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ABSTRACT

Considered is a conceptual analog of buggy algorithms and rule-based mathematical development. The investigations consider whether children's efforts to make conceptual sense of new mathematics instruction in terms of their available knowledge may sometimes lead them to make systematic errors. In particular, the possibility is explored that children overgeneralize concepts from a familiar domain of mathematics in order to interpret a new domain. A total of 113 children in the early phases of school instruction on decimal fractions participated: an American sample from grade 5 (n=17); an Israeli sample from grade 6 (n=21); and French samples from grades 4 (n=37) and 5 (n=38). Children were individually interviewed using described tasks. The findings are discussed in some detail. There seem to be fundamental differences in the kinds of conceptual understanding that produce the Whole Number rule and Fraction rule. Different patterns of rule categorization among the three countries indicate that different curriculum sequences produce different patterns of rule invention. Errorful rules appear intrinsic to learning and cannot be avoided in instruction: they can best be regarded as useful diagnostic tools for instructors, to detect the nature of children's understanding of a mathematics topic. (MNS)

Conceptual Bases of Arithmetic Errors: The Case of Decimal Fractions

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In the past few years, there have been several demonstrations of the phenomenon of systematic but incorrect rules accounting for errors in performance. In the domain of mathematics, the best developed work is on "buggy algorithms" in subtraction (Brown & Burton, 1978; Brown & VanLehn, 1982). Brown and VanLehn's theory of the origin of these algorithms suggests that they come from attempts to repair a procedure by relaxing certain constraints. The repair process operates on a representation of subtraction in terms of symbol manipulation, without reference to quantities and their relationships (Resnick, 1982; Resnick & Omanson, 1987). The theory thus suggests that children construct erroneous rules without reference to the conceptual content or the meaning of arithmetic. Similar accounts of systematic errors in terms of strictly symbolic interpretations have been given for decimal fractions (Hiebert & Wearne, 1985) and for elementary algebra (Matz, 1982; Sleeman, 1982). The work of Siegler and Vago (1978) demonstrating that a changing rule structure provides the basis for children's interpretation of ratios and proportions is somewhat more conceptual in content than the work on buggy algorithms. However, Siegler's work has typically examined only the content of the rules themselves. It has not attempted to situate the rules in a more general knowledge system or to ask about the origin of the rules.

In the present work we look at what might be termed a conceptual analog of buggy algorithms and rule-based mathematical development. We investigate whether children's efforts to make *conceptual sense* of new mathematics instruction in terms of their already available knowledge may sometimes lead them to make systematic errors. In particular, we explore the possibility that children over-generalize concepts from a familiar domain of mathematics in order to interpret a new domain.

Our question requires that we select a topic in mathematics learning in which children commonly invent incorrect rules in the course of learning, and then seek to specify the conceptual sources of these incorrect rules The topic should be one in which there is some conflict between earlier learned concepts and the new concepts to be acquired, so that already familiar ideas may interfere with learning the new ones. The topic of decimal fractions meets these criteria and also offers an opportunity to examine the possibility that different ordering of topics in the school curriculum may differentially affect the kinds of errorful rules that children construct. This is possible because in some countries decimal fractions appear in the curriculum well before ordinary fractions are introduced, while in others ordinary fraction instruction precedes decimal instruction by a year or two.

Using a task in which a child is asked to order decimal fraction numbers, Sackur-Grisvard and Leonard (1985) found that in fourth and fifth grade French classes about half of the children tested used a systematic but incorrect rule to decide which number is greater. There were three different incorrect rules, each used when the numbers to be ordered had the same whole number digit (e.g., 3.214 and 3.8).

According to Sackur-Grisvard and Leonard's Rule 1, the number with more decimal places is the larger one; for example, 3.214 is greater than 3.8 because 3.214 has more digits in the decimal part and because 214 as a whole number is larger than 8. Sackur-Grisvard and Leonard suggest that classroom instruction may support Rule 1 by giving students practice mainly in comparing decimals with the same number of digits, in which case treating decimals as whole numbers always works. Sackur-Grisvard and Leonard found that Rule 1 was common; it was used by 40% of their fourth graders and by about 25% of their fifth graders.

