to continue in the course. He listened and expressed his desire that she do well and his availability to work with her if she chose to continue.. In her journal later that week she revealed her decision.

> In doing alot [sic] of thinking last night and pondering what I am going to do about improving my grade and learning to THINK in the manner that you have set forth. I discovered that I am going to stick with it. I am determine [sic] to work my self [sic] until I achieve in this course What I will expect from you is help, with me as a student you will gein your money's worth as a teacher. I will also get with other people in the class and work as hard as possible to improve my thinking ability.

Georgia had decided to continue in the course and had softened her rhetoric with the teacher. However, in an interview with one of the other research team members, Georgia revealed,

> I think you have to begin to think the way he thinks just, I mean even if you don't want to... and even if you don't believe that's the proper way to think, you have to think or you won't make it in that course. So what I have to do is I have to readjust my thinking just for him. Okay, totally turn my thinking around and then go from there. But I am not saying after this class, I won't go back to the way I thought before

Later in the same interview, she shares,

> I guess I'm just a fighter, you know, and I won't let him, you know, flunk me in this course. Just because I will be darned if I get another zero on the test. And if I do, I think I will take him to the Dean or something because you know. You know I could have not of even taken that test. I could have just gone home, sat in the easy chair, and watched TV and got the same grade I got, you know. And, you know, he said you don't get any points for effort. So you know, I guess effort means nothing to him, you know. An effort is nothing, you know. I mean, if I see, I think in order to be an effective teacher which I think he is not, you have to encourage your kids if they are putting forth an effort, you know.

Georgia had tried resisting the shift in classroom norms which characterized the mathematics class. Her resistance had been unsuccessful. She was now prepared to try compliance, play "his game" so well that she would beat him at it.

The Remainder of the Mathematics Class and the Mathematics Education Class

Georgia's behavior in class began to change. She was becoming more involved, more willing to explore ideas, and seemed more committed to understanding an idea before moving on. Although she still expressed herself openly when she didn't like something, these expressions were about specific issues; the overall attitude of resistance was gone. Slowly, she began to take more leadership in her small groups. Georgia's journal entries considered the mathematics more thoughtfully. She reported that she was spending a lot more time thinking about the mathematics discussed in class and working on problems. She distanced herself from Sherry and developed a regular study group with Emily and Judy, two prospective teachers who were weak in mathematics, but who were enthusiastic about the class and committed to doing well.

What had begun as determination to play the "game" well, had developed into a real interest in the mathematics, and later into an appreciation for the instructional approach. As her newly developed mathematical activity began to give her moments of insight, Georgia began to feel a sense of achievement that was not dependent on praise. Although, overall her mathematical understanding remained weak, she noted her progress and began to value the context which fostered her learning.

In class discussions, Georgia began to wonder about proposed solutions. In one discussion, she encouraged her group to try situation after situation in a search for counter-examples. This emphasis on a solution working continued on into the second semester. In Class 31, she again asked her group, "I mean does that work for every case, like did you already go through these?" Georgia tended to be satisfied if all the examples she tried worked; her approach tended to be inductive rather than deductive. She seemed content if she and/or the group felt that the answers were right. No longer did she look to the instructor, a book, or an outside authority for confirmation.

Beginning to see math as an empirical science, Georgia encouraged her group to mix two types of soda in order to resolve a mathematical dispute about the quantitative relationships involved in concentration. Later, in an interview, she revealed that she had experimented with juice mixtures the previous weekend, but believed that it was important for her fellow students to be involved themselves in the exploration. This seemed to be indicative of a shift in Georgia's beliefs about how mathematics is learned.

In the second semester her tendency to assume a leadership role was apparent in a simulation using base five blocks. She expressed concern that all members of the group were understanding each others' explanations and that they were seeing various ways the problems could be solved. She was fascinated by the problems and pursued some extensions of these problems that she generated herself. Twice during the class period, she commented that she was really enjoying working with the Xmania simulation and intended to use it when she had her own classroom.

In the middle of the second semester, Georgia was asked to repeat the role play she had done before the first semester had begun. In this role play, rather than telling the student the algorithm and asking him if he understood as she had done previously, Georgia asked the student questions designed to help him solve the problem himself.

The following interview excerpts provide additional evidence of Georgia's shifting ideas.

Inter: ... let's say I was to come in [to a hypothetical class] and watch [you teach] for a period of a week or two, what kinds of things would I see you doing?

Georgia: ... you'd probably see a lot of classroom work, a lot of group involvement, a lot of student participation, you'd probably see a lot of, let's see, questions, maybe not always a lot of answers, but maybe a lot of questions.

Inter: How can you have questions without answers?

Georgia: Well, I think I'd try to run it a little bit, my classroom, a little bit like we've been running our math. I'd try to stimulate a lot of questions and let them search a little bit more for the answers instead of just always just giving them the answers.

Inter: Why would you do that?

Georgia: I think that students learn more effectively that way. I know it's worked for me this semester. I think it helps them search and see that there's not just one set way for doing problems...It'll help them understand and it will also help them in their thinking process and I think it will help them later on to develop their thinking in a better way. I mean, you can't, it won't just help them in mathematics it'll help them in all areas I think.

Inter: So, I don't think I understand. You're going to ask your kids lots of questions but not give them any answers.

Georgia: Well, they'll eventually come up with the answers but they won't have the answer right off hand,... they'll have to think about it like we do in class you know. We're not always given answers right away as some of the kids would like us to have the answers right away, we have to think and we have to search, and you know sometimes it's days before we even know whether we're right or not but you know, we've gone through a whole thinking process and I believe my students will have to go through that whole thinking process as well. I think it just makes them a better student and a better thinker. I mean, you have to think about mathematics, you can't just be, you know, I don't know, just all black and white, there's got to be some gray in there, you know.

Inter: What do you mean by some gray?

Georgia: Well, it can't be always right and wrong, there's got to be some area in there about sort of like confusion and maybe frustration and maybe like them searching that gray area. It could be them thinking or maybe some confusion or some questions or things that can't always be right or wrong, you know. It can't always be well this way is right or this way is wrong....Sometimes [not telling the answers] causes a lot of frustration, I mean, sometimes it really causes a lot of frustration. But if he tells you why right away then you're not going to search anymore for anymore answers and you're not going to explore the question anymore. ...

The reader might question whether Georgia was just "playing the game well," speaking the party line. Our strongest evidence that Georgia's shift was real was her perseverance to teach consistent with

this new paradigm of mathematics education in the face of considerable resistance from her cooperating teacher.[3]

Later in the interview, Georgia offered her reconstruction of the change that she had undergone.

> I reached a point where I ... got tired of being a spoiled brat....I wanted it all spoon fed you know, just like all my other classes....[The teacher] just wants us to learn to the fullest capability we can learn. And he wants us to look at a new way of looking at mathematics and that's what we have to do. I guess that's where it all started to click. I can't tell you what day I decided to, like, say, "Okay, math turn on." I guess I just really started working and it just all came together.... There are times in the class where...I look at other people in there and I see that they don't really work as hard as I work but yet they're achieving...beyond me, and, you know, that frustrates me at times because I think I bust my butt in this class to get where I'm at. ...But you have to look beyond that and you have to say, "look, I'm not looking at those people, I'm looking at me and where I'm at, where I was at the beginning of the year and where I've come."...But I can't tell you when it all happened...It was really gradual and I went through a lot of work and a lot of opening up my mind to new ideas.

Summary and Conclusions

In dramatic fashion, Georgia's story highlights some important developmental themes regarding the mathematics teacher education of prospective teachers who are themselves products of traditional mathematics instruction. These include loosening of traditional expectations, participating in a renegotiation of classroom social norms, and developing a personal sense of ability to create mathematics. Georgia had been a weak student, never feeling successful and competent in mathematics. Yet, she clung to her traditional expectations of mathematics class. They represented the known and were therefore relatively safe. Not every prospective teacher resists as strongly as Georgia. However, Georgia's experience invites us to take seriously the emotional content that can be associated with making such a significant paradigm shift. Do our teacher preparation and teacher enhancement programs anticipate and support such emotional upheaval?

The second theme involves engagement in on-going renegotiation of the roles of teacher and students in the classroom, what activities are valued, how mathematics is validated, and what it means to be effective in the classroom. Closely associated with this process is a reexamination of what constitutes mathematics and how it is developed. The third theme refers to the prospective teacher's growing sense that "I can develop and express mathematical ideas, judge mathematical validity, and contribute (as a student) in a classroom mathematics community that is developing shared mathematical understandings." While this brief list is in no way exhaustive, all three of these components seem to be important in the preparation of teachers to carry out envisioned reforms. We would emphasize that we are not merely talking about familiarizing prospective teachers with what goes on in reform oriented classrooms or even giving them a taste of what it is like to participate in them. Rather the issue is one of prospective teachers participating in personal transformation as learners of mathematics and members of mathematical communities Do teacher education programs, particularly those that aim to make an impact with a large number of teachers, adequately plan for such changes?

Although Georgia's story demonstrates important change in these three areas, Georgia's overall knowledge of elementary mathematics improved only slightly, remaining a weak link in her preparation to teach mathematics. A longer, more comprehensive opportunity to learn mathematics[4] in a reform-oriented

classroom is indicated for two reasons. First, the conceptual understanding needed is too vast to be learned in one or two semesters. Second, we suspect that the learning curve starts out quite flat as prospective teachers develop new conceptions of mathematics, develop as problem solvers, and learn to communicate about mathematics. As growth in these areas progresses, the learning curve with respect to conceptual understanding can become steeper. Mathematical interventions of insufficient duration, therefore, risk generating only the flat part of the curve. However, how to quantify sufficient duration will require empirical investigation.

Finally, a large part of learning to be a mathematics teacher takes place in the classroom as the prospective teacher has the opportunity to assume the teachers role. This is beyond the scope of this brief report.

Footnotes

[1]The longer paper is available upon request from the first author.

[2]The CEM project is supported by the National Science Foundation under Grant No. TPE-9050032. Any opinions, findings, and conclusions or recommendations expressed in this material are those of the author and do not necessarily reflect the views of the National Science Foundation.

[3]Data from Georgia's student teaching are reported in the longer version of this case study.

[4]Note the emphasis here is on the teacher's opportunity to learn mathematics. This does not replace the need for teacher's to learn about learning and teaching. Neither can mathematics content knowledge be developed solely through pedagogical courses.

THE EFFECT OF A COURSE EMPHASIZING TEACHING TECHNIQUES CONGRUENT WITH THE NCTM STANDARDS ON PRE-SERVICE TEACHERS' BELIEFS AND ASSESSMENT PRACTICES

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This study focused on the beliefs of pre-service secondary mathematics teachers about the nature of assessment practices, the criteria used to evaluate student performance, the relationship between their beliefs and the evaluation criteria, and the effects that a mathematics teaching methods course may have had on their assessment beliefs and practices. The teachers entered the mathematics teaching methods course with beliefs about assessment consistent with the reform movement and maintained those beliefs throughout the course. Their assessment practices were consistent with their beliefs in areas such as applying holistic evaluation methods and in giving equal value to visual and symbolic representations. With respect to important evaluation criteria, the teachers referred to certain criteria (correct answer, show work, provide explanation) more frequently at the end of the course than at the beginning. Dissimilar application of evaluation criteria to samples of student work raised issues for further investigation.

Achieving the goals set forth in The National Council of Teachers of Mathematics Curriculum and Evaluation Standards for School Mathematics (1989) and its Professional Standards for Teaching Mathematics (1991) depends upon individual teachers changing their instructional practices (Clark & Peterson, 1986; Thompson, 1992) as well as their beliefs about the nature of the teaching and learning of mathematics. Classroom assessment practices, because they reflect beliefs, are an appropriate topic to facilitate reflections by teachers about the teaching and learning of mathematics University courses in methods of teaching mathematics are a logical place for pre-service teachers to begin to think about their beliefs about classroom assessment and to formulate their ideas about assessment criteria (Meyerson, 1978; Schram, Wilcox, Lanier, & Lappan, 1988).

Little is known about pre-service teachers' prior beliefs concerning student assessment or about changes in assessment practices that may result from a teaching methods course that directly addresses assessment issues. The purposes of this study were:

1. to identify the extent to which pre-service teachers' beliefs about and criteria for assessing student performance changed as a result of participation in a course designed to emphasize instructional techniques consistent with the NCTM Standards;
2. to determine relationships between pre-service teachers' beliefs about mathematics teaching and learning and the criteria they use to evaluate student performance;
3. to determine the ways in which elementary pre-service teachers differ from secondary pre service teachers in their beliefs about mathematics teaching and learning and in the criteria they use to evaluate student performance.

Because of page limitations, only portions of the first two purposes will be addressed in this paper.[1]

Method

Subjects

Twenty-two secondary pre-service teachers (hereafter referred to as "teachers)" enrolled in a one-semester Mathematics Teaching Methods course at the University of Pittsburgh participated in the study.

The Course

Two instructors taught a six-credit secondary mathematics methods course two nights a week for 2 1/2 hours each night. Overall, the view presented in the course can be characterized as a combination of what Kuhs and Ball (1986) [cited in Thompson, 1992] term *Learner-focused* and *Content focused with emphasis on understanding* . Learner-focused views the teacher as a facilitator of student learning, creating a learning environment which facilitates student thinking and reasoning. A Content/understanding view makes the mathematics content the focus of the classroom activity while emphasizing students' understanding of ideas and processes. The instructors emphasized the role of problem solving, communication, reasoning, and connections in planning instructional activities and in assessing students' understanding of mathematics. Two class periods were devoted to discussing classroom assessment. The goal of these classes was to attempt to influence the beliefs and practices of the teachers toward developing assessment tasks that would elicit a range of knowledge from factual to higher level reasoning and toward employing a broader range of evaluation criteria than a single correct numerical answer.

Instruments

Each of the following instruments was administered at the beginning and eod of the course.

Beliefs inventory.[2] The beliefs inventory contained 34 statements about the teaching and learning of mathematics, seven of which addressed assessment issues. The teachers indicated the extent of their agreement with the statements on a scale of 1 (strongly disagree) to 5 (strongly agree) and provided brief descriptions of their rationale for the selected choice.

[1] A longer version of the paper which will address all three issues in detail will be available at the conference or can be obtained by writing to the authors.

[2] The Beliefs Inventory is an augmented version of one used by the Documentation component of the QUASAR Project, directed by Dr. Mary Kay Stein. QUASAR (Quantitative Understanding: Amplifying Student Achievement and Reasoning) is a Ford Foundation sponsored project designed to enhance mathematics instruction in middle schools with high percentages of students from economically disadvantaged communities. A summary of the QUASAR Project and papers associated with the Project can be obtained by writing to: Dr. Edward A. Silver, Director, QUASAR, LRDC, 3939 O'Hara Street, Pittsburgh, PA 15260.

Scoring student responses activity. The Scoring Activity packet included three open-ended mathematics tasks[3] (Number Theory, Decimal, Pattern), shown in Figure 1, with each task accompanied by seven or eight sample responses that had been generated by middle school students. Using a range of score levels from 0 to 4, the teachers evaluated each student response and provided a brief rationale for the score level selected.

Criteria for full credit responses. Following the evaluation of each set of student responses for a given task, the teachers wrote descriptions of their criteria for a full-credit response ("4") for that task.

Analysis/Results

Criteria for Assessing Student Performance

Overall approach. To determine changes in the assessment criteria used by the teachers, the overall approach for scoring full credit responses was coded as either Holistic or Analytic. On the average across the three tasks, most of the teachers used a holistic approach both at the beginning and end of the course (93% and 82%, respectively). This change was not statistically significant.

Specific characteristics. Each of the descriptions of score level 4 criteria was coded using categories related to expectations for student work (*Correct Answer*, *Demonstrate Understanding*, *Shows Work*, and *Provide Explanation*), with a single description receiving one or more codes depending on the number of criteria mentioned (e.g., a response of "Must have the right answer with the right explanation" was coded as both *Correct Answer* and *Provide Explanation*). The results of this coding, as shown in Table 1, reveal that for all three tasks references to the category *Demonstrate Understanding* decreased significantly from the beginning to the end of the course. Significant increases occurred for the Decimal task in the categories of *Correct Answer* and *Provide Explanation* and for the Number Theory task in the categories of *Show Work* and *Provide Explanation*.

The results indicate that the teachers were likely paying greater attention to the specific requirements identified in the task instructions. For example, in the Decimal task the significant increases in the categories *Correct Answer* and *Provide Explanation* could be linked to the instructions for this task, which required the choice of a correct answer and an explanation of that choice. Similarly, the instructions

[3] The three tasks, which have been released for public dissemination, were developed by the QUASAR Project as part of its Assessment component, directed by Dr. Suzanne Lane. Information on the assessment component of QUASAR can be obtained by sending an inquiry to the address listed in the preceding footnote

for the Number Theory Task (cf: Figure 1) probably influenced the increase in the references to *Show Work* as an appropriate criteria for a score level of 4.

Table 1 Distribution of Categories Mentioned in a Full-credit Response for the Three Tasks

Category / Task	Correct Answer		Demonstrate Understanding		Show Work		Provide Explanation	
	Begin. of Course	End of Course	Begin. of Course	End of Course	Begin. of Course	End of Course	Begin. of Course	End of Course
Decimal	77%	100%*	64%	41%*	5%	5%	45%	73%*
Number Theory	81%	95%	76%	38%*	14%	67%*	0%	29%*
Pattern	90%	100%	35%	14%*	5%	0%	90%	95%

*$p < .01$

In the case of the Decimal task, a situation emerged that warrants further investigation. Although the teachers indicated that a correct answer and an explanation were important criteria, the score levels assigned to a particular student response to the Decimal Task (which contained the correct answer, but an irrelevant explanation) ranged from 0 to 4. It is unclear as to how the methods course and/or the teachers' beliefs influenced these decisions.

Beliefs About Assessment

On the Beliefs Inventory, seven statements addressed important classroom assessment issues (e.g., assessment for instructional guidance, holistic scoring, alternative solution strategies, assigning grades). Overall, the pattern of responses shown in Table 2 indicates that the teachers' beliefs about assessment were consistent from the beginning to the end of the methods course. In addition, there is some evidence that their beliefs were in alignment with the purposes of assessment as espoused by the mathematics education reform movement (e.g., agreement with statements #1 and #4).

Table 2 Mean Response to the Statements on Assessment Beliefs

Statement	Beginning of Course	End of Course
1. Assessment for instructional decisions	4.0	4.3
2. Written tests as best means to assess progress	2.6	2.3
3. Chief role of assessment is to assign grades	2.0	1.9
4. Some work evaluated in manner similar to essay	4.1	4.3
5. Complete solution may include visual images ...	4.3	4.5
6. Use of English language given equal weight	2.5	2.4
7. Credit for appropriate methods that differ ...	4.5	4.5

Linking Assessment Beliefs to Evaluation Criteria

Statement #5 (A complete solution to a mathematics problem may include visual images or diagrams in addition to equations and numbers.) was one statement chosen to investigate the relationship between teachers' beliefs about assessment and the criteria they used to evaluate student performance because 20 of the 22 teachers either strongly agreed or agreed with this statement on one or both administrations of the Beliefs Inventory. If their beliefs were aligned with practice, then this subgroup of 20 teachers would likely assign similar score levels to a correct response that was based on the use of a diagram or visual and to a correct response that did not contain a diagram. The investigation of this conjecture involved looking at the score levels that the subgroup members assigned to the two sample responses to the Number Theory task shown in Figure 2. Response A included a diagram as part of a correct solution, and Response B did not use a diagram.

For Response A (with diagram), 16 of the 20 subgroup members assigned it a score level of "4" for one of the two administrations of the Scoring Activity, and four members assigned it a score level of "3" on both administrations. This pattern of scoring was consistent with the pattern for Response B (without diagram) in which 18 subgroup members assigned it a score level of "4" for one of the two administrations, and two members assigned it a score level of "3" on both administrations. Thus, the teachers' beliefs in the appropriateness of diagrams and visual images were actualized in that they assigned high score levels to both the response utilizing a diagram and the response expressed in numbers.

Another issue emerged that warrants further study. Of the teachers who assigned score levels of "3" to Response A and B, all of them did so due to a lack of generality of the response (e.g., "[25] is a correct solution, but there are others."). These teachers obviously were reserving the highest score level for a response that went beyond the stated requirements of the problem. The particular aspects of the course and/or their beliefs that influenced this decision are presently unclear.

Discussion

The secondary pre-service teachers entered the course with beliefs and assessment practices somewhat consistent with current reform efforts and maintained those beliefs and some of the practices throughout the course. Further evidence of the alignment between the teachers' beliefs and practices and mathematics education reform emerged from the teachers' use of several criteria associated with

communication and higher level reasoning to evaluate student responses, their acceptance of holistic as well as analytic evaluation methods, and their assignment of nearly identical score levels to correct responses that employed different, but correct strategies. With respect to the assessment component of the methods course, there is evidence that the teachers responded to the idea that teacher expectations of student performance should be made explicit to students in both instructional explanations and assessment practices. The decrease in references from beginning to end of the course to the specific characteristic *Demonstrate Understanding* as a criteria warrants further investigation, as it may reflect a change in the meaning these teachers attributed to the phrase "to know and understanding mathematics" (Wilson, Cooney, & Badger, 1991). Making pre-service teachers aware of their own beliefs and practices with respect to assessment may encourage them to see the value of open-ended mathematics tasks as appropriate measures of students' mathematical knowledge and to include these and other alternative assessment methods in their classroom evaluation practices. Discussions about beliefs and practices can also assist the methods course instructor in designing activities that may influence the pre-service teachers as they re-examine, and perhaps change, their instruction and assessment methods.

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Figure 1. Three assessment tasks used to study evaluation practices of pre-service teachers.

NUMBER THEORY TASK

Yolanda was telling her brother Damian about what she did in math class.

Yolanda said, "Damian, I used blocks in my math class today. When I grouped the blocks in groups of 2, I had 1 block left over. When I grouped the blocks in groups of 3, I had 1 block left over. And when I grouped the blocks in groups of 4, I still had 1 block left over."

Damian asked, "How many blocks did you have?"

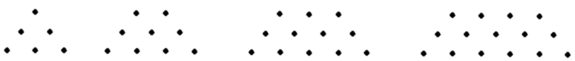
What was Yolanda's answer to her brother's question?
Show your work.
Answer: __________

DECIMAL TASK

Circle the number that has the greatest value.
.08 .8 .080 .008000
Explain your answer

PATTERN TASK

For homework Miguel's teacher asked him to look at the pattern below and draw the figure that should come next.

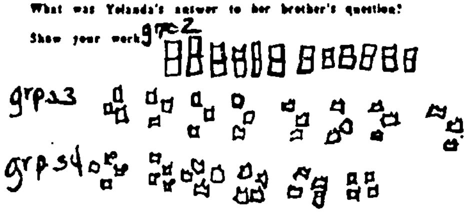


Miguel does not know how to find the next figure.
A. Draw the next figure for Miguel.
B. Write a description for Miguel telling him how you knew which figure comes next.

Figure 2. Two selected responses to the Number Theory task

Response A (with diagram)

What was Yolanda's answer to her brother's question?
Show your work.



Answer: 25

Response B (without diagram)

What was Yolanda's answer to her brother's question?
Show your work.

2,4,6,8,10,12,(13) 1 block left
3,6,9,12,(13) 1 block left
4,8,12,(13) 1 block left
Each time there was one block left

Answer 13

SHARED AUTHORITY: A ROADBLOCK TO TEACHER CHANGE?

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Margaret, a teacher participant in the Atlanta Math Project, is attempting to change her pedagogical practice to reflect the reform recommendations from NCTM (1991). This paper suggests shared authority as a framework to help analyze and interpret the seemingly random patterns of change observed in the first two years Margaret has been with the project. An interpretation of the concept of shared authority is offered and examples from the data are provided.

Mathematics instruction in the United States is experiencing careful scrutiny. In line with current research on learning, The Professional Standards for Teaching Mathematics (NCTM, 1991) is recommending pedagogical practice that supports student communication, reasoning and problem solving. Teachers in the field who are attempting to refine their practice to more closely mirror these recommendations are often experiencing a difficult and complicated struggle as they attempt to change. This paper will tell a piece of one teacher's story.

Background

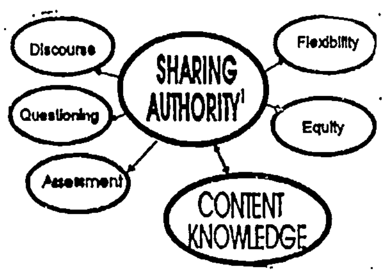
In earlier papers I described the research framework (Hart, 1991) and analysis of year one for Margaret (Hart, 1992; Hart and Najee-ullah, 1992), a teacher participant in the Atlanta Math Project. This paper will continue the discussion through Margaret's second year with the project. Four video tapes were chosen for analysis: two tapes previously discussed for year one (9/20/90, 5/3/91) and two tapes for year two (9/6/91, 3/19/92).

Choosing a Perspective

Perhaps the most difficult aspect of telling Margaret's story is the selection of what to tell and what to leave out. My teacher change paradigm acts as a filter to screen data and may cause me to interpret what I see from only this perspective. But even within the plethora of what is, I must make decisions of what is representative of the evidence of change observed in Margaret's teaching practice and hence offer a subjective view.

It is within this confusion that the notion of shared authority within the classroom has emerged as a piece of the framework to help me understand Margaret's inconsistent behavior over the first two years of the project -- as well as the behavior of other teachers in our project. This invisible data becomes more apparent as I rethink my discussion of Margaret in year one and add to it the observations from year two. Much of the seemingly unrelated behaviors begin to connect and a larger theme begins to emerge.

What do I mean to share authority in the classroom? I picture this as a continuum upon which classrooms might be placed. Even individual events occurring over the course of a single class period might be placed on various points along the continuum. In a lecture-oriented classroom the intellectual authority is almost exclusively in the hands of the teacher. In a classroom where student share their ideas verbally or come to the overhead projector and describe their strategies, there is more sharing of meaning making. Further along the continuum student thinking is genuinely considered and is not always filtered through the teacher, approved by the teacher or given credence by the teacher. In a classroom where the intellectual authority is most fully shared the students would participate in the decision making for acceptable justifications and arguments. As shown in Figure 1, the willingness to share authority impacts many other aspects of classroom interaction and decision making.



Introducing the notion of shared authority in the classroom -- as I believe the Standards (1991) do suggest -- raises issues of propriety, power and ownership. As students begin to take more responsibility for the intellectual life in a classroom they struggle with prevailing customs, situations of who owns an idea and the standing of the person presenting a position. Individuals who are considered to have more power, e.g., the teacher, will receive more unspoken authority. As the teacher moves to open up the intellectual climate, he or she must begin to relinquish his or her control and power and begin to acknowledge the varying positions that might be held. For many, teaching mathematics encompasses their professional identity, a position of authority which they control and orchestrate. Depending on the individual, sharing the power in the classroom may not be an easy task because they may be unwilling to relinquish the authority they associate with being a "teacher". It may help to explain the difficulty many teachers are experiencing as they attempt change.

