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ABSTRACT

This paper addresses the problem of whether two-dimensional solutions with different apparent meanings and different implied interpretations of the number-correct scale could be produced from the same test data set by simply shifting the orientation of the two-dimensional projection plane. An artificial data set of 3,000 response vectors and a real data set (3,153 examinees) obtained from administration of the American College Testing Program 60-item Mathematics Test were used. For the artificial data set, the alternative orientations of the two-dimensional projection plane in the three-dimensional space did suggest different interpretations of the unidimensional score. The relative position of item vectors changed with each orientation, and the definition of the axes in the two-dimensional solution was different in each orientation. Results with the real data were not as clear as those for the artificial data; however, some differences in the dimensions of the solutions were apparent. Results imply that multidimensional exploratory analyses should follow a strategy that emphasizes determining the largest number of dimensions that yield meaningful results, rather than the smallest number of dimensions that come close to reproducing the relationships in the data. Five tables and five graphs illustrate the discussion. (SLD)

Interpretation of Number-Correct Scores When the True Number of Dimensions Assessed by a Test Is Greater than Two

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Paper presented at the annual meeting of the National Council on Measurement in Education, Chicago, April 4, 1991.

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Interpretation of Number-Correct Scores When the True Number of Dimensions Assessed by a Test Is Greater than Two

The motivation for this paper came from attempting to understand the results of some work done to study the replicability of the dimensional structure of the ACT Assessment Mathematics Test (Ackerman, 1990). The purpose of that research was to determine whether multiple forms of tests that had been constructed to be parallel using traditional test construction procedures would still be considered to be parallel when analyzed from a multidimensional perspective. Most tests at ACT are measures of achievement that are fairly rich in the content covered. They are not produced to be factorially pure, but rather to adequately cover the domain of content being assessed (i.e., secondary school mathematics). These tests have been shown to have dominant first factors in a factor analytic sense, but also to have fairly complex multidimensional structures. Therefore, the question of multidimensional parallelism was of special interest.

The particular results that stimulated the research reported here were that two-dimensional, multidimensional item response theory (MIRT) analyses of several forms of the ACT Assessment Mathematics Test seemed to imply that the content areas dominating the number-correct score for the test seemed to be somewhat different across forms. This implication is a result of the finding (Reckase, 1989) that the number-correct score is a function of a linear combination of the dimensions assessed by a test. These dimensions are typically labeled in accordance with items that have high loadings on factors produced by an exploratory factor analysis. In this case, one test form might seem to stress a dimension labeled as geometry slightly more than that dimension was stressed on other test forms, while a dimension labeled as algebraic symbol manipulation might appear to be stressed more highly on another form of the test. This difference in emphasis seemed to occur despite the fact that the tests were constructed to the same content specifications and to the same statistical criteria. Also, unidimensionally, they appeared parallel.

Further work on these mathematics test forms suggested that they were better described by three to six dimensions of content, rather than the two dimensions in the original analyses (Miller & Hirsch, 1991). Since the two-dimensional solution considered in the early studies was thought to be a projection on a two-dimensional space of the higher dimensional solutions, it was hypothesized that the differences found in the two-dimensional solutions could be the result of the orientation of the two-dimensional projection plane in the higher dimensional space. In other words, the test forms could be multidimensionally parallel, but the number-correct scores could seem to be measures of different composites of content areas because the two-dimensional solutions were "looking" at the solutions from different perspectives, thus assigning different interpretations to the axes in the two-dimensional space. The different solutions would be the statistical equivalent of looking into a room through different windows and deciding that the views were not of the same room because the objects seemed to be in different orientation to each other.

The specific problem to be addressed by this paper is whether two-dimensional solutions with different apparent meanings, and different implied interpretations of the number-correct score scale, could be produced from the same data-set simply by shifting the orientation of the two-dimensional projection plane. Of course, considering this problem only makes sense if the data are explained more completely by solutions in a higher dimensional ability space.

Two different approaches were taken to address this problem. First, an artificial data-set with known multidimensional structure was analyzed to determine whether the hypothesized effect could be observed under relatively favorable conditions. The data-set was created to have three, equally weighted clusters of test items that measured hypothetical traits that were only slightly correlated. Two-dimensional solutions were determined that emphasized different pairs of clusters of items.

The second approach was to analyze a real data-set obtained from the administration of the ACT Assessment Mathematics Test to determine whether different two-dimensional solutions could be obtained that implied different meanings for the number-correct score. The expectation was that a real data

replication could be obtained for the result found with the simulated data. A detailed description of these two data-sets and the analyses performed on each is presented in the next sections of this paper.

Method

<u>Data-sets</u>

Two different data-sets were analyzed as part of this study. The first data-set was generated using the three-dimensional, normal-ogive MIRT model to emulate a real test that had three equally important, slightly correlated sections. This model is given by:

$$P(\underline{\theta}_j) = c_i + (1 - c_i)N(\underline{a}_i'\underline{\theta}_j + d_i), \qquad (1)$$

where $\underline{\theta}_j$ is the vector of abilities for person j,
$\underline{a}_i$ is the vector of discrimination parameters for item i,
d_i is a parameter related to the difficulty of item i,
c_i is the lower asymptote for item i,
and N( ) is the cumulative normal distribution function.

The simulated data-set consisted of 3000 response vectors that were generated using the 90 sets of item parameters presented in Table 1. The lower asymptote parameter was set to zero for all items. These item parameters were generated to be clustered around three directions in the three-dimensional ability space. The first 30 items were closest to dimension 1, the second 30 closest to dimension 2, and the last 30 closest to dimension 3.

Insert Table 1 about here

The real data-set consisted of the responses of 3153 examinees to the 60-item Mathematics Test on Form 39G of the ACT Assessment. Items on this test are classified as Pre-Algebra (PA), Elementary Algebra (EA), Intermediate Algebra/Coordinate Geometry (IA/CG), Plane Geometry (PG), and Trigonometry (T). This test was selected because earlier analysis had indicated that the test was assessing several dimensions.

Analyses

The multidimensional structures of the two data-sets were determined using NOHARM computer program (Fraser, 1987). This program provides estimates of the item parameters specified by Equation 1. The program allows the user to determine the orientation of the solution in the ability space by specifying certain items to have zero discrimination on one or more hypothesized ability dimensions. In this study, it was possible to specify the orientation of the two-dimensional solutions by forcing different items to have zero discrimination on the second dimension.

In the analysis of the simulated data, three differently oriented two-dimensional solutions were obtained. Each orientation was selected to illustrate the change in the appearance of the two-dimensional solution. In solution 1, item 1 was forced to have zero discrimination on the second dimension and item 31 had zero discrimination on the first dimension. For solution 2, item 1 again had zero discrimination on the second dimension and item 61 was restricted to having zero discrimination on the first dimension. The final solution required item 31 to have zero discrimination on dimension two and item 61 to have zero discrimination on the first dimension. For each solution, the angles between the direction assessed by each item and the first dimension were computed and the vectors representing these items were plotted using the method given in Reckase (1985). The angles and the orientation of the vectors facilitated comparisons of the three solutions.