Sackur-Grisvard and Leonard's Rule 2 specifies that the number with *fewer* decimal places is the larger. Thus, given the pair 1.35 and 1.2, Rule 2 chooses 1.2 as greater. Rule 2 was the least common in their sample, occurring in less than 6% of their cases. Grossman (1983), however, has suggested that a similar rule (choosing the longest decimal number as the smallest) was commonly applied by large numbers of entering U.S. college students.

Sackur-Grisvard and Leonard's Rule 3 gives a correct judgment when one or more zeroes are immediately to the right of the decimal point in one of the numbers, and otherwise chooses as larger the number with the most digits to the right of the decimal point. Thus, given three numbers to order (e.g., 3.214, 3.09, 3.8), Rule 3 correctly chooses the number with the zero as the smallest, but then uses Rule 1 to order the remaining pair: i.e., 3.09, 3.8, 3.214. Rule 3 was used by about 8% of Sackur-Grisvard and Leonard's fourth graders and by 14% of their fifth graders.

We designed this study to examine the sources of these consistent categories of errors in more detail and more directly than has been done previously. Generally, we were interested in the ways in which children use preexisting knowledge in constructing a mental representation of a new domain of knowledge. Our working hypothesis was that children's errors may derive from their efforts to apply previously learned concepts or notational conventions to a new domain. A child who has just been exposed to instruction on decimals must build a representation of decimal numbers and relate decimals to other well or partially acquired number systems--notably whole numbers (for all children) and ordinary fractions (for children in countries where ordinary fractions are taught earlier than decimal fractions)--as well as to certain conceptions of measurement (cf. Kieren, 1975). Prior knowledge of whole numbers and fractions can both support and interfere with construction of a correct concept of decimals. Table 1 lists elements of knowledge needed for a full decimal fraction concept (column 1) and indicates corresponding elements of whole number knowledge (column 2). For each pair of knowledge elements, column 3 shows whether the elements support or contradict each other.

Table 1 consists of three sections. The first (A) refers to basic conceptual elements of knowledge about column values. The last two (B and C) refer to knowledge of notational and naming conventions that may interact with conceptual knowledge to produce incorrect rules. As can be seen in the table, there are some similarities and some dissimilarities between whole numbers and decimals. The fact that the new decimal concepts are embedded in a structure that shares key features of place value with whole numbers may suggest to children that the new system is identical to the old one and lead them to ignore the differences between the two. This might even be heightened by teachers' attempts to help children use their prior knowledge of whole numbers to facilitate learning the new decimal system.

Table 1 about here

Section A of the table shows that with respect to basic principles of column values the structures of decimals and whole numbers are in accord. Each column is ten times larger in value than the column to its right (A1 and A2). Decimals and whole numbers are also conceptually similar in the way zero functions as a place holder (A3). These conceptual similarities, which are likely to be stressed in instruction, lead to a prediction that children first learning about decimal fractions might attempt to use their knowledge of whole number values in comparing decimal fractions. This would produce Sackur-Grisvard and Leonard's Rule 1 (the longer decimal is larger), which we will henceforth call the *Whole Number* rule.

Fully integrated knowledge of the place value structure would lead children to recognize that, while the decimal point marks a transition from units to fractions, the basic relationship between columns is not changed. Other features of the decimal system, however, might lead children to interpret the decimal point as a fundamental "breaking point" in the digit string. For example, the structure of column names (Section B of the table) encourages the interpretation of decimal fractions as near mirror images of whole numbers. Only a difficult-to-hear <th> distinguishes the names (B1),[1] and if that distinction is ignored it appears that the same sequence of column names moves to the right after the decimal point, to the left before it (B3). Indeed, with respect to column *values*, the values do decrease as one moves right from the decimal point (A5), while they increase as one moves left from the point. Furthermore, a zero at the extreme right on the fraction side of the point leaves the number's value unchanged, while a zero at the extreme left on the whole number side leaves the value unchanged (A4). A further complication is that nothing corresponds to the units column on the fraction side (B2). Failure to learn and apply these differences between whole numbers and decimals could lead to errors such as calling the tenths column a *oneths* column or not recognizing that decimal numbers are actually symmetrical around the units column rather than around the decimal point. The symmetry around the decimal point and the difficulty of distinguishing the whole number and fractional names can be expected to heighten children's tendency to apply a Whole Number rule. The highlighting of the decimal point

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[1] Children who spoke two languages other than English were included in the present study. In French, the difference between <dizaines> and <dixiemes>, <centaines> and <centiemes> is easier to discriminate in writing, but many children confound the terms orally. In Hebrew, the terms <asarot> and <asiriot>, <maot> and <maiot> are easier to discriminate than the corresponding English terms, yet similar enough to cause difficulty for children.

by symmetry would make them recognize that things change after the point, while the name similarity could lead them to believe that, for example, tenths are smaller than hundredths, thus reinforcing the whole number judgments of relative size.

