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ABSTRACT

This study examines the construct of problem representation and the processes used by learners to construct or modify problem representations in problem-solving situations. Students (n=14) from freshman calculus courses at the University of California at San Diego participated in videotaped interviews in which they were asked to think aloud as they encountered dilemmas in solving certain tasks. The videotape protocols for each participant were analyzed and results were reported in the form of case studies. The case studies were considered as a group for the purpose of generalizing results. Three sections summarize the findings of the study. First, the levels of solution activity that the students were inferred to achieve during the interview are summarized. Second, the results of a pair of case studies are presented to illustrate individual differences in solvers' ability to construct representations. Third, the results are discussed in more general terms, including a comparison to other research findings about representation. Results indicate that: (1) traditional views of representation need to be reconsidered, (2) the process of representation appears more dynamic than previously thought, and (3) the solvers' use of increasingly abstract levels of solution activity suggests the need to address qualitative aspects of mathematical performance. Contains 23 references. (MDH)

Representation Processes in Mathematical Problem Solving

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REPRESENTATION PROCESSES IN MATHEMATICAL PROBLEM SOLVING

INTRODUCTION

Many theoretical accounts of mathematics learning include a prominent role for mental representation processes (Greeno, 1980; von Glasersfeld, 1987; Yackel, 1984). In these studies representation processes relate to the learner's ability to develop an understanding of the situation or task at hand.

Mental representations have been used to describe problem solving processes in mathematics. Specifically, a problem representation is "a cognitive structure which is constructed by a solver when interpreting a problem" (Yackel, 1984). In particular, the construct of problem representation has played a central role in describing the knowledge that learners bring to mathematical problem solving situations (Chi, et. al., 1982; Larkin, 1983; Hinsley, et. al., 1977; Mayer, 1985). Research suggests that the success of capable problem solvers may be due in large part to their ability to construct appropriate problem representations in problem solving situations to use as aids for understanding the information and relationships of the situation at hand.

Despite agreement concerning the importance of solvers developing internal representations for use in problem solving situations, the cognitive studies that have been undertaken have seldom focused on the ways that learners actively modify their problem representations when they encounter problematic situations. At the core of the difficulty is the fact that researchers have adopted a variety of theoretical perspectives to study representations in problem solving situations. As might be expected, these differing perspectives can lead to different and sometimes divergent explanations of how representations function in problem solving situations. For example, many of the Expert-Novice studies in algebra word problems have examined whether the solver is able to perceive the "problem structure" of a given task (Mayer, 1985; Reed and Sanchez, 1990). In these studies, a specific problem structure refers to an a priori assignment involving the underlying mathematical relationships of the task at hand (e.g., rate problems, motion problems, mixture). As a result, a solver's ability to recognize similarity across tasks that embody similar "problem structures" is taken as evidence that the solver has developed an appropriate problem representation. These studies seldom include detailed explanation of how the solver's inferred representations function as aids for understanding to develop appropriate solution activity.

There have been numerous challenges to the cognitive models described above, with most questions revolving about epistemological and pedagogical considerations. For example, Roschelle and Clancey (1991) question the underlying assumptions of these models, including "that the world comes pre-represented, already parametricized into objects and features" (p. 11). This concern that the models attribute the environment as a primary source of learners' mathematical knowledge has been voiced by other researchers, most notably by those who reject the process of encodism and argue instead that representations can develop from non-representational phenomena (Bickhard, 1991). Finally, challenges to traditional cognitive models of representation have been made on pedagogical grounds, asserting that instructional implications of these models point to inappropriate instructional activities. For example, critics have described the instructional activities that are suggested by the models as reflective of an "instructional representation approach", which has as its primary goal that "students construct mental representations that correctly or accurately mirror mathematical relationships located outside the mind in instructional representations" (Cobb, et. al., 1992). An implication of the instructional representation approach is that in order to develop appropriate instructional activity, educators must first identify the class of representations we wish our students to have, and find ways to make these representations "transparent" (or explicit) to our students (Cobb et. al., 1992).

All of the critical commentaries cited above challenge the cognitive characterization of representations as static containers of knowledge transported across similar learning situations. In contrast, constructivist theories of learning have considered the act of representation as a dynamic process that contributes to the learner's sense-making actions in problem solving situations. This view includes a focus on both the ways learners actively organize or structure their prior experiences and the conceptual knowledge that results from structuring activity. For example, constructivist models of number development attribute re-presented sensory-motor action as a major source of the mathematical knowledge constructed by children as they develop counting strategies (Steffe, et. al., 1983; Steffe, et. al., 1988; Cobb, 1988). Specifically, the children's ability to mentally re-present their prior mathematical activity (i.e., to "call it up" and "run through it" in thought) is considered to play a crucial role in the mathematical knowledge they construct.

The constructivist view of representation, as conceptual knowledge the learner derives from experience, is consistent with the notion that learners continually operate on the "frontiers of their knowledge" and actively construct new knowledge in problem solving situations "when their current knowledge results in obstacles, contradictions, or surprises" (Cobb, 1988). More precisely, the knowledge employed by learners in specific situations operates as long as it remains "viable" in the sense that it serves the purposes as seen or interpreted from the point of view of the solver (von Glasersfeld, 1987). In this way, problem solving situations not only test the viability and efficacy of the solver's existing knowledge but also can serve as opportunities for solvers "to modify existing representations (models) which may have outlived their usefulness" (Johnson-Laird, 1983).

