DOCUMENT RESUME

ED 341 584 SE 052 751

AUTHOR	Prawat, Richard S.; And Others
TITLE	Experts' Views on the Elementary Mathematics Curriculum: Visions of the Ideal and Critique of Current Practice. Elementary Subjects Center Series, No. 44.
INSTITUTION	Center for the Learning and Teaching of Elementary Subjects, East Lansing, MI.
SPONS AGENCY	Office of Educational Research and Improvement (ED), Washington, DC.
PUB DATE	Nov 91
CONTRACT	G0087C0226
NOTE	70p.
AVAILABLE FROM	Institute for Research on Teaching, 252 Erickson Hall, Michigan University, East Lansing, MI ($6.25).
PUB TYPE	Reports - Descriptive (141)
EDRS PRICE	MF01/PC03 Plus Postage.
DESCRIPTORS	*Cognitive Development; Cognitive Processes; Concept Formation; *Curriculum Development; *Curriculum Research; Elementary Education; Elementary School Mathematics; Interviews; Learning Strategies; *Mathematics Curriculum; Mathematics Education; *Mathematics Instruction; Opinions; Teaching Methods
IDENTIFIERS	*Constructivist Learning; Subject Content Knowledge

ABSTRACT

As part of a larger study regarding an ideal curriculum in five subject matter domains for the elementary school, the views of seven experts in mathematics education were summarized and contrasted. Four experts were nationally and internationally known professors and researchers, and three were elementary school teachers selected by researchers familiar with their teaching and known for their ability to promote mathematical understanding in children. These experts were asked to treat the topic of mathematics education comprehensively by addressing issues of curriculum (goals and objective, selection and organization of content), materials and instruction (presentation of input to students, teacher-student discourse, activities and assignments), evaluation of student learning (formal and informal assessment of student progress toward key goals before, during, and after instruction), and teacher education (subject matter knowledge, professional development). The views of the experts are summarized one at a time, under the categories of: (1) General Approach to Mathematics Learning in School; (2) Learning and Teaching; and (3) Approaches to Curriculum. The overall summary indicated that all the experts were dissatisfied with prevailing mathematics curricula and teaching practice in elementary school with its over emphasis on learning isolated computational skills. Other areas of agreement and disagreement are contrasted, and these comparisons are considered with reference to their implications for ideal mathematics programs. Readers are cautioned that the call for change may be based on multiple, and possibly incompatible, assumptions. (14 references) (MDH)

IRT
price
$6.25

ED341584

Elementary Subjects Center
Series No. 44

EXPERTS' VIEWS ON THE ELEMENTARY MATHEMATICS CURRICULUM: VISIONS OF THE IDEAL AND CRITIQUE OF CURRENT PRACTICE

Richard S. Prawat, Ralph T. Putnam,
and James W. Reineke

Center for the
Learning and Teaching
of Elementary Subjects

Institute for
Research on Teaching

College of Education

Michigan State University

SP033751

Elementary Subjects Center
Series No. 44

EXPERTS' VIEWS ON THE ELEMENTARY MATHEMATICS CURRICULUM: VISIONS OF THE IDEAL AND CRITIQUE OF CURRENT PRACTICE

Richard S. Prawat, Ralph T. Putnam,
and James W. Reineke

Published by

The Center for the Learning and Teaching of Elementary Subjects
Institute for Research on Teaching
252 Erickson Hall
Michigan State University
East Lansing, Michigan 48824-1034

November 1991

This work is sponsored in part by the Center for the Learning and Teaching of Elementary Subjects, Institute for Research on Teaching, Michigan State University. The Center for the Learning and Teaching of Elementary Subjects is funded primarily by the Office of Educational Research and Improvement, U.S. Department of Education. The opinions expressed in this publication do not necessarily reflect the position, policy, or endorsement of the Office or Department (Cooperative Agreement No. G0087C0226).

Center for the Learning and Teaching of Elementary Subjects

The Center for the Learning and Teaching of Elementary Subjects was awarded to Michigan State University in 1987 after a nationwide competition. Funded by the Office of Educational Research and Improvement, U.S. Department of Education, the Elementary Subjects Center is a major project housed in the Institute for Research on Teaching (IRT). The program focuses on conceptual understanding, higher order thinking, and problem solving in elementary school teaching of mathematics, science, social studies, literature, and the arts. Center researchers are identifying exemplary curriculum, instruction, and evaluation practices in the teaching of these school subjects; studying these practices to build new hypotheses about how the effectiveness of elementary schools can be improved; testing these hypotheses through school-based research; and making specific recommendations for the improvement of school policies, instructional materials, assessment procedures, and teaching practices. Research questions include, What content should be taught when teaching these subjects for understanding and use of knowledge? How do teachers concentrate their teaching to use their limited resources best? and In what ways is good teaching subject matter-specific?

The work is designed to unfold in three phases, beginning with literature review and interview studies designed to elicit and synthesize the points of view of various stakeholders (representatives of the underlying academic disciplines, intellectual leaders and organizations concerned with curriculum and instruction in school subjects, classroom teachers, state- and district-level policymakers) concerning ideal curriculum, instruction, and evaluation practices in these five content areas at the elementary level. Phase II involves interview and observation methods designed to describe current practice, and in particular, best practice as observed in the classrooms of teachers believed to be outstanding. Phase II also involves analysis of curricula (both widely used curriculum series and distinctive curricula developed with special emphasis on conceptual understanding and higher order applications), as another approach to gathering information about current practices. In Phase III, models of ideal practice will be developed, based on what has been learned and synthesized from the first two phases, and will be tested through classroom intervention studies.

The findings of Center research are published by the IRT in the Elementary Subjects Center Series. Information about the Center is included in the IRT Communication Quarterly (a newsletter for practitioners) and in lists and catalogs of IRT publications. For more information, to receive a list or catalog, or to be placed on the IRT mailing list to receive the newsletter, please write to the Editor, Institute for Research on Teaching, 252 Erickson Hall, Michigan State University, East Lansing, Michigan 48824-1034.

Abstract

This report summarizes and contrasts the views of seven experts in mathematics education. Four of these experts are nationally and internationally known professors and researchers, and three are elementary school teachers selected by researchers familiar with their teaching and known for their ability to promote mathematical understanding in youngsters. These experts were asked to treat the topic of mathematics education comprehensively by addressing issues of curriculum (goals and objectives, selection and organization of content), materials and instruction (presentation of input to students, teacher-student discourse, activities and assignments), evaluation of student learning (formal and informal assessment of student progress toward key goals before, during, and after instruction), and teacher education (subject matter knowledge, professional development). The experts addressed these issues in the context of ideal programs, as outlined in their responses to a set of questions about what ideal curriculum, instruction, and evaluation practices in elementary mathematics programs might look like, and more typical current practice, as outlined in their responses to questions calling for critique of one of the most widely adopted mathematics curriculum series. This report summarizes the positions expressed by each of the seven experts taken one at a time, then contrasts areas of agreement and disagreement across the more general university researcher-teacher practitioner division. These comparisons are considered with reference to their implications for ideal mathematics programs.

EXPERTS' VIEWS ON THE ELEMENTARY MATHEMATICS CURRICULUM:
VISIONS OF THE IDEAL AND CRITIQUE OF CURRENT PRACTICE

Richard S. Prawat, Ralph T. Putnam, and James W. Reineke[1]

This paper compares the views of two sets of experts--researchers and teachers--regarding ideal curriculum for elementary school mathematics. It represents part of a larger study involving similar analyses in five subject matter domains. The research was conducted at the Center for the Learning and Teaching of Elementary Subjects at Michigan State University, whose mission is to focus on issues surrounding the teaching of elementary subjects in ways that promote students' understanding of their content, ability to think about it critically and creatively, and ability to apply it in problem-solving and decision-making contexts. Review and synthesis of the literature on this topic, both as it applies to subject-matter teaching in general (Prawat, 1989), and as it applies to the teaching of mathematics in particular (Putnam, Lampert, & Peterson, 1990), identified the following as features of ideal elementary curriculum and instruction in various subjects: (a) the curriculum balances breadth with depth by addressing limited content and developing it sufficiently to foster conceptual understanding; content is organized around a limited number of power ideas (basic understandings and principles rooted); (b) teaching emphasizes the relationships or connections between these ideas (integrated learning); (c) students regularly get opportunities to actively process information and construct meaning; and (d) instruction fosters problem solving and higher order thinking skills in the context of knowledge

---

[1]Richard S. Prawat, professor and chair of the Department of Counseling, Educational Psychology and Special Education, and Ralph T. Putnam, associate professor in CEPSE, are both senior researchers with the Center for the Learning and Teaching of Elementary Subjects. James W. Reineke, doctoral student in CEPSE, is a research assistant in the Center.

application, relying on real-world situations for this purpose. The experts interviewed for this study were asked to critique, qualify, and extend these ideas about ideal mathematics curriculum and instruction.

Methods and Data Source

To ensure a range of comment on mathematics curriculum, we recruited four internationally known university experts who represent different, contending views of teaching and learning in mathematics education, and three expert teachers who appeared similarly diverse in their views about and approaches to the teaching of elementary school mathematics. University researchers were selected because of their scholarly contributions and their familiarity with elementary school classroom and curriculum issues. They represented a broad range of views on curriculum within the mathematics education research community. Teachers were selected from among nominees suggested to us by leading university-based scholars (including those who were being asked to participate in the study). Scholars were asked to nominate teachers who were outstanding at promoting understanding of mathematics, including its higher level thinking and problem-solving aspects. These teachers were then interviewed by phone to develop more information about their teaching goals and methods. After stratifying to ensure balance between the primary and later elementary grades, we invited the teachers who seemed most impressive in their phone interviews to participate in the study.

Data were developed from two sources. The first was a detailed, written document in which experts were asked to (a) critique and add to our list of key features of ideal curriculum (see above); (b) indicate how they would address three representative and important goals in mathematics (for this part and the remaining part of the exercise, they were to pretend that they were acting as

consultants assisting the staff of a local school); (c) list important understandings or generalizations related to each goal; and (d) develop a scenario for teaching one of the understandings at each of two grade levels (second and fifth). In addition to written comments, interviews were conducted with each expert. These interviews not only allowed us to further probe experts' views regarding "ideal" curricula, but also afforded opportunities for us to solicit their opinions regarding a widely used current textbook series in mathematics. This material, along with a set of framing questions, had been sent to each of the experts approximately one month prior to the on-campus interviews.

Analyses of the experts' written and interview responses began with transcription of all tape recorded interviews. The three researchers working on this project independently read all the material, focusing first on each informant's views about ideal curriculum, and then on views about the widely used textbook series. Detailed notes were taken and summaries prepared on each expert's views prior to interpretations. The descriptions of experts' opinions that follow thus reflect a shared understanding of what each expert said about ideal and actual curricula.

Summary of Individual Panelists' Positions

All of our experts were dissatisfied with prevailing mathematics curricula and teaching practice. All agreed there is currently too much emphasis in elementary school on learning isolated computational skills. All agreed that major changes need to be made to help students learn mathematics with meaning or understanding. In short, all of our experts were in agreement with at least the broad goals expressed in the current reform movement, as expressed in the National Council of Teachers of Mathematics (NCTM) Curriculum Standards (1989)

and National Research Council's Everybody Counts (1989). Beyond this broad agreement, however, our experts had distinctly different views about mathematics curriculum, teaching, and learning at the elementary school level. Although all agreed that students need more problem solving and developing of understanding or meaning than they are currently getting, their perspectives on what aspects of mathematics should be learned and how that learning can best be facilitated varied widely.

Our university experts ranged from a person who considers himself a radical constructivist (P3) and argues that curriculum can only grow out of actual pedagogical encounters between teachers and students and can, therefore, never be specified in advance, to a person who argues that it should be the responsibility of curriculum developers and educational researchers to meticulously design embodiments for representing powerful mathematical concepts (P1), to a person who argues that all mathematics learning in the elementary school should be firmly grounded in problem-solving situations (P4).

