DOCUMENT RESUME

ED 359 213 TM 019 839

AUTHOR	London, Robert
TITLE	A Curriculum of Nonroutine Problems.
PUB DATE	Apr 93
NOTE	44p.; Paper presented at the Annual Meeting of the American Educational Research Association (Atlanta, GA, April 12-16, 1993).
PUB TYPE	Reports - Descriptive (141) -- Reports - Research/Technical (143) -- Speeches/Conference Papers (150)
EDRS PRICE	MF01/PC02 Plus Postage.
DESCRIPTORS	*Curriculum Development; Field Tests; High Schools; *High School Students; *Mathematics Curriculum; Mathematics Skills; *Problem Solving; *Secondary School Curriculum; Word Problems (Mathematics)
IDENTIFIERS	Authentic Assessment; *Nonroutine Problems; Open Ended Questions; Performance Based Evaluation

ABSTRACT

A secondary mathematics curriculum of open-ended non-routine problems developed by a teacher of high school math over 10 years is described. An open-ended non-routine problem is one that requires problem recognition and orientation, effort, and persistence. It is open-ended in that it allows for various solutions, and requires the student to evaluate a variety of approaches and solutions. Every student can solve the problem, and each solution requires at least a few hours of work. In a sense, it corresponds to an essay in the language arts. A 4-year high school curriculum of non-routine problems has been developed and field tested. The core is a set of 60 non-routine problems, 16 for each year. Each item gives the student a chance to practice problem solving. In field tests over 10 years with the instructor's students, the problems have been refined. Students who have practiced these problems demonstrate "mathematical maturity" in their problem-solving approaches. Appendix A summarizes the non-routine problems for each grade level. Appendix B gives sample problem statements. Appendix C is a draft of teaching suggestions. (SLD)

A CURRICULUM OF NONROUTINE PROBLEMS

1993 AERA Convention

Dr. Robert London
Mathematics Chair
Old Saybrook High School
Old Saybrook, CT 06475
Home Address:
49 Woodland Drive
Old Saybrook, CT 06475
203-388-0423

A CURRICULUM OF NONROUTINE PROBLEMS

INTRODUCTION

"What is essential in the secondary mathematics curriculum?" In this paper I will describe a secondary mathematics curriculum of open-ended nonroutine problems developed over ten years by the researcher as a partial response to this question. I will: (1) define nonroutine problem, (2) describe the curriculum, (3) discuss the rationale for a curriculum of nonroutine problems, (4) identify the theoretical assumptions underlying the pedagogy of the curriculum, and (5) outline the process of field-testing the curriculum, including tentative results and significance of the study. In addition, three appendices are attached: (1) a brief description of the sixty nonroutine problems, (2) a sample of some of the student problem statements, and (3) a draft of the teaching suggestions that will accompany the curriculum.

DEFINITION OF NONROUTINE PROBLEM

In recent years educators have made a major effort to improve the mathematical problem solving abilities of students. We have seen the improvement of problem solving skills identified as an important focus of the NCTM and other professional organizations. Publishers have attempted to increase the emphasis on problem solving in their mathematics textbooks. Preservice and inservice programs have increasingly emphasized methods and materials to improve problem solving skills. In short, in recent years the professionally aware mathematics teacher has been flooded with ideas, techniques and materials to use in the classroom to improve problem solving skills.

This paper is concerned with a type of problem not typically included in even the improved materials of the 1980's and 1990's. I call this type of problem an open-ended nonroutine problem or, more simply put, a nonroutine problem, which, for the purposes of the curriculum described in this article, has the following characteristics: (1) The problem requires three steps to complete: problem recognition and orientation, trying something, and persistence. (2) The problem is open-ended; that is, it allows for various solutions. (3) The problem requires the student to evaluate a variety of potential solutions and approaches to the problem and select one or more to pursue. (4) Every student is able to "solve the problem". Of course, the quality of different student solutions will vary, but students will be able to confront the problem and generate a solution consistent with their ability and efforts. (5) Each problem requires at least a few hours work over at least a week's time and written justification and description of the solution.

In a sense, a nonroutine problem in mathematics corresponds to an essay requiring creative writing in language arts in that the problem requires a significant investment of time; most students can complete the assignment; and the problem requires higher-order thinking skills. To push the analogy one step further, just as writing an essay can be considered the step beyond writing sentences and paragraphs, solving nonroutine problems can be considered the step beyond solving problems involving one strategy or solving a set of word problems involving a variety of strategies.

Before preceding it is necessary to clarify the fourth characteristic of a nonroutine problem: Every student is able to "solve the problem." In my opinion, given the realities of the level of understanding of mathematical content of the typical high school student, it is inappropriate to include nonroutine problems which require significant mathematics content prerequisite skills. For example, the following two problems from Mathematical Problem Solving (Alan Schoenfeld, Academic Press, 1985, pp. 15-6) are examples of problems requiring the three steps of nonroutine problems, but not satisfying the criteria that every student is able to "solve the problem": (1) Given two intersecting lines and a point P marked on one of them, show how to construct, using straightedge and compass, a circle that is tangent to both lines and that has the point P as its point of tangency to one of the lines; and (2) Given triangle ABC, show that it is always possible to construct, with straightedge and compass, a straight line that is parallel to line segment AB and that divides triangle ABC into two parts of equal area.

Notice that many average high school students would not have the content understandings that these problems require. In contrast, the curriculum of nonroutine problems was constructed so as to minimize the mathematical content prerequisite skills required of the student. Therefore, using Schoenfeld's terminology, it should be clear that this curriculum does not directly attempt to improve the category he labels resources ("the facts, procedures, and skills - in short, the mathematical knowledge - that the individual is capable of bringing to bear on a particular problem (p.17)"). It needs to be noted that the curriculum only accounts for approximately 20% of the secondary mathematics curriculum, allowing ample time to focus on this category of resources.

DESCRIPTION OF THE CURRICULUM

This article is primarily concerned with the development and initial field-testing of a four year high school curriculum of nonroutine problems. The core of the curriculum consists of sixty nonroutine problems, involving fifteen problems per year over the four year high school curriculum. Each problem requires one to three weeks to complete and two-thirds of the problems are solved in cooperative groups. Students are required to orally and in writing document the process for solving each problem, including describing all methods which were considered. In addition to the sixty problems, there are two introductory units. One unit describes the three steps of a nonroutine problem, gives over ten examples of the steps in a variety of fields, has the students research additional examples, and requires the students to be able to define and give examples of the three steps. The second unit introduces the students to basic skills necessary to function effectively in cooperative problem solving groups, such as nonjudgmental listing, asking clarifying questions, and brainstorming.

In developing the sequence of problems for this curriculum, many factors were considered to insure a balanced and effective curriculum. One major factor was the inclusion of certain strands of content, with a planned four year sequence of problems within each strand. The seven strands are: (1) introductory problems for each year which help establish a good classroom atmosphere; (2) area and volume problems; (3) problems involving functions with a focus on understanding abstract concepts; (4) prediction problems, including problems involving principles of probability and problems in which students gather information, primarily through questionnaires, and make predictions; (5) problems which attempt to increase the student's appreciation of diversity, especially diversity of ethnic groups and cultures; (6) ecology problems with an emphasis on problems directly affecting their lives; and (7) practical applications affecting their lives, including a sequence of four problems (starting in the third year) that each student defines and solves.

To further clarify what is meant by nonroutine problems and the strands, it is appropriate to briefly summarize a few sample problems from the curriculum: the students (1) determine three different containers without lids that maximize volume; (2) are given an abstract definition of a function ($F(ab) = F(a)\,F(b)$) and three values of the function, F(2), F(3) and F(5) and are asked to determine the values of the function for 1,2,3,...,20; (3) plan an inexpensive, educational and interesting class trip to a nearby city; (4) estimate the number of leaves on a large tree; (5) predict the number of lunches that will be served on a given day; (6) find interesting rituals and customs practiced in their community; (7) identify five ways their school can become more ecologically sound; and (8) approximate pi ten different ways.

To further clarify the curriculum, three appendices are attached: (1) a brief description of the sixty nonroutine problems, (2) a sample of some of the student problem statements, and (3) a draft of the teaching suggestions that will accompany the curriculum.

WHY NONROUTINE PROBLEMS?

Why are open-ended nonroutine problems appropriate for inclusion in the secondary mathematics curriculum? Three reasons will be discussed in this section: (1) each nonroutine problem gives the students practice with three important steps of higher-order problem solving which are typically not covered or emphasized in the traditional mathematics curriculum and are essential to doing mathematics: problem recognition and orientation, approaching a difficult or ambiguous problem by trying something or generating data, and persisting until reaching a satisfactory solution; (2) a sequence of nonroutine problems gives students experience with additional problem solving skills such as finding a pattern and generalizing, developing algorithms or procedures and describing them, generating and organizing data, manipulating symbols and numbers, and reducing a problem to an easier equivalent problem; and (3) most students who have completed a sequence of at least ten nonroutine problems demonstrate a mathematical maturity rarely observed in high school students.

