The probability integral of the multivariate normal distribution (ND) has received considerable attention since W. F. Sheppard's (1900) and K. Pearson's (1901) seminal work on the bivariate ND. This paper evaluates the formula that represents the \( n \times n \) correlation matrix of the \( \chi_{(sub \ i)} \) and the standardized multivariate normal density function. C. W. Dunnett and M. Sobel's formula for the univariate ND function, and R. E. Bohrer and M. J. Schervish's error-bounded algorithm for evaluating \( F(sub \ n) \) for general \( \rho(sub \ ij) \) are discussed. Computationally, the latter algorithm is restricted to \( n = 7 \); even at \( n = 7 \), it can take up to 24 hours for it to compute a single probability with \( 10^{-3} \) accuracy on a computer than is capable of about 1-2 million scalar floating point operations/second. This report presents a fast and general approximation (APX) for rectangular regions of the multivariate ND function based on C. E. Clark's (1961) APX to the moments of the maximum of \( n \) jointly normal random variables. The performance of this APX compared to special cases in which the exact results are known and error-bounded reduction formulae show that the APX's accuracy is adequate for many practical applications where multivariate normal probabilities are required. The computational speed of the Clark APX is unparalleled. The error bound for the APX is about \( 10^{-3} \) regardless of dimensionality, and accuracy increases with increases in \( \rho \). The Clark algorithm provides a generalization of Dunnett's (1955) results to the case of general \( \rho(sub \ ij) \), a natural application of which would be a generalization of Dunnett's test to the case of unequal sample sizes among the \( k + 1 \) groups (i.e., multiple treatment groups compared to a single control group). One data table is included. (RLC)
Approximating Multivariate Normal Orthant Probabilities

ONR Technical Report

Robert D. Gibbons
University of Illinois at Chicago

R. Darrell Bock
University of Chicago

Donald R. Hedeker
University of Illinois at Chicago

May 1990

Dr. R.D. Gibbons
Biometric Laboratory
Illinois State Psychiatric Institute.
1601 W. Taylor St., Chicago, IL 60612, USA.

Supported by the Cognitive Science Program, Office of Naval Research, under Grant #N00014-89-J-1104. Research whole or in part is permitted for any purpose of the United States Government. Approved for public release: distribution unlimited.
Approximating Multivariate Normal Orthant Probabilities

Robert D. Gibbons, R. Darrell Bock and Donald Hedeker

Interim

From 6/01/89 to 6/01/90
June 1, 1990

The probability integral of the multivariate normal distribution has received considerable attention since Sheppard (1900) and Pearson (1901) published their seminal work on the bivariate normal distribution. In the general case, we are concerned with evaluating

\[ F_n(h_1, h_2, \ldots, h_n; \{\rho_{ij}\}) = \int_{-\infty}^{h_1} \int_{-\infty}^{h_2} \ldots \int_{-\infty}^{h_n} f(x_1, x_2, \ldots, x_n; \{\rho_{ij}\}) \, dx_1 \ldots dx_n \]

where \( \rho_{ij} \) represents the \( n \times n \) symmetric correlation matrix of the \( z_i \)'s, and \( f(x_1, x_2, \ldots, x_n; \{\rho_{ij}\}) \) is the standardized multivariate normal density function. Direct evaluation of \( F_n \) is only possible for special cases of \( \{\rho_{ij}\} \). For example, Dunnett and Sobel (1955) have shown that when \( \rho_{ij} = a_i a_j (i \neq j) \), where \( |a_i| \leq 1 \), then
\[ F_n(h_1, h_2, \ldots, h_n; \rho_{ij}) = \int_{-\infty}^{\infty} \left[ \prod_{d=1}^{n} \frac{\Phi \left( \frac{h_d - \alpha_d y}{\sqrt{1 - \alpha^2}} \right)}{f(y) d(y)} \right] \]

where \( \Phi \) represents the univariate normal distribution function. This special case is the basis for much of item-response theory. More recently, however, Bohrer and Schervish (1981), have developed an error bounded algorithm for evaluating \( F_n \) for general \( \{\rho_{ij}\} \). Computationally, this algorithm is restricted to \( n = 7 \), and even at \( n = 7 \), it can require as much as 24 hours to compute a single probability with \( 10^{-3} \) accuracy on a computer than is capable of approximately 1-2 million scalar floating point operations per second.