A final difficulty arises in composing a string of digits into a single decimal quantity (Section C of Table 1). In the whole number domain, one always reads a number as a set of units or *ones*. For example, the number 2,674 is read as "two thousand six hundred seventy-four," with the understanding that the reference is to that many *ones*. In decimals, there is no constant unit corresponding to *ones*. This complicates the task of composing a string of digits into a single decimal number. One strategy for thinking about a decimal as a single quantity is to use the smallest unit of division as the common unit (e.g., thousandths in .374) and to name the number in terms of that unit (e.g., 374 thousandths). One problem with this strategy is that the naming process can further confuse a student about individual column value. For example, saying "three hundred seventy-four thousandths" might suggest to children that there are three hundreds in the number and lead them to think about it as a whole number. And, as we shall see later, naming decimals in this way often makes it harder to compare different length decimals. Decimals can be more powerfully thought about as compositions of tenths, hundredths, and so forth; (e.g., 3 tenths *plus* 7 hundredths *plus* 4 thousandths). However, with this conception there is no convenient, single label for the number. An important element of knowledge needed for understanding decimals as compositions is the equivalences: 1 tenth equals 10 hundredths, which equals 100 thousandths; 1 hundredth equals 10 thousandths. Because of the conflicting naming sequence for whole numbers (1 ten does *not* equal 10 hundreds), children are likely to be confused for some time about such equivalences in the decimal fraction domain. As a result they will invent rules for comparing values that do not depend upon knowledge of these equivalences.

Table 2 specifies the potential connection points between decimal and ordinary fractions. As shown in Section A, the quantity *values*, to which both decimal and ordinary fractions refer, correspond completely. However, important differences in the notational systems could cause confusion (Section B). In ordinary fractions, digits show both the size of the parts and the number of parts; but in decimal fractions, digits show only the number of parts, while the size of the parts are implicit in the column values (B1 and B2). We can predict that some children who have learned ordinary fraction notation before decimal fractions might attempt to apply their conceptual knowledge about the relation between size of parts and number of parts (A2) to an incomplete understanding of the referents of decimal notation. For example, if they know that thousandths are smaller parts than hundredths, and that three digit decimals are read as thousandths while two digit decimals are read as hundredths, they might well infer that longer decimals, because they refer to smaller parts, must have lower values. This kind of reasoning would produce Sackur-Grisvard and Leonard's Rule 2 for comparing decimals, which we henceforth call the *Fraction* rule. We expect use of the Fraction rule to be higher in countries such as Israel and the United States, where fractions are taught before decimals, than in France and other countries where decimals are taught before fractions. In the U.S. and Israel we would expect children who are generally more advanced and whose fraction knowledge is stronger to show more Fraction rule use.

Table 2 about here

Sackur-Grisvard and Leonard's Rule 3 is a special case of the Whole Number rule for numbers with a zero immediately following the decimal point. We believe that this special case rule might be generated by children who become aware of the placeholder function of zero but do not have a fully developed place value structure. As a result they apply their knowledge that zero is very small (i.e., "nothing") to conclude that the entire decimal number must be small. For mnemonic ease we will henceforth call this rule the *Zero* rule. We would expect this rule to appear mostly among children who are advancing beyond the Whole Number rule but are not yet expert. It should be especially frequent among the French children in our study, who, because of the way their curriculum is organized, do not have knowledge about ordinary fractions to draw upon as they advance.