Three Steps of Problem Solving

As stated above, one reason to include nonroutine problems in the curriculum is that each problem gives the students practice with three important steps of higher-order problem solving: problem recognition and orientation, approaching a difficult or ambiguous problem by trying something or generating data, and persisting until reaching a satisfactory solution. It is beyond the scope of this paper to deal with the significance of the three steps. For the purposes of this paper, it is enough to note that although I believe there are strong theoretical reasons to support this particular explanation of the steps of problem solving (see London, "The Three Steps of Problem Solving", in preparation), there is enough agreement between this theory and other theories of the steps of problem solving to proceed without undue concern.

The first step of problem recognition and orientation consists of three components: (1) a sense or attitude that the problem is solvable with the right type of effort (or at least the problem is worthy of attempting to solve); (2) a connection with step two; i.e., consideration of a variety of strategies and a choice of a strategy to pursue; and (3) a connection with step three; i.e., a sense of what a good solution might look like and some sense of the obstacles to a solution. Problem recognition implies that the problem solver realizes that a particular solution is not satisfactory. An example that made a strong impression on me was in a Calculus class I taught a few years ago. To introduce a new technique I asked the students to attempt an integration problem which the students could not solve without this new technique. To my surprise, every student in that class solved the problem - of course, incorrectly! There was no awareness that this was a problem they could not solve with their present techniques and this was the class of the best mathematics students in the school!

A second example of the lack of this step is how many students handle certain problems from the section of the SAT tests which involves quantitative comparisons. If they have not had instruction on this type of problem, many students will answer certain questions quickly and believe that they answered the question correctly. For example, if they are given that $a > b$, they will indicate that $a^2 > b^2$, not realizing that their solution is inadequate and that they should consider cases such as $a = 3$ and $b = -5$.

A clear example of the lack of this awareness is the typical response of students to the nonroutine problem of calculating the area under the curve $y = x^2 + 2$ between $y = 0$, $x = 0$, and $x = 3$. Despite directions to determine the area as best they can, the great majority of the students calculate the area as 15 1/2 square units by "replacing" the curve with the straight lines connecting (0,2) and (1,3), (1,3) and (2,6), and (2,6) and (3,11) and then dividing the area into triangles and rectangles. These students fail to recognize that their calculations could be more accurate by simply increasing the number of divisions. From having worked with these students, I believe they simply do not have experience solving problems which require them to evaluate the adequacy of their solutions. Certainly typical textbook problems do not require this skill. In terms of the first step, the students fail to make an initial connection with the solution; i.e., they have no clear sense at the beginning of the problem what an adequate solution (the best approximation of the area) looks like. Notice their solution is inadequate even though the strategy they picked (trapezoid method) is excellent.

The second step of problem solving emphasized in nonroutine problems is the strategy of approaching a difficult or ambiguous problem by trying something or generating data. The importance of this step was made clear to me at a presentation by Krulik and Rudnick, authors of Problem Solving: A Handbook for Teachers, at a regional NCTM convention. They gave us the following problem to solve:

A census taker comes to the house of a mathematician and asks how many children he has and what are their ages. The mathematician replies that he has three children and the product of their ages is 72. The census taker replies that he has not been given enough information to determine their ages. The mathematician adds that the sum of their ages is the same as his house number. The census taker leaves but returns in ten minutes and tells the mathematician that he still does not have enough information to solve the problem. The mathematician thinks for a short while and then adds that the oldest child likes chocolate ice cream. The census taker replies that he has enough information and leaves.

The problem is to determine the ages of the three children. If one has not seen this or a similar problem a common immediate reaction is that there are two strange parts to this problem: we do not know the house number and what does the fact that the oldest child likes chocolate ice cream have to do with determining the ages? A person comfortable with the strategy of approaching a difficult or ambiguous problem by trying something or generating data will generally start this problem (after the initial reaction of "what's going on?") by trying something, despite the ambiguity of the problem. For example, one could start by generating all the possible products of three numbers which equal 72, even though it may be totally unclear if or how this would help. Indeed, starting this way combined with the third step of problem solving (persistence) can lead to the solution. At this point I will not reveal the remainder of the solution, deferring that discussion to the next section on persistence. The point I wish to make here is the same that Krulik and Rudnick made; that is, they drew our attention to the basic difference between the ways mathematics teachers and students approach this problem. They pointed out that even though only a fraction of the mathematics teachers correctly solved the problem, the great majority of the teachers had one thing in common - they all tried something, using pencil and paper. In contrast, they reported that students when given this problem typically do not write or try anything. Subsequently I have used this problem with several classes to introduce this step of problem solving and can verify that students typically do not try anything.

I will give one example of how this step could be used in solving one of the nonroutine problems. In one problem, students are asked to discover a method for quickly locating a point C on line segment AB such that AB/AC =

AC/BC. By actually trying to locate a point on a sample line segment. students gradually become clearer about how to locate the point. For example, students generally notice after a few experiments that the point C must be closer to B than A. Students inexperienced in solving nonroutine problems are surprised how trying something, even if the problem seems unsolvable, can make the problem easier.

The third step in problem solving emphasized in nonroutine problems is persisting until reaching a satisfactory solution. Returning to the census taker problem, trying something by itself does not necessarily guarantee a solution. An example of persistence in this problem is the students who do not give up when they realize that the listing of all the possible products of three numbers equal to 72 does not lead directly to the solution and does not even seem to help clear up the ambiguity. Rather, a person who persists might eventually take the key step of listing the sums of all the three numbers whose product is 72:

$1 + 1 + 72 = 74$
$1 + 2 + 36 = 39$
$1 + 3 + 24 = 28$
$1 + 4 + 18 = 23$
$1 + 6 + 12 = 19$
$1 + 8 + 9 = 18$
$2 + 2 + 18 = 22$
$2 + 3 + 12 = 17$
$2 + 4 + 9 = 15$
$2 + 6 + 6 = 14$
$3 + 3 + 8 = 14$
$3 + 4 + 6 = 13$

If one looks at this list it is fairly likely that one would notice that there are only two sums which are identical: $2 + 6 + 6 = 3 + 3 + 8 = 14$. Now the original problem becomes easier; first, the house number must be 14; otherwise, the census taker would have been able to finish the problem (e.g.; if the house number is 15, then the children must be 2, 4 and 9). Then the fact that there is an oldest child indicates that the two oldest children cannot be twins, therefore; the children must be 3, 3 and 8.

Many times this step of persistence requires the problem solver to evaluate the adequacy of a solution and to persist until the solution is adequate. For example, in the F(x) problem students are required to calculate the value of F(X) for X = 1 to 20 given that F(AB) = F(A)F(B) and three values of F(X). In solving this problem, two students approached step two similarly by making a chart with the values of $F(x)$ that could be determined directly with the given values, and additional information such as the differences between $F(x)$ and $F(x + 1)$. One student filled in the remaining values based on the "pattern of increase", but observed that one section of his values did not "seem right" yet he left the values as is. Closer inspection of his projections indicate that he had the right idea, but had made a minor error in his projections - what was needed was that he persist with the problem until it was clear that his solution was adequate. The second student, working with the same data and organization of data, initially ran into similar problems, but was persistent until it was clear that the solution was good - and, indeed, his solution was very close to the actual values.

Similarly, two student solutions to an area problem can be contrasted. The first student approached determining the area of an irregular closed curve drawn on graph paper by calculating the average width times the length and ending up with an answer of 286.2 square units. Then the student calculated the area similarly, but by determining the area of the outside first and then subtracting from the total area. Using this method the student obtained an area of 313.6 square units. In both calculations the student generated and organized a good amount of data. However, the student obviously did not evaluate the adequacy of the answers - if the techniques were valid and carried out correctly, how could the student get two answers so different? What was needed was persistence to discover the minor mistake which threw the one calculation off. In contrast, a second student approached the problem of determining the area under a parabola by dividing the area under the figure into rectangles. This student realized that this technique was more effective when the number of rectangles was increased and persisted until it was obvious that the area was approaching 15 square units (the actual area). It should be mentioned that most students divided the figure into three (or, at most, nine) rectangles or trapezoids and considered the solution adequate. The reader might be thinking that perhaps the students realized that more rectangles would result in a better answer but were unwilling to take the additional time to do the needed calculations. My experience indicates that in general the students are not avoiding work - it just really does not occur to most of the students that they could or should persist in the problem - after all, have any of the typical textbook problems they have answered over the years required persistence? I believe that very few if any have encouraged or required that type of thinking.

