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AUTHOR	Cifarelli, Victor
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ABSTRACT

The processes by which conceptual knowledge is constructed during mathematical problem solving were studied, focusing on the cognitive activity of learners (i.e., the ways they elaborate, reorganize, and reconceptualize their solution activity). Underlying this research is the view that learners' mathematical conceptions evolve from their activity as they attempt to resolve situations that they experience as genuinely problematic. Subjects were nine students in introductory calculus classes at the University of California (San Diego) who were interviewed as they solved a set of similar algebra word problems. Experimenters prepared videotaped and written protocols for each subject. Analyses of these protocols are reported within a case study format. Solution activities indicated a gradual construction and elaboration of conceptual knowledge as subjects solved their tasks. Generalizing across the case studies yielded four levels of solution activity: (1) structural; (2) re-presentation; (3) recognition; and (4) instrumental. These levels encompassed three broad categories of conceptual structure: recognition, re-presentation, and abstract. Findings indicate that subjects' developing ability to monitor and plan their solution activities is made possible by their cognitive advances. Two tables and two figures illustrate the discussion. A 19-item list of references is included. (SLD)

Conceptual Structures in Mathematical Problem Solving

Victor Cifarelli

University of California at San Diego

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CONCEPTUAL STRUCTURES IN MATHEMATICAL PROBLEM SOLVING

Victor Cifarelli
University of California at San Diego

INTRODUCTION

The notion of conceptual structure plays a central role in many theoretical accounts of mathematical learning. Mathematics educators who have as their goal the development of "intellectual autonomy" (Kamii, 1985) in the problem solving actions of their students view conceptual structures in terms of their interpretive qualities, as a means by which solvers can organize their problem solving experiences "with a view to making predictions about experiences to come" (von Glasersfeld, 1987) (e.g., making conjectures about the outcomes of one's potential solution activity in new situations).

Despite agreement about the importance of solvers developing such knowledge, there are differing views about how structural knowledge functions as part of the learner's cognitive repertoire in problem solving situations and about how we as math educators can help students develop such knowledge. For example, some schema based theories of problem solving are based on the assumption that abstract concepts exist as fixed, well-defined entities that can be explored in prototypical examples and exercises (see, for example, the work of Mayer, 1985). In contrast, current work in situated cognition challenges this view of concepts and suggests the need to reexamine the traditional view of conceptual structures as "decontextualized formal concepts" which are transferred across learning situations (Brown, Collins, and Duguid, 1989). The idea that learning and cognition are situated suggests that learners build up their conceptual knowledge in the context of ongoing activity. As a result, concepts continually evolve with each occasion of use, "because new situations, negotiations, and activities inevitably recast it in a new, more densely textured form" (Brown, Collins, and Duguid, 1989, p. 33).

The assertion that concepts develop in the course of ongoing activity (and are always under construction) suggests the need for studies that focus on cognitive activity where learners modify or restructure their conceptual understandings. According to Lave (1988) this constructive process commences when a solver gives meaning to, or "articulates their current structure" in a particular learning situation. These situations may become learning opportunities for the solver if, in the course of their activity, they encounter problems or "dilemmas" that were not expected. Resolution of these situations involves an exploration of "the plausibility of both procedure and resolution in relation to previously recognized resolution shapes" (Lave, 1988; p. 165) and can lead to a restructuring or reorganization of one's prior understandings. In other words, solvers in the process of expressing their structural knowledge might face problematic situations that were not expected. The conceptual activity that follows (i.e., genuine problem solving activity) is an ongoing process in which solver compares their current state of knowledge of the problem and their current understanding of what might count as a resolution. The potential result of this reflective activity is that the solver achieves a deeper understanding of their prior activity.

OBJECTIVES

The purpose of the study was to acquire an understanding of the processes by which conceptual knowledge is constructed during mathematical problem solving. The study focused on the cognitive activity of learners with particular emphasis on the ways that they elaborate, reorganize, and reconceptualize their solution activity. Unlike other approaches to studying conceptual knowledge in mathematical problem solving (Mayer, 1985; Chi, et.al., 1982), the study sought to identify and characterize the solvers' conceptual knowledge as it developed. Even though some studies of mathematical problem solving have shown that expert problem solvers possess conceptual or structural knowledge and use it in problem solving situations (Chi, et.al., 1982; Hinsley, et.al., 1977; Schoenfeld, 1985; Mayer, 1985), there has been little in the way of explanation as to how such knowledge is constructed.

Central to the approach taken here is the view that learners' mathematical conceptions evolve from their activity as they attempt to resolve situations that they experience as genuinely problematic (Vergnaud, 1984; Cobb, Yackel, and Wood, 1989). The prominent role attributed to activity as the source of solvers' mathematical knowledge contrasts with other approaches that investigate whether or not the solver perceives the problem structure (Mayer, 1985; Marshall, 1990; Reed, 1990; Silver, 1982). In particular, an emphasis on the cognitive activity of the solver shifts the focus from task analyses where the primary interest is the solver's ability to identify problem structures which have been assigned on an a priori basis (e.g., we as observers might describe two problems as having similar structures because they appear to embody similar mathematical relationships) to contextual analyses where the primary interest is how solvers interpret, or give meaning to, situations in terms of their current understandings (e.g., solvers in the course of solving a pair of mathematical tasks might "see" or interpret them as the same in some way). In the former case the problem's structure is seen as an objective property of the task in question (i.e., the interrelationship of mathematical operations underlying the problem statements), while in the latter case one may speak of the solver's structure as their conceptual organization of mathematical actions and relationships. In these terms, the solver's structure functions as a source with which to interpret, or give meaning to, new situations. The study was therefore conducted to address the following questions: (1) What is structure for the solver? (2) How does this structure evolve in the course of their mathematical activity?

