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ABSTRACT

The efficacy of an early mathematics program aimed at developing number sense and built entirely on children's invented procedures and on their informally acquired quantitative knowledge was studied. To socialize children to think of themselves as reasoners about numbers, the classroom program routinely provided daily conversation about numbers and drew attention to quantitative examples in everyday living situations. The program, initiated in 1988-89, was based on the following principles: draw children's informal knowledge into the classroom, develop children's trust in their own knowledge, use formal notations as a public record of discussions and conclusions, introduce the whole additive structure as quickly as possible, talk about mathematics in addition to practicing it, and encourage everyday problem finding. All children in the first through third grade classes in a school serving a largely minority and poor population participated. First graders were interviewed three times during the year. To assess whether the computational aspects of the standard curriculum were being met, data from the standardized mathematics achievement test that the school annually gives its first graders at the end of March were examined. The test scores of second and third graders who were introduced to a modified version of the program part-way through the 1988-89 school year were also assessed. The program produced large improvements both in number sense and in computational competence across all ability levels. Thinking-based programs successfully teach basic number facts and arithmetic procedures that are the core of the traditional primary mathematics program. An invention-based mathematics program is suitable even for children who are not socially favored or initially educationally able. (RLC)

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FROM PROTOQUANTITIES TO NUMBER SENSE

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The research described in this paper explores the efficacy of an early mathematics program that is aimed at developing number sense and is built enitrely on children's invented procedures and on their informally acquired quantitiative knowledge. In an effort to socialize children to think of themselves as reasoners about number, the classroom program routinely provided daily conversation about numbers and attention to quantitiative examples in everyday living situations. Results from the first year indicate that the program produced large improvements both in number sense and in computational competence across all ability levels.

In the U.S., as in many other countries, there are calls for early mathematics education that focuses less on computational skill and more on mathematical understanding and problem solving. Central among the objectives put forth in this new view of the goals of mathematics education is the development of number sense. According to Sowder (Sowder & Schappelle, 1989), number sense is a well-organized conceptual network that enables one to relate number and operation properties. It can be recognized in the ability to use number magnitude to make qualitative and quantitative judgments and in the use of flexible ways of solving problems involving number. Number sense is as much a habit of thought with respect to numbers and their relationships as it is any particular set of arithmetic facts or skills. It embodies a sense of confidence in one's mastery of numbers, a belief that there are many different ways to use numbers or to solve problems involving numbers, and a sense of empowerment with respect to the world of mathematics and numbers.

From this definition, it follows that a mathematics program aimed at developing number sense must be viewed not just as a program for developing particular forms of knowledge about numbers, but equally for developing dispositions to use this knowledge in flexible, inventive ways. This in turn means that it is not possible to simply add a number sense component to a curriculum that otherwise conveys to children that there are certain "correct" and expected ways to deal with arithmetic problems. Only if children come to believe that there are always multiple ways to solve problems, and that they, personally, are capable of discovering some of these ways, will they be likely to exercise--and thereby develop--number sense. For this reason, we believe, a serious commitment to number sense as an educational goal in mathematics requires that. for a considerable period of time, no specific computational algorithms be taught, but that children instead engage in massive practice in inventing computational and estimation procedures as well as in using them to solve everyday problems that they can understand.

This proposal is far from easy to adopt. Few parents are prepared to risk having their children not be taught to calculate. And teachers are wary of curricula that risk failure in traditional terms as a result of too much experimentation. The result is usually a compromise, at best, in which specific algorithms and

[1]The instructional program described in this paper was developed and implemented in Bill's classroom based on research and evaluation carried out by Resnick and Lesgold. The authors are grateful to the administration, parents, and primary grade children of the St. Agnes School, Pittsburgh, Pennsylvania, for their enthusiastic participation in this work.

procedures are taught (although sometimes with more discussion of their underlying rationales than used to be the case) and practiced, after a relatively short early period of more exploratory number activity.

Our goal was to test whether an early mathematics program built entirely on children's invented procedures could produce sufficient computational skill to meet current societal demands. The route we tock was a calculated gamble. We thought that, by building explicitly on children's informally acquired knowledge about quantities and their relationships and by developing a classroom routine in which conversation about numbers was daily fare, we could provide so much contextualized practice in number use that all reasonable computational objectives would be met without instruction on specific computational procedures.