This section of the report focuses on summarizing and comparing each of the panelists' positions, starting with the professors (P) and proceeding to the teachers (T).

P1: Summary of Approach

General Approach to Mathematics Learning in School

P1 argued that school is the place where children should learn the disciplinary knowledge that our society values, not the more general or informal knowledge that they will learn anyway by participating in other, out-of-school settings (e.g., home, work). Mathematics is viewed as a powerful mental tool--but because it is essentially a formal language in our culture (as opposed to a natural language), we have to think carefully about how to help

students acquire it. Like our other experts, P1 emphasizes that the primary goal of school mathematics is to foster conceptual understanding in students. P1 thinks of this understanding as involving two important elements: First is having an understanding of the semantics of the formal language of mathematics--having mental images of various mathematical constructs. Second is having in place a control system which allows individuals to reflect on their actions. This reflection is important for developing the mental images in the first place and for thinking about how these understandings should be applied in various situations. P1 stated, "I think for understanding we need two things. We need first what you call the semantic basis . . . [and a] control system, which is a mental looking, you come back to that to look from a reflection, reflective abstraction."

Sowder (1989) has drawn some distinctions recently in mathematics that are helpful in understanding P1's approach to mathematics. She argues that there are three different variations on the "constructivist" position in math. The first, she says, centers on the notion of "doing" mathematics. The key issue here is the extent to which students are involved in actually constructing mathematics: abstracting, inventing, providing, and applying. The second position, and she lists P1's approach as one of the exemplars of this perspective, is said to involve "cognitive modeling." The focus here is on providing models that allow students to construct the appropriate representations for important ideas and procedures in mathematics. The third perspective is a social constructivist one.

Interestingly enough, P1 recognized a similar distinction in a recent article laying out her position. She argued that there are two "opposing" approaches to the teaching of mathematics at the elementary level. The first, the "structural" approach, favors the presentation of carefully constructed

mathematical representations that serve to bridge the gap between the physical experience reified by advocates of the second, "natural-environment," approach and the mathematical formalisms that lie at the heart of the discipline. The goal of instruction, according to P1, is to find alternative, "instructionally more efficient," representations that allow direct access to the formal knowledge that we eventually want students to acquire. Her major criticism of the so-called natural-environment approach is that, while it accounts for the genesis of knowledge, it does not account for its structure. P1 emphasized the importance of the structure aspect, perhaps at the expense of the genesis--the personal construction--aspect important to radical constructivists like P3 (see below). P1 seems particularly concerned about the use of ordinary language in the natural-environmental approach.

From an epistemological perspective, P1's approach resembles what Cobb (1989) terms "empiricist-oriented" constructivism. According to this view, students must actively construct knowledge--they cannot just passively ingest it from the outside. However, the knowledge constructed resides independently in the objects in the physical world. Ultimately, according to the empiricist-oriented approach, one can justify mathematical assertions (e.g., that 3 + 5 = 8) by reference to real world objects or events. This type of mathematical verification is particularly important for novices (i.e., children). They are incapable of the formal, deductive reasoning that is used in higher mathematics. Therefore, teachers should provide them with an external reality against which they can check their knowledge. Examples play a key pedagogical role in this regard.

Learning and Teaching

As indicated, for P1, learning mathematics is a matter of coming to know the formal language of mathematics. She wants students to acquire or construct mental "images" of mathematical abstractions that can form the semantic basis for this formal language. Students do not invent mathematics; rather, they construct their own understanding through reflection on their actions, in this case within carefully structured learning systems.

P1 explained what she meant by the term learning system. A learning system is comprised of two components: (a) a knowledge component, which includes the new mathematical relationships and language the student will learn; and (b) an exemplification component, which pertains to the representations to be used in presenting the knowledge component. The knowledge component of this system results from the expert's, in this case the curriculum developer's, knowledge of the discipline of mathematics. Ideally, the identification of certain "units" of knowledge follows from the careful analysis of an entire knowledge domain. These units appear to be built around "big ideas" that encompass a number of specific concepts and procedures. P1 believes that "a unit of knowledge should represent a portion of knowledge that 'stands alone,' that cannot be further dissected without losing its essential concepts." She cites as an example the notion of "additive relation," at the heart of which is the part-whole scheme underlying addition and subtraction in natural numbers.

In designing the learning systems that will form the curriculum, it is important to design examples around features that students will understand (e.g., the color and length of Cuisenaire rods are assumed to be features of objects that students will already be able to attend to and discriminate among). This is the type of knowledge upon which students will build their understanding of the abstract mathematical system. Thus, the attributes of

the materials used in the examples need to be familiar, but the concepts being presented need not be familiar to the students.

During the interview, P1 responded to the charge that she was deciding, a priori, what to teach students; ignoring, for the most part, their own prior knowledge. She replied, "I am declaring that they are coming to school to learn formal knowledge. . . . Schools were designed to make short cuts to learning formal knowledge." She said she agrees that children construct their own knowledge. However, this does not mean that they are free to define what that knowledge is.

P1 pointed out that there are two sides to taking into account previous experience: On the one hand, it is important for teachers to build on students' informal knowledge; on the other, it is equally important to know where previous experience does not coincide with the formal system. "Otherwise," she indicated, "we would not need the formal system. . . . We need the formal because there is some limitation to the informal and we want to give additional tools." Intuitive understanding is not always reliable. P1 cited an example of an addition problem that might be treated as a subtraction problem if the child relies on just his knowledge of language: "I lost five marbles in the morning and three in the afternoon. How many did I lose altogether?" If the child focuses on the word lost, he or she might incorrectly subtract to solve the problem.

According to P1, it is also important to avoid having students develop ideas that will conflict with mathematics they will learn later. One common example of this kind of interference occurs in the teaching of multiplication as repeated addition, which interferes with students' later understanding of "true" multiplicative relationships. P1 argues that we should highlight the differences between multiplication and addition when first teaching them,

rather then trying to make them seem like the same thing to avoid this kind of later confusion.

P1 views the kinds of multiplication problems that children commonly deal with in school as part of a broader multiplicative conceptual field which includes concepts of ratio, vector space, and so forth. According to Vergnaud (1983), whose theoretical views about multiplicative structure P1 endorses, every multiplicative relation is a "four-place relation" consisting of two basic dimensions (e.g., tables and legs) and a mapping function that maintains a constant ratio between the dimensions (e.g., 1 table: 4 legs; 3 tables: 12 legs). Because of the importance of the four-place relation concept, P1 believes that it is important to teach students how to derive the mapping rule in multiplication problems.

Approaches to Curriculum

P1 felt that while we know a great deal about mathematics learning, we do not currently know enough to construct an ideal curriculum. We do know enough, however, to build better curricula. P1 listed a number of understandings or generalizations that should be used as a basis for developing a good mathematics curriculum:

1. **Conceptual understanding:** having mental images to serve as a semantic basis for the formal mathematical system

2. **Syntax:** knowing the symbols of mathematics and how they fit together, including procedures and algorithms

3. **Multiple representations**

4. **A coherent system:** relationships among learned concepts and the entire system of mathematics (e.g., laws of the order of operation)

5. **Uniqueness:** what makes each concept and operation different from others (e.g., multiplication not as repeated addition)

14

6. **Extending the number sets:** making connections between early learning of mathematics and various number sets (natural, integers, rational, etc.) so that it will not be necessary to undo early learning

7. **Applications:** teaching that concepts and operations are applicable in a wide variety of situations--application should be taught explicitly, not just generalized from the mathematical ideas

8. **Rigor, examination, and control:** mathematics as a system always subject to examination, either by formal methods or by modeling and estimating

9. **Creativity and self invention:** never teach mathematics as closed, ready-made

10. **The beauty of mathematics:** encourage by posing open questions, investigating, and so forth

To summarize, in P1's approach, the teacher or curriculum developer begins with an explicit statement about the specific mathematical ideas students should learn; she emphasizes that one must pick a big enough "unit" to foster the learning of a coherent system. One then carefully designs an example to represent the mathematical ideas. The goal is for the example to be isomorphic enough with the mathematical ideas that students can use it to verify their ideas and discover new mathematical relationships. Thus, using manipulatives merely as illustrations of mathematical ideas is not enough.

It is equally undesirable to use whatever happens to be handy because the relationship between the example and the mathematical idea, as well as the language to be used in talking about the example, must be carefully worked out ahead of time. Examples serve as a sort of temporary referent for the mathematical symbols (mathematical language) being learned, but the goal is for the student to build a cognitive representation of the mathematical abstractions. Once the mathematical language and the abstractions underlying them are learned, students learn to _apply_ the mathematics they have learned to

a variety of situations. P1's argument here is that you first build an understanding of the mathematics itself, which has relatively unambiguous meanings, <u>then</u> you learn how to use this tool you have acquired to apply to a variety of situations.

P2: Summary of Approach

P2's approach to school mathematics is similar in many ways to that of P1. As with P1, the goal in education is to equip students with the powerful symbol systems and procedures of mathematics. This formal knowledge is an important tool for construction in our culture, the mastery of which empowers students. The symbols and procedures themselves serve as the starting point and meanings attach to them. The process of attaching meaning cannot be left to chance: the teacher must exercise great care in structuring educational experiences for youngsters. As P2 put it, "Students cannot invent procedures or invent anything in the abstract." If they are to master "complex cultural inventions," they "have to be provided with experiences [that enable them] to construct meanings."

Much of what P2 focused on was learning the algorithms for multidigit operations (+, x) and the accompanying place-value concepts. Her primary goal was for students to learn the procedures with some meaning attached to the symbols. This may represent the focus of P2's research, which has recently dealt with multi-digit algorithms and place-value issues and how to teach them more efficiently.

Teaching and Learning

Like P1, P2 starts with the mathematical concepts or procedures she wants students to learn and then devises powerful embodiments to help them learn. P2

approaches it primarily, however, by arguing that the goal is for students to acquire or construct meanings for the mathematical symbols.

> What I think should be common across all elementary grade levels is a teaching approach that emphasizes the development of meaning for mathematical symbols and the building up of later mathematical concepts from earlier ones. . . . This approach requires using some concrete materials or concerns situations that embodies in easily grasped ways the mathematical structure of the topic being taught. (Written response, p. 8)

According to P2, concrete or physical representations are the key to developing mathematical understanding in children. Once acquired, children can use these representations to determine for themselves if they are operating correctly on mathematical symbols--assuming, of course, that the material has been carefully linked to both symbols and procedures. This involves "re-presenting" the representations to oneself.

P2 has definite ideas about which physical representations or embodiments work best in teaching multidigit addition and subtraction. She favors the use of base-ten blocks to represent the English named-value system, along with the use of cardboard digit cards (i.e., large calculating sheets divided into columns headed by place value descriptors--ones, tens, hundreds, thousands) to represent base-ten positions. These representations provide the opportunity for children to construct meanings in ways that are consistent with the mathematical features of multidigit addition and subtraction. The representations direct student attention to the crucial meanings associated with these operations. As P2 explains, the embodiments enable children to "understand features of both of these systems and connect these features to each other."

P2 believes that the base-ten embodiments help teach both the concepts and procedures associated with multidigit addition and subtraction. In fact, P2 considers these two elements of understanding intertwined. "At least as much

place value knowledge seems to depend upon multidigit addition and subtraction knowledge as vice versa," she concludes. P2 is leery of arguments that suggest that work on place value should precede meaningful learning of multidigit addition and subtraction. It is the "coordinated work" on both--using a size measure named-value (multibase blocks) and a positional base-ten embodiment (digit cards)--that leads to the desired effect. Like P1, P2 does not argue for her particular representations because they map onto children's "natural" understandings. Quite the contrary. The representations she favors "do not for almost all children arise out of children's unitary ways of thinking about number [i.e., as single collections of things]."