A constructivist view of representation (as conceptual structures of knowledge) is adopted for the current study, with the notion that a more precise explanation is needed to clarify how representations are constructed and/or modified in the course of problem solving activity.

PERSPECTIVE

The explanation of ways that learners develop representations in mathematical problem solving was guided by a theoretical perspective which focused on the learner's cognitive activity. Central to this framework is the view that the internal representations constructed by learners are mathematical conceptions which evolve from from the solvers' solution activity in genuine problem solving situations that they face (Vergnaud, 1984; Cobb, Yackel, and Wood, 1989; Pask, 1985; von Glasersfeld, 1987). This belief that learners construct representations in the course of mathematical activity helped to guide the study in the following ways. First, tasks were used which allowed the researcher opportunities to observe solvers as they attempt to resolve what are for them genuinely problematic situations (see Table 1). In contrast to more traditional methods for studying problem representations in problem solving (e.g., the use of cardsorting tasks), the tasks used in the study enabled the researcher to analyze the solvers' mathematical activity, both interpretive and conceptual, across a range of similar, yet different situations. Second, in order to analyze the development of problem representations as a problem solving activity, verbal data was generated by having subjects think aloud as they attempted to solve a set of mathematical tasks.

OBJECTIVES

The purpose of the study was to clarify the construct of problem representation and acquire an understanding of the processes used by learners to construct and/or modify problem representations in problem solving situations. Unlike other studies of problem representations, the study considers representations as structured organizations of actions, built up by solvers in problem solving situations, and serving as interpretive tools of understanding to aid their solution activity. Hence, the study focused on the cognitive activity of the learner with particular emphasis on the ways that they elaborate, reorganize, and reconceptualize their solution activity while engaged in mathematical problem solving.

In an earlier study (Cifarelli, 1991), solvers were interviewed as they solved sets of similar mathematical tasks (see Table 1). From the analysis of video and written protocols, several increasingly abstract levels of solution activity were identified. For example,

Table 1: SET OF LEARNING TASKS

TASK 1: Solve the Two Lakes Problem

The surface of Clear Lake is 35 feet above the surface of Blue Lake. Clear Lake is twice as deep as Blue Lake. The bottom of Clear Lake is 12 feet above the bottom of Blue Lake. How deep are the two lakes?

TASK 2: Solve a Similar Problem Which Contains Superfluous Information

The northern edge of the city of Brownsburg is 200 miles north of the northern edge of Greenville. The distance between the southern edges is 218 miles. Greenville is three times as long, north to south as Brownsburg. A line drawn due north through the city center of Greenville falls 10 miles east of the city center of Brownsburg. How many miles in length is each city, north to south?

TASK 3: Solve a Similar Problem Which Contains Insufficient Information

An oil storage drum is mounted on a stand. A water storage drum is mounted on a stand that is 8 feet taller than the oil drum stand. The water level is 15 feet above the oil level. What is the depth of the oil in the drum? Of the water?

TASK 4: Solve a Similar Problem In Which the Question is Omitted

An office building and an adjacent hotel each have a mirrored glass facade on the upper portions. The hotel is 50 feet shorter than the office building. The bottom of the glass facade on the hotel extends 15 feet below the bottom of the facade on the office building. The height of the facade on the office building is twice that on the hotel.

TASK 5: Solve a Similar Problem Which Contains Inconsistent Information

A mountain climber wishes to know the heights of Mt. Washburn and Mt. McCoy. The information he has is that the top of Mt. Washburn is 2000 feet above the top of Mt. McCoy, and that the base of Mt. Washburn is 180 feet below the base of Mt. McCoy. Mt. McCoy is twice as high as Mt. Washburn. What is the height of each mountain?

TASK 6: Solve a Similar Problem Which Contains the Same Implicit Information

A freight train and a passenger train are stopped on adjacent tracks. The engine of the freight is 100 yards ahead of the engine of the passenger train. The end of the caboose of the freight train is 30 yards ahead of the end of the caboose of the passenger train. The freight train is twice as long as the passenger train. How long are the trains?

TASK 7: Solve a Similar Problem that is a Generalization

In constructing a tower of fixed height a contractor determines that he can use a 35 foot high base, 7 steel tower segments and no aerial platform. Alternatively, he can construct the tower by using no base, 9 steel tower segments and a 15 foot high aerial platform. What is the height of the tower he will construct?

TASK 8: Solve a Similar Simpler Problem

Green Lake and Fish Lake have surfaces at the same level. Green Lake is 3 times as deep as Fish Lake. The bottom of Green Lake is 40 feet below the bottom of Fish Lake. How deep are the two lakes?

TASK 9: Make Up a Problem Which has a Similar Solution Method

Re-Presentation was identified as a level of cognitive activity where solvers could begin to combine mathematical relationships in thought and mentally act on them (e.g., they could reflect on, and "run through" proposed solution activity and have anticipations about the results without resorting to pencil-and-paper actions). The construction of these structures of solution activity was seen as playing a crucial role in the knowledge constructed by the solvers during the interviews. This finding is consistent with what other researchers have observed. For example, that the solvers' were able to reflect on their solution activity as a unified whole suggests they had constructed problem representations that were coherent and connected (Greeno, 1980).

This research extends the results of the earlier study by specifying more precisely the emergence of structuring activity that contributes to the development of formal representations and exploring further some of the individual differences found among the subjects of the earlier study. In particular, episodes from a pair of case studies will be discussed to describe the individual differences between solvers regarding their ability to construct representations in problem solving situations. In addition, the solvers' representations will be discussed in terms of their role as conceptual links to potential actions. In this way, the study will examine the extent to which the solver's representations function to inform their determination of potential solution activity.