In conclusion, the three steps of problem solving identified here are typically lacking in secondary curricula. In contrast, each nonroutine problem involves these steps and the processing of the problems emphasizes these steps.

Other Problem Solving Skills

Each nonroutine problem gives the student the opportunity to practice the three mentioned steps of problem solving. In addition, selective problems in a sequence of nonroutine problems give the student excellent practice with additional problem solving skills important in mathematics, especially (1) finding patterns and generalizing, (2) developing algorithms or procedures and describing them, (3) manipulating symbols and numbers and (4) reducing a problem to an easier equivalent problem. For example, one problem explores the area bounded by $y = 1/x$, $y = 0$, $x = 1$ and $x = b$ and requires the student to answer three questions - what values of b will yield an area of 1, 2 and 1000 (answer: e, e^2 and e^{1000})? The directions for this problem make it clear that for all practical purposes it would be impossible to determine directly the value of b which would give an area of 1000; it is necessary to discover a pattern. There are two aspects of this problem that make it particularly valuable as an inductive problem. Firstly, the student is required to generate the data and this is certainly not an easy task - the student is not familiar with the number e and calculating the areas well is tedious and tricky. Secondly, in processing the problem it is emphasized that to be given full credit for an answer of e^{1000} (e.g.; 2.7^{1000}) the student must have included direct calculations (or educated estimates) for at least three values of b. For example, if only two values of b are calculated, a student who incorrectly guesses that the pattern is e, e^2, e^4, e^8, ... is given the same credit as one who correctly guesses the pattern because both students made the same mistake from a problem solving point of view, generalizing with too little data.

I have observed that students who generalize too quickly on this problem learn from their experience and the processing, and generally do not make that type of error on subsequent nonroutine problems. This fact indicates something about the value of nonroutine problems. When I have emphasized the need for enough examples to generalize in the regular curriculum, most students continue to make that error in lessons in subsequent months. I believe what helps the transfer from the nonroutine problems is the amount of time and effort invested in the solution - after spending two weeks on a problem the student is more interested in the processing and more likely to remember the substance of the processing.

One more example: another problem requires the student to develop a procedure for connecting n random points so that the total distance is minimized. Further, the procedure must be clear enough so that another student can follow the directions and obtain an unique distance (i.e., no judgment should be required). The students seem to learn the most from the process of exchanging procedures, which includes reading one other student's procedure and identifying sections that need clarification, rewriting their own procedure, and carrying out the other student's procedure. Again, it is clear that the quality of experience this type of problem provides is much more likely to help students improve their ability to develop and describe algorithms than what we normally do in the mathematics classroom.

Mathematical Maturity

The best reason I can offer for including a sequence of nonroutine problems in the curriculum is the effect I have observed on the mathematical maturity of the students. Of course, much of this effect is due to the characteristics of nonroutine problems already discussed. However, I believe the whole effect of completing a sequence of nonroutine problems is greater than the sum of the parts. It is as if the student has been transformed mathematically! Instead of acting in all the ways that we normally attribute to most high school and college students, the student acts similarly to a "mathematically mature" person. For example, instead of stopping when an obstacle is encountered, the student will persist. Instead of ignoring obvious contradictions or inaccuracies, the student will actively examine them. Instead of being intimidated by ambiguity, the student will tolerate the ambiguity. Instead of being satisfied with the first solution to a problem, the student will work on a problem until a more satisfactory solution is reached. Instead of staring at a problem that seems unsolvable or confusing, the student will try something until the problem naturally becomes clearer. These are all examples of the type of behavior which is indicative of the gestalt of a mature mathematician. I am not claiming that merely after completing the nonroutine problems the students are mature mathematicians; certainly they lack the experience and knowledge of the mathematician and they still make errors "typical" of students but unusual for a mathematician. However, they do begin to act like mathematicians.

THEORETICAL ASSUMPTIONS AND PEDAGOGY

The curriculum is based on theoretical work described in other works (e.g., London, "Process-Oriented Curriculum", 1974 AERA presentation; and Nonroutine Problems, Janson Publications, 1989). Briefly, the major assumptions of the curriculum include: (1) An effective way to improve students' ability to solve nonroutine problems is to repeatedly put them in a situation in which they are given a nonroutine problem to solve, have the students work on the problem for at least a week and generate their best solution, and then discuss the problem and solutions as a class. (2) In an effective sequence it is normal that each problem is difficult. It is normal that the student would not be clear about how to solve the problem initially; that at times it would feel as if the problem was not solvable; that the first attempts at solution would seemingly lead nowhere; and that one would need to persist until the problem eventually became clear. (3) A cooperative group model is consistent with the purposes of this curriculum. For example, students benefit from being exposed to the strategies and thinking of other students, and group work provides a supportive atmosphere for dealing with the natural difficulty of the nonroutine problems. (4) The curriculum needs to emphasize problems from a variety of fields and problems that directly affect the student's life.

These assumptions and other issues of pedagogy are treated fairly in-depth in the appendix "Teaching Suggestions". Consequently, those issues will not be further addressed here, except to further clarify the fourth assumption: "The curriculum needs to emphasize problems from a variety of fields and problems that directly affect the student's life." This assumption, in my opinion, is directly connected with the driving question in the development of this curriculum: "What is essential in the mathematics curriculum?" In my opinion, most mathematics teachers spend much too much time teaching specific content objectives, if we are trying to teach the students what is essential in mathematics for them to improve the quality of their life. Far too many students leave secondary and postsecondary schools with the belief that most of what they learned in mathematics is of absolutely no use or value to them in their vocational and day-to-day life. On the other hand, I believe if students have an understanding of the three steps of solving a nonroutine problem and how these steps can be used to improve the quality of their lives, then they are learning both the process skills that are essential to doing mathematics well and a way of problem solving which they can use in their day-to-day life and in whatever vocation they pursue. It was clear in developing and field-testing the curriculum that to insure transfer to the student's day-to-day life that it was not enough to have the student solve nonroutine problems involving only typical areas of mathematical content such as algebra, geometry, and probability. For example, in a fieldtest (with above average students) of a curriculum of ten problems involving typical mathematical content, the students demonstrated an understanding of the three steps in their solutions to the last few nonroutine problems, but there was not evidence of significant transfer to the student's day-to-day life. To help insure transfer, the following steps were taken in developing the full curriculum: (1) an unit was developed that describes the three steps of a nonroutine problem, gives over ten examples of the steps in a variety of nonmathematical fields, has the students research additional examples, and requires the students to be able to define and give examples of the three steps; (2) strands of problems not typically associated with mathematics (e.g., multicultural awareness) and/or problems directly affecting the student's lives (e.g., buying an automobile, taking a trip, or discovering how to live in a more ecologically sound way) account for about half the nonroutine problems; and (3) four problems were included in the last two years of the curriculum which require the student to identify and solve a nonroutine problem in his/her life.

A final point needs to be made about the relative role of an emphasis on content and an emphasis on the problem solving skills required to effectively solve nonroutine problems. In developing the curriculum, a variety of measures were taken to minimize the classtime needed to implement the curriculum, but at the same time to insure that enough time was provided to achieve the objectives of the curriculum. Roughly, the curriculum requires less than 20 percent of the classtime over a four year period. In addition, some of the problems directly concern mathematical content typically covered in the mathematics curriculum. During this school year (1992-3), a full set of fifteen problems is being fieldtested in a variety of typical mathematics classes (e.g., geometry, algebra) and in those classes it appears that we are able to also cover well the traditional curriculum.

RESULTS OF THE FIELD-TESTING

The curriculum has been developed and fieldtested over a ten year period. In the first few years the problems were fieldtested primarily in the context of two high school courses: calculus and problem solving (elective course). The problems were developed as individual enrichment problems or introductions to content units. The problems were not viewed as part of an independent curriculum unit. By accident, it was noticed during a "field-testing" of a sequence of ten problems over the period of one year in a calculus course that at the end of the sequence the students demonstrated a mathematical maturity that I had seldom observed in that course. At that point, I viewed the problems as forming the basis of an independent unit and continued in the next few years to develop, fieldtest, discard, and revise individual problems; and began to experiment with sequences of problems. In 1989 I felt satisfied enough with a sequence of ten nonroutine problems I had developed for my calculus class to publish the curriculum for use with students who were in or had completed an algebra two course. I turned my attention to developing a four year curriculum designed for the average high school student. In the summer of 1992 I was awarded an Alden B. Dow Creativity Summer Fellowship to complete the writing of the curriculum. During the 1992-93 school year, all sixty problems are being fieldtested, sequences of fifteen problems are being tested in academic courses, and a sequence of approximately 30 problems is being tested in an elective problem solving course. The curriculum will be revised in the summer of 1993 and the complete four year curriculum will be formally evaluated beginning in the 1993-94 school year.