As the above suggests, P2 does take into account developmental factors in thinking about school mathematics. She suggests that multidigit addition and subtraction be introduced at the second-grade level; the conceptual bases for this understanding are "well within the capacity of most second graders." The relationship between the students' intuitive knowledge and the form and knowledge they must acquire in school is complex; however, P2 emphasizes that students do bring important prior knowledge to mathematics, some of it very helpful and some much less so. As an example of helpful prior knowledge, P2 cites the extensive knowledge children apparently develop on their own about single digit numbers--both in terms of what the symbols stand for and what it means to operate on those symbols. Talking about operations (+, -, x), P2 states,

> I think that all those situations are out there in the real world and kids have experienced them. They've experienced addition, subtraction, multiplication, and division of small number situations, and all you have to do is bring these into the classrooms. . . . So I think that what you do about providing meaning to those basic operational symbols on small whole numbers is bring in a bunch of these situations.

This extensive intuitive knowledge of small numbers is a double-edged sword, however, in that it may interfere with other, more advanced knowledge that students need to acquire: specifically, knowledge about multidigit numbers. As P2 pointed out, students "enter school with this one powerful representation of numbers [a unitary count sequence representation]. Starting with multidigit numbers," P2 added,

> I think we're in the position that we're in with respect to lots of mathematics and that is that culture in general doesn't support that learning; therefore, the schools have to do it. So I think you really have to provide experiences from which students can construct a representation of mu idigit numbers.

P2 elaborated on why she felt our culture doesn't support the learning of this more formal mathematical knowledge. The tendency of children in English-speaking cultures to construe numbers between 10 and 19 in unitary terms may be due to the fact that the corresponding English number words (unlike their Chinese-based counterparts) do not directly name the ten and one values. The arbitrary terms "eleven" and "twelve," for example, do not indicate their composition as "ten and one" or "ten and two." Further masking occurs with the later number words; "thirteen" confuses the situation by pronouncing "three" in an irregular way and putting the modified word for ten (i.e., "teen") after the three instead of before it.

One implication of students coming to view these two-digit numbers in unitary terms is that the students collapse count and cardinal meanings for these numbers--regarding even fairly large collections of objects as unitary entities. P2 argues that this, and the practice of doling out one more place in each successive grade (two digits in second grade, three digits in third grade) accounts for the delay experienced by young children in this culture in coming to terms with our place-value system: "The use of this unitary representation becomes highly automatized in U.S. first and second graders,

and it interferes with their construction and use of adequate representations for multidigit numbers."

P2 also blames our educational system for adding to the language problems that already exist with regard to multidigit numbers. Her research suggests that it makes more sense to start with four-digit numbers than two-digit in teaching place-value concepts and addition and subtraction procedures. (Note: She tends to talk about the procedural and conceptual knowledge as a whole.) The immediate or nondelayed use of four-digit numbers, P2 argues, results from the need on the part of children to see multidigit numbers of several places in order to understand the nature of this sort of number. Part of the argument for beginning with four-digit numbers is a repeated-practice argument: children are exposed to more examples of the underlying tens structure with these numbers.

P2 appears to stress the role of story problems more in the teaching of multiplication than in addition and subtraction, possibly because multiplication maps on to a more complex set of real-world situations than does addition/subtraction. Regardless, she does advocate the use of story problems in teaching the three different interpretations of the times symbol: (a) array (rows and columns); (b) repeated addition (x groups of y); and (c) combinations, derived from combinations of the first two. As P2 puts it, "These meanings need to be introduced through specific examples, and computational work on multiplication needs to be continuously linked to these meanings via student-generated stories and teacher- and text-generated situations." This does not mean that P2 favors teaching multiplication solely through a problem-solving approach, however. As with multidigit addition and subtraction, she believes that physical representations must shoulder much of the burden.

One difference in P2's approach to th teaching of the two types of multidigit operations (multiplication versus addition/subtraction) is worth noting. As with addition/subtraction, P2 apparently has settled on one multipurpose representation. However, this particular way of illustrating multiplication--as an array--does not adequately capture the procedural aspect of the operation. In this sense, it differs from the representation used for multidigit addition and subtraction. Because cf this deficiency, P2 suggests that array representation be used in conjunction with a procedural route: the copy algorithm. In the copy algorithm, the multiplier is decomposed (i.e., 14 x 123 is treated as four 123s plus ten 123s). Students need only understand place value, shift rules (i.e., the idea that multiplying by 10 shifts each digit in the number one place to the left), and addition to use this algorithm successfully. P2 assumes that students will detect the procedural pattern (the process of repeated addition), which is a key aspect of multiplication, if they practice enough with the copy algorithm. This is consistent with her notion that procedural knowledge (i.e., knowing how to use the copy algorithm) contributes to conceptual knowledge and vice versa.

As indicated, P2 believes that a powerful concrete representation should be used in conjunction with the copy algorithm if students are to fully understand what is involved in multidigit multiplication. She suggests using base-ten blocks for this purpose, supplemented with some larger cardboard pieces. An array model can illustrate what is happening. For example, when one multiples 123 by 214, one can illustrate with the concrete materials the fact that one has 214 sets of 123--arrived at by constructing four sets of 123 (one flat, two longs, three units), then taking one set of 123 10 times and then 200 times. This representation provides a semantic base for understanding the copy algorithm.

P2 also believes that the repeated addition approach to multidigit multiplication generalizes to the multiplication of negative numbers. An example may be helpful here: If a equals the size of a particular debt ($3, $4, $5), and b equals the number of such debts that are taken away (these are both negative numbers), the various combinations can be organized as ordered pairs and arrayed in a matrix. It is comparable to repeated addition in that it involves adding "b" number of things "a" number of times. Repeated addition can also be used in the multiplication of decimals. Here an array model is used to convey the sense that .1 x 3 is comparable to 2 x 3; the latter means 2 sets of 3 things; in a comparable way, the former simply means one-tenth of 3 or three-tenths.

In all the above examples, an array representation nicely captures the semantics of what is going on during multiplication.

Approaches to Curriculum

P2, in contrast to P1, apparently thinks about school mathematics in a topical as opposed to a more conceptual or ideational way. In this sense, she packages the material in a fairly traditional way. Her instructional treatment of mathematical content represents a marked departure from what one would normally find in classrooms, however. There is a fair amount of integration across conceptual and procedural boundaries. Place value, for instance, is taught in the context of multidigit addition and subtraction. Similarly, traditional multiplication situations are used to teach the different types of multiplication (repeated addition, array or area, and combinations). This is in keeping with P2's view that conceptual and procedural knowledge interact and are mutually reinforcing. (P1's approach to curriculum, by contrast, is more conceptual than any of the other experts.)

Whereas P1 highlights the importance of a unitary multiplicative structure, P2 appears satisfied with a less ambitious agenda: Simply getting across to youngsters the notion that the times symbol has multiple meanings--array or area, repeated addition, and combinations. All three meanings relate to each other and to counting and can be represented with base ten blocks. For example, an array can be construed as consisting of groups if you focus on each of the columns.

P2 also expressed strong views about the negative effects of the so-called "spiral curriculum"; the low level of mathematical achievement in this country, she believes, partly attributes to the spiralling (or repeating) of substantial amounts of material from one year to the next. Instead of continually circling back on concepts, across the school year and even from one year to the next, topics should be presented thoroughly at one time and expanded upon immediately following the initial presentation. P2 argues that well-developed conceptual understandings will resolve the need for repeated instruction over the years. Curriculum developers need to pick a few key concepts for each grade level. These topics should be taught together in an integrated fashion.

The problem with a spiral curriculum, according to P2, is that no one feels responsible for making sure students get good grounding in each representation system or topic. P2's concern that the teachers commit to teaching certain things at certain grades is influenced by her background in developmental psychology--specifically, her exposure to Piagetian type readiness concepts. Thus, P2 argued, students at the early elementary level are not ready for ratio interpretations of multiplication or rational numbers because they have not yet reached the formal stage of operations. P2 believes that there is no point in teaching things until students are ready, but she

also believes that content that is appropriate can be taught in a more concentrated way. Thus, she advocates teaching more of what kids are ready for at the early grade levels.

P2's views regarding the constructivist nature of learning are similar to P1's. Teachers must play an active role in helping students construct mathematically "correct" meaning. It is important to choose powerful and simple representations that will allow the child to do this. P2 believes it is unfair to expect children to come up with these representations on their own. As she commented during the interview, "Students are going to construct their meanings based on their experiences. They're going to construct some sort of meanings about some sort of things because they're living organisms. The issue really is what kind of experiences you provide." She went on to say,

> If you have a shot at helping kids build a new, more powerful, simpler representation of some mathematical domain, then I think you're being a lot more humane and helpful to [children] to enable them to do it. . . . I think it's cheating them to not provide them with the opportunity to build this other representation.

P2's constructivism contrasts sharply with P3's much more radical version.

P3: General Approach

Kilpatrick (1987), commenting on the idea that mathematical knowledge is constructed by individuals, states flatly that "no mathematics educator alive and writing today claims to believe otherwise" (p. 7). Within the mathematics community, however, disagreement over the nature of what is constructed has created different types of constructivism. Two principles have been identified as the hallmarks of the sort of radical constructivism embraced by P3: (a) the learner actively constructs knowledge, s/he does not passively receive it from the environment; and (b) coming to know is an adaptive process

that organizes one's experiential world--learners do not discover an independent, preexisting world outside the mind of the knower.

Radical constructivists, such as P3, claim adherence to a Pigetian theory of knowledge acquisition. In this view, the learner, through interaction with different environments, constructs his/her own representations of reality; learning is provoked by situations--but knowledge is an individual construction. This set of epistemological commitments is apparent in P3's responses to the curriculum study questions and in tho transcripts of the interviews with him. P3 argues that teachers must recognize and respect the "mathematics of children," rather than specifying in advance the "adult" conceptions students are to acquire. P3 also believes that "learning does not happen unless there is a situation in which learning takes place." The situation described by P3 has two important characteristics. First, it contains a rich set of logical mathematical experiences that learners can immerse themselves in, both physically and mentally. Second, the situation must provide for social interaction relating to the logical mathematical activity. The teacher bears an important responsibility in optimizing both the experience and the interaction that turns the experience into knowledge. As P3 summarized it during the interview, "The teacher is everything."

Epistemologically, P3 represents an interesting contrast with P1 and P2. P3 is critical of their emphasis on representation in the acquisition of mathematical knowledge, arguing that this is an iconic conception of knowledge, requiring a match between what is represented and the individual's cognitive structure. Radical constructivists adhere to a view of knowledge that is based on activity, not likeness or representation. Knowledge results from attempts to build cognitive structures that work. This last point is a bit tricky, however.

(Note: Radical constructivists are well aware that people are constrained in what they are able to achieve cognitively by their prior achievements and experiences. "The child cannot conceive of the task, the way to solv it, and the solution in terms other than those that are available at the particular point in the child's conceptual development" [Van Glasersfeld, 1987, p. 325]. But while this imposes limits on what the person is able to accomplish, it does not prevent the individual from coming up with more workable or viable ways of organizing his/her experience. The test of whether or not a new organization is more or less viable is not some external consequence. Rather, it is the internal, reflective awareness of how neatly things fit together. As Van Glasersfeld argues, the reward comes from the "successful, deliberate imposition of an order" that is "inherent" in the individual's "way of organizing." "Logical or mathematical necessity does not reside in any independent world--to see it and gain satisfaction from it, one must reflect on one's own constructs and the way in which one has put them together" [p. 330]).

P3 resonates with this pragmatic view in talking about mathematics. He writes in his Part II responses, "In my approach, learning is construed as the adaptation of current schemes in problematic situations to resolve perturbations that arise as a result of social interaction or the interaction of a child with a mathematical situation." P3 emphasizes the importance of the child's own activity, arguing that, "One cannot transport conceptual structures from one person's head to another through language, actions, or any other source of perceptual signals." This perspective obviously colors his view of teaching and learning.