In this study, we began by diagnosing children's rules for comparing decimal fractions. We predicted substantial use of the Whole Number rule in all three countries, because all children are familiar with whole numbers when they begin to learn decimals. The Zero rule was expected to replace the Whole Number rule among some more advanced children, especially in France where little ordinary fraction knowledge was available for refining decimal concepts. In Israel and the U.S., we expected heavier use of the Fraction rule, especially among more advanced children and those whose ordinary fraction knowledge is strong. Once children were categorized with respect to rule use, we employed various test and interview items to establish possible sources of these rules.

Method

Subjects

A total of 113 children in the early phases of school instruction on decimals participated in the study. The children's grade levels ranged from fourth to sixth grade. Each sample included children of varying ability levels. The American sample was from the fifth grade in a single school (a middle class, parochial school) with N=17. All children in the class were interviewed by authors of this paper. The Israeli sample was from grade 6 with N=21. The participating children were nominated by their teachers as "average" math students, and all were interviewed by authors of this paper. The French samples were from grades four with N=37 and five with N=38. French children, nominated as participants by their teachers following a request for "average" math students, were drawn from several different schools in the region around Nice. The interviewers were 15 psychology students who earned academic credit for this work but did not participate in the questionnaire design or data analysis. They followed the questionnaire more strictly than the Israeli and American interviewers, therefore, and did little probing. As will be seen, this had some effect on the nature of the data from the French samples.

In both Israel and the U.S., ordinary fractions are introduced at least one year earlier than decimals. Interviews were conducted within three months of completion of a decimal fraction instructional unit; thus, all Israeli and American children had had prior exposure to both ordinary fraction notation and the meaning of decimal fractions. In France, decimal fractions are taught in CM-1 (equivalent to 4th grade), but ordinary fractions are not introduced until CM-2 (equivalent to 5th grade). All French children were interviewed after decimal fraction instruction but before the introduction of ordinary fractions.

Procedure and Materials

Each child was interviewed individually in a semi-structured format. Israeli and U.S. interviewers engaged in some probing for justification or clarification of the child's answers. Each interview was audiotaped, and detailed protocols were obtained from the tapes and from the child's written work. Each interviewer followed a written interview form which varied slightly from country to country but included the same basic content. We describe the American version of the interview here and mention relevant variations in the Israeli and French interviews in the course of reporting data.

Establishing presence of the rules. Our primary task for identifying the children's underlying rules was comparison of two decimal fractions. Let us consider the universe of such comparisons and their possible answers. Given two decimal fractions, A and B, of the form

$$A=0.a_1.a_2...a_n$$
$$B=0.b_1.b_2...b_k$$

A and B can differ in two ways. They might have different lengths, i.e., $n>k$, $n=k$, or $n<k$. Within each column, they also might differ in the actual digit, e.g., $a_1>b_1$, $a_1=b_1$, $a_1<b_1$. Table 3 summarizes the inferences about rules that can be made based on children's judgments for prototype cases in which $n=1$, $k=2$, and there is no zero immediately after the decimal point. As can be seen, if the child chooses A in the first case (i.e., when A is in fact larger but shorter), we cannot know with certainty whether he or she was correct or was applying the Fraction rule. Similarly, a choice of B in the second case may imply use of either the correct rule or the Whole Number rule. Thus, only if the child chooses B in the first case or A in the second can a firm attribution of rules be mad This provides the basic logic for interpreting comparison item responses in our interviews; however, the attribution cannot be made on the basis of a single item. If a child is a candidate for holding a Fraction rule, she or he will reply A in both cases and will be right on the first item (0.8; 0.64) but wrong on the second one (0.2; 0.64). Thus, we attribute a rule to a child on the basis of a series of responses, some of which are correct and some incorrect.

Table 3 about here

When zeroes occupy the first column after the decimal point (i.e., if $a_1 = 0$ or $b_1 = 0$), the Zero rule may be employed. As Sackur-Grisvard and Leonard have shown, this is a specialized version of the Whole Number rule; it specifies that a number with a zero in the first column is smaller than a number with a non-zero digit in that column. The attribution of the Zero rule to a child is made on the basis of a pattern of answers in which the Whole Number rule fits for all cases except for those having zero in the tenths position.

