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ABSTRACT

One of the strengths of the Pennsylvania Council of Teachers of Mathematics (PCTM) is that it gives mathematicians and mathematics educators the opportunity to exchange and contribute to each other's professional growth. The topic for each yearbook is chosen to coincide with the annual PCTM meeting. This 1988 yearbook contains 27 articles which focus on critical issues regarding the goals of school mathematics and related issues, perspectives, ideas, and strategies of interest to mathematics teachers, teacher educators, mathematics supervisors, and curriculum coordinators as they strive to review and improve existing mathematics programs. Topics include: future concerns; changes in elementary mathematics programs; high school calculus courses; change in response to technological development; expert systems and artificial intelligence; preparation of students for secondary mathematics; implications of change for teacher preparation; mathematical thinking; effective textbook use in the elementary school; cooperative learning in mathematics; how to use infinities; and continuing education in the future. (CW)

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MATHEMATICS FOR THE CLASS OF 2000

1988 Yearbook

PENNSYLVANIA COUNCIL OF TEACHERS OF MATHEMATICS

MATHEMATICS
FOR THE
CLASS OF 2000

1988 Yearbook

Robert F. Nicely, Jr.
The Pennsylvania State University

Thomas F. Sigmund
University of Pittsburgh at Johnstown

Co-Editors

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PREFACE

The 1988 PCTM Yearbook — MATHEMATICS FOR THE CLASS OF 2000 — is the fourth yearbook to be developed and distributed to the membership of the Pennsylvania Council of Teachers of Mathematics. The theme was chosen to be congruent with that chosen for the 37th annual meeting of the organization.

The articles in MATHEMATICS FOR THE CLASS OF 2000 focus on critical issues regarding goals for school mathematics and related issues, perspectives, ideas and strategies that should be of interest to elementary teachers, secondary and college mathematics teachers, teacher educators, mathematics supervisors, and curriculum coordinators as they strive to review and improve existing mathematics programs. The articles were written by teachers and researchers from basic and higher education who responded to a call for manuscripts which was sent to all PCTM members in Spring, 1987.

Considerable thanks go to a number of people for their important contributions to the 1988 Yearbook. The ten manuscript reviewers (listed on page iv) shared their insights about the manuscripts that were submitted for consideration and offered many suggestions which were used in the editing process. Glen Blume and Kathleen Heid made significant contributions to the editorial process as we worked to produce a high-quality yearbook. Suzanne Harpster at Penn State also provided valuable editorial assistance. The authors of the manuscripts deserve considerable credit for taking the initiati e and the time to place their ideas in front of their peers. The commercial and institutional advertisers also deserve thanks for their willingness to invest their money by buying space in the yearbook. Last but not least, the PCTM Executive Board deserves credit for its continuing support of the efforts of the Publications Committee.

The co-editors and associate editors were glad to have had the opportunity to work on the 1988 PCTM Yearbook. We hope that the readers will carefully read the articles and implement the ideas that are relevant for them. We invite response from the readers, authors, and advertisers.

Robert F. Nicely, Jr.
The Pennsylvania State University
1988 Yearbook Co-Editor

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SOME CONCERNS ABOUT THE AMERICAN FUTURE

Charles L. Hosler
The Pennsylvania State University

Americans like to win!

If any one word characterizes the American it might be competitiveness. It permeates almost every facet of our national lives and is often the bane of our international friends and foes alike. The competitive spirit has been largely responsible for the great strength and achievement of this country.

Our political structure was designed to foster competition. The struggle between parties, judicial, legislative, and executive branches, states, individuals, and ideas has not abated in over two hundred years. Our legal system is adversarial; our commerce is market based; our schools are competitive; in sports we divide ourselves by every geographic and institutional delineation conceivable and struggle as if our very lives depended upon it It, therefore, seems to be a great contradiction to infer that in some areas of our economy we are no longer competitive.

Considerable time could be spent listing and debating the areas in which we can and cannot currently compete. The list of reasons or excuses would include tax policies, the value of the dollar, the regulatory environment, encouragement of capital formation, natural resources, cost of labor, management skills, leadership, and the like. Although one could argue about the relative importance of these and other factors inherent to our economy, there seems to be no disagreement about one related group of elements that play a strong role in economic competitiveness. Education, research, and the rapidity of conversion of research to products and processes are not only important now, but promise to become increasingly important in the future. This is true not only in the United States but in the entire world.

We, in the U.S., represent only about five percent of the population of the world, but we are a distillation from all of the world's people who wanted an outlet for their energy, drive and ambition and overcame various barriers to get here. Our people represent a rich resource because of their diversity in cultural background. We also have the most diverse set of educational institutions in the world, and these institutions are relatively accessible and available, across the spectrum of our population. For a century the relatively easy access to our educational establishment has transformed the sons and daughters of downtrodden peasants into leaders of industry and government and members of university faculties and national academies.

Our population, our educational institutions, and the intellectual property they generate are basic resources. If we do nothing else to assure that we are competitive in a global economy, we must invest heavily to ensure the quality and accessibility of our educational establishment and the continuous generation of new knowledge. Our concern must be at every level of education.

Although we have developed a large portion of the intellectual potential of our nation, we will have to do better Our competitors are about to surpass us at what we have, up until now, done best. Adaptation to change and innovation in the work place will proceed in proportion to the level of education of every worker and supervisor involved. One has only to look at, or try to work in, some of the lesser developed countries to realize that Ph.D.'s and sophisticated equipment gets one nowhere if the entire work force is not sophisticated enough to be receptive to new ideas, to maintain and repair the equipment or even to read the instruction manuals. Even in the U S. the ignorance of both the public and policymakers stands in the way of improvements in the quality of life. Some of the most promising areas of science, such as biotechnology, may find their utility in improving the human condition greatly hampered by general scientific illiteracy.

The popular press has been filled with alarms about the quality of our entire educational system. Equally alarming are some projections into the future if one feels that high-quality scientists and engineers are an essential ingredient in remaining competitive, or if one is concerned that only an informed public can make good policy decisions. There are some demographic facts and some cultural characteristics extant in America which raise questions about our future technological potential. Despite our focus on scientists and engineers, one must realize that politics, finance, management, arts and humanities all contribute to political stability, competitiveness, and the quality of life. However, at this moment, science and engineering appear to represent future problems which require immediate attention.

The first essential bit of information is that, as a result of low birth rates in the U.S. population in general, the number of 22-year-old Americans peaked in 1983 and will fall rather steadily between now and the year 2000. This will decrease the crime rate, but it also will reduce by one million per year the pool from which college students are drawn and from which young scientists and engineers are recruited. In addition to concern over the drop in total numbers is the fact that of the pool that will be available, a larger proportion will be in racial and ethnic groups whose participation rate has been historically low. By the year 2020, 30 percent of the young population will be black or hispanic. The black and hispanic rate of participation in obtaining B.S. degrees in science and engineering is 14 per thousand as opposed to 56 per thousand for whites and Asians. The fastest growing pool of youths has the lowest participation rate in college and the highest dropout rate from high school, and of those who finish high school, this same group is least likely to study mathematics and science. The latter two facts mean that the pool of young people that represents a growing proportion of the shrinking total pool of 22-year-olds is rendering itself ineligible to provide its share of the science and engineering students in college. This ineligibility stems largely from lack of mathematics. These same individuals will be hampered from playing a constructive role in decision making in a technologically oriented world.

One could assume that the market will clear and that, as shortages occur, attractive salaries will cause more people to choose science and engineering careers and to study mathematics in high school to prepare themselves for those careers. To some extent this may occur, but changes in the rates of choosing science and engineering historically have shown fluctuations only on the order of 10 percent, whereas to simply supply engineers and scientists at the 1984 rate would require a 40 percent increase in the rate of entry in view of the decline in the number of 22-year-olds. Since a growing proportion of the population will be traditional nonparticipants, the probability that any historical increase can be quadrupled is highly unlikely. To stay even at the 1984 production rate of scientists and engineers will require that we quadruple the rate of participation of blacks and hispanics from 14 to 56 per thousand or double the rate of female participation from 28 to 56 per thousand. If participation rates remain as they are today the demographic decline will result in a cumulative shortfall of approximately 600,000 scientists and engineers by the year 2000. It is hard to conceive of a scenario which will require 600,000 fewer scientists and engineers in the year 2000.

In addition to the anticipated decline in the number of B.S. engineers and scientists, we face a projected demand for doubling of the number of Ph.D. researchers and faculty by the end of the century. Since 1972 the number of American Ph.D.'s in science and engineering has been declining in absolute numbers and in percent of degrees awarded. The percent of science and engineering baccalaureates receiving Ph.D.'s seven years later has declined from 11.5 percent in 1972 to 6 percent in 1984. Declines are evident in both the sciences and in engineering, with engineering showing the greatest decline. Fifty-five percent of all engineering doctorates and from 20 to 60 percent of doctorates in other areas of science and mathematics are awarded to foreign students. Fortunately, more than half of these remain in the U.S., contributing strongly to our research and educational programs. In any event, there will be a severe shortage of Ph.D.'s by 1995, not only in science and engineering but in all fields.

We must promote quality education at all levels of the educational enterprise. We must enforce sufficient discipline on the system to ensure that mathematics, physics, chemistry and biology are taught to every high school student capable of benefiting from them. An educational system in which 14- and 15-year-olds are given options to find an easy way through high school, or worse yet, drop out of school, will not enable the U.S. to maintain its status in the world and could lead to Third World status in a generation. By giving options early in secondary education, we have removed options after high school and rendered most of our population ineligible and unprepared for the careers that will enable them to earn a living and the Nation to survive. It is not just a matter of creating more scientists and engineers, but of creating a citizenry capable of playing an informed and constructive role in a democratic nation that employs science and technology. It matters not whether that role is creative, legislative or administrative, or as a spectator and voter, technological literacy is a sine qua non.

Even if we assure that all citizens have the opportunity to achieve their highest intellectual potential, the enterprises will fall short of survival unless we accelerate our rate of support for research, and see to it that new knowledge is brought to bear in the marketplace.

Industry today employs 600,000 research scientists and engineers. The education and research of these 600,000 valued employees was made possible largely through industy- and government-sponsored research grants and contracts to America's universities.

The rigor of our precollege educational system and the quality and accessibility of American universities are fundamental ingredients in the health of American business and industry and in maintaining the spiritual, material, and political well-being of our citizenry. The quality of science and engineering, in particular, portends to be fundamental to our well-being domestically and our competitiveness internationally. However, there must be some very rapid changes in patterns of basic education of young people in America, if science and engineering are to be options when university study is contemplated, and if general technological literacy is to be improved. This translates into increased participation of all students in training in mathematics. Without substantial shifts of talent into science and engineering today, and this requires high quality and early training in mathematics, the next generation could well find itself in a less developed country.

Given a superior and diverse educational system with high accessibility and a high rate of investment in research, together with strong university-industry-government collaboration, we will be able to innovate, invent, and convert our new knowledge to products and processes at rates that keep our nation secure in its economic strength, and we can assure the security, political freedom and quality of life of our citizens for generations to come. We must all labor together knowing that enlightenment through education and research is the surest path to peace and prosperity for all people.

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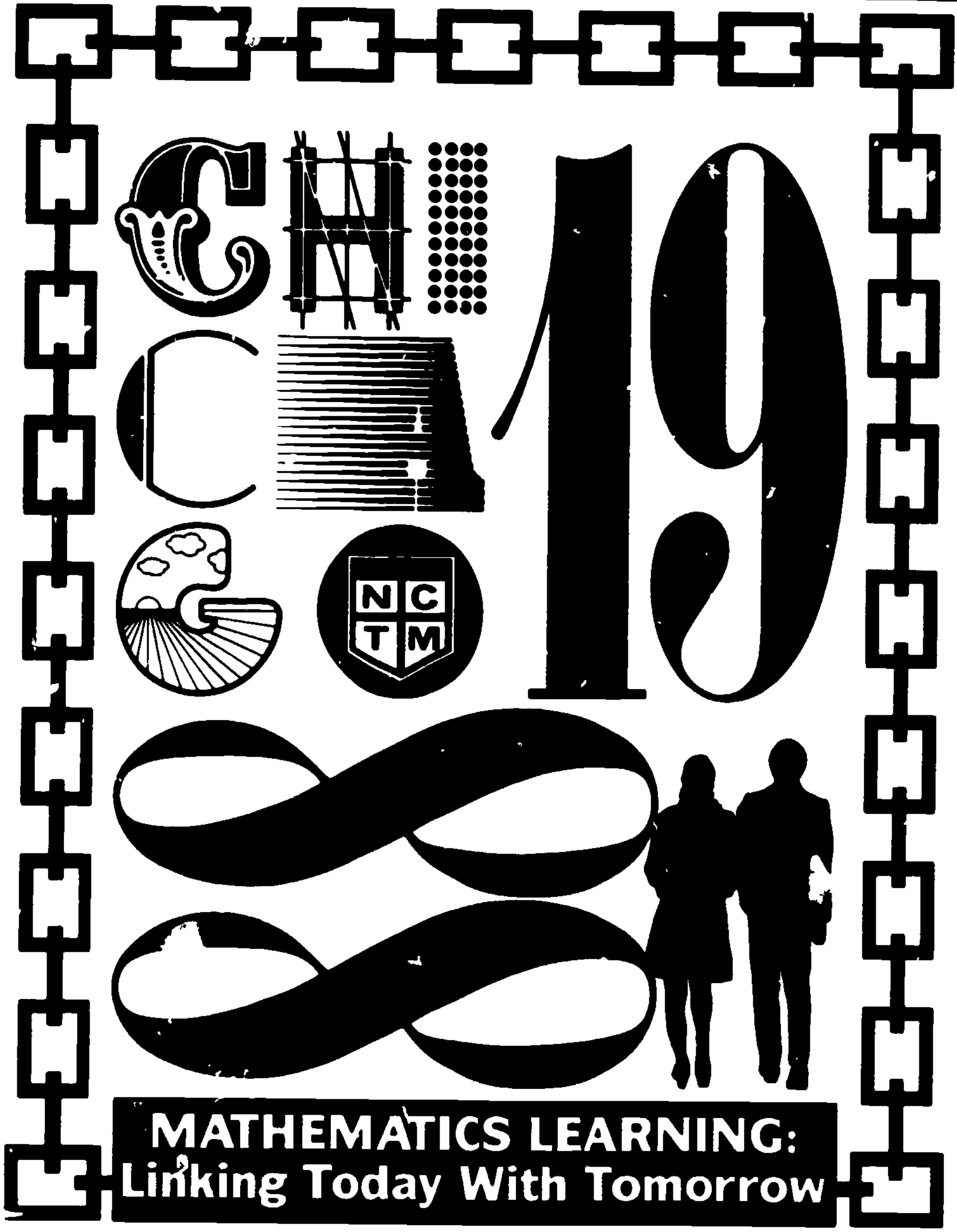
Data sources for this article included the Division of Science Resource Studies of the National Science Foundation, The National Research Council, and the Bureau of the Census.

This article is an adaptation of the address that Charles L. Hosler gave at the graduate commencement at Penn State on August 15, 1987.

ABOUT THE AUTHOR

Charles L. Hosler is Vice President for Research and Dean of the Graduate School at The Pennsylvania State University. His address is 114 Kern Graduate Building, University Park, Pennsylvania 16802



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CHANGE FOR ELEMENTARY SCHOOL MATHEMATICS

David R. Marchand
Clarion University of Pennsylvania — Venango Campus

While society has changed significantly in the past several years, there has been very little change in the elementary school mathematics curriculum since the turn of the century. A declining interest in mathematics, particularly at the elementary school level, has no doubt been caused by what we teach and how we teach it. The constant paper-and-pencil, drill-and-practice approach to teaching mathematics has not been effective in gaining and maintaining student interest as once it might have been. Students need not look far in their own environment to see that skill at pencil-and-paper arithmetic is no longer a valuable skill. At all levels what is important in mathematics has changed. However, the most significant changes have occurred in arithmetic. These changes include the use of technology and the resulting importance of estimation and mental arithmetic.