A second consideration that impacts the notion of shared authority as a major construct in teacher change is the domain of mathematics itself. Long viewed by the masses as the most certain, rigid and fixed of the sciences, the teaching and learning of mathematics has fostered very strong beliefs about what it means to know and do mathematics. To share authority suggests that varying perspectives about mathematics exist and that they may be valuable within the mathematics community of the classroom. This is a potentially controversial position. Perhaps certain individuals (or groups) look at the mathematics differently, not inadequately, just differently.

Margaret's Story

It is within this thinking that I come back to Margaret. In previous papers (Hart, 1992; Hart and Najee-ullah, 1992) I describe some patterns of discourse from the two year-one tapes of Margaret's classroom. I identified teaching behaviors which I believed discouraged student discourse. These included Margaret's tendency to:

- funnel student thinking through directed questioning, e.g., a student would begin with an idea and Margaret would lead him or her in a particular direction through machine gun type questioning;
- mirror student expressions without reacting and interpreting, e.g., a student would say "you get the same answer if you reverse the order" and Margaret would repeat the words verbatim without comment;
- or completing student statements, e.g., a student would start a statement like "not everyone gets singles" and Margaret would add to the thought by saying "so you think it was less?".

I found funneling and completing to be the most discouraging behaviors, with mirroring having almost no effect. As I began to study the tapes from year two, I observed similarities between the beginning of year one and the beginning of year two tapes. At the beginning of each school year Margaret was working very hard to engage the students and to facilitate the discussion. This could be interpreted as the adjustment of the students to an environment that invited their participation rather than Margaret starting over. At the beginning of year two the students were just as resistent to change as the students had been at the beginning of year one. However, in all four tapes I found examples of the behaviors I had initially identified in year one -- but the lessons in year two felt different. Margaret was different but it was not obvious (to me) what was causing this feeling. It was at this point that Jones (personal communication, February 24, 1993) outlined themes she was observing in field notes from the Atlanta Math Project at a project research meeting and the notion of shared authority began to make sense.

The Analysis Revisited

In order to explore shared authority as a larger theme that guided Margaret's practice, I decided to reanalyze the tapes using that perspective. As mentioned before, in the eariler analysis I had focused on behaviors that I perceived had limited or encouraged discourse. I had identified three behaviors:

funneling, mirroring and completing. Because of space limitations I will focus on only on the first for elaboration.

Funneling was evidenced in Margaret's classroom when a student would make a statement that Margaret apparently thought was at the beginning of a "correct" track and would use successive questioning to lead the student to a desired outcome. Margaret showed several examples of this in all four tapes. The following example demonstrates this practice at the beginning of year one. When a student is trying to figure out how to estimate the amount of money taken in from locker rentals at his middle school, Margaret engages him in the following exchange.

Student	It depends on how many lockers there are.
MARGARET	Oh. It depends on how many lockers there are as to how much money was taken in? Or how much money could have been taken in?
Student	Yes. If it was $2.00 you times it by two.
MARGARET	About how many lockers do you think there were?
Student	800
MARGARET	So take the number of lockers . . .
Student	1400 cause I just figured it out
MARGARET	Take the number of lockers times $2 per locker and . . .

In this situation Margaret was leading the student step-by-step to explain his thinking to the class. She appeared unwilling to accept his abbreviated explanation.

An example can also be found in the last tape of year two in a seemingly rote interaction. Margaret is leading a discussion on fractions equivalent to 1/2.

MARGARET	2/4 is equal to 1/2. What else?
Student	4/8
MARGARET	How did you get 4/8?
Student	Multiplied by 2.

MARGARET Multiplied what by 2?

Student Wait, multiplied 1/2 by 4.

MARGARET Multiplied by 4?

Student 4/4ths

Both discussions are funneling students to a desired outcome through successive questioning. On initial observation the first dialogue may seem richer because of the task. But careful examination of the two interaction illustrates some subtle differences. In the locker discussion Margaret is carefully directing the line of discussion even though the student apparently would like to proceed without her. She keeps bringing him back to what she thinks would be the appropriate next step for his explanation. In the later discussion Margaret is asking the student through questioning to explain how she arrived at the answer. She is not feeding her the next lines of an algorithm.

Using the framework of shared authority to re-interpret the two examples, Margaret is giving more authority or freedom to think to the student in the second dialogue. Her questions are less guided. She asks the second student "how did you get it?" -- a question that is not asked in the first example.

Funneling is not new to the mathematics classroom. Almost anyone who has observed teachers has observed the practice. Often seen as a protective mechanism to assure that all the other students in the class "get it" and that the student speaking really does "have it," this practice of taking care suggests that the teacher knows the "it" at which they should arrive. The change in Margaret's teaching practice has been in giving more authority by asking more open questions that let the second student determine some of the outcome of the questioning. She has not abandoned her earlier behaviors, but the essence of her teaching is different. She has relinquished some of her intellectual authority in the classroom.

Discussion

The discourse in Margaret's classroom over the first two years of the project is not markedly different. A closer analysis through the lens of shared authority, however, caught subtle differences in the nature of some of the exchanges. She was beginning to relinquish some part of her power, but the letting go apparently was not easy, given the subtle nature of the changes I observed. It is impossible to say at this point the motivation for Margaret's difficulty in sharing the authority in her classroom. Whether it was taking care of her students or protecting her own position of power in the classroom, the struggle is apparent in the classroom.

Further analysis of interviews with Margaret and field notes of the observations should begin to enhance my understanding of the process of change Margaret is experiencing. It is clear, however, that the process is very complex and not easily reducible to sets of predefined behaviors.

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AUTHORIZING MATHEMATICAL KNOWLEDGE IN A CLASSROOM COMMUNITY

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This paper reports on one aspect of a three year Constructivst Elementary Mathematics (CEM) study of preservice teachers' development of pedagogical and mathematical knowledge in a reform oriented math education program. The paper focuses on the symbolic interactions of classroom participants to seek an understanding of the shared community beliefs about what it means to know mathematics and the location of authority in this mathematical community. The students and teacher co-constructed a classroom practice which emphasized 1) the invention of possible problem solutions, 2) the need to fully explain and justify each solution, 3) the generation of examples and evidence based upon life experience to prove the workability of solutions, and 4), the use of these experiential truths to construct mathematical generalizations. The symbolic interactions created a tension between an inquiry context, in which knowledge construction was a shared responsibility, and an evaluative context for knowing, in which authority rested primarily with the teacher.

> So, sometimes we find out we're right by some authority saying, "yes, you're right," or by some kind of conclusive proof. Other times it's by compiling as much evidence as we can and coming up with our very best model that we can really feel secure in. So I offer you another model of knowing, one which I'm going to push on more in this class. O.K. (Simon, Class 5, 2/5/91)

Simon, the teacher, made this point for his students after playing a hidden object-twenty questions type game. Several of the students were frustrated that he refused to show them the object, arguing that they could ask better questions next time, if he told them what was right. Other students said they would stop thinking and just wait to be shown the right answer if he always revealed the truth at the end of a task. And, one student, Lois, argued that being right should have nothing to do with how deeply you think.

> It's not, it doesn't matter if you're right. The whole point is, you're coming to the conclusion like, how he did it cause at first it seemed kind of like a calculator, you know? . . . then as we thought about it more and we took certain steps then we came to the conclusion that we thought we knew what it was. So, . . . it's not so much you know exactly what it is, I mean, it's nice if you're right. But it's the process leading up to it. I think it could be this, so I'll put one and two together and I'll get this. You know? (Lois, Class 5, 2/5/91)

By the end of the semester, Simon and the students had indeed co-constructed the compiling evidence model of knowing he promised to "push on;" however, along with this more inquiry context, they also co-constructed a hierarchical knowledge community in which the teacher possessed the knowledge, posed all the timely problems and questions, and judged students' work according to mathematical knowledge which he possessed as a member of other mathematical communities. These two contexts--one of inquiry and one of evaluation-- created a community tension. The vast majority of the students entered the classroom experience having understood mathematics learning only within the context of evaluation:

Above all, I think mathematics needs to be clearly explained [by the teacher] in a way that students are able to understand. This involves keeping in mind that students do not have any knowledge, such as yourself, about what is going on. It also entails spending a great deal of time by the teacher analyzing each individual's mistakes and successes. (Linda, Focused Writing #1, 1/15/91)

Initially, Linda's beliefs were common among class members, and influenced the interpretations of Simon's actions as he directed the problem solving activity of the students. However, the community clearly moved from Linda's idea, that mathematics needs to be clearly explained by the teacher, into an inquiry context in which learning is a consequence of sharing ideas about how to solve specific mathematical problems. This development is evidenced by the students' suggestions for changing the format of the final exam.

I don't know, like I can tell though by working in the groups who has developed a more, um, like open way of thinking, like even though they may never get the solution to the problem they still explore many different avenues and you can tell when someone's really exploring a lot and someone that says, "no, this is it." Like, I think when someone says, "no, this is it," then they haven't got a grasp on what the class is all about. (Penny, Class 21, 4/18/91)

I kind of think it should be more like our journals, I mean, in there you got whether or not we understood the problem by what we said. And isn't that what we're doing? (Molly, Class 21, 4/18/91)

I sympathize with you having to evaluate this class cause I think it's very difficult but one of the things that I think everybody is saying is that they have gotten is an idea about how to problem solve... And that perhaps our test could be somewhat set up that way... this is what we've been doing all along is, I mean a few times we've been sent home saying, "O.K., go ahead and work on, do these problems at home and then come in," but it's always come in and talk about it with your group, so that somehow if it could be rearranged so we could work on questions together as a group. (Lilly, Class 21, 4/18/91)

The construction of an inquiry context supports the findings of Wilcox, Schram, Lappan, and Lanier (1990), who "observed a shift in the locus of epistemological authority -- from a reliance on the teacher to their community of classmates and teacher together using mathematical tools and standards to decide about the reasonableness of processes and the results of investigations" (p. 25). However, in the CEM community, the students supported both the social context of learning together how to problem solve in an inquiry community, and the role of the teacher to judge the level of students' individual, mathematical understandings. The symbolic interactions within classroom events provided in this paper, illustrate how members negotiated and constructed these contradictory contexts-- or models of knowing-- within their mathematical community.

The first problem of the semester was given to small groups with the explicit instructions to "come back and tell the group how you went about doing this task. Not just a number that you came up with" (Simon, Class 2, 1/17/91). And groups were told how to work together:

> What I don't mean is a whole bunch of people working independently on the task and checking their answer at the end to see if they got the same answer. . . When I say working as a group, I'm talking about cooperating all through the activity and discussing what you want to do and why you're doing it and making sure that every member of the group understands what's going on. (Simon, Class 2, 1/17/91)

And, when the small groups were ready to report, Simon explained his use of wait time and student paraphrasing of ideas expressed in class:

> The first thing is, I want you to consider what would happen if I asked an interesting question and then somebody answers that question immediately. . . Now I believe that not all questions require the same amount of time to think So I would like to control how much time you have to think. . . So, what I would like you to do is when you want to respond or you want to say something in class, that you raise your hand. . . Second thing, as I started to indicate before, I believe that it's everybody's job in here to understand everything that goes on. . . That has nothing to do with whether it was right or wrong. We can't even respond to it until we have understood it. . . If you don't understand what somebody says I would like you to ask them a question. Sometimes it's as simple as, "Could you just say that again." . . Other times we don't understand what they mean by some things. . . So you ask some question that gets you some clarification. . . And, what I will do at times is I will ask you for a paraphrase, that is that Sara said something and then I turn to Penny and I say, "Penny, will you paraphrase that." . . . Now when I ask for a paraphrase, it doesn't mean that the answer was right or wrong. It only means that I, that I judge it's worthwhile stopping to make sure people understand. (Simon, Class 2, 1/17/91)

After these instructions, the class began their first mathematics discussion. Simon clearly directed the conversation, asking for clarification and paraphrases, with a teacher-student turn-taking pattern.

Simon: All right. Umm, let's, let's hold on to Lilly's question a second. O K. cause I have a feeling that she is going to ask you how do you know it's wrong. O K. So we are not done discussing it yet. But let's make sure that we understand the method that was intended and then we can figure out how to answer this question. Did, did everybody start out by doing something about measuring along one edge and then along the other edge? Is that, is, I saw a lot of that going on. Were there any groups that did not do that? O.K. Now after you measured, Georgia along this edge and along that edge, you did what with your number?

Georgia: Then we multiplied them.

Simon: You multiplied them?

Georgia: To get how many squares there were.

Simon: Were there any groups that did something different than to multiply those two numbers?

Molly: We had to multiply and then add a little section that was like out there

Simon: But the first step was still multiplying?

Molly: Yea.

Simon: O.K. So there seems to be, everybody seems to have done that much. Right? Measure both edges and multiply. Why? The job is to cover this whole table. Why did you multiply? [pause] Bobbie?

Bobbie: All right it seemed like the easiest way to come up with an answer. You know, we tried to get, we tried it both horizontal and we tried it vertical like the cardboard on the table to see if we could get it in, if the cardboard would fit evenly in the table and then all's we would have to do is multiply two numbers and come up with one answer. But we had this, we had a piece left over here so we actually turned the card up the opposite direction. Like perpendicular to the way we had it. . .

Simon: O.K. So that sounds like it is related to what you are sharing Georgia. O.K. Let's backtrack a second. Why is it, you said it was an easy way to get an answer. Is it, is it an easy way to get a correct answer? Now why, why measure along an edge and another edge. How is doing that and multiplying two numbers related to covering this whole table with rectangles? You seemed to all think that was a good way to go about the problem. Why do you think that was? Deb?

Deb: Because, umm, in previous, previous math classes you learned the formula for area is length times width. I guess. So probably everybody has the idea.

Simon: All those evil math teachers you were talking about before and now you are taking their word for it? How do we know if they are right?

Molly: Because they showed us.

Simon: Blind faith?

Molly: The teachers. They showed us how it worked.

Simon: Ahh, what did you think? Jonnie?

Jonnie: They showed, umm, it for example if you put down all the squares, all the rectangles. And then you times like the width and the length together and you got an answer and then you added all of them together and the answer was the same. Both ways. So you knew that that method the length times the width would work because they had us add them up after we times'ed them.

Simon: Do you understand what she's saying?

Class: Ah hum.

Simon: She has gotten something that she is visualizing in her mind. Right? Who can paraphrase what she is saying? Judy? (Class 2, 1/17/91)

Simon symbolized his position as mathematical authority by deciding which statements should receive attention, delaying questions by Lilly, Molly, and Bobbie, and deciding which statements should be paraphrased as in the case of Jonnie's visualization above. The students' initial responses were to comply with his expectations: "We have to say why we did. We have to explain it" (Lois, Small Group #1, Class 3, 1/22/91). Then, the inquiry context for mathematical knowing evolved out of this ever

present demand to explain and justify solutions. Simon emphasized the need to provide evidence in a justification and encouraged students to use all of their knowledge and tools including drawing, life experiences, and manipulatives, to model why a solution works or doesn't work. He often asked questions similar to: "Umm. I don't want to know yet, does it work? I want to know what it would mean to work?" (Simon, Class 6, 2/7/91). But, early on, most of the students did not seem to know how to compile the evidence needed to prove a problem solution right or wrong, interpreting the goal of math activity as coming up with an acceptable verbalization: "Yea. See the thing is we know, we know which one is right. But we just don't know how to say it. So we'll be stuck on this problem for another 10 days till we know how to say it" (Eve, Small Group #1, beginning of Class 15, 3/21/91). The taken-as-shared understanding about how to problem solve began later in Class 15:

Simon: There sees to be a fundamental problem that has existed for several days. .. You're comparing two slopes right. We have a slope here and we have a slope here. . . method A says that this one is steeper and method B says this one is steeper. Is that O.K.? . . . Linda? No? So what do you want to know? [students talking --"which one's right"] And what would tell you which one is right?

Sara: You've got to prove the other one wrong.

Simon: How would you do that?

Sara: Through examples

Simon: Examples of what?

Sara: I don't know [laughter].

Lois: Just draw them, the examples you think of.

Simon: Draw them? O.K. So what would you be exploring when you draw it? (Class 15, 3/21/91)

After small groups, the community then used drawing to finally share sufficient evidence to prove that a ratio formulation worked for determining the steepness of a slope, and an additive approach did not. Before this event students would often go to the chalkboard to illustrate their justifications for a solution (occassionally to generate and compile evidence), but the middle of Class 15 seemed to mark the point where students realized that generating evidence was an essential mathematical process for their inquiry community, and saying or drawing things correctly for the teacher was not the central issue. Later in the class the inquiry was opened up even further when Eve and Lilly posed questions about what a ratio answer of 3 represents and how road signs which indicate a 10% downhill grade are connected to the ratio method. These student originated questions were the first time that someone other than the teacher engaged the class in an extended inquiry (the discussion took the last 38 minutes of the hour and a half

class). The inquiries which consumed the remaining 10 classes included the use of many life related situations and objects to compile evidence and to examine which solution seemed to hold up best through all the analogous workings. They were also marked by more and more student originated questions of each other as the community members sought to better understand each other's explanations and justifications: (colors in the following episode refer to manipulative blocks)

Eve: Yea. Isn't that what we said? Are these two things the same thing? That's what I'm asking.

?: Yes.

Eve: O.K., so if they're the same thing and if you add a strawberry into this one, and a strawberry into this one, then why does that change the taste? I don't understand.

Tammy: Because, the one on the right has a smaller concentration of liquid and you're going to add pure strawberry to that.

Mabel: Say you have two cups of water on the right hand side.

Eve: What's water? White?

Mabel: A cup of water is an orange and a white together, so that's two? Two layers?

Eve: I'm lost.

Mabel: I think you might understand this. Here, let me use the green here. Say this is a cup of water and this is a cup of water. So there's two cups of water, this would be three cups of water, O.K.? Say this is pure water, if you put a drop of green dye into this one and a drop of green dye into this one, this is going to look darker than this one. Because there's less water for the green dye to go on. So the same with the strawberry, it's going to taste more strawberry because there's less.

Eve: O.K., I got it.

Simon: It's a really elusive idea. Alright. Here's a question. . . . (Class 25, 5/2/91)

Simon pressed the community to go beyond just explanations and justifications and move this experiential knowing into mathematical generalizations. He did this in Class 6 when he questioned the class acceptance of Bobbie's suggestion that "when you change its shape you're going to change its area," and in the last class:

Simon: All right, here's the question I've been trying to get to. Can you relate the problem of the cubes four, three versus three, two, to any of the other problems we've worked on this semester?

Georgia: So, we're dividing those two to find the overall taste or percentage or whatever you want to say of strawberry or blueberry in that, and with the slope problem we were dividing height over length of the base to find a percentage of the base or the slope. (Class 25, 5/2/91)

Throughout the semester, the students saw Simon as possessor of mathematical knowledge. However, the data suggests that the inquiry context the students progressively constructed focused primarily upon the

generation of evidence to evaluate student originated solutions, especially those activities which involved the manipulation of materials and concrete visualizations.

Georgia: I have a question. Dr. Simon, how come you always quit when we're doing something fun?

Simon: Because we're always doing something fun!

Georgia: And when we're doing something boring, you don't quit.

Simon: Can you help me with some guidelines. How do I know when we're doing something fun?

Sara: If people are actually volunteering to answer your questions.

Georgia: Yea, I know, look at all the involvement you had today.

? [many people talking]

Bridget: When we have something to play with.

Simon: First of all, are people agreed that what you were just doing is fun?

Sara: I don't want to disagree, but I want to say that this was fun because it's sort of new too. If we're still doing this in four weeks, it's not.

Simon: That's helpful. Emily?

Emily: But it's fun to have like stuff to manipulate, you know what I mean? For me, that's the fun part, to make it different ways, try to look at it different ways instead of just like always thinking of everything in your head, to be able to see it.

Simon: O.K. Judy?

Judy: Well I agree. It just makes it easier for you to visualize different ways of doing things. Instead of just thinking of it, you can actually see it in front of you and try it. See if it works (Class 21, 4/18/91)

The members of this mathematical community experienced a tension between contexts of evaluation and contexts of problem solving inquiry. The teacher described a personal tension between an evaluative role and promoter of students' spontaneous activity. Together, the community members authorized a shared knowledge about ways to go about fully explaining and justifying solutions to teacher posed problems. They simultaneously positioned the authority for generalized mathematical understandings in one community member, the teacher, continually symbolized by his role as the leader of classroom talk, and ultimate judge of each individual's learning ability and progress. A greater understanding of the symbolic actions, and the resulting tension between contexts, should inform future reform efforts which hope to position even greater authority for mathematical knowing in the shared activity of a community whose members are producing, critiquing, and generalizing from models which attempt to represent mathematical relationships in everyday life.

Wilcox, S.K., Schram, P., Lappan, G., & Lanier, P. (1990). The Role of a Learning Community in Changing Preservice Teachers' Knowledge and Beliefs about Mathematics Education. Boston, MA: Paper presented at the annual meeting of the American Educational Research Association.

ON TEACHERS' CONCEPTIONS ABOUT THE ROLE OF ANSWERS IN SOLVING MATHEMATICAL PROBLEMS IN ESTONIA AND FINLAND

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Summary: *The purpose of this paper will be to answer the following specific question: What differences are there in teachers' conceptions in Estonia and Finland, concerning the role of answers in solving mathematical problems? The findings are based on the results of a questionnaire study explored middle school teachers' conceptions (N=100 in each country) of mathematics teaching in Finland and Estonia. The main findings were: The Estonian teachers placed more emphasis on the role of the answer than their Finnish colleagues. But the conceptions found were not coherent with those revealed by Thompson. The Estonians agreed only with the statement concerning routine tasks, all other agreement percentages were low for both countries.*

When considering the theory of constructivism as a basis for the understanding of teaching and learning mathematics (e.g. Davis & al. 1990), it follows that teachers' and pupils' mathematical beliefs take on a key role when trying to understand their mathematical behaviour. During the last decade, many studies on teachers' beliefs were undertaken. Furthermore, Thompson (1992), and earlier Underhill (1988), has compiled a review of results based on teachers' beliefs. However, there are only a few international comparison studies of teachers' beliefs (e.g. Kifer & Robitaille 1989, Moreira 1991).

In this context, we understand beliefs as an individual's subjective knowledge of a certain object or concern for which knowledge there may not necessarily be any tenable basis in objective considerations. Furthermore, we may explain conceptions as concious beliefs, i.e. we understand conceptions as a subset of beliefs. One may find a more detailed discussion of the concept 'belief' e.g. in the paper of Pehkonen (in press).

An international comparison of teachers' beliefs

The findings dealt with here are based on a comparative study which was

carried out during the years 1987–90 and explored teachers' conceptions of mathematics teaching in middle schools in Estonia and Finland. The purpose of the study was to clarify two main questions: What kinds of mathematical conceptions do Estonian and Finnish teachers have? How are these conceptions similar, and how do they differ from each other? Since the international comparison of teachers' beliefs is an unknown field, undoubtedly the first type of research to be conducted should be a survey. And the survey would be best realized through a proper questionnaire, and with a sufficient number of test subjects.

In the questionnaire[1], there were 54 structured items about different situations in mathematics teaching. The teachers were asked to rate their views within these statements on a 5-step scale (1=fully agree, ..., 5=fully disagree). At the end of the questionnaire, there were four open-ended questions about the main difficulties in realizing mathematics instruction.

In both countries, the administration of the questionnaire was done separately by each author. We decided to have responses from about one hundred teachers from each country. The data was gathered in two different ways (EW = Estonia, SF = Finland): One part of the sample consisted of teachers on in-service courses ($N_{EW} = 76$, $N_{SF} = 52$), where the questionnaire was filled in at the beginning of the course. And the other group of teachers were reached by post ($N_{EW} = 30$, $N_{SF} = 34$), i.e. the questionaires were sent to them. Thus, there were altogether 106 responses from Estonia and 86 responses from Finland. The administration in Finland of the questionnaire was carried out in November 1987 (the first phase) and in December 1988 (the second phase), and in Estonia both phases were realized in September 1990.

When considering the results of the questionnaire, the statistics used were mainly on the level of percentage tables. The structure of the questionnaire gave an opportunity to say something about the teachers' conceptions. The reliability of the results was estimated with the test-halfing method. The final report will be published within a year, probably in the research report series of the Department of Teacher Education at the University of Helsinki.

[1]Bernd Zimmermann (Hamburg) developed the questionnaire for the research project "Open Tasks in Mathematics" (cf. Pehkonen & Zimmermann 1990).

The focus of this paper

According to Thompson (1989), the teachers' beliefs she found about problem-solving were condensed into five basic beliefs in which the role of the answer was central. In this paper, we have tried to figure out what might be the conceptions of the Estonian and Finnish teachers about the role of the answer, and to compare them with those found by Thompson (1989). The purpose of the paper will be to answer the following specific question: What differences are there in teachers' conceptions in Estonia and Finland, concerning the role of answers in solving mathematical problems?

Some preliminary findings

We shall look at the teachers' responses to certain statements in the questionnaire which are connected with our research problem.

<u>The answer is most important</u>. The content of the three statements (26, 41, 43) pertain to the first belief *(It is the answer that counts in mathematics, once one has an answer, the problem is done)* in Thompson's (1989) categories:

26 = "When solving problems, pupils should above all reach the right result"

41 = "When checking the class work, one should take into account, above all, the results of the tasks"

43 = "In continuous evaluation, one should take into account above all the solutions of the presented problems"

In Fig. 1, we have the percentage distributions of the responses to these statements.

26: Both teacher groups took a clear stand against Statement 26 (Fig. 1). In addition, the original 5-step frequency distributions were compared with the chi-square test, and the chi-square reached was 20.0 and the error percentage 0.05 %. Thus, the difference between the two distributions was statistically very significant. 41: In Statement 41, the teachers did not take a clear stand for or against the statement (Fig. 1). Here, the difference between the distributions was not statistically significant (the error percentage 5.4 %). 43: In Statement 43, about one quarter of the responses were neutral, and the responses for agreement and disagreement were about fifty–fifty (Fig. 1). The difference bet-

ween the distributions was not at all statistically significant.

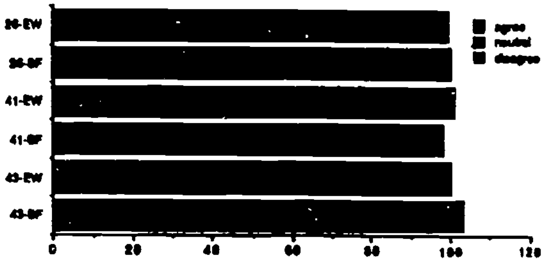

Fig. 1. The percentage distributions of the responses to statements 26, 41 and 43 (EW=Estonia, SF=Finland).