Table 4 shows the comparison items used to detect the rules by extracting the patterns of answers. For each pair of fractions, the child was to indicate the larger number. For ease of interpretation, the table always shows A as the correct answer; although in the actual interview, the position of the correct answer was randomized. A systematic Whole Number rule child would choose the B fraction in the first two sets of items but answer the third sets correctly. A Zero rule child would answer the first set with a Whole Number pattern (B answers) but would answer the second and the third sets correctly (A answers). A Fraction rule child would give correct (A) answers for the first two sets but would give incorrect (B) answers in the third set.

Table 4 about here

Two additional tasks were used to confirm our classification of children to rule-use groups. In the Hidden Numbers Comparison task, two numbers were presented with the actual digits covered, but the number of digits and their relation to the decimal point visible. For example, the child was asked, "Which is larger 0.   or 0.   ?" In each case, the correct answer is that one cannot tell! For these items, Whole Number rule children would select the number with more decimal places, and Fraction rule children would select the number with fewer decimal places. The Zero rule could only be detected if children said specifically that they could not tell which number was larger because they did not know whether or not there was a *zero* immediately after the decimal point.

A Zero Insertion task also served to confirm Whole Number and Fraction rule classifications. In this task, we asked subjects to tell us whether and how the value of 2.35 changed when a zero was inserted in four different places to produce 2.305, 02.35, 2.035, and 2.350.

Once children's rules were established, we used a series of other items--along with children's verbalizations on all of the tasks--to try to uncover children's conceptual frameworks, with particular attention to whether they were thinking of decimals in whole number or ordinary fraction terms. These data were used to infer probable sources of the rules. In one set of items, children were asked the values of digits in different positions in written numbers (e.g., the value of the 5 in 1.54 and 2.45). In a second set, the interviewer read aloud numbers by column name (e.g., "six tenths and two hundredths") and asked the child to write the numbers in decimal notation. A third set of items had children compare two ordinary fractions. Finally, in a fourth set of items, children wrote numbers in decimal form that they were shown in ordinary fraction notation.

Results

Use of the Rules

Table 5 presents response patterns for the basic rule detection items for the U.S., Israeli, and French samples. Individual children are listed at the top of each section of the table. and items (which vary slightly from sample to sample) are listed vertically at the left. Each child's response to each item is shown; a blank space indicates no response or an uncodable response. For purposes of discussion the tables show subjects grouped according to their dominant pattern of response. For the most part, children assigned to the three incorrect rule categories gave responses perfectly consistent with the expected rules. A few children who do not appear consistent on the table (e.g., Israeli subjects 4, 16, 6, 17; French fourth grade subject 9) verbally explained their choices by directly describing the rule into which they are classified. Furthermore, inconsistent responses usually came on specific items that permitted a different judgment strategy than the standard one used by the child. For example, in comparing 0.5 with 0.36, some children recognized that 0.5 was a half and used it as a reference point, saying that .36 was less than half. In comparing 0.25 with 0.100, one child said the two zeroes at the right of 0.100 do not make a difference; therefore, 25 is greater than 1. The other exceptions occurred mainly on items where one of the numbers was an ordinary fraction and the other a decimal fraction. Errors in conversion between the notation systems probably accounted for these responses.

Table 5 about here

Table 6 shows the distribution of rules in the various samples. The general pattern of this distribution for French children is similar to the earlier Sackur-Grisvard and Leonard findings. As expected, there was a low incidence of the Fraction rule overall and a shift from the Whole Number rule toward the Zero rule between 4th and 5th grades. Also as expected, the Israeli and U.S. children show a higher incidence of the Fraction rule, with the Israeli children, who were sixth graders, showing an especially high use of the rule. The absence of Zero rule cases among U.S. children should be noted, although we are not able to offer an explanation.

Table 6 about here

Responses on the Hidden Number Comparison task can be used to confirm the classificati . of children. The relevant data for the combined Israeli and U.S. samples appear in Table 7, which shows frequencies of children in each rule use category who gave each answer.[2] Clearly, responses are largely consistent with the child's rule classification. As stated earlier, Whole Number rule children were expected to choose the longer string as the larger value, and Fraction rule children were expected to choose the shorter string. The "I don't know" answer is correct, but the same answer might be given by children who did not have any idea how to respond. We, therefore, do not consider the "I don't know" responses diagnostic of rule use.