A Theory of the Psychological Origins of Number Sense

We were willing to take this gamble because a substantial body of research, accumulated over the past decade, suggested that almost all children come to school with a substantial body of knowledge about quantity relations and that children are capable of using this knowledge as a foundation for understanding numbers and arithmetic. Two earlier papers (Resnick, 1989; Resnick & Greeno, 1990) review and interpret the research leading to this claim and develop a theory of how informal knowledge of quantities and counting might develop into mathematical knowledge about numbers and operators. We summarize the essential elements of that theory here.

Protoquantitative schemas

During the preschool years, children develop a large store of knowledge about how quantities of physical material behave in the world. This knowledge, acquired from manipulating and talking about physical material, allows children to make judgments about comparative amounts and sizes and to reason about changes in amounts and quantities. Because this early reasoning about quantity is done without measurement or exact numerical quantification, we refer to it as protoquantitative reasoning. We can document the development during the preschool years of three sets of protoquantitative schemas. These are:

- Protoquantitative compare, a schema that makes greater-smaller comparative judgments of amounts of material. Using it, children express quantity judgments in the form of comparative size labels such as bigger, longer, and more. These comparisons are initially based on direct perceptual judgments, but they form a basis for eventual numerical comparisons of quantities.
- Protoquantitative increase/decrease, a set of schemas that allow children to reason about the effects of adding or taking away an amount from a starting amount. Children know, for example, that if mother removes a cookie from one's plate there is less to eat and that if nothing has been added or taken away, they have the same amount as before. These schemas are protoquantitative precursors of children's eventual construction of unary addition and subtraction schemas. They also provide the framework from which conservation of quantity schemas will develop.
- Protoquantitative part/whole, a set of schemas that organize children's knowledge about the ways in which physical material comes apart and goes together, which allows them to make judgments

about the relations between parts and wholes. Children know, for example, that a whole cake is bigger than any of its pieces. This protoquantitative schema is the foundation for later understanding of binary addition and subtraction and for concepts of commutativity, associativity, and the complementarity of addition and subtraction. It also provides the framework for a concept of additive composition of number that underlies the place value system.

Quantification of the schemas

The protoquantitative schemas become the basis for number sense when they become integrated with children's knowledge of counting. Gelman and Gallistel's (1978) seminal work showed that children as young as three or four years of age implicitly know the key principles that allow counting to serve as a vehicle of quantification. Even when children are able to use counting to quantify given sets of objects or to create sets of specified sizes, however, they do not necessarily think of counting as a way of solving problems involving quantity relations. Sophian (1987), for example, has shown that children who know how to count sets do not spontaneously count when asked to solve conservation and similar problems. This means that counting and the protoquantitative schemas exist as separate knowledge systems, isolated from each other.

A first task of the primary school curriculum is to nudge children toward the use of counting--thus exact numerical quantification--in the context of problems that they previously would solve only by applying their protoquantitative schemas. Through such practice, the children not only acquire competence in solving problems about amounts in terms of numerical measures, but they also learn to interpret numbers in terms of the relations specified by the protoquantitative schemas. Eventually, they can construct an enriched meaning for numbers--treating numbers, rather than measured quantities of material, as the entities that are mentally compared, increased and decreased, or organized into parts and wholes by the schemas.

The Instructional Program

With this research base as a grounding for our efforts, we set out to develop a primary arithmetic program that would engage children from the outset in invention and verbal justification of quantity and number operations. Our goal was to use as little traditional school drill material as possible to provide children with a consistent environment in which they would be socialized to think of themselves as reasoners about number. Six principles guided our thinking and experimentation.

1. Draw children's informal knowledge, developed outside school, into the classroom. An important early goal of the program was to stimulate the use of counting in the context of the compare, increase/decrease, and part/whole schemas. This was done through extensive problem-solving practice, using story problems and acted-out situations. Counting (including counting on one's fingers) was actively encouraged, rather than suppressed as it often is in traditional teaching.

2. Develop children's trust in their own knowledge. To develop children's trust in their own knowledge *qua* mathematics, our program stressed the possibility of multiple procedures for solving any problem, invited children's invention of these multiple procedures, and asked that children explain and justify

their procedures using everyday language. In addition, the use of manipulatives and finger counting insured that children had a way of establishing for themselves the truth or falsity of their proposed solutions.