Both groups of teachers were not ready to stress the point that reaching the right answer was most important for the pupils. As a group, they took a neutral stand towards the role of the answer in checking the classwork, and in the evaluation of the presented problems. In both cases, the responses were about fifty-fifty. In the first statement (26), the Finns disagreed more strongly than the Estonians about the importance of the answer, the difference was even statistically very significant.

There is a unique procedure. The three statements (20, 31, 38) are connected with the fourth belief *(Every context (problem statement) is associated with a unique procedure for "getting" answers)* in Thompson'a (1989) categories.

20 = "Pupils should have to experience that one can reach the same result through different methods"
31 = "One should solve, as often as possible, such routine tasks where the use of the known procedure will lead to a sure result"
38 = "Pupils should develop as many different ways as possible of solving a problem, and these should be discussed during instruction"

It is clear that Statements 20 and 38 should be reversed first. In Fig. 2, we give the percentage distributions of the responses to these statements.

20: Both groups of teachers agreed somewhat with Statement 20 (Fig. 2). Between the groups, the difference was not at all statistically significant. 31: In Statement 31, there was no unified view in the responses of the two countries.

The Estonians were clearly for the statement and the Finns more against than for (Fig. 2). The comparison of the original 5-step frequency distributions with the chi-square test gave the chi-square 48.6, and the error percentage under 0.01 %. Thus, the difference between the distributions was statistically very significant. 38: Both groups of teachers agreed with Statement 38, but the Estonian teachers reacted more strongly than their Finnish colleagues (Fig. 2). As we compared the two original 5-step frequency distributions with the chi-square test, we got the chi-square 29.0, and the error percentage under 0.01 %. Thus, the difference between the distributions was statistically very significant.

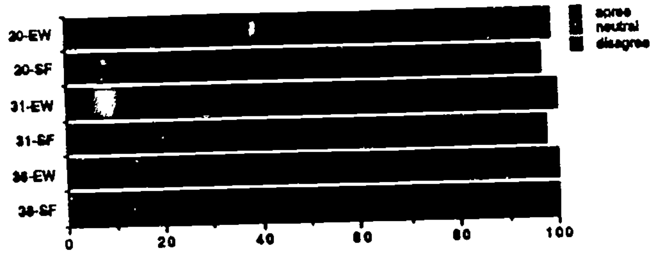

Fig. 2. The percentage distributions of the responses to statements 20, 31 and 38 (EW=Estonia, SF=Finland).

The teachers did not have a unified view on the point of whether pupils ought to have experience using different methods. But they all agreed strongly that pupils should develop different ways of solving problems. The statement (pupils should solve routine tasks) clearly separated the opinions of the teachers: the Estonians were for and the Finns against, and the difference was statistically very significant.

Discussion

The reliability of the questionnaire was calculated using the halfing method (with all items) and the Spearman-Brown formula. This gave a test reliability of 0.77 for Estonia and 0.80 for Finland. For both samples together, the test reliability was 0.83. Combined, these gave an estimate of the consistency of results

(inner reliability) which may be considered to be good enough.

There were clear differences to be found in the conceptions of the teachers in Estonia and Finland. The Estonian teachers placed more emphasis than their Finnish colleagues on the role of the answer. But the difference was not statistically significant in every statement, although the tendency was clear.

As an explanation for the findings, one may give e.g. the following: The Finnish school system has undergone many changes during the last twenty years, whereas the Estonian schools have remained unchanged until the beginning of the 1990s. In Finland, school system changed more than twenty years ago from a parallel school system into a unified comprehensive school system with no special class or school reserved for talented children. In addition, the Finnish society has become much freer which has also had an influence on the school. This has meant changes from subject-centeredness and teacher-directedness to pupil-centeredness which in turn has led to i.a. a liberation in teaching methods. In other words, open teaching methods have found their way into Finnish classrooms.

However, one cannot state that the conceptions of the Estonian and Finnish teachers were similar to those found in the United States (Thompson 1989). If we compare the responses of the six statements dealt with here with the basic beliefs revealed by Thompson, we will have difficulties in finding similarities. It is only in the statement concerning routine tasks that the agreement percentage of the Estonians is high (79 %). All other agreement percentages (agreeing with the belief revealed by Thompson) vary from 1 % to 54 %.

What can we deduce from this finding? First of all, one should note that the teachers' conceptions discussed here are based on the questionnaire data, i.e. they may be seen as "surface beliefs" (cf. Kaplan 1991). And the beliefs found by Thompson were collected with interviews and observations, so they may represent "deep beliefs". Nevertheless, the difference is so great between these two belief groups that something serious should lie behind it. Could this mean that beliefs are strongly culture-bound? The statistically-significant differences between teachers' conceptions in Estonia and Finland, gathered with the same

questionnaire, give an indication in this direction.

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CHANGING VISIONS IN MATHEMATICS PRESERVICE TEACHER EDUCATION

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The main goal of the research described is to investigate pre-service mathematics teachers imagery of life in classrooms, and particularly of themselves as mathematics teachers, in an attempt to understand how changes in these images influence changes in their practice as mathematics teachers. Consonant with this goal, the study is qualitative. Early results suggest that these "visions" are indeed powerful influences in classroom practice, although the images are not often offered spontaneously in interviews: interviewees are sometimes unaware of deep-seated images until asked.

Rationale for imagery as a focus of teacher education.

In many countries, far-reaching changes are taking place in mathematics curricula at all levels as educators recognize the need not only to keep pace with understanding of new aspects of students' learning of mathematics, but also to anticipate the mathematical requirements in a new century (National Research Council, 1989 & 1990). With the notorious lag between the decision to develop a new curriculum and its actual successful implementation in classrooms (Howson, Keitel and Kilpatrick, 1982), it becomes essential that changes be initiated in students' beliefs about teaching and learning mathematics while they are preparing to become mathematics teachers. It is recognised that how teachers think about mathematics learning is a key determinant of how they teach (Simon and Schifter, 1991). Beliefs about mathematics

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and its learning are often deep-seated and unconscious (Thompson, 1992). The process of change, then, is personal and complex.

It is because processes of change in teaching practices are personal and complex that imagery is a significant factor in this regard. It has long been reported in psychological literature that imagery imparts an "affective colouring" to memory (Bartlett, 1932; Arieti, 1976). "Vivid mental images, because they provide psychologically more effective substitutes than do purely verbal encodings for the corresponding external objects and events, have a greater tendency to engage the affective and motivational systems", wrote Shepard (1978, p. 157).

It is not surprising, then, to find imagery in various guises emerging in reports of research on teacher change. Tobin (1992) and Jakubowski and Shaw (1991), identified three components in processes of change, in the study in which these three researchers were engaged in Florida. These aspects of teachers' thinking were perturbation which resulted in reflection on their practice, a vision of what their classroom practice might be, and commitment to the changes implicit in this vision. Weade and Ernst (1990) at a different university, also in Florida, wrote about "Pictures of life in classrooms, and the search for metaphors to frame them", in their description of a study in which pictures and images, and the metaphors which accompanied them, were an integral part of a pre-service teacher education course. In a comparable study during such a course in England, Calderhead and Robson (1991) wrote that "Students were found to hold particular images of teaching, mostly derived from their experiences in schools as pupils, which were sometimes highly influential in their interpretation of the course and of classroom practice" (p. 1). All these studies

suggest that student teachers' imagery of classroom practice is a component of their reflection on teaching and learning mathematics, and that this imagery may be a more powerful influence on their actions in the classroom than the memory of theories which they have learned about in their education courses, but which they have not yet reflected upon in the personal way which develops imagary (Presmeg, 1985; Vernon, 1970; Rokeach, 1960; Thompson, 1984).

Changing visions

The ongoing research study which is the focus of this paper started in August, 1992, in the course, "Teaching Secondary School Mathematics", which is usually the first mathematics education course taken by prospective high school teachers at The Florida State University. Since students' imagery of their future practice as teachers was a focus, the 30 students in the class completed the writer's test for visual preferences in mathematics, which places students on a continuum according to their need for imagery when they solve mathematical problems. Fifteen students of various preferences were chosen. Initially the net was cast wide because it was intended that the study would be longitudinal, and it was recognized that for various reasons not all of these students could be followed through to their experience as beginning teachers two or three years later. It turned out that eight of the fifteen students were also in the writer's class, "Elements of Geometry", in Spring, 1993.

The twofold aim of the study was to investigate the students' imagery of mathematics classrooms they had experienced or hoped to experience (past, present and future), and to learn whether their reflections on their own teaching and learning in the course were related in

any way to these changing images. Reflectiveness was encouraged throughout the course. Students were required to keep a journal documenting their thoughts as they moved through the course. About once a month these journals were read and responded to. Frequent small group discussions followed by class discussions of various aspects of teaching, learning and high school mathematics also gave opportunities for perturbations and reflections on classroom practice in the course of social interaction. (See appendix for course outline.)

Data collected were of three kinds, as follows.

(1) Every four weeks, the class was required to write on the topic, "How I *see* myself as a teacher". Students typically wrote approximately a page of verbal description of their visions of themselves as teachers, how they would arrange their classrooms, the kind of mathematics they would like to teach, even in some cases how they saw their students reacting and relating to them as teachers. These descriptions were collected and kept.

(2) In tape-recorded interviews which ranged from 10 to 55 minutes in length, the selected students were asked to relate "the story" of their learning of mathematics up to that point, including also their reasons for choosing to become mathematics teachers. The narrative mode encourages reflection on past experiences and the feelings associated with them (Bruner, 1990). The students were given several weeks' notice of these interviews, to enable them to see their mathematical histories in retrospect and reflect on which elements they wanted to include in their stories. Most students spoke freely and their stories gave evidence of reflection on what was important to include. Stories typically included anecdotes about teachers (or parents) who had been influential in their learning of mathematics, in

positive and negative instances. The images associated with these experiences were often vivid and carried emotional associations which time had not erased.

(3) An important component of the course is a micro-teaching segment, in a "field experience" of one hour per week. After observing lessons of two experienced teachers with contrasting constructivist and traditional styles (and reflecting in their journals about these), the students interview a middle school or high school learner of mathematics, transcribe the tape, and try to "enter into" the mathematical thinking of their interviewee. They write a report on this clinical interview. After this, each student in the class is video-recorded as they teach a fifteen or twenty minute lesson on a mathematical topic of their choice, to half the class. The segment is then replayed and discussed by students and instructor in a constructive, nonjudgemental atmosphere. The videotaped lessons of the selected students are available for analysis.

These three sources of data will be supplemented by further reports on "How I see myself as a teacher", further interviews, videotapes of some of the students during their internship in schools, and, finally, videotapes and interviews during their initial year as mathematics teachers. The study is still in its early stages, but already there are indications that students' images of themselves as teachers evolve and change as they reflect on their broadening experiences of teaching and learning, informed also by the theories they are exposed to in their education courses. Whether their actual classroom practices will be consonant with these evolving visions, or whether they will revert to earlier school-based images, remains to be seen.

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UNRAVELING THE RELATIONSHIPS BETWEEN BEGINNING ELEMENTARY TEACHERS' MATHEMATICS BELIEFS AND TEACHING PRACTICES

by
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In this qualitative study I investigated six beginning elementary teachers' beliefs about mathematics and mathematics pedagogy and explored the relationships between their mathematics beliefs and practices[1]. *A key question that was addressed is "What are the primary factors that influence beliefs, practices, and the consistency between beliefs and practices?" Data were collected primarily through interviews, observations, a concept-mapping activity, and a questionnaire. In addition, a model of the mathematics beliefs and practice relationships was developed prior to the investigation and was revised based on results of the study. Findings show that prior school experiences and teacher education programs were the key influences on beliefs, and mathematics beliefs and students' abilities were the primary influences on teaching practices. Also, time constraints and lack of resources account for the majority of inconsistencies between beliefs and practice.*

MOTIVATION FOR THE STUDY

In this investigation I explored six beginning elementary teachers' beliefs about mathematics and mathematics pedagogy and I observed their mathematics teaching practices in an effort to better understand the relationships between beliefs and practices. Studies have shown (e.g., Thompson, 1984) that there are times when one's mathematics teaching practice is consistent with one's mathematics beliefs, and there are times when it is not. Upon determining the levels of consistency between beliefs and practice, it is also important to address the question, "What are the primary factors that influence teachers' mathematics beliefs, teachers' mathematics teaching practices, and the levels of consistency between mathematics beliefs and practices?"

There is a need to study teachers' mathematics belief systems. Researchers in mathematics education suggest that the beliefs a teacher has about what mathematics is and what it means to know, do and teach mathematics can be driving forces in that teacher's instruction of mathematical ideas (Cooney, 1985; Kloosterman & Stage, 1989; Lester, Garofalo, & Kroll, 1989). Some mathematics educators have conducted studies and have begun theorizing about the role that beliefs play in understanding mathematics education. Clark and Peterson's (1986) review of literature on teachers' thought processes, for example, notes the importance of understanding teachers' implicit theories and beliefs about education. Thompson (1984) and Peterson, Fennema, Carpenter, and Loef (1989) assert that teachers' beliefs can have profound, but possibly subtle, effects on their mathematics teaching.

[1] This paper is based on my dissertation study done at Indiana University under the direction of Dr. Peter Kloosterman.

Accordingly, research has also indicated that the study of teacher preparation should include not only an examination of content knowledge, but also of thinking processes, attitudes, and beliefs held by preservice and elementary teachers (Peterson et al., 1989).

The existing interpretive investigations of teachers' mathematics beliefs that have been coupled with observations of actual practice have primarily focused on teachers at the junior high (Jones, 19[illegible]; Thompson, 1984) and high school levels (Kesler, 1985; Shaw, 1989), and on preservice student teachers (Owens, 1987). One subgroup of teachers that has been virtually overlooked in interpretive mathematics beliefs studies is the elementary teacher, in particular the beginning elementary teacher.

Beginning elementary teachers' mathematics beliefs are important to explore for four reasons: (1) as previously mentioned, elementary teachers have not been the primary focus of many interpretive studies that explore the beliefs of mathematics teachers; (2) elementary teachers play a vital role in developing students' initial understanding of and beliefs about mathematics; (3) beginning teachers are just starting to build their teaching practice, thus tracking the development of their style of teaching can reveal a lot about their beliefs; and (4) a beginning teacher's beliefs about mathematics and teaching mathematics are likely to be challenged during the first few years of teaching.

Six beginning elementary teachers participated in this study and provided some answers to the questions of (1) what beliefs do beginning elementary teachers have about mathematics and mathematics pedagogy and what do these teachers identify as the key influences on their beliefs, (2) what are these six beginning elementary teachers mathematics teaching practices like, and what do they name as the primary influences on their practices, and (3) how are these six teachers' mathematics beliefs related to their practices, and how do they account for the inconsistencies between beliefs and practice?

In order to help clarify the questions at hand and to attempt to put the pieces of the beliefs-practice puzzle together, I created a concept map (see Novak & Gowin, 1984, for a discussion of concept maps) to help analyze the relationship between one's mathematics beliefs and teaching practice. This concept map led to the development of an initial model of the relationships between mathematics beliefs and practices (see Raymond, [1993] for a detailed description of the initial model). My model of

the relationships between mathematics beliefs and practices helped me to determine the procedures that I followed in this investigation.

METHODOLOGY

The study was conducted via a naturalistic inquiry approach. The participants in the study were six beginning elementary teachers who were all graduates of Indiana University and who had been placed in teaching positions within a 50-mile radius of Bloomington, Indiana. Of the six teachers, five were females and one was a male. The procedures for data collection included seven individual interviews, five classroom observations, an analysis of the participants' lesson planning styles, a take-home beliefs questionnaire, and an activity involving the pieces of my model of the relationships between beliefs and practice.

The analysis of the data took place throughout the data collection phase and beyond, and includes feedback from two study debriefers and comments from the respondents. Each teacher's beliefs about mathematics and mathematics pedagogy were categorized as *traditional, primarily traditional, an even mix of traditional and nontraditional, primarily nontraditional,* or *nontraditional*. Determination of each teacher's beliefs categorization was based on descriptions of traditional and nontraditional perspectives on mathematics and mathematics teaching and learning as described in recent literature (NCTM, 1989, 1991; National Research Council [NRC], 1989).

Similarly, each teacher's practice was categorized on the same "traditional/nontraditional scale " by measuring the extent to which her practice, regarding the classroom environment, types of mathematical tasks, the kinds of discourse, and the means of evaluation, matches the "nontraditional" type of practice described in the *Professional Teaching Standards* (NCTM, 1991). Categorizations of beliefs and practices were compared and discussed in light of the primary influences on beliefs and practices, and the reasons for inconsistencies were identified by the teachers. The final stage of the analysis consisted of the development of a revised model of the relationships between mathematics beliefs and practices based on the findings from the study.

FINDINGS FROM THE STUDY

The findings of the study provide insight into elements that play key roles in the relationships between beginning elementary teachers' mathematics beliefs and practices. They center around the identification of factors that influence elementary teachers' mathematics beliefs, factors that influence elementary teachers' mathematics teaching practice, and factors that cause inconsistencies between beliefs and practices. The findings provide implications for the role that teacher education programs may play in the relationships.

First, the six teachers in the study named past school experiences, prior teachers, their own teaching practice, and their teacher education program as the primary influences on their mathematics beliefs. Several "weaker" influences mentioned were the classroom situation and personal family experiences. These same teachers identified their mathematics beliefs and the abilities of their students as the main sources of influence on their teaching practices. They also indicated that the particular mathematics topic at hand, the school environment, prior teachers, and the mathematics curriculum played a role in determining their practices.

Some results of this study support key findings from other studies (Brown, 1985; Jones, 1990; Thompson, 1984). Specifically, there is a strong relationship between a teacher's mathematics beliefs and teaching practices, and a teacher's mathematics teaching practice is not always consistent with her mathematics beliefs. This investigation also showed that there is a reciprocal influential relationship between beliefs and practice. However, the teachers in this study indicated that their beliefs influenced their practice more than their practices influenced their beliefs. In addition, the teachers displayed a wide range of consistency between beliefs and practice, with two teachers showing a high level of consistency, two showing a moderate level, and two showing only a modest level of consistency. A number of explanations for the inconsistencies that occurred were offered by the teachers, including time constraints, scarcity of resources, classroom management problems, and state standardized testing requirements.

In addition, the teachers in this study felt that it was more the case that inconsistencies occurred when influences (other than beliefs) on practice conflicted with beliefs, dominating beliefs at a particular moment, resulting in teaching practice that was not in agreement with beliefs. This result supports Brown and Borko's (1992) contention that while beginning teachers are being socialized into

the teaching profession. It is often the case that institutional factors, such as time limitations and standardized testing pressures, create conflicts between beliefs held by teachers about the ideal kinds of practice they would like to implement and their actual practice. Conflicting constraints cause teachers to make teaching choices that do not necessarily match their current pedagogical beliefs.

The model of the relationships between mathematics beliefs and practice that resulted from findings in the study is shown in Figure 1. The model represents my vision of the relationships that was influenced by my conversations with, and observations of, the teachers.

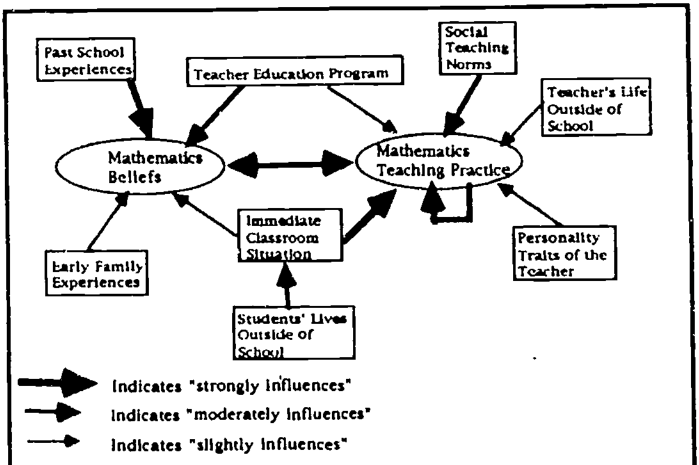

Figure 1. A Revised Model of Relationships Between Mathematics Beliefs and Practice.

As the above model indicates, the results of the investigation showed that these beginning elementary teachers did not attribute much weight to their teacher education programs in terms of their influence on their mathematics teaching practice. However, they thought that their teacher education experience had a moderate level of influence on their beliefs. The teachers involved in the study thought that teacher education programs could do a better job of addressing the issue of the relationship between beliefs and practice, perhaps by offering a forum for helping preservice teachers develop their own philosophy of mathematics education before stepping into the classroom. These

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teachers believed that a stronger sense of self awareness about their mathematics beliefs, in conjunction with a deepening of their mathematics content and pedagogical knowledge, would help to make their teaching practice stronger, more focused, and, perhaps, more consistent with their mathematics beliefs.

IMPLICATIONS FOR MATHEMATICS TEACHER EDUCATION

The findings of this study hold a variety of implications for teacher education. The first and foremost implication is that mathematics beliefs need to be explicitly addressed in teacher education programs. Specifically, preservice teachers should be challenged to discover their own beliefs about mathematics, perhaps by writing their own mathematics diary. They ought to be asked to recount their own school experiences with mathematics as students and to assess their abilities to do mathematics and their confidence in teaching mathematics.

Along with exploring their personal relationships with mathematics, preservice teachers should be asked to describe for themselves what they believe mathematics is all about, and they should debate the issues of what they consider are the best ways to learn and teach mathematics. This should be done within a context of exploring various mathematics learning and teaching styles, allowing students to make their own connections between what methods of teaching mathematics are available and what styles they believe best match their own philosophies of mathematics pedagogy.

I believe that we need to be as realistic as possible with future teachers when preparing them for their profession. Why not talk with them about the socialization process that they will face, and share with them the types of conflicts and choices they are inevitably going to see during their beginning years of teaching? If they are made aware of the fact that their ideal beliefs about teaching may not conform to the realities of teaching, perhaps they will be better able to cope with the conflict if and when it arises. After all, it should be our goal to not only help preservice teachers develop an awareness of their beliefs about mathematics teaching, but to prepare them for the inevitable challenges to their beliefs. This will allow them to confront challenges without upsetting the balance between beliefs and actual practice. Institutional constraints are not likely to disappear in the near

future, so sharing ways of coping with them without abandoning ideals should be pursued in teacher education.

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THE EVOLUTION OF PRESERVICE SECONDARY MATHEMATICS TEACHERS' BELIEFS*

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Three preservice secondary mathematics teachers were followed through their teacher education courses and student teaching. The study is based on interview, written survey, and observation data. Changes and resistance to change in beliefs about mathematics and mathematics teaching and learning are described and placed in the context of theoretical perspectives on belief change. One teacher exhibited permeable beliefs (Kelly, 1955) and experienced significant change in such peripheral beliefs (Green, 1971) as those about the use of technology. Another teacher who held impermeable, dualistic beliefs, became very uncomfortable with his beliefs due to interaction with peers and mathematics education instructors, and then reconfirmed his original views through work with a cooperating teacher. The third teacher assimilated new information and experiences, strengthening his belief system even when confronted with resistance from a cooperating teacher.

Introduction

Research on mathematics teachers' beliefs indicates that beliefs are resistant to change and reinforced through extensive experience (Thompson, 1992). Teacher education programs seem to have little impact on teachers as they fall back to practices consistent with their school experience. This resistance to change makes sense from the epistemological perspective of constructivism (von Glasersfeld, 1987). People develop an understanding of their world through a process of assimilation and accommodation; their beliefs are a product of experiences and reflection. As teachers' conceptions remain viable in the context of experience (e.g., years of sitting in classrooms), beliefs become more strongly held and central elements (Green, 1971) of the person's perception of reality.

The resistance to change in beliefs may be characterized by Kelly's (1955) idea of permeability of constructs and Green's (1971) idea of evidentially and non-evidentially-held beliefs. Rokeach (1968) emphasized the idea that for a person to modify his or her beliefs, the new position must be close to the original. Typically beliefs do not change radically; they evolve through extensive, extended experience (cf. von Glasersfeld, 1987). Thus, it would seem that teacher educators need to provide contexts so that

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teachers adapt to new experiences in a manner that challenges their beliefs about mathematics and teaching and learning mathematics while allowing gradual growth.

Our research focuses on the process by which preservice secondary teachers' beliefs are challenged, affirmed, and adapted. To study this evolutionary process, we focused on the following questions:

- What are their key conceptions of mathematics and learning and teaching mathematics?
- Which aspects of associated belief systems are firmly established and which are more permeable?
- What is the nature of change as the teachers progress through their teacher education program?

Methodology

In the fall quarter of 1992, fifteen preservice secondary mathematics teachers in their final year of a four-year undergraduate mathematics education program were surveyed. The teachers were part of a mathematics education course that emphasized mathematics as a subject to be explored and created. In this survey, the teachers chose and explained similes for mathematics teaching and learning, generated hypothetical student responses to open-ended questions, and gave multiple strategies for investigating mathematical situations.

Based on survey analyses, we selected five teachers exhibiting a range of beliefs about mathematics and mathematics teaching and learning. Using project-developed guides, we interviewed these five teachers to seek elaboration of the reasoning behind their survey responses. The participants completed similar surveys at the end of the fall quarter to document recent challenges to or changes in their beliefs. The participants were then interviewed early in the winter methods course, after a two-week field experience near the end of winter quarter, and at the end of their spring student teaching assignment.

Supplemental data included fall quarter field notes and written assignments, field notes from observations of their winter and spring quarter teaching experiences, and informal conversations with the preservice teachers. Three participants who exhibited an array of resistance, struggle, and change within their belief systems were selected for this paper.

Case Studies

Gregg: Preparing People for Life

Gregg says that the teaching profession is not something he chose, but is a "calling, something I've got to do." Gregg also said that he wanted to be "a teacher with nothing in front," that is, not necessarily a *mathematics* teacher. "Preparing somebody to live" is fundamental for Gregg's beliefs. This idea

shapes his view of the student, what he wants to teach, how he teaches, how he views himself as teacher, and his view of mathematics. "Preparation for life" would be at the core of any quasi-logical or psychological structure (Green, 1971) for Gregg's beliefs. That is, his other beliefs seem to be derived from these strongly-held core beliefs.

Gregg holds three levels of priorities for what it means to prepare people for life. His primary emphasis is on such character qualities as honesty, integrity, and working hard. He says that, "I would much rather you fail something and be honest about it than have to cheat to get by because I can always help you, you know, your math skills." Gregg's second level of emphasis includes thinking, reasoning, problem solving, and being able to adjust to situations. "Hopefully, they'll learn part of that through my class and me giving them different situations they have to adjust to, reason to, and the problem solving . . . and I think that's going to help them in life." The third level reflects Gregg's view of mathematics. An important part of preparing people for life is to give them the skills that they need. In this way, mathematics is a "stepping stone" or a "tool" people need for further schooling and life.