We must take advantage of the computational tools provided by current calculator-computer technology. Since no evidence exists that appropriate knowledge of arithmetic can be developed better by paper-and-pencil than by use of modern computational devices, calculator-computer technology can be used to make the learning of arithmetic less tedious and more interesting. In the past, those students who could not master computational skills were severely limited when developing problem-solving skills, and had virtually no chance for future success in mathematics. For those who were able to develop computational skills, using arithmetic eventually became a boring and menial task. It is true that possession of computational skills does not guarantee the acquisition of problem-solving skills, but no student should be excluded from studying some areas of the curriculum due to difficulty with others. (It should be noted that calculators and computers will not replace conceptual knowledge or mathematical insight. They only replace rote algorithmic calculations.)

Student use of hand-held calculators removes the computation obstacle from problem solving and allows students to focus on a variety of thinking skills. This enables students to "do mathematics." Students can begin to think of problem solving as a process in which solutions often result from exploring situations, guessing-and-checking, stating and restating questions, and developing and applying strategies over a period of time.

Curriculum revision should be an ongoing process that reflects both present and future needs of society. The fact that pencil-and-paper computation traditionally has been part of the mathematics curriculum is not a sufficient reason to keep it there. Although single-digit number facts are still important, as are number facts involving powers of ten, part of the time previously spent on complicated pencil-and-paper computations should be focused on mental arithmetic, rounding, estimation and approximation. Students should be taught to distinguish situations in which calculators are appropriate aids to computation from those in which mental arithmetic, estimation, or pencil-and-paper computations are more appropriate.

Use of the calculator should be integrated throughout the mathematics curriculum beginning at the kindergarten level. However, the calculator should not be used merely to check answers. The calculator can be used as a device to help students explore and learn mathematics. Models for a calculator-integrated mathematics curriculum are being developed at both the state and national levels. The Pennsylvania Department of Education, with the assistance of PCTM, is working on a model for such a curriculum. The questions regarding how and when to use calculators are being answered. lso, publications that provide calculator activities to help students learn mathematics are becoming available. *How to Teach Mathematics Using A Calculator* (Coburn, 1987), is one such publication available from the National Council of Teachers of Mathematics.

A portion of the time previously spent on the development of paper-and-pencil skills should be devoted to interpretation and representation of data and the statistics which accompany data. The constant bombardment in the press and on TV by graphs, charts, tables and the references to data analysis and statistics makes these topics important to current and future citizens. Informal geometry also can be a more significant part of the elementary school mathematics curriculum if less time is spent on paper-and-pencil computation.

No curricular change can be successful without congruence among the written, taught and tested curricula. A written curriculum based on textbooks that emphasize the use of technology, estimation and mental computation can be successful only if the taught and tested curricula reflect similar emphases. As long as standardized tests forbid the use of calculators, teachers are likely to retain the traditional paper-and-pencil arithmetic curriculum. Finally, any significant curricular change must gain the support of teachers, administrators and the citizens of the community which is served. If such support is available, the elementary mathematics curriculum can be changed to emphasize mathematics that is important for the Class of 2000.

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ABOUT THE AUTHOR

David R Marchand is an Assistant Professor of Mathematics at the Venango Campus of Clarion University of Pennsylvania. He is also currently serving as President of the Pennsylvania Council of Teachers of Mathematics.

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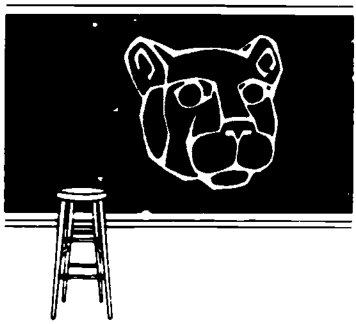

CAREERS IN MATHEMATICS AND SCIENCE EDUCATION FOR EXPERIENCED TEACHERS AND SUPERVISORS

The Division of Curriculum and Instruction at The Pennsylvania State University has instructorships and graduate teaching and research assistantships available for the 1988-89 academic year. These positions provide either part-time employment or a stipend plus tuition. Teachers and supervisors who qualify for a sabbatical leave find these positions to be an ideal way to supplement their income and provide college-level teaching and/or research experience as they pursue graduate work in science and/or mathematics education.

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THE CURRENT STATUS OF CALCULUS COURSES IN PENNSYLVANIA HIGH SCHOOLS

Paul Wilson
Central Bucks High School — East

Imagine an educational system that identifies its brightest math students before they enter high school, plans a mathematics program so that these students will take calculus in their final year of high school, and then encourages students to begin their study of mathematics at the college level with a repeat of calculus. The problem with such a system seems obvious; why would math educators want a system where the best and brightest math students take the same course twice? As undesirable as this may seem it appears to be the system that we have in the state of Pennsylvania.

Background

A single-variable calculus course is now well-established in the twelfth grade at many secondary schools throughout the country and in the state of Pennsylvania. The number of students taking the AP calculus examination nationwide (a small portion of those taking high school calculus) grew by almost 150 percent between 1978 and 1984 (CEEB, 1974; CEEB, 1984). In Pennsylvania 380 of 501 school districts offered a calculus course to 12,997 students in 1984 (Pennsylvania Department of Education, 1986). While this number is small relative to all high school seniors, it represents many of the mathematically talented students.

The Mathematical Association of America (MAA) and the National Council of Teachers of Mathematics (NCTM) have identified three types of high school calculus courses, two which seem to have an "undesirable" impact on students who later take calculus in college.

The first type is a one semester or partial year course that presents the highlights of calculus, including an intuitive look at the main concepts and a few applications and makes no attempt to be a complete course in the subject.

The second type of course is a year-long course that does not deal in depth with the concepts, covers no proofs or rigorous derivations, and emphasizes mechanics over understanding.

The third type is a college-level calculus course, with the expectation that a substantial majority of the students taking the course will master the material and will not then repeat the subject in college. The Advanced Placement program offers a nationally accepted syllabus and a nationwide mechanism for obtaining advanced placement. The MAA and

NCTM strongly recommend that high schools adopt this curriculum for their calculus courses (MAA 1986).

The problems associated with these types of calculus courses have been documented in studies by O'Dell (1983), McKillip (1965), and Neatrour and Mullenex (1973).

O'Dell (1983) compared college achievement in Calculus I of students with no high school calculus to the achievement of those who have studied calculus at the high school level. The study found no significant difference between those who studied calculus for two semesters in high school and those who had no high school calculus. Students with no high school calculus out-performed those students who had taken a semester calculus course in high school. McKillip (1965) found students having one semester of calculus in high school did not earn better grades in the first semester of college calculus than students who had no high school calculus. Neatrour and Mullenex (1973) found that in the state of Virginia 56 percent of the high school calculus students took the course over in college, despite the fact that from the student's point of view the high school course compared favorably with the college course. These studies found no significant difference in achievement between students who had studied calculus at the high school level and students who had no high school calculus experience with respect to their performance in the Calculus I course taken at the college level.

The drawbacks to the first two types of calculus courses would include: lack of adequate preparation for the rigors of college mathematics, inadequate preparation to pass the Advanced Placement examination or a college administered proficiency exam (MAA, 1986), and encouragement of inappropriate attitudes and inadequate motivation in high school students. Students learn to view their twelfth-grade calculus course as an introduction to calculus with the expectation of repeating the material in college.

Recent Research

During the fall of 1986, a survey was constructed to determine how and toward what purpose calculus was being taught to the high school students in the state of Pennsylvania. A pilot study was designed and survey forms were mailed to the calculus teachers in a large suburban county-wide school system in Maryland. Subsequently, the survey was revised based on the recommendations of the 18 (of 31) calculus teachers who returned the survey. The revised survey was mailed to all public school districts in Pennsylvania that reported that they taught calculus during the 1984-85 school year. Of the 380 surveys that were distributed, 121 (32 percent) were returned.

The survey categories and purpose of each category were as follows:

1. To look for differences in the topics which are taught in high school

calculus courses — The course outline section of the survey contained a list of 86 subtopics that were grouped under nine topic headings; properties of functions, circular functions, limits, exponential and logarithmic functions, matrices, derivatives, integration, sequences and series, and elementary differential equations. Teacher responses consisted of information about the amount of time devoted to each topic, opportunity to learn particular subtopics, and the sources for presentations, ideas and problems.

2. To look for differences in the conditions under which calculus is taught — This includes data on classroom conditions and the intent of students to seek advanced placement.
3. To look for differences in teaching materials used in high school calculus — Questions were directed at finding out primary and secondary resources used by teachers to teach the topics and subtopics which comprise the calculus curriculum.
4. To look for differences in backgrounds of calculus teachers — Questions were asked about teachers' educational background, years of experience, years of calculus teaching experience and membership in professional organizations.
5. To look for differences and similarities in the advanced placement syllabus and the high school calculus courses taught in Pennsylvania — Does there seem to be a difference between the content of the high school calculus course and the college calculus course?

The examination of the data generated by this survey has revealed the following picture. The Pennsylvania high school calculus courses that were reviewed are taught by experienced teachers who have been teaching for an average of 20 years and have taught calculus for 11 of those years. Students in the Pennsylvania high school calculus classes that were surveyed do not spend as much time in calculus class as do their United States counterparts. The Pennsylvania high school classes surveyed reported spending 135 hours (180 days x 45 min. per class, on the average); the United States calculus classes surveyed in the SIMS students spend on the average 150 hours (Crosswhite, et al , 1986, p. 175).

Pennsylvania high school calculus teachers perceive the textbook as the primary source for determining the calculus curriculum. Within the parameters established by the textbook, teachers exercise considerable judgment regarding the subtopics that are covered within a particular topic and the time spent on those subtopics. This judgment, exercised by teachers, creates many different calculus curricula and courses among the respondents to the survey.

Many of the subtopics which are a part of the calculus syllabus, teachers report, are taught in courses prerequisite to calculus. This is especially true in the topics of properties of functions and circular functions. To a lesser extent this is also true in the topics of limits and exponential and logarithmic functions.

Whether a particular topic is being taught for the first time in calculus or is being retaught in calculus does not have an effect on the number of class periods devoted to teaching that topic. Teachers are spending the same number of periods reteaching these topics in calculus as the teachers who are teaching these topics in calculus for the first time. Teachers report spending 20 percent of the calculus class periods reteaching material taught in a course prerequisite to calculus.

Eighty-four percent of the teachers indicated that they believed that their students were taking calculus in high school so that they would be better prepared to take it in college. This teacher perception is not supported by student perceptions that are reported in other studies (O'Dell, 1983; McKillip, 1965; Neatrour & Mullenex, 1973).

A difference was found between courses offered to classes where teachers believed the primary reason their students were taking calculus was to seek advanced placement in college and classes where students were planning on retaking calculus in college. Calculus courses where students were seeking advanced placement credit taught students a greater number of calculus subtopics, especially in the areas of circular functions, derivatives and integration.

The implication seems to be that Pennsylvania high school calculus students and teachers feel the purpose of high school calculus is to prepare for "college" calculus. The result in many cases is a course which leaves out many subtopics that would be a part of a college calculus syllabus and spends a significant amount of time reteaching material that is taught in a course prerequisite to calculus. The data from the survey also indicate that many teachers are spending more periods teaching the algorithmic subtopics in calculus and smaller numbers of teachers are teaching the subtopics that deal with concepts, derivations and proof. Moreover, those teachers are not spending as many periods teaching those conceptual subtopics.

This would seem to be in conflict with the 1987 MAA and NCTM assertions that high schools are doing their students a disservice by offering them a year-long course that does not deal in depth with concepts, covers no proofs or rigorous derivations, and emphasizes mechanics over understanding.

Implications

It seems then that many Pennsylvania high schools are offering students a calculus course that does not measure up to the standards of a college-level calculus course and are subsequently not preparing their students to seek advanced placement at the college level. Thus, a situation is created where academically capable students must repeat a year of mathematics because their high school courses have not provided them with the necessary background to continue to the next level of mathematics.

Pennsylvania high schools need to examine the goals of the calculus courses they are offering to their mathematically able students. If advanced placement is the goal, then the course content and coverage should be broad enough and in enough depth to prepare students to achieve this goal. If advanced placement is not the goal, then schools would be wise to establish a meaningful course for these mathematically able students that will not have to be repeated.

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ABOUT THE AUTHOR

Paul Wilson is a mathematics teacher in Central Bucks High School — East His address is Holicong Road, Box 504, Buckingham, Pennsylvania 18912.

Technology — A Force for Change

Terry W. Baylor
Milton Hershey School

The changes in computing technology that take place between now and the year 2000 will dramatically affect many facets of mathematics. The trends of computer miniaturization and per-calculation cost reduction will continue to provide more computing power to more people. As a result of this increased computing power, the smaller and less expensive computers will be programmed with software that is more flexible and easier to use. Sophisticated computer networks will allow a variety of users to access vast amounts of information on other powerful computers. The rate of change in computer technology is not going to slow down. (See the October 1987 issue of *Scientific American.)*

The effects of these changes continue to command our attention in mathematics education, and educators must be prepared to assume leadership roles in addresing a number of important questions. How do we help our students deal with the overwhelming amount of new information and techniques? How can we do more problem solving with less time spent calculating or manipulating equations? What can we do now to help our students as well as ourselves to adapt to changes in the ways mathematics is taught and learned?

In order to make progress in adapting to change, three facets of mathematics education will need special attention: using computing technology in the classroom, unifying concepts in mathematics, and improving skills in problem solving. The first topic to be considered, computing technology, is so widespread that its impact should reflect not only on *what* is taught, but also on *how* it is taught. Until now, mathematics teaching has focused mostly on computational skills. However, with classroom use of calculators and computers, calculating can just become a smaller part of the overall mathematics program. Less time will then be spent teaching pencil-and-paper calculation techniques, allowing more time to teach problem solving.

The style of teaching and the methods of solving problems should take full advantage of the computer's ability to process information. Since the computer quickly and accurately shows numerical relationships, the student can readily discover and easily examine relationships that would otherwise be elusive. Higher-level concepts would be available to more students because they would not be hampered by routine calculations. For example, using the computer to graph equations would allow the student to explore the concepts of graphing as well as the relationships between graphs.

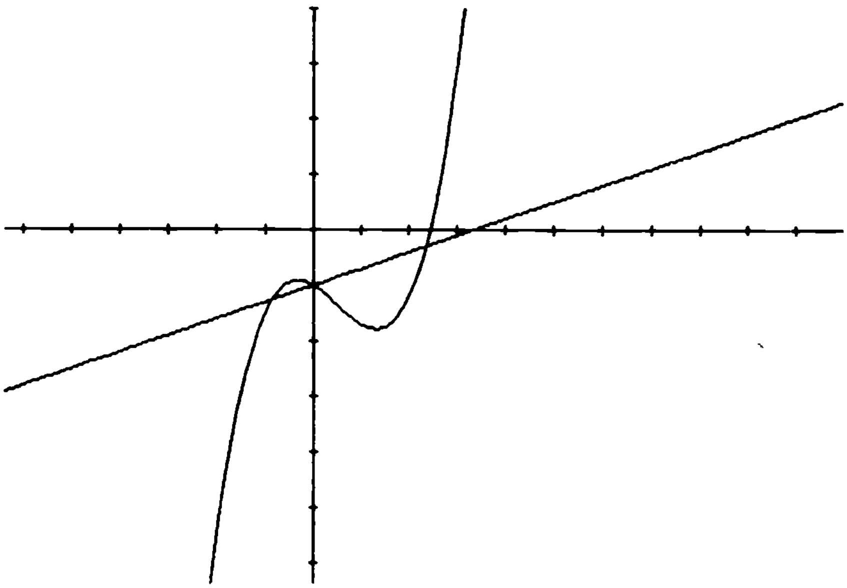

Figure 1
COMPUTER-GENERATED GRAPH

The computer-generated example (Figure 1) shows the relationship between the graphs of $Y = X^3 - 3X^2 - 5X - 20$ and $Y = 3X - 20$. (One tick mark on the X axis is two units, one tick mark on the Y axis is 20 units.) The time and effort saved by using the computer give the student and the teacher more range and flexibility in managing the learning environment.

