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ABSTRACT

This book is intended to bring to a wider audience research concerned with the significant influences on the quality of mathematics learning taking place in schools. The four essays are: (1) "Influences From Society" (Alan Bishop); (2) "The Socio-cultural Context of Mathematical Thinking: Research Findings and Educational Implications" (Terezinha Nunes); (3) "The Influences of Teaching Materials on the Learning of Mathematics" (Kathleen Hart); and (4) "The Role of the Teacher in Children's Learning of Mathematics" (Stephen Lerman). A list of references from the International Group for the Psychology of Mathematics Education conference proceedings contains 77 citations and a list of other references contains 194 citations. (MKR)

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SIGNIFICANT INFLUENCES ON CHILDREN'S LEARNING OF MATHEMATICS

by

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Education Sector

UNESCO

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Introduction

Mathematics has become one of the most important subjects in the school curriculum during this century. As modern soceties have increased in complexity and as that complexity has accompanied rapid technological development, so the teaching of mathematics has come under increased scrutiny.

Research in mathematics education generally has also become more and more important during that time, but has struggled to keep up with the increasingly complex questions asked of it. There was a time when mathematics learning could be conceptualised simply as the result of teaching the subject called mathematics to a class of children in the privacy of their classroom. As long as the concerns were largely *quantitative*, such as how to provide more mathematics teachers, more mathematics classrooms, and more mathematics teaching for more students, that picture appeared to be adequate. We now know that that simplistic picture will not help in answering any of the significant questions of recent decades which predominantly concern the *quality* of the mathematics learning achieved.

Of course it is the case that in many, or even most, countries in the world, there are still quantitative issues surrounding mathematics learning. Indeed, in the developing world there is often little opportunity for the luxury of engaging with subtle issues of quality in the face of horrendous quantitative problems of educational provision. Nevertheless, as the UNESCO publication **Mathematics for all** showed (Damerow et al.,1984), one cannot divorce the problems of mathematics learning from the wider problems of society, and thus there is every reason for all mathematics educators to become aware of the factors which have the potential to affect the quality of mathematics learning taking place in schools. This book is intended to help bring to a wider audience research which informs us about the significant influences on that quality.

In particular, the authors have identified four groups of influences which appear to be of crucial importance for learners of mathematics. Firstly there are the demands, constraints and influences from the society in which the mathematics learning is taking place. These, in a sense, set the knowledge and emotional context within which the meaning and importance of teaching and learning mathematics are established. Recent research has demonstrated that it no longer makes sense to try to consider mathematics learning as abstract and context-free, essentially because the *learner* cannot be abstract or context-free. The research problems centre on which of the many societal aspects are of particular significance, on how to study their influences, and on what if anything to do about them.

The second set of influences concern the knowledge, skills and understanding which the learners develop outside the school setting and which have significance for their learning inside the school. This topic for research has only developed relatively recently, but its findings have surprised many mathematics teachers with its implications for their work. One of the great educational challenges of the present time concerns how school mathematics teaching should take learners' out-of-school knowledge into account.

The third set of influences on children's learning of mathematics come from the teaching materials and aids to learning in the classroom. These have become more subtle and varied - from the textbook to the computer - and have increased in number and importance considerably over the last decades. In the face of such development, the need for continuing research and analysis concerning the significance of these influences has become even more important.

The fourth and final influence is not a new one at all - indeed it could be thought of as the oldest influence on mathematics learning - the teacher. Every learner can quote the memory of a particularly influential teacher, whether good or bad, and every teacher knows the feeling of influence which the position gives them. There has in the last decade been a growth of interest in

research into the influence which teachers can, and do, have on the mathematics learners in their charge, and it is important to bring these ideas to a wider audience.

The four authors have drawn on recently published research which has been reported in publications and at academic conferences, but in particular, as we are all long-standing members of the International Group for the Psychology of Mathematics Education(PME), we have made sustantial reference to ideas which have developed within that context. The major research strands of PME have been fully documented in the following book:

Mathematics and cognition, edited by Pearla Nesher and Jeremy Kilpatrick, Cambridge University Press, Cambridge, 1990

under the headings of: epistemology, learning early arithmetic, language, learning geometry, learning algebra, and advanced mathematical thinking. In addition, the references at the back of *this* book are in two sections - the first, to particular papers in certain Proceedings of the annual conferences of PME, which are referred to in the text as (xxxx,PME,1988) and the second, to other publications.

Interested colleagues can obtain copies of the Proceedings of previous PME conferences by writing to:

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We would like to record our thanks to UNESCO for making this publication possible, and to our many colleagues around the world who share in the development and communication of knowledge in this field.

Finally, we dedicate this book to learners of mathematics everywhere.

Influences from Society

Alan J. Bishop

Schools and individual learners exist within societies and in our concern to ensure the maximal effectiveness of school mathematics teaching, we often ignore the educational influence of other aspects of living within a particular society. It is indeed tempting for *mathematics* educators particularly to view the task of developing mathematics teaching within their particular society as being similar to that of colleagues elsewhere, largely because of their shared beliefs about the nature of mathematical ideas. In reality such tasks cannot deal with mathematics teaching as if it is separable from the economic, cultural and political context of the society. Any analyses which are to have any chance of improving mathematics teaching must deal with the people - parents, teachers, employers, Government officials etc. - and must take into account the prevailing attitudes, beliefs, and aspirations of the people in that society. The failure of the New Math revolution in the 60's and the early 70's was a good example of this phenomenon (see Damerow and Westbury, 1984).

It is therefore my task in this first chapter to explore those aspects of societies which may exert particular influences on mathematics learning. These may happen either intentionally or unintentionally. Societies establish educational institutions for intentional reasons - and formal mathematics education is directly shaped and influenced by those institutions in different ways in different societies. Additionally, a society is also composed of individuals, groups and institutions, which do not have any formal or intentional responsibility for mathematics learning. They may nevertheless frame expectations and beliefs, foster certain values and abilities, and offer opportunities and images, which will undoubtedly affect the ways mathematics is viewed, understood and ultimately learnt by individual learners.

Coombs (1985) gives us a useful framework here. In discussing various 'crises in education' he argues that education should be considered as a very broad phenomenon, rather than being a narrow one, and that there are different kinds of education. In his work he separates Formal Education (FE) from Non-formal Education (NFE), and Informal Education (IFE).

Formal Education, he says, "generally involves full-time, sequential study extending over a period of years, within the framework of a relatively fixed curriculum" and is "in principle, a coherent, integrated system, (which) lends itself to centralized planning, management and financing" (p.24), and is essentially intended for all young people in society.

In contrast NFE covers "any organised, systematic, educational activity, carried on outside the framework of the formal system, to provide selected types of learning to particular subgroups in the population, adults as well as children" (p.23). In contrast to FE, NFE programs "tend to be part-time and of shorter duration, to focus on more limited, specific, practical types of knowledge and skills of fairly immediate utility to particular learners" (p.24).

Finally IFE refers to "The life-long process by which every person acquires and accumulates knowledge, skills, attitudes and insight from daily experiences and exposure to the environment.... Generally informal education is unorganized, unsystematic and even unintentional at times, yet it accounts for the great bulk of any person's total lifetime learning - including that of even a highly 'schooled' person" (p.24).

We shall organise this chapter's contribution around these three different kinds of education, looking particularly at the influences from society on the three kinds of mathematical education. Finally there will be a discussion of some of the significant implications which result from this analysis.

Societal influences through formal mathematics education (FME)

It seems initially obvious that any society influences mathematics learning through the formal and institutional structures which it intentionally establishes for this purpose.

Paradoxically, at another level of thinking it will not be clear to many people just what influence any particular society could have on its *mathematics* learners, in terms of how this will differ from that which any other society might have. Indeed, mathematics and possibly science have, I suspect, been the only school subjects assumed by many people to be relatively unaffected by the society in which the learning takes place. Whereas for the teaching of the language(s) of that society, or its history and geography, its art and crafts, its literature and music, its moral and social customs, which all would probably agree should be considered specific to that society, mathematics (and science) has been considered universal, generalised, and therefore in some way necessarily the same from society to society. So, is this a tenable view? What evidence is there?

The formal influences on the mathematical learners will come through four main agents - the intended mathematics curriculum, the examination and assessment structures, the teachers and their teaching methods, and the learning materials and resources available. The last two aspects which are part of the implemented curriculum (Travers and Westbury, 1989) will be specifically considered in the final two chapters of this book and therefore I will concentrate here on the first two, the intended mathematics curriculum, and the examinations.

The intended mathematics curriculum

If we consider the mathematics curricula in different countries, our first observation will be that they do appear to be remarkably similar across the world. Howson and Wilson (1986) talk of the "canonical curriculum" (p.19) which appears to exist in many countries. They describe "the familiar school mathematics curriculum (which) was developed in a particular historical and cultural context, that of Western Europe in the aftermath of the Industrial Revolution". They point out that "In recent decades, what was once provided for the few has now been made available to - indeed, forced upon - all. Furthermore, this same curriculum has been exported, and to a large extent voluntarily retained, by other countries across the world. The result is an astonishing uniformity of school mathematics curricula world-wide" (p.8).

This fact has made it possible to conduct large-scale multi-country surveys and comparisons of mathematical knowledge, skills and understanding such as those by the International Association for Education Achievement (IEA) whereas such comparisons would probably be unthinkable in a subject like history. Indeed, one of the main research issues in the Second International Mathematics Survey (SIMS) was whether the 'same' mathematics curricula were being compared (see Travers and Westbury, 1989). Some idea of the extent of the agreements found can be gained from Table 1 (below) which shows the relative importance accorded by different countries to potential topics and behavioural categories to be included in the Population A (13 year olds) assessment. Travers and Westbury concluded from their further analysis of the data concerning Table 1 that "The only topic for which there appears to be a substantial problem of mismatch is Geometry. Indeed a major finding of the Study proved to be the great diversity of curricula in geometry for Population A around the world" (p.32).

It could be argued that such international surveys, and the comparative achievement data they generate, encourage the idea that the mathematics curricula in different countries *should* be the same, particularly when, as in this case, they are seeking the highest common factors of similarity. At the very least, this approach could well lead to mathematics educators in many countries anxiously looking over their shoulders at their colleagues in other counties to see what *their* latest curricular trends are. One wonders what would result from a research study which sought to find the *differences* between mathematics curricula existing in different countries. I suspect we might well find a different picture from that portrayed in Table 1. The recent book by Howson (1991) which documents the mathematics curricula in 14 countries, is therefore a welcome addition to our sources.

Indeed societies may well not only influence the *national* intended curriculum in ways in which SIMS would not reveal, but may also influence the intended curriculum at more local levels.

Table 1

	Content topics	Behavioral Categories* Computation	Comprehension	Application	Analysis
000	Arithmetic				
001	Natural numbers and whole numbers	V	V	V	I
002	Common fractions	V	V	I	I
003	Decimal fractions	V	V	V	I
004	Ratio, proportion, percentage	V	V	I	I
005	Number theory	I	I		-
006	Powers and exponents	I	I	-	-
007	Other numeration systems	-	-	-	-
008	Square roots	I	I	-	-
009	Dimensional analysis	I	I	-	-
100	Algebra				
101	Integers	V	V	I	I
102	Rationals	I	I	I	I
103	Integer exponents	Is	-	-	-
104	Formulas and algebraic expressions	I	I	I	I
105	Polynomials and rational expressions	I	Is	-	-
106	Equations and inequations (linear only)	V	I	I	Is
107	Relations and functions	I	I	I	-
108	Systems of linear equations	-	-	-	-
109	Finite systems	-	-	-	-
110	Finite sets	I	I	I	-
111	Flowcharts and programming	-	-	-	-
112	Real numbers	-	-	-	-
200	Geometry				
201	Classification of plane figures	I	V	I	Is
202	Properties of plane figures	I	V	I	I
203	Congruence of plane figures	I	I	I	Is
204	Similarity of plane figures	I	I	I	Is
205	Geometric constructions	Is	Is	Is	-
206	Pythagorean triangles	Is	Is	Is	-
207	Coordinates	I	I	I	Is
208	Simple deductions	Is	I	I	I
209	Informal transformations in geometry	I	I	I	-
210	Relationships between lines and planes in space	-	-	-	-
211	Solids (symmetry properties)	Is	Is	Is	-
212	Spatial visualization and representation	-	Is	Is	-
213	Orientation (spatial)	-	Is	-	-
214	Decomposition of figures	-	-	-	-
215	Transformational geometry	Is	Is	Is	
300	Statistics				
301	Data collection	Is	I	I	-
302	Organization of data	I	I	I	Is
303	Representation of data	I	I	I	Is
304	Interpretation of data (mean, median, mode)	I	I	I	-
305	Combinatoric	-	-	-	-
306	Outcomes, sample spaces and events	Is	-	-	-
307	Counting of sets, P(AB), P(AB), independent events	-	-	-	-
308	Mutually exclusive events	-	-	-	-
309	Complementary events	-	-	-	-
400	Measurement				
401	Standard units of measure	V	V	V	-
402	Estimation	I	I	I	-
403	Approximation	I	I	I	-
404	Determination of measures: areas, volumes, etc.	V	V	I	I

*Rating scale: V = very important; I - important; Is - important for some systems.
A dash (-) = not important.

Moreover in the SIMS study there were striking differences between the intended and the implemented curricula in various countries, and also striking variations in the latter *within* national systems. Some of that variation could undoubtedly be due to local *intended* variations: "With the significant exception of Japan, all systems have relatively high indices of diversity and it may be that such diversity is a reflection of the responses that teachers make to the overall variations of readiness for Algebra that they find at this level" (Travers and Westbury, 1989, p.131).

It is well known that the intended mathematics curricula certainly do vary between schools in differentiated systems, i.e. where there is not a comprehensive (single) school structure. In some highly differentiated systems in W.Europe for example the grammar school, the gymnasium and the lycée have very different curricula from the other schools in those systems. The SIMS study showed (Travers and Westbury, 1989) the following data which gives the 'Opportunity to Learn' figures for Arithmetic and Algebra in the different sectors of the differentiated systems in Finland, Netherlands and Sweden (Opportunity to Learn is a % measure of the availability of that topic in the curriculum).

	Arithmetic median	Algebra median
Finland		
Long course	74	73
Short course	74	67
Netherlands		
VWO/HAVO	87	83
MAVO	83	80
LTO	74	53
LHNO	70	47
Sweden		
Advanced	70	57
General	61	25

As an example of how such curricular differentiation can influence performance, a Hungarian study (Klein and Habermann, PME, 1988) produced the following data:

Type of school and curriculum	Mean test score
Grammar schools Curriculum "Mathematics II" (special)	13.41
Grammar schools Curriculum "Mathematics I' (special)	8.68
Specialised vocational secondary schools (SVSS) Curriculum F (Direction Computing services)	6.03
Grammar schools Basic curriculum	5.41
SVSS Curriculum group A/B/E (Direction Industrial and agricultural professions)	4.87
SVSS Curriculum C (Direction Health and medical professions)	3.39
SVSS Curriculum D (Direction Kindergarten nursing)	2.66

There is however little evidence that the different curricula operating within national systems are anything other than merely subsets of the 'total' national curriculum. One reads phrases like "watered-down", "simpler version" etc. which suggest that differentiated systems will tend towards what Keitel (1986) says was the situation "in England now at the secondary level: a sophisticated, highly demanding mathematics course for those who continue their studies, and next to nothing for the rest" (p.32). The differentiated mathematics curriculum thereby reflects the differentiation which the society appears to want, and which it seeks to perpetuate through its formal educational structures.

In fact there is no *a priori* reason at all why mathematics curricula should be the same in all countries. Mathematics in schools *can* be considered in similar ways to art, history, and religion, where there is no necessity for curricula to be the same in different societies. We now know, from the ethnomathematics research literature which has been gathered in the last twenty years, about the enormous range of mathematical techniques and ideas which have been devised in all parts of the world (see Ascher, 1991). We have evidence from all continents and all societies of symbolic systems of arithmetical and geometrical natures devised to help humans extend their activities within the physical and social environments that have grown to be ever more complex. From the Incas with their quipus to help with their accounting (Ascher and Ascher, 1981), to the Chinese with their detailed geomantic knowledge for designing cities (Ronan, 1981), or from the Igbo's sophisticated counting system (Zaslavsky, 1973) to the Aboriginal Australian's supreme spatial and locational sense (Lewis, 1976) the known world is full of examples of rich and varied indigenous mathe-matical knowledge systems.

So the question needs to be asked: Where does the expectation of a universal mathematics curriculum come from? Of course we would expect there to be recognisable similarities, as Travers and Westbury show, just as we recognise similarities between the curricula of other subjects across different societies. But recognising similarities is totally different from expecting universality. In part the expectation derives from the fact that countries and societies are not isolated; there is much inter-societal communication and there has been for many centuries.

Societies influence the intended mathematics curriculum in most countries through certain nationally or regionally structured organisations. In the case of strong centralized governments the curriculum will be the responsibility of the education ministry, while in more decentralized systems such as the USA, Australia and Canada the power for deciding on the intended curriculum rests with the state or local government. For example, in the United Kingdom the present government has recently instituted a national curriculum, a development in which the highly political nature of national curricular decision-making has been rather obviously demonstrated. We have therefore seen the typical political pressure groups being very active - the 'back to basics' groups led by traditionalists amongst the employers and the government, the teachers and educators concerned about the erosion of their influence by the central government's 'interference', the more progressive industrialists who want to ensure that school leavers can compete with the 'best of the rest of Europe', allied with parents whose genuine concerns about their children's futures are coloured by lurid media reporting about the poor comparative performance by UK pupils in various (and sometimes suspect) international surveys.

Governments, education civil servants, political academics and others with educational power are thus nowadays very aware of what is happening in other countries because they feel that they *have* to be. The competitive economic and political ethic demands it. International conferences, 'expert' visits, exchanges, articles, books and reports, all enable ideas from one country to be available to others. But there is more to it than mere communication however.

The expectation of a universal mathematics curriculum has another basis. As well as communication, there has been gradual acculturation by dominant cultures, there has been the assimilation of new ideas believed to be more important than traditional ones, and there has also been cultural imperialism, practised of course on a large scale by colonial governments (see Bishop, 1990 and Clements, 1989). There has in particular been the widespread development of a belief in the desirability of technological and industrial growth. Underlying this technological revolution has been the mathematics of decontextualised abstraction, the mathematics of system and structure, the mathematics of logic, rationality and proof, and therefore the mathematics of universal applicability, of prediction and control.

What we have witnessed during the last two centuries is nothing less than the growth of a new cultural form, which can be thought of as a Mathematico-Technological (MT) culture (see Bishop, 1988). The growth of the idea of universally applicable mathematics has gone hand-in-hand with the growth of universally applicable technology. Developments within each have fed the other, and with the invention and increased sophistication of computers, the nexus is truly forged.

It is difficult sometimes to understand the extent of the influence of this MT culture, so firmly embedded has it become in modern societies' activities, structures and thinking. We now are in danger of taking the ideas and values of universally applicable mathematics so much for granted that we fail to notice them, or to question them, or to see the possibility of developing alternatives. The acceptance of universally applicable mathematics has had a particularly profound impact on mathematics curricula in all countries and in all societies. The impetus to become ever more industrialised and technologically developed in the so-called "under-developed" countries has been underpinned by the belief in the importance of adopting the mathematics and science curricula of the more - industrialised societies. It is the belief in the power of the ideas of universally applicable mathematics which has created the expectation of the universal mathematics curriculum.

So, all around the world, in societies with different economic structures (see Dawe, 1989), in societies with different political structures (see Swetz, 1978), and in societies with different religious bases (see Jurdak, 1989), learners find themselves grappling, not always successfully, with the intricacies of arithmetical algorithms, algebraic symbolisations, geometrical theorems. Later on, if they pass the required examinations, they meet infinitesimal calculus, abstract algebraic structures, other geometries and applicable mathematics.

Moreover, until perhaps ten years ago, this situation was largely unquestioned. Now it is being seriously challenged from various quarters and we can see different groups within societies trying to exert rather different influences on their school curricula. The major challenges which we can identify so far concern the irrelevance of an industrialised and technological cultural basis for the curriculum in predominantly rural societies, and the rise of computer education in industrialised societies.

The first of these challenges comes principally from within societies having a predominantly rural economy. The growth of the awareness of the economic and societal bases of much of the universal mathematics curriculum has not been as clear as the development of 'intermediate technology' or 'appropriate technology' in many of these countries. However we can now begin to recognise a desire for a more relevant school curriculum which is leading mathematics educators in those countries to search for a more 'appropriate mathematics' curriculum. Desmond Broomes, in Jamaica, is a mathematics educator responding to this perceived concern in the rural society of the West Indies, and his ideas represent this view well. In Broomes (1981) he argues that:

> "Developing countries and rural communities have, therefore, to re-examine the curriculum (its objectives and its content). The major problem is to *ruralize* the curriculum. This does not mean the inclusion of agriculture as another subject on the programme of schools. Ruralizing the curriculum means inculcating appropriate social attitudes for living and working together in rural communities. Ruralizing the curriculum must produce good farmers, but it must also produce persons who would co-operatively become economic com-munities as well as social and educational communities." (p.49)

In Broomes and Kuperus (1983) this approach means developing activities which:

> "a) bring the curriculum of schools closer to the activities of com-munity life and to the needs and aspirations of individuals;
>
> b) integrate educational institutions, vertically and horizontally, into the community so that outputs of such institutions are better adapted to the life and work in the com-munity;
>
> c) redistribute teaching in space, time, and form and so include in the education process certain living experiences that are found in the community and in the lives of persons;
>
> d) broaden the curriculum of schools so that it includes, in a meaningful way, socioeconomic, technical and practical knowledge and skills, that is, activities that allow persons to combine mental and manual skills to create and maintain and promote self-sufficiency as members of their community." (p.710)

In implementing this strategy, according to Broomes (1989) the teacher should pose problems to the learner and these problems should:

- "be replete with the cultural experiences of the learner;
- be, for the most part but not solely, practical and utilitarian;
- Illustrate mathematics in use;
- be capable of being tackled profitably using mathematics;
- not make excessive mathematical demands (in terms of the learner's level of mathematical sophistication);
- provide ample scope for cooperative activity among persons at different levels of mathematics competence."(p.20)

We can find some of these same sentiments shared by educators in parts of Africa, in Southern Asia, in Papua New Guinea and in rural parts of South America and Australia. In some of these countries there is a particularly strong desire to represent the growth of indigenous cultural

awareness through the mathematics curriculum. That is, what we are seeing is not just a rejection of the MT culture which underlies the 'universal' curriculum and underpins modern industrialised societies, but a replacement of that cultural ideology with a rediscovered, indigenous, cultural heritage. Gerdes (1985) in Mozambique and Fasheh (1989), the Palestinian educator, are particularly eloquent in expressing this perspective on the mathematics curriculum, and Nebres (1988) is also adamant about its importance. The new mathematics textbooks in Mozambique, for example, perhaps demonstrate more than any other country's do, just how one can reconstruct the mathematical curriculum within a rural and non-western society.

What is most significant from this kind of curricular activity is not, however, the particular mathematics curriculum developed in any one country, but rather the richer understanding which we can all now share that it *is* possible, and probably desirable in certain situations, to construct and implement alternatives to the canonical universal curriculum. These developments encourage mathematics educators everywhere to explore ways in which the curriculum can become a more responsive and appropriate agent in the mathematics education of the society's learners. The time has passed when it was considered sensible to merely import another society's curriculum into one's own society. It is however sensible to 'import' the *idea* of an alternative mathematics curriculum, as well as the kind of background research necessary to identify the appropriate bases, the strategies for development, and the approaches to implementation. From a rural society's perspective, of course, these ideas and approaches would be most likely to be found in other rural societies.

The second major challenge to the myth of the universal mathematics curriculum is, paradoxically perhaps, coming from within industrialised societies themselves, as computer technology grows in its range of influence. The justification for much of the existence of particular topics in the mathematics curriculum came from their assumed usefulness, both to the individuals and to society as a whole. This was similar to the justification which lay the behind the 'back to basics' movement spearheaded by industrialists and governments who were reacting to the uselessness, as they and many parents saw it, of the New Math of the previous generation. However the validity and relevance of this 'utility' argument is now being seriously disputed within mathematics education as it becomes clear that computers and calculators can take over many of the previously needed skills.

Howson and Wilson (1986) describe some of the features of the new curricular situation brought about by computers:

> "a) algorithms
>
> Algorithmic processes lie at the heart of mathematics and always have done. Now, however, there is a new emphasis on algorithmic methods and on comparing the efficacy of different algorithms for solving the same problem e.g. sorting names into alphabetical order or inverting a matrix.
>
> b) Discrete mathematics
>
> Computers are essentially discrete machines and the mathematics that is needed to describe their functions and develop the software needed to use them is also discrete. As a result, interest in discrete mathematics - Boolean algebra, difference equations, graph theory, ... - has increased enormously in recent years: so much so that the traditional emphasis given to the calculus, both at school and university, has been called into question.
>
> c) Symbolic manipulation
>
> The possibility of using the computer to manipulate symbols rather than numbers was envisaged in the early days of computing. Now, however, software is available for micros which will effectively carry out all the calculus techniques taught at school - differentiation, integration by parts and by substitution, expansion in power series - and will deal with much of the polynomial algebra taught there also. Is it still necessary to teach students to do what can be done on a computer?" (p.69).

Fey (1988) concurs with these views and summarises the situation thus:

> "The most prominent technology-motivated suggestions for change in content/process goals focuses on decreasing attention to those aspects of mathematical work that are readily done

by machines, and increasing emphasis on the conceptual thinking and planning required in any 'tool environment'. Another family of content recommendations focus on ways to enhance and extend the current curriculum to mathematical ideas and applications of greater complexity than those accessible to most students via traditional methods. I distinguish:

1. Numerical comput-ation,
2. Graphic computation,
3. Symbolic computation,
4. Multiple representations of information,
5. Programming and the connections of Computer Science with Mathematics curricula,
6. Artificial intelligences and machine tutors." (p.235)

It is not just the topics within the curriculum which are being questioned or proposed, the whole purpose and aims of mathematics education are now under scrutiny. Dorfler and McLone (1986) present us with the following argument:

> "The dilemma for mathematical educators in considering the place of mathematics in the school curriculum can be described thus. On the one hand, the increased technological demands of society and the development of science require highly trained mathematicians who can apply themselves to a variety of problems; they also require professional scientists, engineers and others to have a greater acquaintance with and competence in mathematical technique. In addition, whilst the increased use of computational aids is likely to lead to less demand for routine mathematical skill from the general workforce, a larger number of technical managers will be needed who can interpret the use of these computa-tional aids for general application. On the other hand the basic mathematical requirement for employment is unlikely to grow beyond general arithmetic skill (often with the aid of calculator) and the interpretation of charts, tables, graphs, etc.; indeed, not much beyond the needs of everyday life mentioned earlier in this section.
>
> Arguments for school mathematics based on its ability to develop powers of logical thinking do not provide sufficient justification as it cannot be clearly shown that such powers are uniquely developed through the study of mathematics" (p.57).

Keitel (1986) argues the same thing from another point of view. She is concerned in that paper with the 'social needs' argument for mathematics teaching:

> "By 'social needs' demands I understand here the pressures urging school mathematics to comply with the needs for certain skills and abilities required in social practice. Mathematics education should qualify the students in mathematical skills and abilities so that they can apply mathematics appropriately and correctly in the concrete problem situations they may encounter in their lives and work. Conversely, social usefulness has been the strongest argument in favour of mathematics as a school discipline, and the prerequisite to assigning mathematics a highly selective function in the school system." (p.27)

She argues in her paper that computer education is now taking over this argument from mathematics education, and that this should release mathematics education from that demand. We need therefore to consider just what the intentions of future mathematics curricula should be. Should mathematics not continue to be a core subject within the school curriculum, but become optional? Should it become more of a critical and politically informed subject serving the needs of a concerned society faced with an environmental holocaust? Should it become more of a vehicle for developing democratic values?

These kinds of curricular debates demonstrate that if we wish to take society's influences on mathematical learning seriously then academic considerations and criteria are not sufficient. In modern industrialised societies mathematics is so deeply embedded in the cultural, economic and knowledge fabric of that society that the mathematics curriculum must also reflect the political, economic and social concerns of that society.

Sociologists concerned with education have long argued about the relationships between educational practices and societal structures. Bowles and Gintis (1976) for example argued that the evidence showed that education, rather than eradicating social inequality (one of the aspirations of education in a democracy), tends to reinforce it. Giroux (1983) however demonstrated that although schools are tied to particular social

features of the society in which they exist, they can also be places for "emancipatory teaching". Bourdieu (1973) distinguishes cultural reproduction from social reproduction, and argues that the first is what is mainly transmitted by the home and the school, with the curriculum and the examinations as the principle instruments for cultural reproduction.

There is however little dispute that the intended mathematics curriculum is an important vehicle for the induction and socialisation of young people into their culture and their society, and as such it tends to reproduce and reinforce the complex of values which are of significance in that society. It is only recently however that such social and political values within the mathematics curriculum have been recognised and discussed (see, for example, Mellin-Olsen, 1987; Bishop1988; Frankenstein, 1989). Hitherto, mathematics, due to the assumptions of universality, was thought to be neutral - i.e.. culture-free, societally-free and value-free. That stance is now no longer valid and those who work in mathematics education everywhere need to recognise the importance of revealing and debating the values aspects of the curriculum which have been largely implicit for many years. Some years ago Swetz (1978) made us aware of some features of the political embeddedness of mathematics education in certain socialist counties, but more recent developments by other researchers are provoking mathematics educators in many other non-socialist countries to consider just how their mathematics curriculum should respond to these different concerns within their particular societies (see, for example, the writings of several people in Keitel et al., 1989). Thus we can see from all these diverse developments that the concept of the universal mathematics curriculum is losing its credibility both in theory and in practice. The intended mathematics curriculum has also been shown to be a vehicle capable of responding to the political and social needs of society.

It has therefore become an object of serious political concern itself - a state of affairs often deplored by many mathematicians. In a sense though this was to be expected. Any subject which becomes as important as mathematics has become in society, cannot be immune to the different pressure groups within that society. The mathematics curriculum is clearly far too important an instrument to be determined by mathematicians alone. Whether it can be as responsive as it needs to be, to accommodate all the influences from society, is however a moot point, and is one which will be taken up in the final section of this chapter.

The examinations

The second major influence from society on the mathematics learners comes through the formal examinations, and the examination system. The examinations seem to the learners to define operationally what is to be required of them as a result of their mathematical education - no matter what their teachers, parents, and other formal and informal educators may say to the contrary. The old adage 'education is what remains with you after you've forgotten everything you learnt at school' may have a certain amount of truth in it, but the fact is that any learner on any formal educational course will be acutely aware of the essential demands of their particular forthcoming examination, or assessment. In particular, as the examinations *operationalise* the significant components of the intended mathematics curriculum, so they tend to *determine* the implemented curriculum.