METHODOLOGY

Subjects

Subjects came from freshmen calculus courses at the University of California at San Diego. This population was of interest to the researcher given the fact that much of the research on representation in mathematical problem solving (Schoenfeld, 1985; Silver, 1982) as well as studies of Expert-Novice differences in problem solving (Mayer, 1985; Larkin, 1983) have focused on the performance of college age students. A total of fourteen subjects participated in the study.

Use of Interviewing Methodology

The use of interviews to gather data was crucial to the goals of the study. It has been stated that most textbook word problems as they are interpreted in typical classroom situations do not serve as genuine problem solving activities because they are not "dilemma driven" (Lave, 1988). The use of interviews helped overcome this difficulty by establishing a social context between the interviewer and the subjects in which dilemmas could arise for the subject's in the course of their ongoing solution activity. Specifically, an interviewing methodology was used which required the solvers to think aloud while solving the tasks. In particular, the researcher wanted the subjects to accept certain obligations during the interview (e.g., explanations of, and justifications for their solution activity). In this way the researcher initiated and guided a social context seldom found in typical classroom situations. As a result, the subjects established their goals and purposes while interacting with the researcher. This approach, together with the nonstandard format for presenting the tasks made possible a focus on the solvers interpretations of tasks (and not the tasks themselves). Consequently it was possible to observe solvers experiencing dilemmas as described by Lave. In other words, dilemmas did arise for the subjects throughout the course of the interviews and these dilemmas provided opportunities for the solvers to further their conceptual knowledge. For example, even though solvers might construct a solution to Task 1, they could conceivably face problems while solving later tasks despite recognizing that similar solution methods might be involved (e.g., solvers could face a problematic situation while solving Task 3 if they try to do exactly the same thing as they did in solving the earlier tasks). Hence, such situations provided opportunities for solvers to develop greater understanding about their solution activity. In addition, the solver's evolving intuitions about "problem similarity" allowed the researcher opportunities to observe how the solvers' newly constructed conceptual knowledge influenced subsequent solution activity in similar situations (i.e., development of control of solution activity).

Data Generation

Data collected in the study took the form of video and written protocols. All of the interviews were videotaped for subsequent analysis. This allowed for an ongoing interpretation and revision of the subject's activity in the course of the analysis. Viewing a videotape of each subject's performance gave the researcher an opportunity to "step back" and analyze the dialogue from an observers perspective. Once something had been "noticed" which might lead to a revision, the tape could be analyzed again in light of the new findings. This allowed for a continual communication between the theory and the data.

In addition to the video protocols that were prepared for each subject, written protocols were used in the subsequent analysis. These protocols took the forms of written transcripts (an ordered record of their verbal statements for each task) and paper-and-pencil records (the written work that the subjects performed as they progressed through the tasks). The written transcripts provided the researcher a means with which to identify and make reference to examples of significant solution activity when they occurred. This method of formatting the verbal responses of the subjects offered an effective yet economical way of reporting results in the analysis that followed. The paper-and-pencil records provided a perspective on the subjects' solution activity different from that of the written transcripts. For example, some records contained examples of perceptual expression used by the solvers (e.g., pictures or diagrams they constructed). In these instances the records helped to clarify the ways that the solvers developed their conceptual knowledge during the interview.

Analysis of Data

The protocols for each subject were analyzed and subsequent results were reported in the form of detailed case studies. The analysis of the protocols proceeded in the following phases.

It was a fundamental hypothesis of the study that solvers construct conceptual knowledge by performing novel activity in situations they find to be genuinely problematic. Hence, the solution activity of each subject was examined in order to identify those situations where they appeared to face such cognitive conflict. This was accomplished through careful examination of the written and video protocols and involved making a distinction between the solvers' novel (genuine problem solving activity) and their routine solution activity (assimilation of the situation to current conceptual structures with no problem experienced). Once this parsing had been made, the subject's novel activity was examined with the goal of identifying instances where major conceptual reorganization may have occurred. Here it was useful to identify qualitative aspects of the subject's solution activity (e.g., processes which enabled them to develop intuitions of problem sililarity during the interview). For example, the solvers were inferred to have experienced problems when their initial anticipations about what to do to solve a particular task proved unviable. In this way, the analysis focused on qualitative aspects of the solvers' solution activity (i.e., solvers' evolving anticipations and reflections) which indicated that constructive activity had occurred.

Based on the results of the qualitative analysis described above, a detailed case study was prepared for each subject. This consisted of the following parts. First, a written summary of the solver's performance was prepared. This portion of the case study focused on the solvers' solution activity with particular emphasis on the ways they actively gave meaning to each task and the novel ways they resolved problematic situations they faced along the way. This meaning making activity involved solvers' interpretation of novel situations in terms of previously constructed solution activity. Second, a macroscopic summary of the subject's performance during the interview was prepared. This summary included both a general overview of the conceptual knowledge the subject appeared to construct while solving the tasks as well as a characterization of the subject's performance expressed as increasingly abstract levels of solution activity.