The purpose of this report is to present a fast and general approximation for rectangular regions of the multivariate normal distribution function, that is based on Clark's (1961) approximation to the moments of the maximum of \( n \) jointly normal random variables. The performance of this approximation is then compared to special cases in which the exact results are known (e.g., \( \rho_{ij} = \rho = .5 \)), cases in which the integral reduces to a unidimensional quadrature evaluation (e.g., \( \rho_{ij} = \alpha_i \alpha_j \)), and finally error bounded reduction formulae for \( \{\rho_{ij}\} \) and \( n \leq 7 \).
ABSTRACT

The probability integral of the multivariate normal distribution has received considerable attention since Sheppard (1900) and Pearson (1901) published their seminal work on the bivariate normal distribution. In the general case, we are concerned with evaluating

$$ F_n(h_1, h_2, \ldots, h_n; \{\rho_{ij}\}) = \int_{-\infty}^{h_1} \int_{-\infty}^{h_2} \cdots \int_{-\infty}^{h_n} f(x_1, x_2, \ldots, x_n; \{\rho_{ij}\}) \, dx_1 \cdots dx_n $$

where \( \{\rho_{ij}\} \) represents the \( n \times n \) correlation matrix of the \( x_i \)'s, and \( f(x_1, x_2, \ldots, x_n; \{\rho_{ij}\}) \) is the standardized multivariate normal density function. Direct evaluation of \( F_n \) is only possible for special cases of \( \{\rho_{ij}\} \). For example, Dunnett and Sobel (1955) have shown that when \( \rho_{ij} = \alpha_i \alpha_j (i \neq j) \), where \( |\alpha_i| \leq 1 \), then

$$ F_n(h_1, h_2, \ldots, h_n; \{\rho_{ij}\}) = \int_{-\infty}^{\infty} \left[ \prod_{i=1}^{n} \Phi \left( \frac{h_i - \alpha_i y}{\sqrt{1 - \alpha^2}} \right) \right] f(y) \, dy $$

where \( \Phi \) represents the univariate normal distribution function. This special case is the basis for much of modern psychometric theory. More recently, however, Bohrer and Schervish (1981), have developed an error-bounded algorithm for evaluating \( F_n \) for general \( \{\rho_{ij}\} \). Computationally, this algorithm is restricted to \( n = 7 \) and even at \( n = 7 \), it can require as much as 24 hours to compute a single probability with \( 10^{-3} \) accuracy on a computer than is capable of approximately 1-2 million scalar floating point operations per second.

The purpose of this report is to present a fast and general approximation for rectangular regions of the multivariate normal distribution function based on Clark's (1961) approximation to the moments of the maximum of \( n \) jointly normal random variables. The performance of this approximation compared to special cases in which the exact results are known and error-bounded reduction formulae show the accuracy of the approximation to be adequate for many practical applications where multivariate normal probabilities are required.
1 Introduction

The probability integral of the multivariate normal distribution has received considerable attention since Sheppard (1900) and Pearson (1901) published their seminal work on the bivariate normal distribution. In the general case, we are concerned with evaluating

\[ F_n(h_1, h_2, \ldots, h_n; \{\rho_{ij}\}) = \int_{-\infty}^{h_1} \int_{-\infty}^{h_2} \cdots \int_{-\infty}^{h_n} f(x_1, x_2, \ldots, x_n; \{\rho_{ij}\}) \, dx_1 \cdots dx_n \]

where \( \{\rho_{ij}\} \) represents the \( n \times n \) symmetric correlation matrix of the \( x_i \)'s, and \( f(x_1, x_2, \ldots, x_n; \{\rho_{ij}\}) \) is the standardized multivariate normal density function. Direct evaluation of \( F_n \) is only possible for special cases of \( \{\rho_{ij}\} \). For example, Dunnett and Sobel (1955) have shown that when \( \rho_{ij} = \alpha_i \alpha_j \), where \( \alpha_i, \alpha_j \leq 1 \), then

\[ F_n(h_1, h_2, \ldots, h_n; \{\rho_{ij}\}) = \int_{-\infty}^{\infty} \left[ \prod_{i=1}^{n} \Phi \left( \frac{h_i - \alpha_i y}{\sqrt{1 - \alpha_i^2}} \right) \right] f(y) \, dy \]

where \( \Phi \) represents the univariate normal distribution function. The probability in equation (2) can be approximated to any practical degree of accuracy using Gauss-Hermite quadrature (Stroud and Sechrest, 1966). It should be noted that when \( \rho_{ij} = \rho \) for all \( i,j \), then

\[ F_n(h, h, \ldots, h; \{\rho\}) = \int_{-\infty}^{\infty} \left[ \prod_{i=1}^{n} \Phi \left( \frac{h + \rho^{1/2} y}{\sqrt{1 - \rho}} \right) \right] f(y) \, dy \]

and if \( \rho = .5 \) and \( h = 0 \),

\[ F_n(0, 0, \ldots, 0; \{.5\}) = \frac{1}{n + 1} \]