The results of the completed field-testing support the following hypotheses: (1) without instruction students do not typically have the skills or attitudes necessary to successfully complete nonroutine problems; (2) a curriculum which primarily requires students to solve nonroutine problems at the appropriate level of difficulty and discuss the process is an effective method of instruction for improving the student's ability to solve nonroutine problems; (3) after approximately ten problems most students demonstrate a clear difference in their ability to solve nonroutine problems and their attitude concerning problem solving; and (4) additional problems help the student to refine his or her skills and improve his or her ability to transfer those skill to different contexts. The significance of the curriculum can probably be best described by reporting the observed effect on the mathematical maturity of the students. Instead of acting in all the ways that we normally attribute to most high school students, the student acts similarly to a "mathematically mature" person. For example, instead of stopping when an obstacle is encountered, the student will persist. Instead of ignoring obvious contradictions or inaccuracies, the student will actively examine them. Instead of being intimidated by ambiguity, the student will tolerate the ambiguity. Instead of being satisfied with the first solution to a problem, the student will work on a problem until a more satisfactory solution is reached. Instead of staring at a problem that seems unsolvable or confusing, the student will try something until the problem naturally becomes clearer.

APPENDIX A
SUMMARY OF NONROUTINE PROBLEMS

Ninth Grade

1. Getting Acquainted. After completing a questionnaire gathering interesting information about them, students develop a similar questionnaire to gather interesting information about their teacher.
2. Axioms of Algebra. Students generate a list of algebraic statements that are always true. The project is evaluated on completeness and conciseness.
3. Scavenger Hunt. Students use a variety of resources to gather certain facts or bits of information (e.g., What size is the largest centipede ever found?). Students must use both written and people resources.
4. Leaves on a Tree. Students calculate the number of leaves on a large tree. They are required to develop at least three methods.
5. Interesting Topic. Using written references including articles, books, and interlibrary loans, students research a topic that they are interested in.
6. Improving the Class. Students generate a list of ways to improve the class based on collected data.
7. Number of Beans. Students estimate the number of beans in a large jar without touching the jar, using at least three methods.
8. Calculating the Area of an Irregular Closed Curve. Students calculate (three methods) the area of an irregular closed curve drawn on 1/4" graph paper.
9. Comparing Products. Students compare the effect on the environment of certain products (e.g., cloth diapers versus disposable diapers).
10. Having Fun in Your Community. Students determine ways to have fun in their local area.
11. Home and the Environment. Students determine how to make their homes more ecologically sound.
12. Persistence. Using questionnaires, students investigate topics (e.g., How do students define a loving relationship?) that require persistence to research satisfactorily.
13. Connecting Points. Students devise a procedure for connecting random points (at least 15) on a sheet of paper so as to minimize the distance required to connect them.
14. Interesting Rituals and Customs. Students find at least five interesting rituals or customs practiced in their community.
15. Buddha. Students calculate the probability of a sea turtle putting its nose through a floating ring in the Pacific Ocean. The problem is based on a quote by Buddha concerning the probability of being born human.

Tenth Grade

1. Collage. Students create a collage that will interest the students in the class. The problem also helps to build a good atmosphere in the class.
2. Improving the School. Students prepare a list of five to ten ways to improve the school. They develop questionnaires to help them with the project.
3. One Inch of Rainfall. Students calculate the volume of water created by one inch of rain in their community.
4. Triangles. Students determine under what conditions two triangles are congruent. They are given two, three or four bits of information about the triangle.
5. Planning a Cultural Trip. Students plan a class trip that is inexpensive, enjoyable, and increases the class' appreciation of the diversity of cultures.
6. Congruency in Quadrilaterals. Students determine under what conditions two quadrilaterals are congruent. They are given four, five or six bits of information about the quadrilateral.
7. Maximum Area. Students determine in a variety of situations how to maximize area given a certain amount of perimeter.
8. School and the Environment. Students determine ways to make their school more environmentally sound.
9. Assigning Grades. Students assign grades given raw scores and two given grades without knowledge of the test or the total number of points on the test.
10. Predicting the Number of Lunches. Students predict the number of lunches that will be served on a given day. They can only use data collected from students to make the prediction.
11. Flipping Coins. Students predict the outcome of tossing ten coins 1000 times (e.g., how many times will the result be 6 heads and 4 tails).
12. Ecologically Sound Lunch. Students plan three lunches that are ecologically sound, relatively inexpensive, and enjoyable to eat.
13. M & M's. Students calculate how many M & M's it would take to fill their classroom. They are required to develop at least three methods.
14. Buying a Car. Students determine the best new car for their values and a given amount of money.
15. Geometric Constructions. Using student developed measuring devices, the students are given a variety of outdoor geometric constructions to complete. Students do not know what constructions they will be given so they must prepare for a variety of problems.

Eleventh Grade

1. Awesome Tape. Students develop a half hour tape of music that the class will like.
2. Dividing a Line Segment. Students devise a method to divide a line in a manner that embodies the Golden Mean.
3. Environmentally Sound Fundraising. Students determine a variety of ways to raise money for school groups that are environmentally sound.
4. Planning and Taking a Class Trip. Students plan a trip that is inexpensive, interesting, and educationally valuable.
5. Problem 1. Students generate a problem of interest and investigate it.
6. Expensive Tape. Students determine three box designs that minimize the need for an expensive tape to wrap it.
7. R(x). Students are given the abstract definition of a function ($R(X) = R(X - 1) + X$) and one value of the function and asked to determine a variety of difficult values (e.g., R(1000)).
8. Area Under a Curve. Students determine the area bounded by $y = x^2$, $y = 0$, $x = 0$ and $x = 3$.
9. Favorite Television Shows. Students predict the favorite television shows of students through the use of a questionnaire.
10. Ecologically Sound Community. Students determine a variety of ways to make their community more environmentally sound.
11. Problem 2. Students generate a problem of interest and investigate it.
12. Finding a speaker. Students find an inexpensive, interesting speaker on the topic of improving appreciation of cultural diversity.
13. F(x). Students are given the abstract definition of a function ($F(ab) = F(a)F(b)$) and three values of the function and asked to determine a variety of difficult values.
14. Making Predictions. Students predict the outcome of certain experiments involving picking objects out of a bag containing 20 green objects, 10 red objects and 5 blue objects (e.g., probability of picking two green objects in a row).
15. Slope. Students develop a method to determine the slope of a variety of nonlinear functions.

Twelfth Grade

1. Generating Nonroutine Problems. Students generate a list of nonroutine problems that the class would be interested in studying.
2. G(x). Given that G(ab) = G(a) + G(b) and three values of G(x), students determine the values of G(x) for x = 1 to 20. Averaging is an unacceptable method.
3. Marking G(x). Students determine a reliable, yet time saving method for grading the G(x) problem.
4. Improving the Community. Students determine actions which would improve the quality of the community in which they live.
5. Increasing Diversity. Students determine how an environmental group can increase the diversity of its membership and better serve a more diverse group of people.
6. Making an Area Equal to One. Students calculate the value of b such that the area bounded by y = 1/x, y = 0, x = 1, and x = b is equal to 1, 2 and 1000.
7. Maximum Volume. Given a piece of construction paper the students construct the solid without a lid that holds the most volume.
8. Profit and the Environment. Students determine a method for calculating profit that takes into account the effect on the environment.
9. Problem 3. Students generate a problem of interest and investigate it.
10. The Best Menu for the School. Students predict the most popular menu for the school based on a questionnaire they develop and administer to a portion of the student body.
11. Environmentally Sound Products. Students develop a list of ten products that they believe would have the most positive effect on the environment if made available to people in the community.
12. Approximating Pi. Students calculate pi in ten different ways. Students are evaluated on accuracy, creativity and variety.
13. Problem 4. Students generate a problem of interest and investigate it.
14. Exploring Functions. Students compare a variety of functions (e.g., log x, x squared, 2 to the x power) in a written project.
15. Planning a Vacation. Students plan an enjoyable vacation with a given amount of money and time.

APPENDIX B
SAMPLE PROBLEM STATEMENTS

EXPENSIVE TAPE
STUDENT PROBLEM STATEMENT

TASK

T e task for this problem is to determine the largest box you can make when you're restricted by (a) the amount of tape you can use to secure it and (b) the way you can secure it. The purpose of this task is to give you practice solving a nonroutine problem involving volume. Specifically:

(1) Your group is a research team for a company. You're given the following problem: the company sends out many packages and the company wants to spend less on packaging. By far the most expensive cost is the special tape you need to use to secure the box. When wrapping the box, you need to tape one length (the longest dimension) and the girth (around the middle of the box - - two widths and two heights). You want to: (a) use 30" of tape per box, (b) have at least two of the three dimensions (length, width, and height) whole number inches, and (c) have the box that can hold the most material. The company wants your three best designs for a box.