The case studies were then considered as a group for the purpose of generalizing the results. For this purpose, only those cases which yielded the most information were included in this phase. Of the fourteen subjects who participated in the study, two chose to withdraw after viewing the videotape of their performance (each subject had the option of withdrawing if for any reason they were dissatisfied with their performance). Of the remaining twelve cases, the eight most interesing cases were chosen for further analysis. This decision was based on several factors including the following. First, it was felt that the subjects of these cases demonstrated high levels of task involvement during the interviews (Nicholls, Cobb, Yackel, and Patashnick, 1990). This concern for the subjects' motivations during the interview is important given the fact that the researcher could only infer when the subjects experienced genuine problems. It was felt that these interpretations could be made confidently for subjects who maintained high levels of interest and motivation throughout the interview. Second, the subjects of these cases were particularly verbal throughout the interviews. There was little need to prompt them for comment about their solution activity. Hence, it was felt that their verbal responses provided an accurate description of their mental activity while solving the tasks. Finally, the researcher felt that collectively, these subjects demonstrated a range of abstraction in their solution activity sufficient to make some general inferrences.

The following sections summarize the findings of the study. First, using the results of the earlier study, the subjects are summarized as a group according to the levels of solution activity they were inferred to achieve during the interview. Second, the results of a pair of case studies are presented to illustrate some of the individual differences demonstrated by solvers in their ability to construct representations. Finally, the results are discussed in more general terms including a comparison to other research findings about representation.

FINDINGS

Representation as Levels of Conceptual Structure

The results of the eight cases are summarized in Table 2. The solvers' peformance was described using the increasingly abstract levels of solution activity identified in the earlier study (Cifarelli, 1991). The levels of solution activity can be viewed as cognitive expressions of the solvers' evolving conceptual structures. Hence, by achieving a particular level of solution activity, the solvers' were seen as expressing a level of conceptual structure in their solution activity. Briefly, four of the solvers demonstrated solution activity at the Abstract level of abstraction, two solvers demonstrated solution activity at the Re-Presentation level of abstraction, and two solvers demonstrated solution activity at the Recognition level of abstraction.

The following section includes discussion of episodes taken from a pair of case studies. These episodes will serve to illustrate some of the individual differences between the solvers in their ability to construct representations.

Individual Differences

As a mechanism for explaining and clarifying some of the individual differences inferred from the solvers' solution activity, episodes from a pair of case studies will be presented. The following paragraphs include episodes from the case studies of solvers Marie and Janet. These serve to illustrate examples of the different levels of conceptual knowledge demonstrated by the solvers as well as suggest a basis for discussing the results in more general terms.

Both Marie and Janet were freshmen in the third quarter of the UCSD calculus sequence with both students undeclared in an academic major at the time of the study. Marie went on to earn a degree in Physics while Janet completed a degree in Mathematics. Further, Janet went on to become a high school mathematics teacher.

Marie completed the interview in 40 minutes. She successfully completed all of the

TABLE 2: LEVELS OF CONCEPTUAL STRUCTURE

LEVEL OF ACTIVITY (solvers achieving level)	ATTRIBUTES	EXAMPLES
Abstraction (4)	Solver can "run through" potential solution activity in thought and operate on its results	Solver can draw inferences from results of potential activity without the need to carry out solution activity
Re-Presentation (2)	Solver can "run through" prior activity in thought	Solver can anticipate potential difficulties
Recognition (2)	Solver encounters new situation and identifies activity from previous tasks as relevant for solving current task	Solver recognizes diagrammatic analysis activity as appropriate for solving Tasks 2-9

tasks, demonstrating a Structural level of abstraction in her solution activity. Even though she was able to construct solution for each task, there was strong evidence that she participated in genuine problem solving activity at several points during the interview.

Janet completed the interview in 42 minutes. She successfully completed six of the nine tasks, and did not appear to demonstrate solution activity above the level of Recognition.

A comparison of the solvers' performance while solving Tasks 1-3, and Task 9 will now be presented.

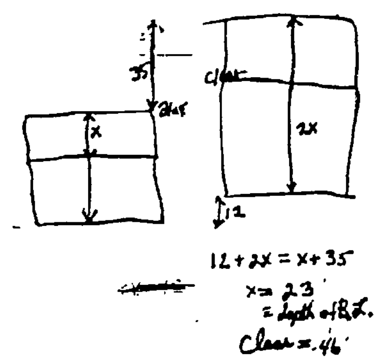
Task 1

Upon reading the problem statement, Janet quickly constructed a diagram with one lake superimposed over the other. After some difficulty arising from the orientation of her diagram, she redrew the lakes so that they were side-by-side. She routinely analyzed the diagram and constructed approppriate algebraic expressions:

> Janet: So Clear Lake is ... this is X, this is 2X, the difference is 12. So we want ... 2X minus ... I'm just trying to make an equation to relate the two.

She generated an appropriate algebraic equation from which she was able to find a solution.

> Janet: This is 2X, so ... we've got 12 plus 2X equals X plus 35. What's the maximum height and what's the maximum depth? And I set those two equations (SIC) to each other ... and solve for X. So, ... X equals 12 minus 35 (SIC) which is 23 feet. So that's the depth of Blue Lake. And the other one, Clear Lake, is 46 feet deep. Am I being timed?



In contrast to Janet, Marie faced much difficulty in constructing a solution. She initially interpreted the task as a routine algebra problem and proceeded to code all information without trying to develop a deeper understanding of the situation.

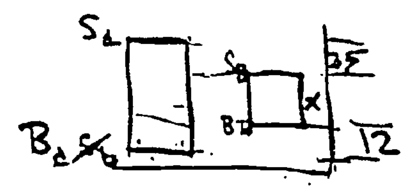
Marie: That strikes me as an algebra with 2 variables. So the first thing I should do is assign variables to everything that is important.