Early on, Gregg was adamantly opposed to using technology in the classroom, a view that he attributes to the college professor he calls his mentor. He said it would "hurt [the students'] reasoning skills." For him, using calculators was "just punching the numbers in and reading it off the calculator." But now he says that in looking for a job, "if I get more than one offer, the amount of money is not the point. It will really come down to the school that has the Mac Lab." Gregg's view of activities that use technology changed significantly, coming more in line with his emphasis on the development of thinking and reasoning skills that he values strongly. This shift came about as a result of participating in open-ended activities with calculators and computers and interaction with instructors and peers who highly valued technology.

Two shifts in Gregg's thinking reflect an increase in the evidential basis (Green, 1971) for his previously held views. The first shift relates to his view of activities. In his fall classes, Gregg saw activities as a way to make class "interesting." In the winter course, he viewed activities as ways to help students "see" concepts, "look at concepts in different ways," and see mathematics as related to life. He believes that activities are important contexts for these ideas. This understanding of the multiple purposes of activities also coincides with a stronger view of the importance of mathematics. Now, contrary to his initial understanding, he says that he is called to be a *mathematics* teacher

Overall Gregg tends to have views with strong evidential arguments. Kelly's (1955) concept of permeability is a reasonable way to characterize his beliefs. Gregg has flexibility in his beliefs that allows him to incorporate new concepts. No fundamental changes occurred, but peripheral ideas changed significantly. Experiences that seemed to have had the greatest effects were interaction with faculty, graduate students, and peers, open-ended problem-solving, and his field experiences.

Todd: Looking for a Plan

Todd's interest in mathematics was punctuated by his studies in engineering. He was amazed to see school mathematics so closely related to the real world. This linking of mathematics to the real world awakened him "just like a light bulb" and was so contrary to what he had experienced in his high school mathematics classes that he became determined to change the situation for other students. His concern over the past year has been to find ways of making this change happen.

Todd's beliefs about teaching and learning are founded in his extensive tutoring experience. He consistently uses the word *coach* when talking about mathematics teaching. He devalues a lecture style of teaching: "I believe a teacher must inspire, motivate, and use all of a student's senses to teach, not just their hearing." Initially he saw activities as means of motivation and later began to look at activities from the learner's perspective--a way of learning mathematics. He thinks that creativity is critical to learning mathematics: "I just want to be able to bring that creativity back to the math classrooms."

Todd seems insecure in his teaching. In discussing his teacher education experience he wants more experience with activity-based learning, observations of exemplary teachers, and help with classroom management. He considers his past experience with mathematics teachers as essentially negative. "I don't think it was bad experience with mathematics necessarily, I think it was bad experience with mathematics teachers that gave me my outlook." In his mind this situation was exacerbated by his student teaching experience. One of his cooperating teachers resisted Todd's desire to deviate from a regimented approach to teaching. Todd believes this teacher's insistence on regimentation denied the students the opportunities "to discover and explore."

Todd has a commitment to teach *his* mathematics and to practice *his* vision, although he worries about his ability to do so given his novice situation. The teacher education experience strengthened his vision but did not give him a plan. His conclusion is: "I'm just at the beginning of that continuum of teaching. I think that I've come a long way but I've still got a long way to go, if that makes sense."

Henry: Organizing for Instruction

As Henry changed his focus from becoming a public relations director for a sports team to becoming a teacher of secondary mathematics, he kept similar goals in mind. Henry believes both are exemplified by the efficient "telling" of the story in a way which will excite the audience. On the first survey, Henry described a mathematics teacher as a "newscaster and a coach because the teacher has to be able to tell the story; yet he/she has to develop motivation."

When describing his notion of ideal teaching, Henry always mentions his former algebra teacher. He believes that her attitude, orderly classroom, and efficient methods of teaching by lecturing, giving examples, and having students practice lists of problems, instill confidence and promote learning. When asked if all her students benefited as much as he did, he explained:

> There were a lot of [students] who didn't like her ... because they felt she was sometimes too abrasive.... If all the students would have followed the pattern that it was designed for and ... kept a positive attitude toward math then they would have done better.

Henry's view of teaching mathematics did not include reaching all students and was very limited until experiencing a variety of teaching methods this past year. As the year progressed Henry spoke more about the teacher as motivator and less about inadequacies of students. Henry sees "new" strategies as ways to make the class less boring but also class time less efficient. His confidence in his beliefs waned about midyear. He became concerned that he was not as prepared for teaching as his classmates because he had never experienced these "new" classroom situations. He became upset during a interactive video activity because he disagreed with many of his fellow students' and especially with the video creator's preferred solutions to behavior management problems. Henry favored immediate disciplinary action and not wasting time on a teacher-student discussion of the nature of the problem. Henry believed "the solutions were too passive.... If you based your teaching around that philosophy you would get run over like a truck by your students." His psychological commitment to this position became obvious as he argued that control was necessary to run an efficient class.

His confidence was rekindled after a short field experience in a methods course. For two weeks, he and his classmates taught small groups of very uncooperative students in a local high school. Henry reflected on the previous behavior management episode as related to the field experience.

[My fellow students] didn't agree with one thing I said and when we got done with that unit over at the high school, I started hearing more people saying, "We did not spend enough time on classroom management. . . ." And I was able to just kick back and [say] "I told you so."

Following ten weeks of student teaching alongside a teacher "whose philosophies of teaching are very similar to mine," Henry expressed confidence in his "telling" methods and his occasional "break from monotony" to capture the students' interest. He explained that his most successful class was designed for a change of pace. After giving students the Pythagorean Theorem and the fact that the angles at the bases of a baseball diamond are right angles, students "explored" the relationships by taking measurements of an actual baseball diamond. He based his idea of success on high scores and explained that students knew this material better "because they figured that if I was going to take the time to drag them out to the baseball field, there had to be something important to learn from it."

Henry listed his former algebra teacher and his student teaching experience as the greatest influences on his thinking about teaching mathematics. He did not mention any of his university course experiences. He believes the courses in the mathematics education department gave him ideas for making classes interesting but that they "under-emphasized" the most important aspect of teaching: "Organization is the key to teaching."

Discussion

An implication of this study is the importance of the teachers' prior beliefs. For example, Gregg and Henry began with views that the use of technology would be more damaging than helpful to students. Both saw their views subtly rejected by peers and instructors, an aspect of Bauersfeld's (1992) social impact on the psychological. Thus, by winter quarter, both were questioning their beliefs. In the spring, Gregg was strongly committed to using technology in the classroom, while Henry had returned to his hard-line opposition stance. Gregg and Henry's prior beliefs concerning mathematics and teaching mathematics and how they held them were very different. During Gregg's teacher education courses he had experiences that allowed him to see technology as a context to promote reasoning skills. Thus, his view of technology came in line with his emphasis on reasoning skills. This shift seems consistent with Rokeach's (1968) claim that people value ideas that are congruent with their own beliefs.

Also, Gregg's and Todd's beliefs seem to be more permeable (Kelly, 1955), that is, open to the incorporation of new ideas, than those of Henry. Henry was very resistant to new ideas in class and in

interviews. When confronted with mathematics problems he had never seen before and various open-ended activities, he sometimes refused to do or discuss the activities. He also incorporated terminology redefining concepts to fit his own views. For example, he praises his cooperating teacher for emphasizing student involvement, which to him means having the students do independent skill practice. Todd differed from Gregg in that Todd had earlier undergone a fundamental shift in his view of mathematics teaching. Thus, he displayed less confidence and sought details of how to put his vision into practice. Gregg, however, held very strong beliefs about mathematics teaching, so is in some sense fine tuning his peripheral beliefs.

This study has several implications for teacher education. To impact on a teacher's beliefs one needs some understanding of the teacher's prior beliefs--including not only descriptions, but also the origins and the way the beliefs are held. Teacher educators need to provide active contexts for teachers to experience learning mathematics in ways contrary to their past experience. Beliefs about mathematics seem to stay constant unless the teacher is actively involved in learning or doing mathematics. Finally, we need to admit and think seriously about the implications of one controversial idea, as uncomfortable as it may seem. Changing beliefs is a significant part of our work as teacher educators.

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EMPHASIZING CONCEPTUAL ASPECTS OF FRACTIONS: ONE MIDDLE SCHOOL TEACHER'S STRUGGLE TO CHANGE

MELVIN R. (SKIP) WILSON, *University Of Michigan*
MICHAEL PAUL GOLDENBERG, *University Of Michigan*

Obstacles to reform in mathematics teaching are explored by focusing on the difficulties a veteran sixth-grade teacher faced in attempting to revise his practice using new materials that stress multiple representations of fraction concepts. Data collection methods included 16 formal interviews, 26 classroom observations, collection of written artifacts, and regular informal interviews of 3 students. The teacher changed his instructional emphasis from rules to concepts but continued to be very directive in his instruction. The teacher's struggle is examined in light of theories that may account for factors inherent in the complexity of the mathematical subject and its applications, as well as in the teacher's intellectual assumptions about mathematical and pedagogical authority.

Driving American mathematics education reform movements is a growing consensus that school mathematics should be portrayed as a useful, humanistic, and vibrant subject to be *understood* and *explored*. This view contrasts with the widely-held image of mathematics as an immutable set of absolute rules and procedures to be *mastered*. Research related to teachers' conceptions of mathematics and mathematics teaching (Thompson, 1992) reveals that many teachers do not think of mathematics education in ways that will enable them to implement teaching practices to support popular reform efforts. But this strand of research has focused primarily on teachers who are not neccesarily motivated to teach in ways consistent with current recommendations. Little information exists about how teachers deal with the complexity of change as they attempt to meet recommendations for reform. Policy makers and teacher educators need a better understanding of how psychological factors such as teachers' conceptions of mathematics and mathematics teaching might influence change. Other factors in this process include the influence of curriculum materials intended to promote change, standardized tests and other forms of assessment, and administrative and school support. Studies of teachers striving to make changes in this context can provide such information. This paper reports the experiences of Mr. Burt, a veteran sixth grade mathematics teacher who was faced with these and other factors as he attempted to improve his teaching.

One issue central to the mathematics education reform movement is the role of authority in teaching and learning. Expecting students to understand and explore mathematical ideas requires them to accept much of the responsibility for determining, for example, appropriate procedures and methods to solve problems. This contrasts with the teacher-centered (and often

dominated) dialogue that takes place in many mathematics classrooms. Perry's scheme (Perry, 1970; Copes, 1982) emphasizes this issue of authority. A dualistic conception of knowledge (by dualistic we mean a belief that all knowledge is either absolutely right or wrong) assumes that there is a single authority, be it a teacher or textbook, thus relegating instruction to an exercise in transmission from the authority. Furthermore, Perry's categories of dualism, multiplism, relativism and commitment provide a useful tool for viewing the extent of teachers' openness to varied viewpoints on mathematical concepts and applications.

Because this study took place while Mr. Burt taught a unit on fractions, we also considered research related to students' and teachers' conceptions of rational number (Ohlsson, 1988, Behr, Harel, Post, & Lesh, 1992; Post, Harel, Behr, & Lesh, 1991). When viewed together with Perry's scheme, these theories of rational number suggest some of the key difficulties teachers face when approaching complicated mathematical strands. Particularly useful is Ohlsson's theory of quotient terms; his work emphasizes that the terminology employed to describe fractions lies in a complex semantic field which he labels quotient terms. Ohlsson suggests that trouble associated with learning fraction concepts may have more to do with confusion caused by overlapping notation and terminology applied to distinct mathematical ideas and applications than with inherent difficulties of either individual procedures or concepts. When a teacher with fundamentally dualistic thinking about mathematics and mathematics teaching is faced with using more relativistic approaches to teaching concepts with inherent ambiguity, the results can be stressful for instructor and students alike.

DESIGN

Methodology

An ethnographic case study design (Stake, 1978) was used to document the experiences of Mr. Burt. Data were collected between September 1992 and May 1993 using interviews, observations, students' and teacher's written work, and teacher's written plans. Two one-hour interviews, conducted during the first month of the school year (September 1992), investigated Mr. Burt's initial views about mathematics, teaching, and his specific understanding of fractions. One of Burt's mathematics classes was observed for a total of 26 days, including 20 consecutive days during a five-week period while he taught a unit on fractions (November-December). Approximately half (12) of these observations were followed by half-hour stimulated recall interviews in which Burt commented on classroom events of that or the previous day. Burt's written notes and plans during this 5-week period were also collected. Three students, chosen to be representative of the class's race, gender, and achievement

diversity, were identified as target students and observed and interviewed periodically during class. Two one-hour interviews at the end of the observation period assessed Burt's reflections on his experiences. Observations of bi-weekly meetings of the sixth grade mathematics teachers constituted another data source. Teacher interview data were audio recorded and transcribed for ongoing analysis. Classroom observations were also audio recorded. Detailed fieldnotes were made of observations (classroom and teacher meetings) and student interviews. Photocopies were made of written artifacts.

Participant and Research Site

Mr. Burt holds K-8 certification with specializations in both science and mathematics and has concentrated on mathematics teaching for several years. He has taught nearly every elementary grade, but has spent the past 12 years teaching sixth grade. He teaches mathematics each day during three 45-minute periods (he teaches language arts during two others). The class in which this study took place met daily from 10:00 a.m. to 10:45 a.m. The class, like the school as a whole, is very diverse in terms of achievement, race, and socio-economic standing. For example the class's (and school's) racial makeup is approximately two-thirds white and one-third African-American. There were 31 students enrolled in this class.

Unlike many 20-year veterans, Mr Burt has a sincere desire to improve his teaching. Although he understood that the researchers would not be at all directive, he was very open to having his teaching studied, since he believed that the opportunity to interact with interested professionals would help him become a more effective teacher. Mr. Burt finds teaching to be challenging yet rewarding. During interviews he frequently compared teaching to his experience in the military, on one occasion claiming that being company commander was an easier job than that of "dealing with 35 individual sixth graders and all of their attendant pressures from outside."

Competing with his feeling of responsibility toward his students are external factors, many of which he views negatively. He reported receiving insufficient administrative support with student discipline, undue pressure to emphasize performance on state proficiency tests (which he believes "drive" the district), and unnecessary classroom interruptions from outside personnel and through the classroom's public address system. However, Mr. Burt cited as his biggest obstacles the extremely diverse student population and the lack of parental involvement. He frequently referred to students with severe disabilities, and students who "need lots of guidance and loving" because they lack a supportive home environment. While

discussing parental involvement, he stated, "If I could get 40 contact minutes a day between parent and child, we'd have the highest achieving kids probably in the world. But if you checked with kids . . . [the actual figure] is around 2 to 3 minutes a day, maybe."

MR. BURT'S EXPERIENCE TEACHING THE FRACTIONS UNIT

In the context of this multitude of competing factors, Mr. Burt experienced success in helping his students understand important concepts related to fractions. He attributed this success to (1) his use of curriculum materials that supported understanding and (2) the fact that he devoted more time to this topic than he had in the past. The curriculum materials consisted of a 20-lesson workbook, in pre-publication form, posing exercises and problems that emphasize fraction concepts, applications, and procedures. The materials (Towsley, Payne, & Payne, 1992) place heavy emphasis on pictorial and physical representations of fractions, as well as on connections among these representations, numerical and verbal representations, and traditional algorithms for operations. Mr. Burt took about three months to teach the 20 lessons (some of this time was spent reviewing topics covered before the fractions unit).

The following episode from an early fractions lesson (Mr. Burt taught this lesson on November 10, 1992) is in many ways representative of his overall approach. Mr. Burt began this particular lesson by modeling (with student input) on the overhead projector the solution to the first exercise in the lesson. He then asked students to complete the next few exercises:

> Do [exercises] 2 and 3 by drawing lines to show equal parts. Put your pencil down and look up when you're done. Don't work ahead. (25 second pause). Please do 4, 5, an 6 and put your pencil down. Do not work ahead (40 second pause while he circulated among students).

Mr. Burt then modeled solutions to the assigned exercises. For example, on the overhead he completed the exercise shown on the left of Figure 1 by drawing a vertical altitude to the middle (isosceles) triangle. One of the students asked if she could show a different way to do the problem. Mr. Burt invited the student to the front of the class and she sketched a solution like the one shown in the right part of Figure 1.

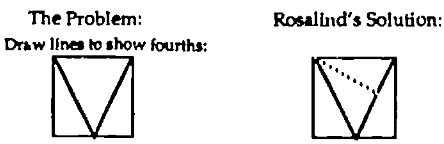

Figure 1. Fraction problem and one student's solution

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Upon seeing the student's solution, Mr. Burt responded, "I don't know if that would be right or not. I'd have to really look at it. I can't tell, I'd have to measure it." He then measured the lengths of the two segments along the base of the triangle she had divided, and announced, "Nope, that won't work because this is longer this way that it is this way. But that was a good idea, you were thinking. Thank you Rosalind."

One thing illustrated by this episode is that Mr. Burt was extremely directive in his instruction. Rarely in his teaching did he attempt to involve students in deciding the appropriateness of strategies or solutions; he was the sole arbiter of correctness. However, we point out that throughout the fractions unit, Mr. Burt's instruction though not student-centered, was conceptually oriented. Obviously, one episode cannot adequately illustrate how Mr. Burt consciously emphasized connections among various common representations of fractions, though he did. During lessons on operations (i.e., addition, subtraction, multiplication, division) he did not introduce procedures until students spent several days on activities designed to help them understand concepts underlying the procedures. For example, before introducing an algorithm to add and subtract fractions, he taught a lesson on estimating sums and differences. In the case of comparing and ordering fractions, he never introduced a procedure (e.g., cross multiplying), but based his instruction entirely on connections to pictorial and physical models. This accent on concepts contrasts with his instructional emphasis both before and after the fractions unit when attention was placed primarily on correct procedures. Although he has recognized for some time that an essential aspect of fractions involves being able to "visualize them," he said he has in past years emphasized rules for operating with fractions because his textbooks have not supported a more conceptual emphasis.

Mr. Burt claimed that his current students understood fraction concepts more clearly than in past years. Both during and after the fractions unit his students were for the most part successful in making important connections among common representations of fractions, and in basing their understanding on conceptual and visual representations of fractional ideas. For example, during interviews two months after completing the fractions unit, all three of the target students, without being prompted, constructed pictorial diagrams to explain how to order 1/2, 1/3, and 5/8.

DISCUSSION

Before teaching the fractions unit, Mr. Burt's dualistic orientation toward mathematics was evidenced by his continual focus on the "correctness" of certain mathematical ideas. For example, he focused almost exclusively on the idea of fractions as division problems. His presentation style throughout the unit, as well as in other mathematics lessons, reflected a

narrow view of mathematics and mathematics teaching. The episode presented earlier in this paper illustrates this dualism. However, his own understanding of fractions was flexible enough to allow him to shift from an emphasis on procedures to an emphasis on concepts. Despite his focus on a single conception, he was *aware* of multiple conceptions of fractional ideas. Even before the fractions unit he spoke of fractions both as parts of units or wholes (a partitive interpretation) and as division problems involving whole numbers (a quotitive interpretation). He also understood the importance of representing fractional ideas in multiple ways: as "pictures," using "numerical representation," with "word problems," and with "concrete materials." Finally, he was sensitive to the idea that students learn better if they apply fractional concepts to their everyday lives.

However, despite an increased emphasis on concepts, during the fractions unit Mr. Burt continued to communicate a very dualistic view of mathematics. The teacher—not the students—determined the correctness of the mathematics. Rather than emphasizing "correct" procedures, he emphasized "correct" concepts. One could certainly argue that this new set of givens (concepts) is more desirable than the old (procedures). Focusing on concepts helps students gain a knowledge base that, to a greater degree, allows them to do their own thinking. But Mr. Burt's approach generally portrayed mathematics as a rigid subject to be mastered and correctly applied, rather than as a way of thinking or as a subject to be explored.

Circumstances, such as a large class with students having diverse needs and interests, no doubt contributed to Mr. Burt's decision to make the classroom primarily a teacher-directed one. But Mr. Burt's dualistic view of teaching in general and of mathematics and mathematics teaching in particular was the primary factor preventing him from shifting away from an environment dominated by teacher-judged "correct" ways of operating, toward a more student-centered environment in which exploration and more relativistic notions of right and wrong (e.g., it is right because it works and makes sense) were valued and emphasized. Mr. Burt claimed that his lack of experience in teaching and learning mathematics from a perspective emphasizing understanding made the experience extremely stressful. In describing his experience, he communicated that he had never felt more discomfort during all of his 20 years of teaching. It is also interesting to note that he felt like he was making tremendous changes in his teaching practice, although the changes seemed less substantial to outside observers. We suspect that repeated encounters with less dualistic approaches to complex mathematical topics may move Mr. Burt further along Perry's continuum toward a more relativistic view of mathematics and mathematics teaching. Because he is fundamentally committed to improving his practice, the discomfort Mr. Burt experienced teaching this unit is

less likely to drive him away from reform than might be the case with a novice teacher or one who is merely flirting with instructional fads. The dissonance between his basic dualism and his willingness to change suggests both the motivation and means for further growth.

These results suggest a need for patience and sensitivity toward teachers as they develop and strive to change. They also suggest that there is more to curriculum reform than simply providing teachers with appropriate curriculum materials, as important as this is. Teachers must also be supported in other ways if they are to implement significant changes in the way they teach. For example, before teachers can teach in ways compatible with national curriculum reform efforts, it might be necessary for them to first *experience* mathematical learning in ways that are consistent with these reform efforts. Moreover, teachers need feedback that will both affirm their progress and give them guidance as they work towards continuing to improve and refine their practice. If we expect the positive effects of reform to have permanence, we must be aware of potential obstacles in the interplay between new curriculum and pedagogy, and teachers' intellectual growth or resistance thereto.

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Short Orals

Title Reshaping Teachers' Perceptions of Instruction and Assessment in Mathematics Through the Use of Performance Assessment Activities

Presenter Miriam Amit, Ph.D. Nancy Katims, Ph.D.

Institution Educational Testing Service.

Educational reform in the 1990's is calling for change in classroom instructional practices and change in testing and assessment. In order for true educational change to occur, instruction and assessment must be linked.

To address this need, the Educational Testing Service (ETS) is developing performance assessment materials for middle school mathematics, known as the PACKETS program. Based on research by Richard Lesh and others, the PACKETS activities are designed to elicit the construction of mathematical models. The program strongly reflects the NCTM Standards, and facilitates the delivery of interdisciplinary instruction, cooperative learning techniques, and the application of mathematical thinking to real-life situations.

An experiment aimed at examining teachers' perceptions about linking instruction and assessment was done with fifteen experienced teachers from middle school grades 5-8. Each participant received a PACKETS activity, which included a newspaper article, a problem related to the article, and support materials. The teachers were asked to act as students and to solve the problem working in small groups of three. Ninety minutes was allocated for the solution. They were expected to document the solution process and to present their solution before the others, defending their approach, procedure, and results. The problem, like the problems in all the PACKETS activities, was carefully chosen to be "real-life," and to encourage interdisciplinary and higher-order thinking. After the presentations, the teachers returned to their role as teachers and were asked to assess and evaluate the work of their peers. Because this exercise was not a typical classroom scenario, issues were raised such as how to evaluate performance, what is fairness, and accuracy, and depth and breadth of knowledge. Out of a heated discussion came new criteria for assessing mathematical performance which were in contrast to earlier perceptions the teachers had held. This led to a new perspective on mathematics instruction in general and problem-solving in particular.

THE FAST FOURIER TRANSFORM:
HOW DOES THE CONTEXT INFLUENCE THE LEARNING OF THIS CONCEPT?

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This article reports an educational study performed with students of the Superior School of Physics and Mathematics of the National Polytechnic Institute, referring to the concept of the fast Fourier transform. We determined some characteristics encountered in the discussions between students and the teacher, generated by the presented material. Resulting from this short course and consultations with professional personnel in the subject of Geophysics, as well as from a bibliographic review in this matter a teaching proposal was obtained within a very special context, directed to one specific problem: the obtainment of geological sections. Students starting the study of Geophysics: Do they learn this mathematical concept within this context or out of it?

Barnett, 415-565-3051

The Implications of Cognitive Flexibility Theory for Designing and Sequencing Cases of Mathematics Teaching

Oral presentation for PME-NA Meeting, 1993

By Carne Barnett
Far West Laboratory for Educational Research and Development

Abstract

Work by cognitive psychologist Rand Spiro and his colleagues (1987, 1988, 1989, 1992, 1993) establishes the theoretical underpinnings for developing a case-based curriculum. This theory proposes a way to promote flexibility of concept use in "ill-structured domains". Such domains are characterized by situations where rules and principles have limited application because what applies in one situation, may be contradicted or confounded in another. Drawing from themes in other prevailing psychological theories such as situated cognition, constructivism and cognitive complexity, Spiro, Coulson, Feltovich and Anderson assert that abstract knowledge is highly intertwined with that of case-centered reasoning. They argue that the best way to achieve cognitive flexibility is by a "method of case-based presentations which treats a content domain as a landscape that is explored by 'criss-crossing' it in many directions".

This session will explore the implications of Cognitive Flexibility theory for a case-based curriculum in the ill-structured domain of rational number teaching. The set of cases described in this session were developed at Far West Laboratory and are being used for the professional development of preservice and inservice teachers. I will discuss how these cases draw on the propositions offered by Cognitive Flexibility Theory for their design and sequence and will also describe some of the findings of studies conducted over the past six years. We have found, for example, that cases can be successful in helping teachers develop a richly connected framework of pedagogical content knowledge related to teaching rational numbers. We will describe how teachers draw from issues discussed in prior discussions to frame and solve problems that arise in new cases and in their own experience. These results will be reported along with insights and questions that have been raised through these investigations for audience discussion.

Title "Confronting the Psychological Aspects of Change in Two Educational Settings"

Presenter Sandra Callis Bethell and Margaret A. Lamb

Institution Holt High School

PROPOSAL

Mathematics educators nationally have called for major curricular reform. This reform movement requires changes in the way teachers and students think about what it means to learn and do mathematics and their relative roles. The psychological changes inherent in this reform are not easily defined nor easily negotiated (Cohen, 1988). Teacher educators must be responsible for helping prospective teachers to think about the ways these changes affect students' propensity to engage in the learning of mathematics. McDiarmid (1992) in a study of preservice teachers' beliefs about individual students' capacities to learn mathematics found that many believe that one needs to have a "mathematical mind" to be good in mathematics, for example. Fullan (1991) in his writings about educational reform indicates that real change involves changes in conceptions and beliefs. Clearly, in order for reform initiatives focused on changing the teaching and learning of mathematics to be successful, they must include an examination of both teachers' own beliefs about students' capabilities for learning mathematics and the psychological effects that are manifested with these kinds of changes in their practice.

A high school mathematics teacher and a special education teacher team taught a General Mathematics class in a Professional Development School and a Master's level mathematics methods course at Michigan State University. The teachers implemented an innovative curriculum emphasizing the development of mathematical questions and different ways of working together in the setting. In examining the data from these two settings, the teachers were struck by the similarities in both populations' reactions to these innovations. Both classes responded with various forms of resistance to this nontraditional content. Each expressed anxiety, frustration and fear as they attempted to learn new ways of mathematical thinking. In both settings the chronicity and intensity of the student resistance increased the practitioners' uncertainty about the validity and viability of implementing the types of changes, thereby risking their commitment to alter their practice.