3. Use formal notations (identity sentences and equations) as a public record of discussions and conclusions. The goal here was to begin to link what children know to the formal language of mathematics. By using a standard mathematical notation to record conversations that were carried out in ordinary language and were rooted in well-understood problem situations, the formalisms took on a meaning directly linked to children's mathematical intuitions.

4. Introduce the whole additive structure as quickly as possible. It is important to situate arithmetic practice in a general system of quantity relationships. This is best done, we conjectured, by laying out the additive structures (addition and subtraction problem situations, the composition of large numbers, regrouping as a special application of the part/whole schema) as quickly as possible and then allowing full mastery (speed, flexibility of procedures, articulate explanations) of the whole system to develop over an extended period of time. Guided by this principle, we found it possible to introduce addition and subtraction with regrouping in February of first grade. No specific procedures were taught, however; instead, children were encouraged to invent (and explain) ways of solving multidigit addition and subtraction problems, using appropriate manipulatives and/or expanded notation formats that they developed.

5. Talk about mathematics; don't just do arithmetic. Discussion of numbers and their relations within problem situations is a crucial means of insuring that protoquantitative knowledge is brought into the mathematics classroom. In a typical daily lesson, a single, relatively complex problem would be presented. After a teacher-led discussion analyzing the problem, teams of children worked together to develop solutions and explanations for those solutions. Teams then presented their solutions and justifications to the whole class, and the teacher recorded these on the chalkboard using equation formats. By the end of the class period, multiple solutions to the problem, along with their justifications, had been considered, and there was frequently discussion of why several different solutions could work. Mathematical language and precision were deliberately not demanded in the oral discussion. However, the equations provided a mathematically precise public record, thus linking everyday language to mathematical language.

6. Encourage everyday problem finding. Children should come to view mathematics as something that can be found everywhere, not just in school, not just in formal notations, not just in problems posed by a teacher. We wanted them to get into the habit of noticing quantitative and other pattern relationships wherever they were and of posing questions for themselves about those relationships. To encourage this, homework projects were designed to use the events and objects of children's home lives: for example, finding as many sets of four things as possible in the home; counting fingers and toes of family members; recording categories and numbers of things removed from a grocery bag after a shopping trip.

Results of the program

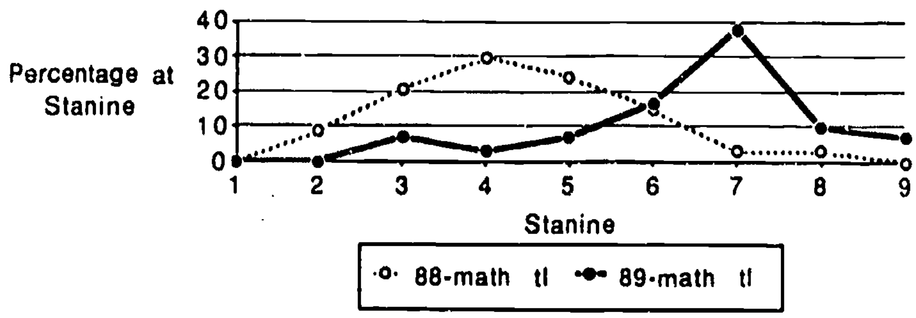
The program was initiated during the 1988-89 school year. The school served a largely minority and poor population. All children in first, second, and third grade classes in the school participated.

Because we had initially intended to introduce the program one year at a time, our data are most complete for the first grade. We interviewed all first graders three times during the year, focusing on their knowledge of counting and addition and subtraction facts, along with their methods for calculating and their understanding of the principles of commutativity, compensation, and the complementarity of addition and subtraction. At the outset, the children were not highly proficient. Only one third of them could count orally to 100 or beyond, and most were unable to count reliably across decade boundaries (e.g., 29-30, 59-60). The size of the sets that they could quantify by counting ranged from 6 to 20. About one third could not solve small-number addition problems, even with manipulatives or finger counting and plenty of encouraging support from the interviewer. Only six children were able to perform simple subtractions using counting procedures. By December the picture was sharply different. Almost all children were performing both addition and subtraction problems successfully, and all of these demonstrated knowledge of the commutativity of addition. At least half were also using invented procedures, such as counting-on from the larger of two addends (the MIN model), or using procedures that showed they understood the principles of complementarity of addition and subtraction. By the end of the school year, all children were performing in this way, and many were successfully solving and explaining multidigit problems.