Unifying concepts is the second facet of mathematics education that must be considered. A true focus on problem solving should rarely be an exercise in using just one concept. By selecting activities that have a wide scope, the teacher can cause the students to bring together concepts from a variety of areas of mathematics. For example, the cost analysis of purchasing and operating an automobile is a comprehensive problem that involves concepts of defining a problem, gathering data, selecting data analysis strategies, doing arithmetic, using descriptive statistics, verifying results, and reporting results. Using a computer for graphic displays makes data easier to organize and interpret. Spreadsheet programs allow data to be aligned, interconnected and graphed. Here concepts of statistics, algebra and geometry intermingle, adding to the student's power to solve problems.

	A	B	C	D	E
1	Initial Expenses for Purchasing a Car				
2					
3		New	New	Used	Used
4		Car A	Car B	Car C	Car D
5	Base Cost	$16,255	$5,999	$4,500	$1,500
6	Accessories	$1,747	$1,750		
7	Dealer Prep	$100	$100		
8					
9					
10	Tax	$1,086	$471	$270	$90
11	Tags	$24	$24	$24	$24
12	Insurance				
13	1 Year	$1,600	$1,200	$900	$800
14					
15	Total	$20,812	$9,544	$5,694	$2,414

Figure 2
SPREADSHEET COMPARING INITIAL CAR COSTS

In this example of a spreadsheet, values in rows 10 and 15 are calculated using the numbers from the appropriate cells in other rows. The method of calculation is determined by the formula created by the person filling in the spreadsheet.

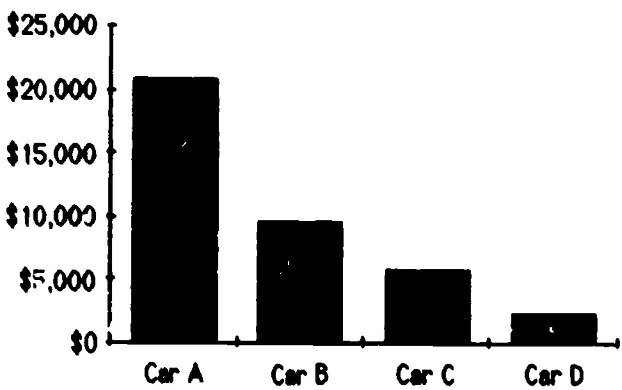

Figure 3
GRAPH OF INITIAL CAR COSTS

The graph in Figure 3 was automatically created by the spreadsheet program.

There are now more tools available to solve problems than ever before.

In order to use these tools effectively, students will need to learn to be flexible in selecting mathematical strategies. Learning to be flexible also requires more emphasis on the relationships between concepts.

Problem solving is the third facet of mathematics education that provides great opportunity for teaching skills that students will need for the future. It should be taught in ways that take full advantage of computing technology. Students should be free to experiment, test and refine various computer-based problem-solving techniques such as successive approximation or using computer-generated graphs to find solutions. Since the world is becoming more complex, skills in understanding complex problems must be developed. To accomplish this, more comprehensive problem solving should be used where the students are involved in *all* of the steps from problem formulation to final evaluation of results. Problems rich in content and containing several levels of complexity can be used to add sophistication to the process. Group problem solving can also serve to prepare students to work with others. At least one way to accomplish this goal would be to form teams of three or four students to solve problems comprehensive enough to sustain a group effort. The teams would be taught how to work together in using group planning and dividing labor. One example of this kind of project would be to create and conduct a student poll. The results of opinion polls or election polls could be presented to the school. Another kind of project would be to have teams of students take measurements and compute areas and volumes of objects in and around the school. For example, students could find how many square yards of canvas it would take to cover the baseball backstop. Indirect measurement can be used to find distances to remote objects. Groups of students could use a transit and tape to triangulate the distance to a distant tower. (See Figure 4.)

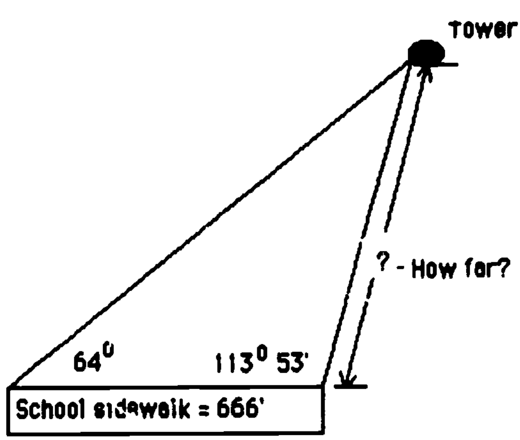

Figure 4
TRIANGULATION PROBLEM

Two or three group problem-solving projects per year would go a long way in teaching the students how to solve problems as a group. To add even more variety, the students of the group would use a mix of solution formats such as graphic display or verbal presentation. Doing this would allow members of the group to develop further presentation and communication skills.

These ideas are not only for the future, but can help to promote positive change today. The first thing that can be done is to prepare students and teachers for dealing with change. Because adjustment to change requires flexibility, a more flexible approach to teaching mathematics is in order. Small adjustments that move in the appropriate direction would demonstrate to everyone change is on the way. Some immediate changes would be: (1) allowing students to do more word processing, graphics and spreadsheet work on computers, (2) spending more instructional time on concept development rather than on symbol manipulation, or (3) adapting to ever-changing computer software and hardware. An additional way to prepare students for change would be to place more instructional emphasis on understanding, generalizing, and adapting to change. Delivery is inseparable from content so teachers must model the kind of learning it takes to adapt to change.

Lessons from changes in the past should temper what is to be done about the future, and it follows that:

> Teachers need . . . opportunities to develop their understanding and their ability to apply their knowledge to new situations as students do and such development does not occur in a one-shot, two-hour workshop on a single topic. Rather, well-planned, extended programs are needed in which teachers have the opportunity to see new techniques demonstrated in classrooms, to try out new methods with their own students, and to reflect upon the changes in the curriculum. (CSPDI, 1985)

Implementation of well-tested unconventional programs faces the difficulty of breaking into the "grid-lock" relationship of curriculum, texts, and tests. Teacher training for innovative programs must provide more mathematics and greater variety of teaching techniques. Adequate time for the transition of programs is necessary so effectiveness is not lost due to lack of time to implement a program.

Although the welfare of the student is the central concern of what is done to prepare for the future, the focus of change is on the teacher because nothing much is going to happen without the full involvement of the classroom teacher. Professional incentives and rewards, as well as more teacher-directed program control, would rejuvenate teacher interest and participation in programs for change. Career ladders that require much professional involvement focusing on change are needed to foster the development of master teachers who are leaders. However, teachers cannot design and implement all of the changes alone. Administrators, school board members, students, parents, and community members will have to work together to provide the best possible resources.

Technology is just one factor that drives the force of change. Mathematics teachers must dramatically alter the content and process of teaching in order to prepare students for the year 2000. In the information age, the change is from merely acquiring information to managing information effectively in order to solve problems.

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ABOUT THE AUTHOR

Terry W. Baylor is a mathematics teacher at the Milton Hershey School, Hershey, Pennsylvania 17033 During the 1987-88 academic year, he is on sabbatical leave to study at The Pennsylvania State University

MATHEMATICS IN THE YEAR 2000: ARTIFICIAL INTELLIGENCE AND EXPERT SYSTEMS

Karen Doyle Walton
Allentown College of St. Francis de Sales

Introduction

In searching for an answer to the question "What mathematics education will be appropriate in the year 2000?", one must first consider what will follow the computer age of the 1980's, which has focused on numerical calculations. *Business Week* has credited artificial intelligence (AI) with creating not only a new computer revolution, but the *important* one. Edward Feigenbaum describes *artificial intelligence* as "the part of computer science concerned with designing intelligent computer systems" which is "emerging from the laboratory and is beginning to take its place in human affairs." He asserts that "knowledge is power" and that "knowledge itself is about to become the new wealth of nations" (Mishkoff, 1985).

In international competitiveness, the stakes are high. Japan's announced Fifth Generation Project is designed to overtake America in the "knowledge industry." Feigenbaum states that "knowledge will be a salable commodity like food and oil" and that America's response to Japan's ambitious goals in the way of knowledge research and development will determine "our role in the post-industrial world" (Mishkoff, 1985).

What is Artificial Intelligence?

Artificial intelligence has been defined in many ways ranging from Elaine Rich's "Artificial intelligence is the study of how to make computers do things at which, at this moment, people are better" to Barr and Feigenbaum's "Artificial intelligence is the part of computer science concerned with designing intelligent computer systems, that is, systems that exhibit the characteristics we associate with intelligence in human behavior." Regardless of the definition, the question "What is intelligence?" must be addressed (Mishkoff, 1985). Douglas Hofstadter (1980) in *Gödel, Escher, Bach: An Eternal Golden Braid* lists the following "essential abilities for intelligence":

- To respond to situations very flexibly
- To make sense out of ambiguous or contradictory messages
- To recognize the relative importance of different elements of a situation

- To find similarities between situations despite differences which may separate them
- To draw distinction between situations despite similarities which may link them.

Bruce Buchanan's definition in the *Encyclopedia Britannica* (1985) identifies "heuristics" as a key element of artificial intelligence:

> Artificial intelligence is the branch of computer science that deals with ways of representing knowledge using symbols rather than numbers and with rules-of-thumb, or heuristic, methods for processing information.

Various other definitions of artificial intelligence focus on other aspects such as intelligent behavior, symbolic (as opposed to numerical) processing, heuristics, pattern matching, and nonalgorithmic procedures.

Artificial Intelligence Research

The Artificial Intelligence Revolution was launched at Dartmouth College in 1956 by a diverse group of scientists from such disciplines as mathematics, neurology, psychology, and electrical engineering, who were joined by the common thread of using the computer to conduct their research. Funded by a $7500 grant from the Rockefelier Foundation, the Dartmouth Conference explored the conjecture "that every aspect of learning or any other feature of intelligence can, in principle, be so precisely described that a machine can be made to simulate it."

Areas of artificial intelligence research to date include:

Natural language processing — enabling people and computers to communicate in a "natural" (human) language

Speech recognition — allowing computers to understand human speech

Computer vision — enabling the computer to receive, interpret, and understand visual images

Robotics — programming electro-mechanical devices to perform manual tasks

Intelligent computer-assisted instruction (ICAI) — computerized "tutors" that adapt teaching techniques to fit the individual student's learning patterns

Automatic programming — programming a computer system that can develop programs by itself which meet the specifications of the user

Planning and decision support — computer construction of formal and detailed plans for realizing a complex goal

Expert systems — designing a computer program to act as an expert in a particular area of expertise (Mishkoff, 1985).

What is an Expert System?

Robert Lewand defines an expert system to be "a computer tool whose purpose is to simulate a human expert in a specific field or domain."

Because human experts are frequently in great demand and short supply, expert systems are developed to assist human experts or provide information if a human expert is not accessible. Although expert systems vary greatly, most share the following principal components:

Knowledge base — consisting of both declarative knowledge (facts about objects, events, and situations) and procedural knowledge (e.g., "if-then" production rules)

Inference engine — which determines which rules to apply to the problem at hand, executes the rules, and determines whether an acceptable solution has been found

User interface — the component of the expert system which communicates bidirectionally with the user (Mishkoff, 1985).

Expert systems have been applied successfully in many areas, among them being agriculture, chemistry, computer systems, education, electronics, engineering, geology, information management, law, manufacturing, mathematics, medicine, meteorology, military science, physics, and space technology.

Thinking Tools for Education

The most common application of expert systems to education has been tutoring systems using artificial intelligence principles to guide CAI. However, new artificial intelligence products now available are aimed at broader uses of expert systems in education. Scholastic, Inc., has developed a program for students entitled *Artificial Intelligence.* A game-playing machine allows the student to invent two-player board games on the computer by choosing rules. The student tells the program only when someone has won the game, but does not divulge the rules of the game. As consecutive games are played, the computer program "learns" the rules and eventually beats the student at his or her own game. *The Intelligent Catalog* published by The Library Corporation helps students who are unfamiliar with research techniques by asking questions such as "What do you want to find?". University of Utah researchers have developed *CLASS LD* which diagnoses learning disabilities in ways similar to medical diagnosis systems (McGinty, 1987).

Expert Systems for Mathematics

Mathematicians, scientists, and engineers use a symbol manipulation program, MACSYMA, in solving complex mathematical problems in differential and integral calculus and for simplifying symbolic expressions. This large, interactive computer system designed originally in 1968 at MIT has undergone continual development and is used throughout the United States by researchers in government laboratories, universities, and private corporations. Although MACSYMA consists of more than 300,000 lines of code in the language Lisp and runs on a Digital Equipment KL-10 computer at MIT, which is accessed through a nationwide timesharing network, *TK! Solver*, a small system with some analogous capabilities, is now available for personal computers (Harmon, 1984). A follow-up to the 1984 introduction of *TK! Solver*, "the first equation solver," is *TK Solver Plus*, which is more friendly and powerful than its predecessor. *TK Solver Plus*, written in the language C, solves individual equations and systems of equations and has built-in functions ranging from the mathematical constants π and e to trigonometric, logarithmic, hyperbolic, Boolean, and complex functions. It also deals with complex numbers and draws graphs, including lines, curves, and surfaces (Galbaldon, 1987).

MathCAD, described as "an engineer's scratch pad," is an excellent tool for engineers, architects, students, and scientists who manipulate formulas. Written for the IBM PC, PC-XT, PC-AT and compatibles, MathCAD handles differential and integral calculus and repetitive solutions. It does not replace an understanding of mathematics or develop formulas to solve problems. Instead it performs number-crunching with grace and speed after a formula has been stated (Green, 1987).

Implications for Mathematics Teachers in the Year 2000

Just as mathematics teachers have been enlisted to teach computer science, it is probable that their focus on problem solving will make them the appropriate instructors for expert systems. Those like MACSYMA and *TK Solver Plus* are tools available "off the shelf" to do mathematics. Expert-system building tools will enable mathematics teachers, as well as teachers from other disciplines, to develop computer applications which enhance students' problem-solving abilities (see Siegel, 1987 and Lewand, 1987). But healthy skepticism is appropriate. Adams and Hamm (1987) report that:

> The best artificial intelligence expert systems give only a mechanistic rule-governed simulation of what people have at the lowest stage of skill development. Anything approaching higher levels of human thought is still not on the technological horizon.

Conclusion

The year 2000 is over a decade away. Just one short decade ago, the personal computer was a rarity rather than a present-day requisite in the mathematics departments of our schools. Any attempt to predict how or what mathematics will be taught in the year 2000 will undoubtedly fall short of the mark. However, one constant remains — the teaching of mathematics is both an art and a science. Although technology will contribute greatly to the latter aspect of mathematics teaching, it is the former, more difficult aspect at which human teachers will continue to excel.

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ABOUT THE AUTHOR

Karen Doyle Walton is the Academic Dean at Allentown College of St. Francis de Sales, Center Valley, Pennsylvania 18034.