For the real data set, the results from a three-dimensional solution were used to select those items which were expected to control the orientation of the two-dimensional solutions in the three dimensional ability space. Table 2 contains the results from that three-dimensional solution. The content classifications for each item are also listed.

Insert Table 2 about here

Items were selected specifically to illustrate the alternative orientations of the projection plane in the three-dimensional solution. For example, items 1, 8, 9, 12, 16, 18, 32, and 33 are pre-algebra, word-problem items. Inspection of their angles shows that these items form a cluster of

items near the θ_1-axis, i.e., they have small α_1 angles with large α_2 and α_3 angles. Therefore, the first two-dimensional solution chosen was to require item 1 to have zero discrimination on the second dimension with no additional requirements placed on the other items. This is commonly referred to as an exploratory solution (Fraser, 1987). It was anticipated that this would produce a solution where θ_1 was defined as some type of problem solving proficiency.

Looking again at Table 2, one can see that items 4, 10, 13, 17, . ., 25, and 44 form a cluster of geometry items located in the θ_1, θ_3 plane. The items have on average an 81° angle from the θ_2-axis with approximately equal angles from the θ_1- and θ_3-axes. Therefore, to obtain a projection plane which differed from that found in solution 1, the second two-dimensional solution fixed item 23 to have zero discrimination on the second dimension. In an attempt to alter the projection plane as much as possible, item 48, which is a trigonometry item, was fixed to have zero discrimination on the first dimension. This item was selected because it is located in the θ_2, θ_3 plane with a 90° angle from the θ_1-axis. It was expected that these two items would define an alternative two-dimensional structure in which the first dimension was measuring an ability described by the geometry items. It was also possible that the second dimension would differed from that found in the first solution (i.e., trigonometry). As with the results of the simulated data, item vectors were plotted to allow convenient comparisons of the two solutions.

Results

As predicted the alternative orientations of the two-dimensional projection plane in the three-dimensional space did suggest different interpretations of the unidimensional score. The relative position of item vectors in the plane changed with each orientation. Thus, the definition of the axes in the two-dimensional solution was different with each orientation, resulting in dissimilar interpretations.

Table 3 contains the MIRT item parameter estimates from the three two-dimensional solutions for the simulated data. Table 4 contains the angle between each item vector and the θ_1-axis in the two dimensional theta plane. These results illustrate that in solution 1 the θ_1-axis is primarily defined

by items 1 - 30 with items 31 - 60 defining the θ_2-axis. In solution 2 item 1 - 30 again define the θ_1-axis with items 61 - 90 defining the θ_2-axis. Finally, solution 3 shows that the θ_1-axis is defined by items 31 - 60 and the θ_2-axis is defined by items 61 - 90. Each of these solutions was derived from the same simulated data-set. The alternative solutions resulted from the different orientations of the two-dimensional projection plane caused by fixing different items to have zero discrimination as described earlier. Clearly, if the different sets of items assessed different content areas, the three two-dimensional solutions would lead to different interpretations of the proficiencies assessed by the test.

Insert Tables 3 & 4 about here

Figures 1 - 3 are the item vector plots for solution 1 - 3 respectively. These clearly show that the definitions of the dimensions assessed by the test differ for each solution. For example, solution 1 indicates that the test assesses a combination of the two dimensions measured by items 1 - 30 (dimension 1) and items 31 - 60 (dimension 2), where as, in solution 2 the two dimensions assessed by the test are defined by items 1 - 30 (dimension 1) and items 61 - 90 (dimension 2).

Insert Figures 1 - 3 about here

With the real data set the results were not as clear. However, some differences in the definitions of the dimensions of the solutions were apparent. Table 5 contains the MIRT item parameter estimates and the angles between each item vector and the θ_1-axis for the solutions using the two different anchoring approaches. In solution 1, item 1, which is a word problem, was set to have zero discrimination on the second dimension. In addition, the set of word problem items clustered with item 1 along the θ_1-axis. Thus, the first dimension could be defined as a word problem dimension. The items most closely associated with the second dimension are intermediate algebra and coordinate geometry items. Therefore, the second dimension could

be defined by those content areas. Elementary algebra items tend to have about 45° angles with the first dimension.

In solution 2, the item with zero discrimination on the second dimension (i.e., item 23) is a geometry item. In addition, the set of geometry items clustered with item 23 along the θ_1-axis. Thus, the first dimension could be defined as a geometry dimension. Item 48, which has zero discrimination on the first dimension, is a trigonometry item. However, the placement of the algebra and coordinate geometry items in relation to the other items on the test generally remained the same. Figures 4 & 5 are the item vector plots for these two solutions. For clarity, only the intermediate algebra items and the items for the content area used to anchor the solutions are plotted. These were the sets of items most closely related to the two axes. Also, a dashed line indicating the average angle for each set of items is included on the plot. Inspection of these plots show that the number-correct score in solution 1 may be considered a combination of word problems and algebra skills, and in solution 2 the number-correct score may be considered a combination of geometry and algebra skills. Note that the use of the single trigonometry item to anchor the second dimension did not have much of an effect on the solution.

Insert Figures 4 & 5 about here

Discussion and Conclusions

Factor analysis and, more recently, MIRT analysis have often been used to determine the underlying structure of educational and psychological tests. One of the traditional goals of factor analysis has been to identify the smallest ability space that yields a reasonable recovery of the relationships present in the data. A critical cause for the problem addressed by this paper is the urge to keep the dimensionality of a solution low so that interpretation is simple and so graphic representations of the results are easily made. The bowing to tradition and the desire to produce simple diagrams led to the use of two-dimensional solutions in early analyses of the ACT Assessment Mathematics Test when higher dimensional solutions were more complete.

This is not to say that the early analyses were flawed. The test data yielded large first factors. Some might argue that extracting two dimensions was overfactoring. Simple eigenvalue rules for the number of factors and the desire for clear orthogonal solutions have resulted in a strong emphasis on low dimensional solutions. This emphasis may have resulted in many examples of the effects demonstrated in this paper appearing in the research literature.

To make the implications of the results of this research clear, the results indicate that the solutions obtained from analyses that underestimate the number of dimensions are probably not unique, even after rotational indeterminacy is accounted for. Many different two-dimensional projection planes can be placed in a higher dimensional space, and the interpretations of a higher dimensional solution projected on those planes can be quite different.

In the case of the simulated data, the axes of the two-dimensional solutions were defined by either dimensions 1 and 2, 1 and 3, or 2 and 3 from the three dimensional solution, depending on how the projection plane was oriented. If the first solution were the only one considered, the conclusion would be that the data represented the interaction of the dimensions defined by items 1 - 30 and 31 - 60. The number-correct score could be interpreted as a weighted composite of those two dimensions. The skills assessed by items 61 - 90 would be considered to require roughly equal amounts of dimensions 1 and 2. Yet, the other two solutions are equally as good as the first, and they result in different interpretations of the relationships in the data. In each case, the number-correct scores could be interpreted as being a composite of proficiencies defined by different pairs of clusters of items.