(2) Each group will prepare an oral presentation of two minutes or less on the gathered information. I will select the member of your group to present the oral report.

(3) In addition, your group will turn in a written report including your three best designs, how you arrived at your designs, and what other designs you considered.

EVALUATION

(1) 30 points: The quality of your three designs, evaluated by the sum of the three volumes.

(2) 30 points: The quality of the written report. Is the report concise and complete?

(3) 20 points: The quality of the oral presentation.

(4) 20 points: Evaluation of your work on the identified group skills.

F(X)
STUDENT PROBLEM STATEMENT

TASK

The task for this problem is to determine some values for a function, given only the abstract definition and a few values of the function. The purpose of this task is to give you practice solving a nonroutine problem involving an abstract function. Specifically:

(1) You are given the following information about a function F(X): For a, b; $F(ab) = F(a) F(b)$; $F(2) \approx 2.585$; $F(3) \approx 4.505$; and $F(5) \approx 9.070$. As best you can, complete the values of F(X) for X = 1 to 20, rounding your answer to the nearest thousandth. Some values you probably will not be able to determine to the nearest thousandth directly from the given information. Part of the assignment is to determine your best approximation for these values.

(2) You will turn in a written report including the strategies you used and how you determined each of your values.

(3) Only a few students will be asked to orally explain their project, but each student should be prepared to give an oral presentation.

EVALUATION

(1) 50 points: Accuracy of your calculations.

(2) 50 points: Quality of your written report, including clarity concerning how and why you made your calculations.

LEAVES ON A TREE
STUDENT PROBLEM STATEMENT

The task for this problem is to estimate the number of leaves on a tree specified by me. Your group can use any method as long as the method does not cause damage to the tree. This task gives you practice developing and evaluating a number of alternative methods to solve a problem. Specifically:

(1) Your group will develop and evaluate at least three methods to estimate the number of leaves on the tree. You will select one of the methods as your best and indicate your final estimate based on that method.

(2) Your group will prepare an oral presentation of two minutes or less on the gathered information. I will select the member of your group to present the oral report.

(3) Your group will turn in a written description of: the process you used to arrive at your final estimation, why you believe your estimate is valid, and a description of all other methods you considered (minimum of two additional methods).

EVALUATION

(1) 30 points: The judged accuracy of your estimate. The accuracy of your methods will be judged by your fellow students (15 points) and a panel of teachers (15 points).

(2) 30 points: The quality of your documentation and the quality and quantity of the alternative methods you considered.

(3) 20 points: The quality of the oral presentation.

(4) 20 points: Evaluation of your work on the identified group skills.

PERSISTENCE
STUDENT PROBLEM STATEMENT

TASK

Your group task is to learn more about the students in your school. The purpose of this task is to give you practice with a problem requiring persistence. Specifically:

(1) Your group will select three research questions, about students in your school, to answer. You may select any of the questions from the attached handout PERSISTENCE: RESEARCH QUESTIONS. You may make up one question not on the handout, provided that I judge the question to be as complex as the questions on the handout.

(2) Your group will develop a questionnaire that a student can fill out in 10 minutes or less that will help you answer your three research questions. The questionnaire needs to be typed or written neatly in black ink. I will copy the questionnaire and give it to 30 representative students. For this task, you may assume that the 30 students represent the school.

(3) After the completed questionnaires have been given back, your group will analyze the data from the questionnaire and you will have up to 30 days to complete the task. You will be given time to work in your group up to twice a week. The task will be completed when you believe you have answered the research questions to the best of your ability. You will be permitted to collect additional data; for example, if you wish to develop a second questionnaire, I will give the questionnaire to 30 representative students.

(4) When all the groups are finished, each group will prepare an oral presentation of five minutes or less on the gathered information. I will select the member of your group to present the oral report.

(5) In addition, your group will be required to hand in a written summary of your conclusions including how you organized and compiled your data, and what data supports your conclusions.

EVALUATION

(1) 20 points: The quality of your answers to the three research questions. Did you discover interesting conclusions about your questions? Was your research thorough?

(2) 20 points: How well you analyzed your data. Were your conclusions valid? Did you fully analyze your data?

(3) 20 points: The quality of the questionnaire(s) you developed. Were the questions likely to give you good data to answer your research questions? Was the questionnaire concise and clear?

(4) 20 points: The quality of the oral presentation.

(5) 20 points: Evaluation of your work on the identified group skills.

Note well: Your choice of topics will not affect the evaluation. Also, the length of time you spend on the project will not affect your grade; that is, if the quality of the work is the same, two groups that spend different amount of time on the task will receive the same number of points.

PERSISTENCE: RESEARCH QUESTIONS

1. How does having a job influence the life of a student?

2. What are the three improvements in the school that students believe are most important?

3. Are there differences between honor students and other students?

4. Are there differences between athletes and other students?

5. What makes someone stand out as a trend setter in this school?

6. What do students think makes a good teacher?

7. How do students define a loving family?

8. Whom do students consider their heroes/heroines?

9. How do students learn best?

10. What do students do to relax?

11. Where do students feel safe?

12. Why do students have pets?

13. What does it mean to be patriotic?

14. What are the world or political issues most on students' minds?

GETTING ACQUAINTED
STUDENT PROBLEM STATEMENT

TASK

Your group task for this problem is to gather interesting information about me, your teacher, by means of a questionnaire. The purpose of this task is to introduce you to the process of gathering information and to help us get acquainted. Specifically:

(1) Each group will develop a questionnaire for me consisting of a maximum of seven questions, each of which asks a question that you think will result in interesting information about me.

(2) After I have selected three of the best questions from each group and responded in writing to those questions, each group will prepare an oral presentation of two minutes or less on the gathered information. I will select the member of your group to present the oral report.

(3) Your group will turn in a description of the process you used to select your questions. The description should not exceed two pages and should include information such as: consultations with people outside your group, the method your group used to generate questions, and your rationale for retaining or eliminating questions from your completed questionnaire.

EVALUATION

(1) 40 points: The quality of your questions. The evaluation will be based on your list of questions and your description of the process you used to select your questions. In determining your points, I will evaluate whether the questions were likely to produce interesting information about me from the perspective of students, and whether your efforts when beyond the ordinary in your search for interesting information.

(2) 20 points: Class evaluation of the interest level of the information in your group's oral presentation. Students will rate the information on a 1 (not interesting at all) to 5 (very interesting) scale.

(3) 20 points: The quality of the oral presentation. I will base my evaluation primarily on the level of preparedness of the presentation (e.g., is the presentation well organized?).

(4) 20 points: Evaluation of your work on the identified group skills.

BUDDHA
STUDENT PROBLEM STATEMENT

" There's a story that once the Buddha walked with his monks by the seaside and he said to them, 'Monks, if there were a blind turtle swimming in the oceans of the world and also a wooden yoke, and this blind turtle came up for air once every hundred years, do you think, monks, that this blind turtle could put her head through that wooden yoke?' The monks said, 'No, sir. That's impossible. They couldn't be in the same place at the same time if they're swimming around in the oceans of the world.' The Buddha said, 'No. It's not impossible. It's improbable, but not impossible.' And he added, 'The same improbability reigns over being reborn a human being.' "

TASK

The task for this problem is to calculate the probability of a blind sea turtle, coming to the surface of the Pacific Ocean, putting its nose through a ring randomly floating on the surface of the Pacific Ocean. The purpose of this task is to give you practice solving a nonroutine problem involving probability. Specifically:

(1) As stated above, the task is to calculate the probability of a blind sea turtle, coming to the surface of the Pacific Ocean, putting its nose through a ring randomly floating on the surface of the Pacific Ocean. You will need to make some assumptions (e.g., the size of the ring), and use some written resources to complete the problem.

(2) You will submit a written report documenting how you solved the problem, including false starts, and your final calculation of the probability (expressed as a decimal).

(3) Only a few students will be asked to orally explain their project, but each student should be prepared to give an oral presentation.

EVALUATION

(1) 50 points: Accuracy of your calculations.

(2) 50 points: Quality of your written report, including clarity concerning how and why you made your calculation.

IMPROVING THE SCHOOL
STUDENT PROBLEM STATEMENT

TASK

The task for this problem is to identify ways that this school can be improved. The purpose of this task is to give you practice solving problems using a questionnaire as a tool and requiring persistence. Specifically:

(1) Each group will prepare a list of between five and ten ways to improve this school. Your work on this task will be in two parts. In the first part, your group will brainstorm ideas for improving the school and, based on those ideas, prepare a list of tentative ideas for improving the school and a questionnaire to gather information from students in this class to determine whether your ideas are good. I will copy and distribute the questionnaires and return the completed questionnaires to your group.