Focusing on the mathematical activity of solvers, the study recognized that there exists an inherent subjectivity in the meanings learners give to mathematical situations (von Glasersfeld, 1987). This subjectivity of meaning has important implications for the ways that the terms problem and problem solving were defined. Specifically, the learner's interpretations of the situation help determine appropriate courses of action for them to carry out, or more precisely, their goals and purposes. For example, the learner's interpretations might lead them to consider whether or not some prior strategy or course of action would be useful. Consequently, it is reasonable to suggest that the problems experienced by solvers are subjective in the sense that they arise in situations where the consideration of prior activity in a new situation calls into question current understandings (i.e., their understandings prove unviable for the situation when actual activity is carried out). In view of this inherent subjectivity, the study adopted a definition of problem solving consistent with the idea that solvers interpret new situations in terms of their current understandings. Solvers were said to have a problem when they were faced with a situation where they could not "see" any way to achieve their goals or purposes (Lester, 1978; Pask, 1985). This characterization of the learner as an active sense making agent suggests broadbased definitions of problem and problem solving closely related to the learner's goals and purposes.

When problem solving is related to one's goals as stated above, a variety of situations qualify as genuine problem solving situations. For example, solvers might face a problematic situation when they attempt to make sense of or understand statements that describe a specific algebra word problem. Alternatively, the solver's problem might be to understand why a particular solution method led to unanticipated success or why two different solution methods led to the same result. These situations arise unexpectedly for solvers in the course of their goal directed activity and can serve as learning opportunities for solvers (Pask, 1985; Cobb, Yackel, & Wood, 1989). Successful resolution of such situations involves the construction of conceptual understanding in the context of ongoing activity (Vergnaud, 1984; Lave, 1988) with the result being that the solvers organize or build structure for their current solution activity (or restructure their prior solution activity). Hence, the goals of the study were to provide clarification for these ideas by observing solvers as they experienced and resolved a range of problematic situations that arose in the course of their activity and to characterize their subsequent construction of conceptual knowledge.

METHODOLOGY

Subjects

Subjects came from calculus classes at the University of California at San Diego.

This population was of particular interest given the amount of research devoted to analyzing problem solving abilities of college age students (Chi, Glaser, and Rees, 1982; Hinsley, Hayes, and Simon, 1977; Mayer, 1985; Larkin, 1983). A total of nine subjects participated in the study.

Use of Interviewing Methodology

Subjects were interviewed as they solved a set of similar algebra word problems (see Table 1).

The use of interviews to gather data was crucial to the goals of the study. It has been stated that most textbook word problems as they are interpreted in typical classroom situations do not serve as genuine problem solving activities because they are not "dilemma driven" (Lave, 1988). The use of interviews helped overcome this difficulty by establishing a social context between the interviewer and the subjects in which dilemmas could arise for the subject's in the course of their ongoing solution activity. Specifically, an interviewing methodology was used which required the solvers to think aloud while solving the tasks. In particular, the researcher wanted the subjects to accept certain obligations during the interview (e.g., explanations of, and justifications for their solution activity). In this way the researcher initiated and guided a social context seldom found in typical classroom situations. As a result, the subjects established their goals and purposes while interacting with the researcher. This approach, together with the nonstandard format for presenting the tasks made possible a focus on the solvers interpretations of tasks (and not the tasks themselves). Consequently it was possible to observe solvers experiencing dilemas as described by Lave. In other words, dilemas did arise for the subjects throughout the course of the interviews and these dilemas provided opportunities for the solvers to further their conceptual knowledge. For example, even though solvers might construct a solution to Task 1, they could conceivably face problems while solving later tasks despite recognizing that similar solution methods might be involved (e.g., solvers could face a problematic situation while solving Task 3 if they try to do exactly the same thing as they

Table 1: SET OF LEARNING TASKS

TASK 1: Solve the Two Lakes Problem
The surface of Clear Lake is 35 feet above the surface of Blue Lake. Clear Lake is twice as deep as Blue Lake. The bottom of Clear Lake is 12 feet above the bottom of Blue Lake. How deep are the two lakes?

TASK 2: Solve a Similar Problem Which Contains Superfluous Information
The northern edge of the city of Brownsburg is 200 miles north of the northern edge of Greenville. The distance between the southern edges is 218 miles. Greenville is three times as long, north to south as Brownsburg. A line drawn due north through the city center of Greenville falls 10 miles east of the city center of Brownsburg. How many miles in length is each city, north to south?

TASK 3: Solve a Similar Problem Which Contains Insufficient Information
An oil storage drum is mounted on a stand. A water storage drum is mounted on a stand that is 8 feet taller than the oil drum stand. The water level is 15 feet above the oil level. What is the depth of the oil in the drum? Of the water?

TASK 4: Solve a Similar Problem In Which the Question Is Omitted
An office building and an adjacent hotel each have a mirrored glass facade on the upper portions. The hotel is 50 feet shorter than the office building. The bottom of the glass facade on the hotel extends 15 feet below the bottom of the facade on the office building. The height of the facade on the office building is twice that on the hotel.

TASK 5: Solve a Similar Problem Which Contains Inconsistent Information
A mountain climber wishes to know the heights of Mt. Washburn and Mt. McCoy. The information he has is that the top of Mt. Washburn is 2000 feet above the top of Mt. McCoy, and that the base of Mt. Washburn is 180 feet below the base of Mt. McCoy. Mt. McCoy is twice as high as Mt. Washburn. What is the height of each mountain?