Table 7 about here

An additional item that confirmed our classification of children according to the rules was the Zero Insertion task, which called for comparing the number 2.35 with 2.305, 2.035, 2.350, and 02.35. Virtually all children in all rule categories knew that inserting a zero before the 2 or after the 5 did not change the value. However, comparisons of 2.35 with 2.305 and 2.035 discriminated among the rule use categories.

[2]These items were administered to the French sample in a way that did not permit discrimination of the malrule, and those results are, therefore, omitted.

Table 8 shows data on these items for the U.S. and Israeli samples combined.[3] As can be seen, Whole Number rule children almost always chose the longer numbers (i.e., 2.305, 2.035) as the larger. This is the expected Whole Number response. A few Whole Number children judged 2.035 to be equal to 2.35. This is consistent with a Whole Number judgment based on the integers' value rather than on the number of digits (i.e., 035=35). All but one Fraction rule child chose 2.35 as having the larger value, as would be expected. Since these are also correct answers, however, this pattern can be considered only weak support for the Fraction rule classification.

Table 8 about here

Conceptual Bases Underlying the Rules

We turn now to the cognitive sources of the errorful rules. We have two kinds of data to draw on in making our inferences: the children's verbalizations as they worked on the comparison tasks, and the patterns of answers for the items that more directly examined place value and fractional knowledge. We will consider these for the Whole Number and the Fraction rules in succession.

Whole Number Rule. We had hypothesized that the Whole Number rule results from children's attempts to apply their knowledge about whole numbers to the new kind of numbers they are learning without integrating information about the fractional values. This was confirmed by typical verbalizations of the Whole Number children as they responded to comparison items. For example, here are explanations by two Israeli children:

S 4: 0.5 < 0.25, "because 25 is bigger."
4.7 < 4.08, "because the zero does not matter and 8 is bigger than 7."

[3] The French formulation of this question did not yield comparable data.

S 16: 4.8 < 4.63, "since 63 is bigger than 8."

When comparing 2.35 with 2.305 and 2.035, Whole Number children often referred to a number's decimal portion as a whole number, saying that "three hundred and five" or "three hundred fifty" was bigger than "thirty five."

Whole Number children also showed confusion about zero's placeholder function. One child, for example, equated 2.35 and 2.035, because the zero is "just a place marker and that would still be thirty five." This generally fragile understanding of the place value system is further confirmed by items assessing explicit place value knowledge. Most Whole Number and Zero rule children could not correctly give the value of the 5s in 1.54 and 2.45 as ten*ths* and hundre*dths* respectively. Some examples from the protocols give a flavor of the difficulty Whole Number children had in labeling the columns in decimal numbers. Four of the U.S. children said that the 5 in 2.45 was 5 *ones*, suggesting that they were thinking of .45 as the whole number 45. (Some subsequently changed their answers and were given credit for a correct response on the table.) Subject 13 said that the 5 in 2.45 was 5 *tens*, but then changed her mind, saying the 5 in 1.54 stood for 5 *tens*. She did not identify the 5 in 2.45 again, but may have been thinking it was in the *ones* column, because in her explanation of why 5 tenths was greater than 5 hundredths she said, "You add 10 more from the *ones* row to get to the second, to get to the *tens* row." Responses of this kind dominated among the Whole Number and Zero rule children. The Fraction rule children and experts were mostly able to answer these questions correctly.

Other items examined children's ability to write decimals from an oral reading. Whole Number and Zero rule children had difficulty in writing "6 tenths and 2 hundredths" or "3 ones and 6 hundredths" correctly, while Fractions rule children, like the experts, mostly gave correct responses. Whole Number children reversed the order of the digits 6 (tenths) and 2 (hundredths), writing 0.026 or 2.6, which would be the correct ordering if the number were 6 *tens* and 2 *hundreds*. Other errors, too, suggest that Whole Number children were thinking of decimals as whole numbers. For example, for "3 hundredths," one subject wrote .300. Another subject, when she heard the interviewer say "6 tenths and 2 hundredths," asked, "You mean 60 and 2 hundredths?" After the interviewer repeated the number, she wrote 2.60. Interpreting 6 tenths as 60 suggests that she was thinking of 6 tens rather than 6 tenths.