National Council of Teachers of Mathematics

Proclamation

Whereas, mathematical literacy is essential for citizens to function effectively in society, and,

Whereas, mathematics is used every day—both in the home and in the workplace; and,

Whereas, the language and processes of mathematics are basic to all other disciplines, and,

Whereas, our expanding technologically based society demands increased awareness and competence in mathematics, and,

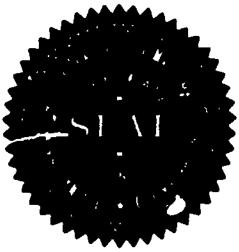
Whereas, school curricula in mathematics provide the foundation for meeting the above needs,

Now, therefore, I, John A Dossey, President of the National Council of Teachers of Mathematics, do hereby proclaim the month of April 1988 as

Mathematics Education Month

To be observed in schools and communities in recognizing the increased importance of mathematics in our lives.

In witness thereof, I have hereunto set my hand and caused the corporate seal of the National Council of Teachers of Mathematics to be affixed on this 1st day of February 1988



John A Dossey

President

PREPARING STUDENTS FOR NEW CURRICULA IN SECONDARY MATHEMATICS

Glendon W. Blume
The Pennsylvania State University

Will emphases in the discipline of mathematics be different in the year 2000 from what they are today? Will students learn different mathematics in secondary schools? Will mathematics be taught in different ways? These and other questions arise when one attempts to predict how teachers should prepare students to learn and use mathematics in the future. This article will examine some of these questions and suggest ways in which teachers can at present address some aspects of the mathematics curriculum of the future.

Does the secondary mathematics curriculum need to change?

Based on changes such as the development of fractal geometry and the emphasis on algorithmic processes in mathematics that have taken place in recent years (Steen, 1986), it is nearly certain that, by the year 2000, new areas in the discipline of mathematics will emerge and emphases in some areas of mathematics will have changed substantially from what they are in 1988. Also, the increasing role that quantification and empirical approaches play in subjects other than mathematics (e.g., the study of demographics and the use of data bases in the social sciences, computer-aided design in industrial arts, and statistical analyses in the physical sciences) suggests that current applications of mathematics will continue to expand and new applications will emerge. Students will need new curricula to be prepared to use mathematics in fundamentally different ways from those of the past, working with today's applications as well as the yet unspecified applications of the future.

Recent national attention to educational reform has generated efforts to redefine and redirect the school mathematics curriculum. Both the Mathematical Sciences Education Board and the National Council of Teachers of Mathematics Commission on Standards (Commission on Standards for School Mathematics, 1987) have drafted new standards for the school mathematics curriculum. These standards specify the nature of mathematical content in various grade levels and provide guidelines for the development of new materials, teaching approaches, and evaluation methods.

What changes are being suggested for the secondary mathematics curriculum?

Major changes in emphases in the revised mathematics curricula center around an emphasis on conceptual knowledge rather than computational proficiency. Mathematics no longer can be viewed as a collection of procedures and techniques, what Steen (1987) refers to as "mimicry mathematics," but as a body of information, information to be communicated to others, reasoned about, and applied to other subject areas. Conceptual knowledge in mathematics must be constructed by students. This suggests that students engage in inquiry; appropriate student activities for a given topic should lead students to examine, conjecture, prove, and analyze or extend the knowledge acquired through this process.

Calls for revision of the mathematics curriculum often have been based on perceived shortcomings in students' capabilities. In contrast, a prime force behind current curricular revisions is the availability of tools that can facilitate students' construction of mathematical ideas and thereby fundamentally change the way students learn and do mathematics. In the past, curriculum developers and teachers have not had access to tools that facilitated such change. However, technology now can continue to provide such tools, both numerical/arithmetic ones such as standard hand calculators, and symbolic/algebraic ones such as symbol-processing calculators and computer software such as muMath (Heid, 1983). The hardware and software tools that can most facilitate inquiry and students' construction of mathematical ideas are those that provide an environment which allows students to make decisions and select the means by which exploration, representation, and data gathering takes place.

What can teachers do now to prepare students for the changes that are being suggested?

If curricula are revised to reflect current recommendations, students will need increased experience with problem solving, reasoning, and communicating mathematics (describing, representing and symbolizing ideas and relationships), and a balance between conceptual knowledge and knowledge of the procedures and techniques that currently dominate the curriculum. Students also will need to select and use a variety of existing tools and be able to adapt to new tools as they become available. Teachers can encourage this by engaging students in the use of existing tools that promote inquiry, exploration of ideas, and the development of conjectures and proofs.

The tools available to students who are learning mathematics will change. Just as many of us could not have anticipated in 1976 the widespread availability of microcomputers and inexpensive calculators in 1988, we cannot now anticipate all of the technological changes that will provide new tools to students by the year 2000. It is possible that advances in the field of artificial intelligence will lead to the development of powerful new tools such as a "word problem solver," a universal "relationship analyzer" that would, for example, generate for students the relationships among various specified algebraic quantities or geometric figures, a geometric "visualizer," and the like. Even though we may not envision the exact nature of these tools, we can prepare students to be capable of using them by providing access to currently available tools such as calculators, function graphing programs, the Geometric Supposers (Yerushalmy & Houde, 1986), spreadsheets, and programming languages such as Logo.

Figure 1 presents a spreadsheet template that could be used to introduce junior high students to the concept of percent. If such a tool were made available to students, they could begin by examining the output from several examples chosen by the teacher. They could then enter their own values of N and generate conjectures about the relationships in the data. Some of these might be ones such as "50% of N will be one-half of N," "200% of N is twice N," "70% of N is 7 times as large as 10% of N," "95% of N is slightly less than N," and the like. Students also could use this spreadsheet template as a tool to explore solutions to questions such as "70% of what number is 91?" Such generation of data and conjectures could also occur with spreadsheet templates for topics such as sequences and series or exponential functions and logarithms in advanced algebra classes.

N	PERCENT OF N						
	10	50	70	95	100	150	200
60	6	30	42	57	60	90	120
5	.5	2.5	3.5	4.75	5	7.5	10
ADDITIONAL STUDENT ENTRIES	(DATA FROM STUDENT ENTRIES)						

Figure 1.
SPREADSHEET TEMPLATE FOR PERCENT

Uses of tools in other fields can provide mathematics teachers with insight into appropriate uses of tools in the learning of mathematics. A business educator's goal may be to prepare students for future use of a variety of word processors by emphasizing fundamental capabilities of word processors rather than the details of a particular word processor used in the school. Analogously, mathematics teachers can emphasize the ways in which mathematical tools can provide help in investigating mathematical concepts and relationships. Mathematics teaching will need to focus less on the development of particular competencies (such as solving equations of a particular type) and more on the capabilities of tools (that assist one when solving such equations) and the development of students' adaptability to new tools as they become available.

Since inquiry and reasoning activities involve higher-level thinking and require time otherwise allocated to other topics, teachers will need to decide which topics in the current curriculum merit reduced attention (NCTM, 1987). The Commission on Standards (1987) provides a suggested list of such topics. Steen (1987) argues that it also will be necessary to have mathematics assessment tests that measure only higher-order thinking skills Teachers can prepare students for this by including in teacher-made tests more questions that require inquiry, data gathering and the generation and proof of conjectures. When being tested, students should have available — and be required to use — the same tools that are available to them each day in the classroom.

The curricular changes suggested in current efforts to revise the mathematics curriculum do not imply the present curriculum contains little of value. However, teachers who are doing a commendable job teaching the current mathematics curriculum need to question whether they are adequately preparing their students for the future. When teachers begin now to move toward implementing a new curriculum for the future, they must ask, "Will what we are doing now be appropriate for students who will use mathematics in the year 2000?" The answer lies in teachers' ability to begin now to prepare students for the ways that mathematics will be learned and used in the next century.

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ABOUT THE AUTHOR

Glendon W Blume is Associate Professor of Mathematics Education in the Division of Curriculum and Instruction at The Pennsylvania State University His address is 177 Chambers Building, University Park, Pennsylvania 16802.

THE REFORMULATION OF SCHOOL MATHEMATICS AND ITS IMPLICATIONS FOR THE EDUCATION OF MATHEMATICS TEACHERS

M. Kathleen Heid
The Pennsylvania State University

Fundamental change in the school mathematics curriculum is on the horizon. Reports appearing in the past few years have recommended radical changes not only in the content of school mathematics but also in the ways mathematics is taught. If these reports are to have any effect on the current school mathematics curriculum, teachers will need to be actively committed to their implementation. This article will explore some of the major changes being discussed and suggest implications for the education of teachers of mathematics — present and future.

Precursor Reports

As early as 1982, national reports suggested the necessity of major revisions in the school mathematics curriculum as well as in the training of teachers of mathematics. The authors of a report to the National Science Commission on Precollege Education in Mathematics, Science, and Technology entitled "The Mathematical Sciences Curriculum K-12: What is still fundamental and what is not" (NSB, 1982) made the following recommendations for elementary and middle school mathematics:

> Calculators and computers: "We recommend that calculators and computers be introduced into the mathematics classroom at the earliest grade practicable. Calculators and computers should be utilized to enhance the understanding of arithmetic and geometry as well as the learning of problem-solving."
>
> Acquisition of skills: "We recommend that substantially more emphasis be placed on the development of skills in mental arithmetic, estimation, and approximation and that substantially less be placed on paper and pencil execution of the arithmetic operations."
>
> Mathematical modelling: "We recommend that direct experience with the collection and analysis of data be provided for in the curriculum to insure that every student becomes familiar with these important processes."

Their recommendations for the secondary school curriculum included:

> Streamlining the curriculum so that topics of new fundamental

importance (including discrete mathematics, statistics, and probability) could receive adequate emphasis.

Reformulating "the content, emphases, and approaches of courses in algebra, geometry, precalculus, and trigonometry" to reflect newly available computing technology.

In *Computing and Mathematics* (1984) Fey, et al, focused specific attention on the impact of computing technology on the school mathematics curriculum. They pointed out a variety of new directions for algebra, geometry, and calculus, more appropriate for computer-rich mathematics classrooms. Among the potential changes suggested for algebra were: a diminished role for manipulative skills; an inversion of the algebra curriculum (with the processes of formulating and interpreting quantitative expressions receiving the initial attention usually reserved for refinement of manipulative skills); and new attention to proportional reasoning, approximate computation, algorithm analysis, matrices, finite methods, and functions. The authors further contended that, with available computing technology, geometry classrooms might become fertile ground for student exploration, generation, and proof of mathematical theorems (instead of the present focus on applying stated theorems to carefully selected exercises).

Current Recommendations

During the past two years, three of the most influential mathematics education organizations (The National Council of Teachers of Mathematics, The Mathematical Sciences Education Board, and the Mathematical Association of America) have established special committees and charged them with creating plans, guidelines and frameworks for the school mathematics curriculum. Reports from two of those committees (NCTM's Commission on Standards for School Mathematics and MSEB's Curriculum Framework Task Force) are now in working draft form, and the third (NCTM/MAA Joint Task Force on Curriculum: Grades 11 13) has just issued its final report. What is most striking about these reports is their singular recognition of the need for fundamental change in the school mathematics curriculum.

Echoing a previous NCTM report (NCTM, 1984), the working draft of NCTM's Curriculum and Evaluation Standards for School Mathematics advocates that calculators be available to all students at all times, that a computer be available in every classroom for instructional purposes, and that computers be available to every student for individual and group work (NCTM, 1987, pp. 4-5).

The technology itself, however, is not enough to ensure the needed reform. Major attention needs to be paid to the new roles demanded of teachers and students alike in the mathematics classrooms of the future.

An important new perspective offered in the Standards draft focuses attention on the desired actions by students and, consequently, on the methods of instruction which would foster these actions. Classrooms envisioned in the Standards document would be "places where interesting problems are explored using important mathematical ideas. . . . in various classrooms one could expect to see students recording measurements of real objects, collecting information and describing their properties using statistics, and exploring the properties of a function by examining its graph." (p. 1) The actions to be expected of students are well-described by the Standards draft's choice of verbs: examine, explore, represent, transform, prove, apply, solve problems, communicate, formulate, conjecture, verify, construct, appreciate, model real-world phenomena, interpret, investigate, translate, analyze, and so on. Notably minimized in the suggestions of the Standards committee is today's rivetted classroom focus on computational skills.

The Standards draft points out a need to re-examine not only the content of school mathematics but also its methods of instruction:

> To provide all students an opportunity to learn the mathematics they will need, the emphases and topics of the present curriculum should be altered. More importantly, methods of instruction need to emphasize exploring, investigating, reasoning, and communicating on the part of all students. In particular, teachers should view their role as guiding and helping students to develop their mathematical knowledge and power. (p. 1)

These new teacher and student roles are largely unexplored in today's classrooms.

New Teacher and Student Roles in Algebra: An Example

Among the clearest impetuses for change is the rather universal call to reformulate the school mathematics curriculum in light of the power and availability of computing devices. The implementation of this recommendation alone has major implications for teacher and student roles in the mathematics classroom. The present Algebra I and Algebra II curricula, for example, center on the acquisition of by-hand facility with the transformations of algebraic expressions. At present there are both computer programs (like Microsoft's muMath and microcomputer versions of Maple) and calculators (like Hewlett Packard's HP 28C) which perform many of these symbolic manipulations.

> With a command as simple as "SOLVE(324 X^2 − 705 X − 700 = = 0, X)," muMath will swiftly respond with (35/12 and −20/27), the truth values to the equation,
>
> $$324x^2 - 705x - 700 = 0$$

MuMath not only solves linear, quadratic, cubic, and quartic equations but also solves systems of linear equations, simplifies algebraic and trigonometric expressions, performs matrix operations, produces exact symbolic derivatives and integrals, and performs infinite precision arithmetic in any of thirty-six bases.

Among the many capabilities of the HP 28C is a facility in performing step-by-step algebraic transformations in accordance with well-known field properties.

A "commutative" key (<- ->) performs the following transformations:

Before	After
(A + B)	(B + A)
(A * B)	(B * A)
(A − B)	(−(B) − A)
(A / B)	(INV(B) * A)

A "distribute" key (D −>) performs reasonable transformations on multiplication, exponentiation, logarithms and antilogarithms:

Before	After
(A * (B + C))	((A * B) + (A * C))
(A ^ (B + C))	((A ^ B) * (A ^ C))
LOG (A / B)	LOG (A) − LOG (B)
ALOG (A + B)	(ALOG (A) * ALOG (B))

In addition to its algebraic transformation capabilities, the HP 28C: calculates derivatives, indefinite integrals, and definite integrals; works with real numbers, complex numbers, vectors, and matrices; plots expressions and statistical data; and calculates statistics and probabilities.

An impressive array of other computing tools put in the reach of every secondary mathematics student the facility for automated production of graphs and function tables as well as immediate numerical res for just about every reasonable procedure.

Given schoolroom access to such computing technology, algebra teachers of the future will need to find ways to enhance student understanding of symbolic form without engaging them in extended forays with by-hand transformations of those forms. In such classrooms, teachers as well as students will need to become comfortable with mathematics courses whose raison d'etre is not refinement of manual symbol manipulation skills.

Implications for the Education of Teachers of Mathematics — Present and Future

Several important themes emerge from both the NCTM Standards draft and the work of MSEB's curriculum frameworks taskforce. Among the more important themes are ones of: problem solving and mathematics modelling, communication, reasoning, new approaches to existing content, and new content areas. The discussion which follows describes the thrust of each of these areas (as put forth in the documents of both committees), concomitant changes in mathematics teaching, and needed related changes in the preparation of school mathematics teachers.

Problem solving and mathematical modelling

Students need to learn to use problem solving techniques, to formulate real-world problems, to verify and interpret results in new problem settings, to create and use mathematical models of the real world, and to become comfortable in using mathematics meaningfully.

The mode of operation in the mathematics classroom of the future needs to be the exploration of mathematical ideas. Instead of units and lessons revolving around the refinement of a well-defined list of manual skills (with casual and after-the-fact attention to applications), they might center on problem situations for which neither the solution process nor the related skills have yet been taught. Teachers and students could spend time analyzing the problem, planning solutions, and interpreting results. Computing tools could be used to execute the routine procedures.