Just as was the case with the idea of the universal mathematics curriculum, so it is with mathematics examinations - they look very similar from country to country. As long as there existed international consensus about the content of the intended curriculum, it also seemed sensible to have similar examinations. There are however variations between countries, and although there has been no systematic research into the variety of examination practices around the world, it appears that the following represent the major aspects of difference:

- proportion of short to long examination questions. Some education systems emphasise short questions more, and may cast them in a multiple-choice format, while others prefer long questions to predominate. The issue seems to relate to how large the content sample is for the examination - short questions allow a greater sampling, long questions less;
- proportion of 'content' to 'process' questions. This relates to the previous aspect, but refers essentially to whether the examination is intended for the

assessment of knowledge and skills which have been 'covered' in the teaching, or for the assessment of processes which are assumed to have been developed;

- extent of teacher role in the examination process. Some systems merely expect the teacher to be the administrator of the examinations produced by 'outsiders', while others take the view that the teacher's assessment is a key part of the examination process. In the latter case there will be more of a moderating role played by 'outsiders';

- proportion of oral and practical work included in the examination. In some systems the examinations are entirely written, whilst in others practical materials will be available and in others pupils will give their answers mainly orally.

- amount of examined work completed *during* the course of teaching, rather than in a final 'paper'. As more process aspects of mathematics teaching are emphasised, so the final paper has decreased in importance;

- extent of pupil choice. In some examinations all questions and papers are compulsory, while in others there are choices to be exercised by the candidates. Such choices allow pupils to emphasise their strengths and minimise their weaknesses thereby allowing pupils to show what they know rather than what they don't know. Individual attributes are valued by some societies more than by others;

- the extent to which the examination is norm-referenced or criterion-referenced. While all examinations inevitably contain both aspects, the emphasis is a matter of decision, and relates particularly to the underlying purpose of the assessment.

Thus even within the framework of the canonical universalist mathematics curriculum there can be marked variations in examination content, procedure and emphasis, related to the particular goals of the society and to the societal demands being made of the assessment.

Moreover as the development of alternative curricula increases so we can expect to see yet more different problems and tasks presented, and different issues raised, and these may well enter the formal examination structure and process. When this happens, it is likely to be a politically sensitive matter, as was demonstrated by an incident in a recent public mathematics examination in the UK. In 1986, candidates in one examination for 16 year olds were presented with some information on military spending in the world, which was compared with a statement from a journal (New Internationalist) that "The money required to provide adequate food, water, education, health and housing for everyone in the world has been estimated at $17 billion a year". The candidates were asked "How many weeks of NATO and Warsaw Pact military spending would be enough to pay for this? (show all your working)". The outcry from the national newspapers and from society's establishment was remarkable, with one headline asking "What has arms spending to do with a maths exam?"(Daily Mail, 14 June, 1986) and public correspondence on this issue continued for some time, illustrating well the societal controls felt to be important on the examinations and the examination system.

Here we are of course talking about summative, rather than formative assessment. Society has little interest in the latter, seeing it as merely a part of the teaching process. Summative assessment, however, usually in the form of an examination, is very much felt to be society's concern and various tensions can be seen to exist in all societies over the formal school examinations. These tensions seem to be particularly sharply defined in mathematics education perhaps because mathematics itself can appear to be a sharply definable subject, but also because of its role in selection for future education or careers.

A major tension is between examinations as indicators of educational achievement, and as instruments for academic control. In a school subject like mathematics, in which there is much belief in the importance of the logical sequences within mathematics itself, the prevailing image projected by the curriculum is of a strongly hierarchical subject - certain topics must necessarily precede others. With such an image, academic progression and control can be felt to be tightly determined, with, for example, performance at one level being seen to be achieved before the next topics in the logical

sequence can be attempted. This image controls much of the current practice in mathematical assessment, with only those who have demonstrated mastery of the required knowledge being allowed access to higher level learning in the subject. It is an image which is favoured by many academic mathematicians, and by those 'meritocratic' traditionalists amongst employers, parents, and government officials.

In contrast to this image is the belief that the examinations should as far as possible enable the learners to demonstrate what they personally have learnt as a result of their mathematical education. A consequence of this belief is that, given the variety of school learning contexts for mathematics, the variety of teachers and materials influencing learners, and the variety amongst the learners themselves, the examinations should be as broadly permissive as possible. As well as perhaps including what might be called standard mathematical questions, the learners should, therefore, be allowed to submit projects, investigations, essays, computer programs, models and other materials, and to be assessed orally as well as in written form. All of these types of assessment do exist in different countries as I have suggested, but their acceptance in an examination system by a society depends strongly on beliefs concerning educational access, the recognition and celebration of individual differences, and the encouragement of individual development through a broad mathematical education.

Under this view, there is no necessary reason why examinations in mathematics at school should be the same from one society to another. Each society makes its choice for its own political, educational and social reasons. This view tends to have strongest support amongst the more progressive educators, parents, and politicians of a more socially democratic persuasion. It is a view which also seems to be gaining strength as 'personal technology' - hand-held calculators of increasing sophistication, and personal computers - becomes broadly accepted within mathematics teaching. Personal technology has negated many of the traditional logical sequences thought to be so essential to mathematical development e.g. calculators are challenging the ideas of sequencing of arithmetic learning, and symbolic manipulation computer software is doing the same for algebra and calculus as we have seen. In addition, personal graphic calculators can now present geometrical displays which are forcing mathematics educators to rethink the orders of geometrical and graphical constructions. Educationally, these personal technologies are putting so much potential mathematical power in the hands of individual learners that developing and examining individual, and particularly creative, talents becomes much more of a general possibility. As the range of possibilities of mathematical *activities* increases so the necessary ordering of mathematical mastery becomes less clear and the former view becomes less tenable.

In the debate between these two views, one suspects that what is really at stake is the more fundamental issue of social control versus educational opportunity, and the role which examinations and the examination systems play in this political struggle. In all industrialised and developing countries, access to the higher and further levels of education is not just academically controlled but is socially controlled as well. The established social order within any society has a vested interest in controlling mobility within that society and academic educational control is an increasingly powerful vehicle for achieving this (see Giroux, 1983 for a lucid discussion of writings on this issue).

Moreover academic control, through the use of mathematical examinations is a particularly well-known phenomenon. It happens throughout Europe, for example, and in many other countries. Revuz (1978) described the phenomenon (mainly from the French perspective) in these terms:

> "Moreover, the quite proper demand for the wide dissemination of a fundamental mathematical culture has boomeranged, and been transformed into a multiplicity of mathematical hurdles at the point of entry into various professions and training courses. A student teacher is regarded as of more or less value in accordance with his level in mathematics. Subject choices which make good sense in terms of their relevance to various jobs have been reduced to the level of being 'suitable only for pupils who are not strong enough in mathematics'; instead of attracting pupils by their intrinsic interest, they collect only those who could not follow, or who were not thought capable of following, a richer programme in mathematics. Success in mathematics

has become the quasi-unique criterion for the career choices and the selection of pupils. Apologists for the spread of mathematical culture have no cause to rejoice at this; this unhealthy prestige has effects diametrically opposed to those which they wish for.

The cause of this phenomenon is undoubtedly the need of our society, as at present constituted, to try to reproduce a selective system which runs counter to attempts at democratisation which are going on simultaneously. Latin having surrendered its role as a social discriminator, mathematics was summoned involuntarily to assume it. It allowed objective assessment to be made of pupils' ability, and it seemed, amidst the upheaval of secondary education in general, to be one of the subjects which knew how to reform itself, which stood firm and whose usefulness (ill understood, truth to tell) was widely proclaimed." (p.177, translated by D.Quadling)

A similar perspective on the phenomenon is offered by Howson and Mellin-Olsen (1986). They say:

"Mathematics, as we have written earlier, enjoys a high status within society at large. This may be due in part to its 'difficulty' and its utility, but it has also been argued by sociologists, such as Weber (1952) and Young (1971), that its status is also due to the way in which it can be 'objectively assessed'. The public, rightly or wrongly, has faith in the way in which mathematics is examined and contrasts the apparent objectivity of mathematics examinations with the 'softer', more subjective forms of assessment used for many other subjects, e.g. in the humanities. Young advanced the view that three criteria which helped determine a subject's status in the curriculum were:

1 the manner in which it was assessed - the greater the formality the higher the status;

2 whether or not it was taught to the 'ablest' children;

3 whether or not it was taught in homogeneous ability groups.

Certainly, mathematics would fulfil all Young's conditions. It is important, then, when considering possible changes in methods of assessment to realise that their chances of acceptance will be greater if they do not contradict society's expectations of what a mathematics examination should be." (p.22)

One example of this 'social filter' role of examinations is that in many European countries it is extremely difficult for some immigrant groups to gain access to higher education - not because of overt racist practices but because of academic control practices which have the effect of restricting access to certain groups, through language and cultural aspects for example.

Where these practices have been recognised by certain immigrant communities one interesting effect has been to concentrate their educational and parental attention on the mathematical performances demanded by the 'social filter'. In the UK, for example, the Asian Indian students achieve significantly higher in mathematics examinations than many other immigrant groups, and they thereby progress more successfully to the higher levels of education (see Verma and Pumfrey, 1988). One suspects that the large numbers of oriental Asian students succeeding in mathematics at higher levels in American universities is a similar response to a perceived societal situation by an ethnic minority community (see, for example, National Research Council, 1989, and Tsang, 1988).

However, generally the control through mathematics examinations still prohibits access to higher educational opportunities for many other minority groups and indigenous peoples in societies like Europe, Australia and USA. Furthermore where the Western European model of examinations has either been adopted by developing countries or in some cases where their students still take European examinations, the same problems occur. For example, Isaacs' (1985) description of some of the changes which the Caribbbean Examination Council is making to its examinations, indicates the problems well:

"The examinations will probably continue to undergo modifications to ensure that they reflect more accurately the abilities, interests, and needs of Caribbean students. It is hoped that they will help reduce the number of students who see mathematics as a terrifying trial in their rite of passage to adulthood, and, at the same time, increase substantially the number who perceive and use mathematics as a tool for effective living." (p.234)

The other example of social control by mathematics and its examinations which is now recognised concerns the barriers to the continuing educational opportunities for girls. There are strong views expressed to the effect that girls and women are penalised in many countries through the use of certain mathematics examinations (see Sells, 1980). Also, since mathematics is not felt to be as popular a subject as others, by girls, to the extent to which mathematics is used for control, it restricts girls' possibilities in education more than boys'.

Educational opportunity could be expected to be a strong goal in any democratic society but, even in such societies, either the examination structure, or the actual examinations, ensure that not all children have the same educational opportunities. Insofar as societies contain inequalities, those will tend to be reflected in unequal educational opportunities, despite steps being taken to remove the barriers and obstacles. The mathematical examination is a well-used obstacle, for example, but it can be altered, its effects can be researched and publicised, and its influence therefore either be reduced or exploited. One suspects however that all that will happen will be that unless fundamental changes take place in the structuring of societies, those societies will create newer obstacles - perhaps indeed the rise in importance of computer education will increasingly come to mean that computer competence will replace mathematical competence as the filter of social and academic control.

Societal influences on children through informal mathematics education (IFME)

It has been important to explore fairly thoroughly the formal and intentional influences which society exerts on mathematics learning through the curriculum and the examinations, because to a large extent it is the public knowledge, or at least perception, of these which determine to a large degree the characteristics of the informal education which societal influences bring to bear on young people. Few adults have access to within-school or within-classroom information except through occasional visits either as a parent, or as an 'official visitor' e.g. as a school governor, or prospective employer. Also, Chevallard (1988) makes us aware of a contradiction within modern industrial societies that is:

> "1 No modern society can live *without* mathematics.
>
> 2 In contradistinction to societies as organised bodies, all but a few of their members can and do live a gentle, contented life *without any mathematics whatsoever*." (p.49)

Mathematics is becoming, therefore, an 'invisible' form of knowledge to many people in society.

On the other hand, most adults will have memories of their own experiences of school learning, including mathematics, which may or may not be pleasant, or accurate, or even relevant to today's situation. Of course in most societies it is still the case that not all adults have been to school, that even those who have may not have stayed very long, and even if they did they may not have experienced much mathematics beyond arithmetic. Nevertheless it would be rare at least to find a parent who did not feel that they had some experiences and beliefs about the school subject with which to influence the young learners. This is of course particularly the case for all adults' experiences and understanding of various mathematical ideas gained through their work, their leisure and through their own informal education, as Nunes will discuss in the next chapter. That cultural and societal heritage provides an influential source of knowledge, particularly in societies where formal education facilities are under pressure.

There are many individuals in a society who could potentially influence a young person, but if we are considering young learners as a whole, the two principal groups of people who seem to have the greatest potential for IFME are the adult members of the family and of the immediate social community, and people working in what I shall term 'the media'. From the young person's perspective, the only other major source of influence, apart from adults exercising a formal educational role, is that of their peer group, and their role will be mentioned in a subsequent section.

Turning first to the immediately available adults for the young learner of mathematics, we can surmise at once that it is their perception, memory and image of how mathematics in school was for them which will colour their influence. As well as therefore being a traditional image, it also appears to be a mainly negative image. The

evidence, such as it is, documents the feelings of inadequacy and inferiority felt by many parents about their knowledge and image of mathematics. In the UK, for example, the influential Cockcroft Committee (1982) commissioned a study into the mathematical knowledge of adults in UK society. The findings were dramatic, as they revealed not just the widespread inability to do what could be called 'simple' mathematical tasks but also the frustrating, unpleasant and generally negative emotions felt by many people about their mathematical experiences. Here are some of the statements made by respondents:

> "I get lost on long sums and never know what to do with the 'left-overs'."
>
> "My mind boggles at the arithmetic in estimation."
>
> "I'm hopeless at percentages really."
>
> "I'm afraid I have to write it down. My brother can do it in his head."
>
> "My husband says I'm stupid."

Even in a study which involved only UK adults who had already obtained first degrees at University (excluding mathematics degrees), many of the same difficulties and negative attitudes emerged (Quilter and Harper, 1988). In both studies the reasons for these problems were mixed, but memories of poor teaching and uncaring teachers figured prominently, as did an image of mathematics as a 'rigid' subject, lacking relevance to their personal lives, and having correct procedures which needed to be performed accurately. Strangely it also appears that for many adults, the negative experience is assumed to be so widespread that to claim mathematical ignorance and inadequacy is socially acceptable, however unpleasant it may be personally.

From the perspective of the previous section also, we can well understand the guilt associations which usually accompany such feelings. If, as was described there, mathematics has been used as a strong academic and social filter by society, many people will have experienced failure as a result. Even those who survived the filtering process can be expected to have some negative feelings about mathematics - they know that they had to do well in it to progress in their education or their work, and that that incentive was,in many cases, their only motivation for continuing to study it.

If the collective parental and adult memory of school mathematics is in fact a largely negative one, this memory can so easily be transmitted as a negative image to the next generation, thereby influencing the mathematical expectations of the children, their motivations for studying mathematics and their predispositions for continuing, or not, to study the subject. For example the evidence from the gender research demonstrates the strength of parental influences (Fox, Brody and Tobin, 1980). While the *patterns* of influence may not be the same across societies, the influences themselves clearly exist.

However we should not assume that all parental influence is of a negative kind, nor that there is nothing to be done about it. The FAMILY MATH project (Thompson, 1989) is a project concerned to help parents to help their children with their school mathematics. As Thompson says: "The EQUALS educators work to encourage all students to continue with math courses when they become optional in high school in the United States - particularly those students from groups that are under-represented in math-based careers. These educators asked EQUALS to develop a program that would also involve parents in addressing this issue" (p.62). The reports about FAMILY MATH indeed suggest that it is possible to affect and mediate parental influences on their children's attitudes and achievements in mathematics.

Turning now to what I call 'the media', we need first to clarify what this might mean in different societies. It refers to the people responsible for disseminating information, images, beliefs, and ideas in general, produced by individuals and groups within society, but received by the young through different media - newspapers, circulars, books, radio, films, TV, advertisements etc. Occasionally, as with for example an academic visitor to the school, a young learner will have a chance to engage *directly* with that individual, but such opportunities are rare, and in general their ideas are mediated by 'the media'.

One important difference between traditional village societies and modern industrial societies is the relative proportion of influence coming from immediate community adults compared with that from the media. Whereas in traditional village

societies the local adults can exert maximal influence over the young learners (see for example, Gay and Cole, 1967), in modern industrial societies the amount of information and imagery coming from the media is much greater and more complex, and has an all-pervading influence both on the young and on the adults themselves. In these societies, the information and images available can include job and career information, the latest scientific and engineering ideas presented through 'popular science' television programmes and in magazines, games and puzzles of a mathematical nature, as well as facts, figures, charts, tables and graphs on every conceivable aspect of human activity. The learners will see mathematical ideas and activities appearing in a variety of contexts; and they will see people engaging with those ideas in various ways and with various feelings. However 'the media' do not take it upon themselves to represent the whole spectrum of human engagement with mathematical ideas - their task is often one of satisfying a customer or client need, as in selling a magazine or advertising a new computer system.

Generally therefore, and unlike the personal knowledge one has from known adults, the media will tend to project 'typical' images and this typicality has become a source of great concern amongst those educational and social activists seeking to increase the educational and employment possibilities of groups within their society who do not enjoy equal opportunities under the present systems. For example, the research on gender, on social class, and on ethnic and political minorities in industrialised societies has often pointed to biased stereotypical images portrayed by the media. In terms of influences on the learner of mathematics, these images often relate to the prestige and importance industrialised societies attach to mathematical, technological, scientific and computer-related careers, to the need for high levels of qualification and skill in these subjects, and to the predominance in these careers of mainly men, who tend to come from the more powerful ethnic or social groups in that particular society (see, for example, Holland, 1991; Harris and Paechter, 1991). Thus the messages which are received by the young people are that mathematics is a very important subject, that it is a difficult subject, and that only certain people in society will be able to achieve well in it.

A recent movement to overcome this largely unhelpful influence is known as the 'Popularization of Mathematics', a collection of attempts to 'improve' the image of mathematics as a subject, by using media of all kinds (TV, radio, newspapers, games, etc.) to reveal the more pleasurable faces of mathematics and to generate interest, enthusiasm and positive intentions, rather than failure, fear and avoidance. The conference of that title, organised by the International Commission on Mathematics Instruction, and its report (Howson and Kahane, 1990) involved educators from many countries in the issues of why popularize, how to do it successfully, and who are the 'populace', amongst others. The ideas ranged from Mathematics Trails (walks involving mathematical problems) through city centres, to mathematical game shows produced for prime-time national television, from breathtakingly beautiful films and videos to mind-bending wooden puzzles, from 'popular' mathematical articles in national papers to special mathematical magazines for children.

The conference, and the report, left one in no doubt that the mathematics education community believes that the influences coming at present from the media are not generally helpful and need to be educated, and enriched.

Societal influences through non-formal mathematics education (NFME)

Adapting Coombs' (1985) definition, non-formal mathematics education would involve any organised, systematic, mathematics education activity, carried on outside the framework of the formal system, in order to provide selected types of learning to particular subgroups in the population, adults as well as children.

We can easily recognise NFME in provision such as adult numeracy programmes, out-of-school 'gifted children' courses, televised learning courses, correspondence learning activities, and vocational training courses of many kinds. As at previous congresses, for example, the Action Group 7 on "Adult, technical and vocational education" held at the International Congress on Mathematics Education (ICME6) in Hungary in 1988,

contained many examples of interesting developments grouped under the sub-headings of 'Vocational/Technical Education', 'Adult Education' and 'Distant Education', (Hirst and Hirst, 1988).

At that congress there was also a Special Day devoted to the topic "Mathematics, Education, and Society" (Keitel et al., 1989) at which various ideas concerning NFME were presented. For example, Smart and Isaacson (1989) described a course for non-science qualified women who wished to change to a career in technology and science, and which involved many mathematical 'conversions'. Sanchez (1989) talked about a radio programme series in Spain designed to popularise mathematics. Also Volmink (1989) described different approaches to "Non-school alternatives in Mathematics Education" which the black communities in South Africa have turned to because of their frustration with the FME provision at present: e.g. private commercial colleges, supplementary programs, and the Peoples' Education Movement which is articulating generalised concepts which "aim to transform the educational institution and will find final embodiment only within a new political structure" (p.60). Thus NFME not only offers opportunities for a different context for learning mathematics, but in some situations it is clearly intended to challenge the precepts and practice of FME. In the same vein, and in the same country, South Africa, Adler (1988) describes a project concerned with "newspaper-based mathematics for adults" who had had little access to mathematics education because of the inadequate provisions within the apartheid system.

Non-formal education has in some way been a marker of educational development in a society, and Freire (1972) showed the developing world the way in which it could play a significant role in a society's development. The Peoples' Education Movement in South Africa is a good recent example of this, and it is clearly very important that mathematics is not left out of the whole process. In industrialised societies also there are significant NFME developments, and a good example is given by Frankenstein's (1990) book which describes a 'radical' mathematics course intended mainly for adult students who have been made to feel a failure at maths. It is a textbook which tries to overcome learning obstacles by developing methods that help empower students. More than most textbooks, it uses examples, illustrations, activities and tasks drawn from everyday experiences and contexts not just to develop mathematical knowledge, but also to develop a critical understanding in her adult students.

Furthermore many of the participants at the previously mentioned ICMI conference on 'The Popularization of Mathematics' (Howson and Kahane, 1990) spoke of mathematics clubs, after-school activities, master classes, and exhibitions, all of which contribute important influences of a non-formal mathematics education kind.

Volume 6 (1987) of the UNESCO series "Studies in Mathematics Education" is also entirely devoted to the theme of "Out-of-school mathematics education" and contains many examples including mathematical clubs, camps, contests, olympiads, and distance education courses. Losada and Marquez (1987) document in a detailed way the many out-of-school mathematics activities in Colombia and demonstrate how significant that form of education can be. As they say "while the teacher-student ratio in schools ranges from 1 to 20 to 1 to 60, traditional classroom education cannot be expected to be particularly effective... In such circumstances out-of-school mathematics programmes can provide more opportunities for more young people" (p.117).

Clearly the borderline between IFME and NFME becomes blurred at times, particularly when there are considerable attempts to enrich both at the same time by, for example, enlisting the help and support of adults and other significant people in the overall mathematical education of the young people. That does not really matter of course. What is important is that educators recognise as *educational*, the influences which come either informally or in a more structured way, non-formally, from the various segments of society.

What then can be said about such influences and what they affect? If we reflect a little more on the informal and non-formal influences described so far, we can detect effects at two levels, both of which have important consequences for mathematics learners; the level of mathematical ideas, and the level of learning mathematics.

Influences on mathematical ideas and on the learning of mathematics

At the first level, there are clearly influences regarding mathematical concepts and skills. Before children come to school, their parents will often have taught them to count, to begin to measure, to talk about shapes, time, directions etc. Neither will this kind of parent's talk and activity cease when the children begin school. Within the immediate adult community particular knowledge, often related to mathematics, will be shared, as Nunes documents in the next chapter. The newspapers and other media reports involving money, charts, tables, percentages will all inform and educate the learner, complementing and supplementing the information and ideas being learnt in school. Now, with sophisticated calculators and home computers becoming more widely available, the young mathematical learner may well be racing ahead of the formal mathematics curriculum and will often outstrip the teacher's knowledge in some specific domains.

Equally the sources for mathematical intuition are frequently images from society, rather than from within school. Popular images and beliefs, for example, about statistical and probabilistic ideas seem not only to come from society but also seem to be relatively impervious to formal educational influences. (Tversky and Kahnemann, 1982; Fischbein, 1987).

Geometrical images will come from physical aspects of the environment, although what is important is how the individual interacts with that environment. Thus, once psychologists thought that living in a 'carpentered environment' (with straight sided houses, rectangular shapes, right angles etc) was very important for learning geometry and for developing spatial ability. Now we know that what is important is how individuals interact with their environment. So for example, to an urbanised European the desert of central Australia may seem to be devoid of anything which could aid spatial development, but the spatial ability of the Aborigines who live and work there is known to be exceptional because of what they have to do to survive there (Lewis, 1976). This is equally the case with Polynesian navigators (Lewis, 1972) and Kalahari nomads (Lea, PME, 1990). Interactions with the physical environment of a society undoubtedly give rise to many geometrical images and intuitions.

The other major source of societal influence on the learner's knowledge of mathematical ideas is the language used. The relationships between language and mathematics are of course extremely complex, and there is no space here to cover all the ground which has in any case already been analysed by others (Pimm, 1987; Zepp, 1989; Durkin and Shire, 1991).

From the perspective of this chapter the two most important aspects for mathematics educators to be aware of seem to be:

- the fact that mathematics is not language-free,
- not all languages are capable of expressing the mathematical concepts of MT culture.

The first point may seem obvious, but it has profound implications. Mathematical knowledge, as it is developed in any society relates to the language of communication in that society. As has already been shown, the mathematics curriculum in many countries has been based largely on the Western-European model and it has a certain cultural, and therefore, linguistic basis. Though this basis is an amalgam of different languages, the principal linguistic root is believed to be Indo-European (Leach, 1973). That particular 'shorthand' omits the important Greek and Arabic connections in the development of universally applicable mathematics, and we should perhaps consider the Indian-Greek-Arabic-Latin chain as being its original language base. From this base, Italian, Spanish, French, German, and English developed its language repertoire during the 17th, 18th and 19th centuries, and it is probably the case that nowadays English is the principal medium for international mathematical research developments. This is an important problem for any country, researcher, or student, for whom English is not the first or preferred language.

In relation to the second point, in many countries of the world there are several languages used, but for national and political reasons, one (or some) are specifically chosen as the national language(s). It is likely that not all the languages being used in a society will necessarily be capable of expressing the

concepts and structures of the MT culture - this will largely depend on the roots of the language. European-based languages, those with Indian roots, and the Arabic family of languages appear to have the least structural differences, although there are always particular vocabulary gaps as international mathematics develops.

Other languages, in rural Africa, Australasia, and those used amongst indigenous American peoples, are being studied and demonstrate their difficulties in expressing both the structures and the vocabulary of the MT culture's version of mathematics (see, for example, Zepp, 1989 and Harris, 1980). This is not of course to say that these languages are incapable of expressing *any* mathematical ideas - they will certainly be capable of expressing the mathematical ideas which their cultures have devised. This is the important linguistic relationship with ethnomathematics - and another reason for seeking to create more independent societally-based mathematics curricula rather than relying on the model from MT-based societies. Thus in this way the societal language(s) can reinforce the societal mathematics which can offer the bases for alternative curricula.

But language issues are extremely complex, particularly from a societal perspective, and the political and social conflicts which different language use can cause, can seem to be of a different order from those which should concern mathematics educators. Nevetheless so much damage has been done to cultural and social structures in many countries by assuming the universal validity of MT-based mathematics that we cannot ignore the language aspects of this cultural imperialism (see Bishop, 1990). If countries, and societies within countries, are to engage in the process of cultural reconstruction then the language element in relation to informal, non-formal, and formal mathematics education is critical.

A final point concerning the informal and non-formal influences on mathematical ideas is that they have a cumulative effect. They build up into an image of 'mathematics' as a subject itself. For example, we have already noted that it is projected as being an important and prestigious subject in both industrial and developing societies and is thereby projected as being essentially a benign subject. Little mention is publicly made of its extensive association, through fundamental research, with the armaments industry, with espionage and code breaking, and with economic and industrial modelling of a politically-partial nature. Little public debate occurs about the questionable desirability of fostering yet more mathematical research to make our societies yet more dependent on even more complex mathematically-based technology (see however, Davis, 1989 and Hoyrup, 1989). The reaction of 'the media' to the examination question about costs of armaments shows the extent of public ignorance of these matters.

Equally mathematics in society is typified, and imagined by most people, as *the* most secure, factual and deterministic subject. There is little public awareness of the disputes, the power struggles, or the social arenas in which mathematical ideas are debated and constructed. Descartes' dream still rules the general societal image of mathematics. For example, in the study by Bliss et al. (1989) concerning children's beliefs about "what is really true" in science, religion, history and mathematics, the majority of children in England, Spain and Greece considered both mathematics and science to be truer than history or religion. As Howson and Wilson (1986) put it "Only in mathematics is there verifiable certainty. tell a primary child that World War 2 lasted for ten years, a.,d he will believe it; tell him that two fours are ten, and there will be an argument" (p.12).

At a second level the informal and non-formal societal influences concern the *learning* of mathematics. We have already noted the beliefs about its difficulty and its motivations, but there are also more fundamental and significant beliefs about how mathematics is learnt. Paralleling the popular image of mathematics as secure factual knowledge is the widespread belief that mathematical procedures need to be practised assiduously and over-learnt so that they become routine, and that this should go hand-in-hand with the memorising of the various conceptual ideas and their representations. Another popular belief concerns 'understanding' as being an all-or-nothing experience, rather than a gradual increasing of meaning and constructed connections. Overall the popular image is of a received, objective, form of knowledge,

rather than of a constructed subject, running contrary to what we know from recent cognitive research. (Lave, 1988; Schoenfeld, 1987).

There is also, as has been mentioned, the impression that because it is reputed to be a difficult subject, only some learners will be able to make progress with it. Thus, rather like artistic or musical ability, young people perceive themselves as either having mathematical ability or not. The research on mathematical giftedness does in fact demonstrate that it clearly is a precocious talent, appearing early rather like musical giftedness, but one suspects that the image from society about mathematical ability is rather more broad in its reference than just to giftedness.

The concept of 'ability' does seem to be a pervasive one in society outweighing 'environment' as the cause of achievement, particularly in the 'clear-cut' subject of mathematics. The accompanying belief is that one is fortunate to be born with this ability since it is (clearly!) innate. The pride of a parent on discovering that their child is a gifted mathematician is probably only clouded by the popular image from society of mathematical geniuses being slightly odd characters living in a remote and esoteric inner world and unable to socialise with other people - a theme to which several contributors to the ICMI conference on 'Popularization' referred.

Equally parents, although not necessarily rating their son's or daughter's abilities any differently, do apparently believe that mathematics is harder for girls and requires more effort to succeed in (Fox, Brody and Tobin, 1980). This belief undoubtedly helps to shape the feeling, prevalent in different societies, that mathematics is not such an important subject for girls to study. Fortunately many womens' groups are now hard at work dispelling this image, and IOWME (the International Organization of Women and Mathematics Education) has been particularly active. Recent research (Hanna, 1989) for example shows that the creation of a more positive and favourable image for girls who are mathematically able is apparently having some beneficial effects.

These then are some of the most significant influences which society exerts on the learners of mathematics. The 'messages' the learners receive are many, and often conflict. Some are intentionally influential, while others are merely accidental. But they all help to shape important images and intuitions in the young learners' minds, which then act as the personal cognitive and affective 'filter' for subsequently experienced ideas. Let us then move on to consider how the learners make sense of, or cope with, these various societal influences which they experience. Are there any research ideas which can help us to understand and interpret the learners' situation?