Thus, the resistant student responses experienced in both a secondary and a university setting have implications for teacher educators, as they strive to develop a deeper understanding of the psychological aspects of changing conceptions of teaching and learning mathematics. Further, these findings need to be considered in the development of curriculum and practicum experiences in the redesign of teacher education programs.

The purpose of this session is twofold: 1) to discuss these psychological aspects of students' resistance to engage in new ways of learning mathematics, 2) to explore the corresponding psychological disposition towards mathematics and mathematics learners that both preservice and veteran teachers must embrace in order to change their practice and surmount this resistance, and 3) to investigate ways in which these issues can be addressed in teacher education programs.

Title A Model for Achieving Equity in Mathematics Teaching and Learning

Presenter Nedina S. Bezuk and Frank A. Holmes

Institution San Diego State University

This short oral presentation will discuss several aspects of a model we have developed whose goals are aimed at achieving equity in mathematics teaching and learning, focused on students from diverse ethnic groups. This model is multifaceted, requiring the collaboration and interaction of persons from many groups, including teachers, parents, students, university professors, college students, and district administrators. This model strives toward improving students' mathematics understanding, achievement, and attitudes by focusing on enhancing teachers' knowledge of mathematics and of mathematics teaching. The following section described the components of the model in more detail.

Collaboration and interaction of persons from many groups:

Many different groups of people are critically important to achieving equity in mathematics. These groups include teachers, parents, students, university professors, college students, and district administrators. The contributions and responsibilities of each group will be discussed.

Enhancing teachers' mathematics teaching:

Some factors that we have found to be effective in changing teachers' approach to teaching mathematics include creating a sustained period of interaction between teachers and mathematics educators, with workshops and discussion groups year-round rather than just for a few weeks in the summer; as well as providing opportunities to "try out" new techniques in a less structured setting, in our case, in before- or after-school mathematics enrichment sessions with children. We will describe in greater detail the benefits of this integration of content knowledge enrichment and experimentation with alternate teaching methods.

Improving students' mathematics understanding, achievement, and attitudes:

We believe that several factors contribute to improving students' mathematics understanding, achievement, and attitudes. These include the following: (a) "beyond the bell" activities, which are before- or after-school mathematics enrichment sessions, which increase the amount of time students devote to mathematics each week; (b) enrichment rather than remediation, aimed at helping students become interested in mathematics, confident in their ability to succeed in mathematics, and viewing mathematics as more than just computation; (c) focus on developing conceptual understanding with students constructing knowledge rather than on drill and practice aimed at memorizing procedures; and (d) starting at an early age-- in our case, in second grade, providing a solid foundation and feeling of success.

Our presentation will discuss the components of our model in more detail.

Developing children's mathematical thinking

George Booker and Denise Bond
Institute for Learning in Mathematics and Language
Griffith University, Brisbane, Australia

Mathematical skills needed in today's technological society have vastly changed. It is no longer enough to be proficient at pencil and paper computations, once so highly valued and essential. Today, employers require employees to apply, interpret and solve mathematics in a variety of situations involving computers, calculators and measuring instruments as well as pencil and paper. Indeed, a capacity to solve novel problems is seen to be integral to all facets of modern society.

Although mathematical problem solving has been emphasised since Polya's work in the 1940s, it is only in the last five years that a multitude of programs concerned with the integration and teaching of problem solving have evolved. The area of assessment has not attracted the same amount of attention and analysis and educators are only now beginning to come to terms with the need to assess problem solving. This has brought about a corresponding concern for students experiencing difficulties. As a result, there is a need for a means to diagnosis difficulties in problem solving and to assist particular individuals to develop more appropriate processes. The research to be reported at this session focussed on assessment and remediation of mathematical problem solving.

The assessment component involved the construction of a diagnostic instrument designed to provide a profile of a student's problem solving ability highlighting both strengths and weaknesses. The data showed that these ranged from an understanding of the underlying computation or numeration skills, through an ability to read or comprehend problem statements, to a capacity to implement appropriate problem solving plans or strategies .

The remediation component involved the development of appropriate teaching methods through the construction of a teaching program. Weaknesses or omissions discovered through analysis of the diagnostic results were incorporated into a weekly teaching program which utilised a cooperative approach to encourage a shared approach to solving problems and emphasised discussion and reflection. An additional feature was the construction of problems by the participants. Each session was video-taped for subsequent analysis.

CONTEXTUALIZED TESTING AND EARLY MATH COMPETENCY

Mary E. Brenner

University of California, Santa Barbara

The purpose of this study was to examine the mathematics performance of children entering school for information about the nonschooled mathematical skills that children bring to the educational process. Two recent emphases in the study of cognition have shaped this project--situated cognition and the role of language in mathematical thought. The study asks whether children will display more mathematical competencies when tested in their own dialect of English and through more situated tasks which are akin to what children do in everyday life. The subjects for this study were four classrooms of Native Hawaiian children just beginning kindergarten (N=107). Many of the children were from low income families and many of them spoke Hawaiian Creole English (HCE). The children were given two distinct tests over three individual testing sessions. Both tests sampled skills that are typically covered in kindergarten including counting, comparisons, money, simple computation and vocabulary. The Paper test was a paper and pencil exam based upon the school curriculum. One section of this test was given twice, once in standard English and once in HCE in counterbalanced order. The Puppet test was designed to test children in a more situated manner using puppets to act out two everyday scenarios-- a party and buying stickers. The Paper test and Puppet test were about equal in measuring children's overall level of mathematical knowledge and showed a similar pattern of strengths and weaknesses. A comparison of twelve similar test items revealed some important differences. The children looked stronger in counting skills on the Paper test because the task of counting real items had more complexities due to a less orderly arrangement of test materials in the course of acting out scenarios. However, the children looked stronger on items testing one-to-one correspondence on the Puppet test because the situation strongly cued the appropriate action. Many of the children, particularly in the rural sample, showed enhanced performance on the HCE version of the test. An item analysis revealed that the difference was due to specific semantic items which differ between HCE and standard English rather than more general differences in syntax or pronunciation. Although the various test methods did not reveal different _levels_ of competency in the children, they demonstrated that these children have basic school readiness skills as well as language-specific skills which are not normally obvious in the school context.

THE FAILURE OF ONE FIFTH GRADE STUDENT TO USE IMAGERY IN MATHEMATICAL UNDERSTANDING

Dawn L. Brown and Grayson H. Wheatley
Florida State University

Over the last thirty years there has been considerable interest in the relationship between spatial ability and mathematics achievement. Because the search for a relationship between these measures was psychometric, the results were equicoval. However, using qualitatiive methods a link has been established between mathematical meaning and the use of imagery. In particular, Brown and Wheatley (1989, 1990) provided evidence that fifth grade students use imagery in the construction of mathematical meaning. In these investigations it is important to recognize that we consider mathematics to be the activity of constructing patterns and relationships rather than the memorization of procedures. Further, our idea of imagery includes not only concrete imagery (pictures in the mind) but abstract and dynamic mental activity.

The purpose of the current investigation, of which only a small part is reported here, was to extend our understanding of how students use imagery in their mathematical activity. Students' imagery and mathematical understanding was probed using qualitative data from a variety of sources. The results to be presented here, however, will be confined to a detailed analysis of one student, Jean. We chose to focus on Jean in this paper because her poor imagery seemed to limit the meaning she could give to mathematics tasks.

Prior to the beginning of the school year in which these data were collected Jean had attended a local elementary school where computational procedures were emphasized in the mathematics curriculum. On a standardized achievement test given during the year of the experiment Jean performed well, particularly on the computation and problem solving subsections. However, her score on a test of mental rotations (WSAT) was relatively low.

During clinical interviews using spatial tasks which required students to construct and represent a mental image, Jean's imagery was found to lack vividness and resolution, but was still better than most students who had been interviewed previously who scored very low. She also had difficulty performing mental transformations. Of particular interest here, however, was her weaknesses in her relational understanding of mathematics. For example, she was unable to interpret diagrams or use them to help her in the solution process. This is what Bishop has referred to as the ability to interpret figural information (IFI). When confronted with a mathematical problem her approach was often to perform some arithmetic operation on the numbers in the problem statement. These calculations were performed mechanically and without any attempt to make sense of the problem. Her understanding of geometry was also extremely weak. She could recognize geometric figures, but had little knowledge of any properties of the figures. She also had little understanding of geometric transformations. Only in one case, that of fractional quantities, did Jean seem to have a visual image that helped her solve problems. She could visualize fractional quantities such as 1/37 or 13/10 and sort them into the categories: "about 0," "about 1/2" or "about 1."

We feel the analysis of these data to be instrumental in helping us understand the role imagery plays in mathematical understanding. In particular, her lack of understanding of information in diagrams (IFI) severely limited her ability to make sense of many mathematical tasks. These results also suggest an interesting teaching experiment for future research. Since Bishop has suggested that this ability to interpret figural information is one type of spatial ability that can be taught it should be possible to structure experiences that would allow Jean and students like her to make sense of figural information and increase their ability to give additional meaning to her mathematical experiences.

Title: Intentional and attentional models of number
Presenter: Stephen Campbell
Institution: Simon Fraser University

The focus of this presentation is to compare and contrast some central aspects of Husserl's theory of intentionality regarding the phenomenological origins of number with von Glaserfeld's attentional model for the conceptual construction of units and number.

Husserl's theory of intentionality was initially inspired through his attempts to elucidate the 'cognitive accomplishment of arithmetic and of pure analytical mathematics in general'. As a student and assistant to Weierstrass, Husserl perceived the need to clarify the concept of whole number as a crucial step in establishing a rigorous foundation for analysis. Further influenced by Brentano's notions of intentionality and his descriptive approach to psychology, Husserl embarked upon a renewed Cartesian quest to ground objective knowledge upon essential and ideal forms accessible through phenomenological analyses of conscious experience.

Although differing from Husserl in ontological commitment, von Glaserfeld has acknowledged Husserl's 'enormous merit in advancing the investigation of the concepts at the very root of arithmetic and mathematics' and having been 'one of the first to realize the role of what we now call "reflective abstraction"'. For von Glaserfeld, Piaget's distinction between empirical and reflective abstraction, combined with Ceccato's interpretation of conceptual structure as 'patterns of attention', forms the basis for his attentional model (Steffe et al., 1983).

Whereas von Glaserfeld's model is elegant, relatively simple and clear, Husserl's intentional analysis is subtle, complex and difficult to unravel. Recently, Husserl's work in the philosophy of mathematics has been rendered more accessible by Miller (1982). Drawing upon Miller's study, this presentation constitutes an attempt to encapsulate central aspects of Husserl's analysis of number with extended von Glaserfeld notation in order to accentuate similarities and differences in these two approaches.

Miller, J. Philip (1982). *Numbers in presence and absence: A study of Husserl's philosophy of mathematics.* The Hague: Martinus Nijhoff.

Steffe, L. P., von Glaserfeld, E., Richards, J., & Cobb, P. (1983). *Children's counting types: Philosophy, theory, and application.* New York: Praeger.

STAFF DEVELOPMENT IN AN URBAN CENTER: CHANGING PRACTICES BEFORE BELIEFS BY PROVIDING STRUCTURE AND EXPERIENCE

Suzanne H. Chapin and Walter Stone
Boston University, Boston, MA

Most staff development programs fail to reach their potential; teachers return to their classrooms following training with new ideas and evident excitement, but do not implement what they have learned. We have found that providing the structure and experience necessary to change the way teachers actually teach produces significant improvements in classroom instruction and leads to fundamental changes in teachers' belief systems. This presentation reports the results of a program that directly models new teaching methods in the classroom, providing a context in which teachers must teach differently and reflect on what they are doing.

Many factors are involved in teacher change: teachers' beliefs, knowledge of content, understanding of pedagogy, in-class performance, ability to translate knowledge into appropriate practice, and perception of support systems. The Classroom Centered Teacher Development Mathematics (CCTDM) Project[1] developed a three-part model for teacher staff development that combines and integrates these variables: 1) instruction in mathematics content and pedagogy; 2) mechanisms that contribute to the implementation of content and pedagogical changes in the classroom (such as modeling of lessons, guiding the planning and constructing of lessons, and providing opportunities for teachers to reflect on the educational experience); and 3) establishment of support systems among project staff and teachers to encourage shared (and supported) risk-taking.

In this model, changes in individuals' beliefs — about mathematics, about themselves as learners of mathematics, about themselves as teachers of mathematics, and about students as learners of mathematics — followed classroom practice and observation, reversing the traditional model of instructional change following changes in beliefs. As teachers witnessed cognitive and attitudinal changes in students' learning outcomes, their own beliefs changed. Changed beliefs reinforced and supported further risk-taking and further changes in the teaching of mathematics.

[1] Funding for the CCTDM Project was provided by the Dwight D. Eisenhower Mathematics and Science Education Act Eight teachers of grades four, five, and six from Chelsea, Massachusetts participated in the project.

Title Unguarded metaphors in educational theorising: Metaphorically modelling the social-personal interface

Presenter David J. Clarke

Institution Mathematics Teaching and Learning Centre
Australian Catholic University (Victoria)
Oakleigh, Victoria 3166, Australia

Our theorising in educational contexts, and in particular in relation to the process of learning, labours under the burden of an inheritance of metaphors which may be both inappropriate and inhibiting with respect to our continued theorising about cognition. In particular, the metaphors of: reflection, negotiation, and abstraction should be subjected to immediate scrutiny. These metaphors have been employed as implicit models of aspects of the process of coming to know in socially-located situations such as classrooms. The unguarded use of each of these metaphors has enabled theorists to gloss over the need to specify any mechanism whereby an interaction in the social domain is translated into a corresponding perturbation in the cognitive domain. The use of such phrases as "the reflective abstraction of experience" or "the negotiation of meaning" compound the uncertainty and confuse social and cognitive phenomena. It is the purpose of this paper to suggest how these and other metaphors might be incorporated with legitimacy into a coherent model of the process of coming to know.

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Title **Implications of Varela's *The Embodied Mind* for mathematics education**

Presenter **A. J. (Sandy) Dawson**

Institution **Simon Fraser University**

The position I wish to advance in this presentation is based on these points:

- The world does not come to us prepackaged upon birth, nor are we parachuted into an already completed world which we then have to develop representations of in order to make *sense* of that world.
- There is no one right way, no one right algorithm, and no non-buggy algorithms. Buggy algorithms, however, are not defective and in need of a fix.
- Evolution is not about the survival of the fittest. Evolution is about *satisficing*:

  ...the second step...is to analyze the evolutionary process as *satisficing* (taking a sub-optimal solution that is satisfactory) rather than optimizing: here selection operates as a broad survival filter that admits any structure that has sufficient integrity to persist. (Varela et al, p.196)
- The important thing about interactions with students is the richness of their answers. Teaching is not paramount. Ascertaining the meanings students have is paramount, not to see if theirs match ours or a computer screens, but to understand how they are making their world. It should be assumed that students will have many ways of doing things, many ways of interpreting things, and many ways of *seeing* what is on a computer screen.
- How is knowledge created? We try things out to see how they work, to see if they fit, to see if they match. If they work, then the *new integrates* what had been there before, *the old*, and *subordinates the old to the new*. It is not a question of accommodation or assimilation!
- Knowledge creation does not take place in a vacuum, and is not an individualistic preoccupation; it's not done by a cognizing agent operating alone:

  .. cognition is no longer seen as problem solving on the basis of representations; instead, cognition in its most encompassing sense consists in the enactment or bringing forth of e world by a viable history of structural coupling. (Varela et al, p.205)
- The future descends--it is not given, it is not preordained, but the interaction of the cognizing agent and the environment exercises options which are viable.

Reference

Francisco J. Varela, Evan Thompson & Eleanor Rosch (1991). *The embodied mind: Cognitive science and human experience.* Cambridge, MA: The MIT Press.

EFFECTS OF TEACHING PROCESSES IN ELEMENTARY SCHOOL MATHEMATICS

John H. Dumin
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Australian Catholic University
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Two studies of teaching problem solving processes were concluded, one with a fourth grade class in Australia and the other with a fifth grade class in the United States. Students in each class were randomly assigned to two groups. One group received instructions on problem solving processes while the other served as a control. Teachers analyzed the subject matter to identify mathematical skills. Problem solving processes of composition, conjunction and inverse, as well as skills, were taught to the students in the problem solving processes groups. Only the content skills were taught to the control groups. Pretests and posttests on content and transfer material were administered. The students in the processes group did as well on the subject matter knowledge as the control group. However, on task involving processes both with content and transfer material, the processes groups performed significantly better. In one study only students in the process group were able to solve a transfer problem.

Title Exploring Mathematics and LOGO in a Costa Rican Classroom

Presenter Laurie D. Edwards

Institution University of California at Santa Cruz

This report will describe work with Costa Rican students in the area of computer-based mathematical explorations, presenting information about the use of a computer microworld for exploring positive and negative numbers. An overview of a national project in technology and education in Costa Rica will also be presented (the project is a joint effort of the Costa Rican Ministry of Public Education and the Omar Dengo Foundation, a non-profit group).

The "mathematical explorations in Logo" study was carried out as one part of the Foundation's Computers in Elementary Education Project. This project, ongoing since 1988, has placed computer laboratories (20 IBM PS/2 computers plus printer) into more than 160 schools, primarily in rural and marginal urban settings. In 1992, a new goal was set, that of extending the use of Logo into the teaching and learning of mathematics. This report provides details of a computer environment for mathematical exploration created by the author and tested in a Costa Rican elementary school.

The objective of the work was to design and program Logo-based learning environments for elementary age students in which the students could use the computers to enhance their understanding of mathematics. The environment described in this report was concerned with addition and subtraction of positive and negative numbers. The microworld consisted of dynamically-linked visual and symbolic representations of positive and negative numbers, with the central visual representation being a number line. The number line was originally displayed with only the origin labeled, and extending to the right. The students were given a set of commands to make a "rabbit" (also displayed) jump along the number line. In doing so, number sentences or equations showing additions and subtractions were also generated and displayed above the number line. If a set of entries by a student resulted in the rabbit jumping to the left of the origin, the number line was extended (in a different color) to include the negative numbers. The students were thus able to use the visual and symbolic feedback from the number line to construct an understanding of the addition and subtraction of positive and negative numbers.

This microworld was tested with a whole class of 6th grade students (32 students, approximately 12 years old), and observational data was collected. Although this was a very small scale pilot of the microworld, there were interesting results concerning the students' attempts to make sense of negative numbers in this context. The informal results will be presented, as well as a discussion of plans for further testing and research with the microworld.

Gender and Ethnic Differences
in the Diverse Environment of an Urban College Mathematics Classroom

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The subjects involved 233 students (52 males, 181 females; 21 African-Americans, 55 Whites, 110 Latinos, 42 Asians and 5 Others) enrolled in mathematics courses required for non-majors. Within this subject pool of varied ethnic and social backgrounds, there were 92 undergraduates, plus 74 junior college transfers, and 67 graduate students, with ages ranging from 19 to 53 years. Fifty-five students were willing to indicate they were undocumented residents, immigrants, or entered the United States on a visa. In particular, 40% are residents of the inner city, with 11% living in South Central Los Angeles.

Among a dozen indicators, e.g., achievement, attitude, etc., the technique for measuring median household income is of special interest to researchers. In preliminary characterizations of data collected over a two year period (1990-1992), the students scored significantly lower on ELM math achievement than other CSU students [$\underline{M} = 427$, $t(96) = -2.3$, $.0238^{*} \leq \underline{p} < .05$], significantly lower SAT math scores than the national average [$\underline{M} = 387$, $t(96) = -10.9$, $1.63e^{-18} \leq \underline{p} < .05$] and had significantly lower incomes than the median household income, or \$34,965, for Los Angeles County [$\underline{M} = \$32,700$, $t(232) = -2.55$, $.0122 \leq p < .05$).

But with regards to gender, this data set yielded no significant differences at the $p < .05$ level. Moreover, investigations of ethnic and immigrant differences yielded mathematics self-concept means, in ranked order, as Asians, Latinos, Whites & African-Americans ($\underline{M} = 22.7, 22.5, 21.8, 21.2$), but the narrow differences were non-significant. But variance within these groups, as measured by regression, found age, achievement and self-efficacy, or expectancy, as significant indicators of mathematics self-concept.

Burton, L. (Ed.). (1990). *Gender and Mathematics: An International Perspective*. New York: Cassell.

Benson, J. (1989). Structural components of statistical test anxiety in adults: An exploratory model. *Journal of Experimental Education*, 57, 247-261.

Hart, L. (1992). Two generations of feminist thinking. [Review of *Gender and Mathematics*]. *Journal for Research in Mathematics Education*. 23, 79-83.

ISSUES IN THE DESIGN OF CURRICULUM-EMBEDDED PERFORMANCE ASSESSMENTS IN MATHEMATICS

Joan I. Heller
Educational Testing Service

In October 1991, the California legislature committed the state to developing a new assessment system which will make assessment "an integral part of the instructional process, pursuant to which all pupils have the maximum opportunity to demonstrate what they know; to think, problem solve, and apply their knowledge in testing situations; and to be assisted and motivated through the assessment process to reach higher levels of learning" (S.B. 662, 1991, Sec. 2). One component of the new system, *curriculum-embedded assessment*, would consist of instructional performance assessments selected by teachers from sets of state-approved tasks, to be given to students at times most appropriate to the instructional flow in their classroom. We are engaged in a collaborative effort with the California Learning Assessment System to develop and formatively evaluate curriculum-embedded assessments at three grade levels.

In this report, we present an analysis of several design possibilities that emerged as teams of teachers worked over the past year to define what it means for state assessments to be instructional and to be embedded in curricula. Major design options correspond in part to different levels of state versus teacher specification of instructional and assessment activities. Deliberations over these options suggest: (1) An assessment framework is critical for identifying adequate samples of student performances. (2) Even if teachers and students are provided with choices within and among menus of tasks, the assessments are not likely to be viewed as "embedded" unless they can be customized to fit into particular classroom contexts. (3) Many teachers strenuously oppose de facto state control of extended instructional activities. The state may prescribe tasks that require days or weeks to complete (e.g., relating certain aspects of two long, complex texts) but should not attempt to control day-to-day instructional processes. (4) In order to support new and inexperienced teachers, however, the state should provide tasks that can be used without modification by teachers who are not able, or don't want, to customize them. These considerations argue for development of well-specified tasks, accompanied by guidelines within which they may be customized.

Investment of Self as a Component of Mathematical Engagement

Kathy M.C. Ivey and Steven R. Williams
Washington State University

Recommendations for reform from Dewey to the NCTM Standards have had as a central concern engaging students in the learning and doing of mathematics. The importance of characterizing those factors which lead to such engagement becomes critical as attempts are made to hand over more responsibility for learning to students—as we expect students to conjecture, justify, and explain their thinking. Many factors affecting engagement have been suggested, among them self efficacy, volition, and orientation toward goals. Whatever the merit of these constructs, the hope implicit in current reform documents is that the proper combination of curriculum and teaching method will result in engagement for most students.

On one level our data provide support for current reform efforts (specifically the NCTM Standards) and indict traditional mathematics teaching. At a deeper level, however, the data suggest that possibly critical factors affecting engagement are missing from the discourse surrounding reform. Specifically, our data suggest that the motivational constructs listed above are not sufficient to characterize engagement for all students. For example, the opportunity for a student to invest "self" in doing mathematics is one critical factor emerging from our data. This investment of self is mediated by conscious decisions regarding what is worthy of such investment, and is therefore unique to each individual.

In this report, based on data from a larger project on discourse in mathematics classrooms, we instantiate the problem of engagement using data from classroom observations, questionnaires, and careful interviewing of ten students from a classroom where norms for discourse, justification, and participation are in flux. Narrowing to a case study of one student, we describe a pattern of engagement across a school year and outline the reasons he gives for his engagement (or detachment) from classroom mathematics.

Our student sees space for creativity and self expression as crucial for engagement. In general, he believed that "in math, you can't think your own way." In part, this was due to a perception that mathematics was "set up," that it was "its own little system." He contrasted this to other subjects in which "you can do whatever you want, you're making it up as you go." We characterize this as the central factor in his judgement of worthiness for engagement. Although it is related to a feeling that math is done by others and delivered to him, it differs in that he feels that "you can't use any of your own thoughts because it's going to be the same all the time." He characterized mathematics as being always the same, repetitive, and reliable. Thus, it doesn't matter who does the mathematics; it will be the same whether he does it or whether others do it and hand it to him. In contrast to writing, where there is something of the author in the written work, there seems to our student to be nothing of the mathematician in the mathematics.

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FEMINIST PEDAGOGY AS A TOOL FOR PROMOTING THE STUDY OF MATHEMATICS BY FEMALES

Judith E. Jacobs

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Women often are blamed for their "inability" to perform as well as men in mathematical and scientific fields. It is my belief that the problem is not with women's ability to do mathematics. The problem lies with their unwillingness to study mathematics as it is currently taught and currently constituted. This presentation will present a conceptual framework for encouraging women to believe that mathematics is something they want to study.

Mathematicians and mathematics educators have much to learn from women studies programs and the feminist pedagogy developed in those programs. We also need to examine what it means to know mathematics. For too long the traditionally valued deductive approach to mathematics has been emphasized to the exclusion, or devaluing, of knowing mathematics inductively and through intuition.

Aspects of feminist pedagogy can be used to make women more willing to study mathematics, and therefore learn mathematics. In addition, a different view of mathematics, how one does it and what it means to know mathematics, that complements the discussion of how women come to know things as presented in *Women's Ways of Knowing* will be presented.

Theoretical Foundation:

Belenky, M. F., Clinchy, B. M., Goldberger, N. R., & Tarule, J. M. (1986). *Women's Ways of Knowing: The Development of Self, Voice, and Mind* . NY: Basic Books.

THE HARMFUL EFFECTS OF ALGORITHMS IN PRIMARY ARITHMETIC

Constance Kamii and Faye B. Clark

The University of Alabama at Birmingham

Piaget's theory, constructivism, states that children construct logico-mathematical knowledge from within, in interaction with the environment. On the basis of this theory, we have been working in a public school with teachers who do not teach algorithms but, instead, encourage children to invent their own procedures. The teachers at this school who did not teach any algorithms in 1991-92 were distributed as follows in grades 1-4, respectively: 4/4, 2/3, 1/3, and 1/3.

The performance of children in grades 2-4 who had and had never been taught algorithms was compared in individual interviews. The students were asked to solve written problems such as the ones below. They were asked to work each problem without paper and pencil, give the answer, and explain their procedures. The interviewer took notes on what each child said.