It will no longer suffice for teacher education candidates to march through a compulsory list of upper division mathematics courses without ever creating and solving an original problem. Teachers of mathematics must themselves become problem solvers and problem creators. In addition, they must be reflective problem solvers — as aware of the process as of the solution. (Being a good problem solver is a necessary but not sufficient condition for teaching others to become good problem solvers.) They must learn to teach all their students (not just the brightest) to become insightful problem solvers. College mathematics courses must emphasize problem solving, even at the expense of coverage of a larger array of topics. College mathematics teachers and mathematics educators must learn to teach problem solving rather than merely to present problem solutions and theorems in finished form. Teacher preparation programs must focus on problem solving and might include courses in mathematical modelling and the teaching and learning of problem solving.

Communication

Students must learn not only how to perform mathematical procedures but also how to communicate mathematical ideas through a variety of representations. They should became adept at working within and between graphical, verbal, concrete, and symbolic representations. They should not be satisfied, for example, with being able to solve systems of equations through mastering a particular algorithm if they cannot represent that solution process graphically or concretely or create an adequate verbal representation for the system and its solution.

More generally, students must learn to read, write and talk about mathematics. Mathematics classrooms should provide students a forum for communicating mathematical ideas and arguments through writing as well as through talking. Students at all levels should learn to read about mathematics and to question and apply what they have read.

Today's mathematics classes pay little attention to developing student facility in working between and among representations — the far more popular stance is to work on mastery of skills within a single representation. Students get little experience in talking about, writing about, or reading about mathematics. In many mathematics classrooms, for example, class "discussions" consist of teacher-guided demonstrations with student input confined to the provision of "fill-in" answers to a carefully chosen set of knowledge-level or comprehension-level questions. In a large number of mathematics classrooms, students are not required to present their written work in complete sentences or to justify their reasoning in well-constructed paragraphs. Far too frequently, the implicit rule in mathematics classrooms is that reading assignments are not meant to be understood. Teachers abide by this rule by presenting the content of the reading assignment the next day in class. Students express their knowledge of the rule by not doing the reading.

Mathematics teachers can no longer shy away from conducting classroom discussions in which original student input plays an essential role, from requiring that students communicate about mathematics in writing, and from expecting that students read about mathematics and understand what they are reading. Teachers must learn to encourage students to initiate talk about mathematics, to write and read about mathematics. They must learn to create appropriate forums for students to share mathematical ideas, discoveries, and arguments.

If teachers are to encourage communication, teacher preparation programs must provide them with the appropriate tools and models. College mathematics classrooms must become places where student input (oral and written) is not only encouraged and expected but also refined. Mathematics education classes must teach prospective teachers to hone their own communication skills and to encourage open communication about mathematics in their own classrooms. Teachers should emerge from their preparation programs fully confident of their ability to use a variety of instructional settings to promote communication about mathematics. Appropriate use of large-group discussions, small-group explorations, and paired learning should all be part of a mathematics teacher's repertoire.

Reasoning

If the students of tomorrow's classrooms are to become able problem solvers, capable not only of producing solutions but also of communicating about those solutions effectively, they must develop refined inductive and deductive reasoning skills. Students must learn to use inductive reasoning to formulate mathematical conjectures, and deductive reasoning to test those conjectures. They must learn to assess the validity of mathematics arguments they encounter in their reading and in their discussions with classmates and teachers. They must be able to monitor their own deductions with classmates and teachers. They must be able to monitor their own deductions through analysis of the logic as well as through search for appropriate counterexamples.

Since reasoning ability develops in qualitatively different stages, the development of that ability in students requires teachers with extensive understanding about the growth of reasoning in adolescents. Tomorrow's strongest mathematics teachers will have considerable expertise in developmental and cognitive learning theory. They will be familiar with neo-Piagetian theories such as Collis' Structure of Learned Outcomes so that they can recognize qualitative differences in students' responses (Is a student capable of making generalizations from concrete instances? Does a student ignore relevant features of concrete examples? Is a student unable to reason from abstract examples?) They will know van Hiele's levels of learning geometry and be able to apply van Hiele's phases of instruction for advancing students along those levels. They will be familiar with current general cognitive theories on memory and on meaningful learning as well as mathematics-specific theory on the learning of mathematical representations and on differences in the acquisition of conceptual and procedural knowledge. They will know what has been learned about students' misconceptions about mathematics, and be able to apply diagnostic and prescriptive techniques for correcting those misconceptions. Teacher preparation programs must provide extensive work in each of these areas. Active classroom teachers must update their knowledge of the theories of learning mathematics.

New emphases and new content

In the mathematics classrooms of the future, familiar topic areas will undergo radical transformations and new topic areas will replace many of the traditional ones.

Transformations of traditional curricula. In algebra, traditional symbol manipulation skills will be deemphasized. Algebraic work will concentrate on formulating and interpreting symbolic, numerical, and graphical representations for problem situations. Functions (from a conceptual and informal point of view) will assume a more prominent role throughout secondary mathematics. Computers and calculators will be used to generate and manipulate symbolic expressions, table of function values, and function graphs. Greater emphasis will be placed on the relationships of functions to real-world situations, the embodiment of functions in tables and graphs, and the graphical and numerical relationships among similar function rules.

Geometry will be integrated throughout all grade levels. Special attention will be paid to computer-based explorations of two-dimensional and three-dimensional figures and to the creation and use of geometric models for problem situations. Some of the traditional emphasis on geometry as an axiomatic system will be replaced with more informal and applied reasoning about the properties of spatial figures. Coordinate and transformational geometry will play a central and integrated role in the study of geometry.

As with algebra and geometry, trigonometry will focus on the analysis of periodic real-world applications. Capacities of computers and scientific calculators for graphing and function evaluation will reduce (or eliminate) the importance of by-hand solution of trigonometric equations and inequalities and of table-reading skills. A decreased emphasis on verifying trigonometric identities will allow time for studying the relationships between different representations of trigonometric phenomena: circular functions, polar coordinate representations, and series representations.

New curricula. Among the new topic areas for the secondary mathematics curriculum of the future are statistics, probability, and discrete mathematics.

The study of statistics and probability should be based on real data, fairly informal and integrated throughout the K-12 curriculum. Elementary school students can acquire initial statistical experiences by collecting and organizing data and can be introduced to probability with activities involving flipping coins, spinning dials, and tossing dice. To extend their experience with statistics, middle school students can construct tables, charts, and graphs of data they have collected, and then draw inferences based on their interpretation and analysis of the data. Their exposure to probability can be enhanced through live or computer simulations of complex situations. High school students can round out their statistical experience by designing their own experiments using data analysis tools such as curve-fitters, measures of central tendency and variability, and correlation. Beginning with the middle school curriculum in probability and statistics, students can learn the importance of appropriate sampling techniques and the unique characters of statistical reasoning and probabilistic reasoning.

Implications of new curricula for mathematics teaching. The implications of these new approaches and new content for mathematics teaching are myriad. First, teachers will need to acquaint (or reacquaint) themselves with the new content. Teacher preparation programs must provide for the appropriate mathematics courses. School mathematics approaches to probability, statistics, and discrete mathematics, however, need to be tailored to the clientele. Teachers and curriculum writers will need to construct materials which actively engage students at appropriate cognitive levels. Mathematics teacher preparation programs should give prospective teachers experience in the creation of such materials.

Conclusion

From every corner of the mathematics education community there are strident calls for a significant reformulation of school mathematics. The changes being suggested cannot be implemented merely with the issuance of new textbooks. In some cases, teachers will need to learn new content. But more significantly, they will need to reconceptualize their own views of the proper content of school mathematics and to master a new array of teaching roles and techniques. Now, more than ever, mathematics teachers need to learn more not only about mathematics but also about how mathematics is learned.

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ABOUT THE AUTHOR

M. Kathleen Heid is Assistant Professor of Mathematics Education in the Division of Curriculum and Instruction at The Pennsylvania State University Her address is 171 Chambers Building, University Park, Pennsylvania 16802

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THINKING IN MATHEMATICS: WHAT MAKES IT DIFFERENT?

Marlin E. Hartman
Indiana University of Pennsylvania

Does mathematics require a different kind of thinking than other disciplines? If the answer is "no," then individuals should be able to reason as well in mathematics as they do in other disciplines. And hence, a student who does well in one discipline should do equally well in mathematics. If the answer is "yes," then individuals who do well in another discipline would not be expected to do equally well in mathematics. If the argument was that simple, we could stop at this point.

For years, we have used as one of the arguments for teaching mathematics that it teaches individuals to think. In the mathematics community, we speak of mathematics as a way of thinking and doing. Mathematical work consists largely of observation, guessing at what might be true, developing a feeling of what ought to be true, formulating and testing hypotheses, seeking analogies, forming mental pictures, and trying out ideas, often without any certainty of success. Unfortunately, the process by which secondary school students often learn mathematics offers little opportunity for them to participate in the aforementioned activities and the thought processes involved.

When learning mathematics, if students miss one concept, it is likely to cause them considerable difficulty in understanding the next. For this reason, mathematics "learning" often rapidly degenerates into the rote learning of responses that are required and approved by the teacher. The student is given the impression that practice, simply doing lots of exercises, is the key to success. And in doing the practice exercises, little attention is given to the development of and practice in a variety of problem solving approaches. Students need to experience exercises that will lead them to the realization that what was gained in solving a certain class of problems can be applied to a much broader class of problems. If they are unable to make this transfer, they will often attack the new problem in a purely random fashion — if they attack it at all.

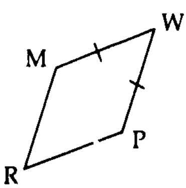
The teacher's task is to provide sufficient effective experiences so that the students can learn the principles upon which a large number of algorithmic forms are based. The teacher must offer students the tools for thinking their way out of difficulty. These tools consist of asking oneself the questions the teacher would normally ask, constructing schemata to assist in organizing data, and eliminating irrelevant information. This is a monumental task — one that requires years of doing. Drill and repetition have no place in this process. In fact, instead of drill and practice, the concept of experience should be advanced because students are not going to master all mathematics concepts on the first encounter. Students must be trained to meet the unexpected.

If we accept this premise, then we can get down to the business of developing the kind of thinking that is essential for success in mathematics. If students are constantly searching for an example to follow or spend a great deal of time trying to recall whether or not they have seen and worked a problem like this before, then they are functioning almost exclusively with recall and recognition. They do this because they have been taught that they need only rely on recall and recognition to successfully complete assigned problems. We must, therefore, provide experiences that permit students to develop an area of the affective domain which is best called a "value complex."

In order to think "mathematically," one must accept the premise on which mathematical systems are built. One must learn to revise opinions when mathematical conjectures which have been demonstrated to be true contradict the ideas held which were based on personal intuition. I am not suggesting that the two domains mentioned function totally independent of each other. In fact, in high school geometry and in all mathematics courses beginning with the upper-level undergraduate courses, the concept of proof requires students to use both the cognitive and the affective domains.

Think about the format of plane geometry proofs. On the left side of the schemata are the statements and on the right side are the reasons or justifications. On the left side, the student is asked to make decisions about what is given, what could come next, what should come next, what must come next, what has been accomplished, and what else must be accomplished. Unlike algebraic manipulation, the step just completed does not necessarily give a clue as to what comes next. Added to this burden is the fact that now each statement must be justified. The student must now jump to his/her file of learned facts; definitions, theorems, corollaries and axioms, and "previously proved" theorems, and recall which justifies the statement he/she just made. Hence, the free association of ideas is interrupted for the purpose of recalling a fact. For example:

GIVEN: $\overline{WM} \cong \overline{WP}$; $\angle M \ncong \angle P$
PROVE: $\overline{MR} \ncong \overline{PR}$



PROOF: Statements	Reasons
1. $\overline{WM} \cong \overline{WP}$	1. Given
2. $\overline{MR} \cong \overline{PR}$	2. Indirect Assumption
3. Let W and R determine $\overline{WR}$	3. Postulate 1
4. $\overline{WR} \cong \overline{WR}$	4. Congruence of segments is reflexive
5. $\triangle WMR \cong \triangle WPR$	5. SSS (Steps 1, 2, 4)
6. $\angle M \cong \angle P$	6. Corres. parts of $\cong$ $\triangle$ s are $\cong$
7. $\angle M \not\cong \angle P$	7. Given
8. $\overline{MR} \not\cong \overline{PR}$	8. Step 6 contradicts Step 7

(This proof appears in the 1987 edition of *Geometry*. The authors are Travers, Dalton and Layton, and the publisher is Laidlaw Brothers.)

I suggest that the process just described is what leads many students to believe that they can't do proofs and hence can't think mathematically. I suggest that if students did the statement side completely — doing all the thinking and free association of ideas before seeking justification for the statements made — they would experience greater success and would have a far better grasp of the thinking process in mathematics.

To support this point, let us examine the background skills that students typically bring to a geometry course. Students usually spend the previous year doing algebra, and algebra does not require justification to any great extent. Perhaps they are asked to justify steps in the algebraic solution of a problem as part of the class discussion but rarely, if ever, on a written examination. The following problem illustrates this point:

$$\frac{2}{x+1} - \frac{4x}{(x+1)(x-1)} = \frac{3}{x-1}$$
$$2(x-1) - 4x = 3(x+1)$$
$$2x - 2 - 4x = 3x + 3$$
$$2x - 4x - 3x = 3 + 2$$
$$(2 - 4 - 3)x = 5$$
$$-5x = 5$$
$$x = -1$$

The problem as presented to the students consists of the first equation. There are a very limited number of options available to the students, but most textbooks and most teachers make a cardinal rule to "clear the fractions." From that point on, what the students must do is dictated to a great extent by what happened in the previous step. This is not the case in geometry! Quite often the steps of a geometric proof can be interchanged and in the longer, more involved proofs, whole subsets of steps can be interchanged.

Let us get back to the algebra problem. How would algebra grades be effected if the students were required to supply reasons for each step in the problem? More important than the grade, how would the progression through algebra and the attendant development of algebraic skills be affected? I'm certain that we would encounter the same difficulties that we now encounter in a geometry course.

One final comment on the algebra example. You have no doubt observed that the "solution" $x = -1$ does not check in the first equation. Thus, the students have experienced an extraneous root without a radical in the equation, see the need to check the solution other than when the result is obvious, and encounter a null equation (which rarely happens for secondary mathematics students until they take trigonometry).

There appear to be two major inhibitors to success in doing mathematics, and both are voiced by the frustrated student in mathematics class. The first is the inability to "see meaning" in mathematical statements. The second is the tendency to "lose the thread" in a mathematical solution or argument.

In the first case, some students feel no association with the mathematical statement because they feel no association with the symbols being used. This happens because the students either cannot or will not "put themselves" into the problem. In short, the students do not visualize doing what is called for in the problem. Related also to the first case is the way we see and categorize things in everyday life. Given an orange, a standard basketball, and an optic-orange golf ball, one person might categorize the objects by their color, while another would categorize them by their shape! In the real world, either categorization would be acceptable; whereas, in mathematics, one could lead to success while the other could lead to frustration. The following problem illustrates the situation in mathematics:

> A train, an hour after starting, has an accident which detains it a half-hour, after which it proceeds at ¾ of its former rate and arrives 3½ hours late. Had the accident happened 90 miles further along the line, it would have arrived only 3 hours late. What was the length of the trip in miles?

I have presented this problem to pre-service and in-service teachers and those who experience difficulty all take the same approach. Since the problem asks for the length of the trip in miles, they attempt to establish equations in terms of distance; whereas, those who are successful realize that the equations must be established in terms of time.