The competing influences on the individual learners

Considering first of all the area of motivation, this is perhaps the most significant aspect needing to be richly understood by mathematics educators, because it provides the essential dynamic for the mathematics learning process wherever and whenever it takes place. A classic dichotomy is usually made between extrinsic and intrinsic motivation, referring to the location of the influence for motivation. In one sense we can think of societal influences as being essentially extrinsic whereas, for example, the 'puzzling' nature of a mathematical problem can seem to be located internally to the learner. The interaction between intrinsic and extrinsic motivation though relates both to the internalisation of the extrinsic, and also to the perception of a familiar external target for the intrinsic. It is clear that societal influences initially exist 'outside' the learner, but the extent of their influence will depend on how internalised they become, and, therefore, that internalisation process is the key to understanding motivation for the learners.

We have claimed in this chapter that societies formally influence mathematics learning through their school curricula and their examination structures. From the learner's perspective, therefore, those influences will motivate to the extent that they are internalised and help to shape the learner's goal structures. There appear to be two significant and interacting aspects to attend to here - the 'messages' being communicated to the learner and the people conveying those messages. In terms of the mathematics curriculum, the principle message will concern the values - explicit or implicit (the 'hidden' curriculum) - which the curriculum embodies, mediated by the teacher, and which may, or may not, be seen

as significant by the learner. The area of values in mathematics curricula has not been well explored in research, but the clues we have from research on attitudes and beliefs certainly supports the above analysis (see for example, McLeod, PME, 1987).

However, the extent to which the mathematics curriculum embodies the values of MT culture (for example) will only partly determine the motivational power of those values. The real determining factor will be each individual's acceptance of those values as being worthwhile or not, which will depend on how they relate to the learner's goal structure, and how different people figure in that structure. To talk of goal structures is not however to suggest that they are fixed and immutable. One recently developing research area - that of situated cognition (Lave, 1988; Saxe, 1990) - reveals the phenomenon of 'emergent' goals. If one understands mathematics learning as something which develops from a social mathematical activity, then through that activity will emerge mathematical goals, to be achieved, which would not necessarily have been apparent before the start of that activity.

Thus, one corollary is that as one learns about mathematical ideas through doing mathematical activities, one also learns about goals, and in a social context. A statement by the teacher to the effect that an important goal of mathematics learning is to become systematic in one's thinking, for example, may well be negated by the learners' shared experiences of trial-and-error approaches which nevertheless succeed in finding appropriate solutions to problems (see Booth, 1981). Moreover, it may well be that the latter message, learnt in activity and in the context of one's peers, or one's adults, will be internalised more significantly than the teacher's message.

This kind of gulf, between what learners are told that they should do and what they actually do to be successful, may well be one source of the well-documented dislike of mathematics, and even of the phenomenon of mathophobia, the fear of mathematics. It is the kind of experience which, if repeated often enough, will lead learners to believe that mathematics, even if it is an important subject, may not be what they personally want to invest their energies in. More research on learners' emergent goals, particularly in the school learning context could provide some very powerful ideas for mathematics educators.

A more significant contributor to the learner's mathematical motivation may well be the messages concerning the examinations, which come both formally and informally from society. The within-school 'messages' concerning final examinations, combined with the experiences of immediate adults will be highly significant in creating a schema of beliefs and expectations with which the learner's own experiences of mathematical success or failure will be interpreted. The internalisation of beliefs and expectations, as with any aspect of motivation, will be dependent on the significance to the learner of the people mediating those beliefs and expectations. Sullivan (1955) refers to 'Significant Others' i.e. people who can exercise that kind of influence - they may be role models, advisers, counsellors, objects of worship and awe, or of scorn and dislike. Their significance is a personally attributable characteristic and will be likely to vary both as the richness of the individual learners varies and as the variety of 'types' in the society varies. From this analysis we can see that the learner's mathematics teachers, parents, mathematical peers and real or mediated 'mathematicians', are likely to be the major sources of mathematical motivation influence.

The learner's interpretation of success and failure does seem to be an important research site for understanding their motivation, particularly in a subject like mathematics, which is believed by many learners to be 'clear-cut', where one always knows (they think) whether the answer is right or wrong. Moreover this act of 'interpretation' can be seen as part of the wider cognitive task for the learner, of understanding mathematical activity as a societally defined phenomenon. Attributing success or failure in the subject relates strongly for the learner to personal questions and concerns like "How will I fit into society - what job will I do - how mathematically qualified should I become?" and "How do I relate to other people in society - am I more or less competent at mathematics than they are?"

Attribution theory can help us here. Wiener's (1986) work is being explored in relation to mathematics learning and the findings are interesting. For example Pedro et

al. (1981) reported that, in US society, males are more likely to attribute their success to their ability than females, and that females are more likely than males to attribute their successes to their efforts. So far, however, we have no research which systematically explores mathematical attributions in relation to the influences of significant others, although the role of the teacher does appear to be fundamental, as might be expected (see Lerman, this book).

Attributional understandings and interpretations like these appear also to be strongly embedded in the learner's meta-cognition about mathematics. Because mathematics at all levels is learnt in socially defined activities, the people who share in influencing those mathematical activities will also play a role in shaping each learner's total perspective on mathematics. Gay and Cole (1967) in their study of the Kpelle, were clearly taken by surprise by one of the college students they were testing:

> A Kpelle college student accepted all the following statements: (1) the Bible is literally true, thus all living things were created in the six days described in Genesis; (2) the Bible is a book like other books, written by relatively primitive peoples over a long period of time, and contains contradiction and error; (3) all living things have gradually evolved over millions of years from primitive matter; (4) a "spirit" tree in a nearby village had been cut down, had put itself back together, and had grown to full size again in one day. He had learned these statements from his Fundamentalist pastor, his college Bible course, his college zoology course, and the still-pervasive animist culture. He accepted all, because all were sanctioned by authorities to which he feels he must pay respect. (p35)

The researchers seemed to expect logical consistency to be an over-riding criterion for that student, and they used that kind of evidence to infer that mathematical reasoning (as they knew it) had little place in Kpelle life. We now know of course that mathematics is not the universal and independent form of knowledge which they assumed. Moreover, and perhaps for us now more importantly, they also assumed that such socially-bound cognitive behaviour was not typical in America (their home). Lave (1988) has certainly demonstrated that not to be the case, and in general the growing awareness of the socially constructed nature of all human knowledge makes us realise that all learners have the task of making sense of other people's messages in a variety of social situations.

Thus although perhaps mathematics educators would like to think that in some way societal influences from the immediate adults and from the media will enhance, support and generally reinforce the messages which the child receives from school, the reality is more likely to be one of conflict. The variety of people involved makes it unlikely that there will be a kind of benign consistency.

The learner is therefore rather like a bilingual, or multi-lingual, child who must learn to use the appropriate 'language' in the appropriate social context. In fact it would be more appropriate to describe the mathematics learner as bi-cultural, or multi-cultural, since so much more than language is involved in learning mathematics. We have already noted the influence of interactions with the physical environment which create so much of an impact on geometrical and spatial intuitions. Also social customs and habits, which give rise to expectations, are not always expressed through language but are learnt through social interactions while engaged in particular activities.

Halliday's (1974) work on 'linguistic distance', for considering the cognitive task for bilingual learners, is an interesting construct here. Dawe's (1983) research used Halliday's idea to conjecture that bilingual learners of mathematics in English at school would have more difficulty the 'further away' the structure of their home language was from English. So, for his study, the order: Italian, Punjabi, Mirpuri and Jamaican Creole, was hypothesised as being the difficulty dimension, with Italian being the 'closest' to English. For logical reasoning in English this proved to be the case, but not for mathematical reasoning, where a gender effect interacted with the language factor.

It is possible to broaden the idea of linguistic distance to that of 'sociocultural distance' between two different principal social situations experienced by the learner. Thus it seems reasonable to conjecture that mathematics learners whose immediate home cultures relate more closely to the structure and character of the school culture,

will have less difficulty in reconciling the messages coming from the two cultures, than will learners whose home cultures are a long 'distance' from their school culture.

Perhaps though we will make more progress in educational terms if we attend less to the "messages received" metaphor which emphasises the role of the learners as predominantly passive receivers, and more to the idea of the learners as constructors of their own personal and cultural knowledge. The essential cognitive task for any learner is making meaning, by creating and constructing sensible connections between different phenomena, and because most mathematical activity takes place in a social context, this mathematical construction will take place interactively. Each new generation of young mathematical learners is actively reconstructing the varied societal knowledge of mathematics with which they come into contact, and in their turn they (as adults) will influence the social context within which the next young generation will engage in their own reconstruction.

Of particular relevance to this point is Saxe's recent research (Saxe, 1990) which demonstrates three aspects of importance to us as educators. Firstly his street candy-sellers in Brazil learned as much from older children as they did from immediate adults in the communities. So we should include 'older children' in the categories of 'immediate adult' in our previous discussions, as they are likely to play a key role in any socialisation process. Also that knowledge was validated and developed within a strong peer-group structure, and it is likely that peer-groups influence far more than has been recognised so far. Finally although there was some evidence of school mathematical ideas and techniques being used out of school, it was very clear that the street selling experiences had furnished a rich schema which informed the children's learning within the school situation. This finding supports, interestingly, the cognitive instructional research findings (see Silver 1987 for a summary) that the most significant contributor to the learning of new information is the extent of the previously learnt information. However it should also encourage educators to realise that the most important prior knowledge may well have been learnt *outside* the school context, and will therefore be embedded in a totally different social structure.

My own example of this, already extensively quoted (see Bishop, 1979), concerns an interview with a student in Papua New Guinea, a situation where the home/school socio-cultural distance is large. I asked him "How do you find the area of this (rectangular) piece of paper?" He replied "Multiply the length by the width". I continued "You have gardens in your village. How do your people judge the area of their gardens?" "By adding the length and width". Taken rather aback I asked "Is that difficult to understand?" "No, at home I add, at school I multiply". Not understanding the situation I pursued the point: "But they both refer to area". "Yes, but one is about the area of a piece of paper and the other is about a garden". So I drew two (rectangular) gardens on the paper, one bigger than the other, and then asked "If these were two gardens which would you rather have?" to which he quickly replied "It depends on many things, I cannot say. The soil, the shade ..." I was then about to ask the next question "Yes, but it they had the *same* soil, shade ..." when I realised how silly that would sound in that context!

Clearly his concern was with the two problems: size of gardens, which was a problem embedded in one context rich in tradition, folk-lore and the skills of survival. The other problem, area of rectangular pieces of paper was embedded in a totally different context. How crazy I must be to consider them as the same problem!

So how did *he* reconcile the conflict which I could see? I cannot answer that question because I firmly believe that for him there was no conflict. They were two different problems set in two entirely different contexts, and it was only I who felt a conflict. As with Gay and Cole above, my cultural background encouraged me to believe that logical consistency demanded a resolution of the conflict which would arise *if* one were to attempt to generalise the two procedures. If one were not interested in doing that however, there is no problem.

Educational implications

What, then, are the educational implications that can be drawn from all of this evidence, analysis and conjecture? The first thing to ask is "What do we mean by education?" since it is now clear that the learner is subject to many influences of a potentially educating kind. Also we can now see that if 'education' is only considered to be

what happens in school then any potentially powerful out-of-school experiences will be denied a legitimacy in that society. As was said earlier, at present the school curriculum and examinations contain what most adults think should be learnt through mathematics teaching and any other mathematical knowledge they have learnt through practice is not "real" mathematics. In fact it would be easier to argue that totally the opposite is true: school mathematics, because it is learnt in the necessarily artificial world of school cannot be 'real' mathematics!

But the issue is not whether one context is real and another not. Both clearly are real, but they are also very different. Therefore the conceptual task is not to create an artificial dichotomy but to begin by looking for the similarities and Coombs (1985) has offered us an important way forward. We can now consider formal mathematics education (FME), informal mathematics education (IFME) and non-formal mathematics education (NFME).

In certain cases the developments we can see in NFME and IFME are motivated by the desire to complement, supplement and generally extend the FME 'diet' offered by the formal educational institutions, while in others there is an objective of challenging the assumptions, the structures or the practices of FME. These kinds of developments will surely have influences on FME to the extent that they can demonstrate their successes to the educational professionals and to the wider society.

From the formal educational viewpoint these developments in IFME and NFME could be considered as unwanted challenges to the educational status quo, and which should therefore be resisted by whatever political and social means are possible. However more positively one can consider these developments as important educational experiments whose effectiveness, if proven, and whose validity, if accepted, compel them to be considered as viable possibilities within the formal educational provision.

This is not to propose their demise as valid educational agents in their own right but to recognise that formal mathematics education can learn from these 'experiments', and can be modified. In some cases they are developing new methods, new materials and new practices, which could be used to extend the range of ideas reaching teachers, and teaching material developers. In others, there could be influences on the intended formal curriculum of the society and on the examination procedures.

The increases, both quantitative and qualitative, in NFME and IFME provision in both industrial and developing societies, certainly demonstrate that FME at present is not meeting the demands which different societies are making. It should therefore be a continual source of challenge to those of us who are responsible for the character of FME in any society to explore its potential for responding to the different demands and influences coming from society. For example, television and other popular media can raise the awareness about certain ideas, but they cannot develop the knowledge of those ideas systematically. Neither can they develop skills effectively. On the other hand skills, such as key-typing, can be learnt effectively through non-formal situations, which could have much pay-off for formal educational work with computers and word-processors. So how should FME be shaped, in the context of rapidly expanding and influential NFME and IFME developments?

Trying to answer that question means that those of us who work in mathematics education more generally need to be much more aware than we have been of developments in NFME and IFME, to accept them as valid educational concerns, to stimulate their active growth and to recognise their growing power and influence not just on FME but on societies' development generally.

This chapter has demonstrated not only the range of influences which society brings to bear on mathematics learners, but also how the learners try to deal with these and how the different educational agents respond to societies' demands. It is a chapter predicated on the belief that a societal perspective on mathematics learning is essential in framing the more narrow and specific concerns of research, development, and practice within formal mathematics education.

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The Socio-cultural Context of Mathematical Thinking: Research Findings and Educational Implications

Terezinha Nunes

R: 8 years old, is solving a problem about giving change in a simulated shop situation which is part of an experiment. He writes down 200 - 35 properly aligning units and tens and proceeds as follows:

R: Five to get to zero, nothing (writes down zero); three to get to zero, nothing (writes down zero); two, take away nothing, two (writes down two).

E: Is it right?

R: No! So you buy something from me and it costs 35. You pay with a 200 cruzeiros note and I give it back to you?

After another unsuccessful attempt on the same computation, the experimenter asks:

E: Do you know how much it is?

R: If it were 30, then I'd give you 170.

E: But it is 35. Are you giving me a discount?

R: One hundred and sixty five.

From Carraher, Carraher & Schliemann, 1987, p.95

Much research has shown that mathematical activity can look very different when it is embedded in different socio-cultural contexts. For example, we calculate the amount of change due in school word problems by writing down the numbers and using the subtraction algorithm. Outside school in a shop we may figure out change differently, using oral methods and decomposing numbers (as R. did, decomposing 35 into 30 and 5 and solving the two problems sequentially) or even by adding on from the value of the purchase to the amount given as the money is handed to the customer. It is my purpose in this paper to discuss the fact that mathematical activities look different in different contexts and to explore the implications of these differences for mathematics education. The review presented here is selective both due to space limitations and to my choice of depth rather than breadth of coverage. Only two topics in mathematics education will be reviewed: arithmetic and geometry. For a broad coverage of topics, other works can be consulted (Keitel, et al., 1989; Educational Studies in Mathematics, 1988; Nunes, in press; Stigler and Baranes, 1988).

This paper is divided into four main sections. The first section relates to arithmetic and analyzes two types of difference between arithmetic in and out of school, differences in success rates and in the activity itself. The second section relates to space and geometry and looks at the interplay between logic and convention in geometrical activities. In the third section, two views of how to bring out-of-school mathematics into the classroom are contrasted. Finally, the last section briefly summarizes the main issues in the study of the socio-cultural context of mathematical thinking and outlines ideas which can be explored and investigated in the classroom.

Arithmetic in and out of school.

Human activity in everyday life is not random but organized, or structured, to use a more mathematical term. Even the simplest interaction, like that between two friends meeting on the street, can be shown not to be totally spontaneous but rather structured and predictable. I will distinguish two types of organization for the purposes of this discussion, social and logico-mathematical organization. Social organization is prescriptive and often implicit; it has to do with what people should do in certain types of situation without their being necessarily able to know why they behave in those particular ways. Logico-mathematical organization is deductive and can often be made explicit; it allows people to go beyond the information given at the moment and to know why the deduced information must be correct (and that holds even in the cases when an error has been made). The logico-mathematical structure pertains to the actions and situations as such while the

social organization pertains to the actions as interpersonal events, that is as interactions.

In any mathematical activity, both social and logico-mathematical organization are involved, regardless of whether it takes place in the classroom or outside (for a different view, which rejects the idea that mathematical skills are based on logical structures, see Stigler & Baranes, 1988). However, it must be emphasized that distinguishing between the social and the logico-mathematical organization for analytical purposes does not assume the independence of these two aspects of organization in human life. In any instance of mathematical activity, be it in the classroom or outside, both forms of organization come into play.

Mathematical activities carried out in and out of school have different social organizations but are based on the same logico-mathematical principles. The kernel of the social organization differences between mathematical activity in and out of school appears to be that everyday activities involve people in mathematizing situations while traditional mathematics teaching focuses on the results of other people's mathematical activities (Bishop, 1983) Thus in school, teachers expect that students will produce a particular solution (related to the application of an algorithm, for example) right from the moment a problem is posed. In contrast, an everyday problem may be correctly solved through many different routes and no particular route is prescribed from the outset.

To take a simple example, Scribner and her collaborators (Scribner, 1984; Farhmeier, 1984) have analyzed how inventory takers solved the problem of finding out quickly and accurately how many cartons of milk were in the ice-box often from viewing points which required them to fill in information about invisible cases in stacks. No particular route is "correct" or "expected" from the outset. In the classroom, a similar problem would be phrased as "how many cartons of milk are there if there are 38 cases and each case holds 16 cartons". The school problem requires in principle counting cases and multiplying number of cases by number of cartons in a case. Scribner and her collaborators found that in real stock-taking, counting single cases to multiply by the number of cartons was not the only strategy available. Several other strategies were used especially because piles were not neat and cases might not be complete. Other methods included the use of volume concepts (for example, counting stacks of cases of known height by looking only at the top of the stacks), skip-counting cartons in incomplete cases (for example, 8, 16, 24, 30 etc), compensating across cases and stacks (for example, counting a half case as full and then compensating for this when coming across another incomplete case), keeping track of partial totals per area in the ice-box etc. In short, actual inventory taking does not have a pre-established path: inventory takers tend to draw freely on known information combined in several ways. In contrast, a school word problem of the same nature is most likely to be used in order to practice the multiplication algorithm, a result of other people's mathematical activity which is to be learned in school.

Does this central difference in the social structure of mathematical activities in and out of school affect the way people represent and understand the logico-mathematical structures? This question will be looked at here from two perspectives. The first relates to the rates of success in mathematical activities carried out in and outside school. The second relates to the way the activity is carried out, its characteristics and methods.

Differences in rates of success.

Two studies have documented systematically very important differences in rates of success when *the same people* carry out basically the same mathematical calculations in and out of school.

The first of these studies was by Carraher, Carraher, and Schliemann (1985), who interviewed five street vendors (9 to 15 years old) in Recife, Brazil. The youngsters sold small items like fruits, vegetables, or sweets in street corners and markets. The study started with the investigators approaching the youngsters as customers and proposing different purchases to the children, asking them about the total costs of purchase and the change that would be given if different notes were used for payment. The study was summarized by Carraher (PME, 1988) as follows:

"Below is a sequence taken from this study which exemplifies the procedure:

Customer/experimenter: How much is one coconut?

Child/vendor: Thirty-five.

Customer/experimenter: I'd like three. How much is that?

Child/vendor: One hundred and five.

Customer/experimenter: I think I'd like ten. How much is that?

Child/vendor: (Pause) Three will be one hundred and five; with three more, that will be two hundred and ten. (Pause) I need four more. That is... (pause) three hundred and fifteen... I think it is three hundred fifty.

Customer/experimenter: I'm going to give you a five-hundred note. How much do I get back?

Child/vendor: One hundred and fifty.

> When engaged in this type of interaction, children were quite accurate in their calculations: out of 63 problems presented in the streets, 98% were correctly solved. We then told the children we worked with mathematics teachers and wanted to see how they solved problems. Could we come back and ask them some questions? They agreed without hesitation. We saw the same children at most one week later and presented them with problems using the same numbers and operations but in a school-like manner. Two types of school-like exercises were presented: word problems and computation exercises. Children were correct 73% of the time in the word problems and 37% of the time in the computation exercises. The difference between everyday performance and performance on computation exercises was significant." (Carraher, PME 1988, p.3-4).

Thus, young street vendors were more successful in their computations outside school than in their efforts to solve school-like exercises presented to them by the same experimenters.

Similar results were obtained by Lave (1988) in a study with 35 adults (21 to 80 years) in California. All of the subjects in this study had higher levels of instruction (range 6 to 23 years) than the Brazilian youngsters. Lave and her colleagues observed adults engaged in shopping in supermarkets and trying to decide which was a better buy comparing two quantities of a product with two varying prices. An example of a problem of this type is transcribed below.

> A shopper compared two boxes of sugar, one priced at $2.16 for 5 pounds, the other $4.30 for 10 pounds. She explains,"The five pounds would be four dollars and 32 cents. I guess I'm going to have to buy the 10-pound bag just to save a few pennies." (Murtaugh, 1985, p.35)

Problems of this nature involve comparisons of ratio-- which one is larger, 2.16/5 or 4.30/10? Lave and her associates observed 98% correct responses in the ratio-comparisons carried out in the supermarket by the adults; in contrast, only 57% of the responses given by the same adults in a maths test of comparisons of ratios were correct. Thus, the results observed by Carraher, Carraher, and Schliemann (1985) for Brazilian street vendors are replicated among American adults: mathematics outside school shows better results than in school-like situations for the same subjects working on the same types of problems.

Differences in the social organization of street and school mathematics and their similarities in logico-mathematical structuring.

Several authors have tried to analyze the activities which are carried out when people are engaged in mathematical problems in and out of school. Nesher (PME, 1988) summarized some of the contrasts between mathematics in and out of school. Citing Resnick (1987), Nesher pointed out the following differences:

> "(1) schooling focuses on the individual's performance, whereas outside school mental work is often socially shared;
> (2) school aims to foster unaided thought, whereas mental work outside school usually involves cognitive tools;
> (3) school cultivates symbolic thinking, whereas mental activity outside school engages directly with objects and situations;
> (4) schooling aims to teach general skills and knowledge, whereas situation-specific competence dominates outside." (p.56)

To these, Nesher added also that:

> "Learning outside school is part of the immediate social and economic system. The goal on the part of the trainer is to

> put the trainee as soon as possible on the production line" (p.56)

while

> "This is not exactly the case at school. Schools aim to pass on knowledge to students to partly be employed at school in further training, but mainly to be employed elsewhere, after leaving school (...) schools have to deal with questions of motivation or with questions of rewarding procedures" (p.57)

which may not come up outside school.

Nesher's contrast of learning in and out of school, however, does not cover what many authors seem to regard as perhaps the two most important differences. These are: (1) that the mathematics used outside school is a tool in the service of some broader goal, and not an aim in itself as it is in school (see Lave, 1988; Murtaugh, 1985); and (2) that the situation in which mathematics is used outside school gives it meaning, making mathematics outside school a process of modelling rather than a mere process of manipulation of numbers.

Carraher (PME, 1988) expanded on this last aspect of the difference by arguing that

> "Mathematics outside school is conducive to the development of problem solving strategies which reveal a representation of the problem situation. The choice of models used in problem solving and the interval of responses are usually sensible [when mathematics is carried out outside school] even though not always correct. Students using school mathematics often do not seem to keep in mind the meaning of the problem, displaying problem solving strategies which have little connection with the problem situation and coming up with and accepting results which would be rejected as absurd by anyone concentrating on meaning." (p.19)

Carraher supported this analysis by referring to a study by Grando (1988), in which farmers' and students' responses to a series of problems were contrasted both in terms of the strategies used and in terms of how sensible the responses provided were in view of the problem. The farmers (n = 14) in the study had little or no school instruction and had learned most of their mathematics outside school; the students (20 in seventh and 40 in fifth grade), even though belonging to the same rural communities, had learned most of their mathematics in school. They were asked, for example, how many pieces of wire with 1.5 meters of length could be obtained by dividing into pieces a roll of wire 7 m long.

> "The farmer's responses, obtained through oral calculation [typically an out-of-school method], fell between 4 and 7 pieces with 93% giving the correct answer. The students' responses fell between .4 and 413 pieces. These extreme answers were given by students who carried out the algorithm for division (correctly or incorrectly) and did not know where to place the decimal point" (p.12)

a difficulty not faced by farmers. who continuously thought about the meaning of the question and would neither accept as an answer to this problem a number which indicated less than one piece of wire nor a number as high as 413.

The role of modelling in the farmers' procedures was clearly identified through their verbalizations, which indicated that they simultaneously kept track of the number of pieces and the amount of wire used up. Their responses could be obtained by a process like the following: "one piece, one meter and a half; two pieces, three meters; four pieces, six meters--then, it's four pieces".

The argument being presented here is not that farmers were more able than students in mathematics. It is possible that students could have solved the problem correctly if they had used the same method as the farmers. All of the students attempted division in this problem, a method which was adequate and perhaps triggered by the idea of dividing the roll of wire into pieces. However, faced with the difficulty of operating with decimals, they tried no other route to solution. Farmers, in contrast, were not restricted to division by any particular set. They were consequently able to avoid the decimal division by adding the pieces up to the desired total length, thereby preserving their ability to monitor their response as solution was approached.

Differences in methods for solving problems in and out of school are also very clearly observed in problems involving more than one variable, like proportions problems. From the mathematical point of view, all proportions problems involve the same structure. Thus it is possible to develop a

general algorithm which can be applied in any proportions problem independently of the type of variable involved, be it money, length, weight or whatever. This is in fact what the teaching of proportions in school aims at: the transmission of a general algorithm, which takes the form $a/x = b/c$. However, outside school the nature of the variables and the relationships which connect them are not set aside for the sake of the development of a general algorithm. The routes to solution tend to reflect in some sense the relationship between the variables, as we can see by contrasting the findings reported by Carraher (PME, 1986) with foremen and by Schliemann and Nunes (1990) with fishermen solving proportions problems in Brazil.

Carraher (PME, 1986) observed that construction foremen solving proportions problems about scale drawings preserved throughout calculation the notion of scale--that is, the notion that a certain unit drawn on paper corresponds to a certain unit of "real life" wall. The 17 foremen interviewed in this study were successively shown four blueprints and asked to figure out the real-life size of one or more walls in each blueprint drawing. The data they used to solve the problem were obtained from the blueprint: a pair of data on one wall (the real-life size and the size on the blueprint) and the size on the blueprint of the target wall. All successful solutions with unknown scales were obtained by foremen who first simplified the initial ratio (size on scale to real-life size) to a unit value and then used this ratio in solving the problem. When the original pair of data on one wall was, for example, 5 cm on the blueprint corresponding to 2 m in real-life, foremen first simplified this ratio in order to find the basic correspondence--"2.5 cm on paper is worth 1 m of wall", as a foreman said--and then used this simplified ratio to find the length of the target wall. This strategy of finding the simplified ratio allowed foremen to carry out the arithmetic in a way that reflected the relationship between the variables, value on paper and real-life size.

Foremen's strategies build an interesting contrast to those observed among fishermen by Schliemann and Nunes (1990) because fishermen hardly ever used as a method the identification of a simplified relationship between the variables. Schliemann and Nunes interviewed 16 fishermen who had to deal with the price-weight relationship in their everyday life in the context of selling the products of their fishing. On the basis of their analysis of the fishermen's everyday activities, Schliemann and Nunes expected that fishermen would develop methods of solution of the type which has been termed "scalar solution" by Noelting (1980) and Vergnaud (1983). Scalar solutions to proportions problems involve carrying out parallel transformations on the two variables (such as doubling, trebling or halving the values) without calculating across variables (and thus finding the functional factor). Schliemann and Nunes observed that scalar solutions were preferred by fishermen even when these solutions became cumbersome due to the fact that the target value was neither a multiple nor a divisor of the data given in the problem (like finding the price of 10 kilos given the price of 3 kilos of shrimp). The strong preference observed among fishermen for scalar solutions cannot be explained in terms of their school instruction for two reasons: (1) scalar solutions are not taught in Brazilian schools, where the Rule of Three ($a/x + b/c$) is traditionally taught; and (2) 14 of the 16 fishermen had less than 6 years of schooling and the proportions algorithm is taught in 6th or 7th grade in the area where the study was conducted.

The preservation of meaning in out-of-school mathematics is clear not only in the modelling strategies involved in problem solving but also in the calculation procedures used. Reed and Lave (1981), who first described the difference between oral and written arithmetic in greater detail, made this point quite clearly by characterizing oral arithmetic, typical of unschooled Liberian taylors, as a "manipulation of quantities procedure" and written arithmetic as a "manipulation of symbols procedure". Carraher and Schliemann (1988) further expanded this analysis by pointing out both differences and similarities between oral and written arithmetic. Oral arithmetic preserves the meaning of quantities in the sense that hundreds are treated as hundreds, tens as tens, and units as units, as can be seen in the two examples below, one involving subtraction and one involving division.

In the first example, the child was solving the problem 252 - 57 using oral arithmetic in a simulated shop. The child said: "Just

take the two hundred. Minus fifty, one hundred and fifty. Minus seven, one hundred and forty three. Plus the fifty you left aside, the fifty two, one hundred ninety three, one hundred ninety five" (from Carraher, PME, 1988, p.10).

In the second example, the child was solving a word problem which asked about the division of 75 marble among five boys. He said: "If you give ten marbles to each one, that's fifty. There are twenty-five left over. To distribute to five boys, twenty five, that's hard. (Experimenter: That's a hard one.) That's five more each. That's fifteen" (from Carraher, PME, 1988, p.7).

It can be easily recognized that the relative value of numbers is preserved throughout in oral computation--for example, the children said "two hundred minus fifty" and "give ten marbles to each boy". In contrast, in the school algorithm for computation children would be taught to say "*seven* divided by five, *one*" (when dividing *seventy* by five and obtaining *ten*) and "two minus one" (when subtracting *one hundred* from *two hundred*), ignoring the relative value of the digits.

Carraher and Schliemann (1988), however, pointed out not only differences but also similarities between oral and written arithmetic, arguing for the coordination of social and logical structuring in any context in which mathematical activities are carried out. The similarities were related to the logico-mathematical principles that constitute the basis for the arithmetic calculations. Detailed analysis of the processes of calculation showed that the properties of the operations used in oral and written computations are the same: associativity for addition and subtraction and distributivity for multiplication and division. Neither users of oral nor users of written arithmetic name these properties of the operations. However, when they understand the procedures they use, they explain the steps in calculation drawing on the properties of the arithmetic operations.