Teaching and Learning

Because P3 sees mathematics instruction not as the transmission or imposition of a particular curriculum, but as the facilitating of the individual child's attempts to construct mathematical meanings, the teacher plays a critical role. For P3, the teacher is essentially an extension of the role he plays as a researcher. Like the constructivist researcher, the teacher's task is to build a model of how a particular student is making sense mathematically, and then to perturb the system by posing questions or new situations that will help the student confront the limitations in his/her thinking. Teaching is thus as much about finding out and understanding what students know and how they are thinking as it is about directly helping them learn. What seems less clear in P3's approach is how this relationship between individual teacher and individual learner might play out in the social setting of the classroom.

Although P3 emphasizes the importance of social interaction in the individual's construction of mathematical meaning, he only outlines the desirable interplay between students and teachers in the educational setting. Essentially, what is required on the teacher's part is an extreme sensitivity to the child's own efforts after meaning. There is a paradox associated with this approach, however. "The teacher has to be sensitive to the mathematical reasoning of children and encourage it in every possible way. But he or she should not demand that the child reason in predetermined, fixed ways, especially when there are alternative ways that are just as good." P3 wants teachers to appreciate how children might reason their way through subtraction problems and what sorts of responses are indicative of more advanced understanding (thus there is both a descriptive and normative element); however, training children to reason in these ways is counterproductive. The

reasoning is valid only if it is an outgrowth of the child's <u>own</u> constructive activity.

P3 applies his perspective to the issue raised in the framing questions (see Appendix A) relating to how one might help students gain a conceptual understanding of multiplication. Like P1, P3 was interested in students understanding the deep structure of multiplication. He also saw multiplication as a four-place relation. He uses the example of pennies per pound. The key is to "transform units from one rank to another." The child can do this when s/he can reason using measurement composite units. This means the child "can take a number word [e.g., six] as referring to a partitioned number sequence." Thus, the child understands that six times three refers to a number sequence which involves units of three taken six times in succession. Prior to this, children can construct hypothetical groups (e.g., threes), but they must use this partitioning scheme on objects to solve multiplicative problems. P3 provided the example of a child who, in determining how many blocks there were in nine rows, each of which consisted of three blocks, had to actually count the rows, saying, "One is three, two is six . . ." "Nine" did not refer to the blocks being already partitioned into groups of three. There are two stages before this stage; in the one immediately prior, the child has to actually count out the composite units first before they can be counted. Thus, P3's approach bears some relationship to P1's. They agree on what sort of understanding mediates one's ability. P3, however, rejects P1's claim that this understanding can be directly taught, using carefully designed examples. P3 believes that knowledge about the four-place relation is arrived at slowly, following a sequence of stages in which the child progressively elaborates his/her understanding of number.

Approaches to Curriculum

Of all our experts, P3 seemed the most uncomfortable with the notion of an ideal curriculum. He was explicit in his opposition to the idea that one can specify in advance what it is that children should learn. He characterized the traditional approach to curriculum as Platonic, meaning that it assumes there is a world of mathematical objects that exist independently of the thinking individual. P3 places his emphasis on coming to understand and accept the "mathematics of children--their ways of making mathematical sense of their experiences."

The problem in attempting to specify the understandings one might attain as a result of a mathematics program, according to P3, is being clear about whose understanding is being discussed. As P3 put it, "It is very difficult for me to think like a child thinks." He cited as an example the notion of subtraction being the inverse of addition. While he can talk about what this understanding entails for him as an adult mathematician, he feels that he is on much shakier ground in specifying what it might mean for a child: "What might subtraction as the inversion of addition mean for a child who successfully carries out the inversion? The question of what a child has to do to construct the operation is even more difficult to assess." It is hard to decenter and construct an understanding of what inversion might involve as seen through the eyes of a youngster. Furthermore, any statement an adult might come up with to describe the child's reasoning would be "nothing more than a model--a simplified version" of how a child might actually think. The problem with presenting teachers with models of this sort, however, is that they get reified. We should learn to value and appreciate the mathematics of children and not try to force everyone to think the same way.

In P3's approach, the teacher is assumed to have more powerful mathematical knowledge and to be able to use this knowledge to help guide and shape the student toward more powerful mathematical constructions. This could be a problematic set of assumptions. Many elementary school teachers may not have sufficient knowledge of mathematics to help students construct meaning in that subject. This issue was pursued during the interview. When posed with the problem of assuming that "you're a 20-year-old elementary teacher who has minimal math background," P3 replied, "I have the confidence that our negotiations would be credible. I don't think the mathematics that we're going to negotiate--that anybody would say that it would be outside of the normal mathematics."

For P3 curriculum is created through the interaction of teachers and students in classrooms; it cannot be written by others and imposed. He argues that the notion of <u>ideal curricula</u> should be replaced with what he calls <u>abstracted curricula</u>. These are models abstracted from the ongoing teaching and learning of particular mathematical topics, rather than curricula that are determined a priori. They contain descriptions of conceptions students are likely to have, mathematical strategies they are likely to use, and problem situations that are likely to be useful in helping students construct more powerful mathematical knowledge. These abstracted curricula would evolve and remain fluid, providing teachers with resources for working with individual children, but never prescribing a particular set of learning goals or activities. In discussing abstracted curricula, P3 used the plural curricula rather than the singular curriculum "to emphasize diversity and variability rather than homogeneity and constancy in educational practice in mathematics." This reinforces the notion that he is reluctant to specify any particular mathematical learning goals for students through curriculum.

The final set of observations relevant to P3's approach to curriculum concerns his views about problem solving. This was an issue that was not mentioned much during the interview. Perhaps its importance is assumed in the radical constructivist approach. In one sense, problem solving is a central part of what it means to learn mathematics. Thus, P3 emphasizes that children construct mathematical meanings for themselves by dealing with problems that they have made for themselves in situations encountered or set up by teachers. However, and this is worth noting, P3 never really deals with problem solving as the application of mathematical knowledge or tools to serve their ends--for example, to use mathematics to solve problems encountered in other subject areas or in the real world. Thus he seems to view problem solving as the means to learning mathematics, not as a goal of that learning.

P4's General Approach

P4 talked about the task of schooling as enculturation--coming to know the culturally and historically accepted problem situations and ways of applying mathematical language (symbols, signs, and rules) that constitute mathematics. Of all our experts, P4 was the most explicit about criticizing the status quo of traditional mathematics teaching and curriculum and offering an alternative vision. Much of the language he used to describe what mathematics education should be like was formulated in contrast to what it currently is like (i.e., algorithmic and abstract).

In terms of the goals of mathematics education, P4 contrasted the existing curriculum's emphasis on the "record of knowledge" with an alternative view of knowledge that entails the doing of mathematics. Although students clearly need to learn the symbols, facts, and procedures of mathematics, it is critical that they learn these in the context of purposeful problem solving.

The ultimate goal is to give students the opportunity to develop mathematical power or mathematical literacy, not just collections of inert knowledge.

Problem solving plays a key role in P4's thinking about mathematics. This is not unusual for a mathematics educator. Unlike some of our other experts, however, P4 considered this activity to be an integral part of the curriculum. According to Stanic and Kilpatrick (1988), three general themes characterize the role of problem solving in the school curriculum. Two themes--problem solving as skill and problem solving as art--are less relevant to this section of the paper than the third--problem solving as context. Another way to describe this third approach is to say that problem situations serve as the vehicle or the mechanism for teaching concepts or skills. This stance toward problem solving pervades P4's approach to mathematics. In his Part 1 response, for instance, P4 talked about how, by grade five, children "should be dealing with a variety of problem situations."

P4 sees the development of conceptual understanding--at least as it relates to computation--as being intertwined with mathematical sense making (i.e., exploring, discussing, and testing mathematical ideas) and mathematical problem solving. "My definition of multiplication," he wrote, "includes all three and more." He elaborates, "Multiplication begins with the assumption that there exists a set of problem situations in which the implied relationships between the magnitudes expressed in the situation can be represented by a multiplicative expression."

P4 appears to emphasize the processes involved in doing mathematics--problem solving, communicating, making connections, seeing patterns, and so forth--in order to sharpen the contrast with the existing curriculum's overemphasis on what he calls the record of knowledge. It is also clear, however, that P4 thinks that there are important mathematical ideas to be

learned. He is harshly critical of the traditional practice of describing these ideas as collections of discrete instructional objectives, arguing that this fragments the curriculum and prevents students from seeing how the knowledge fits together as a whole. P4 prefers to identify a few key topics or domains that are important for students to learn, describing these as richly interconnected networks of formal mathematical symbols and procedures, concepts, and situations described by the mathematical symbols (after Vergnaud's [1983] conceptual fields). Mathematics education should be about helping students construct these rich knowledge structures through the process of actively solving a variety of problems.

P4's brand of constructivism appears to lie near the midpoint of a hypothetical continuum--marked at one end by P3's view of the child bootstrapping his or her way toward greater understanding and, at the other, by P1's view of the child as consumer of carefully crafted and presented external representations. Another way to put it is to say that there are elements of both internal <u>and</u> external structuring in P4's approach to mathematics. For example, P4 argues that the child has "a natural capacity" for visual imagery; through this mechanism, external situations can become linked in the child's mind with mental constructions. External, "situational knowledge" is an absolutely essential element of one's understanding in the mathematical domain. This contextual information is what gives meaning to what would otherwise be abstract concepts and procedures. Eventually, when mathematical concepts and procedures are experienced in enough different contexts, they become divorced from any one context and thus become generalized knowledge. This is a popular view in the literature on educational transfer. P4 stresses that there are motivational advantages to

the use of problem situations as well: "Situations create an investigative spirit, and a questioning, challenging frame of mind."

To summarize, P4 believes that the ultimate goal in school mathematics is to foster connections between mathematical concepts and procedures and the real world.

Learning and Teaching

According to P4, mathematical symbols and rules are best learned by solving problems. Mathematical language is a tool for representing situations. Thus, representing real-world situations is of the highest priority in mathematics instruction. More abstract, physical representations like counters and geoboards should be introduced after students have experienced the real thing. P4's vision of the mathematics curriculum is best defined by example. In the one that was sketched out during the interview, the instruction was, predictably, problem focused: A fifth-grade class views a videotape of the 100-meter dash at the Olympics. Their task is to count the number of steps, estimate the length of the steps, average the time per step for the winner--comparing this with similar data on the second and third place finishers. P4 explained that one of the advantages of the problem-focused approach is that students can come to see mathematics as a tool for representing situations: "They will see mathematics as a process of abstracting quantitative relations and spatial forms from the real world of practical problems." Mathematical ideas are associated in the mind with situations one has encountered in the past.

The objective of the mathematics reform effort, according to P4, is "to develop a collection of activities that should both interest students and give them an opportunity to develop 'mathematical power.'" P4 does insist that it

is insufficient to simply develop a collection of interesting activities--it must be a "program of activities from which knowledge or skill can be developed"; activity must be sequenced in a planned way for this to occur. As P4 puts it, "The knowledge gained must lead somewhere." Problems provide an important context for later learning, however. "When information is presented in a familiar contextual setting [like the Olympics], the transitions and the concepts and procedures are likely to be remembered."

P4 emphasizes the importance of the discourse process in mathematics teaching and learning. Communication and reasoning are seen as key elements in his activity-oriented approach. He argues that different content domains in mathematics are like the "petals of a flower"; at the center, and thus assigned a key role, are the interconnected processes of "problem-solving, communication, and reasoning." As this model suggests, P4 believes that students should play an active role in the learning process:

> [They] should constantly extend the structure of the mathematics that they know by having to make, test, and validate conjectures. As long as students are making the conjectures, their mathematical knowledge will always be restructured, consciously or unconsciously, because conjecture cannot be created from nothing.

The University Perspective: A Summing Up

It might be helpful at this point, before looking at teachers' responses to our questions about the ideal curriculum, to summarize some of the areas of agreement and disagreement among the university-based mathematics education experts.