TASK 6: Solve a Similar Problem Which Contains the Same Implicit Information
A freight train and a passenger train are stopped on adjacent tracks. The engine of the freight is 100 yards ahead of the engine of the passenger train. The end of the caboose of the freight train is 30 yards ahead of the end of the caboose of the passenger train. The freight train is twice as long as the passenger train. How long are the trains?

TASK 7: Solve a Similar Problem that Is a Generalization
In constructing a tower of fixed height a contractor determines that he can use a 35 foot high base, 7 steel tower segments and no aerial platform. Alternatively, he can construct the tower by using no base, 9 steel tower segments and a 15 foot high aerial platform. What is the height of the tower he will construct?

TASK 8: Solve a Similar Simpler Problem
Green Lake and Fish Lake have surfaces at the same level. Green Lake is 3 times as deep as Fish Lake. The bottom of Green Lake is 40 feet below the bottom of Fish Lake. How deep are the two lakes?

TASK 9: Make Up a Problem Which has a Similar Solution Method

did in solving the earlier tasks). Hence, such situations provided opportunities for solvers to develop greater understanding about their solution activity. In addition, the solver's evolving intuitions about "problem similarity" allowed the researcher opportunities to observe how the solvers' newly constructed conceptual knowledge influenced subsequent solution activity in similar situations (i.e., development of control of solution activity).

Data Generation

Data collected in the study took the form of video and written protocols. All of the interviews were videotaped for subsequent analysis. This allowed for an ongoing interpretation and revision of the subject's activity in the course of the analysis. Viewing a videotape of each subject's performance gave the researcher an opportunity to "step back" and analyze the dialogue from an observers perspective. Once something had been "noticed" which might lead to a revision, the tape could be analyzed again in light of the new findings. This allowed for a continual communication between the theory and the data.

In addition to the video protocols that were prepared for each subject, written protocols were used in the subsequent analysis. These protocols took the forms of written transcripts (an ordered record of their verbal statements for each task) and paper-and-pencil records (the written work that the subjects performed as they progressed through the tasks). The written transcripts provided the researcher a means with which to identify and make reference to examples of significant solution activity when they occurred. This method of formatting the verbal responses of the subjects offered an effective yet economical way of reporting results in the analysis that followed. The paper-and-pencil records provided a perspective on the subjects' solution activity different from that of the written transcripts. For example, some records contained examples of perceptual expression used by the solvers (e.g., pictures or diagrams they constructed). In these instances the records helped to clarify the ways that the solvers developed their conceptual knowledge during the interview.

Analysis of Data

The protocols for each subject were analyzed and subsequent results were reported in the form of detailed case studies. The analysis of the protocols proceeded in the following phases.

It was a fundamental hypothesis of the study that solvers construct conceptual knowledge by performing novel activity in situations they find to be genuinely problematic. Hence, the solution activity of each subject was examined in order to identify those situations where they appeared to face such cognitive conflict. This was accomplished through careful examination of the written and video protocols and involved making a distinction between the solvers' novel (genuine problem solving activity) and their routine solution activity (assimilation of the situation to current conceptual structures with no problem experienced). Once this parsing had been made, the subject's novel activity was examined with the goal of identifying instances where major conceptual reorganization may have occurred. Here it was useful to identify qualitative aspects of the subject's solution activity (e.g., processes which enabled them to develop intuitions of problem sililarity during the interview). For example, the solvers were inferred to have experienced problems when their initial anticipations about what to do to solve a particular task proved unviable. In this way, the analysis focused on qualitative aspects of the solvers' solution activity (i.e., solvers' evolving anticipations and reflections) which indicated that constructive activity had occurred.

Based on the results of the qualitative analysis described above, a detailed case study was prepared for each subject. This consisted of the following parts. First, a written summary of the solver's performance was prepared. This portion of the case study focused on the solvers' solution activity with particular emphasis on the ways they actively gave meaning to each task and the novel ways they resolved problematic situations they faced along the way. This meaning making activity involved solvers' interpretation of novel situations in terms of previously constructed solution activity.

Second, a macroscopic summary of the subject's performance during the interview was prepared. This summary included both a general overview of the conceptual knowledge the subject appeared to construct while solving the tasks as well as a characterization of the subject's performance expressed as increasingly abstract levels of solution activity.

The case studies were then considered as a group for the purpose of generalizing the results. For this purpose, only those cases which yielded the most information were included in this phase. Of the nine subjects who participated in the study, two chose to withdraw after viewing the videotape of their performance (each subject had the option of withdrawing if for any reason they were dissatisfied with their performance). Of the remaining seven cases, the four most interesing cases were chosen for further analysis. This decision was based on several factors including the following. First, it was felt that the subjects of these cases demonstrated high levels of task involvement during the interviews (Nicholls, Cobb, Yackel, and Patashnick, 1990). This concern for the subjects' motivations during the interview is important given the fact that the researcher could only infer when the subjects experienced genuine problems. It was felt that these interpretations could be made confidently for subjects who maintained high levels of interest and motivation throughout the interview. Second, the subjects of these cases were particularly verbal throughout the interviews. There was little need to prompt them for comment about their solution activity. Hence, it was felt that their verbal responses provided an accurate description of their mental activity while solving the tasks. Finally, the researcher felt that collectively, these subjects demonstrated a range of abstraction in their solution activity sufficient to make some general inferrences.

The following sections summarize the findings of the study. First, results of a single case study are presented to illustrate examples of solution activity that helped to drive the analysis. Then the results are discussed in more general terms drawing from a subset of the original nine cases (i.e., the four cases as described above).