More recently, however, Bohrer and Schervish (1981), have developed an error-bounded algorithm for evaluating \( F_n \) for general \( \{\rho_{ij}\} \). Computationally, this algorithm is restricted to \( n = 7 \), and even at \( n = 7 \), it can require as much as 24 hours to compute a single probability with \( 10^{-2} \) accuracy on a computer than is capable of approximately 1-2 million scalar floating point operations per second. It is unclear whether vectorization of this algorithm is possible, so that the greatly increased speeds of parallel computing environments could be exploited (e.g., 20-80 million floating point instructions per second). Even still, it is unlikely that this algorithm would be computational tractable for \( n > 10 \).
An alternate approach to approximating \( F_n \), can be obtained by noting that

\[
F_n = Pr(x_1 \leq h_1, x_2 \leq h_2, \ldots x_n \leq h_n) \tag{5}
\]

If \( h_1 \ldots h_n = h = 0 \), and the \( x_i \) follow a standardized multivariate normal distribution, \( F_n^0 \) is a so-called "orthant" probability. Note, however, that this orthant probability is equivalent to

\[
F_n^0 = Pr \{ max(x_1, \ldots, x_n) \leq 0 \} \tag{6}
\]

If \( max(x_1, \ldots, x_n) \) were normally distributed, which it clearly is not, with mean \( E[max(x_1, \ldots, x_n)] \) and variance \( V[max(x_1, \ldots, x_n)] \), then,

\[
F_n^0 = \Phi \left[ \frac{h - E(max(x_1, \ldots, x_n))}{\sqrt{V(max(x_1, \ldots, x_n))}} \right] \tag{7}
\]

where in this case \( h = 0 \). For the more general rectangular region case of \( h_i \), we could set \( h = 0 \) and subtract \( h_i \) from the mean values of each of the \( x_i \), which to this point have been expressed in standardized form.

To use this algorithm, we must first have an accurate method of computing the first two moments of \( max(x_1, \ldots, x_n) \) where the \( x_i \) have a joint multivariate normal distribution with general correlation \( \rho_{ij} \), and some bound on the error introduced by assuming that \( max(x_1, \ldots, x_n) \) has a normal distribution. Such an approximation has been described by Clark (1961), and in the following, we describe its use in connection with evaluating \( F_n(x_1, x_2, \ldots, x_n; \rho_{ij}) \). We begin by reviewing Clark's original formulae.

### 2 The Clark Algorithm

Let any three successive components from an \( n \)-variate vector, \( y_1 \), be distributed:

\[
\begin{bmatrix}
y_1 \\
y_{i+1} \\
y_{i+2}
\end{bmatrix} \sim \mathcal{N} \left( \begin{bmatrix}
\mu_i \\
\mu_{i+1} \\
\mu_{i+2}
\end{bmatrix}, \begin{bmatrix}
\sigma_i^2 & \sigma_i\sigma_{i+1}\rho_{i,i+1} & \sigma_{i+1}^2 \\
\sigma_i\sigma_{i+1}\rho_{i,i+1} & \sigma_i^2 & \sigma_i\sigma_{i+2}\rho_{i,i+2} \\
\sigma_i\sigma_{i+2}\rho_{i,i+2} & \sigma_i\sigma_{i+2}\rho_{i,i+2} & \sigma_{i+1}^2 \sigma_{i+2}^2
\end{bmatrix} \right)
\]

Let \( \hat{y}_i = \text{max}(y_i) = y_i \), and compute the probability that \( y_{i+1} > \hat{y}_i \) as follows:

set \( z_{i+1} = (\mu_i - \mu_{i+1})/\zeta_{i+1} \).

where \( \zeta_{i+1} = \sigma_i^2 + \sigma_{i+1}^2 - 2\sigma_i\sigma_{i+1}\rho_{i,i+1} \).