She constructed a diagram and proceeded to generate all possible algebraic relationships. Symbols representing variables were manipulated in a mechanical fashion as she tried to code and relate information contained in the problem statements without reflecting to the extent necessary to consider whether such assignments were relevant to the solution of the problem. This activity resulted in the generation of algebraic equations which she later found to be inappropriate.

$$S_c - S_b = 35$$
$$B_c - B_b = 12$$
$$(S_c - B_c) = 2(B_c - B_b)$$

Marie: I have 4 unknowns and 3 equations. And that's not good enough for me to solve an algebra problem.

Marie realized she faced a genuine problem and abandoned her initial unreflective approach in favor of a more reflective approach as indicated by the solver's intention to use the drawing as an interpretive tool to aid her conceptualization and elaboration of potential relationships. (This change to a more "sense-making" approach appeared to be an example of "dilemma driven" activity as described by Lave (1988).)



Marie: This is the bottom, this is the surface of Blue Lake and this is the bottom of Blue Lake. This distance is 12 and this distance is 35. And this whole distance is twice that whole distance. (LONG PERIOD OF REFLECTION HERE)

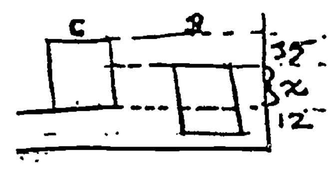
Marie: Okay, if I label this whole distance X ... I can say ... that 12 plus X plus 35, which is the height of Clear Lake, is going to equal twice X. And that's the relation in one variable I can solve.

$$12 + x + 35 = 2x$$
$$12 + 35 = x$$
$$12$$
$$35$$
$$47 = x$$

Marie: And the relation I was missing here is the fact that I'm looking at differences in height, not absolute height.

This constructive activity enabled Marie to generate an appropriate algebraic equation for the problem, albeit an incorrect one (i.e., she made an error in labeling her diagram). This algebraic relationship expressed a wholist interpretation of the task rather than isolated relationships that corresponded to fragments of the problem statement. Upon discovery of an error in her diagram, the solver reconstructed algebraic expressions and generated a new algebraic equation which led to a correct solution.

Marie: The bottom of Lake, ... and this lake is 12 feet above the bottom of that lake. So I didn't draw it that way. I drew it 12 feet below.



Marie: That means that my geometrical solution is probably off.

Marie: So, the distance between these two is still 35. The distance between these two is 12. Yeah, but X doesn't mean the same anymore.

$$35 + x = 2(12 + x)$$
$$35 + x = 24 + 2x$$
$$11 = x$$

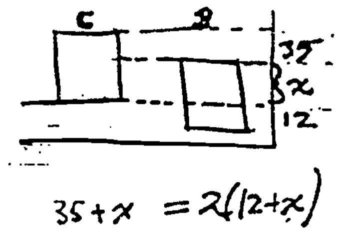
Marie: So, 35 plus X equals 24 plus 2X. So 35 minus 24 equals ... X. So Clear Lake is equal to 35 plus X which is 46. And Blue Lake is equal to 12 plus 11 which is ... 23. That's the solution!

$$C = 35 + 11 = 46$$
$$B = 23$$

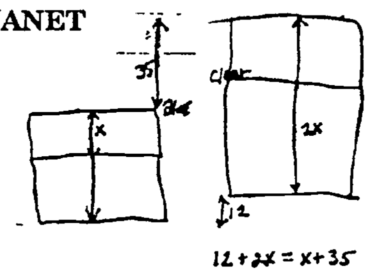
The solvers' solution activity for Task 1 involved the construction of novel relationships in the course of which they each developed an initial conceptual structure. This activity was novel in the sense that it involved meaning making activity in genuinely problematic situations. Given this initial implicit structure, solution activity performed while solving Tasks 2-9 gave rise to opportunities for the solvers to elaborate and reconceptualize the relationships they constructed while solving Task 1.

There appeared to be differences between Marie and Janet in their interpretations of the task and the subsequent strategies they developed to construct solutions. For example, a comparison of the diagrams they used to construct their solutions indicates that Marie's diagram was the more complex of the two, assigning the variable X to signify the length of a quantity not directly stated in the problem statements (i.e., the overlapping segment when the lakes are positioned side-by-side). In contrast, Janet's diagram more closely followed the information presented in the problem, assigning X and 2X to signify the depths of the two lakes.

MARIE



JANET



A second difference between the solvers' performance concerned the extent to which they found the task to be problematic. Janet's solution activity indicated that the task was routine for her while Marie needed to resolve 2 problematic situations in order to construct a solution. Specifically, Marie initial strategy of coding all the information contained in the probelm statements did not work for her and she proceeded to construct a diagram to use as an interpretive aid. Even though this approach enabled her to construct appropriate algebraic relationships, she nevertheless faced a second problematic situation because she had made errors in the labeling of her diagrams.