FINDINGS

As a mechanism for explaining and clarifying the levels of solution activity described above, the results of a single case study will be presented. The following paragraphs include episodes from the case study of solver MB and serve to illustrate examples of the different levels of conceptual knowledge demonstrated by the solvers.

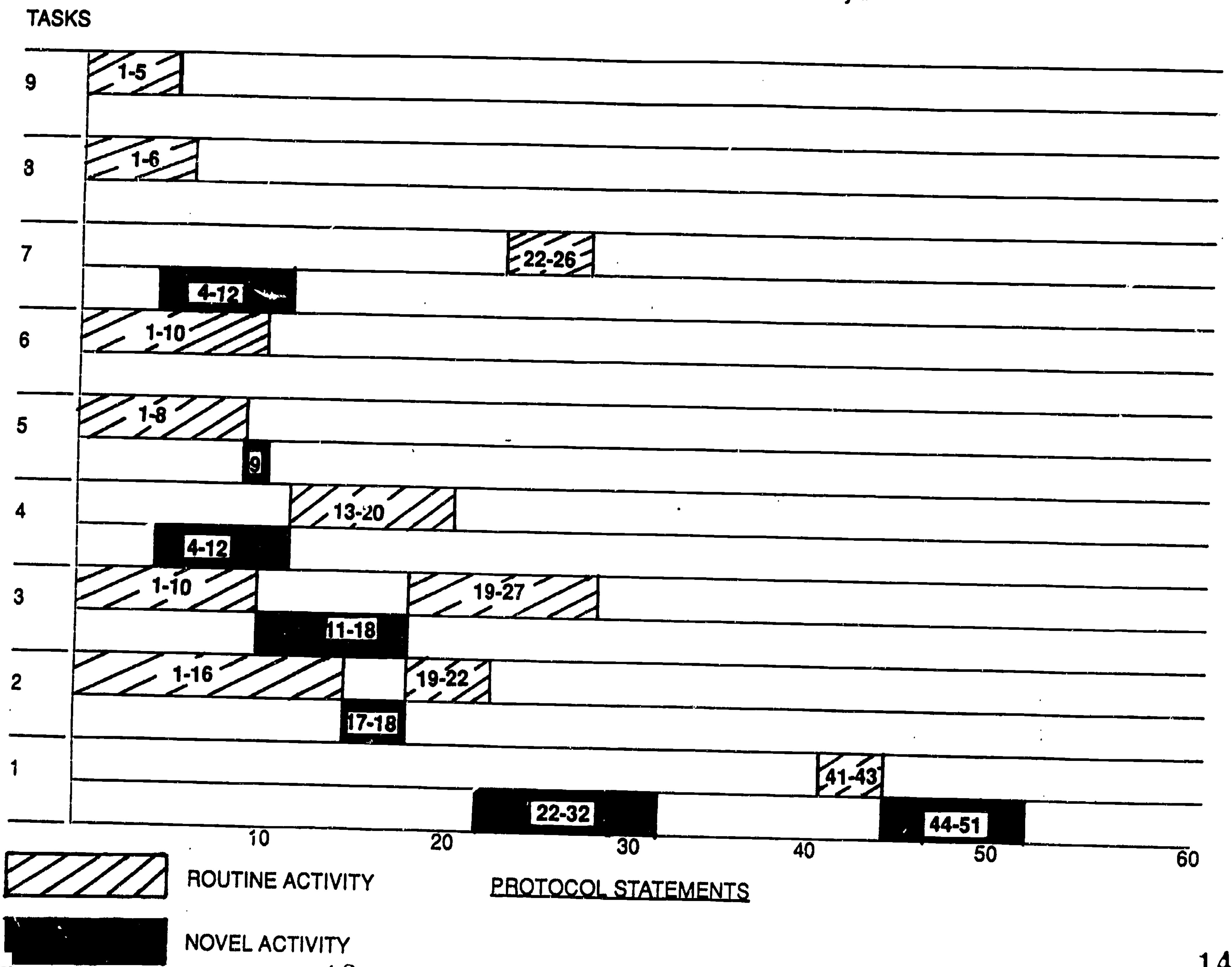
Case Study #1: MB

Solver MB was a female in the third quarter of the UCSD calculus sequence. Though undeclared in her academic major at the time of the interview, she eventually pursued and completed a degree in Physics.

The interview lasted a total of 40 minutes. Even though the solver was able to successfully complete all of the tasks, there was strong evidence that she participated in genuine problem solving activity at several points during the interview. Figure 1 is a task-by-task parsing of the solver's solution activity during the interview and identifies occasions where it was inferred that the solver faced genuinely problematic situations which she attempted to resolve (i.e., her current understandings did not work for her; she had a problem). The solver's novel solution activity (i.e., activity where it was inferred that the solver faced a genuine problem) is distinguished from her routine solution activity (i.e., activity where it was inferred that the solver assimilated the situation to her current conceptual structure and did not face a problem).

The solver's performance during the interview can be summarized as follows. The solver struggled to construct a solution to Task 1. She initially attempted to code all information contained within the problem statements. When she realized that this approach would not lead to a solution, she pursued an alternate solution method incorporating a graphical approach (i.e., diagrams of the lakes were constructed and relevant lengths from the diagrams were translated to a vertical axis which served as a reference aid in constructing relationships). This solution activity led to a correct solution and resulted in the construction of an initial recognitionary structure (i.e., as a result of her solution activity, she had a structure which allowed her to recognize the

Figure 1: Constructive Activity for Tasks: 1-9: Case Study #1



relevancy of using similar solution activity on subsequent tasks). Solution activity performed while solving Tasks 2-9 enabled the solver to elaborate and refine the initial structure, achieving higher levels of abstraction and control in her solution activity with each successive task. The following paragraphs include episodes from the solver's performance and help to illustrate the conceptual developments she made during the interview.