P1 starts with a focus on formal mathematics. Mathematics is viewed as a powerful mental tool in our culture--a highly formalized language used to represent aspects of the world and to manipulate ideas. School is the place where children should learn and understand this formal language of

mathematics, with understanding meaning essentially having cognitive images of the abstract mathematical constructs to which various mathematical symbols refer. It is important that these cognitive representations closely map onto the formal mathematical system, rather than being highly idiosyncratic, either for individuals or to particular situations. Thus, students should acquire a cognitive representation of addition that transcends the many particular meanings addition might take on in particular contexts (e.g., joining collections of objects, counting on). The focus for P1, then, is on equipping students with particular ways of thinking about mathematical constructs; once internalized, these then become powerful conceptual tools.

Like P1, P2 wants students to acquire meanings for various mathematical symbols. But whereas P1 emphasizes the importance of a single unifying representation for a particular domain (e.g., addition and subtraction), P2 is more comfortable with multiple meanings with a focus on links among the formal symbol system, conceptual understanding, and computational procedures.

For P3, the mathematical thinking of the individual student is the starting point for thinking about the nature of mathematical knowledge. For P3, who labels himself a radical constructivist, mathematics is better viewed as part of an individual's abstracted experience, acquired more from interacting with the world than through formal instruction. Because it is impossible for one person to fully know what another is experiencing, we should try not to impose our adult mathematical concepts on children in school. As educators, we should be more accepting of the children's mathematics--their ways of making mathematical sense of their experiences. This knowledge is just as valid as those of adults and mathematicians. In summary, P3 believes that the goal of mathematics instruction is to help children build upon and extend their own personal mathematical meanings.

For P4, knowledge of mathematics is tightly linked to the problem situations in which it is used and acquired. Like P1 and P2, he regards mathematics as powerful tools developed by society, but he places as much emphasis on the culturally agreed upon problem situations and ways of thinking about them as on the agreed-upon meanings to be given to the formal symbols of mathematics. P4 also emphasizes, in addition to learning the concepts and procedures of mathematics (which he terms the written record of mathematics) the importance of students coming to think like mathematicians, being able to engage in mathematical conjecturing and argument, because these are also critical aspects of what it means to know mathematics. Teachers should encourage students to explain how they know something and to try to convince others that their arguments are valid--skills which are "fundamental to the notion of proof."

In the next section of this report, the focus will be on our teacher experts' views regarding the ideal elementary school mathematics curriculum.

T1's General Approach

T1 is a teacher with an unusual amount of experience in mathematics. She has 14 years of teaching under her belt, both at the junior high and elementary level. Shortly after beginning to teach, she completed work on a master's degree in special educati[illegible] She started a learning disabilities clinic in reading and mathematics--[illegible] this led to her developing a greater interest in different approaches to the teaching of both of these subjects. She went to a number of mathematical conferences and in that way got drawn into the statewide network of mathematics educators. She served on the committee that drew up the K-8 curriculum guidelines in mathematics for the state of California. She was also involved at the state level in reviewing

elementary math textbooks to determine if they were consistent with the new, conceptually oriented math framework developed at the state level. These efforts were aimed at improving the quality of mathematics education in California. Clearly T1's thinking about mathematics education and the "ideal curriculum" in mathematics is closely aligned with the work of these groups.

T1 wants students to become fluent users of the powerful tools of mathematics. Like the university-based experts, she views mathematics as "a tool to organize information and make decisions about real problems." She believes that thinking, problem solving, and sense making should permeate the elementary school curriculum, including mathematics. T1 emphasized that the important skills and concepts in mathematics are not limited to those involving number--the ubiquitous arithmetic skills and procedures that have pervaded elementary mathematics curriculum. Children should learn concepts and skills from all strands of mathematics, with key understandings being interwoven in rich mathematical activities, rather than being taught as discrete concepts or subskills.

T1 also feels that it is important that students enjoy working with mathematics, that they experience mathematics as playful and interesting. This is important, in part, so that students will not opt out of pursuing more mathematics later in school because they erroneously find it boring or frustrating.

Teaching and Learning

T1 writes that she "views learning from an information processing or constructivist perspective." T1 clearly does represent a cognitive-psychological perspective in her written comments and her interview. In both places she takes issue with the traditional view of the teacher as "dispenser

of knowledge." Both the teacher and student must play active roles in the instructional process. Students bear a special responsibility for "making sense of situations, procedures, key understandings." This view of the role of teacher and student is consistent with emerging constructivist thinking.

Like most constructivists, T1 discards the notion that learning is hierarchical--that "if a child acquires the requisite set of subskills, this will lead to the acquisition of general concepts." She favors, instead, a view that emphasizes connections between different pieces of information. T1 rejects the notion that skills and facts are unimportant in such teaching. However, they do not constitute the basis for all later learning the way we once thought they did. Skills and concepts (such as knowing basic facts, computational algorithms) should help students become "critical thinkers and users and doers of mathematics." T1 adds, "Skills and number facts need to come AT THE END of a sequence of instruction (or learning opportunities) perhaps over several years, not be the substance of the entire learning every time a 'topic' is covered every year." In her own teaching, T1 strives to ensure that no child leaves her classroom "without being made to think and reason during the day."

T1 believes that students learn important mathematics by their personal efforts at sense making and problem solving, although, as indicated, she also believes the teacher plays an important role. The teacher is responsible for engaging students in mathematically rich activities within which they can solve problems and search for patterns, making connections to mathematical knowledge that they already have. (This seems to be what P1 calls the "natural-environmental" approach in which students are expected to abstract their mathematical understandings from dealing with rich problem situations and a variety of representations of the same mathematical constructs.) Her

goal is for students to construct mental images and symbols, which become powerful mathematical tools to be used in a variety of situations.

It is of prime importance to T1 that students understand the mathematics they are learning. She emphasizes that it must connect in meaningful ways to what they already know:

> If there's not some sort of an interaction . . . between the child and the mathematics, if there's not something that connects for a child then they're merely going through empty exercises to please teachers and I question how much they're able to take in.

T1 never wants a child carrying out a symbolic procedure, like the addition algorithm, without first having established meanings for what is being done. Thus, when working to help students learn a particular topic or set of ideas, T1 feels it important always to start with concrete activities. As students solve problems within this concrete environment, they begin to see patterns and connections and to represent these in various ways. This is what T1 calls the connecting level of activity. Finally students begin to use and learn about the traditional symbols and procedures for manipulating them--the symbolic level. T1 argues that learning virtually always proceeds best when done through these levels: concrete-connecting-symbolic. This is true whether the learner is a child or an adult; it is almost always helpful to begin to understand an idea by working with some kind of concrete representation of it. (Note: It wasn't clear from the interview just what the criteria are for something to be labelled as concrete--whether this term refers to actual physical objects that a child can touch and move, or if it can also refer to some context or situation that is real and familiar to the child.)

While arguing the importance of the levels talked about above, T1 emphasized that they are not always pursued in a lock-step linear fashion. Rather, one can productively move back and forth among the concrete,

40

connecting, and symbolic:

> It's a sequence that I always kind of have in my head. . . . I want to get ultimately to the symbolic. But maybe once we get to the symbolic, we dip back down to the concrete to help us with understanding.

T1 is quite opposed to thinking of mathematical knowledge as hierarchical or as decomposable into many discrete learning objectives. She emphasizes the rich interconnections among various mathematical ideas. For example, in thinking about multiplication, she emphasized the importance of connecting ideas of place value, multiplication as grouping, and multiplication as area (which is connected to geometry). She also doesn't think of "thinking skills" or "problem-solving strategies" as being higher order skills that are to be learned <u>after</u> students acquire more basic knowledge. She criticizes textbooks and teachers who approach the teaching of problem solving either by teaching specific strategies such as "guess and check" or "make a table or graph" as separate skills that can be applied in a recipe-like fashion to solving problems. Rather than thinking of problem solving as a matter of applying basic skills and knowledge already acquired to new situations, she thinks that children should learn about various operations and numerical relationships through the process of solving problems. This problem solving should take place in mathematically rich settings so that different students can come to different understandings within it. She is opposed to the idea of using problem settings in which the teacher has in mind one right answer.

<u>Approaches to Curriculum</u>

T1 comes at the issue of mathematics curricula from a constructivist perspective. She seems to have a good grasp of what this entails. In her written comments, she emphasized that her approach to ideal curricula is not

"narrow and linear," involving "discrete pieces of mathematics to be learned," but rather can be defined as "fluid and interactive with the children for whom it is designed." She argued that curriculum in mathematics should represent a "vision of an environment committed to providing children with experiences which allow them to construct their meaning from materials, questions, tasks, and interaction--to solve problems which are complex and filled with key understandings from several different strands of mathematics." Problem solving should be a vehicle by which concepts and skills are presented "in contexts which engage students actively in their own learning."

To our statement about the depth versus breadth issue, T1 also added a developmental component: Some ideas <u>are</u> more powerful than others and thus are deserving of more in-depth treatment; however, children differ to some extent in their developmental readiness for certain of these ideas. This needs to be factored into the decision about which ideas to stress when. She uses place value as an example, suggesting that we might delay intensive experience of these concepts until children are in second or third grade, emphasizing instead measurement, pattern, and geometry before that time.

T1 also had trouble (as do some of the other constructivists) with the notion of "ordering or organizing" the key understandings; she said this "makes it sound as though the teacher holds the power for making understanding happen in children by the proper sequencing of key ideas." The representative goal of "developing a conceptual understanding of computation [multiplication]" is equally problematic for her. She felt that this puts too much emphasis on understanding computation so one can become facile at it. She pointed out that there is a difference between knowledge of an algorithm and understanding the concepts that underlie the algorithm; she would prefer to stress the ideas behind the computation (e.g., area concept of multiplication).

T1 listed a cluster of 11 key understandings from the California Mathematics Model Curriculum Guide (California State Department of Education, 1987), including concepts from the strands of number geometry, patterns, and algebra. T1 believes that students should deal with multiplication in a variety of problem settings and from a variety of perspectives, gradually coming to make general abstractions or, in her words, "developing a cognitive structure about multiplication." T1 talked about three big ideas that were important for students to develop through this searching for patterns in multiplication. First is place value, a notion which she considers "a key issue in children's developing an understanding of abstract number concepts." In her written response, T1 listed the following description of the key understanding of place value from the Model Curriculum Guide:

> Any number can be described in terms of how many of each group there are in a series of groups. Each group in the series is a fixed multiple of the next smaller group. (p.19)

As she discussed place value, T1 emphasized the first part of this description, the idea that place value requires counting groups as single objects. The other two big ideas that T1 emphasized in discussing multiplication were viewing multiplication as grouping and multiplication as area. She considers it important for students to come to understand both of these views of multiplication, one coming from the strand on number and the other from the strand of geometry. According to T1, these two views are connected to a superordinate concept, the idea that "the same patterns can emerge from a variety of settings."

T1 also expressed strong views about not introducing multiplication facts too early in the elementary school curriculum. T1 felt that this was a mischievous practice in two ways: First, it creates the impression that learning the multiplication facts is the same as developing a cognitive

structure about multiplication; second, it detracts from the real task in the early grades--that of developing a deep understanding of place value.

T1's .pproach to teaching the two different views of multiplication appear consistent with her constructivist philosophy. The groups concept of multiplication is developed over time by having children work together to generate lists of things (i.e., eyes, feet) that come in groups. Children also learn to represent groups by means of squares of construction paper. This seems comparable to what P1 does with the Dienes blocks. A similar lesson is described at the fifth-grade level to develop the area concept of multiplication. Students are given different numbers of multilink cubes, for example, and asked to make all the rectangles possible for each number.

T2's General Approach

T2 was the only middle school teacher we interviewed (she had been an elementary teacher). Consequently, she brought a somewhat different orientation to the task. For example, she argued that we need to raise the ability level in elementary schools so students can have three productive years in middle school. As with some other expert teachers we interviewed, T1's discussion centered on pedagogical issues, assuming that the content would be determined by the curriculum that is presented to teachers.