To sum up the contrasts presented hitherto between mathematics in school and out of school, we saw that mathematics outside school is a tool to solve problems and understand situations while school mathematics involves learning the results of other people's mathematics. As a consequence of this difference, mathematics outside school tends to be more like modelling, in which both the logic of the situation and the mathematics are considered simultaneously by the problem solver. In contrast, school mathematics typically focuses on mathematics per se, resorting to applications not as a basis for the development of understanding but as occasions for practising specific procedures. Despite the differences, arithmetic in and out of school relies on the same properties of operations. The social organization of the mathematical activities varies in and out of school while the logico-mathematical principles which come into play remain invariant.

Logic and convention in geometrical activities.

Arithmetic is a type of mathematical activity related to the quantification of variables and operations involving quantities. Geometry can be seen as the mathematization of space--or the "grasping of space" through mathematics (Freudenthal, 1973). It involves a whole range of activities which can be carried out in and out of school like determining positions, identifying similarities (in shape, for example), analyzing perspectives, producing displacements, quantifying space, and deducing new information through operations. Some of these activities are involved in cultural practices outside school. These practices may have to do with everyday needs of whole populations or they may be specific of subgroups carrying out particular activities. Determining position is a simple example of a geometry-type activity in everyday life which may be shared by whole populations. In order to walk about in a village or a city, everyone needs some ability to determine one's position in relation to the place one wants to go to - especially if the place to be found was never visited before or if one is using a map. Orientation in space appears to be accomplished by a mixture of logical and socio-culturally determined ways of representing space.

The logic of determining positions involves relating objects to each other according to an imaginary system of axes which can be applied in any situation. Piaget and his collaborators (Piaget and Inhelder, 1956; Piaget, Inhelder & Szeminska, 1960) have made a significant

contribution to our understanding of this logic of space by tracing the development of the understanding of vertical and horizontal in Swiss children. They were able to show that children's understanding of the vertical and horizontal axes does not stem directly from the empirical experience of standing upright and lying horizontally. Piaget's work suggests that this under-standing is constructed as children relate objects to one another in space.

Piaget and his colleagues did not concern themselves with variations in the use of these systems of coordinates either within a culture across different situations or across cultures. More recently other authors have looked at these variations and pointed out interesting cultural inputs into this logical system. For example, Harris (1989) notices that in most Western cultures we use the framework of "left" and "right", "in front of" and "behind", "above" and "below" when describing positions. "Left" and "right" are descriptions derived from an imaginary axis related to the two sides of the body as "in front" and "behind" are related to the front and the back of the body; "above" and "below" are related to high and low, as head and feet relate to each other. When this organization is represented on paper, conventions are that the left-right axis on paper corresponds to a horizontal line while the perpendicular axis from the nearest to the furthest point on the page corresponds to a vertical line.

However, as Harris further points out, other cultures may not readily use the body-related system of axes (like left-right) either for everyday practices or when space is represented on paper. Several Aboriginal groups in Australia seem to prefer to use compass points even over very small distances. "For example, when one Aboriginal woman was looking for a particular picture on a wall covered with photos, her companion directed her to look to the *west* side of the wall" (Harris, 1989, p.12) - a situation in which Westerners would have used "left/right" instead. Similarly, when teachers were giving instructions on how to do letters in a handwriting lesson, they would use the compass points; "to write the letter 'd' the instructions (spoken in Walpiri) might be: 'Start here on the north side, go across south like this and down across to the north and up, and then make a stick like this" (Harris, 1989, p.12). Bishop (1983) also reports the description of a system of sloping axes by a student from Manu, an island in Papua New-Guinea, as "a line drawn horizontal in the northwest direction". The description of a sloping line as "horizontal" is incongruent with Western conventions and could be interpreted as incorrect. However, if we consider it together with the description of the slope as "in the northwest direction", this student's description of the diagram reveals the use of compass points in the representation of relationships in a plane.

Piaget and his collaborators (1960) stressed the importance of this system of axes not only for locating a point in space but also as part of devising a means for measuring angles and defining figures in a plane. They described children's success at copying angles as a function of their understanding that the slope of a line could be described by two points simultaneously related to a horizontal and a vertical axis. Similarly, Carraher and Meira (1989) observed how youngsters learning to use LOGO employed a system of coordinates when trying to determine the angle to be turned by the turtle in order to copy a target figure. The imposition of this imaginary set of axes on the manipulations in space with LOGO is remarkable because no such a system had occurred to the youngsters' instructors at the outset. The LOGO learners seemed to be bringing to this new situation a way of organizing space which they had developed independently of that particular experience. The use of such static reference systems is even more interesting when one reflects upon the fact that this is not the only way that the youngsters could have approached the metric of angles in a dynamic situation. As Magina (1991) suggests, such a metric is available in our culture in the reading of traditional clocks but not a single subject in the Carraher and Meira study used the clock analogy when quantifying angles in LOGO.

A different type of geometric activity which was also analyzed by Piaget and his co-workers deals with the concept of the straight line as "the line of sight", a concept which is involved in projective geometry. As Freudenthal (1973) points out, this is a complex definition of a straight line and may be preceded by other ways of understanding the straight line. It may also be developed more clearly in connection with certain activities of specific groups of people who

have to take into account perspective in drawing or who have to carry out navigation.

While interviewing fishermen, I obtained the following explanation for how they found their way in the narrow channel which they took through the barrier of reefs off the coast where they fished:

> "You just have to look at the church and the tall coconut tree (a particularly tall tree which stands out in the region). When the church runs in front of the tree, that's how you get into the channel".

When I asked what he meant by "the church running in front of the tree", the fisherman explained:

> "The things that are closer to the ocean move faster as you move and look at them. They don't move, you move, but what is behind them changes and makes it look like they move. That's what we mean when we say that the church runs in front of the coconut tree".

Despite the anecdotal nature of this example, it helps us understand why it is that people concerned with navigation turn out to do so well in projective geometry tasks: they rely on coordinations of lines of sight in order to determine their course. In contrast, when we orient our routes on land we rely more on roads and distances than on the line of sight.

Bishop's (1983) results comparing three groups of students in Papua New-Guinea seem to support the idea that projective geometry is related to specialized activities. He showed the students a series of photographs of small abstract objects and asked them to identify the place from which the camera took the photograph. This is reportedly a difficult task, which was performed much better by the students who came from an island (who averaged 51 correct responses) than by the students who came from the highland or the urban environments (who averaged 34 and 26 correct responses, respectively). The specificity of the activity is even more clearly understood when one looks at the results of some of the other tasks that were presented to the students. These results do not reveal a general superiority of the islanders in all spatial tasks. For example, the islanders showed a more restricted spatial vocabulary than the other two groups of students and drew maps picturing the route from their own room to the university office which were less accurate than the maps drawn by urban students.

In conclusion, similarly to what was observed in the field of arithmetic, everyday practices involving the mathematization of space reflect both socio-cultural experiences and invariant logical relations. Locating objects on a plane may vary both within a culture and across cultures in the use of body-anchored or compass-related systems of axes. However, it is possible to identify an invariant logic on which these variations are based, which is constituted by the very notion that an imaginary system of axes can be imposed on the to-be-represented space. Everyday practices of particular groups can also give rise to specific ways of understanding space which may not be as important to other people not involved in such practices. The conception of a straight line as "the line of sight" seems to emerge more readily among people involved in navigation than among people who do not concern themselves with navigation. This concept of the straight line does not appear to be a routine that people memorize but a real activity because the same ability can be used in rather difficult and different tasks such as finding the position from which a picture was taken.

Bringing street mathematics into the classroom.

In the previous sections we have discussed examples of mathematics learned outside school without discussing how to build connections between everyday mathematics and teaching. In this section we will explore how everyday practices can be brought into the classroom and what effects this may have on school learning.

The idea of bringing out-of-school mathematics into the classroom is not new. The following arguments, amazingly current both in their nature and in the goals proposed for arithmetic teaching, are taken from a manual prepared for teachers by Brideoake and Groves as early as 1939:

> We felt that in the past, many children who failed to achieve success in "sum lessons" showed considerable grasp of the subject when shopping in the High Street or taking the milk. They were not devoid of number sense, but the school approach seemed to be faulty (p.5).

> In the earlier period it was "sums" which mattered. *Class teaching* was the method , with its visualising and memorising, and the ideal was that the children should learn the four rules and be able to get the daily "six sums" right. Then a reaction in favour of *individual work* set in. This was led by the great educationist, Madame Montessori, and resulted in a flood of "number apparatus", extensive exercises in counting, and pretentious written sums; but still there was, very often, little mental alertness or real number power. Now many teachers are feeling their way to a *realistic approach*, with its underlying idea that development of number sense is what matters, and that "sums" are only the expression of this (Brideoake and Groves, 1939, pp.9-10).

To want to use a realistic approach or bring children's number sense into the classroom is one thing; to be successful at it is quite another. In this context, an informal observation by an educator, Herndon (1971), is very informative. He describes how he met in a bowling alley a student who had difficulty in school arithmetic but who could keep track of eight bowling scores at once in the alley. He then realized that all of his students had some use of mathematics outside school in which they were successful and tried to make them solve exercises in school which were similar to what they could do outside school. This is how he describes his attempt:

> Back in eighth period I lectured him [the boy from the bowling alley] on how smart he was to be a league scorer in bowling. I pried admissions from the other boys, about how they had paper routes and made change. I made the girls confess that when they went to buy stuff they didn't have any difficulty deciding if those shoes cost $10.95 or whether it meant $109.50 or whether it meant $1.09 or how much change they'd get back from a twenty. Naturally I then handed out bowling-score problems, and naturally everyone could choose which ones they wanted to solve, and naturally the result was that all the dumb kids immediately rushed me yelling. "Is this right? I don't know how to do it! What's the answer? This ain't right, is it?" and "What's my grade?". The girls who bought shoes for £10.95 with a $20 bill came up with $400.15 for change and wanted to know if that was right? (Herndon, pp.94-95).

Bringing out-of-school mathematics into the classroom is not simply a matter of finding an everyday problem and presenting it as a word problem for the application of a formula or an algorithm already taught. This approach does not change the main difference between mathematics in and out of school pointed out earlier because students would still be in the same position of trying to learn the products of other people's mathematical activities. Bringing out-of-school mathematics into the classroom means giving students problems which they can mathematize in their own ways and, in so doing, come up with results (methods, generalizations, rules etc.) which approach those already discovered by others.

Different PME authors have explored routes to mathematics teaching related to the idea that out-of-school mathematics can be brought into the classroom with positive results. Some of these studies will be reviewed here and can be used as starting points for further explorations. I would like to distinguish two types of teaching approaches in which out-of-school mathematics is brought into school.

The first approach starts from a particular aspect of mathematics which one wants the students to learn--notation systems, methods, theorems etc.--and then searches for everyday problems which instantiate that aspect of mathematics. In this case, the teacher will create constraints in order to ensure that the specific aspect of mathematics comes up in the analysis of the everyday situation and will not consider teaching successful unless the specific goal is achieved.

The second approach involves bringing sensible problems from everyday life into the classroom without a pre-established idea about which particular method of mathematizing the situation is to be the end-result of the lesson. Although the teacher chooses the problem because there are interesting mathematical relations which can be explored in the situation, there may be no ready-made solution process which the students are expected to learn by analyzing the problem situation and many ways of solving the problem will be considered legitimate. Teaching will be considered "successful" if students analyze the situations, use mathematical concepts in

their analyses, build relationships between different concepts and improve on their own methods as they repeatedly employ them on solving further related problems. Teachers may introduce mathematical notations as the pupils devise their solutions but the specific teaching of these notations is not the aim of the activities from the outset.

In short, the first approach is task oriented. It aims at teaching concepts, notations, methods etc. and can be evaluated on the basis of whether or not pupils accomplished the specific target. The second approach is global and open-ended. It aims at developing "mathematical minds". Its evaluation must take into account pupils' progress over time and determine whether or not they become more sophisticated in their mathematical analyses of situations as they progress in school.

This section is divided into three parts. In the first two, I will present work and ideas from both of these approaches. I leave to the reader the task of evaluating them and deciding whether they can be combined into a unified philosophy for the teaching of mathematics or not. In the third part, some of the teachers' reactions to bringing out-of-school mathematics into the classroom are presented.

Teaching specific aspects of mathematics with the support of everyday life experiences.

We have seen that people learn much about mathematical concepts outside school. However, it does not follow that school mathematics can profit from children's knowledge of out-of-school mathematics. It is quite possible to recognize that mathematical ideas are often learned outside school but still be sceptical about the importance of bringing out-of-school mathematics into the classroom. The need for studies which show whether bringing everyday concepts into the classroom can actually contribute to the development of school mathematics was clearly argued by Janvier (PME, 1985). His arguments are summarized in the following points:

"(1) certain mathematical ideas have natural phenomenological counterpart since they result by a process of abstraction from objects and observable entities" (p.135);

(2) these ideas formed from everyday experience represent models which give meaning to the mathematical ideas;

(3) two positions can be formed about the role that these models may have in mathematical education: (a) models are restricted in their generalizability and *should* therefore be forgotten as soon as correct and efficient handling of abstract symbols is achieved; and (b) a model is irreversibly a model for life and can be brought into play at any moment when there is a need for the mathematical relationships and rules to be rediscovered.

The question posed by Janvier is thus essentially whether everyday mathematics is so bound up with the context in which it was learned that it cannot be "pulled up" from the context and transformed into a more general model. Models for understanding money, for example, could remain defined as such and the relationships between numbers which were learned in money contexts would not be recognized in other contexts. A different but related concern is expressed by Booth (PME, 1987), who raises the possibility that students may not link up the specific representations used in teaching with the more general mathematical ones, and will thus not profit from the experiences with the particular situations. Below we will review a sample of studies in which children learned mathematics in specific contexts and were tested for their learning in more general ways. They do not represent an exhaustive review of the literature; they are described here as samples of attempts to teach particular aspects of mathematics on the basis of situations which pupils may already understand from their everyday experience.

Higino (1987) analyzed the use of children's knowledge of money as a support for learning place value notation and the written algorithms for addition and subtraction. She worked with a task previously used by Carraher (PME, 1985; Carraher and Schliemann, 1990) in which children play a shopping game and are asked to pay different amounts of money for the items they purchase. The children are given *nine tokens* of two different colours which represent coins of different denominations, one and ten. They are then asked to pay for items they purchase in the play shop using these tokens. The amounts of money can be obtained (a) only with singles

(for example, six); (b) only with tens (for example, 30); and (c) mixing singles and tens (for example, 17, a sum which can only be paid if the child uses one 10 and 7 singles). The 60 children in Higino's study were 7 to 9 years of age and were attending second grade in state schools in Recife, Brazil. They were matched at the outset of the study with respect to their knowledge of place value notation and then randomly assigned to one of four groups:

(1) an *unseen control group* that received only the regular classroom instruction on place value notation and the addition/subtraction algorithms;

(2) a *symbolic teaching group*, that received experimental instruction on place value and the algorithms without recourse to any materials other than explanations about the decades and the rules of the addition/subtraction algorithms;

(3) a *sequential everyday-symbolic group*, that first practised counting money and working out change in the pretend shop and then received instruction on place value and on the addition/subtraction algorithm;

(4) a *parallel everyday-symbolic group*, that practised counting money and giving change and received parallel instruction on place value notation and the addition/subtraction algorithms.

Results showed that all four groups progressed with instruction according to an immediate post-test. However, in a delayed post-test given approximately one month later, children from the symbolic teaching group no longer differed significantly from the unseen control group while the other two experimental groups still did significantly better. Moreover, children from the parallel everyday-symbolic group showed a small improvement in their mean number of correct responses to the addition/subtraction problems after one month of instruction while the other groups showed slight decrements. Thus Higino was able to show that connecting everyday experiences with classroom learning of addition and subtraction produces better results than teaching without regard for children's previous knowledge.

Another study which attempted to analyze the role of everyday situations in the development of basic concepts of addition and subtraction was carried out by De Corte and Verschaffel (PME, 1985). They argued that most instructional approaches use word problems as applications of addition and subtraction but it is questionable whether word problems representing real situations only have this role in elementary arithmetic.

> "Indeed, recent work on addition and subtraction word problems has produced strong evidence that young children who have not yet had instruction in formal arithmetic, can nevertheless already solve those problems successfully using a wide range of informal strategies that model closely the semantic structure and meaning of the distinct problem types." (p.305)

Working from this assumption, De Corte and Verschaffel compared a control group, instructed in the traditional way described above, to an experimental group, who learned addition and subtraction concepts mostly from solving word problems and learning several ways of representing the semantic relations in the problems. They observed that the experimental group made twice as much progress on word-problem solving than the control group after one year of instruction, a difference which was statistically sig-nificant. De Corte and Verschaffel concluded that this line of investigation is worth pursuing further although they recognize that the problems used in the experimental program were relatively poor in content and did not approach outside-school situations as much as desired. Similar positive results were reported also by van den Brink (PME, 1988) after a year-long experiment on children's learning of addition and subtrac-tion concepts either by modelling from everyday experiences or from teaching with-in a traditional approach.

Looking at somewhat more complex situations, Janvier (PME, 1985) analyzed the role of models in teaching negative numbers by contrasting two groups of students, one who had learned negative numbers from a symbolic model and a second one which had learned from what he calls a "mental image" model based on particular experiences. The symbolic-model group learned negative numbers by having the symbolism introduced only loosely related to an initial situation and then concentrating on rules for manipulating numbers. Different colours

were used for negative and positive numbers and rules were of the type: "when you add two numbers of the same colour you make an addition and keep the same colour; when you add two numbers of different colours, you subtract the small one from the big one and take the colour of the big one" (Janvier, 1985, p.137).

The mental-image group attempted to solve problems about a hot-air balloon to which bags of sand or helium balloons could be attached, the first having the effect of lowering the balloon while the latter brought it higher. Balloons and bags were initially of one unit but several of either could be used at once so that symbolism could be developed for larger numbers. Adding meant tying an extra-element and subtracting was identified with removing an element from the dirigible. Janvier found that the mental-image group did significantly better on addition items than the symbolic-model group; no significant difference was found for subtraction. Furthermore, when students were interviewed later, those from the mental-image group could correct the mistakes they had initially made by going back to the model; however, the symbolic-model students simply repeated within the model the same mistakes and were unable to revise their wrong answers.

Other studies which must be mentioned and which deal with more advanced mathematical concepts are in the realm of functions and algebra. Janvier (1980) and Nonnon (PME, 1987) have investigated the use of situations represented graphically in the teaching of functions. Both studies analyze students' difficulties with graphic representation and improvement after teaching. However, no systematic comparisons between the experimental groups and others taught without the support of the everyday situations represented graphically are available and clear conclusions are thus precluded. Filloy (PME, 1985) studied the use of addition/subtraction of packages of unknown weights onto two- plate balance scales as images for algebraic manipulations with unknowns. Although he reports some success, he seems to be cautious about the results of this experience in a later paper. Filloy and Rojano (PME, 1987) argued that subjects could achieve a good handling of the concrete model but developed a tendency to stay and progress within this context only, showing difficulty in abstracting the operations to a syntactic level and in breaking with the semantics of the concrete model.

To summarize, it seems possible to use everyday situations in the classroom as instances of particular aspects of mathematics and obtain better learning of this mathematics in context than in a traditional teaching schema relying primarily on symbolic manipulations. Further research is undoubtedly still necessary since only a very narrow range of mathematics has been systematically investigated.

Real-life mathematics used in an open-ended fashion.

Streefland (PME, 1987) presented in very clear terms the instructional principles which guide the open-ended type of realistic mathematics education which we are distinguishing from the modelling of specific mathematical aspects discussed above. These instructional principles are:

a) context problems (or situations) occupy a dominant role in mathematics education, serving both as a source and as a field of application of mathematical concepts;

b) a great amount of attention must be paid to the development of situation models, schemas and symbolising;

c) children can make significant contributions to classroom work through their own productions and constructions in the process of moving from informal to formal methods;

d) this learning process is of an interactive nature;

e) different concepts become more clearly interrelated in this progressive mathematization of situations.

Streefland (PME, 1987; PME, 1988) has produced several examples of how children can learn mathematical concepts in the progressive mathematization of real situations. In one PME research report, Streefland (PME, 1987) describes the process of mathematization leading from division in the sense of equal sharing to fractions. The children start from a situation in which they are asked to work out how to share three candy bars among 4 children. Examples of how the distribution is performed and what

children learn from this distribution in this first phase of teaching involve:

> dividing each bar of chocolate into 4 equal parts and taking one part for each child, a procedure which leads children to describe each child's share as 3 x 1/4 or 3/4;
>
> realizing that each child could have a half chocolate, using up two whole chocolates, and the last chocolate would then be cut into for equal parts, each child's share being described as 1/2 + 1/4 (this description helps children understand that 1/2 and 2/4 are the same amount).

In the second phase of teaching, "concrete stories of fair sharing gradually melt into the background (...). The activities change into composition and decomposition of real fractions" (Streefland, PME, 1987, p.407) using methods of decomposing fractions into units and finding equivalences, but it is still best "not to wipe out all the traces of the real situations referred to even at this stage" (p.407). For example, 5/6 can be decomposed into 1/6 + 1/6 + 1/6 + 1/6 + 1/6 and then other equivalences can be found like 3/6 + 2/6 and then 1/2 + 1/3 and so on.

In the third level of teaching, "the rules for the composition and decomposition of fractions will become the objects of mathematical thought (...) the methods of production which turn out to be the most efficient might become standard procedures or algorithms" (p.407). "For example, children work with questions like: 1/2 + 1/3 are probably pseudonyms for other fractions with a common name; produce these other fractions".

Streefland reports that children working with this approach to the teaching of fractions show increasing skills in producing equivalent fractions, in operations with fractions with the exception of division, and a general dominance of the elementary laws for operation as methods of production.

Streefland (PME, 1988) presented further examples of his instructional approach including topics like subtraction, division and further examples about fractions. He also strengthened his theoretical principles on teaching, by arguing that:

- when students work with their own constructions for mathematizing reality, they have the opportunity of strongly interconnecting several topics which are related but taught separately in other approaches to mathematics education because "genuine reality can be organized mathematically in various ways" (Streefland, PME, 1988, p.89);
- in this learning process, children acquire a variety of aids and tools in the progressive mathematization of situations becoming competent in the use of different terminologies, symbols, notations, schemas, and models;
- organizing and structuring the mathematics should be as much as possible the business of the children themselves, a principle which stresses the importance of interaction and cooperation in this way of approaching mathematics education.

Other significant aspects of this realistic approach to the teaching of mathematics are emphasized by De Lange (1987; in press), who discusses the importance of using real mathematics also in the assessment of children's success in learning mathematics, and by Boero (PME, 1989), who points out that there are misconceptions of real situations which may creep up in mathematics classes when external contexts are used for conceptualizing. According to Boero, these misconceptions do not necessarily hinder the process of conceptualizing mathematics; they are rather problems to be faced by students "if one wants the pupils to gradually understand that there are levels of 'intuitive evidence' and 'intuitive' ways of thinking which must be exceeded if a rational working command of certain phenomena is to be reached, and that mathematics may have an important role in this passage from intuition to rationalization" (Boero, PME, 1989, p.69).

The evidence produced by Streefland on pupils' progress in the process of mathematizing situations is provocative. Mathematics appears as a way of representing and mentally manipulating situations, bridging the gap between a problem and a solution and allowing children to go beyond the information they started the task with. Streefland's descriptions of how children progress from close modelling the situations to developing abbreviated and general procedures as they interact with each other and compare the efficiency of the different procedures suggest the importance of exploring this approach further with a

new range of topics. It is especially interesting to note how children seem to understand quite clearly the mathematical equivalence of their different procedures on the basis of their modelling properties but still go on to evaluate their efficiency. They do not stop working when they "find the answer".

To summarize, the open-ended realistic approach to mathematics teaching involves the search for everyday situations which can be treated as sensible problems in mathematics classes. The choice of situations is not geared to the teaching of particular mathematics topics. Pupils success is not evaluated on the basis of whether a specific bit of mathematical knowledge was accomplished. Instead, pupils success is evaluated on the basis of the progressive sophistication of their methods and their building of relationships between concepts, symbols, notations and terminologies. This is essentially the basis of Streefland's (PME, 1988) contrast between the teaching approach developed by "realistic instruction" to the "structuralist approach" proposed by Dienes. While the realistic approach starts from pupils good sense in a problem situation and progresses to more formal procedures, in the structuralist approach "vertical mathematizing is overstressed and formal procedures are imposed" (p.89). In other words, concrete materials devised for instruction within the structuralist approach do not aim at bringing out-of-school reality into the classroom. Instead, mathematical objects are devised for the purpose of concretely exemplifying mathematical procedures and representations already formalized. The mathematical objects thus devised may have no meaning outside the classroom. This means that pupils may not have models which they can readily call upon to understand their experiences in the classroom.

This open-ended realistic approach is not restricted to arithmetic but has also been tried out with geometry. Goddijn and Kindt (PME, 1985) strongly criticized traditional geometry teaching as stereotyped and restricted to "the flat world of textbook". They propose to give geometry teaching a new approach by working from the three-dimensional world. Examples of questions in their programme include working with proportion and scale comprehension as students look at pictures and take into account distances and viewing points. Students attempt to find the position from which a picture was taken and are encouraged to use explanatory drawings in this process. In so doing, they "discover" the line of sight and the changes in perspective and size of figures as the "viewer" moves about. They look at shadows and reflect on related phenomena such as the phases of the moon. They watch a student enact the movements of the moon around the earth in front of a bright light and ask each other how is moon seen, lit-up or in shadow, from the perspective of pupils is different parts of the room. They use the "line of sight" from the point where they stand, imagine it from the point where someone stands, and try to figure out where someone must stand in order to have a particular viewpoint.

Goddijn and Kindt describe how students (and teachers alike!) become surprised at how well their methods using lines of sight (which may even be concretely represented by strings) work in drawing and solving these problems. But the results of this exploration of space are not restricted to perspective drawing; students found, for example, a totally new manner of doing the classic assignment of constructing the line through a given point which intersects two skew lines. Thus, by teaching geometry as "grasping space through mathematics", Goddijn and Kindt were able to observe a progressive sophistication of students' methods for dealing with spacial and geometrical problems.

Teachers' reactions to bringing out-of-school mathematics into the classroom.

Many mathematics teachers seem to resist at least initially, the suggestion of bringing informal knowledge of mathematics into the classroom. Several reasons are pointed out in the literature. First, Schultz (1989) observes that teachers seem to expect that they will carry out the same pedagogy they were themselves the recipients of. They learn from modelling their teachers as much as (if not more than) they learn from theories in teacher education programmes. Second, they are usually concerned with covering the standard curriculum material, which is determined in terms of traditional mathematics topics, and not in terms of mathematization of situations. To this concern is added the resistance to the use of intuitive methods in the classroom because

these methods conflict with teachers' views of mathematics as a formal system of knowledge (see Ernest, 1989) and as "something superior" (Rosenberg, 1989).

Strategies which have been used to deal with such resistance include both teachers' observations of children's spontaneous mathematical activity outside the classroom and teachers' experience of the same teaching approach when they themselves mathematize experiences of a more complex nature. Rosenberg (1989) reports that "Aha!-experiences happened very often when they (the teachers) started from informal problems and finally got to familiar formal structures" (p.83).

A last source of resistance comes from teachers' beliefs about what students will/will not do when faced with a new educational experience. In this context, it is interesting to report briefly on an informal observation with a group of Brazilian teachers, who found it hard to believe that pupils would experiment with and improve upon procedures and mathematical models once they had already found an answer to the problem. In traditional mathematics teaching, finding an answer usually determines the end of the activity. For this reason, teachers expected that students will stop working as soon as an answer is found. They were also convinced that some answers are correct and others are wrong and that shifting mathematical models within a problem simply means "finding the right answer versus finding the wrong answer". A little later in the day I asked the teachers to develop a simple way of determining the rate of inflation, a problem of everyday significance in Brazil. They quickly came up with the idea that inflation could be viewed as the average of all price increases. They drew up a list of items, wrote down their estimations of increases in prices in percentages and calculated the average increase. They were not satisfied with their solution to this problem and did not stop working. They realized that housing and grocery shopping might affect people's everyday expenses more markedly than entertainment, for example. One teacher suggested that they could give weights to the items which would be proportional to the impact the item had on a worker's monthly spending. They worked out that the wage could be represented as 100% and they could then estimate what percentage of it was spent per month on each item. They calculated a weighted average of the increases as their new solution. At this point, the teachers seemed to have an "Aha! experience". They realized that, in situations such as the one they had just worked on, students might well look at the same problem in different ways by trying out different mathematical models on the same problem.

Finally, it is interesting to add Schultz's (1989) comments on the impact that the use of real-life mathematics in the classroom had on the student-teachers she worked with. Although no systematic data are presented, she observed that they became more aware of what they had learned and the way they learned and seemed to develop new methods and ideas for teaching their own students. If modelling from real-life situations turns out to have a positive impact on future mathematics teachers, this may be just as important as the effects it has directly on pupils. However, no research data seem yet available on this issue.

Final comments

This paper reviews research which shows that the social context in which mathematics is carried out influences the way people approach mathematical problems. Mathematics in many of today's classrooms is taught as an abstract form, a set of symbols, procedures, and definitions to be learned for perhaps some later application. Mathematics used outside school is a type of modelling: it is a way of representing reality so that further knowledge about the reality can be obtained from the manipulation of the representations without the need to check these new results against reality (D'Ambrosio, 1986). These differences in the social situations and their corresponding mathematics have an impact on rates of success and types of procedures used by the problem solvers. Research has shown that bringing out-of-school problems into school as a way of teaching mathematics is not only possible but also a beginning of a more successful story about mathematics teaching. Further research is still needed and more clear theorizing so that successful isolated experiences can be transformed into an effective educational theory.

Bringing out-of-school mathematics into the classroom means facing questions which may not be addressed in the traditional

forms of teaching. Some of the questions which emerge within this approach to mathematics teaching are presented below.

We know that many concepts can be learned outside school and that this knowledge can be useful in school. But we cannot be casual about which concepts and situations we choose to bring into school. An implicit route in this choice appears so far to be to look at present mathematics curricula and choose problem situations appropriate for present day curricula. However, is it possible to identify some "core" concepts which will guide this choice in a new way?

A second question has to do with the introduction of mathematical representations. Pupils solving real problems do not need to reinvent mathematical representation. When and how is mathematical representation to be introduced?

A third question which remains to be investigated is which factors affect the passage from informal to more formal mathematical representation in the process of mathematizing real problem situations. Much of the success of the realistic approach depends upon interaction between pupils and we have achieved relatively little understanding of what in these interactions brings about children's progress. Is it conflict between different perspectives or is it the convergence of different ways of representing the same problem? This is not a trivial question but one of great theoretical significance, as Bryant (1982) has pointed out.