The analyses of the real data also gave opportunities for multiple interpretations -- though as is always the case with real data, the results were not as clear. Depending on the analysis, one dimension of the solution was either geometry or problem solving. The other dimension remained constant with an emphasis on algebra. The implied meaning of the number-correct score could be quite different depending on the solution that happened to be obtained.

The real data solutions are a result of obtaining similar projection planes when either the geometry or problem solving items were used to anchor the solutions, even though those items were located in different regions of the three dimensional space. Thus, either set could be used to anchor the solution, resulting in an ambiguity in the interpretation of the skills assessed by the test. It was hoped that a more pronounced effect would be observed, but the strong emphasis on algebra in the test tended to favor a particular subset of two-dimensional solutions.

These results were obtained using MIRT analyses of matrices of dichotomous item scores. Yet, MIRT is strongly related to traditional factor analysis and similar findings would be expected using factor analytic methodology. It would not be surprising if some of the disagreements in findings in the educational and psychological literature where a result of underfactoring a data matrix and basing interpretations on different views of the same phenomenon.

These results imply that multidimensional exploratory analyses should follow a strategy that emphasizes determining the largest number of dimensions that yield meaningful results, rather than the smallest number of dimensions that come close to reproducing the relationships in the data. Underestimating the complexity of the structure of a data matrix seems to cause more interpretive problems than overestimating that complexity. After all, seldom has science found that reality is simpler than first thought. We are constantly being reminded that we underestimate the complexity of the areas that we study.

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Table 1

Item parameters used for simulation.

Item	a_1	a_2	a_3	d
1	1.47	.60	.60	2.93
2	1.55	.58	.53	-2.85
3	1.47	.56	.52	-.52
4	1.47	.49	.64	2.19
5	1.54	.56	.59	4.29
6	1.47	.50	.48	1.78
7	1.51	.59	.57	.41
8	1.60	.51	.58	-.05
9	1.56	.53	.57	-.01
10	1.45	.48	.48	1.83
11	1.55	.61	.45	-.83
12	1.49	.52	.58	-.22
13	1.49	.52	.52	2.63
14	1.58	.59	.61	-2.80
15	1.46	.62	.54	-.87
16	1.46	.48	.50	-.08
17	1.58	.51	.52	.97
18	1.45	.55	.60	-1.06
19	1.52	.48	.55	2.15
20	1.55	.60	.54	.71
21	1.51	.61	.64	-1.00
22	1.46	.55	.56	-.26
23	1.48	.58	.61	1.32
24	1.54	.60	.54	-.65
25	1.47	.59	.55	-1.80
26	1.45	.61	.52	-2.94
27	1.48	.45	.61	-.39
28	1.55	.53	.57	2.83
29	1.49	.54	.56	1.85
30	1.52	.49	.53	-.62
31	.60	1.47	.60	2.93
32	.53	1.55	.58	-2.85
33	.52	1.47	.56	-.52
34	.64	1.47	.49	-2.19
35	.59	1.54	.56	4.29
36	.48	1.47	.50	1.78
37	.57	1.51	.59	.41
38	.58	1.60	.51	-.05
39	.57	1.56	.53	-.01
40	.48	1.45	.48	1.83
41	.45	1.55	.61	-.83
42	.58	1.49	.52	-.22
43	.52	1.49	.52	2.63

Table 1 continued

44	.61	1.58	.59	-2.80
45	.54	1.46	.62	-.87
46	.50	1.46	.48	-.08
47	.52	1.58	.51	.97
48	.60	1.45	.55	-1.06
49	.55	1.52	.48	2.15
50	.54	1.55	.60	.71
51	.64	1.51	.61	-1.00
52	.56	1.46	.55	-.26
53	.61	1.48	.58	1.32
54	.54	1.54	.60	-.65
55	.55	1.47	.59	-1.80
56	.52	1.45	.61	-2.94
57	.61	1.48	.45	-.39
58	.57	1.55	.53	2.83
59	.56	1.49	.54	1.85
60	.53	1.52	.49	-.62
61	.60	.60	1.47	2.93
62	.58	.53	1.55	-2.85
63	.56	.52	1.47	-.52
64	.49	.64	1.47	-2.19
65	.56	.59	1.54	4.29
66	.50	.48	1.47	1.78
67	.59	.57	1.51	.41
68	.51	.58	1.60	-.05
69	.53	.57	1.56	-.01
70	.48	.48	1.45	1.83
71	.61	.45	1.55	-.83
72	.52	.58	1.49	-.22
73	.52	.52	1.49	2.63
74	.59	.61	1.58	-2.80
75	.62	.54	1.46	-.87
76	.48	.50	1.46	-.08
77	.51	.52	1.58	.97
78	.55	.60	1.45	-1.06
79	.48	.55	1.52	2.15
80	.60	.54	1.55	.71
81	.61	.64	1.51	-1.00
82	.55	.56	1.46	-.26
83	.58	.61	1.48	1.32
84	.60	.54	1.54	-.65
85	.59	.55	1.47	-1.80
86	.61	.52	1.45	-2.94
87	.45	.61	1.48	-.39
88	.53	.57	1.55	2.83
89	.54	.56	1.49	1.85
90	.49	.53	1.52	-.62

Table 2

Item parameters and angles for the three-dimensional solution of the real data set.

Item	a_1	a_2	a_3	d	α_1	α_2	α_3	Content
1	0.51	0.00	0.00	1.45	0	90	90	PA
2	0.42	0.28	0.00	1.19	33	57	90	EA
3	0.46	0.29	0.13	1.12	35	58	77	IA/CG
4	0.38	0.09	0.38	0.83	46	80	46	PG
5	1.34	0.19	0.44	0.51	20	82	72	PA
6	0.55	0.37	0.28	0.65	40	59	67	EA
7	0.53	1.58	0.89	-0.62	74	33	62	IA/CG
8	1.09	0.16	0.16	0.45	12	82	82	PA
9	0.72	0.39	0.33	0.13	36	64	68	PA
10	0.94	0.27	1.51	0.24	59	82	33	PG
11	0.66	0.54	1.07	-0.93	61	67	39	PG
12	0.69	0.11	0.47	0.10	35	83	56	PA
13	0.85	0.16	1.41	0.57	59	85	31	PG
14	0.64	0.65	0.50	0.01	52	51	61	PA
15	0.71	1.16	0.83	-0.83	63	43	59	EA
16	0.63	0.37	0.40	0.03	41	63	61	PA
17	0.94	0.17	0.79	-1.75	41	82	50	PG
18	0.74	0.23	0.39	-0.40	31	74	63	PA
19	0.64	0.43	0.45	-0.35	44	61	60	PA
20	0.52	0.63	0.67	-1.11	61	53	51	T
21	0.57	1.40	0.81	-0.94	71	35	62	IA/CG
22	0.77	0.71	0.47	-0.44	48	52	66	IA/CG
23	0.32	-0.01	0.38	-0.35	50	91	40	PG
24	0.77	1.12	0.50	-0.67	58	39	70	EA
25	0.70	0.36	1.00	-0.63	57	74	38	PG
26	0.40	0.96	0.58	-0.49	71	36	61	IA/CG
27	0.66	1.08	0.70	-0.83	63	42	61	IA/CG
28	0.65	0.56	0.31	-0.12	44	53	70	EA
29	1.14	1.55	0.84	-1.25	57	42	66	EA
30	1.10	0.34	1.46	-0.29	54	79	38	PG
31	0.67	1.08	0.53	-0.75	61	38	68	IA/CG
32	0.58	0.29	0.18	-0.66	30	65	75	PA
33	1.07	0.40	0.63	-0.44	35	72	61	PA
34	0.27	0.31	0.63	-1.13	69	66	33	T
35	0.57	0.35	0.16	-0.88	34	60	77	EA
36	0.78	0.97	0.72	-1.05	57	48	60	IA/CG
37	0.29	0.25	0.47	-1.08	61	66	39	PG
38	0.60	0.89	0.62	-1.10	61	44	60	IA/CG
39	0.70	0.64	0.16	-0.79	43	48	80	EA
40	0.82	0.97	0.77	-1.16	56	49	59	IA/CG
41	0.84	0.33	0.49	-0.50	35	71	61	PA
42	0.82	0.47	0.47	-0.70	39	63	64	EA
43	0.77	0.68	0.25	-1.87	43	50	76	IA/CG