T2 had several opportunities to re-examine pedagogical issues in mathematics. She was involved in a study conducted at the Educational Technology Center in Cambridge, Massachusets. One finding of this study, according to T2, was that students often appeared to know mathematical algorithms, but were unable to apply them in situations where it wasn't clear which algorithm should be used. This work clearly has influenced T2's thinking. More recently, T2 has been associated with the Regional Math Network

at Harvard. She was involved in a curriculum development project at the junior high school level with this organization.

T2 mentioned that she entered the field of education late and became a mathematics teacher by accident. Her training was in bilingual education, but she had trouble finding a job in that area. The first position offered her was in middle school. She was concerned that her mathematical knowledge was limited and began an independent study of the discipline to prepare herself for the upcoming year. T2 spoke at length about reading different textbooks to develop her own knowledge of what she was to teach. Throughout this discussion, T2 stated that she, personally, needed to know the "why" of the mathematics--the "how" wasn't good enough. This need to know why has influenced her instructional approach in mathematics.

What T2 wants for her students can be summarized in terms of three major goals. First, she wants to empower students mathematically so that mathematics becomes "a pump, not a filter." This is especially true for the low SES and minority students with which she works. For T2, learning mathematics is, in part, a political issue. It is an important means for minority students to advance. T2's second goal is for students to see mathematics as <u>useful</u>--that they know when and how to apply mathematical skills in a variety of situations. In other words, students need to connect their mathematics knowledge with real-world and scientific situations. Finally, T2 wants to communicate a sense of wonder and curiosity to her students through her own attempts to learn and understand. "If I'm curious about stuff and I wonder about stuff, I assume that some kids might also. . . . I don't want them to absorb a rule and just assume it has to be because I said so and not think about it at all."

Teaching and Learning

The importance of sense making is at the core of T2's views about teaching and learning. This teacher strongly believes that it is important for students to conceptually understand what they are learning. She wants students to know why they invert the second fraction when dividing fractions, for example. She argues that the "why" should be presented even when the students don't care to know. T2 does not expect that all students will develop conceptual understanding at the same time. She commented that she tells students, "'I know some of you won't understand this, but walk through it. If it's going to confuse you a lot, forget that I brought it up. For those of you who do get it, that's fine. You'll see it again next year and the year after. One of these years it will click.'" Like many of the teachers in the overall curriculum project, T2 takes an individual differences approach.

T2 is clear about her ultimate goal, which is to have students understand mathematics well enough to be able to transfer the knowledge from one situation to another without each new situation being taught. To this end, she speaks of making connections between mathematical concepts. However, she argues that the teacher needs to make these connections if the students can or will not do it on their own. Giving students answers such as "You need it to become an architect" aren't enough. You need to know how and why architects use mathematics in their work. She argues that problem solving should be employed as the means of providing the "why" of mathematics. Because of this concern, she is leery of the NCTM standards being enforced in all school districts. Changing teachers' conception of mathematics and the way they teach will not be enough without changing the students', and perhaps their parents', views of why mathematics needs to be learned. "Knowing that you need to pass Algebra I is not enough, you need to know why you need to pass Algebra I."

As indicated, as far as T2 is concerned, "connections" are the key to conceptual understanding and to having knowledge that can be used and transferred. "The more connections they can make, the longer they'll remember and be able to apply it. I want them to be able to understand it sufficiently so that they can then apply it or use it and transfer that knowledge without it having to be another taught thing that they may segment." An important site for developing these connections is rich interdisciplinary problem contexts in which students can explore a variety of mathematical ideas. Two examples of such contexts that T2 discussed were the Voyage of the Mimi (1985) multimedia materials and a unit on the solar system that she and a collaborative group of teachers developed. These are settings that extend across weeks or months of instruction, and within which a wide variety of mathematical problems can be posed and solved.

T2's involvement with the Mimi project took place nearly four years ago. At that time, she was teaching a sixth-grade class and decided to team up with a teacher who taught a second-third combination class. The Mimi project lasted approximately two months. Students were organized into "crews," with three sixth graders and two second-third graders on each crew. The classes met three times a week. Students were involved in solving real-world problems--reading charts, calculating distances. At the end of this project, T2 indicated, students were asked to evaluate it as a learning experience. She was struck by one student's comments: "One of my best math students, straight A's all the way through sixth grade said, 'I always knew, because I was told that one half of an apple and one half of an apple equalled one apple, but I never was sure of that because to me it looked like two halves of an apple. Now I understand better.'" This quote is important because it nicely captures T2's views about

the advantages of a problem-solving curriculum: When students deal with real-world type problems, the mathematics is made less abstract, more concrete.

The sort of rich problem-solving contexts discussed above play key roles in this ideal curriculum. They serve at least three important functions:

1. They serve as sites for making the all-important connections between mathematical ideas and skills and the contexts of their use. Through these rich problem settings, students come to know when and how various mathematical procedures and concepts can be used. They also make connection between various abstract numbers and real-world settings; for example, getting a sense for the magnitude of different numbers.

2. They play an important role in motivating students to think about and learn mathematics. These rich, interesting contexts help students realize that mathematics is useful and relevant.

3. The problem contexts serve as a source for representing important mathematical concepts that need to be taught (e.g., the Red-Line Subway in Boston as a representation for thinking about integers). T2 is always on the look out for problem settings that will lend themselves to exploring a variety of mathematical ideas.

A variety of problem settings and approaches to problem solving also allow the teacher to address individual differences in students. T2's concern about individual differences reflects her belief that individual learners have different learning styles (e.g., auditory, visual) and are motivated by different things.

Approaches to Curriculum

T2 is a lot like T3 (see below) in her firm commitment to a problem-solving approach to the teaching of mathematics. She describes her goal as being one of helping students develop problem-solving thinking skills. She believes that there are certain strategies that one can use during problem solving, and this may be what she has in mind when she talks about thinking skills. She provides several examples of these strategies: drawing solutions, using diagrams, visualizing, showing visually how the solutions came about. In addition to

these problem representation strategies, T2 wants students to be able to use other tools, such as partitions, defined as the breaking down of numbers into more basic elements that can be more easily operated on (e.g., 31 x 2 = 30 x 2 + 2 x 1), and other "mental-math short cuts." These specific tools are also taught with the aim of getting students to be better at problem solving.

T2 distinguishes between what she calls problem-solving and textbook word problems. The latter, she believes, are solved algorithmically:

> If the kid is smart enough, he knows that these two pages deal with multiplication, and chances are that if you try multiplication on all these word problems you're going to get ahead and you're going to score high. It doesn't really require the kid to think about, What's this problem about? and What should I use? What tools should I pull out for this? Why should I use addition or multiplication or division or whatever?

Often, students are taught to solve word problems by searching for a key word that indicates which operation to use. Word problems are opportunities to practice certain operations--but that is not what she means by "problem solving."

As indicated, T2 emphasizes the motivational and social aspects of problem solving in her definition: The problems students attempt to solve should be personally meaningful and lend themselves to more than one approach. This latter criterion provides a rationale for letting students work in groups. She believes that students can learn from one another: "Part of what helps them focus on areas is other kids' conversation. . . . Someone else's question may trigger something." If the problems can only be solved in one way, however, "then there really isn't a whole lot you have to talk about other than what did you get." T2 facilitates conversation about math by structuring the small groups--asking students to share their solution strategies, to evaluate the strengths and weaknesses of each, to generate a certain number of different approaches. The key to all of this, according to T2, is for the teacher to

exercise care in the selection of problems: "You have to be careful how you choose the problems," she says. "I have to be careful always that I don't make up problems that will be contradictory in results somewhere, or that become too difficult, or that will have in it something that I don't see that can lead to that one right answer only." She regards this as a major problem in this approach: "It's not always easy. I would like very much to have a lot of time to just look at materials . . . to find a place where there is a whole lot of stuff that you can look at that is geared to different kinds of concepts."

T3's General Approach

T3 writes that her ideal mathematics curriculum would be interdisciplinary; that is, it would show how mathematics applies to different situations both in and out of school. The curriculum also should be relevant; that is, there should be a need for this knowledge in arenas other than the classroom. Finally, the curriculum must take into account the developmental stage of the learner. This last point apparently reflects T3's background in early childhood education. Her position on mathematics education, and perhaps education in general, is based on a strong developmental component, more so than the other teachers we interviewed. T3 talked about the developmental level of students and their capabilities at each of the different levels. In the interview, for example, she mentioned that if students "can't conserve, there's not a lot of point. Sometimes you're wasting your time trying to teach until you get through some of those things." In her written contribution, T3 argued that teachers must begin to employ "developmental processes that cater to the needs of individual children."

T3 wants kids to view mathematics as fun and easy and to realize that it is useful and relevant to their out-of-school lives, now and in the future. T3 designs her classroom activities to capitalize on familiar contexts and she creates activities in which students can see the applicability and inter-relatedness of what they are learning. In particular, she has had her third-grade students create a small town in their classroom--a town which they design, build, and participate in throughout the school year. These activities become the site for important learning experiences in social studies, mathematics, art, science, and language arts.

Teaching and Learning

T3's pedagogical approach to mathematics is best illustrated by describing her "classroom city" project, which she has done on three occasions with her third-grade classes. In this project, students take responsibility for designing and building a minicity. Typically, this city consists of five to six wooden structures (e.g., post office, city hall, fire and police station, bank). Students wind up with various jobs in the city. T3 uses this simulated situation to introduce a number of mathematical problems. For example, students must punch in before going to work; a time clock borrowed from the office is used for this purpose. Because salaries are based on the amount of time worked, this becomes an opportunity for students to figure out the number of minutes between two points in time; they also must use their multiplication skills to calculate the amount of money each employee has earned based upon the number of minutes worked (they are "paid" 3 to 5 cents per minute). As she indicated in her written response to Part 1, a number of measurement concepts are taught in conjunction with the actual building of the city; students are expected to make scale drawings of the classroom prior to designing the city,

for example. T3 also mentioned balancing one's check book and making change as other real-world occasions for students to use math skills.

Not surprisingly, given this commitment to situated, authentic learning, T3 also was a strong advocate for interdisciplinary learning. She believes that children need to be aware of how mathematics impacts on science, social studies, music, art, and so forth. This, and her insistence on relevance, defined as understanding how content can be used in the real world, helps to explain her unique problem-solving approach to instruction. She combined two of our criteria for ideal curriculum when she commented, "The opportunities for students to actively process information and construct meaning should include this interdisciplinary approach with relevant and interesting content presented at an appropriate level of difficulty." This statement obviously provides the pedagogical justification for her classroom city approach to instruction.

T3's commitment to a real-world approach to teaching and learning extends to the types of objects she uses when introducing mathematical content. For example, she uses money (dollars, dimes, and pennies) rather than base-10 blocks to introduce regrouping in subtraction (especially with lower socio-economic status kids) because "they can understand it with the money where they can't with the blocks. The blocks have no relevance for them but money has relevance for them."

Pedagogically, T3 appears similar to T2 in her approach to mathematics. She values the use of situations from which the students can gain mathematical knowledge. Like T2, she views the teacher's role as one of presenting ways in which school mathematics can be used in different situations. Like T2, she draws on a fairly traditionally defined knowledge base in designing these experiences. For example, during her presentation of multiplication as repeated addition, she mentioned that she embodies the operation through the

use of objects and cups (e.g., 8 cups with 3 objects in each would be written 3+3+3+3+3+3+3+3, or 3 eight times). She also mentions presenting multiplication as an array of items. While she uses both discrete (cups and objects) and array representations of multiplication, she talks about it mainly in terms of repeated addition. The ultimate goal for the students is to learn the traditional multiplication "super shortcut."