The differences between the solvers' solution activity described above are minor in the sense that they only describe differences in strategies and the extent to which they had to overcome problematic situations in constructing solutions. Several more interesting differences emerged while they solved later tasks. These differences relate to the abstractness of the knowledge they constructed during the interviews as inferred from the conceptual developments they demonstrated while solving later tasks. For example, Marie was able to develop her solution activity to the extent that while solving later tasks, she could anticipate what it was she needed to do, and the presence of potential problems, prior to carrying out her solution activity. These anticipations resulted from from her reflections on the appropriatenes and efficacy of prior solution activity when she faced problematic situations. In contrast, Janet did not appear to demonstrate a similar level of abstraction in her solution activity. She did not appear to reflect on and "distance" herself from her potential solution activity as did Marie. As a result, her evolving anticipations were low level in the sense that she could only identify or recognize the appropriateness of prior solution activity in new situations, and could not anticipate the presence of potential problem prior to carrying out her solution activity.

The following sections include episodes from the solvers' performance on Tasks 2, 3, and 9. These episodes serve to illustrate further the emerging conceptual differences between the solvers while they solved later tasks.

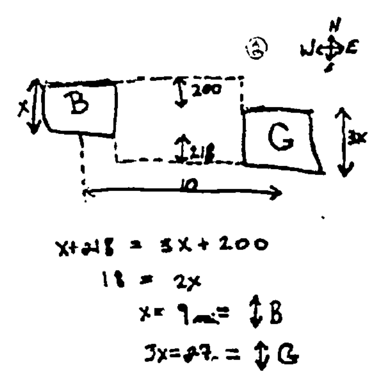
Task 2

Both solvers constructed solutions to Task 2 after noting that the task was very similar to Task 1. For example, Marie stated that the tasks were similar and her actions indicated that she anticipated similar solution activity.

Marie: The first thing that strikes me is that this problem is alot like the previous one. And I think it would ...**(ANTICIPATION)**... I think it would serve me well to start off in this one by just drawing a picture.

Similarly, after reading the problem statements, Janet routinely constructed a diagram very similar to that which she constructed in solving the previous tasks. Interestingly, both solvers were puzzled upon discovering the superfluous information, suggesting that their initial anticipations of similarity were based on recognizing that diagrammatic analysis activity of the type performed while solving Task 1 was appropriate to the new situation and that they could not anticipate potential difficulties prior to carrying out their solution activity.

Janet: Um. A line drawn due north through the city center falls 10 miles east of the city center of Brownsburg. How many miles across ... **(ANTICIPATION)**... So, I don't know that this has anything to do with anything? ... So, just immediately I'm thinking you're throwing a curve in there. Let's find out! Okay, so ...**(ANTICIPATION)**...if this just like the lake problem in that I end up with 2 equations (SIC).



Janet: (AFTER LONG PERIOD OF REFLECTION) ... I'm thinking that this line drawn due north has nothing to do with the problem. So, ... **(ANTICIPATION)** ... I'll just look at the other relationships first.

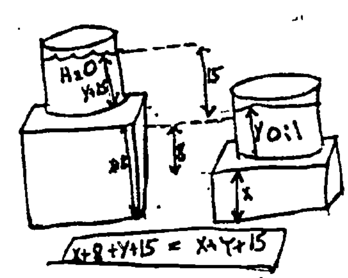
The solvers demonstrated solution activity similar to one another with regards to level of abstraction while solving Task 2. Their interpretations indicated they could recognize new tasks as requiring solution activity similar to that they demonstrated while solving Task 1 even if they could not anticipate potential difficulty prior to carrying out their solution activity.

The solvers' solution activity while solving Task 3 indicated that they were now operating at different levels of abstraction. Specifically, Janet continued to anticipate potential solution activity as similar to her prior activity of solving the Task 1. These anticipations were low level in the sense that she still needed to fully carry out her solution activity and could not anticipate potential problems that might arise.

Janet: Okay, so here's my stand ... So, if we ... from here to here ... from the top of the oil surface to the top of the water surface in the tank is 15 feet. What is the depth of the oil in the drum? So, how much oil, what! The depth? ... of the water!? ...(REFLECTION)...

Interview: What are you thinking?

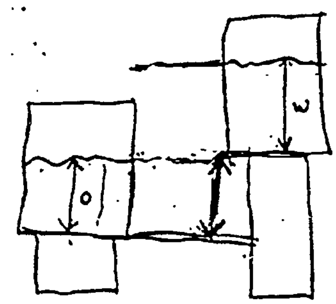
Janet: I'm thinking that this is one of those wonderfully impossible problems, that I either got to think about this a bit, or I'm stupid.



Janet: If the water level is 15 feet, I'm thinking that maybe it's not possible, that I would need two variables, and of course I don't need two variables. (ANTICIPATION) I just have to relate them somehow. Let's see ... (REFLECTION) ...(ANTICIPATION)... there is no relationship! I'm thinking this is a crazy problem! I have no idea! ... (REFLECTION) ... Oh maybe I do. ... Wait a minute! ... (REFLECTION)... Nope! I have no idea. I give up. Next!

In contrast, Marie demonstrated abstract solution activity while solving Task 3. In particular, she anticipated the presence of a potential problem soon after reading the problem statements and proceeded to make a conjecture about the nature of the problem she faced.

Marie: And here's the water level, here's the oil level. And the waterlevel is 15 feet above the oil level. So solve it ...(ANTICIPATION)... the same way. Impossible! It strikes me suddenly that there might not be enough information to solve this problem. So I better check that. (LONG PERIOD OF REFLECTION) I suspect I'm going to need to know the heights of one ofthese things. But I could be wrong so ... I'm going to go over here all the way through.