Table 2 continued

44	0.67	0.33	0.88	-1.34	54	74	40	PG
45	0.74	0.65	0.57	-0.83	49	55	60	PA
46	1.00	1.04	0.97	-1.47	55	53	56	EA
47	0.81	1.13	1.19	-2.31	64	52	50	PG
48	-0.00	0.20	0.46	-2.43	90	67	23	T
49	0.63	1.00	0.81	-2.20	64	46	55	IA/CG
50	0.50	0.79	0.63	-1.66	63	45	56	IA/CG
51	0.66	1.05	1.37	-2.25	69	55	42	PG
52	1.32	1.34	1.83	-3.53	60	59	46	IA/CG
53	0.75	0.83	0.55	-1.25	53	48	64	IA/CG
54	0.30	0.76	0.73	-1.71	74	46	48	PG
55	0.59	0.88	0.87	-2.15	65	50	50	IA/CG
56	1.33	1.19	1.12	-4.02	51	56	58	IA/CG
57	0.74	1.26	1.06	-2.90	66	46	54	IA/CG
58	0.85	0.78	0.71	-2.14	51	55	58	PG
59	0.40	0.31	0.45	-1.42	54	63	48	PA
60	0.75	0.85	0.79	-2.92	57	52	55	T

Table 3

Item parameters from three two-dimensional solutions to the simulated data.

	Solution 1			Solution 2			Solution 3		
Item	a_1	a_2	d	a_1	a_2	d	a_1	a_2	d
1	3.48	0.00	4.21	3.05	0.00	3.74	0.44	0.43	1.48
2	0.70	0.22	-1.48	0.79	0.12	-1.50	0.49	0.38	-1.45
3	0.95	0.03	-0.35	0.95	0.00	-0.34	0.48	0.38	-0.32
4	0.79	0.09	-1.18	0.85	0.01	-1.19	0.46	0.34	-1.12
5	0.52	0.32	2.09	0.52	0.34	2.12	0.38	0.44	2.09
6	0.79	0.08	1.00	0.83	0.03	1.01	0.47	0.31	0.95
7	0.94	0.06	0.22	0.90	0.06	0.21	0.50	0.38	0.20
8	0.98	0.06	0.01	0.94	0.05	0.01	0.50	0.41	0.01
9	0.92	0.04	0.01	0.86	0.05	0.01	0.44	0.40	0.01
10	0.81	0.05	1.06	0.79	0.04	1.04	0.41	0.37	0.99
11	0.97	0.04	-0.49	1.01	-0.02	-0.49	0.51	0.37	-0.45
12	1.05	0.06	-0.16	0.98	0.07	-0.16	0.48	0.48	-0.14
13	0.70	0.18	1.41	0.67	0.18	1.40	0.37	0.45	1.36
14	0.70	0.29	-1.49	0.75	0.22	-1.49	0.50	0.45	-1.47
15	0.87	0.09	-0.46	0.87	0.06	-0.46	0.49	0.38	-0.43
16	0.92	0.02	-0.08	0.90	0.00	-0.08	0.47	0.36	-0.08
17	1.00	0.05	0.54	0.95	0.05	0.53	0.47	0.44	0.49
18	0.89	0.11	-0.62	0.92	0.06	-0.62	0.50	0.39	-0.57
19	0.79	0.10	1.19	0.84	0.04	1.20	0.49	0.33	1.13
20	0.92	0.18	0.39	0.91	0.15	0.39	0.55	0.43	0.37
21	0.94	0.15	-0.56	0.88	0.16	-0.55	0.47	0.51	-0.52
22	0.93	0.09	-0.14	0.89	0.09	-0.13	0.48	0.43	-0.13
23	0.86	0.11	0.73	0.75	0.16	0.70	0.38	0.48	0.68
24	0.96	0.12	-0.41	0.99	0.06	-0.41	0.56	0.40	-0.38
25	0.76	0.15	-0.99	0.81	0.09	-1.00	0.48	0.37	-0.95
26	0.65	0.25	-1.56	0.73	0.16	-1.58	0.49	0.37	-1.54
27	0.93	0.02	-0.22	0.84	0.05	-0.21	0.42	0.41	-0.20
28	0.72	0.13	1.47	0.67	0.16	1.45	0.32	0.45	1.40
29	0.87	0.06	1.01	0.86	0.04	1.00	0.44	0.39	0.94
30	0.91	0.08	-0.38	0.91	0.05	-0.38	0.50	0.38	-0.35
31	0.00	3.24	4.05	0.38	0.46	1.49	3.56	0.00	4.41
32	0.21	0.67	-1.44	0.44	0.40	-1.41	0.74	0.11	-1.45
33	0.09	0.90	-0.24	0.52	0.39	-0.22	1.04	-0.06	-0.25
34	0.26	0.67	-1.17	0.57	0.34	-1.16	0.86	0.08	-1.22
35	0.40	0.48	2.16	0.43	0.44	2.17	0.47	0.38	2.16
36	0.09	0.79	1.01	0.42	0.40	0.95	0.81	0.03	1.00
37	0.11	0.85	0.26	0.44	0.45	0.25	0.86	0.05	0.26
38	0.12	0.87	0.09	0.56	0.36	0.08	1.08	-0.08	0.09
39	0.07	0.91	0.00	0.48	0.41	0.00	1.00	-0.05	0.00
40	0.05	0.75	1.01	0.40	0.34	0.96	0.85	-0.07	1.02
41	0.08	0.86	-0.41	0.43	0.43	-0.38	0.89	0.01	-0.41