Then \( P(y_{i+1} > \hat{y}) = P(y_{i+1} - \hat{y} > 0) \)

\[
= \Phi(-z_{i+1})
\]
the value of the univariate normal distribution function at the standard deviate

$$-z_{i+1}.$$ 

Now let $\hat{y}_{i+1} = \max(y_i, y_{i+1})$ and assume (as an approximation) that $(y_{i+2}, \hat{y}_{i+1})$
is bivariate normal with means,

$\mu(y_{i+2}) = E(y_{i+2}) = \mu_{i+2}$
$\mu(\hat{y}_{i+1}) = E(\hat{y}_{i+1}) = \mu_i \Phi(z_{i+1}) + \mu_{i+1} \Phi(-z_{i+1}) + \zeta_{i+1} \Phi(\zeta_{i+1}),$  \hspace{1cm} (8)

variances

$$\sigma^2(y_{i+2}) = E(y_{i+2}^2) - E^2(y_{i+2}) = \sigma_{i+2}^2$$
$$\sigma^2(\hat{y}_{i+1}) = E(\hat{y}_{i+1}^2) - E^2(\hat{y}_{i+1}) = \sigma_{i+2}^2,$$ \hspace{1cm} (9)

where

$$E(\hat{y}_{i+1}^2) = (\mu_i^2 + \sigma_i^2) \Phi(z_{i+1}) + (\mu_{i+1}^2 + \sigma_{i+1}^2) \Phi(-z_{i+1}) + (\mu_i + \mu_{i+1}) \zeta_{i+1} \Phi(\zeta_{i+1}),$$ \hspace{1cm} (10)

and correlation

$$\rho(\hat{y}_{i+1}, y_{i+2}) = \frac{\sigma \rho y_{i+2} \Phi(z_{i+1}) + \sigma^{i+1} \rho y_{i+1} \Phi(-z_{i+1})}{\sigma(\hat{y}_{i+1})}.$$ \hspace{1cm} (11)

Then,

$$P(y_{i+2} = \max(y_i, y_{i+1}, y_{i+2})) = P((y_{i+2} - y_{i+1}) > 0) \cap (y_{i+2} - y_i > 0))$$ \hspace{1cm} (12)

is approximated by

$$P(y_{i+2} > \hat{y}_{i+1}) = P(y_{i+2} - \hat{y}_{i+1} > 0)$$
$$= \Phi \left( \frac{\mu_{i+2} - \mu(\hat{y}_{i+1})}{\sqrt{\sigma_{i+2}^2 + \sigma^2(\hat{y}_{i+1}) - 2\sigma_{i+2} \sigma(\hat{y}_{i+1}) \rho(\hat{y}_{i+1}, y_{i+2})}} \right).$$ \hspace{1cm} (13)

Assuming as a working approximation that $\hat{y}_{i+1}$ is normally distributed with the above mean and variance, we may therefore proceed recursively from $i = 1$ to $i = n - 1.$ where $y_{n+1}$ is an independent dummy variate with mean zero and variance zero (i.e. $y_{n+1} = 0$). Then, for example.
\[ P[y_{n+1} = \max(y_1, y_2, \ldots, y_{n+1})] \]
\[ = P[(y_{n+1} - y_1 > 0) \cap (y_{n+1} - y_2 > 0) \cap \ldots \cap (y_{n+1} - y_n > 0)] \]
\[ = P[(-y_1 > 0) \cap (-y_2 > 0) \cap \ldots \cap (-y_n > 0)] \] (14)

approximates the probability of the negative orthant of the specified multivariate normal distribution. In the case of \( n \) correlated standard normals, the negative and positive orthant probabilities are identical. The probability of any other orthant can be obtained by reversing the signs of the variates corresponding to 1's in the orthant pattern. Of course, \( \hat{y}_{i+1} \) is not normally distributed. Errors produced by substituting normal approximations for the moments of \( \hat{y}_{i+1} \) are discussed in the following section.

More generally, to compute a multivariate normal probability over an \( n \) dimensional rectangular region, for example.

\[ \int_{-\infty}^{h_1} \int_{-\infty}^{h_2} \ldots \int_{-\infty}^{h_n} f(x_1, x_2, \ldots, x_n; \{\rho_{ij}\}) dx_1 \ldots dx_n \] (15)

we compute the negative orthant setting \( \mu_{n+1} = h \). Finally, to approximate the integral for general \( h \), we compute the negative orthant by setting \( \mu_{n+1} = 0 \) and \( \mu_i = \mu_i - h_i \).