Finally, it would be desirable to accomplish better descriptions of the process of progressive mathematization which takes place when students interact and solve real mathmaticss problems together. Is there a single route in the development of each particular concept or are there many ways to get to the same place? Does natural language, which is of course used in the students' interactions, play a part in this process or not? Is the progressive mathematization of situations a process of reconstruction of solutions at ever- higher levels of abstraction or does it involve dropping old models which are then replaced by new ones? Is there a moment when real problems are no longer important or do they retain their role of giving initial meaning to formalizations even at the highest levels of abstraction?

Hopefully, further research and fruitful cooperation between researchers and teachers will bring clearer answers to many of these questions in the future.

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The Influences of Teaching Materials on the Learning of Mathematics

Kathleen Hart

This is an excerpt from a conversation between a teenage girl and a mathematics educator: (I: Interviewer; A: Alison)

I: You need four cubes for eight people so for four people...?

A: Yes, but then if it was soup cubes, you'd want the same amount of taste wouldn't you so you would have to put the same amount in the soup to get the same taste to it, if you put less it wouldn't taste as nice.

I: Really, even if you were only making half as much soup?

A: Yes, 'cos you need the same amount don't you, else it would taste horrible, it wouldn't taste as strong.

I: But let's say you make enough soup for 8 people and you put 4 cubes in, now the next day you are making soup for 4 people. How many cubes would you put in then?

A: I would put in two but it would not taste as nice would it?

I: Don't you think so?

A: No.

I: You don't think it would taste the same?

A: Because if you are going to make soup you are going to have to use the cubes anyway to make it taste aren't you?

I: Yes

A: So if you want it to taste just as good as it did when you made it for 8 you would have to put the same number of cubes in it. Otherwise it won't taste the same.

I: I think it would. I think if you are going to make half as much soup, you want half as many cubes.

A: I suppose you could, but I wouldn't if I was making the soup.

Frank is only ten years old and he and his class have been 'taught' about the volume of a cuboid.

(I: Interviewer; F: Frank) $l \times b \times h = V$

F: Well, a, b, c, d, e, f, g, h, i, j, k, l - 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12. That'll be 12 times 2 and ...

I: Can you tell me how you know that?

F: Well, l is ... um... the tenth letter of the alphabet.

I: Yes.

F: So, 10.

I: Ah... who told you that?

F: ... (long pause)...

I: Did you just make it up?

F: Think so, yes.

I: Okay, and 'b' will be? And 'h'?

F: H... um... eh 4... a, b, c, d, e, f, g, h, ... 1, 2, 3, 4, 5, 6, 7, 8, 9, 10 ... two's is twenty... eight, twenty eights will be one hundred and ... one hundred and ... eight ... oh, two hundred and ... oh, one hundred and forty.

I: Alright, and then it says equals V.

F: Equals V? 140.

I: How do you know that that's not right, shaking your head?

F: V isn't the 120th letter of the alphabet (laughs)...

Frank's common sense came to the rescue!

These children have been in school mathematics classes for six to ten years but there are obvious misconceptions held by them. This chapter is about research concerning materials which are available to the teacher to assist in the process of teaching. Materials can include books, computer software or languages, concrete materials (manipulatives) or other aids. The emphasis is on research which led to the production of the material or which sought to produce evidence of its effectiveness. The main source of the research references is the work of members

of the international group Psychology of Mathematics Education.

Most children learn their mathematics in school although they use it and obtain their first ideas of the topic in the world outside the classroom. Within the classroom they may be grouped with others of the same age, other pupils of the same attainment level or simply all the other children who live in that village or street. The teacher who provides the experiences from which they are supposed to learn has a difficult task no matter which way the children who share the classroom are selected. The researches of PME take as a basic premise that the teacher is striving for the children to understand what they are doing. PME reports which describe teaching experiments in which a form of written material, a set of concrete manipulatives or a computer program have been tried so that learning is more effective are not numerous. Many are based on the desire to provide empirical evidence of the validity of a learning or psychological theory and in only a few cases are the results overwhelmingly supportive. However, books are written and materials produced and the research results must shed some light on how this can be better achieved so that learning takes place. Nobody has yet provided a solution for the question 'how can I best teach this child'. Throughout the world, with very few exceptions the performance of children on mathematics tasks is considered unsatisfactory by educators in their country. Is this because educators set unreasonable tasks in the discipline of mathematics or because the expectations are entirely reasonable but those who teach the subject singularly inefficient?

Learning theories and their influences

Most research is carried out within a theoretical framework. The choice of framework influences the assumptions, the premises suggested and tested and the outcomes. Sometimes materials are invented to conform to the theory and the research is perhaps to test their effectiveness or indeed provide some evidence of behaviour which supports the theory itself. Often existing books or teaching aids are criticised because their development contradicts a learning theory. At the moment most mathe-matics books for young children which are based on language and words only would be condemned as "too abstract", developmental psychology having pro-vided theories which state that a young child is incapable of abstract reasoning.

Piaget's developmental theory outlined stages of mental development loosely tied to age. The theory divided intellectual development into four major periods: sensorimotor (birth to 2 years); pre-operational (2 years to 7 years); concrete operational (7 years to 11 years); and formal operational (11 years and above).

Piaget's interest was in the psychology of the child but many educators have extended and interpreted his writings in order to apply the theory to teaching as is shown here by Ginsburg and Opper (1969) in their book *Piaget's Theory of Intellectual Development:*

> "While Piaget has not been mainly concerned with schools, one can derive from his theory a number of general principles which may guide educational procedures. The first of these is that the child's language and thought are different from the adult's. The teacher must be cognizant of this and must therefore attempt to observe children very closely in an attempt to discover their unique perspectives. Second, children need to manipulate things in order to learn. Formal verbal instruction is generally ineffective, especially for young children. The child must physically act on his environment. Such activity con-stitutes a major portion of genuine knowledge; the mere passive recep-tion of facts or concepts is only a minor part of real understanding."

The theory of Piaget has for many years influenced those who teach mathematics, mainly through the advice given to teachers in training. The definition of the concrete operational level and the features which distinguish it from the formal operational level have led often to a belief that children below the age of eleven years should be given manipulatives to assist them in learning mathematics or indeed in some cases to replace the abstraction of mathematics with only those experiences which could be regarded as 'real'.

This has further become the suggestion that the use of concrete materials is good in

itself irrespective of the topic being taught. Researchers have started to discuss the gulf between manipulatives and formal school mathematics and PME members have considered this with respect to the learning of Algebra.

Ausubel (1963) rather than having an age-dependent theory provides a theory of learning which emphasises the dependence of new knowledge on old knowledge and is perhaps better suited to the needs of mathematics teaching which traditionally builds on previous teaching. As Novak (1980) says in quoting him:

> "However, we must also consider the cognitive functioning and psychological set of the learner as new knowledge is internalised. Here we must distinguish between rote learning wherein new knowledge is arbitrarily incorporated into cog-nitive structure in contrast to meaningful learning wherein new knowledge is assimilated into specifically relevant existing concepts or propositions in cognitive structure. Since the nature and degree of differentiation of relevant concepts and propositions varies greatly from learner to learner, it follows that the extent of meaningful learning also varies along a continuum from almost pure rote to highly meaningful".

In Europe successful learners of mathematics in schools and universities have always been few in number, successful linguists or engineers are more numerous. This state of affairs did not improve proportionally with the introduction of universal secondary education. This has often meant that those who succeed have had the reputation of possessing a superior (or certainly different) type of intelligence, their success being judged not attributable to the skill of the teacher or the suitability of materials but to the make-up of their brains/intelligence. Psychologists and educationalists can only measure performance on tasks and therefore a theory which discusses the potential for learning as does that of 'information processing' is of value.

Those who have espoused this theory tended to produce tasks which they said were useful for judging the amount of 'space' available in one's cognitive make up. More recently the emphasis in mathematics education research has moved to the constructivist point of view.

A large number of PME members espouse Constructivism, indeed the XIth Conference in Montreal had Constructivism as its central theme. Kilpatrick (PME, 1987) describes some fundamental points:

> "The constructivist view involves two principles:
>
> 1. Knowledge is actively constructed by the cognizing subject, not passively received from the environment.
>
> 2. Coming to know is an adaptive process that organizes one's experiential worlds; it does not discover an independent pre-existing world outside the mind of the knower."
>
> "The first principle is one to which most cognitive scientists outside the behaviourist tradition would readily give assent, and almost no mathematics educator alive and writing today claims to believe otherwise. The second principle is the stumbling block for many people."
>
> "Radical constructivism adopts a negative feedback, or blind, view toward the 'real world'. We never come to know a reality outside ourselves. Instead, all we can learn about are the world's constraints on us, the things not allowed by what we have experienced as reality, what does not work. Out of the rubble of our failed hypotheses, we continually erect ever more elaborate conceptual structures to organize the world of our experience."
>
> "Von Glasersfeld has identified five consequences for educational practice that follow from a radical constructivist position:
>
> (a) teaching (using procedures that aim at generating understanding) becomes sharply distinguished from training (using procedures that aim at repetitive behaviour);
>
> (b) processes inferred as inside the student's head become more interesting than overt behaviour;
>
> (c) linguistic communication becomes a process for guiding a student's learning, not a process for transferring knowledge;

(d) students' deviations from the teacher's expectations become means for understanding their efforts to understand; and

e) teaching interviews become attempts not only to infer cognitive structures but also to modify them. All five consequences fit the constructivist stance, but they appear to fit other philosophical positions as well."

At all levels of the constructivist belief the child is active in the learning process and not a passive receiver of knowledge. Radical constructivists would advise teachers to provide rich learning experiences for the child from which he can construct his own mathematics. This is a viewpoint different from that which says the teacher armed with materials is the source of the major part of mathematics information in the classroom.

Ample evidence has been provided, over many years, that children change the information they are given by adults and remember it in a different form. Children also invent their own methods for carrying out mathematical tasks. Street children in Brazil for example have adequate methods of calculation of profits on goods they sell. Many English children ignore the generalised, formal methods of solution they are taught in school, in favour of more specific, inven'ed methods (Booth, 1981). Whether this is because the child's unsullied intuition is at work or whether it is because of desperation arising from a lack of understanding of the teacher's presentation, is unknown. We know that many children start school knowing how to count, to sort and to put objects in order. Children are able to construct mathematics but nobody has yet shown that they can construct all that is needed to be a mathematically competent adult in the year 2000. Research which shows a teacher can be most effective with talk and a blackboard is not often carried out in mathematics education. Brophy (1986), writing in the Journal for Research in Mathematics Education, quoting from the general research on teaching said:

> "Mere engagement in activities will not facilitate learning, of course, if those activities are not appropriate to the students' needs. Thus, the teacher's desire to maximize content coverage by pacing students briskly through the curriculum must be tempered by the need to see that the students make continuous progress along the way, moving through small steps with high, or at least moderate, rates of success and minimal confusion or frustration. Process outcome research suggests that teachers who elicit greater achievement gains from their students address this dilemma effectively partly by
>
> selecting activities that are of appropriate difficulty levels for their students in the first place and partly by preparing those students thoroughly for the activities so that they can handle them without too much confusion or frustration.
>
> This research also indicates that the academic learning time that is most powerfully associated with achievement gains is not mere 'time on task', or even 'time on appropriate tasks', but time spent being actively taught or at least supervised by the teacher. Greater achievement gains are seen in classes that include frequent lessons (whole class or small group, depending on grade level and subject matter) in which the teacher presents information and develops concepts through lecture and demonstration, elaborates this information in the feedback given following response to recitation or discussion questions, prepares the students for follow-up assignments by giving instructions and working through practice examples, monitors progress on those assignments after releasing the students to work on them independently, and follows up with appropriate feedback and re-teaching when necessary. The teachers in such classes carry the content to their students personally rather than leaving it to the curriculum materials to do so, although they usually convey information in brief presentations followed by opportunities for recitation or application rather than through extended lecturing."

Generally both theory and research seem to point to the importance of the teacher and her interaction with the child. Materials cannot of themselves provide enough for the child to learn. Increasingly the suggestion is that teachers adopt the methods of research and learn from listening to and observing children.

Progression in mathematical ideas

Written materials, textbooks, worksheets, computer programs are produced with a particular progression in mind. The writer presupposes a logic which dictates what should precede what other piece of mathematics teaching. The structure of mathematics often dictates what must be considered a pre-requisite. For many years Bloom's taxonomy provided a theoretical underpinning for progression in what teachers should present to children. More recently the suggestions of van Hiele concerning the structure of Geometric progression in the school curriculum has become popular in research projects and has been extended to other parts of mathematics. It is quite general and the increasing complexity of demand is obvious. These definitions are from Shaunessy (1986):

> "Level O. (Visualization) The student reasons about basic geometric concepts, such as simple shapes, primarily by means of visual considerations of the concept as a whole without explicit regard to properties of its components.
>
> Level 1. (Analysis) The student reasons about geometric concepts by means of an informal analysis of component parts and attributes. Necessary properties of the concept are established.
>
> Level 2. (Abstraction) The student logically orders the properties of concepts, forms abstract definitions, and can distinguish between necessity and sufficiency of a set of properties.
>
> Level 3. (Deduction) The student reasons formally within the context of a mathematical system, complete with undefined terms, axioms, an underlying logical system, defined terms and theorems.
>
> Level 4. (Rigor) The student can compare systems based on different axioms and study various geometries in the absence of concrete models."

Biggs and Collis (1982) have formulated a taxonomy to provide tasks which become structurally more difficult and to assess the quality of children's answers. The advantage of this is that the child is not labelled and conveniently assigned to a level of performance as is the danger with neo-Piagetian assessment but the quality of response can be seen to vary depending on the task.

Gagne was one of the first theoreticians to analyse topics for the pre-requisites at each stage (especially in mathematics). Such an analysis is vital for any production of materials.

Curriculum developers have ample advice on progression but little empirical evidence of a teaching sequence which is more effective than another because of the order of presentation. The research scene in education is woefully lacking in longitudinal surveys where the same children are investigated as they grow older. We have details of the per-formance of cohorts of children of different ages but a statement about the performance of a particular age group cannot reflect the diversity within that group. Age is not a very good predictor of performance both because of the diversity in one age level but also because there is ample evidence of young children being capable of learning quite complicated mathematics in a novel setting. Here the questions that must be asked are:

(1) Although young children can learn this topic, why should they?

(2) They learn in a research setting, but are they going to learn with the ordinary teacher in the ordinary classroom?

Hierarchies in mathematics attainment

In the absence of a detailed longitudinal study following the same children for many years, the empirical evidence of progression we have is based on the performance of children of different ages on various mathematical topics. These data take on greater significance if the questions are written according to some postulated hier-archical demand. National and indeed international surveys may provide information on levels of difficulty but they are usually carried out in order to monitor whether a representative sample of the nation's children are performing any differently to the representative sample tested in a different year. The testing of children using the van Hiele levels in Geometry has given some

evidence of levels of difficulty in Geometry (often within the U.S. setting).

A progression of easy to hard items gives information on some aspects of children's understanding but is of more value if there is evidence that children who can successfully deal with items at a stated level of difficulty can also demonstrate success on those that are regarded as being at a lower level. Such a check can be carried out on the single occasion of testing by written questions or interviews. Statistical techniques to quantify scalability are available. In the case of methods of solution used by children to solve problems one can investigate their use at different times in the child's school career. Karplus and Karplus (1972) was able to show that strategies used by children to solve Ratio and Proportion problems (or in most cases fail to solve them) formed a progression. In his longitudinal study, as the sample grew older (and the pupils learned more mathematics) they moved from one strategy to another. Those methods of solution that were abandoned were considered to be at a lower level being replaced by more sophisticated ones.

For example he showed that children who reasoned that one amount was an enlargement of another simply because it was bigger were, two years later, reasoning in a different fashion. Some incorrect methods used, however, were very persistent and showed no change or as many children moved into using them as moved out and on to other strategies.

Concepts in Secondary Mathematics and Science

The aim of the CSMS research project (Hart, 1981) was to inform teachers about what secondary school children found difficult in their mathematics and what they found easy, with some indication of why. The sample was large (1000 children aged 12-16) and the intention was to provide teachers with a view of the mathematical performance of the vast majority of children over four years of their secondary schooling, rather than at 16 years of age, when they complete a national examination. The methodology was to collect data from interviews and also from written tests. The interviews provided information about how children attempted to solve problems and about what errors they made. They were used to inform and enhance the written test results. Interviewing children one-to-one has become an increasingly popular research method during the last ten years. The belief is that a child will answer more freely and truthfully if allowed to describe orally and in its own words what is being done. The CSMS project formulated a set of ten subject hierarchies in which groups of items were shown to be scalable, in that success at harder levels presupposed success at easier levels. A child was deemed to have succeeded at a particular level if it had correctly answered two-thirds or more of the items in that level and in all easier levels.

It was apparent from the results that there was more variation within an age group than between age groups. The 14-year-olds' performance on a given item was only about five to ten per cent better than that of the 13-year-olds. Nor did there appear to be any large jump in understanding at fifteen, although more of the older children solved the harder items successfully. A longitudinal survey of 600 children produced evidence that certain errors committed by pupils at age 12 years were likely to persist, and be still apparent at age 15. Two hundred children were tested on questions in Algebra, Ratio and Graphs at the end of their second, third and fourth years in secondary school.

The research is dependent on the items and is likely to be influenced by whatever mathematics the pupils had previously studied. All data are influenced by many variables - previous mathematical experiences, teachers, books, socio-economic class, the items used and the evaluation of the answers. The social factors were spread in the CSMS data in that with such a large sample chosen on the criterion of approximation to the normal distribution of non-verbal I.Q. scores, the influence of any one teacher or textbook was blurred. However, simply because children are more successful on certain mathematical problems than on others does not mean that this is necessarily the best order for the presentation of material. In the absence of research showing that there is a 'best order', a case can be made for the premise: if the pupils score badly on this topic which they have normally been taught, then it is

harder than this other on which they did well, therefore in the material being written it should come later. The alternative is to rely on an analysis of the mathematics involved or to base the order of presentation on intuition in which reflects teaching experience.

The CSMS research was carried out in English schools but the items have been used by mathematics education researchers from many other countries and appear to produce valuable information for them. Lin (1988) has replicated all the CSMS test papers in Taiwan with large samples. His results tend to support the existence of the levels and show no great discrepancy although Taiwanese children tend to be very successful or fail badly on the items. Lin's interviews show that the Taiwanese sample do not resort to child methods but are usually seeking in their memory for an algorithm they have been taught.

The performance pattern of a representative sample of Taiwanese children was very different however, perhaps reflecting the two cultures, as shown in Figure 1.

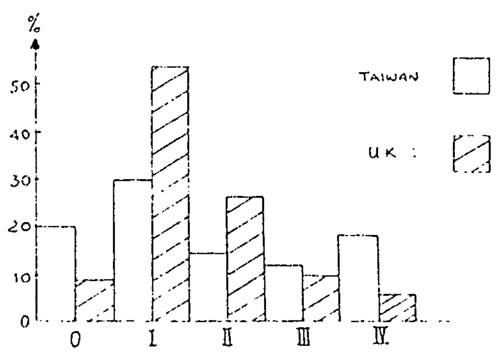


Fig. 1 Ratio Results UK/Taiwan

Progression in number complexity

Children's skills with number usually start with the ability to enumerate the contents of a set, whether it is simply by repeating number words whilst climbing the stairs or giving a number to the contents of a group of toys, cakes or people. It has been long recognised that fractions and decimals are much more difficult for children than the counting numbers. When there is a non-integer value in a question it is made very much harder, not slightly more difficult. On the current mathematical diet provided in schools many British children reject all but whole numbers and exist throughout their secondary school years trying to make sense of their mathematics whilst using only whole numbers. The nature of the understanding of Fractions has occupied many mathematics education researchers over many years. The symbolic representation a/b hides at least seven different interpretations, the following is provided by Kieran (1976).

His seven meanings are listed below, we have illustrated each with an example:

1. Rational numbers are fractions which can be compared, added, subtracted, etc.

This aspect concentrates on the meaning of a fraction, usually the model employed for teaching is a region or piece of a rectangle or circle. The operations on fractions emphasise recognition, conventions and rules such as 'you can only add two fractions if they have the same denominators'.

2. Rational numbers are decimal fractions which form a natural extension (via our numeration system) to the whole numbers.

With the advent of metrication in Britain the use of decimal fractions has increased and fractions as parts of a whole (5/8, 3/14) which were common in imperial measures have ceased to be important. The calculator of course employs decimal fractions and the increased usage of this aid in the classroom means a greater emphasis on decimal notation.

3. Rational numbers are equivalence classes of fractions. Thus {1/2, 2/4, 3/6 . . . .} and {2/3, 4/6, 6/9 . . . .} are rational numbers.

The common use of equivalence is in the comparison of two fractions, addition and subtraction, e.g. 1/2 = 2/4, if the second form is more apposite we use that rather than the first. Children often think the fraction has changed in size and refer to 2/4 as 'bigger' because 2 and 4 are bigger than 1 and 2, whereas of course the two ratios name exactly the same amount.

4. Rational numbers are numbers of the form p/q, where p, q are integers and q is not 0. In this form, rational numbers are 'ratio' numbers.

This is a large step from meaning (1) as the ratio a/b is not used to label a part of a whole. The form a/b is often used in mathematics problems, e.g. in enlargement where a scale factor of 2/3 cannot be seen as part of a pie/circle at all.

5. Rational numbers are multiplicative operators.

Interpreted by some as a function machine, others as a sharing or partitioning e.g. 1/6 of 12.

6. Rational numbers are elements of an infinite ordered quotient field. They are numbers of the form $x = p/q$ where x satisfies the equation $qx = p$.

In school mathematics this can be viewed as using fractions to answer questions which are impossible within the whole number system, e.g. 3/5 or $3 = 5n$.

7. Rational numbers are measures or points on a number line.

The number line is a model often used for teaching mathematics, fractions can be placed on the number line just as can whole numbers and negative numbers. In particular, the measurement of length essentially uses a number line.

All these meanings are at some time employed in school mathematics, the emphasis given to each in teaching varies considerably. A child is dependent on teaching for the realisation of the use and conventions and cannot be expected to 'discover' the meaning of what is an abstraction. The results of the CSMS research (Hart, 1981) shows that although many children can give the fraction name of a region and can cope with the introductory ideas of the topic, at least half the secondary school population refuses to work with fractions as numbers, e.g. with fraction dimensions in an area problem. This of course restricts the mathematical attainment of these children since they cannot solve ratio problems, fail to see the set of values which can be taken by a letter and are very restricted in the use of area or volume formulae.

Such a range of interpretations of a/b is not adequately catered for by the simple introduction through "a" regions of a pie which has been subdivided into "b" pieces. The model is of little use to illustrate the multiplication or division of fractions. You cannot multiply a piece of pie by another piece and you certainly cannot enlarge a photograph by multiplying by a region. We tend to teach the region fraction names when the child is quite young, eight or nine years old and the image is powerful, reinforced in the case of 'one half' by common usage. The fact that the operation of division is complete when one has a/b is not recognised by at least half the 11-12 year olds tested by the CSMS team.

The CSMS Fractions results and subsequent interviews pointed to the child's refusal

1) to see a/b as the result of division,

2) to accept that a/b was a number, and

3) to appreciate the equivalence of fractions which were often seen as equivalent and multiples of each other at the same time.

For example in the CMF research (Hart et al., 1989) a group of 12-year-olds were taught 'equivalent fractions' by their teacher.

All the six children (boys) present at the delayed interviews, three months after the teaching, were asked whether 10/14 was double 5/7 or equal to it. Four children declared the two fractions equal, appealing to the fact that two times five gave ten, etc. Mick thought saying they were equal was a better answer but one could be double the other. Matt however thought that 10/14 was both equal to and double 5/7. His reasons were interesting in that they did not include multiplication:

(I: Interviewer; M: Matt)

I: So they are both true are they: 10/14ths is double 5/7ths and 10/14ths is equal to 5/7ths?

M: Yes.

I: What about this one? 10/14ths is more than 5/7ths?

M: That's false.

I: How do you know that?

M: Because that can't be the same as that. If that is true then that would be false.

I: What about this one?

M: 10/14ths, you just add them: add that and you get 10 and add that you get 14.

There is a large gap between being able to give a name to a region and being able to manipulate fractions with operations. The fraction 'one half' is used by young children and seems to be accessible much earlier than any other. The ability to compu'. one half of a quantity or enlarge by a factor of two are not indicative of an understanding of other fractions and ratios.

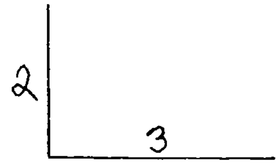
Other difficult topics

Other topics which have been shown to be more difficult than has traditionally been believed are multiplication, proof, ratio and proportion. The CSMS results demonstrated that many British pupils did not use the operation of multiplication, replacing it with repeated addition. Fischbein et al. (1985) considered 'repeated addition' intuitive. Children who interpret multiplication in this way, need to see a unit repeated, so "1/3 x 3/8" does not fit happily within this format. PME researchers have shown that the nature of the multiplier is an important factor affecting the difficulty of the problem. Children perform better if the multiplier is an integer, less well if it is a decimal larger than 1, a multiplier which was a decimal less than 1 was even more difficult. The nature of the multiplicand has scarcely any effect but further recent research has investigated whether the demand of 'symmetric' problems such as "If length is x metres and the breadth y metres, what is the area?" is greater than 'asymmetric' ones such as "one litre of milk costs x francs, how much for y litres?". The evidence was clouded by pupils' use of the formula for area and so the investigation continues.

There has, for many years, been a PME working group looking at the problems children have in dealing with Ratio and Proportion. Piaget stated that proportional reasoning was appropriate to the formal operational level, it is a topic often called upon by science teachers and has been a rich source of research problems. As we have seen Multiplication and Fractions have both been shown to be difficult ideas for children, computations associated with Ratio and Proportion require the use of both. Children as young as six years of age have a qualitative understanding of proportion and can provide for themselves a comparison measure as reported by Streefland when he described a young boy stating that a cinema poster had an inaccurate picture of a whale because he had seen a whale the previous year and the whale was not that much bigger than a man.

The quantification of these ideas is much more difficult and the type of number involved is important. Enlargement by an integer scale factor is easier than by a non-integer. Ad hoc methods are used by both adults and children to solve proportion problems. One very prevalent incorrect method known as "the incorrect addition strategy" accounts for up to 40 per cent of the errors in some (usually geometric) problems in the CSMS Ratio study. It has been shown to occur even more often in the USA.

This strategy (referred to as "the incorrect addition strategy") stemmed from the belief that enlargement could be produced by the addition of an amount rather than by the employment of a multiplicative method. The child, in this example, reasons that since the base line had been increased by two units so must the upright be so increased.



Enlarge with a base of 5

The error was persistent over time and seemed not to be part of a continuum which eventually brings success. A subsequent research project "Strategies and Errors in Secondary mathematics (SESM) found that those in the British sample who consistently used this method tended to replace multiplication by repeated addition on items which they solved successfully. An SESM teaching experiment showed that the error can be eradicated in enlargement problems and greater success attained with short-term intervention which addresses the basic problems:

(i) the recognition by the child that the 'method' he uses produces strange figures

(ii) the need for multiplication

(iii) facility with fraction multiplication

(iv) the possession of a technique for finding the scale factor.

A teaching scheme which included the teaching of multiplication of fractions was tried with groups of pupils known to use the "incorrect addition strategy". There was initially great success with this but on delayed post tests the Fraction manipulation proved to be the barrier. A modified version of the teaching module, in which the Fraction work was replaced by the use of the calculator proved to be more successful (Hart, 1984). The intervention material was designed to meet the perceived problems listed above, and when used by classroom teachers with intact classes resulted in the abandonment of the incorrect addition strategy. Successful performance on Proportion problems was demonstrated by about 60 per cent of the total sample (n= 80) three months after the teaching. The classes had been chosen by their teachers, as likely to benefit from the material, a fact which was borne out by the pre-tests. Of the children classified as persistent users of the incorrect addition strategy, 22 out of 23 had abandoned it by the time of the delayed post test. The CSMS longitudinal data on the other hand had shown that half of the 'adders' at the age 13 years were still using this strategy two years later. It does seem worthwhile to identify an error and design corrective material which tackles those aspects symptomatic of the erroneous reasoning.

Other research on Ratio and Proportion has investigated whether the child, when faced with a problem such as that below, uses the ratio (from Vergnaud, 1983): (xb or xa).

Richard buys 4 cakes, at 15 cents each.

How much does he have to pay?

$a = 15$, $b = 4$, M_1 = number of cakes, M_2 = cost.

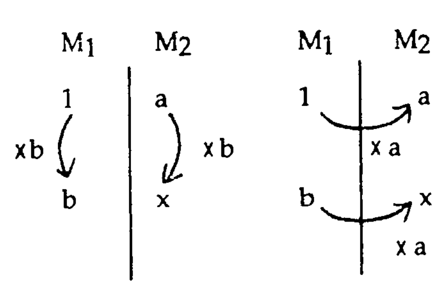


Fig. 2

Vergnaud refers to the ratio xb as 'Scalar' and xa as 'Function'. Other authors call them 'between' and 'within'. The scalar ratio seems to be that most favoured by pupils in different countries but whether this is by choice or the effect of teaching is not known. It would seem sensible to encourage children to use both since this gives them greater flexibility in choosing the computation they intend to do.

Through research we have found that in problems involving Ratio and Proportion,

(i) the nature of the scale factor is important, often methods other than the algorithm can be used,

(ii) some errors are very common but can be corrected by intervention and

(iii) 'within' and 'between' ideas are not equally accessible.

All such considerations should be taken into account in the writing of materials. Firstly the examples given to the child for solution should be such that the 'within' view sometimes appears to be the most natural and on other occasions the 'between' numbers being compared are the easiest. One should certainly avoid examples which require only doubling. Multiplication is such a neglected operation it seems sensible to deal with it again at secondary level regarding it as 'related' to division and not addition and viewed as the total of an array rather than repeated collections. Generally an effort must be made to convince children that the method for solution taught by a teacher is supposed to cover numerous cases and be generalisable. The project 'Children's Mathematical Frameworks' (CMF) and subsequent research reported at PME in 1989 have monitored teachers in their own classrooms. At no time did the teachers in the sample (n = 36) tell the children why a generalisable method was important. It seems sensible for a teacher 'to sell' what is very much more powerful in the context of a problem which cannot be solved by more naive or ad hoc strategies. It is often the case that a very powerful tool is introduced to solve a problem for which the child can see an answer immediately and being sensible rejects the teachers' method.