(2) In the second part, you will analyze the data from the questionnaires and prepare your final list. To prepare your final lis' your group may construct and administer a second questionnaire as long as the questionnaire addresses items on your first list or new items not on other group's original list. Your group's final list can only contain ideas on your original list or new ideas not on any other group's original list.

(3) Each group will hand in a written report including the group's final list of improvements and documentation of the group's process of selecting the final list. Each group will prepare an oral presentation of two minutes or less on your work.

EVALUATION

(1) 40 points: The quality of your items, based on whether your ideas are practical (e.g., can it be carried out? Is it counter to school policy?) and how students and I rate the ideas on their likelihood to improve the school.

(2) 20 points: Completeness of your written description of how you developed your final list.

(3) 20 points: The quality of the oral presentation as rated by the students (10 points) and the teacher (10 points).

(4) 20 points: Evaluation of your work on the identified group skills.

PLANNING A CULTURAL TRIP
STUDENT PROBLEM STATEMENT

TASK

The task for this problem is to plan a trip that will increase the class' appreciation of other cultures. The purpose of this task is to solve a problem that requires you to use written and people resources, and will increase the class' appreciation of other cultures. Specifically:

(1) Each group is to plan an one day class trip. The trip should have three characteristics: (a) it has educational value and will increase the class' appreciation of other cultures, (b) it is interesting and enjoyable from the point of view of students, and (c) it is inexpensive. Your group's plan should include transportation, food considerations, schedule, costs and a description of the day's activities.

(2) The first activity in your groups will be to brainstorm what resources your group can use to plan this trip, including written and people resources. In addition, your group can prepare a questionnaire to gather data from the students concerning their interests in various ideas.

(3) Each group will prepare an oral presentation of two minutes or less on the gathered information. In your presentation, you are trying to convince the class that your group's proposed trip is the best. I will select the member of your group to present the oral report.

(4) In addition, your group will turn in a written report including what resources you used, how you decided on the details of the trip, and a description of the trip.

EVALUATION

(1) 30 points: The quality of the trip. I will judge whether the trip has educational value and the class will judge the quality of the trip from the point of view of students on the three criteria listed above (educational value, interest level and cost).

(2) 30 points: The quality of the written report. Were a variety of resources used? Is the process of deciding on the details of the trip clear and concise?

(3) 20 points: The quality of the oral presentation. The class will rate their interest in the trip based on your presentation.

(4) 20 points: Evaluation of your work on the identified group skills.

BUYING A CAR
STUDENT PROBLEM STATEMENT

TASK

The task for this problem is to determine the car in a given price range which would be most consistent with what you value in a car. The purpose of this task is to give you practice solving a nonroutine problem involving a real life application and the use of resources. Specifically:

(1) You will select a price range from the choices I give you and determine the car in that price range which would be most consistent with what you value in a car.

(2) You will turn in a written report including what resources you used to make your decision, how you arrived at your decision, and why you feel your decision is a good one. You should indicate what features and characteristics in a car you value and how your choice is consistent with those values.

(3) Only a few students will be asked to orally explain their project, but each student should be prepared to give an oral presentation.

EVALUATION

(1) 40 points: Quality of the resources you used in making your decision. Were you persistent in finding resources to help you make your decision?

(2) 60 points: Quality of your written report, including clarity concerning how and why you made your decision.

INDIVIDUAL PROBLEM, NUMBER 1
STUDENT PROBLEM STATEMENT

TASK

The task for this problem is to define and solve a problem which is significant to you. The purpose of this task is to give you practice solving nonroutine problems in your own life. Specifically:

(1) You will need to define your problem and get the problem approved by me. You will submit a written summary of the problem, how you intend to work on solving it, and how long you believe it will take you to complete the project. The problem will be approved if I am convinced that the problem is meaningful to you and the problem is significant (i.e., the problem would require the three steps to solve it).

(2) When the project is completed, you will turn in a written report including your solution to the problem, and a description of how you arrived at your solution. You should include anything you learned about problem solving in your description.

(3) Only a few students will be asked to orally explain their project, but each student should be prepared to give an oral presentation.

EVALUATION

(1) 60 points: The quality of your solution. Did you fully investigate your problem? Is your solution consistent with your data?

(2) 40 points: Quality of your written report, including clarity, conciseness, and comprehensiveness of your written report.

ENVIRONMENTALLY SOUND PRODUCTS
STUDENT PROBLEM STATEMENT

TASK

The task for this problem is to develop a list of ten products that would have the most positive effect on the environment if made available to the people of your community. The purpose of this task is to give you practice solving a nonroutine problem affecting the environment and your life. Specifically:

(1) For this problem, each group will research at least ten products that would have a positive effect on the environment if made available to the people of your community. You will probably need to use a variety of resources, including written and people resources.

(2) You want your products to be ecologically sound (minimize negative effects on the environment and maximize positive effects on the environment), but also ones that people would buy.

(3) Each group will prepare an oral presentation of two minutes or less on the gathered information. I will select the member of your group to present the oral report.

(4) In addition, your group will turn in a written report including a final list of your ten products, a description of how you arrived at your final list, and what resources you used, including resources that did not result in any new ideas.

EVALUATION

(1) 30 points: The quantity and quality of your final list. The quality will be judged by how likely it is that people would buy the products and how positive of an effect the products would have on the environment.

(2) 30 points: The quality of the written report. Is it clear how you arrived at your final decision and why? Does the report indicate the resources you investigated and how they were used?

(3) 20 points: The quality of the oral presentation.

(4) 20 points: Evaluation of your work on the identified group skills.

APPENDIX C
DRAFT: TEACHING SUGGESTIONS

STRANDS

In developing the sequence of problems for this curriculum, many factors were considered to insure a balanced and effective curriculum. One major factor was the inclusion of certain strands of content, with a planned four year sequence of problems within each strand. The seven strands are: (1) introductory problems for each year which help establish a good classroom atmosphere; (2) area and volume problems; (3) functions with a focus on understanding abstract concepts; (4) prediction, including problems involving principles of probability and problems in which students gather information, primarily through questionnaires, and make predictions; (5) multicultural awareness, problems which attempt to increase the student's appreciation of diversity, especially diversity of ethnic groups and cultures; (6) ecology with an emphasis on problems directly affecting their lives; and (7) practical applications affecting their lives, including a sequence of four nonroutine problems (starting in the third year) that they define and solve.

DIFFICULTY LEVEL

In my opinion, the most effective way to improve students' ability to solve nonroutine problems is to repeatedly put them in a situation in which they are given a nonroutine problem to solve, have them work on the problem for at least a week and generate their best solution, and then discuss the problem as a class. For this method to be effective, it is <u>essential</u> that the problems be at the right level of difficulty. The problem has to be "appropriately challenging". Practically speaking, this means that the problem cannot be too easy (e.g., a student reads the problem; after a few moments "sees" a solution; and then writes up the details), nor too difficult (e.g., the student has no idea how to even start the problem and finds the problem completely frustrating). Typically, the best results occur under the following type of circumstances: the student reads the problem, has no clear idea what the solution is but has a few ideas about how to start, tries something and becomes clearer about the solution, eventually finds a good solution, and, from the processing of the problem, realizes some ways he or she could have improved the process or solution.

In developing the sequence of problems, I have attempted to arrange the problems in a sequence that gradually increases the difficulty level of the problems. However, the sequence needs to be adjusted for every group. The major ways you can affect the difficulty level of the problems are: (1) Change the grouping for the problem. The problem becomes easier as you move from problems assigned to an individual, to groups, to the class as a whole. In the curriculum, there are no problems completed as a class; however, this is an appropriate option if a problem seems frustrating to a large number of the students. (2) Brainstorm different possible strategies for a problem as a class. (3) Give the students specific help or hints on appropriate problems. (4) Have the students complete progress reports part way through the project and give the students feedback on their strategy. In addition, a useful general policy is to let the students know that they can see you individually (or as a working group) when the problem becomes frustrating, and that you will give them the assistance necessary to make the problem at the appropriate level of difficulty.