Table 3 continued

42	0.14	0.86	-0.07	0.51	0.41	-0.07	0.96	0.01	-0.08
43	0.26	0.67	1.47	0.50	0.38	1.44	0.77	0.14	1.50
44	0.21	0.69	-1.40	0.42	0.45	-1.37	0.72	0.15	-1.40
45	0.12	0.83	-0.50	0.43	0.45	-0.47	0.83	0.07	-0.49
46	0.12	0.82	-0.02	0.48	0.39	-0.02	0.93	-0.01	-0.02
47	0.04	0.91	0.58	0.47	0.39	0.53	0.98	-0.07	0.58
48	0.14	0.84	-0.61	0.44	0.46	-0.58	0.86	0.07	-0.61
49	0.19	0.64	1.14	0.44	0.36	1.12	0.73	0.08	1.16
50	0.08	0.95	0.42	0.46	0.46	0.39	0.97	0.01	0.42
51	0.21	0.82	-0.57	0.52	0.44	-0.55	0.91	0.09	-0.58
52	0.10	0.81	-0.15	0.49	0.35	-0.15	0.95	-0.06	-0.16
53	0.24	0.75	0.78	0.53	0.41	0.76	0.85	0.12	0.79
54	0.14	0.86	-0.35	0.47	0.45	-0.33	0.89	0.06	-0.35
55	0.22	0.78	-0.96	0.43	0.51	-0.92	0.76	0.18	-0.95
56	0.22	0.60	-1.52	0.37	0.44	-1.50	0.60	0.19	-1.52
57	0.09	0.79	-0.18	0.44	0.38	-0.17	0.86	-0.00	-0.18
58	0.17	0.68	1.51	0.46	0.35	1.47	0.78	0.04	1.54
59	0.11	0.73	1.09	0.44	0.36	1.04	0.81	0.01	1.10
60	0.09	0.87	-0.29	0.47	0.40	-0.27	0.94	-0.02	-0.29
61	0.44	0.47	1.49	0.00	3.07	3.78	0.00	3.86	4.67
62	0.43	0.48	-1.42	0.17	0.75	-1.47	0.17	0.74	-1.47
63	0.37	0.52	-0.26	0.03	0.91	-0.28	0.00	0.92	-0.28
64	0.35	0.50	-1.13	0.01	0.87	-1.21	0.02	0.83	-1.19
65	0.37	0.38	2.07	0.33	0.46	2.13	0.31	0.44	2.10
66	0.40	0.43	0.98	0.02	0.83	1.04	-0.03	0.88	1.04
67	0.41	0.53	0.17	0.09	0.88	0.18	0.05	0.91	0.18
68	0.41	0.55	-0.02	-0.05	1.11	-0.02	-0.09	1.13	-0.02
69	0.39	0.50	-0.04	0.03	0.92	-0.04	-0.01	0.94	-0.04
70	0.37	0.46	0.97	-0.01	0.89	1.05	-0.06	0.93	1.05
71	0.43	0.47	-0.43	-0.01	0.97	-0.47	-0.09	1.04	-0.48
72	0.44	0.54	-0.13	0.13	0.88	-0.14	0.08	0.91	-0.14
73	0.35	0.43	1.32	0.09	0.70	1.37	0.07	0.71	1.37
74	0.43	0.52	-1.49	0.16	0.78	-1.54	0.18	0.75	-1.53
75	0.41	0.50	-0.49	0.05	0.91	-0.53	0.01	0.94	-0.53
76	0.38	0.49	-0.02	-0.10	1.07	-0.03	-0.17	1.13	-0.03
77	0.36	0.52	0.52	-0.04	0.99	0.57	-0.07	1.00	0.57
78	0.43	0.51	-0.55	0.13	0.85	-0.58	0.09	0.87	-0.58
79	0.37	0.49	1.11	0.01	0.90	1.19	-0.01	0.90	1.19
80	0.41	0.49	0.38	0.01	0.94	0.41	-0.05	0.99	0.41
81	0.40	0.62	-0.58	0.12	0.94	-0.62	0.13	0.90	-0.61
82	0.38	0.54	-0.13	-0.01	1.00	-0.14	-0.04	1.00	-0.14
83	0.42	0.52	0.71	0.11	0.85	0.75	0.08	0.87	0.75
84	0.46	0.53	-0.38	0.08	0.96	-0.41	0.02	1.02	-0.41
85	0.42	0.50	-0.94	0.08	0.88	-1.00	0.05	0.89	-1.00
86	0.41	0.50	-1.54	0.20	0.70	-1.57	0.23	0.65	-1.56
87	0.37	0.57	-0.20	0.02	0.99	-0.22	0.01	0.96	-0.22
88	0.38	0.46	1.43	0.07	0.80	1.51	0.05	0.80	1.50
89	0.39	0.52	0.99	0.03	0.94	1.07	0.01	0.93	1.06
90	0.38	0.49	-0.32	-0.05	0.99	-0.35	-0.10	1.03	-0.35

Table 4

Angle from the θ_1-axis from the three two-dimensional solutions for the simulated data.

	Solution		
Item	1	2	3
1	0	0	44
2	17	9	38
3	2	0	38
4	6	1	36
5	32	33	49
6	6	2	33
7	4	4	38
8	3	3	39
9	2	3	42
10	4	3	42
11	2	1	36
12	3	4	45
13	14	15	51
14	23	16	42
15	6	4	38
16	1	0	38
17	3	3	43
18	7	4	38
19	7	3	34
20	11	9	38
21	9	11	47
22	5	5	42
23	7	12	52
24	7	3	36
25	11	6	37
26	21	12	37
27	1	3	44
28	10	13	54
29	4	3	41
30	5	3	37
31	90	50	0
32	73	42	8
33	84	37	3
34	69	31	6
35	50	46	39
36	83	44	2
37	83	46	4
38	82	32	4
39	86	40	3
40	86	41	5

Table 4 continued

41	85	44	0
42	81	39	1
43	69	37	11
44	73	47	12
45	82	46	5
46	82	39	1
47	88	40	4
48	81	46	5
49	74	39	6
50	85	45	0
51	76	40	6
52	83	36	3
53	72	38	8
54	81	44	4
55	75	50	14
56	70	50	17
57	84	41	0
58	76	38	3
59	81	39	1
60	84	41	1
61	47	90	90
62	48	77	77
63	54	88	90
64	55	89	88
65	46	54	55
66	47	88	92
67	52	84	87
68	53	93	95
69	52	88	91
70	51	91	94
71	48	91	95
72	51	82	85
73	51	83	84
74	50	78	76
75	51	87	90
76	52	95	98
77	55	92	94
78	50	82	84
79	53	90	91
80	50	89	93
81	57	83	82
82	55	91	92
83	51	82	85
84	49	85	89
85	50	85	87
86	51	74	71
87	57	89	89
88	50	85	87
89	54	88	89
90	52	93	95

Table 5

Item parameters and angles for the two two-dimensional solutions for the real data sets.