The solver's initial attempt to solve Task 1 could be described as an unreflective, instrumental approach (i.e., she did not appear to reflect on or think about the nature of potential solution activity prior to carrying it out). She initially interpreted the task as a routine algebra word problem and proceeded to code all information without attempting to develop a deeper understanding of the situation.

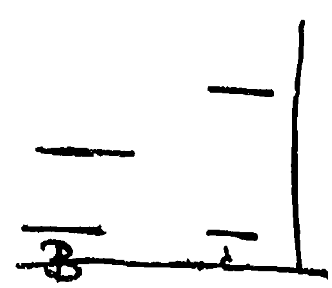
> S: That strikes me as an algebra problem with 2 variables. So the first thing I should do is assign variables to everything that is important.

She constructed a diagram and proceeded to generate all possible algebraic relationships. Symbols representing variables were manipulated in a mechanical fashion as the solver tried to code and relate everything in the problem without reflecting to the extent necessary to consider whether such assignments were relevant to the solution of the problem. This activity resulted in the generation of algebraic equations which she later found to be inappropriate.

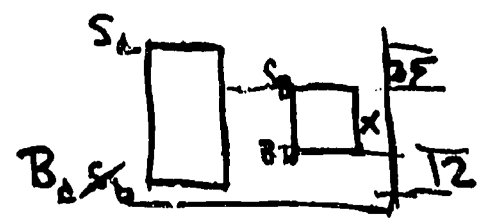
S_c = surface of clear lake
S_b = surface of blue lake
B_c = bottom of clear lake
B_b = bottom of blue lake

> S: I have 4 unknowns and 3 equations. And that's not good enough for me to solve an algebra problem.

$$S_c - S_b = 35$$
$$B_c - B_b = 12$$
$$(S_c - B_c) = 2(B_c - B_b)$$



The solver realized she faced a genuine problem and proceeded to develop an alternate method of solution. She abandoned her initial unreflective approach in favor of a more relational approach where reflection upon entities signified by the symbols led to the construction of a viable solution method. (This change in her approach appeared to be an example of "dilemma driven" activity as described by Lave.) This reflective approach was indicated by the solver's conscious intention to use the drawing as an interpretive tool that would aid her conceptualization and elaboration of potential relationships.



S: I am going to look for a geometrical relationship for my drawing which I am going to redraw because this is not accurate.

S: This is the bottom, this is the surface of Blue Lake and this is the bottom of Blue Lake. This distance is 12 and this distance is 35. And this whole distance is twice that whole distance. (LONG PERIOD OF REFLECTION HERE)

S: Okay, if I label this whole distance X ... I can say ... that 12 plus X plus 35, which is the height of Clear Lake, is going to equal twice X. And that's the relation in one variable I can solve.

S: And the relation I was missing here is the fact that I'm looking at differences in height, not absolute height.

$12 + x + 35 = 2x$

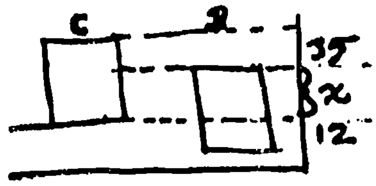
$12 + 35 = x$

12
35

$\boxed{47 = x}$

This constructive activity culminated with the generation of an appropriate algebraic equation for the problem, albeit an incorrect one (i.e., she made an error in labeling her diagram). This algebraic relationship expressed a wholist interpretation of the task rather than isolated relationships that corresponded to fragments of the problem statement. Upon discovery of an error in her diagram, the solver reconceptualized the problem and generated a new algebraic equation which led to a correct solution.

S:	The bottom of Lake, ... and this lake is 12 feet above the bottom of that lake. So I didn't draw it that way. I drew it 12 feet below.
S:	That means that my geometrical solution is probably off.
S:	So, the distance between these two is still 35. The distance between these two is 12.
S:	Yeah, but X doesn't mean the same anymore.
S:	So, 35 plus X equals 24 plus 2X. So 35 minus 24 equals ... X.
S:	So Clear Lake is equal to 35 plus X which is 46. And Blue Lake is equal to 12 plus 11 which is ... 23. That's the solution!