Additional evidence suggests that the program was having many of the desired effects on children's confidence in their mathematical knowledge. Many of the children sang to themselves while taking the standardized tests that the school regularly administers. When visitors came to the classroom, they offered to "show off" by solving math problems. They frequently asked for harder problems. These displays came from children of almost all ability levels; they had not been typical of any except the most able children during the preceding year. Homework was more regularly turned in than in preceding years, without nagging or pressure from the teacher. Children often asked for extra math periods. Many parents reported that their children loved math or wanted to do math all the time. Parents also sent to school examples of problems that children had solved on their own in everyday family situations. Knowing that the teacher frequently used such problems in class, parents asked that their child's problems be used. It is notable that this kind of parental engagement occurred in a population of parents that is traditionally alienated from the school and tends not to interact with teachers or school officials.

To assess whether the computational aspects of the standard curriculum were being met, we examined data from the standardized mathematics achievement test that the school annually gives its first graders at the end of March. We used as a control group the children who had studied mathematics with the same teacher the preceding year. The following figure shows this comparison.

California Achievement Test
Math—Total Subscore



As can be seen, there is a massive improvement (equivalent to 1 1/2 standard deviations) from 1988 to 1989. Of particular importance, the statistical change was not achieved by improving the performance of the higher ability children, while leaving lower ability children behind. Rather, the entire distribution shifted upward. To check on whether these differences might have been due to a population difference, rather than an instructional program difference, we also compared scores on the "readiness" test that the school had administered at the end of kindergarten. The control group had performed slightly better than the program group in kindergarten.

The test scores of second and and third graders who were introduced to a modified version of the program partway through the 1988-89 school year provided additional evidence of the program's effectiveness. The second graders in the program showed an improvement equivalent to 1 1/2 standard deviations, the third graders to 1 standard deviation, compared with children who had not been in the program.

Conclusion

Our data show that an interpretation- and discussion-oriented mathematics program can begin at the outset of school by building on the intuitive mathematical knowledge that children have when they enter school. Such a program appears to foster the habits and knowledge that signal developing number sense. Our standardized test score data show that this kind of thinking-based program also succeeds in teaching the basic number facts and arithmetic procedures that are the core of the traditional primary mathematics program. Assuming that we can maintain and replicate our results, this means that a program aimed at developing sense can serve as the basic curriculum in arithmetic, not just as an adjunct to a more traditional knowledge-and-skills curriculum. Finally, our results suggest that an invention-based mathematics program is suitable even for children who are not socially favored or, initially, educationally able. This kind of program, if present in schools at all, has usually been reserved for children judged able and talented--most often those from favored social groups.

This apparently successful program presents some fundamental challenges to dominant assumptions about learning and schooling. Both educators and researchers on education have tended to define the educational task as one of teaching decontextualized knowledge and skills. An alternative view of the function of school in society is to think of schools as providing contexts for knowing and acting in which children can become <u>apprentices</u>--actual participants in a process that is socially valued, even though they are not yet skilled

enough to produce complicated performances without support. In this project, we were trying to create an apprenticeship environment for mathematical thinking in which children could participate daily, thus acquiring not only the skills and knowledge that "expert" mathematical reasoners possess, but also a social identity as a person who is able to and expected to engage in such reasoning. Several lines of theoretical work (e.g., Collins, Brown, & Newman, 1989; Lave, 1988; Rogoff, in press) inspire our thinking about learning as apprenticeship.

Our work addresses questions of how the apprenticeship metaphor can usefully inform early learning in a school environment. Among the problems to be solved is that of insuring that necessary particular skills and knowledge will be acquired, even though daily activity focuses on problem solving and general quantitative reasoning. Our first year standardized test results suggest that we have not done badly on this criterion, but we need to understand better than we do now what in our program has succeeded and what the limits of our methods might be. A second, even broader issue is the nature of the master-apprentice relationship. In traditional apprenticeship, teaching is only a secondary function of the master in an environment designed primarily for production, not instruction. Future work will analyze the role of the teacher in maintaining an apprenticeship environment specifically for learning purposes.

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