T3 stressed that she tries to focus on what multiplication _is_ rather than how it is done. She uses a careful sequence of activities to help students understand not only the concepts (e.g., the notion of set, the commutative property of number), but also the language involved in multiplication. She might begin, she indicated, by giving students a number of counters and asking how many groups they could make with the same number in each group. She explained, "I would try to lead the children to see when you have equal numbers in groups, you can say 4+4+4+4 . . . is 4 six times. I would use the verbal sentence 'four times six equals 24,' but I would not use the written '4 x 6 = 24' at this point."

Like many teachers, T3's approach to mathematics represents an interesting blend of the innovative and the tried and true. Thus, although she emphasizes the importance of students knowing the meaning of mathematical operations, and having ample opportunities to apply the ideas (especially in the context of the town), T3 also considers it important for students to master their basic facts. She does drills ("mad minutes") throughout the year, insisting that students learn to recall facts without relying on counting.

Approaches to Curriculum

T3 appeared to buy into all our criteria for ideal curriculum, although she indicated that she had trouble understanding what we meant by "emphasizing the

relationship between powerful ideas . . . so as to produce knowledge structures that are differentiated yet cohesive." She particularly agreed with our notion of fostering problem solving in the context of knowledge application. As indicated, this is consistent with her real-world problem-solving orientation to the teaching of content. Her enthusiasm for this sort of approach appears to be based, in large part, on her belief that it is motivating and enjoyable for students to participate in activities like those related to the classroom city. One of the most important goals in mathematics, according to T3, is that students come to view that subject as fun and easy, something they can succeed at. The focus for T3--and for T2 as well--is broader than that presented by the university-based experts. T3 takes into account youngsters' affective as well as cognitive needs. For her, the problem in mathematics teaching relates more to <u>how</u> the content is typically presented than to <u>what</u> it is that we are asking children to learn. Teachers simply need to be more sensitive to the developmental needs of students. An important part of this sensitivity is knowing when students are most receptive to new learning. T3 described these situations as "teachable moments." T3 commented, "Teachable moments--I guess maybe that's the kind of word for it. Things first happen that enable you to deal with issues and with objectives and things in a natural kind of way that I think makes a lot more sense to the kids than the work in the books sometimes." This, "natural kind" of learning is much more likely to happen in an activity-oriented classroom according to T3.

<u>The Teaching Perspective</u>

The teacher experts appeared to be less explicit than the university experts about how mathematical knowledge should be thought of for elementary

school. This is not surprising, since their task in teaching is to deal with teaching particular content to students, rather than thinking in the abstract about the nature of mathematics curriculum and learning. Of the teachers, T1 was the most explicit about how knowing mathematics should be conceived for thinking about elementary curriculum and teaching, arguing, like P4, that mathematics should be thought of as "a tool to organize information and make decisions about real problems." T2 and T3 emphasized the importance of students having opportunities to connect mathematical knowledge to its uses in real-world settings by engaging students in rich interdisciplinary settings.

Overall, the teachers were much more explicit than the professors about the goal of developing positive attitudes toward mathematics. For example, T1 commented,

> Most first graders and kindergartners love mathematics--they love school. Most fourth graders say they hate mathematics. I worry that there is an affective part of everything we do, whether it's reading or mathematics. If somehow as a teacher I don't convey a love of literature, a love of books, then no matter what skills I teach--skills are important--if that affective part is not there, they're never going to do it. In mathematics, if there's not an affective part, many of those children are going to self-select out of advanced math and science courses.

T1 went on to describe one memorable lesson she had conducted in mathematics. The class was engaged in problem solving using pattern blocks, and students became so involved in the task that they wanted to give up recess. She commented further,

> I'm on the floor with the pattern blocks, and all of a sudden it hit me. My fifth graders had voluntarily given up P.E. for mathematics! And I had accepted it as a normal course of events. . . . Now, it may never happen in my entire teaching experience again, but I thought, Isn't that nice--that on at least one occasion the level of involvement was so intense, they were so interested in what they were doing, they were willing to make that commitment.

This concern for the affective or noncognitive aspects of teaching comes as no surprise (Prawat, 1985); it may reflect the fact that the teacher experts

approach curriculum planning more with the whole child in mind. The university-based experts, in contrast, are more caught up in the knowledge-related debates within their subject matter communities. These debates--like whether or not teachers should introduce alternative number bases to students--might seem esoteric to many practitioners.

Overall Summary

All of our experts were dissatisfied with prevailing mathematics curricula and teaching practice. All agreed there is currently too much emphasis in elementary school on learning isolated computational skills. All agreed that major changes need to be made to help students learn mathematics with meaning or understanding. So all agreed with at least the broad goals expressed in the current reform movement, as expressed in the NCTM Curriculum Standards (1989) and National Research Council's Everybody Counts (1989).

But beyond this broad agreement our experts had highly variable views on what mathematics curriculum and teaching in the elementary school should be like. Although all agreed that students need more problem solving and developing of understanding or meaning than they are currently getting, their perspectives on what aspects of mathematics should be learned and how that learning can best be facilitated varied widely.

Our university experts ranged from a person who considers himself a radical constructivist and argues that curriculum can only grow out of actual pedagogical encounters between teachers and students and therefore can never be specified in advance, to a person who argues that it should be the responsibility of curriculum developers and educational researchers to meticulously design embodiments for representing powerful mathematical concepts, to a person who argues that all mathematics learning in the

elementary school should be firmly grounded in problem-solving situations. How do we account for these different views on curriculum? The scholars and the teachers we talked with differ in a lot of ways. In the remainder of this paper, we use three key features of their perspectives to characterize their positions: (a) their beliefs about the nature of mathematical knowledge and what should be learned in elementary school, (b) their beliefs about how mathematics is or should be learned--their views on teaching and learning, and (c) how their beliefs about mathematical knowledge and about teaching the learning come together in their views on the role curriculum should they play in elementary school classrooms.

The Nature of Mathematical Knowledge for Elementary School

All of our experts, both from the university and the classroom, firmly believed that mathematics entails more than the isolated computational procedures that dominated current school curriculum and practice. All thought mathematics instruction should focus on helping children establish meaning or understanding of mathematics. But they held different views on what it means to know mathematics and what mathematics students should be learning in elementary schools.

Three aspects of mathematical knowledge are useful in characterizing the views of the experts. In describing the nature of the kind of mathematical knowledge we want students to acquire in elementary schools, the experts considered important to varying degrees (a) the formal symbol systems of mathematics and their underlying meanings or semantics, (b) the mathematical understandings (sometimes informal understandings acquired in out-of-school settings) of the individual, and (c) the various settings in which mathematics is useful for solving problems. If asked, each of the experts would argue that

all three of these are important to consider in thinking about what mathematics elementary school children should learn. But the experts highlighted these aspects to varying degrees. What was figure to one expert was ground for another.

Learning as Active Construction of Knowledge

Virtually all of our mathematics experts, the teachers a well as the university-based researchers, espoused some version of a constructivist view of learning. They all emphasized the fact that learners had to be actively involved in the process of meaning making if significant learning is to take place; they also appreciated the need for the teacher to attend carefully to what students are saying and doing in mathematics. Beyond this broad level of agreement, however, the experts had dramatically different views about the learning process and how to facilitate it through teaching or instruction.

Not surprisingly, our university-based experts were the most explicit in support of the notion that the learner must be actively involved in the learning process. P3, who considers himself a radical constructivist, placed the most emphasis on the active involvement of the child in the construction of meaning, arguing that

> The teacher has to understand that it's not possible just to tell the child mathematics; the child has to be actively involved in the learning of mathematics--the activity of the child is critical. . . . One cannot transport conceptual structures from one person's head to another through language, actions, or any source of perceptual signals.

Even P1, who placed so much emphasis on students coming to understand the accepted semantics of the formal language of mathematics, insisted that learning is ultimately a constructive process of reflection by the individual:

> You can't teach for understanding; it is something that happens to the students. You can supply the condition, but it's up to the

> students. . . . So I don't think that we can take a general explanation, tell it to a child and he understands it. From that point of view I am a constructivist. Knowledge is something that one has to construct himself. . . . It's up to him to use the environment. I cannot reflect instead of him.

Our teacher experts also appeared to be committed to constructivist views of learning, although only T1 was very explicit in this regard. She emphasized that her ideal curriculum would provide students with experiences that allow them to construct their own mathematical meanings:

> The emphasis needs to be on students actively processing information and constructing meaning. All instruction should support this goal. . . . Student's thinking and reasoning are the critical attribute, not an add-on to facilitate concept development.

T1 contrasted her approach to curriculum with a nonconstructivist view, which she characterized as "narrow and linear," involving "discrete pieces of mathematics to be learned." T1, who had been active in the mathematics reform movement in California, favored curriculum that was "fluid and interactive with the children for whom it is designed." Clearly, her vision of mathematics was informed by constructivist notions of teaching and learning.

Although these views were made less explicit by the other two teachers, elements of constructivist thinking were evident in the language they used to rationalize particular approaches to the teaching of mathematics. T3, for example, favored a cross-disciplinary, problem-solving approach to mathematics, having children design a city in her classroom each year, complete with wooden structures representing stores and municipal buildings. Tasks associated with the construction and operation of this city became occasions for integrating the teaching of mathematics with subjects like social studies and science. T3 used constructivist language in talking about how important it was for each new group of students to begin anew:

> Many teachers and parents have asked why I don't leave the city up for the next class. I believe much of the value of the project would be

lost if this was done. The students would not feel it was their city, and the lack of investment would seriously undermine the success of the city.

T2 also highlighted the importance of getting students actively involved in their own learning. Like T3, she stressed the importance of connecting mathematics with the natural environment. This more concrete understanding is constructed by students in the process of their dealing with real-world problems.

Thus, across our seven experts was widespread support for the general constructivist notion that students must fashion their own understanding in mathematics. But beyond this general agreement, the experts had quite different views about the constructive nature of the learning process and the role of the teacher in helping students construct their understandings of mathematics.

At one extreme among our experts was P3, for whom mathematical knowledge is the personal meanings the individual learner is able to construct. For P3, "learning is construed as consisting in the adaptation of current schemes in problematic situations to resolve perturbations that arise as a result of social interaction or the interaction of a child with a mathematical situation." The teacher plays a key role in the learning process, not by presenting or modeling particular mathematical ideas, but by providing the perturbations to help the child structure and interpret problematic situations. Like the constructivist researcher, the teacher's task is to build a model of how a particular student is making sense mathematically, and then to gently prod the student into new ways of thinking by posing questions or new situations that will help the student confront limitations in his or her thinking. Through experience, the teacher will develop a personal theory of "children's mathematics." This theory, rather than the discipline of

mathematics, guides in assisting the teacher in predicting the direction the student's thinking will go and deciding what to do next, instructionally.

In sharp contrast to P3's radical constructivist perspective is P1, who as we saw earlier also believes that students' understanding is ultimately a result of their reflections on their own actions. P1 argued strongly for the position that by carefully structuring the environment with which the child interacts, one can greatly facilitate and shape the kinds of understandings the child constructs. To help students construct the desired understanding of the semantics of formal ideas--usually involving a concrete manipulative such as Cuisenaire rods, also with a carefully specified set of rules for acting on the objects. This carefully designed example serves as a sort of temporary referent for the mathematical symbols being learned, but the goal is for the student to build a cognitive representation of the mathematical abstraction. A good example is not merely an <u>illustration</u> of a mathematical idea, but is isomorphic enough with the mathematics that the students can use it to verify their ideas and discover new mathematical relationships. It is the students' reflection on their actions within this carefully designed system that leads to their construction of appropriate mathematical understandings.

Much of the difference between P3's and P1's views of the constructive nature of learning mathematics, as well as those of our other experts, is captured by the extent to which they regard learning as an <u>internalization</u> process (Cobb, 1989). Cobb argues that the internalization view of learning underlies much of the work on instructional representations. According to this view, which he characterizes as environmentally driven, mathematical relationships are internalized from concrete materials, such as base ten blocks, pictures, diagrams, or other exemplifications. The images associated with this concrete material serve as the semantic basis for more formal

mathematical language and relationships. Cobb contrasts the internalization view with one that places more of a premium on the socially mediated construction of meaning. Thus, some theorists believe that the meaning of instructional materials must be negotiated by the teacher and the students. According to this second view, which represents a more radical or extreme form of constructivism, mathematical meaning emerges from a dialectical process that is both individual and social.