3 Accuracy of the Clark Approximation

The errors of the Clark approximation result from the replacement of non-normal distributions by normal approximations. For example, suppose that we are interested in the maximum of four standard normal variables, i.e., \( \max(y_1, y_2, y_3, y_4) \). By assuming that \( \hat{y}_2 \) is normally distributed with expected value \( E[\max(y_1, y_2)] \) and variance \( V[\max(y_1, y_2)] \), we can then use the moments of \( \max(\hat{y}_1, y_2) \) as an approximation for those of \( \max(y_1, y_2, y_3) \). Next, we assume that \( \hat{y}_3 \) is normally distributed with expectation and variance equal to the corresponding moments of \( \max(\hat{y}_2, y_3) \), and can therefore use the moments of \( \max(\hat{y}_3, y_4) \) as an approximation for those of \( \max(y_1, \ldots, y_4) \). In this example, of course, \( \hat{y}_2 \) and \( \hat{y}_3 \) are not normally distributed. Furthermore, this is a rare case in which the distribution of a statistical variate diverges from normality as sample size increases. Tippett (1925) first showed that skewness and kurtosis of the maximum of \( n \) standard normals goes form .019 and .62 respectively for \( n = 2 \) to .429 and .765 for \( n = 100 \) to .618 and 1.088 for \( n = 1000 \). In terms of expected values of \( n \) standard normal variables, the effect of this non-normality is quite small. For example, for \( n = 10 \), the true value is
1.5388 and the approximation yields 1.5367. Even for $n = 1000$ the expected value is 3.2414 and the approximated value is 3.2457.

The effect of non-normality on the accuracy of the approximation is also dependent on the difference between $E(y_{i-1}, y_i)$. For example, suppose we wish to approximate the moments of $\max(y_1, y_2)$ where $y_1$ and $y_2$ are not normally distributed. Clark (1961) points out that if the difference $E(y_1) - E(y_2)$ is large relative to the greater of $V^{1/2}(y_1)$ and $V^{1/2}(y_2)$ the random variable $\max(y_1, y_2)$ is almost identical to $y_1$. Certainly the first two moments of $\max(y_1, y_2)$ would be minimally affected by replacing $y_1$ and $y_2$ by normal approximations. However if $E(y_1) - E(y_2)$ is small relative to the respective standard deviations, then the use of normal approximations could conceivably result in significant errors in the approximation of the mean and variance of their maximum.

In light of this, the following illustrations of the accuracy of the Clark approximation are, in fact, the worst case results, since they represent the case in which the expected values of the $y_i$ are equal. These results indicate that the lower bound for the Clark approximation is approximately $10^{-3}$, as illustrated in the following section.

## 4 Illustration

To evaluate the performance of this algorithm, we have examined a series of equa-correlated multivariate normal distributions for which exact results are known (see Gupta, 1963) and those considered by Schervish (1984). Table 1A displays results for 3 to 7 equa-correlated standard normal random variables with selected values of $\rho = .2, .3, .8$, and .9, and upper integration bounds of 0, 1, and 2. Inspection of the tabulated probabilities reveals that the Clark algorithm is generally accurate to at least $10^{-3}$ and that computational times are a linear function of dimensionality. The speed of the Clark algorithm does not depend on $\rho$. In contrast, the speed of MULNOR (Schervish, 1984) is exponentially increasing with both dimensionality and $\rho$. In the 7-variate normal case with $\rho_{ij} = .9$, MULNOR required almost a day to compute a probability which was accurate to $2 \times 10^{-5}$, whereas the Clark algorithm computed the same probability with $4 \times 10^{-4}$ accuracy in less than three thousandths of a second. Inspection of these results and others not reported here, suggest that the accuracy of the Clark approximation increases with increasing $\rho$.

Table 1B displays results for orthant probabilities of higher dimensional integrals ($n = 10, 20$, and 40), for the special case of $\rho = .5$, where $F_n^\rho = 1/(n + 1)$. Again, results are accurate to at least $10^{-3}$, and computational times are linear in $n$. MULNOR could not be used to evaluate integrals of this dimensionality.
Finally, Table 1C displays results for some tail probabilities of the multivariate normal distribution. In this case, the upper bound of the integration was -2.5, n = (3,5,10), and ρ = (.5,.9). These probabilities ranged from $10^{-3}$ to $10^{-5}$ and accuracy of the Clark approximation was $10^{-5}$ in all cases.