The suddeness with which Marie was able to anticipate a potential difficulty together with her reflections exploring the nature of the problem she now faced suggests she had attained a level of reflective activity not demonstrated while solving prior tasks. More precisely, she could anticipate the potential conflict because she could "run through" the potential solution activity in thought and "see" difficulties that might arise. This reflection on potential solution activity was interpreted as an an act of re-presentation -- the solver "ran through" her potential solution activity and "saw" the results as problematic.

Additional differences between the solvers in their solution activity were inferred while they solved while they solved Tasks 4 through Task 9. Even though Janet continued to construct correct solutions, she never demonstrated solution activity above the level of recognition. For each task she always started by constructing a diagram similar to that which she constructed to solve Task 1. Her anticipations of what to do to solve each task amounted to recognitions that she would essentially do the same thing as she did to solve the lakes problem and she never demonstrated that she could anticipate results of potential solution activity prior to carrying it out with paper and pencil. Hence, problems for her were experienced only when imitations of her prior activity carried out in the new situations did not work. In other words, she never developed an abstract level of understanding of the efficacy of those prior actions to the extent she could reflect on their viability in interpreting new situations.

In contrast, Marie continued to develop her understandings and demonstrated increasingly abstract solution activity while solving Tasks 4 through Task 9. She could reflect on her potential solution activity to the extent that she could "run through" the activity in thought and produce a result.

The differences between the solvers' understandings are best exemplified by their solution activity while solving Task 9. The task required that they make up a problem similar to those they just solved.

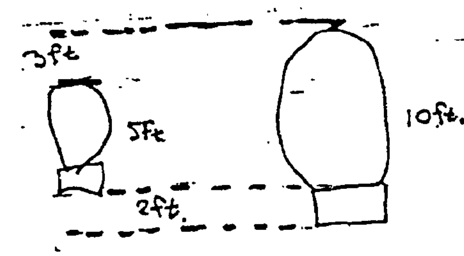
Marie's solution activity in Task 9 indicated that she had reorganized her conceptual understanding (at a higher level of abstraction) to the extent that she could reflect on her potential solution activity and anticipate its results and evaluate the usefulness of the results for the current situation without the need to carry out the activity with paper and pencil. In other words, she could reflect on her potential solution activity and determine appropriate relationships.

> Marie: Okay, ... (ANTICIPATION) ... I'm thinking of something with different heights. Oh, ... (ANTICIPATION) ... bookshelves in a bookcase. No, ... (ANTICIPATION) ... that's no good. ... (ANTICIPATION)...How about hot air balloons!

The solver ran through potential solution activity for the particular situation she proposed (i.e., bookshelves) and anticipated its results (i.e., that it would not work for "bookshelves" but that she could solve it for "hot air balloons"). So, her structure allowed her to run through potential solution activity in thought, produce its results, and draw inferences from the results. Her subsequent actions in completing the task (i.e., leading to a formal statement of a similar problem) were routine and indicated she was very confident that she had constructed a correct solution.

Marie: Okay if I were going to draw a picture of the problem I'd have one hot balloon that looks like ... (DRAWS BALLOON). And a bigger hot air balloon that looks like that. And I'll make this distance ... 3 feet.

Marie: I'll make this distance 2 feet. And I'll make this height 10 feet high 'cause that makes this 12 feet and that makes this one twice this one which is useful.



Marie: So, I'll just say the top of one hot air balloon HAB being the abbreviation for that, is 3 feet,

Marie: I'll make it a yellow hot air balloon which will make it easier, above a green hot air balloon.

Marie: The bottom of the yellow hot air balloon is 2 feet below the bottom of the green hot air balloon. The yellow balloon is twice the height of the green balloon. Let's make that a lake. What are the heights of the balloons?

In contrast to Marie's highly abstract solution activity, Janet did not appear to construct significant conceptual knowledge while solving Tasks 4 through Task 9. Even though she demonstrated the ability to identify, or recognize when prior solution activity was appropriate for solving a particular task (i.e., draw a diagram and reflect on the ways that she found potential solution activity similar to her solution activity while solving tasks), she could not advance to a more abstract level where she could reflect on her potential activity in a cohesive way (i.e., she could not reflect on her potential activity and anticipate results prior to carrying out the activity with paper and pencil).

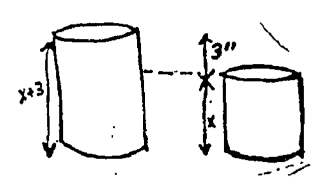
Janet's inability to develop more abstract solution activity appeared to involve conceptual limitations. Specifically, her understanding of the concept of a variable was limited in the sense that she had great difficulty utilizing variables in novel situations such as while solving Task 9. Since Task 9 gave the solver an opportunity to make up a similar problem, it was hypothesized that she would have been able to construct such a problem statement, incorporating a direct proportional relationship between heights as she had done while solving prior tasks. However, this was not the case. She experienced great difficulty as she reflected on the appropriateness of situations as to whether they would yield appropriate algebraic relationships and equations.

Janet: Let's see ... yes. Indeed something simple algebraic, something that sits still. So, if two kids have different size, different glasses, but of the same circumference ... Two kids and you have Kool Aid to parcel out. Let's see ... ah ... the same circumference but different heights ... I don't have two of same glasses. So you're going to fill them to a certain point.

Okay, so the height of this little one is X and ... let's see ... you know that ... let's see ... you know that this glass is twice as ... is as tall as the small one, ... plus 3 inches. Assuming you can pour to the rim you can get ... X plus 3 of fluid in the large glass, you only get X in the small.