7 + 52 + 186	504 -306	13 x 11 (presented only to third and fourth graders)

It was found at every grade level that children who had never been taught algorithms produced more correct answers and that their wrong answers reflected better number sense. In second grade, for example, 45% of the No-Algorithm Class got the correct answer to 7+52+186 compared to 12% in the Algorithm Class. The wrong answers given by the Algorithm Class were: 29, 29, 30, 198, 200, 295, 838, 906, 938, 986, 989, 1000, and 9308. (The answers of 29 were obtained by adding all the digits as ones [7+5+2+1+8+6]. Totals in the 900s were obtained by adding 7 to the 1 of 186 and carrying 1 from the tens column.) The following wrong answers found in the No-Algorithm Class reflected better number sense: 138, 235, 236, 243, 246, 255, and 617.

These and many other data support the conclusion that algorithms make students give up their own ways of thinking and "unteach" place value, thereby hindering children's development of number sense (Kamii, C. (in press). Young children continue to reinvent arithmetic, 3rd grade. New York: Teachers College Press).

Title HOW DOES OBSERVED STUDENT UNDERSTANDING COMPARE WITH THE ULTIMATE RESULT OF CLASSROOM ASSESSMENT, THE SEMESTER GRADE?

Presenter JUDITH KYSH

Institution UNIVERSITY OF CALIFORNIA, DAVIS

How does observed student understanding compare with the ultimate result of classroom assessment, the semester grade?

The College Preparatory Mathematics: Change from Within Project has developed replacement texts for Algebra, Geometry and Algebra 2. The overarching goal of the materials and the teaching methods they are designed to support is to develop student understanding of the mathematics they are learning to use. But no one is quite sure how to evaluate the development of understanding, so learning more about and developing better means of assessment is the goal of a second major project. The study I am reporting is a preliminary study for the further work we will be doing in assessment, and its purpose has been to help us better focus some of our questions.

Twice a week during the spring semester of 1992, I visited three Algebra 1 classes, in which the CPM materials were used. In the classrooms I was an observer when the teacher was leading the whole class, but when the students were working in groups, about 25-40 minutes in the classes where the teachers were using the materials as intended, I circulated among the groups, as did the teacher, and responded to questions. The three teachers and I selected a diverse group of about twenty students to focus on more closely. The teachers selected students they felt could do better than they were showing on classroom assessments, students whose grades were in the middle range, B to D. I selected a subgroup of that group for more practical reasons. They were the thirteen students whose work I came to know best as I circulated and worked with groups.

In addition to the classroom work, I held 35-50 minute interviews with six of the students in late May. In the interviews we "walked through the book" and paused to examine problems they chose that they did not fully understand.

Based on the examples of the work we did together, four categories or levels of understanding/non-understanding emerged. These categories have some interesting relationships with the students' grades and have lead to a number of additional questions in regard to ethnicity and gender as well as about what our testing and other written assignments measure.

Title Relationships Between Anxiety for Teaching Mathematics, Prior Student History, and Expectations for Mathematics Teaching Style Among Prospective Elementary School Teachers

Presenter Gavrielle Levine, Ph.D.

Institution C.W. Post Campus, Long Island University

Elementary school teachers are expected to teach all content areas. Recent emphasis has been placed on the way in which mathematics is taught, in addition to the mathematical content itself (NCTM, 1989). Several factors may influence a teacher's presentation style, particularly with respect to mathematics, including anxiety about teaching mathematics and the way in which the teacher was taught mathematics as an elementary or secondary school student. The present study examines the relationship between anxiety for mathematics and mathematics teaching style and prior student-experience and teaching style in pre-service elementary school teachers.

Prospective elementary school teachers enrolled in a course concerned with current methods and materials of teaching mathematics completed questionnaires designed to measure their anxiety about teaching mathematics, their prior experience as a student, and their anticipated style of teaching. Anxiety about teaching mathematics was measured by an adaptation of the State-Trait Personality Inventory (STPI) (Spielberger, 1979). The STPI includes two sets of questions, one measuring state anxiety, which can vary with time, and the other measuring trait anxiety, which is hypothesized to remain relatively stable over time. The heading of the trait anxiety measure was modified to read "How do you feel about teaching mathematics?" to focus responses. No other changes were made in either the questions or administration of this instrument. Both the state and trait measures were administered to all participants. A background survey was developed to assess the way in which the prospective teachers had been taught mathematics in elementary and secondary school, and the way in which they planned to teach mathematics in their classrooms. All of the measures were administered twice: first during the initial meeting of the course and then following completion of the final examination at the end of the course.

A relationship was found between prior student experience and anticipated teaching style. Anticipated teaching style changed as a result of participating in a mathematics methods course the focus of which was the use of manipulative materials and the reduction of anxiety. Implications for mathematics teacher education will be discussed.

FROM ROLLER COASTERS TO LIGHTENING BOLTS: STUDENTS FINDING FUNCTIONS IN THE WORLD AROUND THEM

Susan Magidson
University of California at Berkeley

Mathematics teachers have long struggled to respond to students' frequent refrains of "When are we going to use this?" and "What does this have to do with my life?" Recent mathematics education reform in the United States (e.g. NCTM Curriculum and Evaluation Standards for School Mathematics, 1989; Mathematics Framework for California Public Schools, 1992) has emphasized the critical role of real world situations in mathematics instruction, with the implicit assumption that the responsibility for conceiving these contexts rests with teachers or curriculum designers. But what would happen if we turned the task of connecting mathematics and the real world back to the students?

I addressed this question in two Foundations of Algebra classes, a class designed to help students develop a conceptual base that would help them to make more sense of their Algebra I classes. My goal had been to teach the conceptual foundations of algebra in a meaningful way — to encourage and support students as they searched for patterns and connections, resolved inconsistencies, made generalizations, and otherwise worked to make sense of the mathematics — but I had not worked to situate the mathematics in real world contexts. Since I was having difficulty myself coming up with interesting real-world examples of linear functions, I turned the task over to the students. Using the abstract definitions of functions with which we had been working, I gave the students the task of coming up with real world examples and expressing them using the four representations we had been studying (words, tables, graphs, equations). The students produced a wealth of examples, far broader than I could have come up with myself and more relevant to their lives.

I will present the results of my analysis of students' examples (n=200) in terms of context and mathematical characteristics and discuss implications for teachers, curriculum designers, and mathematics cognition researchers. Preliminary results show that students' examples were overwhelmingly first quadrant, linear, multiplicative (y=mx) relationships. Examples were fairly evenly divided between examples of continuous functions (e.g. the distance you are from a lightening bolt is a function of the elapsed time between when you hear the thunder and when you see the lightening) and discrete functions (e.g. the number of persons who can ride a roller coaster at any given time is a function of the number of cars, assuming a constant number of persons per car). The wide range of contexts provides insights into the connections students make between mathematics and their experiences in the world.

Violation of Complementarity & Piaget's Conservation of Liquid Quantity Experiment

Egon Mermelstein Kiang Chuen Young

The importance of understanding investigator bias and their effects on research findings is demonstrated with Piaget's conservation of liquid quantiy experiment

Failure of the investigator to incorporate Bohrs' complementarity principle, the acknowledgement of two equally valid distinct points of view, which cannot be apprehended simultaneously, but only sequentially, it is hypothesized, is the source of confusion and ambiguity in conservation of liquid quantity research.

Violation of complementarity, the acknowledgement of one point of view and the denial of another equally valid point of view, as expressed in logical compensation and also in empirical compensation, signals a defect in the construction of Piaget's conservation of liquid quantity experiment. To uncover this defect, attention is focused on the distinction between a transformation of the liquid and the effect of this transformation.

The transformation in Piaget's experiment is demonstrated to be a geometric transformation of position and not a transformation of shape. The change in container shape creates the illusion of a transformation in shape and its' effect of a change in water level A conservation of liquid quantity experiment, which demonstrates the complementarity principle, is presented

EXPLAINING EXPLAINING: EXPLANATIONS AND COGNITIVE CONSTRUCTIONS

Jean Mitchell

Washington State University at Vancouver

This paper presents observations from some students' explaining behavior, and offers a conceptualization of "explanation" within a constructivist framework.

The students (7th graders doing story problems, singly or in pairs) were induced to explain only by a question from a second person. Explaining why they did something was not easy for them, even when it seemed clear from their actions that they had explicit reasons for their actions. Often, the closest they could get to explaining what they had done was to tell what they had done, a phenomenon noted by other researchers such as Schoenfeld (1985). However, the students did at times undertake to explain. Sometimes the act of trying to articulate reasons would help them detect errors. Often, however, it did not, and in these cases the students' language process seemed to reflect the error in their model in an interesting way: when they got to the point of the error, they would pause, then "jump" the gap in their model and go on as if the connection had been made. In at least one case, a conceptual explanation jumped a gap, while a subsequent procedural explanation revealed the error to the student.

This evidence, while thin, suggests that many cognitive connections may be implicit, allowing for the construction of loosely coupled structures. Explanations may be a process of tracing connections. Any gaps in the cognitive structure are either revealed by the trace or "jumped," thereby leaving the connection implicit and preserving the defective structure.

Cognitive constructions, being dynamic, have both procedural connections, which unfold in time and have action descriptions, and structural connections within the model. This offers two types of connections to trace: procedural and structural. It may be difficult for younger students to distinguish the two, resulting in their tracing actions when asked for an explanation.

REFERENCE: Schoenfeld, A. H. (1985). Mathematical Problem Solving. Orlando, Florida: Academic Press.

Title: Using Reflective Practice to Understand Children's Thinking in Mathematics

Presenters: Deborah Muscella and Evelyn Gibbel

Institutional Affiliations: TERC, Cambridge, MA. and Santa Rosa School District, Santa Rosa, CA

Can reflective practice provide teachers with more insight into children's mathematical thinking? What role does a researcher play in supporting the reflective practice as a vehicle to understand how children's mathematical ideas develop? We are addressing these two questions in a two-year research study with 12 elementary teachers. During this presentation, we will share the reflections of one teacher as she traced how her primary-aged children (Grades 1-3) developed their understanding of sorting and classifying. By sharing journal entries the teacher kept over the course of a semester, the teacher will document how her ideas changed about what the children were learning; the researcher, will describe her interpretations of the children's mathematical thinking and the evolution of the teacher's thinking.

In using the term reflective practice we refer to teachers' ability to be circumspect about their learning, to ask questions of themselves and of others, to challenge their present practice, to consider alternatives to teaching (Schon, 1992). In examining reflective practice, we are using narrative method, which is the description and restorying of the narrative structure of educational experience (Clandinin and Connelly, 1991). In this paper, the teacher describes how her primary students develop their understanding of a mathematical concept, sorting and classifying. During the first few lessons, she reflects on how solid her students' concepts about sorting and classifying are. In fact, she wonders if the investigations that she presents are too elementary for the students. Over the course of the next few weeks, however, she realizes that the students' understanding of sorting and classifying is less secure. She begins to see considerable developmental variation among the students. She has come to this realization by writing her reflections and looking more deeply into how her teaching influences what the children are learning.

The researcher looks at the teacher's reflections as a way to understand how one teacher assesses children's mathematical learning. She reflects on the teacher's learning, examining how her pedagogy evolves in response to her assessment of what the students are learning. The teacher's reflections serve as a tool to understand more clearly how reflective practice adjudicates teaching and learning in classrooms.

Title

Presenter

Institution

TITLE: Exploring Concrete Approaches to Algebra

PRESENTERS: Bill Parker and John Dalida

INSTITUTION: Kansas State University

Our nation's goals and priorities for schools have changed from increased attendance at the turn of century, to equality of educational opportunity in the 1950s and 1960s, to academic achievement for all students in the 1990s (Graham, 1993). With this emphasis on outcomes and success for all students in the 1990s, many people are rethinking the traditional algebra course which many consider the gatekeeper to success beyond high school. As a result, several commercial publishers have introduced a variety of packages to introduce or to teach algebraic concepts and procedures. We will report on initial investigations of preservice elementary teachers learning algebra using concrete materials. This investigation focuses on the value of these materials in overcoming the discontinuities between arithmetic and algebra.

The Mismatch Between Pedagogical Innovations
And Student's Willingness To Change: A Case Study

Vânia Maria Pereira dos Santos
Universidade Federal do Rio de Janeiro

The purpose of this case study was to examine one teacher candidate's reluctance to change her conceptions and beliefs about mathematics learning and teaching in the setting of the mathematics classroom and to question teacher educators' attempts to implement changes in mathematics teacher education. From January to May 1991, a class of twenty-six teacher candidates was observed for fifteen weeks (three days a week for two hours each day) in the context of a newly designed mathematics content course for preservice elementary teachers. Students were observed while exploring, doing, and talking about group problem-solving activities. The main emphases of this course were the use of problem solving, cooperative learning, reflectiveness and communication. As a natural consequence of these emphases, there was a need to align assessment with instruction and to evaluate students' growth in a variety of ways (e.g., written reflections, individual and group tests, group problem solving). The main study examined the influence of this course on eight students' knowledge, beliefs, and metacognitive awareness about mathematics. Classroom observations, four interviews with each of the eight students from this class, and document analysis of students' work formed the bulk of the artifacts which were examined in the main study.

This case study addresses the dilemmas experienced by one student (from the eight students interviewed) enrolled in this course as an attempt to understand which were the constraints to altering her belief systems about mathematics instruction. From August to December 1991, I gained more insights from this student through sessions of member checking of her story and sessions of mathematics tutoring. The closer contact with this student suggested that she was reluctant to accept the challenges proposed by the novelties of the course. She did not enjoy (a) to experience the alterations in students' and teacher's roles, (b) to communicate her approaches to mathematics problems and to explain why certain procedures work in mathematics, (c) to reflect about her mathematics learning process and her mathematics difficulties, and (d) to be responsible for her own learning. The classroom and interview episodes as well as other events that illuminated this research study were analyzed and interpreted by using selected daily field notes of mathematics lessons, open-ended interviews, and notes from tutoring sessions and member checking sessions. The analyses indicated that the novelties and challenges of the course almost did not affect her beliefs about the (a) teacher as the dispenser of knowledge and the mathematics knowledge authority in the class, (b) student as a passive participant on the learning process, and (c) nature of mathematics learning and teaching as well as assessment. The group work was the only positive aspect of the course that was acknowledged by this student. Being a student who was comfortable with learning mathematics in a more traditional lecture style (that matched with both her learning style and desire of acquiring instrumental understanding) and who was not willing to learn mathematics in a meaningful way, she did not alter her belief systems about mathematics and mathematics instruction, did not broaden her mathematics understanding, and did not enhance her metacognitive awareness of herself as a learner. This case study calls the attention of educators engaged in altering mathematics teacher education programs because it suggested that one semester of innovations was insufficient to alter beliefs and to develop metacognitive awareness of students at varying levels of (a) awareness of themselves as learners, (b) motivation, and (c) willingness to engage in innovations as well as to learn mathematics in a meaningful way

Cognitive exploration on concepts of Calculus in two variables. Concept of function, notion of movement, and graphic representation.

Evelia Reséndiz B., Francisco Cordero O.

Departamento de Matemática Educativa, CINVESTAV - IPN, Mexico

This study reports partial results of an investigation on the construction of mathematical knowledge in the classroom, i.e.,Calculus in two variables.

Due to the objectives of the investigation, we had to join form and content in one knowledge. Form, because different approximations related to the teaching practices of advanced mathematics appeared. And content, because these forms led to new arguments, more related to its use than to the attributes of the methematical object.

Thus, it was important to identify the construction patterns and the most significant representations in their cognitive processes through the environment developed in the classroom. In this case we considered the knowledge in the following relations: teacher-student, student-student, and student-teacher.

The event took place with thirty students, whose ages fluctuated between 19 and 23. It consisted in transmitting the typical themes of a course in various variables, orientating them toward movement arguments. We covered the mathematical contents, observing the discourse generated by the student facing specific situations, based on simple models of continuous variation.

One of the most significant mathematical activities was the project-problem, as a medium where we consider as catalysts the meanings that each student attributes to the knowledge taught and as didactic tools.

Regarding the mathematical content of this course, we could say that the organizing axis was the exploration of functions as forms that describe the slanted behavior of the graphs. We found here a relevant aspect on the cognitive structure: the conception of object function and its attributions of movement considered by the student, they joined them to its graphical and analytical representations and derived in arguments that showed notions of slanted behavior, where the students found meaning to the concepts of limit, differential and integral. - We named this fact, due to its importance in the investigation, Model of Behavior: $F(x,y) = P(x,y) +$ "linear expression", where "linear expression" $= Ax + By$, finding a geometric meaning of "translation" from the original surface $P(x,y) = z$. - $F(x,y) = f(x) + g(y)$ was significant for the elaboration of their Calculus arguments, where a need of separating the variables x and y to analyze any function in two variables $F(x,y) = z$ prevails. This situation appears to us as an important aspect of comprehension transference in the student, where the conception of function (one variable) is transported to an idea of separating the variables to conceive a functional relation of two variables, configurated by the expression $F(x,y) = f(x) + g(y)$. We found three interrelated elements that could orient an explanation on the construction of this knowledge: slanted behavior, representations, and transference.

PLANS FOR KINDERGARTEN GEOMETRY LESSONS: ERRORS BY PRESERVICE TEACHERS?

Cheryl Boyle Rolph

University of Delaware

An analysis of lesson plans prepared by preservice elementary teachers for a unit on geometry for kindergarten students identified serious content and pedagogical errors.

The NCTM Standards recommend that K-4 mathematics content be expanded to include non-arithmetic topics such as geometry; however, research indicates that preservice teachers lack adequate knowledge to teach elementary school mathematics for understanding. Ball & Feiman-Nemser (1988) found that published teachers' guides scaffolded beginning teachers' knowledge by helping them understand more about a topic and how it is learned.

Preservice teachers enrolled in a math methods course were provided the teachers' edition of a kindergarten geometry chapter for use in planning lessons for a 3-day unit on either plane or solid shapes.

Analysis of the lesson plans of 8 students revealed that half the students did not clearly differentiate between 2 and 3-dimensional figures. Content errors included mistakes in dimensionality and incorrect vocabulary and properties. Pedagogical errors consisted of choosing inappropriate activities and materials for the type of unit prepared.

These results indicate that even at the kindergarten level the lesson plans of preservice teachers frequently have both content and pedagogical errors.

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BEYOND ACTIVITIES PROJECT:

A PROFESSIONAL DEVELOPMENT MODEL FOR GRADES 4-6

Sharon Ross

California State University, Chico

The purpose of the Beyond Activities Project was to design, implement, and evaluate a professional development model in mathematics. Eighteen teacher-authors were selected on the basis of their potential for providing leadership in elementary mathematics in a time of change. They were responsible for developing and teaching thematic curriculum units of instruction to heterogeneous classes in summer Young Mathematicians Programs for students in grades 4-6. In a 4-week residential institute setting, the teachers worked collaboratively in teams of three or four--planning and writing in the afternoons and evenings, and teaching the newly developed material in the mornings.

The mathematical themes of the three units are whole number division, decimals, and three-dimensional geometry. Each of the teacher-authored units represents an attempt to support teachers who are learning to use a constructivist paradigm in the teaching of mathematics. In the units one finds emphases on strategies for whole-class processing and discussion after cooperative groups have completed a problem-solving task. No attempt is made to teach algorithms; multiple student-generated solution strategies are preferred. Calculators are available to students throughout. Each unit has a contextual theme and is rich with connections to other mathematical topics and other content areas.

After additional piloting and refinement, the teacher-authors prepared the three thematic units of instruction for dissemination. A one-week seminar prepared 12 additional teacher-leaders to help in the work of leading two-day dissemination workshops in their local regions. For each of the three units, 12 workshops, led by two project teachers, were conducted at sites throughout California. The workshops enrolled more than 2000 additional teachers, providing each with a unit to be used to replace a chunk of traditional textbook instruction.

Another one-week seminar enrolled 40 additional workshop leaders who will, in addition to the original project teachers, be available to do workshops after the funding period of the grant. The project was funded by Eisenhower moneys from the U. S. Department of Education and the California Postsecondary Commission.

How Fourth Graders Construct Area Models for Fractions

Andee Rubin, Cornelia Tierney
TERC, Cambridge, MA

To most students, the mathematical term "area" evokes the automatic response "length times width." Many students who can rattle off this formula do not know that it applies only to certain geometric figures, can't identify the width and length of a rectangle, and don't have a solid intuitive sense of area. What is missing in particular is a qualitative visual sense of area. Because we believe an understanding of area is a basis for important mathematical concepts such as fractions and multiplication, the elementary mathematics curriculum we are developing at TERC includes a significant amount of work on qualitative manipulations of area, connected with activities that conceptualize fractions as ratios of areas.

An area model for fractions makes certain important concepts about fractions particularly accessible. The idea of the "whole" is one such concept. In creating or understanding a fraction, a student must first understand what the "unit" or "whole" is. By working with different sizes and shapes of wholes, students develop a concept of whole that is not attached to a single geometrical figure.

Using an area model for fractions emphasizes three other central concepts:

-Fractional parts of a whole are equal parts. Students create their own fractional divisions of area representations, providing them the opportunity to focus on the equality of the fractional parts.

-Equal parts of shapes are not necessarily congruent. Students encounter and create situations where the two halves or four quarters of a square do not look the same, but are equal in area.

-Two fractions (relative to the same unit whole) can always be compared, even if they have different numerators and denominators.

The session will consist of a brief description of the fourth grade curriculum unit that uses this area model, but will focus on the presentation of students' work, in particular their use of drawings to investigate equalities and inequalities among fractions.

Title A Constructivist Theory of Mathematics in Practice

Presenter Erick Smith

Institution University of Illinois - Chicago

As a theory of learning, radical constructivism received increasing levels of attention during the 1980s. As opposed to earlier tabula rosa theories of learning, it offers a vision of the learner as an active agent, confronting problems and resolving them through goal-directed actions. More recently, constructivism has come under question as over-emphasizing the individual and ignoring the social/cultural setting. Partly as a result of these concerns, there has been an increase in interest in what might be called social constructivist theories, including socio/historic approaches based in Vygotsky, social interactionism, and situated learning.

This paper argues that in many cases, the social constructivists are focusing their question from a different perspective than radical constructivists. Whereas radical constructivists have taken an individual perspective, how is it that an individual can organize and make sense of her/his experiential world, social constructivists have often taken a more social perspective, how do individuals function in their social/historical setting. When we keep these two perspectives and the associated questions separate, the criticisms of radical constructivism seem less valid. Radical constructivism does take social aspects into account when modeling the individual's construction of her/his world. In addition, some might argue that it also does account for the world of social interactions. I suggest, however, that this latter claim, might be analogous to attempting to explain the motion of the planets by using molecular theory - while potentially possible, it may not be the most appropriate. Thus the paper argues that radical constructivism does need a complementary social theory and that both the individual and social perspectives need to be a part of a research program.

Advocates of the notion of practice (Lave and Wenger, etc.) have challenged the cognitive view that knowledge can be encapsulated and transferred to another, arguing instead that knowledge grows out of goal-directed actions within a community of practice. Communitarians (MacIntyre) offer a complementary view of practice that argues that virtues and ethical considerations also grow out of participation in practice, and like knowing in practice, are not subject to 'rational' analysis. In the paper I contend that combining these visions of practice offers the complementary social perspective that radical constructivism needs without undermining its basic emphasis on the individual sense maker. Practice offers both a social perspective for understanding knowledge construction and an ethical basis for knowing. Radical constructivism offers a theory of the individual learner and a means to protect the individuality and diversity among individuals. Putting them together, I argue for a constructivist theory of mathematics in practice. As a specific instance related to education, I develop a vision of what a practice of learning might look like (as opposed to practices of schools) and of mathematics in the practice of learning.

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THE NOTION OF VARIATION IN PHYSICAL CONTEXTS.

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In the didactic of Calculus, at college level, it use in a systematical form terms such as increments and differentials, either in geometric and analytical situations or, later, in applications. Therefore phrases such as " take a differential element" or "considering a differential of..." are in the texts. We can perceive in them that the meaning, of these terms, to be transmitted to the students is independent of the context in which it is formulated.

The object of this study was to study whether the notion of variation held by the students is affected by the context in which it is formulated and by their scholastic level. Therefore, we explored the differences that might exist between the variational thinking of a child and an adult. In other words, we asked ourselves if the differential, for example, will have the same meaning when we refer to a differential of ground, of temperature, of area, etc.

The main sources for the results in this study were epistemological and cognitive. The epistemological study was done trying to understand the construction of the related concepts to the notion of variation in the physical context studied, in the scientific thinking through history, and how these are transmitted as collective knowledge through texts, dictionaries or through lectures.

The cognitive study provided information on how students perceive the notion of variation in a determined physical context. We chose for this study the phenomenon of heat dissemination. It was performed through clinical interviews with three students of 6, 14 and 20 years of age. A laboratory was prepared and four experiments related to the phenomenon were designed and the students had to answer questions derived from these experiments. The interviews were videotaped for later analysis and none took more than 40 minutes.

The results of this study show that the notion of variation depends on the context where it is used. Even more, it emerges in the context, that is, it is shaded by the context. The idea of variation, emerged in an environment of movement of a particle, allows us to talk about a differential of space; our study showed that the idea of a differential of temperature does not come naturally in the subjects of the study nor in the epistemological analysis. It is the handling of these ideas of variation, that makes a phenomenon such as heat dissemination to be more complex than others in terms of its mathematical description. This creates a new problem of cognitive nature: Which contexts of significance, and under which circumstances, are the most adequate to develop notions of variation?

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Title Learning Style and its Role in the Formation of Student Structural Knowledge
Presenter Mary M. Sullivan
Institution Curry College

Researchers report that ability, maturity level, motivation, and attitude contribute to success in mathematics and related disciplines, including elementary statistics. Success has been measured largely by performance on teacher prepared achievement assessments which may or may not measure conceptual understanding, also referred to as structural knowledge. Success and understanding may be only slightly related, especially at the elementary levels in mathematics and statistics.

Tall (1991) says that true understanding, especially at abstract levels, is for the most able. While this may be accurate, it is important to remember that it is the average to above average students who populate many college classrooms. Solid instruction, also essential for effective learning, is more critical for these students. Could the traditional instructional method-- lecture--be sufficiently at odds with students' learning style that their learning and structural knowledge is hampered?

The psychological literature defines declarative knowledge as domain specific knowledge of concepts, facts, and principles, often referred to as the "what" of learning. Procedural knowledge is understanding how to do something such as follow a procedure, and is sometimes referred to as "knowledge how." It is a strategy knowledge. Structural knowledge, also referred to as cognitive structure, bridges the continuum between declarative and procedural knowledge. A consideration of structural knowledge in conjunction with achievement measures offer more information concerning actual understanding than the achievement measure alone.

Students who have what is commonly called an abstract learning style often achieve higher grades in college than those students who learn more concretely. Is it a factor of their intelligence or their learning style? One approach from which to consider the issue might be an examination of the cognitive structure formed by students of different learning styles. Since students are individuals, it is expected that they process information in a manner somewhat unique to themselves. Researchers in the tradition of Witkin, Myers-Briggs, and Kolb have looked at learning style as a variable in the learning process. However, little attention has been given to the role of learning style in connection to acquiring structural knowledge (Jonassen, Beissner, & Yacci, 1993).