5 Discussion

Clark's (1961) formulae for the moments of the maximum of n correlated random normal variates can clearly be used to obtain a fast and accurate approximation to multivariate normal probabilities. Examination of a series of examples involving special cases in which the true results are known, reveals that the error bound for the approximation is approximately $10^{-3}$ regardless of dimensionality, and that accuracy increases with increases in $|ρ|$. These results are conservative in that we would expect the ill effect of using normal approximations to be greater when $ρ_i = ρ$, (i = 1, n) which is the case used in the illustrations.

In terms of computational speed, the Clark approximation is clearly unparalleled. A reasonable estimate of the speed of the Clark algorithm is given by,

$$\text{speed} = \frac{0.0004(n)}{\text{megaflop}} \text{ seconds}$$

where megaflop is the number of scalar floating point instructions per second that the computer is capable of performing.

Numerous applications of the Clark algorithm suggest themselves. Some preliminary work in this area has already been conducted by Daganzo (1984), in the context of discrete choice models of consumer behavior, and by Gibbons, Bock and Hedeker (1987) in item-response theory. Other potential applications include multivariate generalizations of probit analysis (see Ashford and Snowden, 1970 for the bivariate case), and random-effect probit models (Gibbons and Bock, 1987), where the Clark approximation was used to estimate first-order autocorrelation among the residual errors.

Another area of potential interest is in the approximation of multivariate $t$ probabilities, which can be considered as the joint distribution of n variates $t_i = z_i/s_i$, (i = 1, 2, ..., n) where the $z_i$ have a multivariate normal distribution with zero means and unknown variance $σ^2$, and known correlation matrix $\{ρ_{ij}\}$, while $σ^2/σ^2$ has a $\chi^2$ distribution with $ν$ degrees of freedom and is independent of the $z_i$. Dunnett (1955) has evaluated this joint density for the case of $ρ_{ij} = ρ = \frac{1}{n}$ by obtaining $F_n(z_1, z_2, ..., z_n; \{ρ_{ij}\})$ and integrating out $s$. Use of the Clark algorithm would provide a generalization of their result to the
case of general $\{\rho_{ij}\}$, a natural application of which would be a generalization of Dunnett's test to the case of unequal sample sizes among the $k + 1$ groups (i.e., treatment groups and a single control).

**References**


Table 1
Probability that \( n \) Standard Normal Random Variables with Common Correlation \( \rho \), are Simultaneously \( \leq h \).

### A. Comparison with MULNOR

<table>
<thead>
<tr>
<th>( n )</th>
<th>( h )</th>
<th>( \rho )</th>
<th>True(^1)</th>
<th>MULNOR</th>
<th>Clark</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td></td>
<td>Prob</td>
<td>Time(^2)</td>
<td>Time(^3)</td>
</tr>
<tr>
<td>3</td>
<td>2.0</td>
<td>.9</td>
<td>.96170</td>
<td>.96170</td>
<td>.196</td>
</tr>
<tr>
<td>4</td>
<td>2.0</td>
<td>.5</td>
<td>.92854</td>
<td>.92845</td>
<td>7.275</td>
</tr>
<tr>
<td>4</td>
<td>2.0</td>
<td>.8</td>
<td>.94759</td>
<td>.94758</td>
<td>13.735</td>
</tr>
<tr>
<td>4</td>
<td>2.0</td>
<td>.9</td>
<td>.95708</td>
<td>.95707</td>
<td>18.557</td>
</tr>
<tr>
<td>5(^4)</td>
<td>1.0</td>
<td>.3</td>
<td>.52111</td>
<td>.52113</td>
<td>40.161</td>
</tr>
<tr>
<td>7(^4)</td>
<td>0.0</td>
<td>.9</td>
<td>.32967</td>
<td>.32965</td>
<td>NA</td>
</tr>
<tr>
<td>7(^4)</td>
<td>0.0</td>
<td>.2</td>
<td>.04043</td>
<td>.04038</td>
<td>NA</td>
</tr>
</tbody>
</table>

\(^1\) Gupta (1963)
\(^2\) Seconds on a DEC 2060
\(^3\) Seconds on a COMPAQ 386-25, Weitek 3167, SVS FORTRAN
\(^4\) Accuracy set to \( 10^{-3} \) instead of \( 10^{-4} \) for MULNOR