Even though she constructed an appropriate situation, she faced a problem when she assigned variables, realizing that the algebraic equations were not the same as those she generated while prior tasks. In particular, she let X signify the height of the smaller glass. Even though she introduced the relationship that the larger glass was twice as high as the smaller glass, she chose to let (X+3) signify the height of the larger glass reasoning that the capacity of the larger glass could be achieved by adding 3 inches to the smaller. As a result, she was not able to construct a pair of algebraic expressions for the top-to-bottom length.

Janet: So you have to figure out ...(ANTICIPATION)... oh this is ridiculous. Um ... let's see ...(POINTING WITH HER PENCIL AT THE VARIABLE EXPRESSIONS ON HER DIAGRAM)... this is too simple. I think I would have to work on this to make it harder. Let's see. The height of one glass, we know that one glass, is two times as large, so you'd have to fill this one twice to get as much. You have to give this guy another 3 inches of Kool Aid to get as much as the first child. Um ... I could pose another problem about lakes, about buildings and mountains or something cause this is not ...(LONG REFLECTION HERE)



At this point the Janet senses that she has a problem even though she is convinced tha the proposed problem situation is similar to the previous problems. She pauses to reflect on the situation and is prompted by the interviewer:

Int.: Is this problem similar to those earlier problems?

Janet: No, not really. Yes, only in the idea that you look at the maximum height here. The difference in these two is, the height of these two glasses is 3 inches, so how much more Kool Aid does he have to be given to have the same amount of liquid as the first child. But ... (POINTS AGAIN TO VARIABLE EXPRESSIONS)... it's just too simple! Those had ... just had a little more to think about cause we had more differences. They're all the same though ... it just seems like there must be something else!

In stating that the problem is similar to the previous tasks, only too simple, it appeared that the source of her difficulty was due to the fact that she could not see a way to construct an algebraic equation from her diagram as she had while solving earlier tasks. In trying to resolve the difficulty she faced, Janet reflected on her solution activity of solving Task 8, and finally decided to give up.

Janet: Well this is similar to that last one, in that the last lake one had the same heights, the same surfaces. They were the same height, the surfaces of the two lakes were the same height and one was deeper, three times deeper than the other one or something like that. ... It has to have been given, the difference in these, the depth of these two which I don't remember. ... And that's similar to this. .... That's as close as I'm going to get to a lake problem. I'm done.

DISCUSSION and CONCLUSIONS

The results of the study will now be discussed in more general terms. Drawing from the results of the eight cases, the following paragraphs describe the conceptual knowledge the solvers appeared to construct during the interviews.

Analysis of the solvers' solution activity indicated a gradual building up and elaboration of their conceptual knowledge as they solved the tasks. Procedures constructed by solvers while solving the earlier tasks were elaborated as they solved later tasks. This development of conceptual knowledge was indicated by the solvers changing anticipations and reflections. In particular, the solvers demonstrated conceptual knowledge when, in interpreting the task, they could reflect on their potential solution activity (and generate anticipations about its results) without the need to actually carry out the particular actions (see Figure 1).

The levels of solution activity identified in the study can be viewed as cognitive expressions of the solvers' evolving conceptual structures. These structures can themselves be described as organizations of the solvers' solution activity that provide order for their experiences and form to their interpretations when faced with new situations. In other words, the solvers' structures were purposeful organizations of their prior experiences that subsequently served to organize their future experiences in ways compatible with their goals.

The results of the study indicate that the process of representation influences problem solving activity in the following ways. First, the data suggests that we need to reconsider traditional views of representation and adopt a perspective which acknowledges both the constructive function of representation in the development of conceptual knowledge and the resulting mental objects upon which solvers can then reflect and transform as they interpret problem situations they face. Previous theories have tended to adopt a single perspective in studying representation with the result being an incomplete profile of what we know to be a very complex process. For example, some studies of representation as a process in mathematical problem solving have chosen to adopt the latter perspective of representations as objective knowledge which the solver transports across various learning situations (Mayer, 1985; Reed et. al., 1990). The results of the study suggest that these views are incomplete and need to address situations when the solver's representations do not work and need to be modified through novel solution activity. Second, the process of representation appears much more dynamic than previously thought as articulated by traditional theories of mathematics learning. The data suggests that representations that solvers construct function as tests of viability of the

Figure 1: Summary of Solution Activity

TASK 1 Solves the target task	--------->	**TASKS 2-9** Solves variations of original task

<u>EMERGING STRUCTURE</u>

(Evolving Anticipations)

<u>PRIMITIVE STRUCTURES</u>	--------->	**<u>ABSTRACT STRUCTURES</u>**
Early Tasks (Low Level Anticipation)		Later Tasks (High Level Anticipation)
Solver must carry out all solution activity		Solver can reflect on potential activity
Solver cannot reflect on potential activity and cannot anticipate its results		Solver can mentally "run through" re-presentation of potential activity

solver's mathematical knowledge in genuine problem solving situations. For example, while solving Task 3 solver Marie was able to anticipate a problematic situation by running through her potential activity in thought. This action served to make her potential activity an object upon which she could reflect. The result of this act of re-presentation was nothing less than a critical examination of the efficacy of her prior knowledge in the current situation. Third, the finding that the solvers' demonstrated several increasingly abstract levels of solution activity while solving the tasks suggests the need to address qualitative aspects of mathematical performance seldom considered as important in the study of representations in mathematical problem solving. For example, with the exception of Larkin (1983), researchers have tended to ignore the idea that solvers' mental representations exist within levels of abstraction.

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