	Solution 1				Solution 2			
Item	a_1	a_2	d	α_1	a_1	a_2	d	α_1
1	0.42	0.00	1.41	0	0.38	0.08	1.39	12
2	0.27	0.31	1.15	48	0.30	0.32	1.15	47
3	0.39	0.34	1.10	41	0.42	0.36	1.10	41
4	0.51	0.16	0.83	17	0.53	0.19	0.83	19
5	1.02	0.38	0.44	21	1.07	0.44	0.44	22
6	0.55	0.44	0.65	39	0.60	0.47	0.65	38
7	0.71	1.77	-0.62	68	0.87	1.77	-0.61	64
8	0.72	0.29	0.38	22	0.75	0.34	0.38	24
9	0.69	0.49	0.13	35	0.73	0.53	0.13	36
10	1.42	0.43	0.21	17	1.44	0.52	0.21	20
11	1.04	0.65	-0.87	32	1.11	0.71	-0.87	33
12	0.79	0.23	0.10	17	0.82	0.24	0.10	19
13	1.34	0.32	0.50	13	1.35	0.41	0.50	17
14	0.72	0.75	0.01	46	0.80	0.79	0.01	45
15	0.89	1.27	-0.81	55	1.01	1.33	-0.81	53
16	0.67	0.47	0.03	35	0.72	0.51	0.03	35
17	1.19	0.35	-1.75	16	1.22	0.42	-1.74	19
18	0.75	0.35	-0.40	25	0.79	0.39	-0.40	26
19	0.71	0.53	-0.35	37	0.76	0.57	-0.35	37
20	0.71	0.74	-1.10	46	0.79	0.79	-1.10	45
21	0.73	1.47	-0.91	64	0.87	1.55	-0.92	61
22	0.78	0.81	-0.43	46	0.86	0.86	-0.43	45
23	0.49	0.06	-0.35	7	0.57	0.00	-0.36	0
24	0.74	1.19	-0.65	58	0.86	1.25	-0.66	55
25	1.08	0.49	-0.61	25	1.13	0.55	-0.60	26
26	0.52	1.04	-0.48	63	0.62	1.07	-0.48	60
27	0.79	1.19	-0.82	57	0.90	1.24	-0.82	54
28	0.61	0.62	-0.11	45	0.68	0.66	-0.11	44
29	1.17	1.68	-1.22	55	1.35	1.80	-1.24	53
30	1.63	0.54	-0.27	18	1.69	0.63	-0.27	21
31	0.69	1.17	-0.74	59	0.81	1.22	-0.74	56
32	0.49	0.37	-0.64	37	0.53	0.39	-0.64	37
33	1.13	0.56	-0.43	26	1.19	0.62	-0.43	28
34	0.54	0.38	-1.08	36	0.57	0.42	-1.08	36
35	0.46	0.42	-0.86	43	0.50	0.45	-0.86	42
36	0.91	1.10	-1.05	50	1.02	1.16	-1.05	49
37	0.48	0.32	-1.06	34	0.50	0.35	-1.06	35
38	0.72	1.00	-1.09	54	0.82	1.04	-1.09	52
39	0.53	0.65	-0.74	51	0.59	0.68	-0.74	49
40	0.98	1.12	-1.16	49	1.09	1.18	-1.16	47

Table 5 continued

41	0.88	0.46	-0.49	28	0.93	0.51	-0.49	29
42	0.84	0.59	-0.69	35	0.90	0.64	-0.69	35
43	0.62	0.72	-1.78	50	0.69	0.76	-1.78	48
44	0.99	0.47	-1.31	26	1.04	0.53	-1.31	27
45	0.83	0.77	-0.83	43	0.91	0.82	-0.83	42
46	1.22	1.24	-1.47	46	1.34	1.33	-1.48	45
47	1.14	1.28	-2.19	48	1.27	1.36	-2.21	47
48	0.16	0.24	-2.26	56	0.00	1.25	-3.47	90
49	0.83	1.16	-2.20	54	0.94	1.24	-2.21	53
50	0.65	0.91	-1.66	54	0.74	0.96	-1.66	52
51	1.04	1.12	-1.96	47	1.16	1.19	-1.96	46
52	1.62	1.46	-3.01	42	1.75	1.60	-3.02	42
53	0.78	0.96	-1.25	51	0.88	1.00	-1.25	49
54	0.54	0.85	-1.63	58	0.62	0.89	-1.64	55
55	0.83	1.03	-2.11	51	0.94	1.07	-2.10	49
56	1.37	1.43	-3.82	46	2.93	2.78	-6.66	44
57	1.01	1.46	-2.86	55	1.20	1.74	-3.09	55
58	0.96	0.95	-2.14	45	1.06	1.01	-2.14	44
59	0.54	0.41	-1.41	37	0.57	0.44	-1.41	37
60	0.88	1.08	-2.94	51	1.03	1.13	-2.97	48

Figure Captions

Figure 1. Plot of item vectors for two-dimensional solution of three-dimensional simulated data anchoring axes using items 1 and 31.

Figure 2. Plot of item vectors for two-dimensional solution of three-dimensional simulated data anchoring axes using items 1 and 61.

Figure 3. Plot of item vectors for two-dimensional solution of three-dimensional simulated data anchoring axes using items 31 and 61.

Figure 4. Plot of item vectors for the problem solving and the algebra items for the real data-set.

Figure 5. Plot of item vectors for the geometry and the algebra items for the real data-set.

Figure 1: Plot of item vectors for two-dimensional solution of three-dimensional simulated data anchoring axes using items 1 and 31.