In the second case, the failure to use "parallel progression" can cause one to "lose the thread." For instance, the "adding" and "taking away" processes can be demonstrated with a physical object. Fractionization can also be demonstrated physically. However, when the symbolic language is abstract, there is little opportunity for "parallel progression."

To answer the question that I posed initially, "Yes, mathematics does require a special kind of thinking; and indeed, it is a way of thinking and doing." If mathematics is viewed as a set of rules to be memorized and a few "tricks" which are applied at opportune times, then the probability is high that an individual will join the large segment of society which appears to be perfectly capable and competitive in other ways but who readily concede that they are "no good at mathematics." Unfortunately, this is a ready-made role that our society accepts.

ABOUT THE AUTHOR

Marlin E. Hartman is Professor of Mathematics at Indiana University of Pennsylvania. His address is Mathematics Department, Stright Hall, Indiana University of Pennsylvania, Indiana, Pennsylvania 15705.

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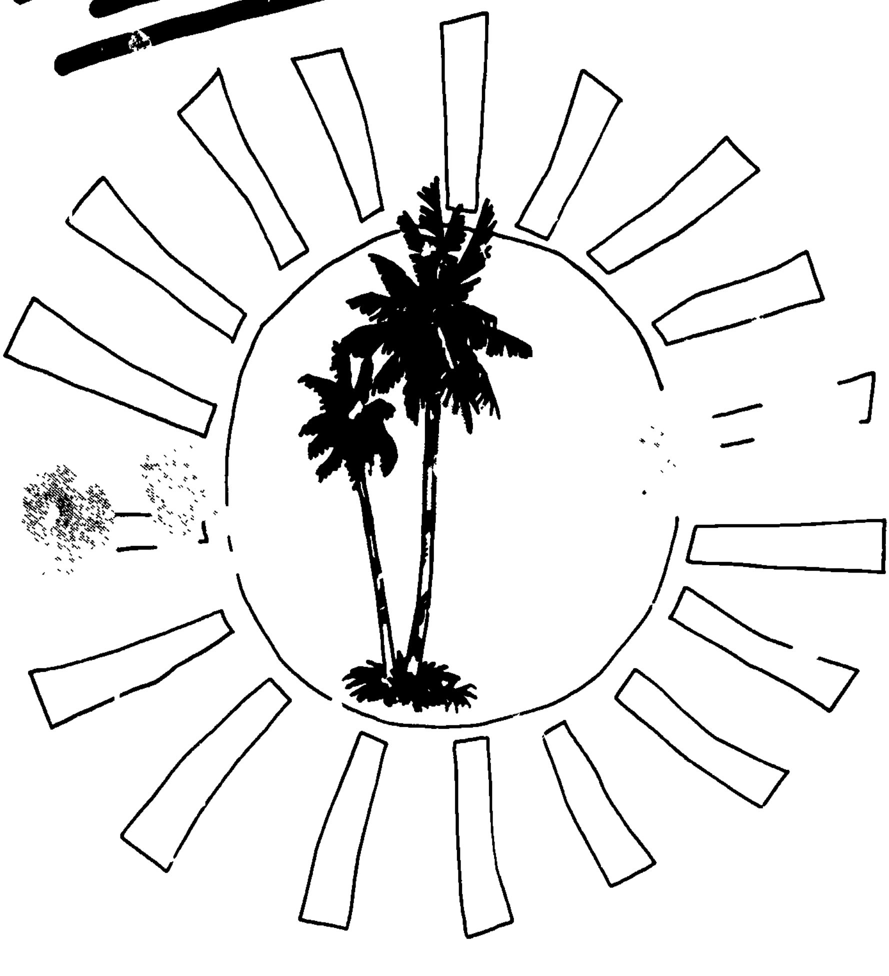

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THE EFFECTIVE USE OF ELEMENTARY MATHEMATICS TEXTBOOKS: A CHALLENGE FOR TEACHERS

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Robert F. Nicely, Jr.
The Pennsylvania State University

Most of the students who enter first grade in September, 1988 will be members of the high school graduating class of 2000 A.D.! *Now* is the time for mathematics educators to focus on the critical curricular and instructional elements of the mathematics program for these first "young adults" of the next century.

The National Council of Teachers of Mathematics, in its 1980 *Agenda for Action* recommended that "problem solving be the focus of school mathematics in the 1980s." This concern has certainly not diminished, and has become even more important for the future. The recent working draft of the "Curriculum and Evaluation Standards for School Mathematics" which was prepared by the Working Groups of the Commission on Standards for School Mathematics of the National Council of Teachers of Mathematics (1987), notes that "problem solving should be the central focus of the mathematics curriculum." *Standard One — Mathematics as Problem Solving* stresses that problem-solving should "permeate the mathematics curriculum so that students can use problem-solving processes in their learning of all mathematical content, use strategies in solving a wide variety of problems for many contexts, discuss alternate solution strategies in relationships among problems, formulate problems, develop and apply a variety of strategies to solve one-step, multistep, and nonroutine problems, verify and interpret results with respect to the original problem situation, and generalize solutions and strategies to new problem situations."

In order to accomplish this ambitious goal, attention must be focused on the mathematics curriculum and associated instructional materials and activities to ensure that they deal effectively with problem solving and other higher-order intellectual behaviors. The textbook continues to play a major role in the mathematics curriculum. It is still the most important factor in determining what mathematics is taught (Johnson & Rising, 1967; Brandt, 1978; Willoughby, 1984). Furthermore, the National Advisory Committee on Mathematics Education (1975) asserted that (1) students read very little of the textual material in their mathematics textbooks and (2) mathematics textbooks are used primarily as a source of problems. Therefore, it is important for teachers, supervisors and other decision-makers involved in the selection and use of textbooks to know the intellectual levels of the problems in textbooks so that they can take the necessary steps to ensure that students will be able to work on problems that are likely to help them become competent problem solvers.

Recent Research Results

During the Fall of 1987, the authors conducted a study to determine what thinking skills fifth grade students would be exposed to as they solved selected problems in the current (1987 and 1988 copyright) textbooks published by five major commercial publishing companies. Using an analytic tool which had been developed by one of the authors (Nicely, 1970), we analyzed all of the decimal problems involving multiplication and division. Each problem was classified by cognitive level and stage of mastery.

The cognitive level list consisted of twenty-seven primary verbs which were identified to describe covert tasks. These verbs were grouped into nine categories and arranged in an ordinal scale. Figure 1 lists the cognitive verbs and their associated levels.

Levels	Verbs
Level 0	No Task; Observe; Read
Level 1	Recall; Recognize; Repeat; Copy (Imitate, Reproduce)
Level 2	Iterate
Level 3	Compare; Substitute
Level 4	Categorize (Classify, Group); Illustrate (Exemplify)
Level 5	Apply; Relate; Convert (Translate); Symbolize; Summarize (Abstract); Describe
Level 6	Justify (Support); Explain (Interpret); Analyze
Level 7	Hypothesize (Theorize); Synthesize (Organize, Structure); Generalize (Induce); Deduce
Level 8	Prove; Solve; Test (Experiment); Design
Level 9	Evaluate

Figure 1
COGNITIVE VERBS AND LEVELS

Stages of mastery consisted of six descriptors beginning with "readiness" and proceeding through "development," "practice," "demonstrate," "overlearning," and concluding with "enrichment." (See Nicely, 1970, for complete definitions of these terms.)

When all of the problems had been analyzed, solved and classified, totals and percentages were computed. Extra practice problems in the back of each book were included in the study, as well as all of the problems that were included in reviews and tests. Problems in which decimals were computed with percentage and/or fractions were not included in this study. (For purposes of this research report, the books are identified only as A, B, C, D, and E.)

Book A had 675 decimal problems involving multiplication and division of decimals. Book B had 402 such problems, Book C had 619, Book D had 560, and Book E had 580. In every book, more than fifty percent of the problems fell into the "practice/iterate" category. Book A had 59.7% of its problems in this category. Book B had 55.9%, Book C had 65.3%, Book D had 52.7%, and Book E had 52.4%. (In the "practice/iterate" category, the student typically looks at a given problem and solves it by mechanically repeating the process. An example of "practice/iterate" would be 39 6 x 2.3 = ______.) Such problems require a relatively low level of cognitive involvement.

To simplify our reporting of the data, we totaled the number of problems that were classified as "justify," "hypothesize," "prove" and "evaluate" (the top four levels of the cognitive scale). We found that Book A had only 1.6% of its problems in these top four cognitive levels, Book B had only 2%, Book C had only 1.8%, Book D had only 2.7%, and Book E had only 5%.

These disappointing results are quite consistent with the results of earlier studies. (See Nicely 1980-81; 1985a; 1985b; 1987; Nicely, Bobango and Fiber, 1984; Nicely, Fiber and Bobango, 1986) If these newly copyrighted textbooks are representative of all the textbooks on the market, then students will likely spend a lot of time on rote-type problems and will not have many opportunities to develop and practice higher levels of cognitive thinking.

The Challenge

If we want students to learn how to operate at higher intellectual levels, and if textbooks are the major source of the written curriculum, what can teachers do to quickly, easily, and effectively upgrade the curriculum? Charles and Lester (1982) provide one solution. They suggest that given problem statements "can be modified in several ways to obtain new, related problems. Variations in a basic problem can be formed easily by following a set of five principles." They suggest that teachers can (1) change the problem context/setting, (2) change the numbers, (3) change the number of conditions, (4) reverse given and wanted information, and/or (5) change some combination of context, number, conditions, and given/wanted information. By starting with one basic (and often low-level intellectual) problem, teachers can use these principles to make such problems more intellectually challenging for their students. The following examples illustrate some of these principles. In each example, the initial problem was adapted from one of the problems in the textbooks that we analyzed.

Example 1 is similar to a problem in Book A.

Chicken is 79¢ per pound. What is the cost of 2.6 pounds of chicken?

A teacher can easily change this basic problem into one in which the students must justify their answer.

> A recipe calls for 2.6 pounds of chicken. The corner convenience store sells chicken for 79¢ per pound. The supermarket sells chicken for 79¢ for the first pound and 75¢ for each additional pound. Where will you get the better deal? Why?

A different rewording of the initial question has the students operate at still another higher-order level.

> The supermarket sells chicken for 79¢ for the first pound and 75¢ for each additional pound. If you pay the clerk with $10.00 and receive $3.21 in change, how many pounds of chicken did you buy?

This same problem could be modified even further.

> List five different ways the clerk could give you $3.21 change without using any dollar bills.

Example 2 is similar to a problem in Book C.

> Diane will babysit 18 hours this weekend. She is paid $1.75 per hour. How much money will Diane make this weekend?

This problem can easily be modified to become a more complex problem.

> Diane will babysit 18 hours this weekend. She is paid $1.75 per hour. Diane wants to buy her sister two sweaters for Christmas. Each sweater will cost $14.98. Will Diane be able to buy the sweaters?

Or the problem could be reworded in this way.

> Diane wants to buy a $24.00 sweater for her sister for Christmas. She has two weeks in which to earn the money. Diane babysits every Friday and Saturday evening from 9:00 p.m. until 1:00 a.m. She makes $1.50 the first hour and $1.75 for every hour after that. Will Diane be able to buy the sweater? Why?

The students could be asked to solve the problem for Diane's rates.

> After one year of babysitting every Friday and Saturday evening for three hours each evening, Diane has earned $559.00. Diane is paid more on Saturday evenings than on Friday evenings. Diane receives a lower rate for the first hour. What could Diane's rates be?

Example 3 is similar to a lower-level problem in Book D.

> A driver paid $1.15 per gallon for 13.5 gallons of fuel. What was the total cost?

Since most students have traveled in a car, this problem could be related to experiences the students have had. It could be set up something like this

> A state is 304.5 miles wide at one point. The road crossing the state is a toll highway. A driver averages 23 miles to a gallon and has a full

13.5 gallon tank. Knowing that gas is $1.15 per gallon and the toll is $.06 per mile, the driver begins his trip across the entire state with $50.00. How much money should he have when he arrives?

By including a second driver in another car, the teacher could create even more of a challenge.

A state is 304.5 miles wide at one point. The highway crossing the state has a toll of $.06 per mile. Driver A averages 20 miles to a gallon and has a 14.5 gallon tank. Driver B averages 23 miles to the gallon and has a 13.5 gallon tank. Gas is $1.15 per gallon. Crossing the entire state, each driver begins the trip with a full tank of gas and $50.00 Which driver should arrive with more money?

This same problem could be reworded to make it more intellectually challenging.

A driver paid $1.15 per gallon for 13.5 gallons of gas, which filled his tank to capacity. If he averages 23 miles to the gallon and travels 931.5 miles, how many stops must he make for gas and what will be the total cost?

Students might need or want to draw pictures, work with partners or even role-play these problem situations to arrive at solutions. Techniques which will raise children's cognitive skills to the "justify," "prove," "hypothesize," and "evaluate" levels will be well worth the effort.

Summary

Acknowledging that thinking skills tend to resist precise forms of definitions, Resnick (1987) lists some key features of higher-order thinking. She contends that higher-order thinking (1) is nonalgorithmic, (2) tends to be complex, (3) often yields multiple solutions, (4) involves nuanced judgment and interpretations, (5) involves the application of multiple criteria, (6) often involves uncertainty, (7) involves self-regulation of the thinking process, (8) involves imposing meaning, and (9) is effortful. As teachers strive to help their students develop and practice higher-order thinking skills, they will need to develop expertise in analyzing and classifying their present instructional resources so that they can go beyond them to create meaningful learning environments for their students. Resnick (1987) and Charles and Lester (1982) provide useful frameworks for creating these new learning opportunities from the low-level materials that are commercially available.

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ABOUT THE AUTHORS

Colleen A. Conlon is an elementary school teacher in Altoona Area School District. Her address is Irving Elementary School, 110 Cherry Avenue, Altoona, Pennsylvania 16603. Robert F. Nicely, Jr. is Professor of Education and Assistant Dean of the College of Education at The Pennsylvania State University. His address is 77 Chambers Building University Park, Pennsylvania 16802.

COLLEGIAL (COOPERATIVE/TEAM) LEARNING IN HIGH SCHOOL MATHEMATICS CLASSES

Carl A. Guerriero
Dickinson College

Three centuries ago, Comenius (Cole, 1950) suggested that large classes be broken up into groups of ten or so pupils, each with a leader, to take advantage of the increased pupil interaction possible in smaller groups. The Bell and Lancaster system during the 19th century used bright students as apprentice teachers to instruct small groups of pupils within the larger class (Cole, 1950). Early in the 20th century, researchers began to systematically study grouping or teaming strategies. These practices have evolved to become team learning, peer tutoring, cooperative learning and collegial learning.

During the 1960's and 1970's, educators examined learning styles and their relationship to achievement in a team learning mode through aptitude-treatment interaction studies (Guerriero, 1971; Chronbach and Snow, 1967). The University of California at Berkeley established a Center for Team Learning (Poirier, 1970), the University of Minnesota established a Cooperative Learning Center (Johnson and Johnson, 1987), and Johns Hopkins University supported an extensive cooperative learning project (Slavin, 1987).

Still, with all of this activity, few teachers have heard of team/cooperative/collegial learning. Many have, however, tried their own grouping methods. Why, then, is the strategy not more popular? Could it be a lack of support? Do school administrators understand why the team learning classroom often looks chaotic and is a bit noisy? Do they realize achievement will probably be higher in a cooperative atmosphere? Do they know that students' social skills and self esteem will be improved?

Research has provided tentative answers for the achievement, self esteem, and social skills questions (Slavin, 1987). Students do achieve academically at least as well as they would in conventional classrooms, their self esteem is higher and they get along better with other students as a result of cooperative learning. While much of the research has been done at the elementary school level, there are secondary level and adult (Johnson and Johnson, 1987) studies that support the cooperative learning strategy.