The current reform movement in education proposes that teachers teach for mathematical understanding; that teachers conduct their classes so that students' knowledge of mathematics grows as they grow; that they conduct their classes with attention to the development of students' knowledge structure and thinking processes (Tall, 1991). A consideration of students' learning style may enhance the teaching process.

U. S. MIDDLE GRADE STUDENTS' UNDERSTANDING OF PROBABILITY

Jane O. Swafford

Illinois State University

Piaget and Inhelder (1975) concluded that probability concepts are acquired in stages in accordance with their theory of development. To test this, Green (1983) conducted a large study involving nearly 3000 British adolescents and found that probability concept levels increased with age but that most students had not attained the Piagetian level of formal operations by age 16. The purpose of the present study was to explore the development of probability concepts among U. S. middle grade students.

Method Fifty-six children from grades 3 through 8 were presented 9 probability tasks individually in clinical interviews. Tasks used by Green or Piaget and Inhelder and tasks adapted from those used to reveal misconceptions among older subjects were administered in an interview format by trained interviewers who recorded the students' answers and explanations. The data sources consisted of the interviewers' field notes and reports.

Results U. S. middle grade students' performance was much better than that reported by Green for British students. A correct answer was given by over fifty percent of the U. S. students to all but one item. Middle grade students who missed an item gave reasons similar to those given by students in other countries and exhibited some of the judgmental heuristics identified among adults. Overall, the performance of the grade 3 students was lower than that of the older students. However, on some items older students, most of whom had been introduced to probability in school, scored lower than their younger peers.

Conclusions To the extent that the sample was representative, U. S. middle grade students can easily handle the recommendations of the NCTM *Standards* with respect to probability. However, the study does raise the question of why the performance of U. S. students was so much better than that reported for British students.

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CONCRETE REPRESENTATIONS AND LANGUAGE IN DEVELOPING A CONCEPT OF MULTIPLICATION OF FRACTIONS: A CASE STUDY

Susan B. Taber

University of Delaware

Mrs. Adams, a fifth-grade teacher attempted to "teach the concept" of multiplication of fractions by having students represent multiplicative situations with fractions using a paper-folding activity described in her new textbook. During the next two weeks the teacher and students spent 80 percent of their class time constructing and discussing physical representations for problems with fraction multipliers.

This case study documents the students' classroom experiences, examines changes in students' thinking, and makes connections between those changes and specific instructional experiences. The post-test of 20 multiplication word problems revealed that many students had made large gains in their ability to solve multiplication word problems of various kinds. Examination of their written strategies, however, indicates that most students did not associate the operation of multiplication with the kinds of problems they had worked and discussed in class using both concrete models and mathematical expressions.

Close analysis of the language used in class and of the kinds of operations that students actually performed on the models, which were assumed to represent the operation of multiplication, suggests why the classroom experiences actually encouraged students to associate the operation of division with multiplication problems with fraction multipliers.

Title Classroom Research and Classroom Practice: Blurring the Boundaries

Presenters Patricia P. Tinto, Barbara A. Shelly, Nancy J. Zarach
Institution Syracuse University, Westhill High School, Nottingham High School

This case study of two classrooms, one urban, one suburban, describes the evolution of research tools into classroom assessment instruments.

Over a four year period, two secondary mathematics teacher-researchers, cognizant of the NCTM *Curriculum and Evaluation Standards*, began making changes in their teaching practice. They focused on the use of cooperative groups, the use of technology, and descriptive and reflective writing as instructional strategies. The expectation was that the students would gain new perspectives not only about how they learned but also about what they learned. The teachers' intent was to create a shift of authority within the class from teacher to students. Both teachers asked themselves, "How will I know that change is occurring?"

To look at both small group and whole class dynamics, four data sources were used: whole class and small group video tapes, small group audio tapes, individual student journals, learning logs and student reflections that were regularly written as part of each homework assignment, and field notes (audio) from the teacher-researchers giving observations and reflections.

As the two teacher-researchers collected data about their classes, they noted that the research process of systematically collecting and analyzing data gave them insights into their teaching and the students' learning that had immediate impact on their teaching. They heard student voices not previously heard in the more traditional classroom setting. As they listened to the tapes of their groups, they became aware of conversations that they were not part of during the class day. One noted that "It is easy to keep track of a lecture classroom because everything is passed through the teacher, whereas in a cooperative setting, it is impossible for the teacher to listen to every conversation when there are eight groups and eight ideas being talked about." She noted, "Listening to the tapes has helped me see things I should do differently. It allows me to be in on all the conversations." She stated that she heard students modeling her language and her behaviors and it made her more aware of her interactions with students. The tapes and journals originally intended as research tools had, over time, become assessment tools that improved her teaching.

In such collaborative settings, the boundaries between research and teaching blur. One becomes the other. Unfortunately, we are too accustomed to a language that talks about research and teaching as if they are separate entities. Yet both are related forms of reflective inquiry that call for the individual, teacher and researcher, to put herself and her students in a situation of reflective learning. Research as reflective inquiry leads the teacher as researcher to a deeper understanding not only of student learning but also of her role in that reflective process. In this way, the act of carrying out reflective inquiry in the classroom about student learning by the teacher leads quite naturally to more effective teaching.

This presentation is part of an National Science Foundation SGER funded project, 1992-93.

STUDENTS' DIFFICULTIES IN APPLYING MORE THAN PROCEDURAL KNOWLEDGE TO SOLVE MATHEMATICAL PROBLEMS

Manuel Santos Trigo
CINVESTAV, México

This study focuses on the analysis of the students' approaches to problem solving by considering information gathered mainly through task-based interviews (college level). The results show that in order to help students to improve their ways of solving mathematical problems, it is necessary to pay attention to the mathematical content, cognitive and metacognitive strategies, ways of validating and using mathematical arguments, and ways of extending the problems. When all these ingredients consistently become part of mathematical instruction, then the students may develop their own frames for solving mathematical problems. These frames may resemble what people in the field of mathematics do while working on mathematical problems.

Title: Changes in Elementary Preservice Teachers' Beliefs About Mathematics Teaching

Presenter: Nancy Nesbitt Vacc (main presenter) and George W. Bright
Institution: University of North Carolina at Greensboro

This study examined changes in preservice teachers' (PTs) beliefs about mathematics teaching. Since beliefs influence how teachers teach (Kagan, 1992; Calderhead & Robson, 1991), understanding the process of changing behaviors is essential to the current reform in mathematics education.

This study is part of a larger project that is investigating whether cognitively guided instruction (CGI) (e.g., Carpenter, Fennema, Peterson, Chiang, & Loef, 1989) can be incorporated effectively into preservice teacher education. With CGI, teachers use knowledge of students' cognitions as the basis for planning instruction. Effects of CGI have been documented through use of the Teacher Belief Survey (Carpenter, Fennema, & Loef, 1990) which was employed in the current study along with written and oral responses to questions about mathematics teaching. Subjects were 56 undergraduates in a two-year professional education sequence and two groups of graduates (N=23 and 17) in a one-year program. Undergraduate data were gathered across 1991-93, and data from the graduate groups were gathered across 1991-92 and 1992-93, respectively. Analysis of baseline data indicated no significant differences in subscale or total scores on the Beliefs Scale for undergraduate PTs and the first group of graduate PTs. At the end of one year, however, graduates' beliefs were significantly more constructivist than undergraduates'. While both groups cited the importance of knowing mathematics content and the use of manipulatives, undergraduates cited the need to know how children learn whereas graduate PTs cited knowing how to relate mathematics to children's lives and recognizing that children need to explore concepts in developing understanding and to share solutions to problems. Differences in beliefs appear attributable to graduate PTs having had a mathematics methods course (the undergraduates' methods course is taken during the second year), more time in the schools working with students, six weeks of full-time teaching, and more opportunity to develop a language for talking about mathematics instruction.

Based on these data, beliefs about teaching mathematics can be changed during one year of preservice education. Whether the changes are fundamental ones having lasting impact or superficial ones that respond only to demands of the certification programs remains uncertain.

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WHAT IT LOOKS LIKE TO WORK IN A GROUP AT COLLEGE LEVEL MATHEMATICS CLASS

DRAGA VIDAKOVIC
Duke University

Research done in the last two decades shows that small-group task-oriented work in mathematics classes offers learning opportunities that do not typically occur in traditional classrooms. Different types of cooperative learning have been used in traditional lecturing and discovery learning classroom situations. Most of the research on cooperative learning in mathematics has been done in elementary and secondary schools and almost all ot it is quantitative. The purpose of this qualitative study was to examine the experience of working in group in a cooperative learning environment in a freshman calculus class at Purdue University. Specifically, the purpose was to investigate how people interact in order to achieve the academic goal. To achieve the goal of the study one particular group (four members) was chosen and followed over two weeks. The following research questions were considered: (i) How do the students interact? Is there a pattern? ii) What is the social structure of a group? (iii) What are students' attitudes towards the group work and do they change over time? (iv) What does working in groups mean to the students?

Four main categories of communication emerged from the data: one-way, two-way, multiple-way and non-verbal communication. In this specific case multiple-way communication was the dominant mode of conversation.

Working together students established some roles in the group which were not assigned in advance. For example, the only male in the group was the "intellectual" and overall group leader. He was the speaker also. One girl was the "grapher", she always drew graphs for the group. Another girl was the manager for the group. She would divide up the homework assignments and take care that the complete homework assignment was submitted for grading. Third girl was a helper for every one. For the particular group the data showed that the attitude towards group work was positive and pretty stable for all its members. The study also showed a lot of features of small-group work that desirable to the students, including a) the availability of immediate feedback and explanations, b) use of language that fellow students can understand, and c) the shared understanding of the difficulties that fellow students are having with mathematical problems. Also, the study showed some of possible disadvantages of the group work. The biggest concern is the evidence of establishment of the roles within the group. On that way the group members may specialize only certain concepts and not master the rest.

Title The Constructivist Teaching of Constructivist Teaching

Presenter Virginia W. Warfield

Institution University of Washington

In the course of listening to, reading about, and discussing the current theories on the actions and non-actions required of a person who wishes to help someone become an effective, active, interested and lively learner of mathematics I have been struck by the parallels with the actions and non-actions that seem to me to be required of a person who wishes to help someone become an effective, active, interested and lively teacher of mathematics. It is a truism that lectures on the ineffectuality of teaching by lecturing are at best unproductive, but I maintain that the connections are far deeper and wider than just that.

My own perspective on this issue comes from twenty years of directing a program in which classes are taught by "modified group discovery" (translation: the instructor leads the class through a structured sequence of questions, constantly adjusting the ratio of repetition to concept introduction on the basis of student response; the students are responsible for answering not only the instructor's questions but those of their classmates, and for spotting and correcting any errors made by either.) Before they undertake discovery teaching, all of the instructors agree, at least in theory, to follow this format. Many, in fact, are not merely willing, but actively enthusiastic about the idea. They also have in hand a manual in which the lesson plans are laid out in detail: "Have them evaluate 5-(3-8). Give them several more at that level. Once they are comfortable, try 3-[2-(4+1-7)]." Occasionally, all of this produces someone who walks into the classroom and actually does "discovery teaching". Far more often what turns up is a patchy mixture of discovery, lecturing and pure floundering. It is my job to see that the former swiftly replaces the latter two. My vehicle is the notes that I make while sitting at the back of the classroom observing, and the discussions of these notes that I subsequently have with the instructors.

It is in this process that I see compelling parallels with that of teaching in a group discovery classroom--or a classroom anywhere in the constructivist range. If an instructor--or a class--is told too much, the opportunity for insights and the kind of involvement that produces internal growth is lost. If an instructor--or a class--is told too little, the sense of direction is lost, generally (in the case of the instructor) to be replaced by a sense of impending doom. In neither context is there a single "right" amount of information to give, and in both one key is a constant readjustment tuned to the needs of the recipient.

I feel strongly that a study of these parallels would enrich both of the forms of instruction.

Title A THREE STAGE LEARNING MODEL FOR TRIGONOMETRIC FUNCTIONS BASED ON COMPUTER GRAPHS

Presenter ELFRIEDE WENZELBURGER

Institution UNIVERSIDAD NACIONAL AUTONOMA DE MEXICO.

In this study a learning model for functions was used to describe the learning of trigonometry functions with a function plotter on a microcomputer. According to the model, the learning process has three stages:

Stage 1: Free exploration (FE)

Stage 2: Analysis and Comparison (AC)

Stage 3: Experimentation and Practice (EP)

The model is a dynamic and recursive process model and was used as the background for a didactical strategy which was developed in a graphical computer environment. The learning activities suggested by the strategy, lead students to the construction of concepts related to the basic characteristics of trigonometric functions $y = a\ \sin bx + d$ and $y = a\ \cos bx + d$ amplitude, period and position of the graph in the coordinato system.

The learning model was developed on the assumption that learning is a constructive activity (Glaserfeld, 1987, Goldin, 1992) and that different representations are the basis of reasoning. Enphasis was put on visual reasoning (Dreyfus, 1991) and the transition from graphical vs. algebraic representations. Some authors speak in this context about multiple linked representations (Goldenberg, 1988).

Related studies have shown that students can find analogies between graphical and algebraic representations if they are trained to do so (Schwarz, Bruckheimer, 1988, Schwarz and Dreyfus, 1989).

In previous studies (Wenzelburger, 1989, 1990a, 1990b, 1991, 1992) results were found which favored the learning of function concepts of the experimental groups in a graphical computer environment which has the characteristics of a "generic organizational system" (Tall, 1985)

The pretest showed that students participating in the study had some previous knowledge of functions of type $y = a\ \sin bx$ and $y = a\ \cos bx$. Therefore gainscores were computed and a student's t was calculated.

The delayed retention test was given to all students four months later without prior notice.

The reliability of all tests was computed for both groups with the splithalf-method and turned out to be good (values between 0.83 and 0.92). Also the indices of item ease were determined for all items and found to vary between 0.21 and 1.00 (percentages of correct responses of total responses given).

The results reported show that in the posttest, students in the computer groups did somewhat better than the control group, while in the delayed retention test the difference between both groups was more marked and favored the computer group. This is in agreement with results found in previous studies by the same researcher. (Table 1, 2)

TABLE 1

AVERAGE GAINSCORES IN %

	NUMBER OF STUDENTS	AVERAGE	STANDAR DEVIATION
COMPUTER GROUP	14	78.57*	16.48
CONTROL GROUP	11	70.54*	13.25

*Significant at the 0.10 level.

POSTTEST

TABLE 2

AVERAGE SCORES IN %

	NUMBER OF STUDENTS	AVERAGE	STANDARD DEVIATION
COMPUTER GROUP	12	78.83**	13.61
CONTROL GROUP	9	65.88**	17.76

**Significant at the 0.05 level.

RETENTION TEST

BEST COPY AVAILABLE

METAPHORS WE GRAPH BY:

A QUALITATIVE STUDY OF STUDENTS' DIFFICULTIES

AS THEY LEARN TO USE A GRAPHING CALCULATOR

Carol G. Williams

Pepperdine University

As graphing calculators become more prevalent in secondary and college math classes, teachers must become more cognizant of the special difficulties they present. This study looked at four students in a business calculus class as they worked with TI - 81 graphing calculators to solve problems typical of those in their course work. The researcher interviewed each student four times over the course of the semester. The data consists of videotapes of the interviews and the transcriptions of relevant sections.

An analysis of the data shows that these students have difficulty in using the calculator's graphing utility, particularly adjusting the viewing window. One student seems to adhere to a mental model consistent with using the RANGE features of the calculator as a magnifying/reducing machine on which she can set the magnification factor. The researcher elaborates this metaphor and shows how the student's actions are consistent with its features despite the fact it inhibits successful completion of the task. All of the students exhibit use of portions of the metaphor and initially have difficulty with the tasks. Interviews conducted near the end of the course show the students have developed a better understanding of the calculator's functions, although they occasionally revert to their old ways of thinking.

This study also includes a review of current research on using graphing calculators and research on computer graphing utilities that show similar difficulties for the students. The researcher shows how this study relates to the other work in the field. The report also contains implications for classroom teachers and areas for further research.

Title

High School Math Group Projects Using Managers

Presenter Judith Moore Young

Institution College Park High School , Pleasant Hill, Calif

Projects are assigned quarterly in my high school math classes. These are to be done over a two week period outside of class with a group of four students. One student is appointed project manager who develops supervising skills, coordinates and evaluates the other group members.

INDUSTRY FOCUSED EMPHASES

- o Teamwork
- o Communication skills
- o Problem-solving applications
- o Professional work habits
- o Presentation skills -(oral and graphic)

EDUCATIONAL LEARNING OBJECTIVES - in addition to mathematics

- o Relevance - real local data used
- o Research and library work
- o Higher level of persistence than normal is expected
- o Apply graphic, writing and oral talents
- o Use alternative method of evaluation - wholistic grading

ADDITIONAL BENEFITS WITH PROJECT MANAGERS

- o Management and people skills
- o Group dynamics
- o Leadership development
- o Time management
- o Motivational methods

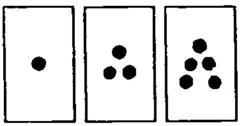
RESPONSIBILITIES OF PROJECT MANAGERS

- o Equitable work assignments using available students & their talents
- o System design - how do all parts fit together
- o Scheduling and projecting timeline
- o Status reporting
- o Adapting for unexpected changes
- o Quality assurance and adherence to standards
- o Presentation performance
- o Evaluation and grading of individuals
- o Communication with "Big Boss" (Teacher)

INFERRING CHILDREN'S INTERNAL STRATEGIC PROBLEM REPRESENTATIONS

Richard A. Zang and Gerald A. Goldin
Rutgers University, Piscataway, New Jersey 08855-1179 USA

We seek to infer children's internal strategic or heuristic representations from overt behavior in a task-based, problem solving interview. This is the first of 6 such interviews in a longitudinal study of 22 elementary-school children over 3 years. The clinician begins by laying out three cards, one at a time:



After a brief pause to allow for spontaneous response/pattern detection, the child is asked, "What do you think would be on the next card?" Our goal is always to elicit a *complete and coherent verbal reason* and a *coherent external representation* (with materials provided) before proceeding to the next question. The child is then asked (in slow succession): • "What card do you think would follow that one?" • Do you think this pattern keeps going? • How would you figure out what the 10th card would look like? • Here's a card (showing one with 17 dots in the shape of a chevron) ... can you make the card that comes *before* it? • How many dots would be on the 50th card? For example "Jacqueline" (age 9, 4th grade, Spring '92) initially added two dots to the bottom of the chevron for her representation of the fourth card, but for the fifth card she "filled in" the chevron pattern. She never drew another "triangular" representation from that point on in the interview. Instead, she made a table with a *linear* pattern of dots; i.e., 1 •, 2 •••, 3 •••••, 4 •••••••, etc. We had expected children to adopt an "add two" strategy. However, we had not anticipated the extent to which the children abandoned the visual/geometric pattern. Although Jacqueline's specific behavior was unique, it was also representative in that eventually almost all the children relied on numerical over geometric representations.

Reference: Goldin, G. A., DeBellis, V. A., DeWindt-King, A. M., Passantino, C. B., and Zang, R. A. (1993). "Task-Based Interviews for a Longitudinal Study of Children's Mathematical Development." *Proceedings of the 17th International Conference of PME* (in press).

Posters

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Mathematics Mada Maaningful Through Instructional Games

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Abstract

An increasing awareness and acceptance of child cantred learning approaches and teaching is growing amongst educators, researchers and members of the general public. Consequently children are being encouraged to talk about and share their ideas and conjectures with peers and important others without fear of rejection or reprimand. The development of environments that encourage children to discuss and justify their thoughts and conjectures is increasingly seen to be important in resolving conflicting view points and establishing communally shared understandings. It is also an area which has been prominent in recent constructivist perspective on learning.

Play is a fundamental part of children's learning. It is their way of making sense of the world around them through situations which are engaging, familiar and non-threatening. Furthermore, games create a social setting that is rich with real world experience. They allow children to use and understand mathematical language and social organisation in a purposeful way.

A research project consistent with constructivism and child centred learning and teaching approaches is presented where instructional games were implemented into a classroom setting. The project allowed the children's mathematical concepts to be developed and their understanding consolidated and reinforced by creating an environment which used language as a bridge between the children's informal mathematical knowledge and abstract mathematical concepts.

Title The Effect of Symbolic Processing on Calculus Students' Mathematical Beliefs

Presenter William Horton

Institution University of Northern Colorado

UNC is a mid-sized university whose audience for Calculus is primarily Liberal Arts majors, many of whom plan to be public school teachers. For this audience, the reduction of tedious calculation, a compelling visual presentation of Calculus concepts, and collaborative discursive projects which are facilitated by a symbolic processing course should pay substantial dividends. In particular, I expect students who use symbolic processing as an essential part of their Calculus experience to believe more strongly than students in a traditional course that mathematics is a sense making activity in which they actively participate. This presentation will discuss the results of research investigating how these beliefs evolve for students in both traditional and symbolic processing based courses.

I have developed a questionnaire which was administered to Calculus students at the beginning and end of the 1992/93 academic year. I will discuss the development and validation of the questionnaire, the statistical results about change in beliefs, pedagogical implications of the study and plans for future research.

Title Linking Instruction and Assessment Through Performance-Based Activities

Presenter Nancy Katims, Ph.D, Miriam Amit, Ph.D.

Institution Educational Testing Service

New views of teaching and learning call for new forms of assessment to validly reflect the nature of learning outcomes and to properly link assessment with curriculum and instruction.

The PACKETS Program for Middle School Mathematics represents an example of an innovative educational program in which such a linkage is created. Based on research by Richard Lesh and other mathematics educators, PACKETS materials include a sequence of activities in which students' learning is being encouraged while the teacher is able to document this learning. The activities promote the use of interdisciplinary instruction, cooperative learning techniques, and the principles of the NCTM Standards.

A sequence of PACKETS activities begins with the students reading a data-rich newspaper article, based on a real-life situation, and adapted for middle school students. A set of readiness questions accompanies the article, to encourage the students to read it with a "mathematical eye." The students then work in groups, usually for one or two class periods, on a model construction activity. The students write up their solutions and present them to the class. Follow-up exercises are designed to show the students the variety of mathematical approaches that are possible and to explore the underlying mathematics in greater depth. Assessment guidelines help the teacher and students to capture the richness and multi-dimensional nature of the student work at both a descriptive and an evaluative level.

In this poster presentation, participants will have the opportunity to see the conceptualization framework upon which the program is based, a sequence of activities from the program, and examples of student work that represent different mathematical approaches to an activity. The discussion will address issues at several levels, including the effects on students, teachers, and classroom instructional practices when materials like PACKETS activities are introduced into middle school mathematics classrooms.

MATHEMATICS OF THE ENVIRONMENT: AN EXPERIMENTAL CURRICULUM

Mathew T. Mitchell

University of San Francisco

Mathematics of the Environment (or MOE) is a new high school curriculum created to address two widely perceived problems in current secondary mathematics instruction: low mathematical achievement and negative student attitudes toward the subject. In order to address these dual problems, MOE firmly places the content to be learned within a real, everyday context that is important and challenging to high school students.

The poster presentation presents both an overview of the curriculum as well as the initial evidence (both qualitative and quantitative) assessing the success of this program. Evidence will be gathered from evaluations conducted during the Summer of 1993 (in two schools) and partial data from evaluations conducted during the Fall of 1993 (from two additional schools). The evaluations will include results from surveys, focus group transcripts, and sample portfolios of student work. The emphasis of the poster presentation will be on MOE's effectiveness in enhancing student interest in mathematics.

MOE attempts to put into practice the results of current research on how to enhance student interest in the mathematics classroom (Mitchell, 1993). Towards this end, MOE was designed to be a mathematical adventure which would be perceived as both meaningful and involving to students. The MOE instruction hinges around the central problem of students assessing the environmental "health" of a nation. Students divide up into groups, each group responsible for investigating a different country. The students' final outcome is to find and propose a solution for resolving an environmental problem within their nation. During the six to eight week period that MOE runs, students need to learn and use specific mathematical skills presented in the course in order to accomplish this final outcome.

References

Mitchell, M.T. (1993). Situational interest: Its multifaceted structure in the secondary mathematics classroom. Journal of Educational Psychology (in press).

Title <u>The Function Minute--An Effort to Teach Basic Function Concepts More Effectively</u>

Presenter <u>James Olsen</u>

Institution <u>University of Northern Colorado</u>

The function concept is unifying and central to the understanding of modern mathematics and mathematical modeling. The function concept is composed of many aspects. This complex concept is further complicated by the many representations a function can have. The difficulties and misconceptions students have with the function concept are now pretty well documented. One reason for the difficulties is that the notions are conceptual in nature, and emphasis and attention is often placed on procedural skills.

The teaching of the function concept and related notions at the College Algebra level is particularly arduous because students arrive with a variety of conceptions, misconceptions, and phobias concerning functions. Students often also come with the belief that functions are a mathematical concoction and of no real-world use.

Jim has developed and implemented, in his College Algebra course (spring semester 1993) , a program to teach basic function concepts. The program, called the *Function Minute*, attempts to show students how functions appear (repeatedly) in the real world, teach the many notions surrounding function gradually (starting early the semester), emphasize multiple representations, and make the function concept a central thread of the course.

SECOND GRADE CHILDREN'S SCHEMES TO COORDINATE TWO UNITS

Tad Watanabe

Towson State University

Children's coordination of units has been shown to influence their understanding of arithmetic (Steffe and Cobb, 1988), fractions (Watanabe, 1992), Proportional Reasoning (Lo and Watanabe, 1993), and geometric reasoning (Wheatley and Reynolds, 1993). However, very little is known about their schemes to coordinate units.

Four second grade chik ~n's attempts to coordinate two different units were investigated. The analysis of the data revealed that there were four different schemes were used by these children. They were: one-as-one, one-as-many, many-as-one, and many-as-many. These schemes appear to reflect the participants' mathematical sophistication. Further investigations of children's unit coordination may be fruitful.

References

Lo, J. & Watanabe, T. (1993). *Children's image-based schemes for solving proportion-related tasks*. Paper presented at the Annual Meeting of the AERA, Atlanta, Georgia.

Steffe, L.P. & Cobb, P. (1988). *Construction of arithmetical meanings and strategies*. New York: Springer-Verlag

Watanabe, T. (1992). *Coordination of units: A case of rational numbers*. Paper presented at the Annual Meeting of AERA, San Francisco.

Wheatley, G.H. & Reynolds, A. (1993). *The construction and coordination of units in mathematics*. Paper presented at the Annual Meeting of the AERA, Atlanta, Georgia.

Discussion Groups

THE OPERATIONAL-STRUCTURAL DUALITY OF MATHEMATICAL CONCEPTS IN ADVANCED MATHEMATICAL THINKING

The A.M.T. Working Group of P.M.E.