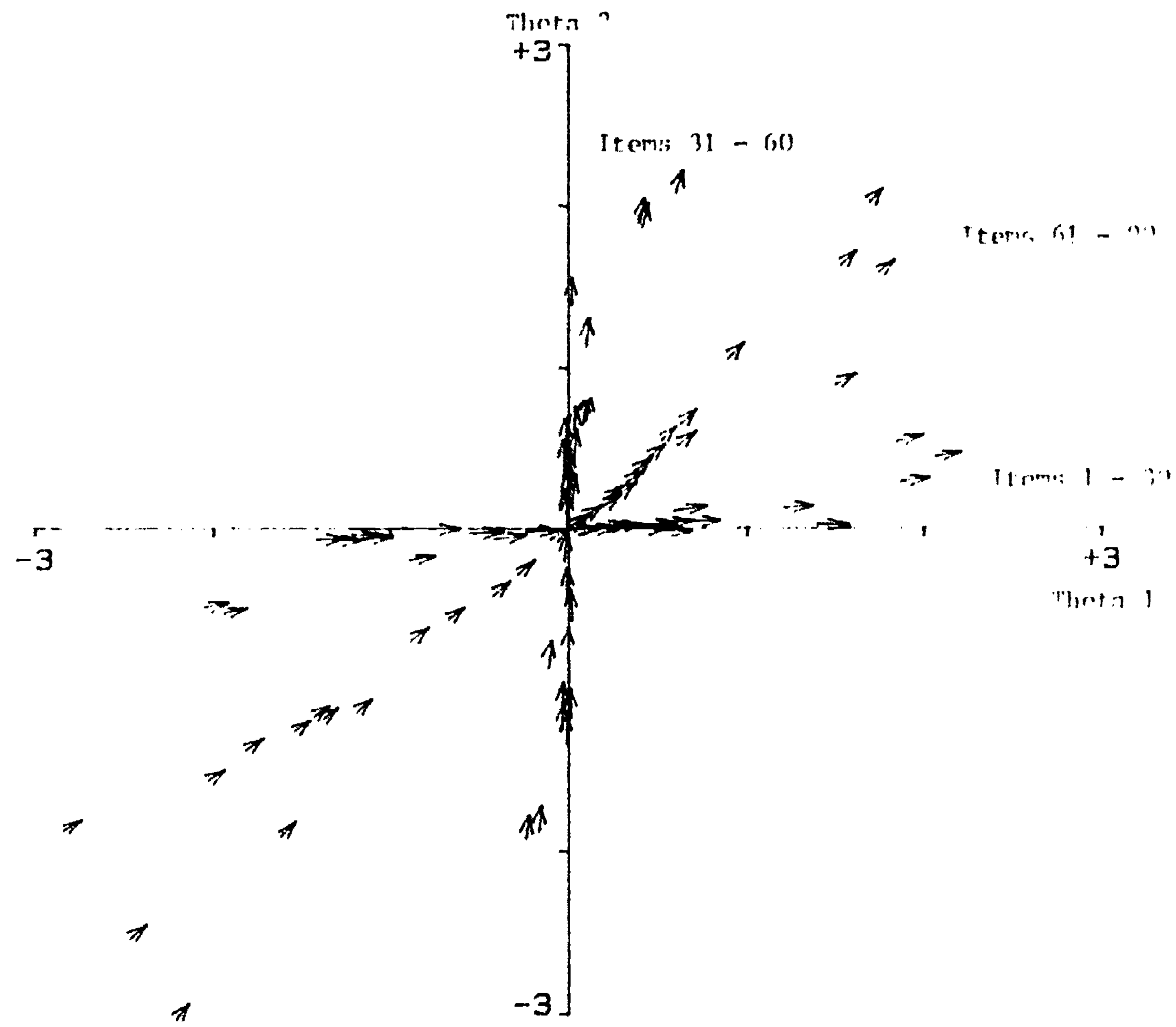

Figure 2: Plot of item vectors for two-dimensional solution of three-dimensional simulated data anchoring axes using items 1 and 61.

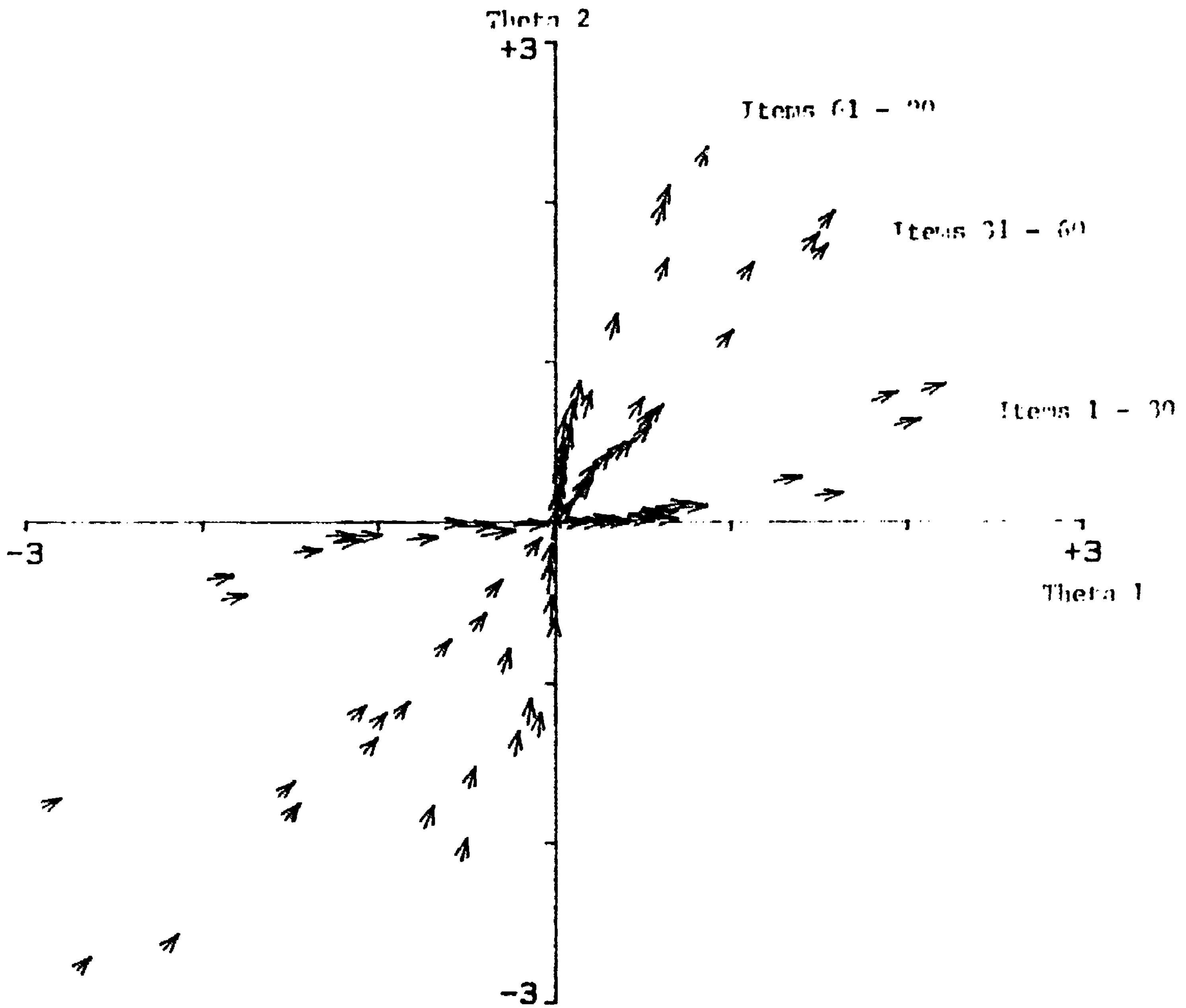

Figure 3: Plot of item vectors for two-dimensional solution of three-dimensional simulated data anchoring axes using items 31 and 61.