At the same time, it needs to be clear that in an effective sequence it is normal that each problem is difficult. It is normal that the student would not be clear about how to solve the problem initially; that at times it would feel as if the problem was not solvable; that the first attempts at solution would seemingly lead nowhere; that one would need to persist until the problem eventually became clear - - these are normal features of nonroutine problems which students have to experience to become good problem solvers! These features differ from unproductive frustration and need to be recognized and encouraged as normal. It is my experience that it usually takes about ten problems before students begin to understand these features and the steps of a nonroutine problem At that point it is as if the student has been transformed mathematically! Instead of acting in all the ways that we normally attribute to most high school and college students, the student acts similarly to a "mathematically mature" person. For example, instead of stopping when an obstacle is encountered, the student will persist. Instead of ignoring obvious contradictions or inaccuracies, the student will actively examine them. Instead of being intimidated by ambiguity, the student will tolerate the ambiguity. Instead of being satisfied with the first solution to a problem, the student will work on a problem until a more satisfactory solution is reached. Instead of staring at a problem that seems unsolvable or confusing, the student will try something until the problem naturally becomes clearer. So, for probably at least ten problems, we need to be patient and supportive, and allow the students to experience and learn from difficult, yet not frustrating, problems.

STUDENT PROBLEM STATEMENT: OVERVIEW

For each problem, the student is given a student problem statement which summarizes the problem, outlines the steps needed to complete the problem, and lists the criteria for evaluating the student's work. The following sections clarify different sections of the student problem statement.

INDIVIDUAL VERSUS GROUP PROBLEMS

Each problem is identified as being completed either individually or in small groups. Approximately 2/3 of the problems are completed in groups and 1/3 completed individually. From an instructional point of view this distribution seems appropriate. If well organized (see the section on group skills), group work provides a rich environment for stimulating the improvement of problem solving skills. Individual problems give the student an excellent opportunity to internalize and apply what is experienced in the group context. Also, the individual problems give the teacher a chance to more clearly see what progress individual students have made.

As mentioned earlier, a given problem is usually more difficult when tackled individually versus by a group. Therefore, if you believe the class or some individuals are not ready for solving a nonroutine problem individually, it would be appropriate to allow them to work in groups (or provide enough hints to convert the problem into one more appropriate).

WRITTEN REPORTS AND ORAL PRESENTATIONS

Each problem requires the students to submit a written report documenting the process of solving the problem. The written report provides the basis for organizing the processing of the problem; consequently, from an instructional point of view, the written report needs to clearly and concisely document the process of solving the problem, including all strategies considered (even strategies that did not seem useful), the steps taken to solve the problem, and why the steps were taken. It has been my experience that it takes some time before students can document their work well in the described manner. Therefore, for the first few problems you need to pay particular attention to helping students make the transition to adequately documenting their work. One way to accomplish this is to focus attention on the written reports during the processing of the first few problems. This can be done in a number of ways. For example: (1) you can give back the written reports with specific comments on what needs to be added or clarified and have the groups rewrite the reports; (2) you can select two portions from each report, one positive, one requiring improvement, and discuss the reports as a class during the processing; (3) you can take one or a few typical written reports and discuss them in depth as a class, or (4) you can require an outline of the written report before it is completed. In any case, an investment of time in the beginning of the curriculum will prove fruitful later in the year.

Each problem solved in groups includes an oral presentation and each individual problem has a provision for selective oral presentations. In addition to giving the students experience communicating orally concerning their work, the oral presentation provides an opportunity to insure that each student in a group understands the group's solution, the steps in arriving at that solution, and why the steps were taken. Normally the teacher randomly selects the person from the group to give the oral presentation and the presentation affects 20% of the grade. This provision is usually enough to insure that most, if not all, students will at least understand the group's work (measures to insure that each student participates in the solution are discussed later).

Students should be encouraged to vary their oral presentations by including visual aids, involving the class, or finding creative ways to present their material. Generally, the problem statements allow for two minute presentations. Usually this is an adequate amount of time, allowing for a fair view of the group's work, yet preserving precious class time.

The oral presentations can be evaluated by you only, the students only, or both you and the students. The directions leave this open to your discretion. Personally, I find varying the method of evaluating the oral presentations effective. A sample form for evaluating the oral presentations is included with the curriculum.

As was mentioned, each problem provides for an oral presentation and a written report. However, once students are preparing these presentations and reports adequately, it is not necessary to include both for each problem. For example, for some problems, perhaps just an oral presentation is required; for others only a written report. Also, you can omit the provision for an oral presentation with the understanding that you might call on a few students to check for understanding of their group's work.

GROUP SKILLS

I am convinced that a cooperative group model is very consistent with the purposes of this curriculum. Three major reasons for this opinion are: (1) students need to learn to work effectively in groups--most real life problems lend themselves to group problem solving and most jobs require you to work effectively in groups; (2) students benefit from being exposed to the strategies and thinking of other students; and (3) group work provides a supportive atmosphere for dealing with the natural difficulty of the nonroutine problems. The curriculum is constructed so that approximately two thirds of the work is done in cooperative groups. It is beyond the scope of this curriculum to provide comprehensive directions for effectively using cooperative groups. However, since the proper use of groups is an integral part of this curriculum, I have provided an introductory unit in the curriculum on basic group problem solving skills including nonjudgmental listening, paraphrasing, asking clarifying questions, constructing I-messages, and brainstorming, plus a section consisting of a few one day group problem solving tasks. The unit is meant to be completed before the students start work on the nonroutine problems.

In the evaluation section of the student problem statements, there is an indication that the evaluation of the group's work on the identified group skill is worth, in most cases, 20 percentage points. The group skill or method of evaluation is not identified to allow you to pick the most relevant skill. While this curriculum is structured to emphasize work in cooperative groups, the curriculum itself does not contain specific instructions concerning the sequencing of group skills and specific exercises or group skills to focus on in each problem. It is my experience that the appropriate group skill(s) to focus on naturally presents itself for each individual class of students. Therefore, it would be inappropriate and counterproductive to identify the specific group skill for <u>your</u> class to focus on during any given problem. However, I will include here one list of group skills from a handout based on Johnson and Johnson's work. The list is divided into four levels of skills, and, in my opinion, provides an excellent framework for planning what skills to focus on with your class. The skills are: (1) Forming Skills: moving into groups quietly, staying with the group, using quiet voices, encouraging everyone to participate, keeping hands and feet to self, looking at the group's paper, using people's names, looking at the speaker, and using no put-downs; (2) Functioning Skills: stating and restating the purpose of the assignment, setting or calling attention to time limits, offering procedures on how to most effectively do the task, expressing support and acceptance verbally, expressing support and acceptance nonverbally, asking for help or clarification, offering to explain or clarify, paraphrasing and clarifying other member's contributions, energizing the group with humor, ideas, or enthusiasm, and describing feelings when appropriate; (3) Formulating Skills: summarizing the material aloud, seeking accuracy by correcting and/or adding to summaries, seeking elaboration by relating to other learning or knowledge, seeking clever ways of remembering ideas and facts, demanding vocalization of other member's reasoning processes, and asking members to plan aloud how to teach material to others; and (4) Fermenting Skills: criticizing ideas without criticizing people, differentiating where there is disagreement, integrating different ideas into a single position, asking for justification of others' conclusions or ideas, extending other members' answers or conclusions, probing by asking questions that lead to deeper analysis, generating further answers, and testing reality by checking group's work against instructions.

In addition, I want to discuss the following characteristics of cooperative group work that are integral to this curriculum: (1) There is a consistent emphasis in this curriculum on specific skills necessary to work effectively in groups; these skills form a relatively independent curriculum that develops parallel to the development of problem solving skills. Although independent, development of these group skills affects both the quality of the solutions and learning for the nonroutine problems. A suggested list of group skills and exercises to develop them are included with the curriculum. (2) The nature of the group work should be interdependent. Some of the group skills stressed above focus on developing interdependence in the group; however, there is certainly an additional need to focus on this issue. Some advocates of cooperative groups recommend the use of roles (e.g., recorder, paraphraser, observer) in the group to help foster interdependence. Brainstorming also helps to start the group out sharing ideas. In addition, you can have students work on the problem perhaps one night individually before groups are formed and then have each person take a few minutes to share his or her ideas in the group. In general, you need to be aware of the level of interdependence in the groups and take steps to insure a gradual improvement in these skills. (3) Students need to be held accountable for their individual understanding of the group's work. The oral presentations provide the simplest way to check accountability. In addition, you need to monitor informally the quality of the contribution of each student to his or her group's work, and, when you observe difficulties, take steps to improve the situation. (4) The groups are expected to work outside of class and many times need to meet as a group. Therefore, in forming the groups you need to insure that students are in a group which can get together outside of class time. In forming groups, I ask students to specify what times they can meet outside of class. (5) Varying the composition of groups is useful. If you divide the class according to ability to solve nonroutine problems (as best you can...) into three groups: above average, average, and below average, then groups can be composed in a variety of ways, including groups with one student from each ability group; students all from the same ability group; or some groups with highs and average, others with average and below average. Students learn different types of things from each type of grouping. In addition, after students have mastered the basic skills of working with a teacher generated group of students, I allow the students to pick their own groups for at least some of the problems.