How can one instructional strategy do all of these things? First, as Slaven (1987) notes, there is nowhere for a student to hide in a four- or five-member team. They must participate! Frequently, many students, whether because of a personality trait (Lawrence, 1979), a lack of ability or a worry about one of the "teenager problems," are only passive observers in the classroom. (No doubt some are mentally a thousand miles away. Others, even if attentive, can simultaneously be daydreaming.) Students actively engaged in learning must concentrate on the task, but if they are passive listeners, only a small portion of their mental capability is taxed. (One can read 500-700 words per minute but one can speak at only about 125 words per minute. If the mind can absorb 500-700 words per minute, then, while listening to a speaker, less than twenty percent of that capability is needed, leaving eighty percent for daydreaming.)

Second, students think like students and teachers think like teachers. How many times has a teacher answered a student's question about mathematics only to be greeted with a puzzled look. A different explanation evokes a similar response. Ultimately another student will say "What she means is . . ." and the puzzled look is replaced by an understanding smile. Teachers think like teachers and students think like students.

A third characteristic of cooperative learning is that it provides opportunities for students to express opinions, make mistakes, and contribute to solutions. The most important people in a teenager's life are peers. It is necessary for students to be accepted as equals and not to be embarrassed in front of their cohorts. Team learning provides a small, collegial group, conducive for interactions and contributions without fear of making a mistake. A friendly environment is where even the smallest contribution may be the key to solving a geometric proof, proving a trigonometric identity, or designing a method to measure some inaccessible distance. In team learning, the whole is greater than the sum of the parts — where individually four or five students could not solve the problem — together they can. Sadly, too little opportunity is provided students to recognize this phenomenon.

Finally, a variety of rather well-established components of learning theories can be incorporated into the small group strategy. One specific format, described later in this paper, suggests beginning each day with an advance organizer (Ausubel, 1960). This is not the typical "set" which either tells the class the objective for the day, reviews what was taught yesterday, or notes an interesting relationship between the day's work and the student's experience. Rather it is, as Ausubel describes, a high level presentation which will later subsume (act as a "coat hanger" for) the new material from the lesson making the learning "meaningful." This organizer may be sufficient instruction for the high achievers. However, for the majority of the class, this five or ten minute mini-lesson, will be just enough to make the work for the day "meaningful."

Behavioral theories form the foundation for the contingencies necessary to encourage students in the heterogeneous teams to help one another and to desire to learn themselves. A one-question quiz at the beginning of each class, on the previous day's objectives, is the key. Each student works independently on the quiz but a team score — the sum of the correct answers — is recorded each day. If all five members answer correctly, the score is 5 and the team receives an A. A score of 4 is a B and so on. This information, a 1 or 0 for each individual and the composite team grade, is recorded. This is *not* time consuming and it provides a wealth of diagnostic information to help plan future lessons as well as to individualize when the need arises.

There is more to this contingency. At the end of the unit of work, typically several weeks, a "comprehensive" unit test is administered and graded. Again, each student works alone. If a student's grade on the unit test is greater than the mean team grade (the average of the daily quizzes for the team) then the student is awarded his or her unit test grade. However, if the unit test grade is lower than the mean team grade, then the individual's unit test and the mean team grade are averaged to produce a higher grade. This sounds complicated, but it isn't. Students quickly understand the process and realize that the team grade can only improve their grade — not reduce it. The system of contingencies provides motivation for the better student to teach the less able, and motivation for all students in the team to do well. It also replaces competition with cooperation.

Advocates of Gestalt and humanistic theories of learning suggest that the ultimate goal of education is to free the learner from the teacher — to have students learn how to learn (Bruner, 1962). Team learning does just that. In fact, teachers feel unneeded when teams are busily working on the day's assignment. It is one thing to say we want to free the student from the teacher, but yet another for the teacher to walk around the classroom hoping a student will ask a question. There is a tremendous urge to sit down and help a team in apparent distress, but restraint is in order. More often than not the team will solve its own problem.

Such instructional strategies require planning and organization. The teacher becomes a coordinator and resource person, while students shift from a competitive, individualistic mode to a cooperative and collaborative attitude. These new roles are not easily accommodated. Students will resist the process. They often do not know how to work cooperatively on a learning task. It will take time.

The following steps outline one model for team learning. If used, it should be used completely. As both teacher and students become comfortable, modifications can be made. Initially, however, the heterogeneous groups, the advance organizers, the daily one-question quiz, the unit test and the grading process — all the result of many years of classroom use — should be tried in their entirety.

Step 1. The teacher must identify and select team leaders. These students should be chosen for their mathematics ability and their leadership capability. The number of leaders will depend upon the size of the class, assuming five members per team.

Step 2. Each student in the class should list ten students, in preferential order, with whom they would like to work in a small group situation. (Unknown to the students, only the lists of the team leaders will be actually used.)

Step 3. The teacher will construct the teams using *only* the team leaders' lists, assigning students to teams in cyclic order — one from each team leader's preference list, until the five-member teams are formed.

Step 4. Enter team rosters in the roll book.

Step 5. Each team should be assigned a location in the room.

Step 6. "Team learning" should be explained to the class, including teacher activities, student activities, availability of materials (other texts, answer books, etc.) and the grading system.

The following sections identify the critical teacher and student activities and the grading system.

Teacher Activities

For each lesson in the unit, a brief but concise lecture should be presented to the class prior to the team attacking that lesson. This "mini-lecture" will not be sufficient for most of the students to reach criterion. Further study, practice and discussion will be incorporated within the team meeting.

The usual instructional pattern will be to present the mini-lecture at the beginning of the period. The teams will use the remaining time to study the lesson as a team and to prepare for a one-question quiz the following day. The teacher will usually pick a problem from the exercises in the text but on occasion may design a new problem for the quiz. Results are recorded for each student (1 for correct, 0 for incorrect) and for the team.

Included in the mini-lecture will be information regarding reference materials relevant to the lesson and available in the classroom. Answer books to the text should circulate among the teams. The teacher will be in the uncomfortable position of not being needed and should help only when asked.

Construct a unit test with items similar to those presented in the text for each lesson.

Student Activities

The students will be expected to take notes as necessary from the mini-lecture.

During the team-learning phase the team members may interact between members, use reference materials in the classroom or request help from the teacher. Occasionally the team will further subdivide into teams of two or three students.

Homework will be limited to the amount of study that the student considers to be sufficient preparation for the one-question quiz the next day.

Each student will work alone on the one-question quiz. The nature of the question will allow the teacher to decide upon what constitutes a correct answer. In some cases it may be desirable to require the student to include the work as well as their answer. The student should have his/her name and team number on the answer sheet to facilitate recording procedures.

Each student will work alone on a unit test which is to be administered at the completion of the unit. The test will be composed of questions similar to the exercises presented in the textbook.

The Grading System

A team grade will be calculated for each day on the basis of the one-question quiz. For a five member team, five correct answers, i.e., each member was present and had the correct answer, will represent a grade of A. Four correct answers a B, three a C, two a D and one or fewer correct an F. Since this is the daily quiz, no makeup quiz will be permitted and the absentees will be accorded an incorrect answer. In cases of an extended illness (more than several days), a revision of the grading scale is suggested for that team, e.g., four correct represents an A, three correct a B, etc.

At the end of the unit an average team grade will be computed by simply averaging the daily team grades.

Each student will receive a grade for his/her individual performance on the unit test. The student's overall grade for the unit will be the higher of (1) that student's grade on the unit test or (2) the average of that student's grade on the unit test and the mean quiz grade of the team to which the student was a member. Using this process, the team grade cannot depress the individual student's grade obtained on the final test, only improve it where the team grade was higher than the student's unit grade. Borderline grades may be resolved by observing the student's performance on the daily quizzes.

Collegial (cooperative/team) learning will play an important role in the curriculum of the future. Teachers, however, must find it successful and rewarding. Remember, "cooperation" will probably not come naturally, because schools are traditionally designed for competition. It will take time to implement this strategy, but the rewards will be worth the effort.

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ABOUT THE AUTHOR

Carl A. Guerriero is Assistant Professor of Education at Dickinson College. His address is Dickinson College, Box 28, Carlisie, Pennsylvania 17013.

INFINITIES: HOW CAN WE USE THEM IN THE MATHEMATICS CURRICULUM?

John Selisky
Coudersport Area Junior-Senior High School

Georg Ferdinand Ludwig Phillip Cantor, the supremely gifted and creative Russian-born mathematician, received his doctorate in mathematics from the University of Berlin at the age of 23 in 1868. Two years later he began a teaching career at the University of Halle which lasted 30 years and launched his investigation and publication of the theory of infinite sets and transfinite numbers. This investigation so astonished and disturbed the scientific establishment, that no less an authority than the French mathematician Henri Poincare condemned it as a disease, from which mathematics would ultimately be "cured."

Happily, Cantor's brilliant and original ideas prevailed, and cast a light upon the mathematical landscape which led to a flowering of research and investigation that continues to this day. His famous diagonal proof is today a standard tool for the study of infinite sets. Its genius becomes apparent when, upon its reading and application, one is captivated by its clarity and self-evidence.

Cantor's greatest achievement was to illustrate, and rigorously prove that the concept of infinity is not undifferentiated. That is, not all infinite sets are the same size. As a consequence, infinite sets can be compared to one another. For example, the set of all points on a line and the set of all rational numbers are both infinite. Cantor proved that the first set is larger than the second. This notion, even now, seems so paradoxical that Poincare's distaste for it is almost understandable.

The failure of mathematicians to describe comprehensively the infinite had plagued the science for centuries. In antiquity the famous paradox of Zeno asserted that motion is impossible because of the necessity for an object to pass through an infinite number of points in a finite time. The paradox is exemplified in a story about a race between Achilles and a tortoise who had been given a slight headstart. According to the paradox, Achilles could never catch the tortoise, as the distance between them, after the race was begun, could only be diminished by half, then one fourth, one eighth, and so on ad infinitum. Intuitively, mathematicians knew that Achilles would catch and overtake the tortoise. However they were disturbed by the elegance and clarity of Zeno's argument. It was not until Archimedes' work on infinite series (150 B.C.) that the problem could be understood through mathematics, and not until the development of calculus in the 17th century, that the problem of the relationship between velocity and position was solved

The infinite also had difficult theological connotations. Religious philosophers were not a little dismayed by the possibility that, one day, mortal man would have the capacity to comprehend the infinite. St. Thomas Aquinas regarded the idea as a direct challenge to the unique and absolute infinite nature of God.

The Arithmetic of Infinity

Cantor's voyage to and beyond the infinite began when he was working on a particularly difficult analysis of trigonometric series. Like many of his contemporaries, he was searching for an arithmetic replacement for Euclidean geometry.

Cantor began by defining a finite set as a "collection of separate and definite objects of our intuition or thought." As he was trying to define an arithmetical foundation of mathematics, he intended sets to consist solely of numbers, though the collection could contain other objects as well. He asserted, logically, that one set is equivalent in size to another set if the elements of the first set could be matched, one for one, with the elements of the second set. This definition does not require us to count, or even to be able to count the elements of either set in order to determine their equivalence. For example, consider a container filled with white and black marbles. The simplest way to determine if the whites are equivalent to the blacks is to remove the marbles from the container in pairs. If every marble can be paired with one of a different color, the blacks are equivalent to the whites. If any marbles are left after the pairings are complete, the color of those marbles represents the larger set.

No matter how large a finite set is, a still larger one can be constructed. This larger set is called a power set and is comprised of all the subsets that it is possible to derive from the set before it. Each subset will consist of none, any, or all of the set's elements. The set with no elements is the null set, and the set with all of a set's elements is an improper subset. For example, a set of two elements ⟨A,B⟩ is associated with a four-element power set, comprised of its four subsets: ⟨A⟩, ⟨B⟩, ⟨A,B⟩, and the null set. A three element set ⟨A,B,C⟩ has eight subsets: ⟨A⟩, ⟨B⟩, ⟨C⟩, ⟨A,B⟩, ⟨A,C⟩, ⟨B,C⟩, ⟨A,B,C⟩, and the null set. It can be shown that the number of subsets derivable from a set of X elements is 2^X.

Now that Cantor had established a basis for the generation of finite sets, he carried the argument to infinite sets, using exactly the same logic. That is, the existence of an infinite set implies the existence of another larger set whose cardinality, or size, is represented by 2 raised to the power of infinity.

To describe the number of elements in an infinite set Cantor selected the symbol $\aleph_0$, aleph sub null, the first letter of the Hebrew alphabet, and the subscript zero. By the logic of finite sets, therefore, an $\aleph_0$ set, such as the set of all integers, has exactly $2^{\aleph_0}$ subsets. Just as a three-element set has 8 (2^3) subsets, so also a $\aleph_0$ element set has two raised to the $\aleph_0$ subsets. The fact that $2^{\aleph_0}$ is not a countable number (the size of a set which can be put into a one-to-one correspondence with a set of positive integers) does not diminish the power of the argument. Cantor further described $2^{\aleph_0}$ as equal to another class of infinity, the set $\aleph_1$, or aleph sub one. $\aleph_1$ differs from $\aleph_0$ similar to the way that 2^2, the cardinality of a two-element set, differs from 2^3, the cardinality of a three-element set. $\aleph_1$ is the first transfinite number, Cantor's first step beyond infinity.

As Cantor did more detailed work with transfinite numbers, he made some totally unexpected discoveries, and proved them. The first surprise was that the infinite set of all the fractions is equal in size to the infinite set of all whole numbers. Though it seems as counterintuitive now as it must have to 19th century mathematicians, Cantor proved that the one-to-one correspondence necessary for the pairing of the elements of two sets holds just as well for these two (fractions and whole numbers). Since there are $\aleph_0$ whole numbers, Cantor's proof meant that there are $\aleph_0$ fractions.

We can extend the logic of this argument to even more surprising discoveries. For example, the number of prime numbers is an infinite set with cardinality $\aleph_0$. It is known that there is no largest prime, that is, the number of primes is infinite (Euclid's famous proof). Another oddity about the list of primes is the absence of any noticeable pattern as one ventures further into the realm of large numbers. For example there are nine primes between 9,999,900 and 10,000,000. But among the next 100 integers (10,000,000 to 10,000,100), there are only two. A large tabulation of primes reveals that they finally become less and less frequent as the numbers counted increase. In fact the ratio of the number n divided by the number of primes up to n (π (n)) increases by approximately 2.3. This is the prime number theorem, discovered by Gauss around 1792 (at age 15).

What is remarkable about the primes is that they can be paired one to one with the integers. Both are $\aleph_0$ sets. Again this seems terribly counterintuitive. How can there be the same number of primes as numbers, when the primes diminish so quickly? By rigorous application of Cantorian logic the problem is reduced to a triviality.

Perhaps Cantor's most important discovery was that the set of irrational numbers is an infinite set larger than the infinite set of rational numbers. Cantor also stated this assertion as the number of points on a number line, or continuum, exceeds the number of whole numbers, or repeating and terminating decimals (fractions). Here again, our conventional perspective makes this appear not to be so. How can one squeeze all those irrational points in between the points of rational numbers on a number line which is already infinitely dense? Cantor's proof of this one is even today, a century later, considered one of the most brilliant of all proofs. It is the famous diagonal proof.

Consider the interval between 0 and 1, consisting of real numbers. If such a list could be completed, each positive integer X could then be matched up with a real number r(x) between 0 and 1. Real numbers are denoted by infinite decimals, so the table might contain a list, which looks, in part, like this:

r(1): .1 4 1 5 9 2 6 5 3...
r(2): .3 3 3 3 3 3 3 3 3...
r(3): .7 1 8 2 8 1 8 2 8...
r(4): .4 1 4 2 1 3 5 6 2...
r(5): .5 0 0 0 0 0 0 0 0...