Our experts fell along a continuum between these two views of learning roughly as shown in the figure below, with most of our experts leaning toward an internalization new of learning, described most explicitly by P1.

```
 Socially mediated                                              Internalization
  construction of                                               of understandings
     meaning

<------------------------------------------------------------------------------->
P3                          P4  T1            T3  T2                 P2     P1
```

P2, like P1, argued that students should be provided with carefully designed examples, or embodiments, to help them acquire or construct particular meanings for mathematical symbols:

> What I think should be common across all elementary grade levels is a teaching approach that emphasizes the development of meaning for mathematical symbols and the building up of later mathematical concepts from earlier ones. . . . This approach requires using some concrete materials or a concrete situation that embodies in easily grasped ways the mathematical structure of the topic being taught.

But where as P1 emphasized the importance of a single abstract representation of a particular mathematical domain, such as addition and subtraction, P2 argued that embodiments should be used to help students build a variety of interconnected understandings of the domain. Thus she advocated the use of different embodiments for different situations involving multiplication.

T2 and T3 emphasized the use of problematic situations ways that resembled the recommendations of the remaining universit_ and teacher experts. However, their reasons for using these situations differed from those of the other experts. For both T2 and T3, problem situations presented opportunities to identify "teachable moments." These moments occurred when students found themselves in a situation where their mathematical knowledge was not vast enough to allow them to solve the problem confronting them. Teachers, these experts argued, are responsible for presenting to their students the mathematics that applies in the situation. The teacher has the obligation of helping students see when and why various mathematical procedures are useful, and to present new mathematical ideas when appropriate. We placed these teachers toward the internalization end of our figure because they seemed implicitly to hold the belief that the mathematical ideas need to be presented by teachers in the form of modeling or embodiments to be internalized by students.

For P4 and T1, students should learn mathematics in the context of solving rich and varied problems. It is through solving problems with the support of a knowledgeable teacher that the students construct increasingly sophisticated mathematical knowledge. T1 wrote in response to one of our questions, "I believe that problem solving should be the focus, not knowledge application or skills. All skills and concepts should be approached from a problem-solving context." She explicitly contrasted her perspective with the application approach, in which students are expected to master skills and algorithms first, only then applying them to the solving of problems. "I disagree with that totally," she said. "I think that you have a better chance of getting children to learn math facts or improve number sense by working on problems first and

seeing need for it than to drill and practice to death before they ever get into using it."

Similarly, P4 talked about problem settings being the site for learning mathematics, arguing that "knowledge emerges from problems rather than the other way around." Mathematics, according to P4, is a tool for "representing situations." Situations serve as contexts which give meaning to the "signs, symbols, and roles" of mathematics. Furthermore, authentic situations provide occasions for students to think like mathematicians: "Situations create an investigative spirit, and a questioning, challenging frame of mind." Teachers can use problem situations to encourage students to make conjectures and engage in mathematical arguments, trying to convince others as mathematicians do. Because mathematics learning takes place in the context of these rich problem settings, the interactions between teacher and students play an important role in what mathematical understandings students take from the situation. That is why we placed T1 and P4 toward the left of our continuum. They are not as far to the left as P3, however, because they both argue that the teacher (or the curriculum) should carefully select problem situations so as to support the development of particular mathematical understandings in students. It is insufficient to simply develop a collection of interesting activities, P4 cautioned: "The activities must be sequenced in a program if knowledge or skill is to be developed." T1 reinforced this view: "A lot of what I see in problem solving," she said, "is, 'Here's another cute activity and, wow, this is really fun.'" She contrasted this with her more planful approach: "I don't give them random activities and then just hope that understanding happens."

As can be seen, although generally subscribing to the view that the learner must ultimately construct his or her understandings by being actively involved in the learning process, our experts differed considerably in the details of

this belief and in the implications they drew from it for the role of the teacher. The experts also differed considerably in their beliefs about the role of individual differences among learners in the learning process.

Role of Individual Differences

The importance placed by our experts on the active role of the individual in the learning process suggests that attention to what individuals bring to the learning situation--various differences among individual students--might play an important role in the instructional process. As with their views of the learning process described above, our experts varied considerably in the importance they placed on individual differences and how they conceptualized them.

Of all our experts, P1 put the least emphasis on what the individual learner brings to the instructional setting, keeping her focus more on the mathematical ideas that students are to learn. P1 said she did believe students' informal knowledge influenced their subsequent learning of mathematics, but unlike most of the other experts, she stressed the differences between this knowledge and the more formal mathematical knowledge to be learned in school. The other experts were more inclined to want to accommodate to students' individual differences in instruction. They viewed these differences less as impediments to formal instruction and more as givens that must be taken into account in one's teaching. Even here, however, they held diverse views. Several of our experts stressed the importance of developmental differences, citing Piagetian theory as the justification. For example, T3 speculated that students encounter difficulties with existing curriculum in part because they lack cognitive operations like conservation. During our interview she mentioned that "if they can't conserve, for example, there's not a lot of

point. Sometimes you're wasting your time trying to teach until you get through most of those things." P2 similarly argued that notions of multiplication related to rates, ratios, and fractions are perhaps best left to seventh and eighth grade because they are "formal operational" in nature. What is most noticeable about these examples is the extent to which some experts tend to think of developmental differences in relatively general and fixed ways. This is in marked contrast to P3, who also drew on Piagetian theory but derived a different set of instructional implications from it. In fact, P3 was quite specific about how unhelpful Piaget's general stages are for educators. Educators who are interested in developmental issues, he stressed, would do better to look on development as a specific art, thus focusing on change in a circumscribed domain like mathematics or science.

One thing that was evident in the written and interview responses of our experts was the importance teachers assigned to individual differences in comparison to the university-based experts. It is not surprising that teachers, more than researchers, focus on individual student needs in thinking about curriculum. In a recent survey of American teachers, Stevenson (1989) found they assigned greater importance to the teacher's ability to take individual differences into account in teaching than to any other variable. Asian teachers, in contrast, assigned more importance to content-related factors such as the ability to explain concepts clearly.

It may be helpful to summarize some of the comments about individual differences made by our teacher experts. One, who worried a great deal about issues of equity and access in mathematics, talked about the negative effects of ability grouping. "After a couple of years of being assigned to the lowest group," she said, "those kids' expectations are not high. By the time I get them, they know they're no good in math and they're going to fail. It's very

difficult to turn that attitude around." She went on to explain how important it is to use fundamentally different approaches with these students; by this, she meant more than attempting to accommodate to different learning styles: "We may be thinking about whether they're auditory or visual, but we're not thinking in terms of changing methodology."

The Role of Curriculum

We have seen thus far that although our seven experts share some general assumptions about the nature of mathematical knowledge for elementary school and about teaching and learning, they vary considerably in their beliefs and assumptions. These varied perspectives on knowing mathematics and on learning and teaching come together in these seven experts' views on the role the curriculum should play in elementary school mathematics instruction. In this section, we will provide brief portraits of the ideal curriculum envisioned by each of our experts. We begin with the views of the university experts, because they were most explicit about what the curriculum should be like, the teachers being more likely to accept the curriculum as a given. Whereas the university experts accepted our task of describing desirable curriculum features, the teacher experts tended to view the curriculum as something that is given to them by the school district. The teachers' responses to our questions about curriculum focused on the pedagogy they currently used in the classroom and how best to teach the ideas in the curriculum guidelines they were given.

For P3, the radical constructivist, curriculum is created through the interaction of teacher and child. Our goal as educators is to facilitate the child's construction of ever more powerful and useful mathematical knowledge, but to specify the particular mathematical concepts on children. Thus for P3,

curriculum cannot be specified in advance, but must emerge out of the interaction between teacher and student. From these interactions over time can emerge what P3 calls abstracted curricula--dynamic collections of typical meanings that children construct for particular mathematical concepts and activities that might help teachers and students in their efforts to construct meaning.

For P4, the curriculum should consist of series of rich problem situations, developed around important clusters of mathematical ideas. By working in these problem contexts with the teacher serving as "informed helper," students become increasingly sophisticated in their capabilities for doing mathematics--solving problems and making mathematical conjectures and arguments--and in their knowledge of powerful mathematical ideas and tools.

For P2 and P1, with their focus on having students acquire accepted meanings for the formal symbols of mathematics, the key to good curriculum is carefully designing embodiments and accompanying activities to represent important mathematical ideas. These activities should be research based and should relieve the teacher from having to choose representations for various mathematical ideas. Along with embodiments for teaching particular mathematics, the curriculum should contain activities to help students learn to apply the mathematics they have learned to various problem-solving settings.

With the exception of T1, the teachers were less explicit than the professors about what ideal curriculum should be like. Like P4, T1 argued that curriculum should consist primarily of problem settings organized around important mathematical ideas. T3 and T2 tended to accept the topics of the mathematics curriculum as given, focusing instead on how to integrate mathematical activities with problem-solving settings involving other subject matters.

Final Comments

What should we make of these diverse views among experts on what elementary mathematics curriculum would be like? The problem for curriculum developers and teachers is that underneath a seemingly unified call for major changes in the way mathematics is taught actually lie a number of strikingly different assumptions and images of what good mathematics teaching should be like. We argue that for teachers to make sense of the advice and calls for change that bombard them, they need to realize that they may be based on multiple--and possibly incompatible--assumptions.

References

California State Department of Education. (1987). Mathematics model curriculum guide: Kindergarten through grade eight. Sacramento, CA: Author.

Cobb, P. (1989). Experiential, cognitive, and anthropological perspectives in mathematics education. For the Learning of Mathematics, 9(2), 32-42.

Kilpatrick, J. (1987). What constructivism might be in mathematics education. In J. C. Bergerson, N. Herscovics, & C. Kieran (Eds.), Proceedings of the 11th International Conference for the Psychology of Mathematics Education (Vol. 1, pp. 3-27). Montreal: International Group for the Psychology of Mathematics Education.

National Council of Teachers of Mathematics. (1989). Curriculum and evaluation standards for school mathematics. Reston, VA: Author.

National Research Council. (1989). Everybody counts: A report to the nation on the future of mathematics education. Washington, DC: National Academy Press.

Prawat, R. S. (1985). Affective versus cognitive goal orientations in elementary teachers. American Educational Research Journal, 22, 587-604.

Prawat, R. S. (1989). Teaching for understanding: Three key attributes. Teaching and Teacher Education, 5, 315-328.

Putnam, R. T., Lampert, M., & Peterson, P. L. (1990). Alternative perspectives on knowing mathematics in elementary schools. In C. Cazden (Ed.), Review of research in education (Vol. 16, pp. 57-150). Washington, DC: American Educational Research Association.

Sowder, J. T. (Ed.). (1989). Research agenda for mathematics education: Vol. 5. Setting a research agenda. Hillsdale, NJ: Erlbaum.

Stanic, G. M. A., & Kilpatrick, J. (1988). Historical perspectives on problem solving in the mathematics curriculum. In R. I. Charles, & E. A. Silver (Ed.), Research agenda for mathematics education: Vol. 3. The teaching and assessing of mathematical problem solving (pp. 1-22). Hillsdale, NJ: Erlbaum.

Stevenson, H. (1989). Math achievement: Beijing and Chicago. Paper presented at the biennial meeting of the Society for Research in Child Development, Kansas City, MO.

Vergnaud, G. (1983). Experiential, cognitive, and anthropological perspectives in mathematics education. For the Learning of Mathematics, 9(2), 32-42.

Von Glasersfeld, E. (1987). The construction of knowledge: Contributions to conceptual semantics. Seaside, CA: Intersystems.

Voyage of the Mimi. (1985). New York: Holt, Rinehart & Winston.