EVALUATION CRITERIA

Each student problem statement lists the criteria for evaluation of the problem, describing each component of the evaluation and how many points each component is worth. There are no directions given for determining exactly how to assign points to each category - - that is left to each individual teacher. Personally, I believe the experience of working on the nonroutine problems should be a positive experience which suggests that unnecessary concern for the grade that will be given would be inappropriate. For me, this implies a grading system such as: 90 - 100 % means excellent work, indicating insight and/or effort beyond what one would normally expect; 80 - 89 % means good to very good work, indicating that the student spent a good amount of time on the problem, documented the work and gained reasonable insight into the problem; 70 - 79 % means fair work, indicating that the student put in at least the minimal acceptable effort but clearly could have gone at least a step further ; and unacceptable, meaning that the student clearly did not spend enough time on the problem to learn anything significant and that the work is incomplete. The primary criteria for the grade in this system is the amount of effort the student puts into the problem, the quality of the effort primarily affects the grade within each range. For example, a 80 and 85 might indicate the same general level of effort but the 85 would mean that the quality of that student's or group's insight was better. In practice, approximately 10 - 20 % of the students receive a grade in the 90 - 100 % range; 40 - 60 % in the 80 - 89 % range; 20 - 30 % in the 70 - 79 % range; and, at most, one or two students in the unacceptable range. At times, if I feel that the quality of the class' work (or a significant subset of the class) is not up to par, I give back the assignments and allow additional time to work on the problem, giving some hints if appropriate.

I believe that the nonroutine problems should be seen as an integral part of the grading process for the course. Therefore; I see them worth about 20% of the total grade. For me, this is approximately equivalent to one test grade per quarter; consequently, I average the grades for the nonroutine problems for the quarter and count them as one test grade. Other teachers might prefer assigning a grade and weight for the nonroutine problems, separating that portion of the grade from the other content.

BRAINSTORMING

The ability to generate a variety of ideas and strategies is a key skill for many of the problems; therefore, the introductory unit on basic group problem solving skills includes a section on brainstorming and many of the problems include brainstorming as a first step. In addition, students should be encouraged to use brainstorming whenever they seem stuck for ideas or how to proceed. You might want to occasionally check the quality of the brainstorming of the groups to insure that groups are using the tool properly.

PROGRESS REPORTS

Some of the key nonroutine problems include a provision for progress reports. The progress report is just a simple summary of what has been done so far. It allows the teacher not only to check that the student has been working on the assignment, but also gives the teacher the opportunity to give assistance or hints to a few appropriate students. In addition to when the progress report is suggested in the directions, a progress report would be appropriate whenever you believe it is needed to insure a reasonable level of quality. Also, you might assign a progress report for certain groups or individuals whose work has not been as consistent as other students' work.

PROCESSING

To help the students, the teacher needs to have an overall sense of the purpose and nature of the sequence of problems. Specifically, the teacher needs to be familiar with the three steps of problem solving (recognition that a problem exists, trying something, and persistence) and other problem solving skills (finding patterns, finding and using a variety of resources, developing procedures, manipulative skills and problem reduction) which have already been discussed. When processing individual problems, the teacher should focus attention on these steps and skills whenever possible, especially when a student's solution is a positive or negative example of how to use the steps or skills.

The teacher also needs to realize that typical students require experience working on eight to ten problems before they can be expected to achieve the type of quality in solutions that we might hope for in the beginning of the curriculum. Therefore, we need to have reasonable expectations and patience when processing the problems. It has been my experience that patiently contrasting different student solutions or contrasting teacher prepared solutions with students' solutions is quite beneficial over the course of a school year.

As can be deduced from what I have said so far, the processing of the problems is very important and needs to be carefully prepared. To be valuable it almost necessarily needs to respond to how your students approached the problem. Therefore, I find it helpful in preparing the processing to review each solution and try to pull out selections which will help focus attention on key points. I must emphasis that the focus of the processing needs to come from the students' solutions and attempts.

What else is important in the processing? In addition to emphasizing the already mentioned problem solving skills, the teacher can use the processing to draw attention to well documented solutions, the clever use of resources, or unique approaches. Also, the processing can be an opportunity to examine common errors that students make such as not persisting long enough with a problem, generalizing with too little data, carelessness and not exploring conflicting data or obvious errors.

EXTENSIONS AND ENRICHMENTS

After the initial focus on the students' solutions, it is appropriate to then discuss classical or other outstanding solutions and to discuss the historical significance of the problem. An outline for this part of the processing has been included with the directions for each nonroutine problem. Also, extensions of the problems have been suggested when appropriate.

Obviously, each teacher has to decide how much time, if any, is available for any given problem for enrichment or extensions. Typically, due to time restraints, the teacher will have to pick the extensions which seem most likely to interest his or her students. There are a number of formats for enrichment or extensions that should be considered, including: whole class discussions; extra credit assignments; a second related nonroutine problem completed individually or in groups; or a requirement that students select a certain number of enrichment activities or extensions in a given period of time to complete.

A CURRICULUM OF NONROUTINE PROBLEMS: SOME CONSIDERATIONS

A curriculum of nonroutine problems is not like the traditional mathematics curriculum, and cannot be effectively taught with the same expectations. At a certain point in the curriculum most students feel comfortable with the three steps of a nonroutine problem and even feel a sense of adventure when given the next problem. In contrast, in the beginning of the curriculum, many students feel uncomfortable with the three steps and wish that the problems were "easier" or not part of the curriculum at all. I do not believe that these initial feelings are natural, but rather the result of instructional practices over many years which prevent students from experiencing the three steps, and, in fact, encourage behaviors contradictory to what is required to solve a nonroutine problem. The curriculum is a gradual process of helping the student feel comfortable with the process of solving a nonroutine problem. For the curriculum to work, it is essential that the teacher understand this aim of the curriculum and feel comfortable supporting the students through the gradual process of changing the way they view mathematics and solve problems. I can guarantee that the required patience and gentle insistence on the aim of the curriculum will reap a bountiful harvest!

For the first nonroutine problems, it would be wise to spend some time discussing expectations. For example, you might discuss the following points: (1) the problems require at least a few hours work spread over the period of one to three weeks - they will not be able to answer the problems adequately if they save the assignment for the last day or two; (2) part of the assignment is to document all nontrivial strategies and solutions attempted as well as the final solution, including justification for selecting the final strategy; (3) there is not one "best" solution for these problems, but rather a variety of good solutions they could come up with, including some novel solutions; and (4) it is quite common with a nonroutine problem that they will "solve" the problem one way and then realize a method to refine the solution - point out that you encourage this and want them to document it as part of the assignment.

There are probably a variety of ways to effectively structure the assignments. A structure which has worked well for me is to assign one problem every two weeks. The problem is introduced on a Monday and collected two weeks later. On occasion I will give the students a week or two off either to allow for a more intense focus on the content of the course or just to give them a break from nonroutine problems. In giving the specific directions for each problem it is important to give the students enough information so that they can engage the problem but, at the same time, not give them so much information that the problem becomes straightforward. If the directions are good, the solution will not be straightforward yet each student will be able to work on the problem without unproductive frustration and will be able to generate a solution - in summary, the problem will be challenging yet not impossible.

The teacher should encourage the students to work at least some on the problem the first night, emphasizing that this type of start allows ideas to develop even when the student is not directly working on the problem. Also, I like to communicate to students that they are encouraged to use other resources. For example, for each problem I like to make sure that the students realize I am available as a sounding board for ideas. When students do approach me for assistance I certainly do not tell them how to do the problem but rather try to give the students feedback or hints which will allow them to work productively. Also, they are encouraged to use outside references and resources when appropriate such as computers, science equipment, and math books from previous courses. Of course, a distinction is made between using a resource as an aid in solving a problem and using a resource to solve a problem - it is inappropriate to ask a mathematician for a solution or to try and look up a solution in a book.

CONCLUDING REMARKS

In this introductory chapter I have attempted to give the reader a sense of what is meant by a curriculum involving a sequence of nonroutine problems. Hopefully, I have conveyed my enthusiasm for this curriculum - I believe that a sequence of nonroutine problems has a richness to it that allows the student to reach a mathematical maturity not typically aimed for in other materials. Though I have been enthusiastic about the wide variety of new materials which seem to improve basic problem solving skills, these materials only seem to touch the surface of skills that have been understressed.

Obviously, the sequence is more than sixty separate problems. This chapter tries to provide the theory and suggestions necessary for the reader to see the connections among the problems, thereby facilitating the creation of a learning environment conducive to mastering the objectives of the sequence. I have tried to be complete in the suggestions in this chapter and the remainder of the book, but one essential suggestion has been omitted - allow yourself to be open to the richness of the problems, to the variety of solutions and to the opportunities for growth that the student efforts will create.