The digits that run down the diagonal are: 1,3,8,2,0. These digits will now be used to create a new special real number, d, which is between 0 and 1, but which cannot be on the current list. To create any d take the diagonal digits in order, and change each one of them to some other digit. Any change will do. Of course there are any number of ways to create d. One could, for example, subtract one from each diagonal digit. In this case our new d will be .02719... Because of the uniqueness of the construction of d, d's first digit is not the same as the first digit of r_1; d's second digit is not the same as the second digit of r_2; d's third digit is not the same as the third digit of r_3... etc. Therefore, d is different from r_1, from r_2, r_3, etc. In other words, d is not on the list. Cantor had, in this manner, shown that the set of real numbers could not be put into a one-to-one correspondence with the set of positive integers. He was also able to show that the set of rational numbers could be put into a one-to-one correspondence with the set of positive integers. The set of real numbers was, unlike the set of rational numbers, not countably infinite.

Comprehending The Incomprehensible

One of the most striking revelations of Cantor' work was to show that the set of all natural numbers can be paired one for one with the set of all even numbers, the set of all fractions, the set of odd numbers, etc. Strangely enough, an infinite set can be put into one-to-one correspondence with (or "is equivalent to") one of its subsets. In fact, it is provable that a set is infinite if, and only if, it is equivalent to one of its proper subsets. This is analogous to stating that the number of grains of sand at the beach is infinite if, and only if, each grain can be paired with each grain of sand in some smaller area on that beach.

This seeming paradox highlights the limitations of our senses in trying to perceive the infinite in terms of our finite environment. What physical objects within our universe are expressible in terms of infinity? Consider the number of stars in the Milky Way, about 10 billion; the number of cells in the body, about 60 trillion; the estimated number of protons in the universe, one followed by 79 zeros By any human standard of measurement these are very large numbers. Yet, by the measure of transfinite numbers, they are scarcely noticeable. If it were possible to write $\aleph_0$ in numerical form, it would consist of one, followed by an infinity of zeros.

Cantor himself was startled by his next discovery. He suggested the problem that it might be possible to correspond a surface, such as a square, with a straight line, so that each point on the surface could be paired with a point on the line. Cantor was certain that it could not be done. In 1877, after working on the problem for three years, he reported that the pairing could be done. Additionally, he discovered that such a pairing could be done between the points in a finite volume, such as a sphere or cube, and the points on a line. Once again our visual senses fail us in trying to gain an accurate perspective on these elusive sets of points. Cantor's proof shows that the number of points in any finite dimensional space is equivalent to the number of points on the line.

The question remains as to whether there are any infinite sets whose cardinality is greater than $\aleph_0$ and less than $\aleph_1$. That the question occurred to Cantor is an understatement. He agonized over it for more than ten years. The conjecture came to be known as the continuum hypothesis, and it has never been proved. In 1938, Kurt Gödel proved that the hypothesis could not be disproved. This was the best the mathematics community could do, until 1963, when Paul J. Cohen showed that the continuum hypothesis could not be proved by the axioms of standard set theory.

But what about representations of infinite sets whose cardinality is beyond $\aleph_1$? Isaac Asimov suggests that, perhaps, this possible endlessness may be the number of different possible curves that can be drawn on a plane. Could this be $\aleph_2$? There is still no proof. As to the alephs beyond $\aleph_2$, no series has been found to correspond to it, nor are there any physical models which suggest a derivation.

Other intriguing mysteries lurk inside the continuum. If the number line is infinitely dense with the rationals, and between each rational there exists an infinity of algebraic irrationals (irrationals which are the roots of algebraic equations), where is there room for the transcendental numbers? That transcendentals, those numbers which cannot be algebraic roots such as π and e, exist is a certainty. But where do they live? Is there another infinity which describes the set of transcendentals which exist in some infinitely minute space between each algebraic irrational and rational? Can one of the alephs, possibly $\aleph_3$, be ascribed to the infinite set of the transcendentals? Perhaps these questions are more interesting than the answers may turn out to be.

Infinity in the Mathematics Curriculum

Students typically begin to deal with formalized, numerical concepts of the infinite when first learning the names of the large numbers beyond the hundred thousand place value. They usually have a fascination with the names of the very large numbers, and many of the more motivated students have already learned the names billion, trillion, and, perhaps quadrillion before leaving elementary school. This suggests a reasonable learning objective for elementary students: teaching the place value names to the trillions place for all, and to, perhaps, the sextillions place for some. This will have to be accompanied by ongoing discussion that, while we can name a huge number of numbers, we run out of names long before we get anywhere near an infinity of any order.

The following list gives names and exponential notation for some of the large numbers:

10^{2}	hundred	10^{12}	trillion	10^{24}	septillion
10^{3}	thousand	10^{15}	quadrillion	10^{27}	octillion
10^{6}	million	10^{18}	quintillion	10^{30}	nonillion
10^{9}	billion	10^{21}	sextillion	10^{33}	decillion

As numbers grow to such sizes, they tend to become more and more meaningless. For example, how large is one million? It's easy to write it, but physically difficult to count it. In fact, it would take one approximately 12 days, counting nonstop, one number per second, to reach one million. A million days is roughly about 2800 years, and a million inches is about 16 miles. One billion seconds is about 32 years, one billion days about 2,800,000 years, and one billion inches about 16,000 miles.

One of the largest named numbers, the googol, is 10^{100}, that is, one followed by 100 zeros. When compared to a really large naturally occurring number, 10^{79}, the estimated number of protons in the universe, it is seen how little sense we can make out of such a number.

Another infinite set lesson is the familiar large container of marbles of two colors. The students are asked to determine which color marble is largest in number, without counting either color. The students quickly get the idea of pairing the marbles, selecting one of each color. The exercise will lead to interesting variations, such as placing the marbles in two separate containers, and using three or more different colors. The principle of one-to-one correspondence is the same, whether selecting two colors or n colors. The first color to be exhausted is smallest in number, the last to be exhausted is largest.

Using different colored marbles is another way children can be introduced to the idea of the generation of subsets from a given set. Begin with the set of two marbles, a black and a white. Present the problem, how many different sets can be created from only these two? Have students suggest subsets that may be acceptable. The students will likely name all the subsets from two colors very quickly. Then proceed to the more difficult problem of three colors, then perhaps four. The derivation of the subsets proceeds as follows (in any order):

ORIGINAL SET	SUBSETS	NUMBER SUBSETS
black, white	black	
	white	
	black, white	
	null	4
black, white, red	black	
	white	
	red	
	black, white	
	black, red	
	white, red	
	black, white, red	
	null	8
black, white, red, blue	black	
	white	
	red	
	blue	
	black, white	
	black, red	
	black, blue	
	white, red	
	white, blue	
	red, blue	
	black, white, red	
	black, white, blue	
	white, red, blue	
	red, blue, black	
	black, white, red, blue	
	null	16

It is not realistic to expect elementary school children to discover the 2^n pattern of the derivation of subsets, mostly because they lack the notational skills to represent the idea. It is however, worthwhile for the teacher to point to the pattern, and to emphasize that it follows for sets of any size.

Geometry gives elementary students many opportunities to explore infinities. Two which have been used successfully are described as follows: Supply the students with compasses and straightedges. Ask them to construct a circle. Next ask them to pick any three points on the circle, and label them points a,b, and c. Then have them connect a, b, and c, forming a random triangle. Ask the students to compare the triangles to see if any are exactly congruent. After the students have done the exercise, ask them how many triangles it is possible to draw inscribed in the circle. Then how many circles is it possible to draw? The lesson can be varied by having the students draw a diameter, and use only one of the semicircles. This can also show them that there is another infinity of triangles which can be inscribed in each semicircle, and the whole circle. Thus the concept of an infinite set having an infinite proper subset is appropriately reinforced, using the most elementary of geometric visual processes.

Another even more simple construction uses the creation of two supplementary angles by a random line drawn to any point q on a straight line xy. As the students are directed to do this exercise, they are likely to copy the teacher's example, even closely matching the angular measure of the two supplementary angles. A suggestion is to have them pick any point z above the line xy, then connect it to point q. The resulting two supplementary angles can be compared in much the same manner as with the inscribed triangles.

What are the physical computational limits of the infinite? In short, no one knows. Recently the computer scientist, Yasumasa Kanada, of the University of Tokyo calculated the value of π to more than 134 million decimal places. The printout generated enough digits to paper a gymnasium. As π is an irrational number, and has no pattern of repeating or terminating decimals, the process simply encrypts the digits, and therefore has little practical value, except as a curiosity. But isn't being a curiosity enough? A mathematician who examined the results of this effort, Dr. Peter Borwein, suggested that the search for more digits in π will not stop. He further conjectured that no one will ever know the 10^{1000} digit of π. Apparently the only way to know that digit would be to calculate all the digits before it. Since the entire universe does not have that many electrons, it does not appear physically possible. Or does it?

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ABOUT THE AUTHOR

John Selisky is a mathematics teacher in Coudersport Junior-Senior High School, Coudersport, Pennsylvania 16915. During the 1987-88 academic year, he is on sabbatical leave from Coudersport and is studying at Columbia University.

THE RETURNING ADULTS — THE COLLEGE STUDENTS IN THE YEAR 2000: HOW CAN WE HELP THEM?

Dorothy S. Mullin
The Pennsylvania State University — McKeesport Campus

Have you noticed a change lately in the age of the students in the undergraduate classrooms? There's been a gradual change from the so called "traditional" student to the adult student. Just as society has been observing the graying of America, we now have the graying of the classroom.

The Office of Planning and Analysis at The Pennsylvania State University reported that one-fifth of the Fall 1985 undergraduate enrollment at Penn State's seventeen Commonwealth Campuses was over twenty-five years of age (Cutright, 1986). This phenomenon is occurring in colleges and universities all over the country and has extended the trend that began about 14 years ago. By 1992, students under 25 years old are projected to comprise just 51 percent of the total enrollments, down from 61 percent in 1982 (Shannon, 1986).

With this influx of new and returning adult students to the classroom, what should colleges and universities be doing to meet their varied needs? Faculty and administrators in institutions of higher education need to address the following questions. What needs and experiences are these adult students bringing to the classroom? What do they expect from the teacher? How does one teach these adults effectively? Specifically, how does one teach mathematics to these adults?

Needs and Experiences Brought to the Classroom

Adults usually know what they want out of college. Yet, many times they return to a structured learning situation with painful memories of earlier schooling. Such memories often trigger feelings of insecurity and fears that they are not equipped to handle the learning tasks (Darkenwald and Merriam, 1982).

Malcolm Knowles is credited with publicizing the term andragogy which has been defined to mean "the art and science of helping adults learn." Knowles (1984) has found through his andragogical model that adults bring to the classroom their experiences, a readiness to learn, a need for relevancy and a need to be self-directing.

Adult Students' Expectations of Teachers

Teachers of adults should serve as "facilitators" rather than as "repository of facts." Teachers who allow the adults a chance to help themselves and others will probably have more satisfaction and better results than teachers who do not use this method (Godbey, 1978).

Adult students prefer instructors who 1) are student-oriented, 2) are organized, but not overly structured, 3) have enthusiasm, 4) know their subject matter, 5) are well-prepared, and 6) are able to stimulate interest in the topic (Apps, 1981).

How Adults Learn

Even though the literature suggests that a collaborative learning environment is the most effective, Conti (1985) found in a study that General Educational Development (GED) students learned more in a teacher-centered environment. (This may be because GED students tend to focus on the immediate tasks involved with passing the GED examination.)

As many as 70 to 80 percent of adults say that they would prefer to learn by some method other than classroom lectures; however, lectures usually rank first or second in overall popularity out of the five to ten methods of teaching that are generally presented in questionnaires (Cross, 1981).

Techniques to Help Adults Learn Mathematics

The literature shows that adults learn well with individualized learning techniques, collaborative methods, and lectures. There should be a direct relationship between the general methods of teaching adults to those methods which can be used to help adults learn mathematics.

Studies done at the University of Georgia and the University of Wisconsin-Green Bay have shown that adult students do not do as well on the SAT mathematics tests and other standardized mathematics tests as do younger students. Lack of practice and mathematics anxiety have been forwarded as explanations of why this happens.

To help students overcome mathematics anxiety, some colleges offer counseling with various psychological techniques, small-enrollment non-credit courses to help students build confidence in their mathematics ability and test/retest methods.

Chang (1985) studied the test/retest method in a college remedial mathematics course at Augusta College, Georgia and concluded that retesting provided students with a chance to re-learn the material and also an opportunity to reduce their test anxiety. Some suggestions to implement the test/retest method as an instructional technique include:

1) Do not overemphasize test grades. Students should be informed that the objective of retesting is to identify their weaknesses and strengths in the learning of mathematics.
2) Do not spend the entire period on new material. Students should be motivated to find answers themselves rather than be shown an instructor's solution.
3) Do not cover the material too fast. More attention should be given to each student's individual needs.

Strategies that can help adults feel less anxious in an introductory mathematics classroom are: a supportive classroom environment with an encouraging teacher, extensive review in the beginning, techniques designed to lower anxiety to a manageable level, using concrete manipulatives to help bridge the gap between concrete operations and abstract ideas, individual and group tutoring sessions, using a variety of teaching techniques, providing frequent feedback and addressing students' attitudes about mathematics (Taylor and Brooks, 1986; MacDonald, 1978; Brundage and MacKeracher, 1980). Many articles have been written about the adult woman who is returning to college. Most of the authors conclude that the aforementioned techniques, which will help women, will also help the returning adult male student.

Faculty at the University of Missouri-Kansas City conducted an experimental study that compared an introductory mathematics class that contained only women students with other classes that were coeducational. The women in the experimental section were more successful than the women in the other sections in several areas: grades earned, higher rate of retention in mathematics, and changed attitudes about mathematics (MacDonald, 1978).

A pre-calculus course at Mills College, a liberal arts college for women near San Francisco, was designed to prepare students with weak mathematical backgrounds, not to fill in all the gaps in their educational background. Blum and Givant (1980) describes the course as follows:

> Our course has two components. Component A deals with functions and emphasizes concepts that will be encountered in calculus; component B consists of basic algebraic material. Each component is taught in a different setting, component A in the regular class with a regular instructor, component B in small workshops taught by other undergraduates. (p. 787)

Enrollment in pre-calculus has tripled and students enrolled in beginning calculus have nearly doubled. Even though the emphasis has been on the education of women, the program has helped increase mathematical competence and confidence. The main features of the program are the carefully designed curriculum, positive teaching and the supportive and encouraging environment.

A study was done at the University of Lowell comparing calculus classes taught in the conventional manner to calculus classes taught using an individualized mode of instruction. These calculus classes were for continuing education students who were "typical" adults with heterogeneous backgrounds. The individualized method was mastery-oriented but teacher-paced rather than student-paced. The weekly three-hour class was sub-divided into three parts. The first hour consisted of a lecture on the topic of the week. The next two hours were used for individual and group help. The quizzes were given with immediate feedback to the students. The conclusion of the study was that the students who had taken the classes taught by the individualized mode of instruction were more successful than the students who had taken the conventional mode of instruction. Taylor (1978) reported that "76 percent of the students who took the individualized mode of instruction were ready to take the next mathematics course; however, only 54 percent of the students who were enrolled in the conventional mode of instruction were ready to take the next mathematics course" (p. 12).

There are still many techniques that can be investigated as to the best way to teach mathematics to adults. What methods would help mathematics-anxious adults change their attitudes and also help them master their understanding of mathematics? Teachers and administrators in colleges and universities need to become familiar with the results of the research that has already been conducted so that they can implement the best procedures and programs available to help the growing numbers of returning adult students succeed.

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ABOUT THE AUTHOR

Dorothy S. Mullin is Instructor in Mathematics at the McKeesport Campus of The Pennsylvania State University. Her address is Penn State-McKeesport Campus, University Drive, McKeesport, Pennsylvania 15132