$S_c = 35+x+B_c$

$S_c - B_c = 35+x$ $\qquad$ $S_b = 12+x+B_b$

$S_b - B_b = 12+x$ $\qquad$ $35+x = 2(12+x)$

$35+x = 24+2x$

$11 = x$

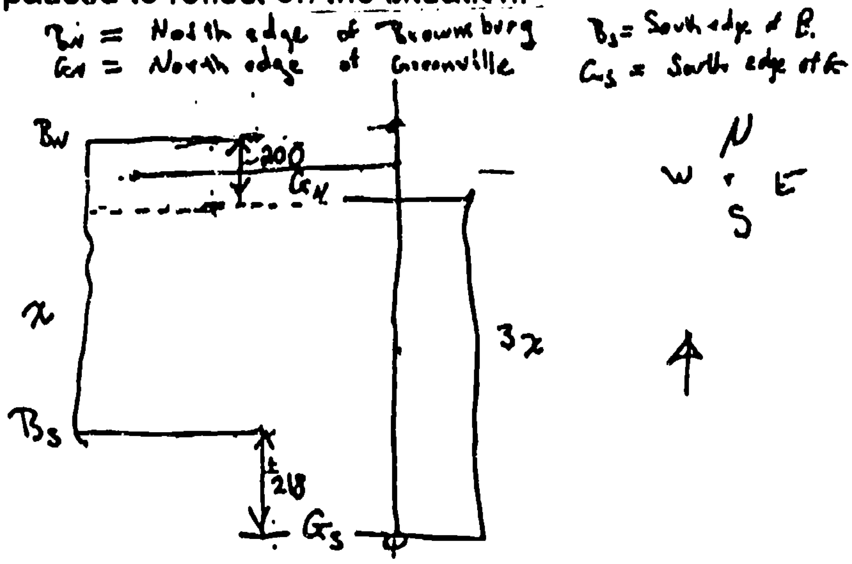
$$\boxed{C = 35+11 = 46 \quad B = 23}$$

The solver's solution activity for Task 1 involved the construction of novel relationships in the course of which she developed an initial conceptual structure. This activity was novel in the sense that it involved meaning making activity in genuinely problematic situations. Given this initial implicit structure, solution activity performed while solving Tasks 2-9 gave rise to opportunities for the solver to elaborate and reconceptualize the relationships she constructed while solving Task 1.

The claim that the solver constructed a conceptual structure for her solution activity while solving Task 1 is supported by her initial anticipations as she solves Task 2. At this point her structure was relatively unsophisticated in the sense that while she could recognize the appropriateness of using similar solution activity, she could not anticipate a potential problem suggested by the additional information contained within the problem statements.

S:	The first thing that strikes me is that this problem is alot like the previous one.
S:	And ... I think it would serve me well to start off in this one by just drawing a picture.

The gradual discovery of the superfluous information puzzled the solver, suggesting that her initial anticipation was based on a <u>recognition</u> of the relevance of activity similar to that which she had just completed (i.e., at best she could only recognize diagrammatic analysis of the type performed in Task 1 as appropriate to the new situation and could not anticipate potential difficulties). She paused to reflect on the situation.

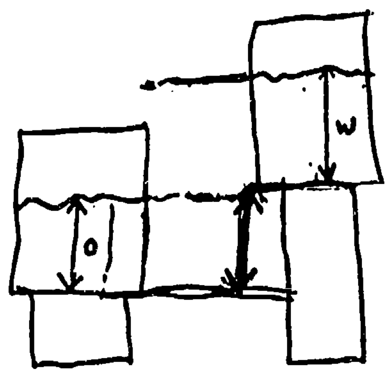


I: What are you thinking?

S: I'm thinking that this line drawn due north doesn't seem to have anything to do with the problem.

While the situation appeared to constitute a minor problem for her, she was not able to state with certainty that the added information was indeed irrelevant. She eventually chose to ignore the information ("So I'll just look at the other relationships first") and developed a solution.

Solution activity performed in Task 3 indicated that additional constructive activity had occurred while solving Task 2 and that the solver had reorganized her structure. After reading the problem statements, she proceeded to construct a diagram. The solver initially anticipated that she would use the same procedures that she had used while solving earlier tasks. However, she anticipated a potentially problematic situation soon after constructing a diagram.



S: And here's the water level, here's the oil level.

S: And the water level is 15 feet above the oil level.

S: So solve it ... (ANTICIPATION) ... the same way. ...(ANTICIPATION)... Impossible!

The suddeness with which she was able to anticipate a potential difficulty suggests that she had attained a level of reflective activity not demonstrated while solving prior tasks (more precisely, she could "run through" the potential solution activity in thought and could "see" difficulties that might arise). Further, this reflective activity served as a driving motivation for subsequent solution activity.

S: It strikes me suddenly that there might not be enough information to solve this problem. So I better check that. (LONG PERIOD OF REFLECTION HERE)

S: I suspect I'm going to need to know the heights of one of these things.

S: But I could be wrong so ... I'm going to go over here all the way through.

The solver spent much time and energy pursuing the elusive information. She finally concluded that the problem, as stated, could not be solved.

Tasks 2 and 3 presented opportunities for the solver to reflect on, elaborate, and developed a deepened awareness of the procedures she developed while solving Task 1. In each case, the solver gave initial meaning to the task she faced by assimilating the new situation to a conceptual structure that functioned at the level of recognition (i.e., she recognized that the activity she performed in solving Task 1 might be relevant for solving Tasks 2 and 3). In resolving problematic situations while solving Task 3, the solver was inferred to have internalized the structure and thus attained a higher level of abstraction in her conceptual understanding (i.e., at the level of Re-Presentation -- she could "re-present" her potential solution activity in thought, run through it and "see" the results as problematic). The solver appeared to make further abstractions as indicated by her solution activity in subsequent tasks.

Task 4 required the solver to construct a problem she could solve. In constructing a problem, the solver reflected on potential solution activity in a powerful way which was not evident in earlier tasks.

S: The things they could ask for are things like ... (ANTICIPATION) ... the height of one of the buildings but ... (ANTICIPATION) ... there's not enough information to get that. ... (ANTICIPATION) ...

S: The only thing we have information about is ... (ANTICIPATION) ... Ah, the relative heights of the two facades.

S: So, if I were ... if somebody wanted me to solve any problem, that's probably what they're asking for.