### B. Higher Dimensional Integrals

<table>
<thead>
<tr>
<th>( n )</th>
<th>( h )</th>
<th>( \rho )</th>
<th>True(^1)</th>
<th>Clark</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td></td>
<td>Prob</td>
<td>Time</td>
</tr>
<tr>
<td>10</td>
<td>0</td>
<td>.5</td>
<td>.09091</td>
<td>.0907</td>
</tr>
<tr>
<td>20</td>
<td>0</td>
<td>.5</td>
<td>.04762</td>
<td>.04657</td>
</tr>
<tr>
<td>40</td>
<td>0</td>
<td>.5</td>
<td>.02439</td>
<td>.02390</td>
</tr>
</tbody>
</table>

\(^1\) \( F_n(0.0, \ldots, 0; \{.5\}) = \frac{1}{n+1} \)

### C. Tail Probabilities

<table>
<thead>
<tr>
<th>( n )</th>
<th>( h )</th>
<th>( \rho )</th>
<th>True(^1)</th>
<th>Clark</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td>-2.5</td>
<td>.5</td>
<td>.00017</td>
<td>.00021</td>
</tr>
<tr>
<td>3</td>
<td>-2.5</td>
<td>.9</td>
<td>.00230</td>
<td>.00231</td>
</tr>
<tr>
<td>5</td>
<td>-2.5</td>
<td>.5</td>
<td>.00003</td>
<td>.00004</td>
</tr>
<tr>
<td>5</td>
<td>-2.5</td>
<td>.9</td>
<td>.00157</td>
<td>.00156</td>
</tr>
<tr>
<td>10</td>
<td>-2.5</td>
<td>.5</td>
<td>.00000</td>
<td>.00000</td>
</tr>
<tr>
<td>10</td>
<td>-2.5</td>
<td>.9</td>
<td>.00099</td>
<td>.00098</td>
</tr>
</tbody>
</table>

\(^1\) Gupta (1963)
Distribution List

Dr. Terry Antonaccio
Educational Psychology
220 Edstrom Bldg.
University of Illinois
Champaign, IL 61820

Dr. James Aigner
1485 Norman Hall
University of Florida
Gainesville, FL 32601

Dr. Erling B. Anderson
Department of Economics
University of Illinois
Champaign-Urbana 61820

Dr. Ronald Armstrong
Rutgers University
Graduate School of Management
Newark, NJ 07102

Dr. Em L. Babbo
UCLA Center for the Study of Emotion
145 Moore Hall
University of California
Los Angeles, CA 90024

Dr. Laura L. Barnes
University of Toronto
2660 St. Bahndt St.
Toronto, OH 4046

Dr. William M. Bartt
University of Minnesota
Dept. of Educ. Psychology
1145 Searcy Hall
Minneapolis, MN 55455

Dr. Isaac Bejar
Max Corp.
Use T. Babcock
R antagon Road
Prairieville, NJ 08541

Dr. Menachem Benbenes
School of Education
Tel-Aviv University
Ramat Aviv 94778
ISRAEL

Dr. Arthur S. Blalock
Code N712
Naval Tugboat Systems Center
Orlando, FL 32813-7100

Dr. Bruce Blaisdell
Defense Manpower Data Center
99 Pacific St.
San Jose, CA 95154

Dr. Robert Breznik
Code 35
Naval Tugboat Systems Center
Orlando, FL 32816-3322

Dr. Robert Breznik
American College Testing Program
P.O. Box 168
Iowa City, IA 52243

Dr. Gregory Carol
CITE/McGraw-Hill
3049 Green Road
Washtenaw, MI 48194

Dr. John B. Caroll
496 Eelke Rd.
Northfield, IL 60093

Dr. John M. Carroll
IBM Watson Research Center
1000 E. Main St.
Yorktown Heights, NY 10598

Dr. Robert M. Carroll
Chief of Naval Operations
O" 1STW
Washington, DC 20500

Dr. Raymond E. Carrol
UES LAMP Science Advisor
AFRL/NOEL
Brooks AFB, TX 78235

Mr. Hue Hue Chang
University of Illinois
Department of Surgery
101 Illini Hall
721 South Wright St.
Champaign, IL 61820

Dr. Norm O. Cott
Department of Psychology
Univ. of So. California
Los Angeles, CA 90089-1041

Dr. Norbert O. Cott
Director, Maneuver Program
Center for Naval Analysis
481 Ford Avenue
Arlington, VA 22202-0358

Dr. Stanley Cohen
Office of Naval Technology
Code 522
800 N. Quincy Street
Arlington, VA 22217-5000