From concrete materials to formalisation

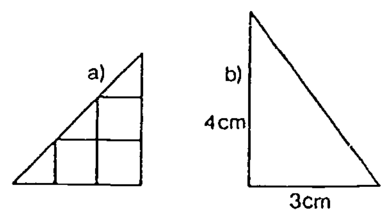
Young children, particularly those under about the age of 11 years, very often experience in school a more practical or applied type of mathematics than their older brothers and sisters. Influenced by the theories of Piaget, teachers in many countries have been advised to base mathematics for young children on the use of concrete materials (manipulatives). Such mani-pulatives can be put to many uses. They can be used to provide the base from which concepts are developed. For example, to know what is meant by the word 'triangle' one needs to have seen, touched and explored a lot of triangles. Manipulatives can be the essence of a problem, for example real or plastic money being used for shopping. One use of manipulatives which has been advocated over a number of years is the provision of well-structured experiences which lead to the 'discovery' (albeit guided discovery) of a generalisation or rule. This can then become part of the repertoire of the child as he moves into the more abstract or formal type of mathematics commonly assumed and built on in the secondary school. An example of this type of teaching is when the child is working towards the acquisition of the formula for finding the area of a rectangle. The classroom experiences recommended by many texts and teacher trainers include:

1) Covering space with various two dimensional shapes in order to be able to quantify how much areas there is.
2) Covering rectangles with squares.
3) Moving squares around to form differently shaped rectangles.
4) Drawing rectangles on squared paper.

These activities are supposed to be enough to prompt the recognition that the area of any rectangle can be found by multiplying the number of units in the length by the number in the width. It is this rule, often written A=l x b, which is carried into the secondary school and which is then used whenever the area of a rectangle is sought.

The assumption that all secondary-aged children have available a 'formal' method of finding the area of a triangle can be shown to be false by the following example from C.S.M.S.

Find the area



	12+	13+	14+
a):	76 %	87 %	91 %
b):	31 %	39 %	48 %

Fig. 3 Area examples CSMS

The pupils tested were 986 in number and came from different British schools. These two questions were attempted by the same children on the same occasion. The formula for the area of a rectangle would have been taught in their primary school mathematics and most of the children would have met the extension to the area of a triangle in their lessons.

The theory is that the child has a firmer understanding of the rule because s/he has seen from where it came and because the discovery was his/hers, the idea will be retained. This view fits well with the constructivist philosophy. Many children throughout the world have learned (or failed to learn) in a different environment as Dorfler (PME, 1989) says

> Mathematics for many students never gets their own activity, it remains something which others have done (who really know how to do it) and devised and which can only be imitated (for instance by the help of automized algorithmic routines at which the student works more like a machine than like a conscious human being). In other words, mathematics mostly is not part of the personal experience and the reason for this very likely is that the mathematical knowledge of the students was not (or only in an insufficient way) the result of structuring and organizing their own experience.

'Guided discovery' seems an attractive alternative, however, there are drawbacks when the ordinary teacher puts theory and interpreted theory into practice in the

classroom. There is also a shortage of detailed and long term monitoring of most forms of teaching, to see whether those that are attractive in the abstract are effective with teachers and children as they are and not as we might wish them. Evidence of the lack of effective learning has become available from the research project 'Children's Mathematical Frameworks' (Hart et al., 1989). In this research, volunteer teachers were asked to prepare a scheme of work for the teaching of a rule or formula, in which the pupils started with concrete material and the formulation was the synthesis of the practical work. Six children in each class were interviewed by the researchers

(i) before the teaching started,

(ii) after the concrete experiences and just before a lesson(s) in which the pupils were expected to 'discover' or be guided to 'discover' the formula,

(iii) just after this lesson(s) and then three months later.

The interviews were to find the child's knowledge of the basic concept, whether he was about to discover (or had just discovered) the formula and then whether he retained for future use a method with concrete materials or a known algorithm. The form of the child's knowledge is important since if given tasks which assume too high a level of sophistication then he is doomed to fail.

The topics investigated were: Equivalent Fractions, The Subtraction Algorithm, Formulae for the Area of a rectangle, the Volume of a cuboid and that for the circumference of a circle as well as the solution of simple algebraic equations, all with children aged 8-13 years. The data were collected in the form of audio-tape recordings and the results showed that the transition from concrete material to formal mathematics was by no means easy or straightforward. Very often the concrete materials did not mirror the mathematics and imperfectly modelled the ideas involved. Sometimes they were sufficient for the child to carry out a computation in a "concrete" manner but did not in any way lead to the rule or algorithm intended as shown here:

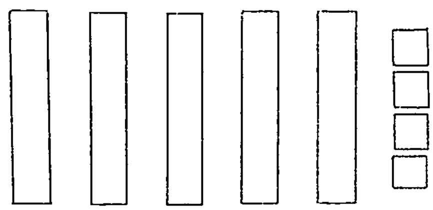


Fig. 4

The child Anne used 'Unifix' blocks to carry out the subtraction 65-29. Anne said "Then I would have taken two tens and another ten, but on the other ten, the third ten, I would have taken one unit off, so I would have 29, oh! - - - oh yes, so then I would take the 29 away and add up all the others and see how much is left. And its 36."

This is an effective way of carrying out a subtraction using bricks but such a method does not lead to the algorithm (which was the intention). This rule requires the child to start with the 'ones' and to decompose 'a ten' into 'ones' thus:

T	O	
6	5	9 from 5 we cannot 'do'.
- 2	9	Go to the tens, change one of these into ten ones and collect these with the 5 ones.
3	6	Take 9 from 15, leaving 6 ones.
		Take 2 from 5 tens, leaving 3 tens.

The primary school teacher who taught 'equivalent fractions' used Cuisenaire rods, discs, a fraction board and the children drew and cut out circular discs. The secondary teachers, in the same investigation, used diagrams of regions. In each case the 'family' of fractions was built around the factors of 12 and the number of parts in a region was dictated by the teacher. There was no general method that a child could adopt in order to provide himself with equivalent regions. Indeed, it is difficult to see how anybody can choose the appropriate number of discs or bricks with which to work unless one already knows the equivalent fractions. Asked to

use regions or discs to find an equivalent for 3/7, how is the child to decide on the number of discs to take from the box unless the rule is already known? Terence in this interchange tries to use diagrams as he has seen his teacher do.

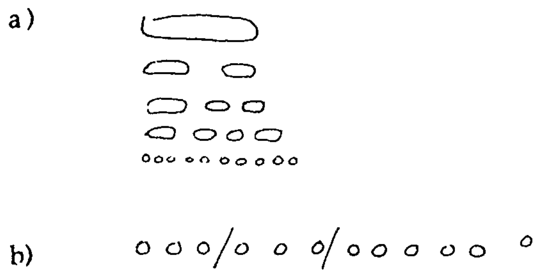


Fig. 5 The drawings of Terence

I: You've drawn 12 circles.

T: I marked in 6/12. Three-eighths ... 1, 2, 3, 4, 5, 6, 7, 8. So the whole one would be in between. These two are not the same.

I: They're not the same, right. 3/8 and 6/12 ... we've knocked out because ... now what did you do? I have to tell the machine what you did. You put your hand over the last four of those circles.

T: Yes, because there's 12 here, but if I put my ... hand over there to show that was 8 and 3/8 would be there and it isn't the same as that [pointing to 6 circles]. Try 6/11, is the same (counts from left hand side of circles and gets to same point).

I: As what ...

T: Er ... 6/12.

A common exhortation of teachers, seen during the research, was to those pupils who found it difficult to find and remember the rule or algorithm. They were told to return to the concrete materials. This however requires them to invent the modelling procedure unaided. Urged to take out the bricks 'to help', those children who did so simply used them as objects for counting.

A significant part of the belief in this type of teaching is that the child's understanding is stronger because the discovery is his. Three months after the teaching the children in the CMF study, when asked how they had come to know a formula, said that their teacher had told them or that a clever person had invented it. They did not say 'we discovered it'. They seemed to make no connection between the work with concrete materials and the formula they used in calculations. If one considers the most important features of a rod (a commonly used manipulative), they are: the material of which it is made (wood or plastic), its colour, its weight, length or volume. To the teacher, who is using this rod for a particular mathematical purpose, its outstanding quality may be that when put end to end with another the two are the same length as a third. The mismatch between the two conceptions is obvious.

Further research built on the CMF results, has pursued the idea that since the materials and mathematical formalisation were so different in nature, a transitional phase (a bridge) was needed. This 'bridge' could be in the form of diagrams, tables, graphs or discussion and should be distinct from both types of experience but linking them. More forms of the 'bridge' are being tried but the first results were disappointing because often the teachers who were asked to put 'bridge' activities into their teaching gave very little time to them (five or six minutes) or in some cases put this activity after the formalisation.

Concrete materials are of course used by teachers for other purposes. Teachers of younger children very often use manipulatives to convey the basic principles of a topic. Research results from the early 70's for example tended to show that the idea of a fraction being a region could be effectively taught when children cut and folded rectangles of paper.

Teaching experiments

The philosophic and theoretic frameworks adopted by most researchers within PME have led to an interest in monitoring 'processes' and problem-solving skills rather than investigating performance in computation. Usually many variables are at work in the classroom and it is difficult to apportion effectiveness to the efforts of the teacher, the nature of the material (text, concrete materials or computer) or the fact that the contents are process-orientated. Harrison et al. (PME, 1980) reported on teaching experiments in

Calgary in which 'process' enriched material on Fractions was taught to 12-13 year olds. The teaching took about eleven and a half weeks and control groups were always good results when they teach congruence? Is this pattern of success the same for other teachers? Is it the teacher or the topic which is the crucial variable?

Test or Subtest		Max. Score	Experimental Mean Score	Regular Mean Score	Probability (Evs R)
Number of Students			N=207	N=179	
CSMS Fractions	Pre	84	35.85 (43%)	34.75 (41%)	
	Post	84	51.57 (61%)	47.76 (57%)	<0.01
Problems	Pre	46	9.95 (21%)	9.35 (20%)	
	Post	46	16.40 (36%)	14.39 (31%)	0.01
Computation	Pre	12	3.15 (26%)	2.43 (20%)	
	Post	12	6.68 (56%)	5.68 (47%)	0.34

Fig. 6 Teaching Experiment (Harrison et al., PME, 1980)

taught from the current textbooks in the usual manner. The results are shown in fig. 6 above.

The experimental groups perform slightly better than the control groups at the end of the teaching. The enthusiasm of the children was greater after working with the process-enriched material, however. Perhaps what is more significant is how little gain there is after the teaching. If the teacher's best efforts over $11\frac{1}{2}$ weeks still leaves the children performing at only 60 per cent success level, perhaps we should reconsider what we expect of the pupils. Research teaching experiments are of necessity short-term and usually for less time than the Calgary study and often the teaching matter is given to children without consideration of their previous knowledge of pre-requisite ideas. Before the topic is taught the pre-test is administered in order to find out how much of the specific teaching points they already know. This is very seldom nil. The pupils usually have quite a lot of knowledge to start with but seem to acquire less than expected during a focussed teaching sequence. A worth-while long term study which could be carried out by teachers in their own classrooms is that of analysing exactly which sections of their teaching are successful. Do they always get poor results when they teach addition of fractions and

Should we not be more concerned about matching the material to the child's level of knowledge?

The computer

The computer used as a teaching aid in the classroom has gained considerable prominence since 1980, the emphasis in research being on how children's thinking skills or deeper understanding have been improved. Tall and Thomas (PME, 1988) for example quoted a study in which children aged 12-13 years were given a "dynamic algebra module". This provided a "maths machine" which evaluated formulae for numerical values of the variables. The teacher, however, was a vital component of the work. Results tended to show that the Control group performed better on skill based questions, for example:

Simplify $3a + 4 + a$
Exp. 38% Control 78%

On questions which required higher levels of understanding the experimental group performed better, for example:

The perimeter of a rectangle 5 by F.
Exp. 50%. C. 29%.

Again we see that the teaching experiment has produced success but half of the children are still not successful.

The use of LOGO in classroom computer work has been increasingly reported. The microworlds are often used as a basis for discussion of what children say and the interpretation they place on situations. Comparison of performance between children in a LOGO environment and those in a control group seldom provides an overwhelming success on the side of the experimental group. Often closer scrutiny of individual marks shows a greater degree of progress for some of the LOGO group. Hoyles (PME, 1987) in a review, of PME presented papers described three areas of research within the context of 'Geometry and the Computer Environment'.

I. Research which explores the development of children's understandings of geometrical and spatial meanings and how progression (for example, from globality to increased differentiation) might be affected by computer "treatments".

II. Research which investigates the "training" influence of computer environments on different spatial abilities.

III: Research which takes as a starting point the design of geometric computer-based situations which confront the students with specific "obstacles" and seeks to identify student/ computer strategies, the meanings students construct and how these meanings relate to the representations made available for the computer "tools". Such research more or less explicitly uses the computer to create didactical tools to facilitate the acquisition of specific mathematical conceptions or understandings."

More recently Hoyles' group have been reporting on particular topic work carried out with small groups of children, for example the concept of a parallelogram. It seems possible that children are assisted in learning by the use of the computer but it is not a panacea.

Illustrations and their use

Textbooks used in mathematics lessons are now extensively illustrated with pictures, diagrams and graphs. It is worthwhile considering why we use these devices and what benefit they are to the children. Do teachers buy colourful, heavily illustrated books because their pupils learn from them more effectively or because they, the teachers, like the look of them?

By 'illustration' we mean non-word material. This includes various categories:

1) Line diagrams which are:
   a) the objects themselves - a triangle;
   b) representing an object so a convention is involved;
   c) representing actions or steps, such as a flow diagram;
   d) representing by a picture.
2) Pictures which show reality as seen by the artist.
3) Photographs.
4) Recording diagrams, tables, graphs, matrices, Venn diagrams.

(The children will be required to draw these themselves later.)

The illustrations can be used:

1) To convey information which is not best conveyed in any other way.
2) To present a problem in non-verbal form.
3) To convey information when the recipient cannot have access to knowledge through words, for example those who cannot read.
4) To provide a focus for discussion so that the children can all give their views about the same thing. (It can also be used to assess).
5) To present the exercise in context e.g. shopping.
6) To form a bridge between reality and abstraction.
7. To record data in a more effective way.
8. To make the books attractive.
9. To (perhaps) motivate.

Children do not automatically realise how to use a diagram or what its intended message is. Its special features need to be taught just as other aspects of mathematics need to be taught. The distinction between diagrams which show a relationship (such as a graph) and a picture which represents

reality, is lost on many children. For example time-distance graphs such as that shown in figure 7 are interpreted not as representative of a journey but as showing hills and plateaus. This was so in about 14 per cent of cases in the CSMS study. (Sample n = 1396)

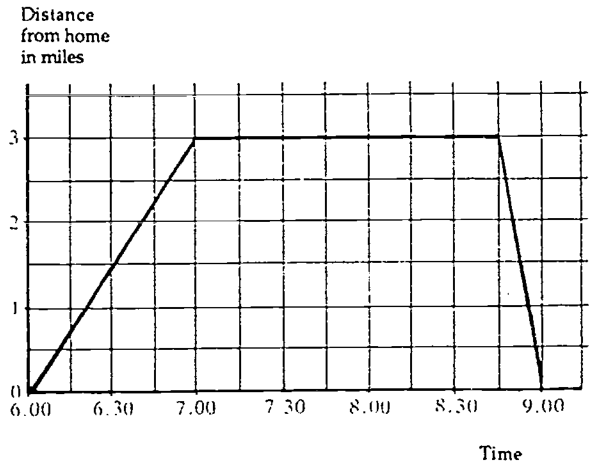
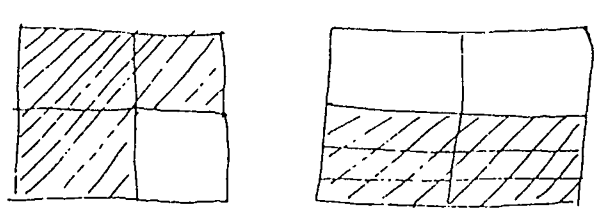


Fig. 7

Often teachers use a diagram to add weight to an argument. Thus when teaching about fractions they are likely to draw, freehand on the blackboard, a circle divided into sections. The purpose of this is difficult to see, since if the pupils were not already knowledgeable of fractions and their equivalents, the diagram would confuse rather than help. For example, unless drawn accurately it cannot provide evidence that $3/8 = 6/16$.

The CMF researchers found that children provided the same inaccurate diagrams when trying to convince the interviewer of equivalence, as seen in this effort of Joe.

a) Matt's diagram for $3/4 = 6/8$



b) Joe's diagram for $3/5 + 7/10 = 10/15$

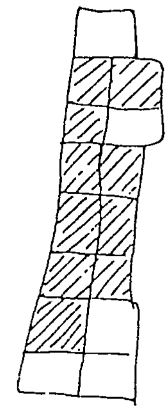


Fig. 8

Textbooks are nowadays often lavishly illustrated, yet there is scarcely any research on the effect these pictures have on the learning of the mathematics. Shuard and Rothery (1984) provided a review of the existing research on reading mathematics. In so doing they analysed pages from textbooks where information was partly given in the picture, partly in the text and sometimes on lists which appeared beside the picture. For example in shopping questions which are essentially testing addition, there are pictures of the articles which can be bought, perhaps labelled with prices, a shopping list and then a sentence or two in the text. The eye movement needed to obtain the necessary information is excessive.

There is some research in science education on the use of illustration in text books and other types of teaching material but very little in mathematics.Much of Geometry is said to be concerned with the improvement of children's spatial visualisation, yet little of it positively addresses this issue. There has been some recent research, resulting in books for children, which attempts to improve pupils' view of three dimensional objects and their interpretation in two dimensions. One study was part of 'The Middle Grades Mathematics Project' from Michigan State University (Winter et al., 1986).

The Spatial visualization instructional material includes ten carefully sequenced activities. The activities involve

representing three-dimensional objects (buildings made from cubes) in two-dimensional drawings and vice versa, constructing three-dimensional objects from their two-dimensional representations. Two different representation schemes are used for the two dimensional drawings. First an "architectural" scheme involving three flat views of the building-base, front view and right view. After students are comfortable with this scheme, they are introduced to isometric dot paper and hence to a representation consisting of a drawing of what one sees looking at a building from a corner. Similarities, differences, strengths and inadequacies of the two schemes are explored.

It was concluded from the analysis of the pre-post data that after the instruction:

(1) sixth, seventh, and eighth grade boys and girls performed significantly higher on the spatial visualization test; however, no change in attitudes toward mathematics occurred;

(2) boys and girls gained similarly from the instruction, in spite of initial sex differences;

(3) seventh grade students, regardless of sex, gained more from the instruction than sixth and eighth graders.

In addition, the retention of the effects of the instruction persisted; after a four-week period, boys and girls performed higher on the spatial visualization test than on the post test.

Conclusion

We now know, through research in mathematics education, that certain topics are difficult for children to learn because they contain many different aspects each of which brings its own degree of complexity so that the total is composed of layers of difficulty. Proportion is such a topic. It seems wise to write materials which build up the different understandings rather than produce the topic all at once. It is also wise to illustrate and give examples which require the use of the idea or method and cannot be easily solved by lesser and more naive strategies. There is then an incentive for the child to take on the more powerful tool.

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A founder member of PME, Dr. Hart is currently its President.

The Role of the Teacher in Children's Learning of Mathematics

Stephen Lerman

There is no doubt that the teacher plays a major part in creating the environment in which children can best learn mathematics. This is not to claim too much for the role of the teacher. Stronger claims have been made, for example the linguistic philosophers of education would want to say that unless learning is taking place one cannot be said to be teaching. Certainly governments in many countries count teachers as solely responsible if there is evidence of children not learning. Thus the statement that begins this chapter is relatively minimal and uncontroversial. Nevertheless, there are many issues to be investigated, and the questions begin when one attempts to establish what part the teacher plays, what is the best environment and how can one create it, and what teachers understand about children's learning of mathematics. In this context, therefore, it is interesting that it is only in recent years that one finds a growing body of research that focuses on the teacher. For example, at the Tenth Meeting of PME in 1986, there were no more than a small handful of research reports concerning teachers, but by the Fourteenth Meeting in 1990 there were three Working Groups and a Discussion Group, and at least 20 research reports.

There are perhaps two major reasons for the relatively late interest in the role of the teacher. The first is the dominance of the notion that if we can specify how children learn, and produce the most appropriate learning materials informed by that knowledge, the teacher's role becomes largely that of an intermediary in the process of transmission. If, further, we can specify what the classroom should look like, how long children should study, and various affective factors, we can tell the teacher what to do to achieve the goal of children learning mathematics. Many would claim that a paradigm which incorporates these is no longer tenable. The transmission metaphor is no longer seen as adequate, and, indeed, the nature of the learning process is seen as less and less determinate. The affective and the cognitive may not be simply separable categories as was previously claimed, and this has led to research on the social context of the classroom which has considerably confused and enriched our observations and ideas. In all of these issues, the teacher is a central figure.

The second reason that research on teachers has only recently playing a significant part on the agenda, is the nature of that research. This is an issue that is the subject of on-going debate (Scott-Hodgetts, in press, b). Most research questions that are concerned with teachers will not be analysable by significance tests. As Mason and Davis (PME, 1989) suggest, "As with our previous work, validity in our study lies, for us, in the extent to which it resonates with experience, and to which it awakens awareness of issues which they might otherwise have overlooked" (p.280-281).

There are some studies that draw on reasonably sized populations for quantitative analysis, such as those examining pre-service teachers' knowledge of aspects of mathematics (e.g. Vinner and Linchevski, PME, 1988). In the main, however, the research methods most used are qualitative, based often on only a small number of teachers (Jaworski, PME, 1988, Lerman and Scott-Hodgetts, PME, 1991, in press), and the methodology and generalisable outcomes of such research are not accepted by all in the mathematics education community.

This is a theme which will arise frequently in this chapter, namely the kinds of methods that are most informative for research on the teacher's role. Two aspects in particular will be emphasised, firstly that a holistic perspective of interactions in the classroom is more fruitful than a fragmented one, and secondly, that case studies, perhaps stimulated by findings from quantitative

studies, provide information that resonates widely with teachers and other researchers.

Expectations of the fruitfulness of research on teaching vary considerably. In a recent volume (Grouws et al., 1988) one finds both the positivist expectations of Grouws: "No longer can we uncritically rely on the folklore of the classroom; rather we must sort out fact from myth and begin to build conceptual networks that will help us to understand and improve the complex process of teaching" (Grouws, 1988 p.1), as well as the sceptical thoughts of Bauersfeld: "... there seems no prospect of arriving at universal or overall theory of teaching-learning processes that we will all accept" (Bauersfeld, 1988 p.27).

However, as can be seen by the growing interest in the role of the mathematics teacher in research reports at PME meetings, the research community is now engaging with these issues, and the themes which will be reviewed in this chapter will indicate some of the important ideas emerging.

Following this introduction, the chapter is divided into four sections. In the first section we will examine some aspects of the teacher's work, including notions of effective teaching, the teaching of problem-solving processes and the role of meta-cognition, the power relationships in the classroom, some remarks on textbooks, and on what constructivism, an alternative theory of the way people learn, might imply for teaching. As with the remainder of the chapter, the review will draw, in the main, on research reported at PME meetings as well as on other widely available literature.

One of the major themes that emerges from the review is the significance of teachers' beliefs about mathematics, and about mathematics teaching, and the influences of school and society on teachers' actions in the classroom. In some senses, this theme is fundamental. Whether one is considering society's influences on the classroom, textbooks or other teaching materials, cultural influences, curriculum change, technology or whatever, they will all be mediated through the teacher, and specifically through the teacher's beliefs about her role in her students' learning of mathematics. Work in this area will be reviewed in the second section.

At the same time, the teacher is an actor in a particular setting, within which the relationships and dynamics are constructed by the actors, but in turn the actors and their roles are constructed by the classroom relationships and dynamics. Therefore, in the third section we will look at the teaching and learning of mathematics in the context of the language and practices of the classroom.

In the final section, we review research on teacher education, including action research, and teachers' knowledge of mathematics. The concluding remarks will return to the issue of the nature and role of research on teaching, and will attempt to highlight major themes for future research.

Aspects of the work of the mathematics teacher

Effective teaching

It may seem that the most appropriate place to begin any survey of the actions of the teacher is to attempt to characterise what is effective teaching, as well as which aspects of teaching make the teacher ineffective. If this is possible, one could design pre-service and in-service courses so as to bring about development and change in those areas. Albert and Friedlander (PME, 1988) reported on an in-service course focused on counselling, and commenced their paper with an analysis of the flaws in unsuccessful teaching and a definition of effective teaching, the latter including "good class management, clear and correct mathematical content, well-planned lessons resulting in a feeling of learning, good learning environment ..." etc. (p.103). There is the danger that an analysis of the actions that identify an effective teacher can become merely a list of desirable attributes, without any real evidence that such attributes contribute significantly to school students' learning of mathematics. Also, phrases such as "good learning environment" are used which themselves require elaboration.

Some researchers have thought it more appropriate to make one's characterisation more general, and to offer it in terms of broad goals. Peterson (1988), in discussing teaching for higher-order thinking, offers such a list:

"(a) a focus on meaning and understanding mathematics and on the learning task;

(b) encouragement of student autonomy, independence, self-direction and persistence in learning, and

(c) teaching of higher-level cognitive processes and strategies" (p.21)

and in the same volume Berliner et al. (1988) go even further in broadening the characterisation of effective teaching, almost to the point of stating the obvious:

> "... there are two separate domains of knowledge that require blending in order for expertise in teaching to occur. These are (1) subject matter knowledge and (2) knowledge of classroom organization and management, which we call pedagogic knowledge." (p.92)

It may be a more fruitful approach to focus on what happens in the classroom, in the context of the goals set. Hanna (PME, 1987) approaches the problem by attempting to identify those teaching strategies "that seemed to have contributed most to greater achievement gains" (p.274) amongst eighth grade students, and concludes that these were:

"(a) an extremely organized approach to teaching, wherein material is taught until the teacher feels it is mastered, thus reducing the need for frequent review, and

(b) an approach in which every presentation of material is followed by extensive practice in applying the new material to new situations." (p.274)

Steinbring and Bromme (PME, 1988) have developed a technique to monitor the process of knowledge development in the classroom, and present this pattern graphically, enabling comparisons over time, student group, teachers etc. Kreith (1989) presents the idea of master teachers acting as role models and working with intending teachers as interns, an issue to which we will return when considering teacher education.

There are dangers of hidden assumptions when one focuses on the effective teacher, or the master teacher. Nickson (1988), for example, suggests that "the notion of the 'expert' is a contentious one and means different things to different people." (p.247) One cannot talk about 'effective' without determining what it is that one values in the teaching of mathematics. Classroom management, for example, is one kind of notion when one is teaching a piece of content, in a transmission model, and quite another when the focus is on group problem-posing work. Again it means quite different things if one is teaching a class of 60, with little or no resources, and quite another with a group of 20 in a well-resourced situation.

We may also, in the end, have doubts about the value of breaking down classroom activities into teachers' actions, students' actions and affective conditions, when what is taking place is a complex process of interactions, intentions, relationships of power, expectations, and so on (Porter et al., 1988). Bishop and Goffree (1986) make a similar point when they argue for a shift from what they call the 'lesson frame' to the 'social construction frame'. The former is characterised as follows:

> "(the mathematics lesson) is construed as an 'event' with a definite beginning, an elaboration and a definite end. It has a fixed time duration. Typically all children will be engaged in the same activities which are planned, initiated and controlled by the teacher." (p.311)

They indicate that research, dissatisfaction with outcomes, and alternative constructivist theories of the way children learn have led to a widening of the frame:

> "This orientation (social construction frame) views mathematics classroom teaching as *controlling the organisation and dynamics of the classroom for the purposes of sharing and developing mathematical meaning.*" (p.314)

They include in their list of features of the frame notions such as "(2) it emphasises the dynamic and interactive nature of teaching; ...(4) it recognises the 'shared' idea of knowing and knowledge, reflecting the importance of both content and context;" (p.314).

Studies based in the classroom provide perhaps the most fruitful approach for research in the area of effective teaching, since an essential component is the recognition of the theory-laden nature of describing an effective teacher, and thus of the need to build suitable research methods in the light of this.

For instance, the teacher might develop her notion of student achievement, and criteria of successful teaching, relevant to the specific context. Through action research, or a collaborating observer in the classroom, teacher strategies that appear to help the students and the teacher to realise that achievement might then be identified. What is important here is that 'desirable attributes' become something specific to the particular teacher, in relation to her own values, and change is seen as an individual's progress in her own terms, rather than something that teacher-educators or researchers do to others.

An illustration of this approach can be seen in the following extract from the reflections of a teacher, who had been critically examining his teaching in the light of some reading he had done, and some subsequent observations of his students (Lerman and Scott-Hodgetts, PME, 1991, in press):

> "So my reflections led me to this point. I began to believe that if the didactic approach was not good enough for Claire, it was not good enough for any pupil; that instruction and explanation by the teacher had its place in outlining the problem, but not somehow giving mathematical ideas."

Another perspective on the issue of effective teaching, is given by Scott-Hodgetts (1984), (discussed also in Hoyles et al., 1985), in a paper that describes her research on childrens' ideas of what is a good teacher. In a paired-comparison test of characteristics that go to make a perfect teacher, the ranking emerged as follows:

> "is able to explain the maths clearly and thoroughly;
>
> has a patient and understanding attitude when people find the work difficult;
>
> is prepared to spend a lot of time with people who need more help;
>
> has a relaxed friendly relationship with the pupils;
>
> keeps up to date with each pupil's work, and lets them know how they are getting on;
>
> makes the lesson interesting and enjoyable;
>
> knows the subject really well;
>
> has a good sense of humour."

Alibert (PME, 1988) found that first year university students valued aspects of the teachers' work that differ from those that the teachers might value. In her study, 52% of the students considered that a good mathematics teacher is first "interesting, convincing, clear, quiet" and in only 11% of answers were there such things as "helps students to reflect, to participate" (p.114). Two of the most interesting aspects of these studies are the following: first, students emphasise quite different aspects of 'good' teaching than teachers themselves might emphasise, and this information could be a valuable stimulus for reflection and research by teachers, and second, that the hidden messages of one's beliefs are often conveyed to students, and reflected in what they value most in their teacher.

Meta-cognition and teaching mathematical 'processes'

One of the major changes in the role of the teacher during the last decade or so, has been brought about by the shifting of the focus of attention to the introduction of problem-solving work into the classroom. It has raised many questions and engendered many debates. An early question was: what are problems, and how do they differ from puzzles and investigations? In fact we often use all three words during any particular piece of work. We may start by being interested in a problem, and begin to investigate it. On the way we solve several puzzles, become stuck on many more, and generate new problems to investigate! Different interpretations of the term 'problem-solving' remain, however, as shown in the study described below.

The issue of 'process' versus 'content' has been discussed, in the literature and at conferences, and we have become aware that

we are at least partly discussing the philosophy of mathematics. To some extent, for the formalists, mathematics is its content and its structure, whereas for the quasi-empiricists (Tymoczko, 1985) mathematics is its processes. In recent years we have been developing ways of assessing students' work on investigations, in particular in Britain where school students have to submit a number of individual course works for the General Certificate of Secondary Education (GCSE), a national examination for 16 year olds. There is still, however, the issue of how one avoids assessment-led investigation work in classrooms (Lerman, 1989a), that is whether the potential for creativity and engagement disappear under the need to have problems whose answers can be 'marked'.