Presented by Gontran ERVYNCK, K.U.Leuven, Belgium

Our purpose is to present a topic, taken from theories of cognitive development appropriate for undergraduate learning of advanced mathematical concepts, which has been a major focus of discussion in the AMT Working Group in the last two years. In doing mathematics there is a permanent interaction from *a process* to *a concept* and back. Processes are encapsulated into concepts (Ed Dubinsky); as such, thorough study of the encapsulation process itself may give us an operational approach to mathematics teaching. A typical example is *function*. There are clearly two approaches, as a process or as an object, but which one is more fruitful ? The definition of a *limit* is given as a process, encapsulation toward a concept is often difficult. More elementary examples are *sum* and *product* in a set of numbers; in elementary algebra almost everything is procedural. Another viewpoint considers the need to introduce a concept as a *tool* for solving problems before it becomes the object of reflection at a higher level. The *symbols* of mathematics play a key roll; often the same symbolism is used to represent the process and the concept. In order to translate a process into an object you have first to give a name to a procedure and then to condense it. The term *procept* has been devised (David Tall) to point toward a combination of the three components involved: a process, an object produced by that process, and a symbolism to denote either of these. This allows for a distinction between *operational* procepts such as $3 + 2$, *template* procepts such as $3x + 2$, and *structural* procepts such as $\sum_{n=1}^{+\infty} \frac{1}{n^2}$.

The point of AMT is now: what meaning may be conferred to the adjective *advanced* ? The presentation of advanced mathematics is currently through definition and proof. If one cannot define what f.ex. a group is, how can one make general statements about it? We may moreover raise the question of how do definitions function in the learning process ? Definitions act as descriptions in a dictionary, evoking a context which has to be interpreted. Why do students have difficulties in AMT ? The process has to become an object, which is difficult; in proofs, or solving problems, often the object has to be transferred back into a process, a difficulty on top of the first one. Which teaching strategies are appropriate to help students make the difficult transition to a definition-proof construction of knowledge?

A tentative conjecture to characterise the difference between elementary and advanced mathematics is that in the former the keyword is *procedures*: students start from problems and examples, performing operations to answer questions they move toward the concept; in the latter the keyword is *inferences*: students start from definitions and proofs to build the concept; examples, when used, are closely tied to a definition.

RESEARCH IN THE TEACHING AND LEARNING OF UNDERGRADUATE MATHEMATICS: WHERE ARE WE? WHERE DO WE GO FROM HERE?

Organizers: Joan Ferrini-Mundy, University of New Hampshire, Ed Dubinsky, Purdue University, and Steve Monk, University of Washington

There is increased interest, particularly in the community of mathematicians, in questions about the teaching and learning of mathematics at the undergraduate level. Professional organizations such as the Mathematical Association of America and the American Mathematical Society have begun to encourage attention to this emerging research area within their conference and publication structures. This discussion session is organized by members of the Mathematical Association of America/American Mathematical Society's Committee on Research in Undergraduate Mathematics, to promote a more sustained focus on this area of research, and to continue last year's PME dialogue. We will address the following questions:

Can we summarize major research areas and methodologies concerning the learning and teaching of undergraduate mathematics, and what are the most appropriate vehicles for sharing this work with a wider audience?

How can mathematicians and researchers in mathematics education collaborate to formulate and investigate significant questions about the teaching and learning of undergraduate mathematics?

How can we encourage more systematic and widespread interest in this area of research, while also maintaining high levels of quality for audiences of mathematicians, mathematics education researchers, and others?

What mechanisms can be developed for sharing work that has implications for practice, in terms of instruction and curriculum, with the community of college mathematics teachers?

Is it viable to propose a PME Working Group on the Teaching and Learning of Undergraduate Mathematics? What might be the relationship with the Advanced Mathematical Thinking Working Group?

A wide range of research has been undertaken concerning the teaching and learning of undergraduate mathematics. There are serious challenges in considering how this work might be summarized and organized so that it can be accessible and helpful to interested researchers and practitioners. Several working reference lists and bibliographies will be assembled for this discussion session, and participants are encouraged to supply additional material. Certainly the monograph produced by the PME Working Group on Advanced Mathematical Thinking provides a very useful organization. Additional compilations and formats might be helpful to various communities.

College and university teachers of mathematics often have serious and important questions concerning issues in student learning and in teaching. *Communicating Among Communities*, the final report of a fall, 1991 conference sponsored by the MAA, includes as one of its recommendations that "those faculty whose professional work is devoted to research in mathematics education, as well all those whose work centers on curriculum development or educational practice" should be appropriately rewarded. Issues in this area also will be raised.

Beyond the relatively well-developed body of work in advance mathematical thinking, there certainly are other research directions and emphases in the area of undergraduate mathematics learning and teaching. These include various intervention-type studies to test curricular innovation or instructional strategies, studies of teaching processes, and studies about the mathematics preparation of preservice teachers. We hope to expand the discussion to determine the ways that these other lines of research, many of which have more profound implications for practice, may be extended and communicated.

Title	Developing Understandings of the Struggles and Issues Related to the Study of Teacher Change
Presenter	Megan Franke, Catherine Brown, Margaret Smith
Institution	UCLA, LRDC University of Pittsburgh, LRDC University of Pittsburgh

Just as teachers are struggling to change the way they teach mathematics, we as researchers are struggling to understand the processes and parameters of this change. For the past few years, researchers, supported in part by the National Center for Research in Mathematical Sciences Education at the University of Wisconsin and the Center for the Development of Teaching at EDC, have met to consider conceptual and methodological issues involved in understanding teachers as they grapple with transforming their mathematics teaching. The participants in these meetings have addressed a number of themes including: (1)understanding patterns of change in teacher's knowledge, beliefs and action, including the relationships among those patterns, (2)identifying the elements of the system of teaching (including teachers, students and the broader school context) in which one would expect to find change; (3) the ethical ramifications of such study including the role the teacher plays; (4) theoretical issues, particularly in connecting our work to work both in and outside the field of mathematics education; and (5) delineating and discussing the methodological perspectives of the various projects.

The purpose of this session is two fold. First, we would like to open this discussion of teacher change and the related issues to others in the mathematics education community. Second, we would like to outline and discuss the struggles and issues related to understanding teacher change. To begin the session we will briefly outline the issues given above. We will ask that the participants identify an issue falling within one of the outlined areas that they currently find themselves struggling with in their own work. After participants have identified their issues for the group we will divide into at least 5 small groups for more focused discussion. These groups will reflect the five general themes cited above: patterns of change in beliefs, knowledge and action; understanding the parameters of the system of change; ethical concerns; theoretical concerns; and methodological concerns. Depending on the number of participants it may be necessary to further divide these groups. In some cases the group will consist of participants who are all struggling with the same issue. In other cases, the groups will consist of participants sharing a range of concerns all falling within the broader category of issues. Our hope is that the participants will not only learn about other participants research on teacher change but also, that the participants will begin and in some cases continue the discussion on how to address some of the issues facing the field of teacher change.

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Title	*Developing Effective Learning Environments and Tasks for Prospective Middle School Mathematics Teachers*
Presenters	*Elizabeth Jakubowski, Grayson Wheatley, and Jenny Erhlich*
Institution	*Florida State University, Tallahassee, Florida*

This presentation will describe a three year National Science Foundation funded middle school teacher education project at Florida State University. Mathematics and pedagogical courses are being developed by teams consisting of mathematicians, mathematics educators, teachers, and graduate assistants. A total of five mathematics courses and two teaching and learning courses will developed by the end of the project. Tasks and learning environments which provide potential learning opportunities for the participants are being developed with the view that learning is changes made in learners' thought patterns to neutralize perturbations that arise through interactions with our world. Learning is accomplished by constructing and elaborating schemes based on experiences; it is very much a personal matter.

Providing opportunities for prospective teachers to construct discipline specific pedagogical knowledge is a priority of this project. Many prospective middle school mathematics teachers have not constructed an adequate meaning of mathematical concepts central to middle school mathematics.

A geometry course for prospective middle school teachers developed by a team will be described in this session. The presentation will include computer micro worlds, video segments of students engaged in problem centered learning, and selected mathematics activities. Novel units developed include spherical geometry, fractals, abstracting, topology and measurement. The presentation will be designed to generate discussion by the audience--reactions to the proposed topics, instructional strategies, and assessment procedures will be sought.

--Discussion Group--

SHAPING REFORM IN ALGEBRA EDUCATION

David Kirshner
Louisiana State University

Jim Kaput
University of Massachusetts

This Discussion Group continues a debate initiated at the 1993 NCTM Presession and the AERA Annual Meeting under the auspices of the Algebra Working Group of the NCRSME. The debate examines the linkages and disjunctions between the current algebra curriculum and the rich empirically oriented approaches being developed by researchers at various sites across the continent. One motivation for the debate is political and strategic: If systemic curriculum initiatives are to be introduced, then it is necessary to consider not only the makeup of the new curriculum, but also the nature of the changes that teachers and schools will be asked to accommodate to. The Discussion Group presents a forum for facing these crucial issues up front.

The substance of the debate concerns alternative visions for a renewed algebra curriculum that inspire the current research enterprise: What is an appropriate balance between formal/structural algebra and empirical algebra? While past meetings have demonstrated considerable consensus within the research community as to the directions that reform needs to push, there are differences to be explored and resolved. There is also a major long term reform-strategy question: Should algebra continue to be embodied in a series of middle and secondary school courses, or should it be integrated within the K-8 curriculum? The session will begin with Jim Kaput's brief summary of the issues and consensus, and David Kirshner's brief analysis of some differences. An energetic and productive debate is anticipated. We look forward to your participation.

Title Cultural support for mathematics understanding

Presenters Irene T. Miura and Yukari Okamoto

Institutions San Jose State University & UC Santa Barbara

There is growing interest in the role that cultural processes may play in the understanding of mathematics. Social and economic activities (Rogoff, 1990; Saxe, 1988; Saxe, Guberman, & Gearhart, 1987) and the use of culturally developed tools, such as language and numerical systems, are examples of cultural processes that may affect mathematics understanding (Cobb, 1993; Kaput, 1991; Miura, Okamoto, Kim, Steere, & Fayol, 1993; Nunes, 1992; Rogoff, 1990).

Mathematical notation systems and the language of mathematics cannot be separated from the cognitive processes of their users. In the proposed session, we will examine the possible effects of Japanese number language characteristics on the mathematics understanding and performance of Japanese children. We provide five examples of language supports for mathematics: 1) the number counting system; 2) certain mathematics terms; 3) the use of numerical classifiers; 4) the syntax of word problems; and 5) the lack of a plural morpheme.

Number counting system. The counting system in Japanese is organized so that number names are congruent with the traditional base 10 numeration system. Japanese children's concrete representations of number reflect this organization and show an understanding of place-value concepts (Miura et al., 1993).

Certain mathematics terms. Fractions, for example, may be easier to understand in Japanese. One-third is spoken as "san bun no ichi" which is literally translated as "of three parts, one." The language directly supports the fractional concept of thirds.

Numerical classifiers. Numerical classifiers are used when enumerating objects, for example, eight "pon" pencil (pon being the classifier for long, thin objects). The classifier adds a coherence to the items being enumerated and integrates them into a set, rather than treating them as individual items. In word problems, classifiers also act as place holders. They may make problems more concrete. For example, the statement, "Tom has 5 pencils; 2 more than Jon," would be translated as "Tom has 5 'pon' pencils; 2 'pon' more than Jon." Numerals in isolation (e.g., the 2 in the previous statement) are not an abstract quantity, and this may engage children in a stronger visual representation of what the word problem is asking.

Syntax of word problems. In Japanese, word problems, in general, are stated in briefer form than they are in English. This brevity may act to limit the cognitive overload which can occur when solving arithmetic word problems. The wording also varies so that the arithmetic operation required is inherent in the phrasing of the problem. For example, the question, "How many marbles does Tom have more than Joe?" is translated as "Tom more than Joe has how many marbles?" The comparison between Tom and Joe, and the search for the amount of difference between them, is explicit in the wording of the question. In subtle but significant ways, the syntax of arithmetic word problems suggests what operations should be used in the solution, eliminates extraneous (and sometimes confusing) information, and keeps the reader focused on the information pertinent to the solution.

Lack of a plural morpheme. Japanese lacks a plural morpheme; there is no verbal agreement for number, and articles are not used to denote number. Children may learn to pay close attention to numbers because they are used only when number is important. This would have important implications for mathematics which is dependent upon numbers.

Title	Teaching, Learning and Understanding Common and Decimal Fraction Concepts
Presenter	Douglas T. Owens
Institution	The Ohio State University

The Purpose of this presentation is to describe responses on two tasks of grade six children involved in a classroom teaching experiment. Research suggests that the emphasis of classroom instruction is often on symbols and procedures, which has been called procedural knowledge. In this study we created a learning environment which would promote the construction of conceptual knowledge Hiebert and Lefevre (1986). We undertook to provide situations in which children would learn relational mathematics (Skemp 1976) as opposed to instrumental mathematics.

The teacher was invited to be a collaborator contributing her expertise on teaching children and knowledge of her class. An instructional sequence was planned collaboratively. The principal investigator and graduate assistant designed and prepared discussion sheets and manipulatives to be used in class. The instructional materials were prepared in advance, but decisions about timing and use of materials were made by the teacher using her professional judgment. The interviews and videotaping drew heavily on the methodology of the small-group teaching experiment (Cobb and Steffe, 1983).

The first interview task reported was set in the context of a metre rule to get at the students' connections between common and decimal fractions, especially in thousandths. Three children could give immediate explanations. One of these saw one eighth as $12\frac{1}{2}$ hundredths and five eighths as $\frac{625}{1000}$. Two children used money, pointing out that a quarter is 25 cents. Thus $\frac{1}{4}$ of a metre is 25 cm and $\frac{1}{8}$ is $12\frac{1}{2}$ cm. One profited by a hint that a quarter is 25 cents. One profited by a hint to start with one half, which she saw as 50 cm and was able to determine that three eighths is 625 thousandths. Two children could determine that one fourth of the metre is 25 cm, but could not come up with one eighth.

The second task is the "balls in the box" task, in which students were asked to indicate a fraction of a set of eight balls of varying sizes and shape. Students were asked to show one eighth of the balls. If they protested that the balls were not the same size, the interviewer asked about one half the class of 24 students. They often indicated 12, in which case the interviewer pointed out that the children were not the same size. Two students agreed that neither the balls nor the people need be the same size. One first said that size doesn't matter, but became unsure when questioned. Five thought that the balls must be the same size, but kids in the class can be different sizes. One other indicated it is *easier* to find one eighth of the balls of equal size. Three thought that balls and people must both be the same size in order to determine a fraction of the set. During class sessions in the current year, these students had not encountered fraction concepts based on sets of discrete objects. The teacher, through the instructional sequence had emphasized that in the measurement situations they had encountered that equal parts were necessary.

USING OPEN-ENDED PROBLEMS IN CLASSROOM

Erkki Pehkonen, University of Helsinki (Finland)

The method of using open-ended problems in classroom for promoting mathematical discussion, the so called "open-approach" method, was developed in Japan in the 1970's (Shimada 1977, Nohda 1988). About at the same time in England, the use of investigations, a kind of open-ended problems, in mathematics teaching became popular, and the idea was spread more by Cockcroft-report (1982). In the 1980s, the idea to use open problems (or open-ended problems) in classroom spread all over the world (see Pehkonen 1991), and research of the possibilities of using open problems is especially now very vivid in many countries.

In some countries, the idea of using open-ended problems in mathematics teaching has also been written in one form or other into the curriculum. E.g. in the new mathematics curriculum for the comprehensive school in Hamburg (Germany), about one fifth of the teaching time is left content-free, in order to encourage the use of mathematical activities (Anon. 1990). In California, they are suggesting open-ended problems to be used beside the ordinary multiple-choice tests (Anon. 1991).

In the discussion group, the following questions will be dealt with: What are "open-ended problems"? Why use open-ended problems? How to use open-ended problems? For structuring the discussion, there will be one or two short presentations (about 10–15 min) from different parts of the world.

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Title Discussion Group on Imagery and its Uses in Metaphors, Metonymies and Beliefs in Mathematics Education.

Presenter Norma C. Presmeg

Institution The Florida State University

1. Imagery is central to all human reasoning, and therefore to all of mathematics.
This statement is made tongue-in-cheek, provocatively. This view is implied in Mark Johnson's writings on metaphor and metonymy (1987 and 1982) and expressed by others, e.g., Senechal in Steen (1990). But then we are all visualizers, and a test such as my preference for visuality in mathematics instrument, which places teachers and students on a continuum from nonvisualizers to visualizers, makes no sense. But individual differences in preference do exist in this regard, as do different types of imagery. Questions emerge concerning the philosophical and psychological aspects of beliefs which underlie teaching and learning of mathematics. What is mathematics? What is imagery?

2. There are at least two different ways in which imagery may be used in mathematical abstraction:
a concrete image ("rich" image - Johnson) may become the bearer of abstract information; or the image itself may be of a more abstract type (pattern imagery, Presmeg, 1985; image schematic structures, Johnson, 1987).
Kant's formulation, based as it is on dichotomies between transcendental reasoning and bodily experiences and emotions, would undermine his own categories if taken to its logical conclusion. Without assuming the dichotomies, Johnson is able to argue the bodily basis, mediated by imagery, of mathematical abstraction, concept formation and meaning-making. Metaphor and metonymy are key constructs in this regard. How may increased understanding of these processes be used to improve classroom learning of mathematics?

3. Nested, transparent metaphors with constituent images form a basis for cultural beliefs (Quinn and Holland, 1987).
What role do cultural images play in the learning of mathematics? Imagery is also an important constituent in teacher change: Tobin (1991) suggested that teacher reflection, a vision of what might replace traditional teaching, and commitment to that vision, were essential elements for change to take place in classrooms. What is the role of imagery in this complicated process?

4. Imagery is ubiquitous, intuitive, often subconscious, and engages the deepest levels of our being.
Imagery is much more than the "visualization" taken over by computer graphics advocates, valuable as these graphics may be in facilitating, e.g., dynamic imagery. Some of the assumptions of writers in Zimmermann and Cunningham (1991) may be challenged in this regard. Images are constructed, representations are not appropriated directly from computer screens by individuals. How do the deeply personal aspects of imagery influence this process and others involved in the teaching and learning of mathematics?

Title Teacher Change in School-Based Mathematics Reform

Presenter Mary Kay Stein, Hilda Borko

Institution LRDC, University of Pittsburgh & University of Colorado at Boulder

This discussion group is associated with the ongoing work of the Teacher Change Working Group (sponsored through the NCRMSE [University of Wisconsin] and the Education Development Center), a national collection of researchers interested in examining methodologies for studying teacher change in the reform of school mathematics.

The purpose of this discussion group is to explore ways in which collections of mathematics teachers working together within a school influence the process of teacher change. Increased movement toward school-based innovation within mathematics reform is exemplified by a number of projects that (either by design or evolution) assist and study teachers as they attempt to work with their school-based colleagues and externally based staff developers to implement change (the projects in which the two organizers are involved-QUASAR [Silver, 1993] and Studies in Improving Classroom Teaching and Local Assessments—are two examples). The emerging sense is that teacher change within this type of improvement effort possesses some characteristics that may distinguish it from individual teacher change. For example, given time together to reflect on newly-constructed mathematical ideas and to work on pedagogical implications of those ideas, school-based colleagues may socially construct a common understanding of warranted practice. Such a common understanding can provide direction and motivation for continued efforts as well as a common ground from which to assist and to critique one another's work.

The organizers will begin the session with a brief discussion of issues that they have encountered in trying to understand how group participation in a reform effort influences the phenomenon of teacher change. Then, contributions from the audience will be solicited with respect to: (a) descriptions of various settings and activities that school-based colleagues use to work together; and (b) commentary on how such collective work influences teachers' knowledge, beliefs, and practice. In addition, the ways in which both individual (e.g., von Glaserfeld, 1983) and social-interaction (e.g., Lave & Wenger, 1991) models of knowledge construction can contribute to an understanding of teacher change in school-based mathematics reform will be explored.

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ON THE MATHEMATICAL PREPARATION OF TEACHERS

Organizer: Eva J. Szillery, Rutgers University and Academy for the Advancement of Science and Technology

Respondent: Frank Lester, Indiana University and JRME

Our foremost educational outcome has shifted from developing skills to developing concepts. What are the impacts of this transition for teacher training in mathematics?

This discussion group is designed to be linked to the Plenary Session by Thomas Cooney. My ten years of experience in observing teachers in classrooms, in teacher training programs and in teaching college mathematics (to non-mathematically oriented students and to electrical engineering majors) forms the basis for proposing that:

> Mathematics teachers in their training should learn more about structures, one of the most fundamental concepts of modern mathematics. The notion of structure and related notions acquired central position in mathematics, from technical as well as philosophical and methodogical points of view.

Why?

a. From the philosophical point of view, the widespread use of structures and of isomorphisms emphasizes one of the main features of modern mathematics - that is, that the nature of mathematical objects is not the important thing but rather the relations which exists between them.

b. While teachers would not teach mathematical structures it is important to see this unifying concept in order to show "the links between the pieces". The author's experience confirms that students have less difficulty with algebra, if arithmetic was taught in an algebraic fashion: emphasizing the operations carried out rather than concentrating on the final answer.

My hypothesis is that on the school level mathematics is effective if the students learn to make connections. For this goal the teachers should know the unifying concepts of structures and through that knowledge the teachers could learn the art of linking elements in Set Theory to Geometry, to Algebra, to Arithmetic, to Statistics, etc. In context of the isomorphic structures it is always easier to learn new structures.

Discussion questions will be posed concerning open-ended problems, guided discovery and the role of proof.

HOW CAN WRITTEN CURRICULUM FACILITATE CHANGE?

Cornelia Tierney, Rebecca B. Corwin
TERC, Cambridge, MA

A number of math educators have told us that there is little or no audience for new kinds of curriculum such as the NSF sponsored Investigations in Number, Data, and Space we are developing for elementary school. They have suggested that time is better spent working with groups of teachers to change their ideas of math education so that they will want curricula like ours. However, our experiences with piloting this curriculum and other recent curricula has lead us to believe that written curriculum can make a difference, not only for teachers who are already seeking new things to do in their classrooms, but also for teachers who are generally accepting of current approaches to teaching math. We have seen this second group of teachers renewed by trying new roles for themselves and their students, new topics and new ways of looking at old topics.

In this discussion at PME-NA, we are interested in learning from others what their experiences are as teachers and with other teachers using new curricula. We will exchange and analyze instances of classroom change that result at least partially from teachers and children grappling with good curricula. What is it about curriculum that may facilitate change in classrooms?

What moves students to make conjectures? to make generalizations? to pose problems? to be eager to communicate their ideas? How do teachers support and include more children in active engagement with mathematics?

What moves teachers to think about their mathematics teaching? to see new strengths in their students? to look at the more typical content in creative ways? to enjoy teaching mathematics?

In the first part of the discussion, participants will describe examples of curriculum that made a difference. Please bring, if you have one, an investigation, a problem, a classroom process, a way of writing for teachers, etc. and the story of the classroom changes you believe were prompted by it. The rest of the discussion time will be spent in considering what these stories tell us about the attributes of engaging curricula. How can written curricula provide a context for teachers to examine the interrelated factors that affect their students' learning, their learning, and their teaching of mathematics?

Title The Mathematical Preparation of Elementary School Teachers

Presenter A. I. Weinzweig

Institution Univ of Illinois at Chicago

MathPet Abstract

We present a model for the preparation of elementary school teachers being developed by the MathPET Project, funded by the NSF. One of its principal objectives is to change the negative belief systems about the nature of mathematics and the process of learning and teaching of mathematics that most prospective teachers bring with them. Recent literature has described the importance of beliefs in the learning and utilizsation of mathematics and has indicated the lack of attention that affective variables receive in teacher preparation. Using problem situations we provide prospective teachers with an opportunity to deepen their understanding of mathematics in a learner-oriented environment, discuss how mathematics is learned and how this learning can be facilitated thus integrating issues of cognition, pedagogy and content in one context. These problem situations are structured so that problem solving and applications are integral parts of the learning process and not merely addenda to a curriculum. We contend that intellectualized discussions of problem solving and metacognition are largely worthless without the participants first having real problem solving adventures of their own and a focussed examination of their own metacognitive processes.

We have developed and are continuing to develop problem situations on a broad variety of topics: counting; addition and subtraction; place value; multiplication and division; fractions; ratio and proportions; symmetry; similarity; properties of solids and plane figures; data collection, organization and interpretation; elementary number theory; measurement of length, area and volume; etc. These problem situations will always provide a context and a need for the particular concept.

Title Capturing Classroom Complexity Through Multiple Frameworks

Presenter Steven R. Williams and Verna Adams

Institution Washington State University

It is increasingly recognized that educational researchers' attempts to describe the complex nature of the classroom require an eclectic and sometimes even untidy assembly of data collection techniques and analytical frameworks. These frameworks can be placed on a continuum ranging from an almost exclusive focus on cognition (as with early studies of counting or solving addition and subtraction word problems) to an almost exclusive focus on social interaction (as with recent studies of everyday mathematics or social construction). Traditionally the cognitive frameworks have held a privileged position in educational research, but recent arguments for the importance of a more socially or culturally oriented view have convinced most researchers in mathematics education that a more balanced view is necessary.

Achieving this balance is our primary concern in proposing a discussion group. Recent work making use of situated cognition or Vygotskian theory suggest that frameworks paying proper respect to both cognitive and social factors are emerging. Still, many questions remain which have both practical and philosophical interest. Of primary interest are the comparative advantages of each perspective, including what each perspective is more likely to reveal about classroom learning and teaching. This has an impact on the kind of help most likely to be offered from within each perspective in solving pressing educational issues. Of comparable importance is determining what each viewpoint might obscure or overlook. In part, this reflects the obvious truth that research questions and analytical frameworks are mutually dependent. But granted that certain research questions will more obviously call for cognitively or socially oriented frameworks, of great practical interest is the question of what insights might be gained by bringing to bear alternative frameworks of the same set of data. It is also important to examine the relationship of data collection techniques with analytical framework; specifically, we wonder whether socially-oriented analytical frameworks necessarily call for more ethnographic methods, as opposed to (for example) the structured interviews often used in more cognitively-oriented studies. And finally, it seems important to ask whether socially- and cognitively-oriented frameworks be reconciled, or whether they are mutually incompatible paradigms.

In order to make a start in addressing these issues, we propose a discussion group with the following format. A short data segment from a middle school classroom will be provided, and we will initiate a brief discussion of what we believe each perspective offers in understanding the mathematics learning and teaching in that classroom. We then hope to examine as a group what each framework obscures, and to widen the discussion to larger issues of what multiple passes through the data (with differing frameworks) might offer, and of the consequences of methodological decisions on both choice of analytical framework and success in addressing issues of importance.

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