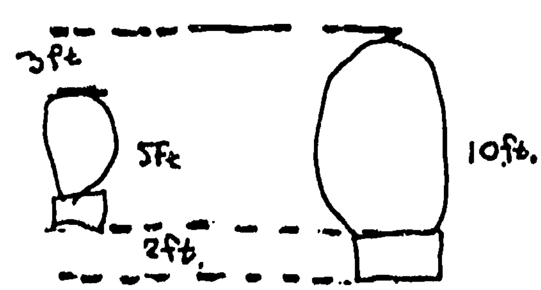
This episode illustrates the solver's developing flexibility and control of her potential solution activity. Her verbal responses above suggest that she could re-present or "run through" her potential solution activity in thought and use the results in an evaluative way, as a means with which to evaluate the viability of her potential solution activity (i.e., a more appropriate problem to solve involved the relative lengths of the two facades and not the heights of the two buildings). This development continued throughout the remainder of the interview.

The solver's solution activity in Task 9 indicated that she had reorganized her conceptual understanding (at a higher level of abstraction) to the extent that she could reflect on her potential solution activity and anticipate its results and evaluate the usefulness of the results for the current situation without the need to carry out the activity with paper and pencil. In other words, she could reflect on her potential solution activity and determine appropriate relationships. The task required the solver to make up a problem which had a solution method similar to the prior tasks.

S: Okay, ... (ANTICIPATION) ... I'm thinking of something with different heights.
S: Oh, ... (ANTICIPATION) ... bookshelves in a bookcase.
S: No, ... (ANTICIPATION) ... that's no good. ... How about hot air balloons!

The solver ran through potential solution activity for the particular situation she proposed (i.e., bookshelves) and anticipated its results (i.e., that it would not work for "bookshelves" but that she could solve it for "hot air balloons"). So, her structure allowed her to run through potential solution activity in thought, produce its results, and draw inferences from the results. Her subsequent actions in completing the task (i.e., leading to a formal statement of a similar problem) were routine and indicated she was very confident that she had constructed a correct solution.

S: Okay if I were going to draw a picture of the problem I'd have one hot balloon that looks like ... (DRAWS BALLOON). And a bigger hot air balloon that looks like that. And I'll make this distance ... 3 feet.
S: I'll make this distance 2 feet. And I'll make this height 10 feet high 'cause that makes this 12 feet and that makes this one twice this one which is useful.



S: So, I'll just say the top of one hot air balloon HAB being the abbreviation for that, is 3 feet,

S: I'll make it a yellow hot air balloon which w'll make it easier, above a green hot air balloon.

S: The bottom of the yellow hot air balloon is 2 feet below the bottom of the green hot air balloon. The yellow balloon is twice the height of the green balloon. Let's make that a lake.
What are the heights of the balloons?

Discussion

The results of the study will now be discussed in more general terms. Drawing from the results of four cases, the following paragraphs describe the conceptual knowledge the solvers appeared to construct during the interviews.

Analysis of the solvers' solution activity indicated a gradual building up and elaboration of their conceptual knowledge as they solved the tasks. Procedures constructed by solvers while solving the earlier tasks were elaborated as they solved later tasks. This development of conceptual knowledge was indicated by the solvers changing anticipations and reflections. In particular, the solvers demonstrated conceptual knowledge when, in interpreting the task, they could reflect on their potential solution activity (and generate anticipations about its results) without the need to actually carry out the particular actions (see Figure 2).

The constructive activity described above was subsequently characterized in terms of distinct levels of solution activity. By generalizing across the nine case studies, a total of four increasingly abstract levels of solution activity were inferred from the solvers' performance. The levels are summarized in Table 2.

The levels of solution activity identified in the study can be viewed as cognitive expressions of the solvers' evolving conceptual structures. These structures can themselves be described as organizations of the solvers' solution activity that provide order for their experiences and form to their interpretations when faced with new situations. In other words, the solvers' structures were purposeful organizations of their

Figure 2: Summary of Solution Activity

TASK 1 ---------> TASKS 2-9

Solves the target task

Solves variations of the original task

Solvers **Anticipations** and **Reflections**
changed as they progressed through tasks

Early Tasks

Low Level Anticipations & Reflections

Solver must carry out solution activity in earlier tasks; They can't reflect on potential solution activity and anticipate its results without carrying out the activity.

Later Tasks

High Level Anticipations & Reflections

Solver can reflect on potential activity in ways; They can "run through" a re-presentation of potential activity in thought and anticipate its results without the need to carry out the activity using paper and pencil.

Problematic Situations
(Learning Opportunities)

THEORETICAL INTERPRETATION

Solver builds initial structure for their solution activity

Solver elaborates, refines, and generalizes the original structure

Table 2: Levels of Solution Activity

LEVEL	DEFINING PROPERTIES	EXAMPLES
Structural	Solver can "run through" potential solution activity in thought and operate on the results	Solver can draw inferences from results of potential activity without the need to carry out solution activity
Re-Presentation	Solver can "run through" prior solution activity in thought	Solver can anticipate potential difficulties prior to carrying out solution activity
Recognition	Solver encounters new situation and identifies activity from previous tasks as relevant for solving current task	Solver recognizes diagrammatic analysis as appropriate for solving Tasks 2-9
Instrumental	Solver demonstrates fragmented, unreflective solution activity	Solver uses mechanical coding activity as part of a translation strategy

prior experiences that subsequently served to organize their future experiences in ways compatible with their goals.

In functional terms, the solvers' structures served to established for them a wide range of possible understandings and patterns of reflective activity. In particular, the solvers' structures functioned as facilitators for their solution activity in the sense that they enabled the solvers to generate anticipations while in the act of interpreting new situations. The solvers' anticipations served as channels for their potential solution activity in the sense that they provided a form or structure for subsequent solution activity. For example, when solver MB re-presented her potential solution activity while solving Task 3, she anticipated a potentially problematic situation and made a conjecture that the task did not contain enough information to ensure a solution. In this instance, her anticipation served to establish an orientation for her subsequent novel solution activity whereby she explored the viability of her conjecture. In this way her solution activity was constrained and enabled her to focus on the efficacy of her solution activity, which in turn made possible further conceptual development.