Problem-solving has led on to problem-posing (Brown, 1984), and opened up the activity to the wide and stimulating possibilities of ethnomathematics, empowerment and critical mathematics (Frankenstein, 1990). But where does it fit in to the school mathematics curriculum? Is it to be added on at times, or can one teach the main part of the curriculum through such work? The nature of mathematics is the essence of the problem here, and it finds expression through the process/content debate. All these questions are the foci of current research work in mathematics education, and we will examine some of them here, as they impinge on the role of the teacher.

In a study of teachers' ideas of what is problem-solving and how to teach it, Grouws et al. (PME, 1990) found that teachers were using four categories of meanings:

"(1) Problem-solving is word problems;

(2) Problem-solving is finding the solutions to problems;

(3) Problem-solving is solving practical problems; and

(4) Problem-solving is solving thinking problems." (p.136)

In their survey of 25 teachers, they found that 6 identified with the first definition, 10 with the second, 3 with the third and 6 the fourth. However, their lesson goals and related instructional methods did not relate to their definitions, and neither did the format of their lessons. Teachers felt that there was not enough time for problem-solving instruction, standardized testing being a major interference. Students had low success and little self-confidence. Grouws et al. conclude:

> "We now know that we must carefully describe what is meant when a teacher gives critical importance to problem-solving and its instruction in her classroom." (p.142)

To some extent, aspects of the study are probably culture-specific since in many countries problem-solving would not be considered to mean word problems. Teachers may well respond by offering the other definitions, however, and many of the other findings may well be supported widely. The final comments of Grouws et al. are an important caution to researchers and a reminder of the influence of teachers' beliefs about mathematics and mathematics teaching. One must also take into account the tendency to respond to certain jargon words, particularly 'problem-solving', which has been a major theme of the 1980's. In this context it is therefore particularly significant to find both that teachers have quite different ideas of what it is, and that their teaching often bears little relationship to those ideas.

Flener (PME, 1990) reported on a study of the consistency among teachers in evaluating solutions to mathematical problems, and of their reaction to solutions showing insight. Hypothetical students' solutions to mathematical problems were sent to 2200 teachers, of whom 446 responded. Flener found a wide range of evaluations, that teachers credited methods that were taught in school and not problem-solving ability at all, and in general teachers gave no credit for insight or creative solutions. Morgan (1991), presented experienced teachers with three hypothetical solutions to a particular problem: the first, (labelled PV) used diagrams and full verbal description, using spoken rather than written language; the second (ND) used numbers and diagrams and a table, with text that presented the work in the order in which it was done, and the third (BS) was concise and symbolic, with words

only as labels, without sentence structure. All the teachers placed ND the highest, and most placed PV second. This is in marked contrast with the usual presentation of pure mathematics in academic journals, and suggests that teachers are developing particular and specific ways of assessing school mathematics problem-solving. She suggests that teachers are developing criteria for what constitutes school mathematics, rather than mathematics in general, and that these criteria are not made explicit either amongst teachers or to school students.

Again, we are led to consider what teachers understand by mathematics, in the context of problem-solving. This issue will be discussed further below, but whatever teachers and researchers understand by problem-solving, a major interest for mathematics educators is the question of whether problem-solving skills can be taught. There are two inter-related and interacting levels to be considered, heuristic methods and meta-cognition. Hirabayashi and Shigematsu (PME, 1986, PME, 1987, PME, 1988) have developed a powerful metaphor for meta-cognition: that of the Inner Teacher. They suggest that the teacher acts as the inner self of the students and thus the task of teachers is to encourage the growth of students' own inner teacher. Mason and Davis (PME, 1989) indicate how useful they find this metaphor for thinking about the shifts in attention required in problem-solving and in mathematics education in general. In discussing some theoretical aspects of teaching problem-solving, Rogalski and Robert (PME, 1988) suggest that it is possible to "design methods related to a specific conceptual field", and they go on to outline how and when such instruction is appropriate:

> "... such methods can be taught to students as soon as they have some available knowledge and the ability to make explicit meta-cognitive activities in a precise way, and to take them as object for thought, and (3) that students benefit from such a teaching. Didactical situations which appear as good 'candidates' for supporting such a methodological strategy involve: work in small groups, open and sufficiently complex problems and a didactical environment giving a large place to students' meta-cognitive activities such as discussion about knowledge and heuristics, and elicitation of meta-cognitive representations on mathematics, problem-solving, on learning and teaching mathematics." (p.534)

The importance of making knowledge of heuristics and self-reflection explicit to students is a theme developed by a number of researchers. In a study specifically on heuristics teaching, van Streun (PME, 1990) concludes that "Explicit attention for heuristic methods and gradual and limited formulating of mathematical concepts and techniques in mathematical education achieve a higher problem-solving ability than implicit attention for heuristic methods and late and little formulating of mathematical concepts and techniques." (p.99) Mason and Davis (PME, 1987) argue for the introduction of particular and specific vocabulary, such as 'being stuck' and 'specialising', in helping the process of learning mathematics. They also claim a connection between "the effectiveness of teachers' discussions of their teaching" and "of students' discussions of their learning" (p.275). This notion of the analogy of the teacher's reflections and the student's will be developed in the final section of this chapter.

Lester and Krol (PME, 1990) examine the "relative effectiveness of various teacher roles in promoting meta-cognitive behaviour in students and the potential value of instruction involving a wide range of types of problem-solving activities" (p.151). These roles are: the teacher as external monitor, as facilitator and as model. They observed the teaching and learning of students in seventh grade classes, and drew the following conclusions:

> "Observation 1: Control processes and awareness of cognitive processes develop concurrently with the development of an understanding of mathematical concepts.
>
> Observation 2: Problem-solving instruction, meta-cognitive instruction in particular, is likely to be most effective when it is provided in a systematically organized manner, on a regular basis, and over a prolonged period of time.

Observation 3: In order for students to view being reflective as important, it is necessary to use evaluation techniques that reward such behaviour.

Observation 4: The specific relationship between teacher roles and student growth as problem solvers remains an open question.

Observation 5: Willingness to be reflective about one's problem-solving is closely linked to one's attitude and beliefs." (p.156-157)

These studies highlight many important issues, such as how to evaluate problem-solving work and how to develop the inner teacher. They also leave open many questions, including what teachers understand by 'problem-solving' and how their interpretation influences the way they teach, and how problem-solving fits into the the whole task of teaching mathematics. Answers to these questions are often implicit unstated assumptions of teachers and researchers, but they are crucial factors. For example, a study designed to examine students' feelings about problem-solving, where that activity is an occasional one in the classroom and when regular textbook-driven mathematical work is set aside, is likely to lead to one set of responses from students. Where that activity is the style of learning that is usually used by the students in their learning of mathematics, one may well find quite different responses. Similarly, concerning teachers' attitudes to problem-solving, the 'hidden messages' conveyed by the teacher of the importance or significance of problem-solving are picked up by students (Lerman, 1989a).

All the studies emphasise making heuristics and reflection explicit to students, which can be expressed in a rich way as the development of the Inner Teacher. It is interesting to reflect on the role of the teacher in this. It has been argued (Kilpatrick, PME, 1987) that the constructivist view of children's learning does not offer anything new in the sense of research methods, or teaching methods. This may be a difficult point to argue when one focuses on childrens' learning of particular content of mathematics, [although both sides are argued (e.g. Kilpatrick, PME, 1987; Steffe and Killion, PME, 1986)] but an interpretation of each student's construction of her own Inner Teacher, in its fullest sense, may be seen as more difficult to formulate outside of a constructivist programme. For example, if one is considering a child learning the procedure of adding two fractions, it could be described as the child constructing concepts and processes that 'fit' the situation, i.e. give the expected answers, make sense in relation to other concepts, can be coherently explained to others etc., or it could be described as the child internalising something taught by the teacher and therefore 'matching' an established schema. Both interpretations could be descriptions of what takes place, and the relative merits of the two descriptions could be argued. From a constructivist view, the 'Inner Teacher' notion will by its very nature have a different meaning for each individual; it has no explicit identity that the teacher could teach, and it will be a meaningful notion to the extent to which it works for the individual. It is a very 'fuzzy' notion, involving specifiable aspects, such as generalising, but also unspecifiable aspects, such as recognising that one is tiring, or stuck, etc. The Inner Teacher notion can of course be specified in a cognitive psychology perspective, and measured against developing schema, as a sort of 'external' teacher replaying its voice in the individual's head. However, if a teacher is considering how to develop the inner teacher in her students, seeing learning as a constructive process may well provide a depth of meaning that other theories do not adequately offer.

In the research literature, one does come across doubts about whether problem-solving methods as such can be 'taught'. Such doubts are expressed often by researchers from outside of mathematics education when investigating mathematical problem-solving. The recent debate in the Journal For Research in Mathematical Education is an illustration of those concerns (Owen and Sweller, 1989, Lawson, 1990, Sweller, 1990). Owen and Sweller (1989) make a case for the lack of evidence of the effectiveness of teaching heuristics. They suggest that the failure of students to use newly learned principles intelligently to solve problems

may not be due to their lack of problem-solving strategies, but what they call "a lack of suitable schemas or rule automation" (p.326). In any event, they claim that there is no evidence that learning problem-solving strategies actually helps in further problem-solving, i.e. are transferable. Lawson (1990) argues to the contrary, that there are small but growing signs of evidence to support the "consideration of different types of problem-solving strategies ... in mathematics classes" (p.409).

In his reply, Sweller (1990) reveals what may be the essence of the differences expressed in the debate, in reviewing the work of Charles and Lester (1984). Sweller is looking for the transfer of problem-solving strategies, and where researchers such as Charles and Lester have found enhanced problem-solving ability after instruction, but have not looked for Sweller's interpretation of transfer, he considers his case reinforced. The issue seems to be what is meant by domain-specific knowledge and skills, and transferability. We would probably argue that the strategies Charles and Lester worked upon, "such as trying simple cases, creating a table, drawing diagrams, looking for patterns, or developing general rules" (Lawson, 1990, p.404) are broad enough to apply to the domain of mathematical problem-solving and at the same time focused enough to be achievable. Since Sweller is dissatisfied with the transferability of these skills, one can only assume that he expects transfer outside of mathematics, to everyday life, or to other school subjects. However, mathematics is a language game, with its own meanings, styles, concepts and so on. One can perhaps articulate a notion of general problem-solving skills in this wide interpretation, but only in the sense of task orientation strategies, or executive strategies (Lawson, 1990, p.404). We would support Lawson in claiming that there is a growing body of research suggesting that within mathematics, problem-solving strategies are a fruitful focus for work in the classroom, and have a positive effect at least on further problem-solving work in mathematics.

In the problem-solving literature, there is some mention of the role of algorithmic methods, that is to say precisely defined procedures for solving problems. A typical example would be solving linear equations by appropriate steps such as 'cross-multiply' or 'take to the other side and change the sign'. Van Streun (PME, 1990) suggests that "Being more successful in problem solving is attended by more frequently employing algorithmic methods" (p.98). Sfard (PME, 1988) discusses the distinction between operational and structural methods:

> "People who think structurally refer to a formally defined entity as if it were a real object, existing outside the human mind. Those who conceive it operationally, speak about a kind of process rather than a static construct." (p.560)

Sfard demonstrates that teaching concepts through operational methods via algorithms can have some success e.g. with the learning of induction. There are dangers in the reliance on algorithmic teaching (Lerman, 1988, Kurth, PME, 1988, Steinbring, 1989) but its role as an heuristic is in need of further research.

Finally, there remains the question of how problem solving should be integrated into the whole mathematics curriculum. To some extent this is a curriculum issue, which may be manifested in the particular textbook or scheme used in the school, and there are many countries and schools where the individual teacher does not have a choice. In those countries and schools where perhaps the content is specified but not the way it is taught, or where there is the goal of a standard to be reached, but the style of work is dependent upon the teacher, how and when problem-solving appears is a matter of choice. As with so many questions in mathematics education this really depends on the teacher's view of the nature of mathematics, and of the process of learning. If the teacher sees mathematics as identified by ways of thought, ways of looking at the world, as processes, then mathematics should be learned through problem solving. If on the other hand, mathematics is thought of as a body of knowledge, a specified amount of which students must acquire, and then apply in problem solving situations, then the teacher will have to find a suitable time and place to teach problem solving processes.

An important element in any discussion about the role of problem-solving in learning mathematics, and one that will perhaps ensure that the debate will remain, is the difficulty of comparison. It is fundamental that if one intends to design some research to compare achievement in mathematics through the two different approaches, one has to make the tests appropriate to the kind of mathematical work. Lester and Krol (PME, 1990) emphasise this issue of the difficulty of comparing different styles of learning in their third observation, precisely because they require different styles of assessment. Again, the most fruitful research may well come from studies by teachers, of children engaged in problem solving work of different kinds, in the context of the learning goals set by the teachers and the school.

The power relationships in the classroom

The classroom is characterised by power struggles and domination. This is not always in the teacher's favour, as most starting teachers will report. Indeed Walkerdine (1989) reports on a situation in which a woman teacher is suddenly dominated by two little boys aged 4 or 5, using sexually abusive language:

> "Annie takes a piece of Lego to add to a construction she is building. Terry tries to take it away from her to use himself ... The teacher tells him to stop and Sean tries to mess up another child's construction. The teacher tells him to stop. Then Sean says: 'Get out of it, Miss Baxter paxter.'
>
> TERRY: Get out of it, knickers Miss Baxter.
>
> SEAN: Get out of it, Miss Baxter paxter.
>
> TERRY: Get out of it, Miss Baxter the knickers paxter knickers, bum.
>
> SEAN: Knickers, shit, bum.
>
> MISS BAXTER: Sean, that's enough. You're being silly.
>
> SEAN: Miss Baxter, knickers, show your knickers.
>
> TERRY: Miss Baxter, show off your bum (they giggle).
>
> MISS BAXTER: I think you're being very silly.
>
> TERRY: Shit, Miss Baxter, shit Miss Baxter.
>
> SEAN: Miss Baxter, show your knickers your bum off.
>
> SEAN: Take all your clothes off, your bra off..." (p.65-66)

Walkerdine adds: "People who have read this transcript have been surprised and shocked to find such young children making explicit sexual references and having so much power over the teacher. What is this power and how is it produced?" (p.66)

In the main, however, the imbalance is in the teacher's favour, and as Bishop (1988) comments:

> "The first and most obvious principle is that the teacher's power and influence must be legitimately used - perhaps we can say that it should be used and not abused" (p.130).

Hoyles (1982) for example gives instances of both:

> "I: What happened then?
>
> P: Well the teacher was always picking on me.
>
> I: Picking on you?
>
> P: Yes, and in one lesson she jumped on me; I wasn't doing anything but she said come to the board and do this sum - fractions it was. My mind went blank. Couldn't do nothing, couldn't even begin.
>
> I: What did you feel then?
>
> P: Awful, shown up. All my mates was laughing at me and calling out. I was stuck there. They thought it was great fun. I felt so stupid I wanted the floor to open up and swallow me. It was easy you know. The teacher kept me there and kept on asking me questions in front of the rest. I just got worse. I can remember sweating all over."
>
> "P. Yes, once, in the second year (and) we had this teacher, she was a really good teacher, maths it was, and I've never been any good at maths. She never pushed you or nothing but let me get on with it at my own pace.
>
> I: What do you mean exactly when you say she never pushed you?

P. Well, she was nice. I had tried and she realised it and didn't keep picking on me. I used to try really hard in her lessons and just get on with it...

I: Can you tell me how you felt during her lessons? What did you feel inside?

P: Well really good, it was really nice to be there." (p.353)

It is of course too simplistic to present the classroom situation as one where the autonomous teacher exercises control, and can choose the power to be benevolent or despotic. The extent to which the discursive practices of the classroom construct the relationships of power is discussed more fully in the section on the language and practices of the classroom below. The relationship to mathematical knowledge is the key factor that will be discussed here.

Mathematics has a special role in society. It is seen as cultural capital for the individual, that is, success in mathematics suggests financial success in the future. It is also used by society to legitimate policies and decisions. Whether it be the billions of dollars of international debt of one nation, or the high rate of inflation of another, the figures become indisputable and can be used to justify policies that make the lives of the majority miserable. This places teachers of mathematics in a unique position, one that can result in reinforcing the powerlessness of the individual in modern society, or can enable people to question and challenge information given. The alternatives can be called 'empowerment' and 'disempowerment'. Cooper (1989) claims that "There is evidence that teachers see mathematics as a crucial subject for reproducing existing social values, and that they modify mathematics curriculum material accordingly" (p.150). Cooper's point is that teachers accept the status quo tacitly, and this results in negative power "where actions that are not in the interest of those in charge are suppressed, thwarted, and prevented from being aired without the elite having to initiate or support any actions or exercise any powers of veto" (p.153). Perhaps the most developed alternative curriculum for mathematics, one that aims to empower students, is that by Frankenstein (1990). She draws on Freire's distinction between a problem-posing curriculum and a 'banking' metaphor, where the individual is thought to store knowledge for retrieval. She offers situations, such as statistical data on employment according to ethnic group in the USA, and invites students to pose questions to investigate mathematically.

Robinson (1989) takes the notion of empowerment into the area of teacher education, and compares two competing models for bringing about teacher-change, the management paradigm and the empowerment paradigm:

> "Rather than seeing change in schools as a finite process with externally specified objectives, as the management paradigm does, the empowerment paradigm sees change as an on-going activity generated within the school by teachers, parents and students as part of an organic process of professional renewal." (p.274)

Teachers are significant people in the lives of children, and the effects of teachers' expectations on individual children is an aspect of the teacher's power that is a focus for research, particularly in the field of gender, ethnicity or class bias. In this context, an important aspect of the impact of teachers' expectations of children is the range of factors to which teachers attribute students' success or failure in mathematics. Fennema et al. (1990), for example, make some strong claims about such impact:

> "(1) A teacher's causal attributions are important because perceptions of why his/her students succeed or fail in achievement situations has an impact on the teacher's expectancies for students' future achievement success.
>
> (2) Teachers' attributions influence students' attributions through teacher behaviour". (p.57)

There has been one study (Kuyper and van der Werf, PME, 1990) reported at a meeting of PME which 'releases' teachers from the responsibility for such influence. They claim:

> " ...the differences in achievement, attitudes and participation cannot be attributed to (characteristics of) individual

math teachers. There is some evidence, however, that the gender of the teacher influences the perception by girls and boys of their teacher's behaviours in a way that might be labelled 'own sex favouritism'. The observed teacher behaviours do not influence this perception." (p.150)

They end their study by suggesting "In our opinion the results fit very nicely into a general pattern, which can be verbalized as follows: *the gender differences in math are not the teacher's fault*" (p.150). Their study involved more than 5800 students and teachers, and clearly the limitations on the size of papers for PME proceedings did not permit a full description of their methodology and results. Their claim that the teacher behaviours do not influence students perceptions, however, is focused on the individual. Some of their own evidence highlights factors that are significant for teachers in general. They found that teachers attribute tidiness more to girls, for example, and industriousness too, whereas 'disturbing order' is more typical of boys in their view. Scott-Hodgetts (PME, 1987) points out that these teacher perceptions and the resulting teacher behaviours have significant cognitive effect, and are not merely affective observations. They serve to reinforce serialist strategies, i.e. a step-by-step approach, which more girls than boys are predisposed to adopt. The result is that more boys tend to develop a broad range of learning styles, whereas more girls remain as serialists. There is substantial evidence in support of the Fennema et al. (1990) claims, as Hoyles et al. (1984) describe (p.26).

This brief discussion is an example of an issue to which we will return in the final section, namely to what extent does research reach teachers and offer them the possibility of changing their work? Some teachers, having read literature that suggests that teachers, irrespective of their gender, spend much more time answering and dealing with the boys in their class than the girls, have been stimulated to examine their own practice (Burton 1986). It should be of some concern if such research remains in learned journals to which most teachers do not have access (Lerman 1990a).

Textbooks

The school mathematics text book is a familiar element in our work, as is well illustrated in Hart's chapter in this volume. However, the teacher's use and dependence on the textbook is a well known phenomenon, but is under-researched in the teaching of mathematics. Laborde (PME, 1987) has looked at students reading texts, and Grouws et al. (1990) mention that the textbook is the most important factor influencing students' attitudes. Van Dormolen (1989) demonstrates how texts can play a role in reflecting real life situations in the classroom, and Frankenstein (1990) takes this further, focusing on the potentially emancipatory role of the material offered in class. But the effect of the dominance of textbooks in mathematics teaching has not been well investigated. When the standard mathematics lesson begins with some initial teacher exposition and is followed by the students working through an exercise in their textbook, and homework is a further exercise from the book, the text, the textbook and the textbook writers constitute an authority in the classroom. Teachers often talk of "they are asking you to..." when attempting to clarify a question that appears in the text, and that students do not understand, and it is interesting to consider what the relationship is between the teacher and the text that is conveyed by this expression.

Social messages that are hidden in texts are unquestioned by teachers and students, partly because the textbook is an illustration and manifestation of the authority implicit in the classroom. This is particularly the case in mathematics, perhaps because the sterile axiomatic form of the presentation of academic mathematics papers reinforces the authority and status of the mathematical text. In the study of history, for example, students are encouraged to draw from a number of sources, highlight the ways in which those textbook writers disagree with each other, and even to explain the context in which different writers hold competing views (this is not the case in all countries of course!). It is quite the opposite in the mathematics classroom. There are also more overt reasons for textbooks being unchallenged in the mathematics classroom,

as mathematics is seen by many teachers as "... a crucial subject for reproducing existing social values" (Cooper, 1989 p.150).

An illustration of the hidden messages of school mathematics textbooks, and a suggestion of ways that teachers could challenge those assumptions, was given by Lerman (1990b). There are different ability level texts in the most popular series in use in British schools, the School Mathematics Project, where the same topic, paying income tax, is dealt with by asking the top ability students to calculate tax on an income of £50,000 whereas the bottom ability text requires calculations for only £9,000. The authors are assuming, and thus are conveying the message to the students that low ability in mathematics, whatever that may mean, correlates with low intelligence, and/or poor career prospects etc. Lerman suggests that the dynamics of the classroom may well change if students were encouraged to critically examine their textbooks, and the examples used, by comparing the pages from the two books. The author has offered such a notion to teachers on an in-service course, and one teacher commented that it would be too dangerous (Lerman, 1990b), perhaps because it threatens the safety of the authority which the textbook establishes in the classroom.

Is the relationship to 'authority' in mathematics any different if the teacher uses a workcard individualised scheme, or individual worksheets? What would be the implications and outcomes of developing materials as and when they are needed, perhaps by the students themselves? Mellin-Olsen (1987) proposes classroom work based on projects:

> "A dormitory town outside Bergen. High, grey concrete blocks of flats. The area was declared ready for occupation as soon as the garages had been built. It was not thought scandalous until later it was discovered that they had forgotten to provide the children with leisure areas.
>
> The teachers of three classes (age group 10) prepared a project. What can we do about the situation?" (p.218)

There are of course implications for the empowerment of students in an example like this, especially when contrasted with a common investigative task on a similar theme, to make a model of a bedroom or classroom. Frankenstein gives many examples of projects for use in mathematics classrooms (Frankenstein 1990). Brown and Dowling (1989) propose what they call a research-based approach as an alternative to the textbook, for teachers to use in the classroom:

> "Our method has been to propose a question - say "who does the best at school?" - as the basis for a research project." (p.37)

These aspects of the use of texts and their influence on students, and the dependence of teachers on texts, is in need of considerable research and investigation. It will be interesting to follow reports of teachers' use of the innovative, perhaps revolutionary ideas of Mellin-Olsen, Frankenstein, Brown and Dowling and others, in changing the function of the text.

Constructivism and the teacher

As a learning theory, constructivism is described in Hart's chapter in this volume, and there are many important research programmes that draw on the paradigm of constructivism, that have been reported in the literature. In this section we are concerned with what the theory might suggest for the role of the teacher.

The Eleventh Annual Conference of PME, held in Montreal in 1987, has been the only such annual meeting to be centred on a theme, that of constructivism. The term 'constructivism' had appeared in earlier meetings, and it was presumably felt that the mathematics education research community in general were unsure of the meaning and of the relevance and/or implications for mathematics education. Consequently that theme was chosen for the PME meeting, to encourage debate on the issue. In his plenary presentation, Kilpatrick (PME, 1987) challenged the claim that constructivism is an alternative paradigm for learning, and suggested that the putative outcomes of constructivist research and teaching are consistent with cognitive psychology. Since that meeting, many papers presented at PME

conferences have been couched in the language of constructivism, and drawn on methodologies of research and teaching, as well as theoretical frameworks, that claim to have distinct interpretations in the constructivist paradigm. The critique, however, has been only infrequently continued (Goldin, PME, 1989), and this is perhaps to the detriment of the debate, since we can only develop our understanding of the links between theory and practice, and in particular alternative learning theories and the consequences for teaching mathematics, through critical discussion.

In relation to the role of the teacher, there have been a number of research reports concerned with the preparation of constructivist teachers (e.g. Simon, PME, 1988). However, the implications of constructivist learning theories for the role of the teacher, and for the kinds of activities that the teacher might initiate as a consequence of a constructivist view of learning, have still not been clearly elaborated, and certainly not well tested, either through theoretical critique or through classroom case studies of teachers. This makes talk of 'constructivist teachers' less meaningful than might be the case were there to be some elaboration of what that implies. There is also the question of the distinction that has been made between 'weak constructivists' and 'strong constructivists', the former being those that subscribe only to the first hypothesis quoted by Kilpatrick (PME, 1987), and the latter being those that subscribe to both:

> "(1) Knowledge is actively constructed by the cognizing subject, not passively received from the environment.
>
> (2) Coming to know is an adaptive process that organizes one's experiential world; it does not discover an independent, pre-existing world outside the mind of the knower." (p.3)

It was suggested at the PME meeting in 1987 that everyone could subscribe to hypothesis (1). There have been some attempts to discuss the implications of both positions. Lerman (1989b) and Scott-Hodgetts and Lerman (PME, 1990) have suggested that it is difficult to see how one can accept the first hypothesis and not the second. They also maintain that the second hypothesis, far from leaving one unable to say anything about anything, as Kilpatrick suggested in his presentation (PME, 1987), is an empowering position, in the sense that ideas are continually open to negotiation and development and we do not expect to arrive at ultimate truths. However, it is clear that more debate and research are needed.

Teachers' beliefs about mathematics

Ideas such as 'the teachers' values', 'what the teacher believes about problem solving' and so on, have occurred often in the review above. The issue of the influence of teachers' beliefs on their actions has been and continues to be of great interest and importance in mathematics education. Nesher (PME, 1988) discusses the effects of teachers' actions and also reveals her own beliefs, in her plenary lecture at the twelfth meeting of PME. In one part of her paper she focuses on the ways that the behaviour of the teacher can have an effect on the child's conception of the nature of mathematics, and she warns of the danger of the teacher being the ultimate judge of whether the child is correct or not:

> "This does not let the child construct for himself the mathematical notions and concepts. Nor does it enable him to realize that the truths of mathematics are objective and necessary." (p.63)

Thom's often quoted claim that "all mathematical pedagogy, even if scarcely coherent, rests on a philosophy of mathematics" (1972, p.204) was one of the factors that stimulated interest amongst educators in epistemologies of mathematics and their influence on teaching, and in the past ten years there have been a number of studies in this area. There have been two different directions of research: a 'top-down approach', so-called because it starts from a consideration of the current state in the philosophy of mathematics and the possible alternative perspectives of the nature of mathematics (e.g. Lerman, 1983; Ernest, 1989), and a 'bottom-up' approach, which begins with teachers' views and behaviour, as in the work of Thompson (1984) for example.

The 'top-down' approach recognises the role of implicit theories, not only in teacher behaviour, curriculum development etc., but also in research perspectives, hypotheses and methods. Thus it aims to reveal the implicit epistemological perspectives which underlie curriculum decisions (e.g. Nickson, 1981), influence the practice of teaching mathematics (e.g. Lerman, 1986), and also determine research directions. As Hersh (1979) wrote:

> "The issue then, is not, what is the best way to teach, but what is mathematics really all about ... controversies about high school teaching cannot be resolved without confronting problems about the nature of mathematics." (p.33)

Scott-Hodgetts (PME, 1987), in her discussion of holist and serialist learning strategies, demonstrates the significance of the teacher's philosophy of mathematics to the extension of various learning strategies. In their study of mathematicians and mathematics teachers views, Scott-Hodgetts and Lerman (PME, 1990) set their analysis of psychological/philosophical beliefs in a radical constructivist perspective, and propose that it provides a powerful explanatory potential:

> "What A *has* been engaged in, without doubt, is what is described as "an adaptive process that organises one's experiential world". In doing this he has brought to bear different models of the nature of mathematics, picking and choosing in order best to 'fit' his particular experiences at different times." (p.204)

Siemon (1989a) suggests that "what we do ... is very much dependent on what we know and believe about mathematics, about the teaching and learning of mathematics, and about the nature of our particular task as mathematics educators" (p.98). Dougherty (PME, 1990), in an on-going study of the influence of teachers' beliefs on problem-solving instruction, characterises teachers' cognitive levels, a set of psychological attributes, on a concrete-abstract continuum, from A, the most rigid, concrete formalism, through B, social pluralism, and C, integrated pluralism, to D abstract constructivism. In interviews and observations of 11 teachers, all on problem-solving lessons so as to standardise the lesson content, Dougherty placed 8 in A and 1 in each of the other categories, and she found strong connections between these positions and conceptions of problem solving.

Methods of analysing and interpreting teachers' actions in the classroom, that engage with the issue of teachers' beliefs about mathematics, have been reported in the literature. Jaworski (PME, 1989) discusses students developing a 'principled' understanding of mathematical concepts, when comparing the inculcation of knowledge, as against its elicitation. She says:

> "Successful teaching of mathematics involves a teacher in intentionally and effectively assisting pupils to construe, or make sense of mathematical topics." (p.147)

She demonstrates the dangers of 'ritual knowledge' in a wonderful example of the reasoning of a child who regularly achieves good marks in mathematics.

> "I know what to do by looking at the examples. If there are two numbers I subtract. If there are lots of numbers I add. If there are just two numbers and one is smaller than the other it is a hard problem. I divide to see if it comes out even and if it doesn't I multiply." (p.148)

Ainley (PME, 1988) has looked at teachers' questioning styles, noting the power relationships, by analogy with the parent/child or drill sergeant/recruit. She suggests that assuming mathematics answers are right or wrong is a danger for the teacher in using questioning, and that teachers assume that questioning is better than exposition. In her study, she looked at the different perceptions of the children and the teachers, of the teachers' questions. She classified questions into: pseudo-questions; genuine questions; testing questions, and directing questions, the latter sub-divided into structuring, opening-up and checking. Her focus on the language of the classroom is further discussed in the following section.