Dr. Hans F. Croxton
Faculty of Law
University of Leuven
P.O. Box 616
Leuven, Belgium

Ma. Carolyn E. Cronle
Johns Hopkins University
Department of Psychology
Charles & 34th Street
Baltimore, MD 21218

Dr. Timothy Davey
American College Testing Program
P.O. Box 168
Iowa City, IA 52243

Dr. C. M. Dayson
Department of Measurement
Sociology & Evaluation
College of Education
University of Maryland
College Park, MD 20742

Dr. Ralph J. Deak
Measurement, Socior, and Evaluation
Benjamin Bldg., Room 1112
University of Maryland
College Park, MD 20742

Dr. Les Diebold
CSER
University of Illinois
100 South Mead Avenue
Urbana, IL 61801

Dr. Douglas Ding
Center for Naval Analysis
481 Ford Avenue
P.O. Box 168
Alexandria, VA 22312-0358

Mr. Hue-Ho Dong
Bell Communications Research
Room 3-122
P.O. Box 1229
Pinehurst, NY 13125-1230

Dr. Finn Drayton
University of Illinois
Department of Psychology
400 E. Daniel St.
Champaign, IL 61820

Dr. Stephen Dunster
224 F. Landau Center
for Measurement
University of Iowa
Iowa City, IA 52242

Dr. James A. Earle
Air Force Human Resources Lab
Brooks AFB, TX 78235

Dr. Susan Eversburg
University of Kansas
Psychological Department
436 Fraker
Lawrence, KS 66045

Dr. George F. Exner, Jr.
Division of Educational Studies
Emory University
115 Franklin Bldg.
Atlanta, GA 30322

Dr. Benjamin A. Fawcett
Operational Technologies Corp.
5800 College Park, Suite 225
San Antonio, TX 78228

Dr. P.A. Fiscus
P.O. Box 24
San Diego, CA 92115-2400

Dr. Leonard Field
Landau Center
for Measurement
University of Iowa
Iowa City, IA 52242

Dr. Richard L. Fergason
American College Testing
P.O. Box 168
Iowa City, IA 52243

Dr. Gerhard Fischer
Leopoldiger Str.
1010 Vienna
AUSTRIA

Dr. Myron Fischel
U.S. Army Headquarters
D.A.C.S.OMR
The Pentagon
Washington, DC 20330-0300

Prof. Donald Fitzgerald
University of New England
Department of Psychology
Armidale, New South Wales 2351
AUSTRALIA

Ms. Paul Foley
Naval Personnel R&D Center
San Diego, CA 92132-4699
Dr. Brig. Wiseman
Huntsville
1154 S. Washington
Arlington, VA 22214

Dr. David J. Wiseman
1424 E. Effingham Hall
University of Wisconsin
750 E. Wisconson Road
Madison, WI 53705-0344

Dr. Ronald A. Westerman
Box 146
Coral, CA 92212

Major John Weise
AFHRL/MOAH
Brooks AFB, TX 78235

Dr. Douglas Wessel
Code 31
Navy Personnel R&D Center
San Diego, CA 92152-6800

Dr. Randal B. Weisss
University of Southern California
Department of Psychology
Los Angeles, CA 90089-1061

German Military Representative
ATTN: Wolfgang Widgrube
Stasiabteilung
D-3300 Bonn 2
4530 Bemhleplatz, NW
Washington, DC 20016

Dr. Bruce Wilkinson
Department of Educational Psychology
University of Illinois
Urbana, IL 61801

Dr. Hada Wang
Federal Aviation Administration
800 Independence Ave., SW
Washington, DC 20591

Mr. John H. Wolfe
Navy Personnel R&D Center
San Diego, CA 92152-6800

Dr. George Wong
Bonaventure Laboratory
Memorial Sloan-Kettering Cancer Center
1233 York Avenue
New York, NY 10021

Dr. Wallace Wulfald, III
Navy Personnel R&D Center
Code 31
San Diego, CA 92152-6800

Dr. Kentaro Yamamoto
22-T
Educational Testing Service
Roselle Road
Princeton, NJ 08541

Dr. Wendy Ye
CTB/McGraw Hill
Del Monte Research Park
Monterey, CA 93940

Dr. Joseph L. Young
National Science Foundation
Room 330
400 G Street, N.W.
Washington, DC 20550

Mr. Anthony R. Zare
National Council of State Boards of Nursing, Inc.
425 North Michigan Avenue
Suite 1544
Chicago, IL 60611