In more specific terms, the levels of solution activity identified in the study encompassed three broad categories of conceptual structure: Recognition, Re-Presentation, and Abstract.

Recognition Structure. Solvers achieved this initial level of conceptual structure when they had organized their cognitive actions in such a way that enabled them to make rather superficial interpretations of task similarity. At this level their structure was primitive in the sense that even though they could recognize similar situations while solving new tasks, they did not demonstrate a high level awareness of their potential solution activity (i.e., "problem sameness" meant they would do the same thing as they did in solving prior tasks with little idea of why or how well the procedures would work in the new situation). Specifically, they were able to recognize the usefulness and relevancy of prior solution activity when they attempted to solve new tasks (e.g., construct a diagram and use it to form alternate algebraic expressions for the same length). This level of structure limited the solvers to rather low level understandings of the efficacy of their prior solution activity in new situations. In particular, they could reflect on actual solution activity only and could not reflect on their potential solution activity. For example, while solving Task 2 solver MB initially anticipated she would perform solution activity very similar to that which she performed while solving Task 1 ("The first thing that strikes me is that this problem is alot like the first one. So I think it will benefit me to start by drawing a picture"). Her anticipation did not include any consideration of potential problems that might occur if she were to actually carry out the activity. When she commenced to carry out her solution activity she eventually faced a problematic situation (albeit a minor one: "This line drawn due north doesn't seem to have anything to do with the problem; I'll just look at the other relationships first") upon discovery of the superfluous information.

In some models of problem solving, recognition is the highest level of cognitive functioning identified (Mayer, 1985). Further, many traditional textbook exercises on problem solving (e.g., the stereotypical algebraic word problems found in most high school textbooks) appear to have been devised with the goal of developing this level of cognitive functioning.

Re-Presentation. Solvers achieved this level of conceptual structure when they had organized their cognitive actions in a way that enabled them to re- present their solution activity. At this level solvers could not only recognize the appropriateness of prior solution activity in the new situation but also anticipate potentially problematic situations prior to carrying out their solution activity. In particular, at this level of structure the solvers could reflect on their potential solution activity and anticipate potential problems prior to carrying out the actual activity. For example, solver MB while solving Task 3 anticipated that she would perform solution activity similar to that which she performed while solving Tasks 1-2. However her initial intuition that the task was similar was quickly followed by an anticipation of a problematic situation ("I construct a diagram ... so ... solve it the same way ... Impossible!"). Unlike her solution activity while solving

Task 2, she was able to reflect on her potential solution activity and anticipate a potential problem. In order to accomplish this she needed to put potential relationships in thought and then traverse through them. In general, this reflective activity by the solvers indictates their conceptual structure as consisting of tightly coordinated actions with which relevant mathematical relationships could be constructed and carried out sequentially.

Abstract. This level of conceptual structure is characterized by a high level of abstract mathematical activity. Solvers achieved this level of conceptual structure when they had organized their cognitive actions in such a way that enabled them to re-present their potential solution activity and operate on it to the extent that they could anticipate its results without carrying out the activity (i.e., they could "run through" a re-presentation of their potential solution activity in thought and generate inferences from its results). This level of conceptual structure was operative in the sense that the solvers' potential solution activity functioned as an object upon which they could reflect and draw inferences from. More precisely, solvers achieving this level had constructed a mathematical object (i.e., they could reflect on potential activity and operate on it to the extent that they could carry it out in thought and use the results to evaluate particular conjectures). This level of functioning where solvers appeared to reflect on their imagined solution activity as novel mathematical objects appears related to operative activity involving a "mental re-presentation of an action upon a representation" (Thompson, 1985).

CONCLUSIONS

The study was exploratory and future work needs to further clarify the findings presented here. Still the results of the study contribute to research in mathematical problem solving in the following ways. First, the characterization of conceptual structures as actively constructed by solvers suggests the importance of self generated solution activity. Problematic situations were not given to solvers. Rather, they were self generated in the sense that they arose as solvers tried to achieve their goals and purposes. In addition, the solvers' ability to transform initial conceptual structures into more abstract forms was made possible by their ability to reflect on their actual or potential mathematical activity as they attempted to cope with such situations. Second, many information processing models of problem solving rely on solvers' ability to categorize problem types and identify recognition of specific problem types as an example of sophisticated problem solving activity (Chi, et al., 1982; Mayer, 1985). The present study found recognition to be but one level of solution activity and identified several more abstract levels of problem solving activity. In addition, the levels of conceptual structure that were identified are compatible with Larkin's (1983) finding that problem representations constructed by solvers while performing problem solving activity range from naive to more abstract structures. The results of the present study extend these results by clarifying the operative qualities that abstract structures possess. Third, the results of the study suggest a relationship between cognitive and higher level mathematical activity. The solvers' cognitive act of interpreting new situations in terms of their structure and the ways that they resolved problematic situations that they faced along the way had a powerful influence on their subsequent solution activity. More precisely, they were able to anticipate what it was they were to do and the result of doing it before they carried out the activity. In metacognitive terms it can be said that planning and monitoring activity (i.e., anticipations about potential activity) developed as a result of the solvers performing specific cognitive acts (i.e., the expressing of their structural knowledge in new situations and the resolution of problematic situations in which they found themselves). The crucial point here is that their developing ability to monitor and plan their solution activity was made possible by their cognitive advances. This calls into question the notion that metacognitive skills can be treated as a separate level of cognitive functioning (Brown, 1988; Silver, 1985).

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