Various studies show that there are a number of variables that mediate, in an investigation of the influences of teachers' beliefs. Thompson (1984) and Lerman (1986)

and other researchers report how the ethos of the school is a major element in the teacher's approach in the classroom. Siemon (1989b) illustrates this with comments from teachers who, when asked "What are the main pressures on you as a mathematics teacher?" replied:

> "School policy...you can't just be the one to break the system ... if you give all your students an A you will be questioned on it...you can't. You've got to work within the system as such.
>
> The kids have their activity book. If the parents look at that at the end of the year and they haven't done one activity ... they're going to question it and the Principal's going to question whether you are following the program." (p.261)

Noss et al. (PME, 1990), in discussing a study of teachers on an in-service course in computer-based mathematics learning, comment that:

> "... the extent to which a participant was able to integrate the computer into his or her mathematical pedagogy (theoretically and/or practically) appeared more related to the direction in which a participant's thinking was already developing and with his or her commitment to change, rather than the style of teaching approach, view of mathematical activity, or rationale for attending the course." (p.180)

A global view of factors affecting the teaching of mathematics reveals clearly the major influence of the conditions and resources in the schools in different countries. Nebres (1989) calls these the Macro problems, as against the Micro problems which are internal to mathematics education:

> "... in many developing countries, the problems that merit most thought and research are those due to pressures from outside society. The purpose of study regarding these pressures is to provide some scope and freedom for the educational system so that it can attend to the internal problems of mathematics education." (p.12)

Nebres emphasises the importance of the role of cultural values, in understanding both the Macro and Micro environments in mathematics education, and suggests that his awareness of that perspective is focused by the "sharp contrast of traditions and cultures in East Asia" (p.20). He writes:

> "... we have realized that questions like the status and salaries of teachers are not simply a function of the economy but also of cultural values. Similarly the rise of what we might call the internal force of genius, wherein mathematics talent springs up even under difficult economic circumstances, seems to be fostered by the cultural environment." (p.20)

Suffolk (1989) gives a detailed study of the situation in Zambia, as seen through the role of the teacher. Most such studies, quite naturally, focus either on the children or on the situation of the schools, and more studies of the effect on the teacher are needed. One approach, as a consequence of a recognition of these variables, as reported by Nolder (PME, 1990), is to undertake a detailed analysis of the mathematics teachers in one particular school which is undergoing some curriculum changes, in an attempt to reveal the nature of these influences. She found, for instance, that such changes led to the teachers experiencing a good deal of uncertainty, which affected their perceptions of their competence and confidence. Stephens et al. (1989) report on mathematics education reform programmes, and state the following principle for involving teachers:

> "Teachers involved in change initiatives need to know where they have come from, where they are going, why they are going there, and should have access both to the best professional thinking of their colleagues, and to theoretically sound, supportive environments." (p.245)

These studies indicate that 'change' is always socially embedded, and cannot be isolated from the cultural context, from the political milieu, from the social context of schools and certainly not from the teachers in the classroom. Ignoring any of these elements mitigates against any improvement in the teaching of mathematics.

A stimulus in the development of a social perspective of the nature of mathematics has come from teachers working in situations of deprivation and/or oppression. If one tends to think of mathematics as value free, and much the same all over the world, the work of some

of these researchers and teachers provide a strong case for rethinking. Fasheh (1989), in describing mathematics education in schools on the West Bank in Israel, argues that what matters is "... whether math is being learned and taught within the perspective of perceiving education as praxis or within the perspective of perceiving it as hegemony" (p.84). Gerdes (1985) demonstrates how mathematics education in Mozambique changed from pre-revolutionary colonial domination to post-independence construction of a socialist society:

> "During the Portuguese domination, mathematics was taught, in the interest of colonial capitalism, only to a small minority of African children ... to be able to calculate better the hut tax to be paid and the compulsory quota of cotton every family had to produce ... to be more lucrative "boss-boys" in South African mines ... Post-independence ... Mathematics is taught to "serve the liberation and peaceful progress of the people". (p.15)

D'Ambrosio (1985), in developing the idea of ethnomathematics, sets the growth of mathematics in a social context, and aligns the development of modern science and, in particular, mathematics and technology with colonialism. There is a growing body of work in the area of ethnomathematics, at the macro level and at the micro level, with profound implications for teachers in mathematics classrooms, both in the so-called developed world, and also the under-privileged world, that engages with ideas of empowerment and liberation.

The language and the practices of the classroom

Recently, there has been an interest in the discursive practices of mathematics teaching (e.g. Keitel, et al., 1989; Weinberg and Gavalek, PME, 1987), influenced in the main by the work of Walkerdine (1988, 1989, PME, 1990). The focus of this critique is a shift away from the individual to the social, and is thus a critique of psychology, which can be characterised as an individualistic way of interpreting people's actions. Investigating the interactions in the classroom (aspects such as discussion between students, teachers' questions, open-ended questions, the teacher's response to novel interventions from students), leads one rapidly into aspects of the power relationships of the classroom. Pimm (1984), for example, talks about the use of the word 'we', and indicates just how much is actually being said and construed and structured by the use of that small word:

> "Given a class of pupils all doing something incorrectly, a teacher can still say, 'No, what we do is ...' Who is the community to whom the teacher is appealing in order to provide the authority to impose the practices which are about to be exemplified?" (p.40)

Of course the dominant paradigm within which research, testing, and theorising on the meaning of understanding takes place is a psychological one. Understanding is seen as some process that the individual undergoes, some transition from not understanding a concept to understanding that concept. There are two elements in this transformation from a pre-understanding state to a state of understanding, the individual and the concept. The latter is seen as an objective, external element which in the educational environment is usually 'possessed' or held in some way, by the teacher. The teacher is seen to perform the intermediary role of introducing the concept, attempting to enable and facilitate its transmission from teacher to student, and measuring the outcome, - that is whether the state of 'understanding' has been achieved. The former element, the individual, is largely a closed unit to the teacher, making the business of determining whether the transformation has taken place or not, a very difficult one. As Balacheff (1990) points out:

> "It is not possible to make a direct observation of pupils' conceptions related to a given mathematical concept; one can only infer them from the observation of pupils' behaviours in specific tasks, which is one of the more difficult methodological problems we have to face." (p.262)

Research in this area in education thus focuses either on interpretations of behaviour that might constitute evidence of the processes of concept acquisition, or on the conditions that will bring about 'understanding'.

Mathematics is seen as hierarchical, and conflated with this is the notion that learning mathematics is hierarchical too, from the basic notions of one-to-one correspondence, setting and combining sets, to more abstract levels of, for example, the calculus. It is seen by educationalists as 'very difficult' (Hart, 1981). Other research shows, however, that children's unrestricted and undirected thinking does not follow the supposed hierarchy (e.g. Lave, 1988; Carraher, PME, 1987). It may just be the case that our assumptions about the nature of mathematics and the nature of the learning process become self-fulfilling. We interpret, theorise, teach, test, assume, expect, measure and thus confirm our initial expectations of children.

The psychological model of learning, which can be characterised as a process of transformation from a pre-understanding state to a state of understanding, determines the style and intention of the various stages of the educational process. The process begins with the teacher assessing the initial cognitive state of the individual. This is followed by the prepared lesson, whose content is determined by research into hierarchies of mathematical knowledge and levels of difficulty of understanding, and whose method is determined by large-scale tests using well-established quantitative procedures. Research on affective factors influence the classroom setting, the appearance of the text, etc. On-going assessment enables the identification of misconceptions, misunderstandings or partial understandings. These can be corrected by suitable re-explanations, reinforcement materials and so on. The final stage is the test that measures the successful acquisition of the particular skill or concept, that is interpreted as understanding, and again that test will have been trialled over large groups of students, in well-established ways.

Whilst the above is somewhat of a caricature, the intention is to highlight the manner in which the psychological paradigm dominates and determines the whole structure of the educational process. It is the essentially private nature of the psychological interpretation of understanding that pervades the educational process from this perspective. Teachers are left to try as best they can to recognise when that process has occurred, and in the case of the recent development of a National Curriculum in Britain, with a long list of attainment targets and levels to be assessed, tick the appropriate box to register that achievement target for the student.

At the same time, there is an assumption, usually implicit, that we all arrive at the same understanding of particular *things*, because of the absolute nature of those *things*, whether physical objects, such as tables, and unicorns, or mental objects such as circles, or 'three'. A critical analysis, therefore, needs to work on both aspects of the understanding process in the psychological paradigm, namely the notion of the individual, and also the notion of knowledge and its posited absolutist nature.

In the psychological paradigm, the major focus of attention is in how the individual, the *subject*, functions in discourses. However, *subject* is simultaneously the apparently autonomous self-constituted individual who articulates a discourse and also the *subject of* that discourse. That is to say, the individual is constituted by a discourse. Thus, for example, in the classroom we are simultaneously the teacher, who initiates and perpetuates a discourse, and at the same time we are constituted by that discourse, and by the relations of power that pertain. To some extent this has been looked at above, specifically related to the power of the teacher. It may appear, though, from that section, that teachers consciously choose positions of power and can reject them. In analysing the discourse of the classroom one can begin to see the subtle historically-rooted relationships and meanings carried by the interactions in the classroom. Pimm's (1984) discussion of the use of 'we', and the position in which the accepted and unchallenged role of the mathematics textbook places the teacher and students are examples of the insights that can be gained from an analysis of the discursive practices of the classroom.

An analysis of how languages operate within discursive practices indicates the way in which knowledge, understanding, meaning and ourselves as subjects are constituted in and

bounded by our language, and that language is specific to discursive practices. Meaning is about use, no less but also no more. In Walkerdine's analysis of the home practices in which the notions of 'more' and 'less' occur (Walkerdine, 1988), she demonstrates the context in which those words have meaning for those children, and the practices through which those meanings are constructed. The classroom teacher who is unaware of the meanings the children are bringing with them into the classroom may well make judgements about children's 'abilities' that are entirely inappropriate.

The alternative view of 'understanding' is one that sees the process as social, rather than psychological. That is to say, the individual does not go through some transformation from *pre* to *post* state of understanding in isolation, acquiring a given well-formed concept in some hidden way. It is necessarily an interaction of the child and his/her multitude of meanings, through language, with others and with experiences. That process of interaction may be largely silent, a shift taking place without being openly articulated, but this does not mean that it is in some way 'private'. Sometimes the understandings do not fit with those of others, including the teacher, and only talk, discussion, suggestions and conjectures and refutations, through cognitive conflict, or shifts of thought through resonance, enable further growth.

In the literature of mathematics education, one finds an emphasis on providing a context for mathematical concepts, and clearly this is essential if mathematics is to be about something. There are two aspects to 'context', however. It is a matter, not only of attempting to embed mathematics in situations that may be 'relevant' or 'meaningful' for students, but also an analysis of the discursive practices within which mathematical terms and concepts appear, both for students in their own experiences and for the mathematical community.

The language used in the mathematics classroom, particularly in relation to bilingualism, has been an important focus of study for a number of years (Dawe, 1983; Zepp, 1989). Most such studies have been concerned with children, and their learning. Once again, the role of the teacher, and here particularly the language of the teacher, has only recently become an issue, as researchers and teachers have come to realise the function that language serves in deep aspects of classroom interactions. Khisty et al. (PME, 1990), for example, reported on an exploratory study of the language of the teacher, in a bilingual classroom, to investigate the reasons for the underachievement of Hispanic students in the USA. They review the language factors that might play a part, emphasising an interactionist perspective, which attempts to bring together the cognitive and social dimensions of learning. They refer to how the use of language structures the classroom, the development of a mathematics register, syntactic and semantic structures in mathematics, and context. They conclude their introduction as follows:

> "The importance of these points is that language issues in the teaching and learning of mathematics may be more crucial than previous research would suggest." (p.107)

As reported above, Ainley (PME, 1988) has looked at the implications of teachers' questions, and revealed both a wide range of purposes, and also a discrepancy between children's perceptions of the teacher's question and that of the teacher. Wood (PME, 1990) focused on discussion in the classroom, and highlights the all too frequent consequences of what may be called 'teacher-centred' learning:

> "The patterns of interaction become routinized in such a way that the students do not need to think about mathematical meaning, but instead focus their attention on making sense of the teacher's directives." (p.148)

In a different direction of attention, Bliss and Sakonidis (PME, 1988) looked at the technical vocabulary of algebra, and the everyday language that is sometimes substituted for these terms, in an attempt to discover whether it helps or hinders communication of algebraic ideas.

Analyses of the language of the classroom also make connections between the concepts

within the 'language game' of mathematics and in everyday use. Pimm (PME, 1990) discusses metonymic and metaphoric connections, and in doing so highlights a central aspect of the ways mathematical language and thought work. For example, metaphoric links are those which would highlight the meaning of 50% in different contexts, whereas metonymic links would be those between 50% and 0.5 and 1/2 and 4/8 etc.

> "Being aware of structure is one part of being a mathematician. Algebraic manipulation can allow some new property to be apprehended that was not visible before - the transformation was not made on the meaning, but only on the symbols - and that can be very powerful. Where are we to look for meaning? Mathematics is at least as much in the relationships as in the objects, but we tend to see (and look for) the objects ...
>
> Part of what I am arguing for is a far broader concept of mathematical meaning, one that embraces both of these aspects (the metaphoric and the metonymic foci for mathematical activity) and the relative independence of these two aspects of mathematical meaning with respect to acquisition." (p.134-135)

One can expect significant further research in the deconstruction and reconstruction of the language and practices of the mathematics classroom.

Teacher education

Many educational researchers in most countries of the world are involved in teacher education, either pre-service or in-service or both, and many, if not all of the issues concerning teaching apply equally to teacher educators. Students on a SummerMath programme were reported by Schifter (PME, 1990) as demanding: "We need a math course taught the way you're teaching us to teach." (p.198) Jaji et al. (1985), in an article whose main focus is on providing pre-service teachers in Zimbabwe with a repertoire of roles from which to choose, rather than replicate the ways in which they were taught, offer this transcript of a teacher educator in action:

> "Now what I am about to say is very important. It will almost certainly come up in the Examinations, so I suggest that you write it down.
>
> (The group took out their pens ...).
>
> In the new, modern approach to teaching ...
>
> (They wrote it down as she spoke ...).
>
> In the new, modern approach to teaching, we, as teachers, no longer dictate notes to children. Instead we arrange resources in such a way as to enable children to discover things for themselves." (p.153)

If we are to suggest that teachers should become involved with research on their practice, as will be discussed below, teacher educators are teachers too and one would expect that teacher educators would be involved in research on their own practice. One can in fact see evidence of a small but growing interest in research on teacher education in recent meetings of PME.

A further consequence of the analogy between students in schools learning mathematics and research on that, and student teachers learning to teach mathematics and research in that area, is that just as we cannot talk of making students learn mathematics, so we cannot talk of making teachers change their teaching. One does however come across such discourse in the literature, in particular in relation to in-service courses. As teacher educators we can offer ideas and possibilities, with activities that encourage reflection and awareness and personal growth (Jaworski and Gates, PME, 1987; Lerman and Scott-Hodgetts, PME, 1991, in press), but to try to impose teacher change is both unrealistic and perhaps arrogant.

There are other reasons for reflecting on the practice of teacher education. In Britain, and probably many other parts of the world too, teacher education programmes are at a turning point, under the influence of political pressures. There is a strongly held view that teachers can best learn "on the job" rather than in college, with the metaphor of apprenticeship in training held to be more appropriate than the combination of theory and practice that might be implied by the metaphor of a profession. According to this view, at secondary age in Britain, (students

aged 11 to 18) anyone can teach, provided one has a degree in a suitable subject, given an appropriate system of apprenticeship. That such a system is vastly less expensive than spending between one and four years in a college is almost certainly the main but unstated rationale. Whatever the motives, teachers and teacher-educators must be clear about the nature of the process of teaching, and therefore of teacher education. There is no doubt that one can learn a great deal by watching other teachers, and not necessarily 'master-teachers', which is a designation used in some countries. Given a theoretical framework in which to critically evaluate and assimilate one's observations, observing others can be an essential and valuable aspect of learning to become a teacher. But learning to become a successful teacher is not a matter of copying someone who has been designated an expert. It is a matter of finding one's own way of teaching, developing ways in which one can identify what is happening in the classroom, and of drawing on experience and theory to make decisions on action, followed by reflective evaluation of those actions. There are frequently conflicting demands on the teacher (Underhill, PME, 1990), and research on factors that help prospective teachers to develop their own understanding of teaching (e.g. Brown, PME, 1986; Dionne, PME, 1987) are needed.

In what follows we will look first at the mathematical content of teacher education courses, and the content/process debate as it manifests itself at this level. A brief review of some of the main issues that arise in the development of in-service courses appropriate to the needs of teachers will then be followed by a development of the notions of the teacher as reflective practitioner and as researcher.

There have been a number of studies that have focused on prospective teachers' or practising teachers' misconceptions of topics in mathematics. Concerning teachers of elementary age children, Linchevski and Vinner (PME, 1988) looked at naive concepts of sets, Vinner and Linchevski (PME, 1988) on the understanding of the relationship between division and multiplication, Tierney et al. (PME, 1990) on area, Simon (PME, 1990) on division and Harel and Martin (PME, 1986) on the concept of proof held by elementary teachers. In the high school age range, Even (PME, 1990) has examined misconceptions in understanding of functions.

What are we to make of these findings, and what might be the possibilities for improving the situation? It is probably the case that most people have negative experiences of learning mathematics at school, and as in the main, prospective elementary teachers are not mathematics specialists, they too are generally at least unconfident about mathematics. If pre-service courses teach mathematics in the same way that those people learned mathematics at schools, patterns of lack of confidence, under-achievement and feelings of failure are likely to remain into their teaching career (Blundell et al., 1989). Since children's early experiences of mathematics occur in interaction with elementary teachers, this is a major problem for mathematics education. Ways of breaking this cycle must clearly be developed, and one significant element, as mentioned above, is for students to be taught the way that we expect them to teach. Schifter (PME, 1990), who came across that demand from her students, reports on a programme of in-service education which attempts to effect such a transformation:

> "... the notion of "mathematics content" as the familiar sequence of curricular topics is reconceived as "mathematics process": at once the active construction of some mathematical concepts - e.g. fractions, exponents - and reflection on both cognitive and affective aspects of that activity. The work of the course is organized around experiences of mathematical exploration, selected readings, and, perhaps most importantly, journal keeping." (p.191)

The journal writing is a most effective way of demonstrating the changes that the teachers felt themselves to have gone through, during the course. The main intention of the journals, however, is to encourage and enable the reflective process that is an essential component of breaking out of the repetitive cycle of lack of success. It is not that such reflection will guarantee that no misconceptions in mathematical understanding will occur, but that where

cognitive conflicts arise in teachers' experience of mathematical activities and concepts, or a recognition of aspects of mathematics that they do not understand, they are more likely to be able to confront them, rather than remain with their misconceptions.

Blundell et al. (1989) also draw on students' reflections as expressed in journals, in their evaluation of the nature of their students' experiences and learning, through the course. In this study the course was a pre-service one for prospective primary (elementary) school teachers, and as in the Schifter (PME, 1990) study, the focus was on mathematical processes and learning processes. They describe the aims as follows:

> "The fundamental belief which underlies this course structure is that a recognition and mastery of these general concepts and strategies will enable students to approach *any* new content area with confidence and competence, and that the efficient acquisition of *meaningful* mathematical knowledge is thus facilitated. Furthermore, this acquisition is not dependent upon the skills of the "teacher", but rather upon those of the "learner" - leading to much greater independence and self sufficiency." (p.27)

Thus the other aspect of 'being taught the way we want them to teach' is that when studen. teachers focus on their own learning reflectively, during the course, they become aware of what their school students are going through in their learning too (Waxman & Zelman, PME, 1987). As the student teachers gain confidence, independence and self sufficiency, they can recognise the potential for their own school students to do the same. The students commented on their early learning experiences, and then on their recent growth as mathematics learners:

> "The only maths teacher I remember, although only vaguely, is the one who kindly allowed me to spend most of my time outside the classroom door ..."

> "September 1965. I hate maths ... I recall heading a page of my maths book "funny sums". For this I was sent to stand outside the class, told I was a "bloody imbecile" and that I was the worst student he had ever had the misfortune to teach ..." (p.27)

> "We're never afraid of anything now are we? Whatever you give us we'll jump in and have a go."

> "I really love doing maths. I really look forward to Tuesdays and Fridays. All this time I've been going on thinking I'm no good at maths - and now I'm doing it." (p.29)

The authors reflect on their own learning as teacher educators, from this analysis:

> "Clearly the students have gained in confidence by this stage of the course, but it is not yet an unequivocal confidence. It seems as though they are still ambivalent, feeling confident when they focus on their process abilities, but doubting when concentrating on content in mathe-matics ... However, they do not overtly acknowledge and value the acquisition of process skills to the extent to which we might have hoped." (p.30)

Journal writing as an empowering activity in reflection, whether it be on one's mathematical work or on one's learning during a course is itself the subject of study (e.g. Brandau, PME, 1988; Hoffman and Powell, 1989; Frankenstein and Powell, 1989).

In-service courses provide the opportunity for teachers to meet with others, reflect on their experiences, re-interpret their work in the light of theories which they may meet during the course and develop areas of their expertise. Fahmy and Fayez (1985) attempt to outline guidelines for in-service education based on their work in Egypt. They emphasise the professionalization of teacher education, orientation towards self-education and research activities and careful consideration of the mathematical needs of the teachers. There are situations in which the in-service programme has to be geared towards major curriculum developments, and thus are firmly focused on these needs. Xiangming et al. (1985) describe a network of in-service courses in institutions throughout China to implement a 12 year programme from a 10-year one in schools. Similarly, Roberts (1984) describes a programme of school-based teacher education in Swaziland, balanced with in-college sessions, and Vila and Lima (1984) a distance-learning programme in Brazil. Eshun et al. (1984) pick up on the problems

that these latter experience, and emphasise the constraints for in-service courses for primary teachers that result from the large numbers involved, and the few higher education institutions that can provide that support.

An increasingly recurring theme in these and other literature concerning teacher education is the notion of teachers as 'reflective practitioners'. Lerman and Scott-Hodgetts (PME, 1991, in press) discuss Schön's characterisation of the nature of teaching and develop his interpretation:

> "Schön ... maintains that teaching is reflective practice, in his discussion of the unsatisfactory distinction drawn between theoreticians and practitioners.
>
> In describing his notion of reflection-in-action, Schön ... starts from what he calls the practitioner's knowledge-in-action:
>
> "It can be seen as consisting of strategies of action, understanding of phenomena, ways of framing the problematic situations encountered in day-to-day experience."
>
> When surprises occur, leading to "uncertainty, uniqueness, value-conflict", the practitioner calls on what Schön terms reflection-in-action, a questioning and criticising function, leading to on the spot decision-making, which is "at least in some degree conscious".
>
> In our situation, we would suggest that we are extending the idea of the reflective practitioner. We would agree with Schön that recognising that a teacher is much more than a practitioner is essential, both for teachers themselves (and ourselves), and for teacher educators, administrators and others. However, when we use the term reflective practitioner, we are also describing meta-cognitive processes of, for instance, recording those special incidents for later evaluation and self-criticism, leading to action research; consciously sharpening one's attention in order to notice more incidents; finding one's experiences resonating with others, and/or the literature, and so on. We are concerned with the transition from the reflective practitioner in Schön's sense, to the researcher (Scott-Hodgetts 1990), who has a developed critical attention, noticing interesting and significant incidents, and turning these into research questions. By analogy with some recent views of the nature of mathematics (e.g. Lerman 1986; Scott-Hodgetts PME, 1987), 'Mathematics Education' is most usefully seen, not as a body of external knowledge, recorded in articles, papers and books, that one reads and uses, but as an accumulation of work upon which a teacher can critically draw, to engage with those questions that concern and interest her/him (Scott-Hodgetts, in press, a)."

The notion of teacher as researcher, although around the education world for some years, has only recently appeared in the literature of mathematics education (e.g. ATM 1987; Scott-Hodgetts 1988; Fraser et al., 1989). An early statement of the potential of bringing together research and teaching was given by the Theme Group on the Professional Life of Teachers at the Fifth International Congress on Mathematical Education (Cooney et al., 1986):

> "It was emphasised that the desirability of having teachers participate in research activities may extend beyond the question of what constitutes research and relate to the professional development of the teacher by virtue of engaging in such activity. That is, not only can the teacher contribute to the creation of grounded theory but the teacher can mature and elevate his/her own expectations and professional aspirations by participating in a reflective research process." (p.149)

Research is often seen as the prerogative of full-time people based in institutions of higher education. This is to misunderstand the nature of teaching and the educational experience. There is no doubt that full-time researchers have had the opportunity to develop skills and perspectives in research that class teachers have not had, but the separation of research from practice is one of the major factors that leads to suspicion of researchers by teachers, and the well-known phenomenon mentioned early in this chapter, that the findings of research do not often reach teachers (Scott-Hodgetts, 1988; Lerman, 1990a). In a comparison study of teachers' perceptions of their classrooms, and pupil perceptions of the same classrooms, Carmeli et al. (PME, 1989) found that there were statistically significant differences, with the teachers being far more positive

that their pupils. There was no mention in the report of whether the teachers were shown the outcome of the study, and what effect that may have had on their own practice, although that may of course have taken place later. Research on teachers, carried out by outsiders, and then reported in the academic literature, is of much less value than research in which the teachers are involved, and which they can use in their own growth as teachers.

Conclusion

There are of course many questions about the influence of the role of the teacher in childrens' learning of mathematics, and in this chapter we have described attempts to engage with some of those questions, and answer them. Two fundamental problems remain, which were referred to in the introduction to this chapter: what do 'answers' look like, in this domain, and what kinds of research methodologies are most suitable to investigate those questions?

As with so many issues in mathematics education, the place to commence this discussion is with attitudes and beliefs. For example, one of the most systematic and developed methodologies for examining the practices of the teacher, in order to achieve children's understanding of mathematics is that developed by the group engaged in *recherches en didactique des mathématiques* in France. It is interesting to examine the assumptions about the nature of classroom interactions that underpin their system, and to see how the alternative view described in this chapter offers another methodological approach. Brousseau, Chevallard, Balacheff and others have developed notions of the didactical process, theory of didactical situations, didactical transposition etc. Balacheff (1990) describes his basic assumptions and the fundamental problem, as he conceptualizes it:

> "So if pupils' conceptions have all the properties of an item of knowledge, we have to recognize that it might be because they have a domain of validity. These conceptions have not been taught as such, but it appears that what has been taught opens the possibility for their existence. Thus, the question is to know whether it is possible to avoid a priori any possibility of pupils constructing unintended conceptions...
>
> I have suggested that pupils' unintended conceptions can be understood as properties of the content to be taught or the way that it is taught." (p.262-263)

This approach to research on teaching is a positivist one, which anticipates the possibility of solving these problems. It is one which focuses on the individual's construction, in the light of the content and what the teacher does. At the same time, it appears that the construction is to be of an object that exists in some a priori sense. The methodology that Balacheff describes assumes that it is possible to specify all the conditions of the classroom, the mathematical content, and so on. Brousseau (1984) recognises the enormity of the task and despairs of the possibility of success:

> "Research into didactics is itself doomed to naiveté and/or failure, since it is becoming more and more difficult, as the body of knowledge concerned with didactics grows steadily more complicated and technical, for a good student of mathematics to devote enough time to absorbing it before he sets out to break fresh ground in this field." (p.250)

A methodology of research on teaching would look different from a perspective which focuses on the construction of knowledge within an analysis of discursive practices, knowledge as signs that are slippery, not fixed, that is signs that appear clear and certain, but shift when analysed and then shift again. If we cannot specify all aspects of the teaching situation in advance, it is essentially because the actors in that setting bring with them their cultural experiences, in the widest sense. These experiences are not 'private', just not completely specifiable. And of course the classroom itself is a whole new social situation with all its own practices. As a consequence, the teaching process is best seen as one which creates situations in which students construct their own knowledge, and where students' constructions are valued, even if they are unintended. There are conceptions and misconceptions, but there are also partial-conceptions and other-conceptions

too. Valuing these contributions recognises that learning is what an individual does for herself, not something the teacher does to the student, and also places responsibility on the student for her own learning. It also avoids the student dependence on the teacher as judge, and enables the development of self-critical skills in students. All conceptions offered can be brought into the 'objective' arena of the classroom for examination, to search for 'proofs and refutations'.

In the context of research methods, one learns something important about mathematics teaching when one discovers that in a survey of 32 teachers and 1338 pupils the teachers had a significantly more positive view of a number of elements of the classroom environment (Carmeli et al., PME, 1989). One learns much more, perhaps, in a case study of a teacher who takes that information and attempts to modify and develop the relationships, communication, discussion and other interactions in her classroom, and develop ways of evaluating the changes and the consequences for herself and her students, through action research. Thus, qualitative research is perhaps most useful when it provides insights that stimulate research by teachers themselves, and where the generalisability is found through other teachers and researchers recognising, and reflecting on, the implications for their own work. That is to say, 'proof' by resonance, where an idea or an analysis of a classroom experience or event 'resonates' with the experience of the reader or listener, may be more fruitful and more widely applicable than 'proof' by significance testing, at least in relation to teaching.

As mentioned above, there are many questions for research and investigation about teaching that have been identified in this chapter that can generate valuable insights into the teaching of mathematics. Some recommendations of important areas for research and investigation follow, and pursuing the theme of the final section, there is an overlap of the concerns of the teacher, the researcher and the teacher educator. These concerns will be emphasised by people according to their needs and interests as well as circumstances, rather than their particular task in mathematics education. As they have been elaborated in the body of the chapter, they will be merely listed here.:

(1) the influence of the language and the practices of the mathematics classroom on learning;

(2) ways of rooting the mathematical activities of the classroom in the experiences of the children, whilst at the same time finding ways of drawing them into mathematical discourse;

(3) identifying teachers' beliefs and the influence of those beliefs on practice. There are methodological problems, such as the impact of society's demands, the particular school ethos, mathematics education jargon, and there are theoretical frameworks being developed in which to set such studies;

(4) developing the Inner Teacher, the meta-cognitive functioning that is an essential part of learning and mathematising;

(5) the nature of problem-solving and problem-posing, the influence of teachers' perceptions of problem-solving, the teaching of problem-solving processes and the relationship between the teacher's role and student growth as problem-solvers;

(6) classroom-based studies focusing on how teachers define and attempt to realize the goals that they set, through action research;

(7) investigations on the implications of alternative learning theories for teaching and teacher education;

(8) the role of the textbook in the social relations of the classroom, and the relationships to mathematical knowledge, and alternative forms of text;

(9) the study of the practice of teacher education in an analogous way to that of the teacher in the classroom.

The mathematics classroom is a confusing but rich environment. Students bring their worlds into it, and express those worlds through language. Our role as teachers is to enable them to interact with and integrate the language game of mathematics, in a

manner that empowers rather than disempowers, and that enables them to continue to keep mathematics as their own, not to see it as belonging to us, the teachers and the authority. It can be maintained that not being able to arrive at absolute and certain answers to how that can be done, makes the science of mathematics teaching just like all sciences, an open-ended and exciting process of experimentation and theory-building.

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