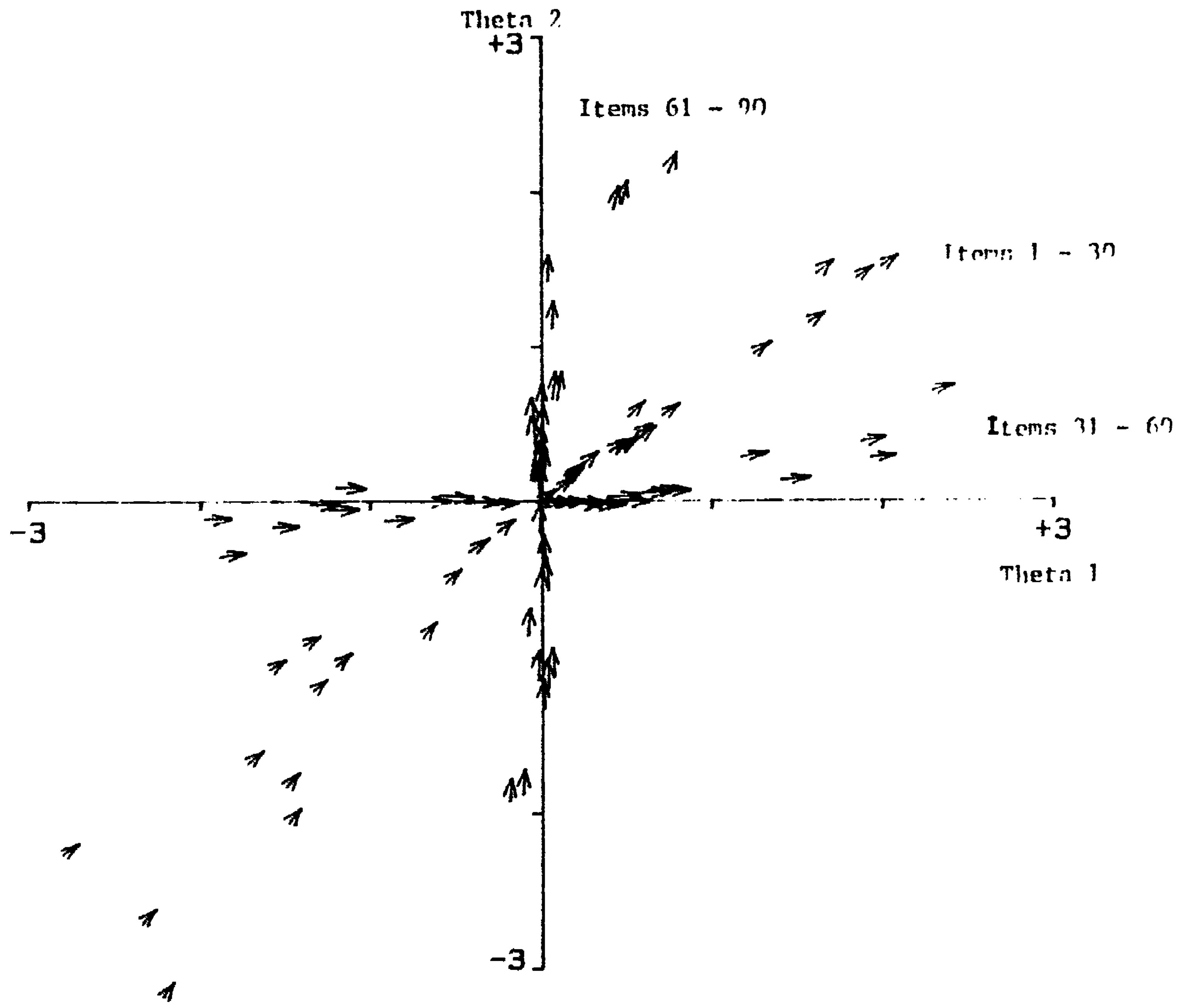

Figure 4: Plot of item vectors for the problem solving and the algebra items for the real data-set.

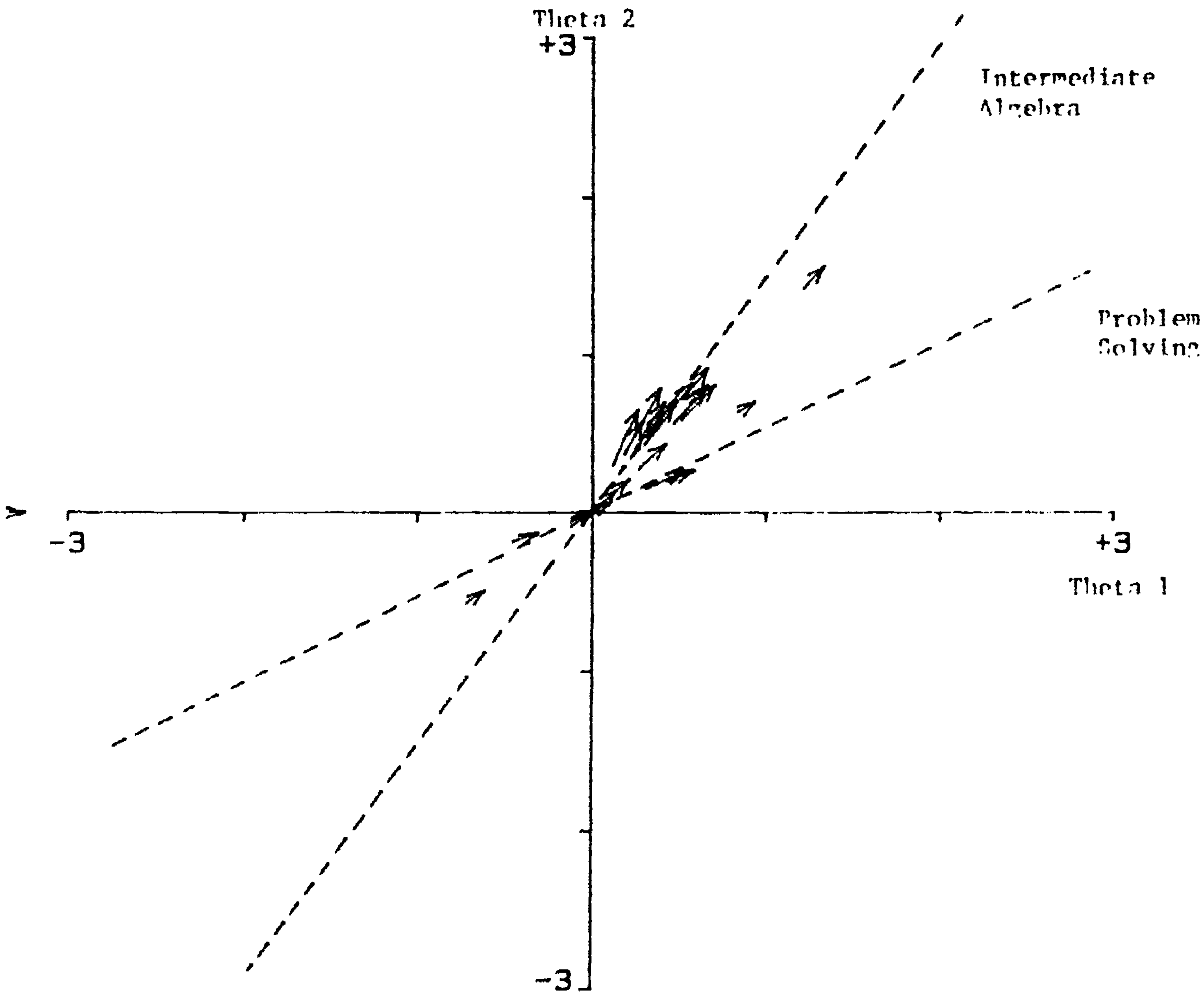

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Figure 5: Plot of item vectors for the geometry and the algebra items for the real data-set.