Fraction Rule. We hypothesized that the Fraction rule results from children's efforts to integrate knowledge about fractional parts and ordinary fraction notation with their place value knowledge. In particular we expected them to know that if a number is divided into more parts, the parts are smaller. We expected, however, that Fraction rule children might have some difficulty figuring out whether the digits stated explicitly in the decimal form correspond to the numerator or the denominator of an ordinary fraction (i.e., to the number of parts or to the size of the parts).

Several aspects of the data confirm these expectations. First, a few Fraction rule children directly stated their bases for comparing decimals. Here is a typical explanation from an Israeli subject:

S 20: 4.7 > 4.08, "since this has 8 hundredths and this has 7 tenths."
7.457 > 4.4502, "because this is hundredths and this is ten-thousandths."

Interviewer: "So what?"

S 20: "Hundredths are bigger than ten thousandths."

To further understand how fraction knowledge might affect decimal understanding, we searched the protocols of Fraction rule subjects for overt evidence that they were applying their fraction knowledge to decimals. We found two such instances. Both were cases where fraction reasoning produced correct judgments. U.S. Subject 5, comparing 4/100 and 0.038, said that 4/100 was bigger because, "One hundred is smaller than one thousand in the decimals. Smaller numbers make the larger pieces." U.S. subject 11's explanation, although less clear, also seemed to use fraction knowledge to guide correct decimal comparisons. When ordering ten numbers, she said that 1.4 is larger than 1.067, "because 1 and 4 tenths is more of a part than 1 and 67 thousandths . . . . Four tenths is like more of a part . . . . It's like 5 tenths is a part, so you almost have a part. But on this [1.067] you don't even have a part."

In addition, Fraction rule children showed rather good knowledge of place value, as already discussed. Some of their explanations on the zero insertion task (2.35 compared with 2.035 and 2.305) confirmed this. Of the Fraction rule children, U.S. subject 5 was the most vocal. She said that 2.35 is greater than 2.305, "because the 0 would take hundredths place, and 5 would get pushed down to thousandths." This child gave a similar justification for choosing 2.35 as greater than 2.035, saying, "Zero takes the tens place, and 3 and 5 get pushed down." More generally she explained that zero makes a difference in the middle of a number but not at the beginning or at the end, where it is "just a placeholder." Subject 5 thus limited the placeholder *language* to the beginning and end of the numbers, but applied the placeholder *concept* to all positions. U.S. subject 7 gave no justifications for his individual comparisons but did give a childlike version of the full placeholder theory. He said that zero matters in between other numbers because it shows that, "Nothing is home in the middle space." (His teacher had used a similar formulation.)

Table 9 shows the frequency for U.S. and Israeli children of responses on items comparing 1/333 with 1/334 and 1/3 with 1/4. These items test children's knowledge of a feature of ordinary fraction notation that is frequently confusing to beginners: as the digit in the denominator grows, the size of the unit fraction shrinks. As can be seen, Whole Number rule children had great difficulty while Fraction rule children succeeded to a larger extent (although not perfectly). In particular, Whole Number children judged as greater fractions in which the number in the denominator is larger. These items were not scored for the French sample because the children had not yet encountered ordinary fractions in their school instruction. Many French children simply refused to respond; others treated the slash in the fraction as equivalent to a decimal point (e.g., 3/10 = 3.10). The French children did know, however, certain common benchmark fractions and their decimal equivalents (e.g., .5 = 1/2).

Table 9 about here

Table 10 shows results on the items requiring children to translate common fractions with denominators other than 10 or 100 (i.e., 3/4, 1/5, 3/2, 2/3) into decimals.[4] None of the children could do this reliably, but their attempts to do so revealed their ideas about how numerators and denominators of regular fractions correspond to decimal notation. There were three main categories of incorrect translation:

1. Encode only the numerator in the decimal, and ignore the denominator. For example, 3/4 becomes .3 or .003, 3/100 becomes .3. This translation reveals knowledge of the correspondence between the written digits in ordinary and decimal fractions (element B1 in Table 2). It fails, however, to find a correspondence between the notation of the denominator in ordinary fractions and the place value system of decimal fractions (element B2 in Table 2).

[4]These items were not included in the Israeli interview and were not interpretable in the French sample because of the French children's limited prior exposure to ordinary fraction notation.

2. Encode only the denominator in the decimal, and ignore the numerator. For example, 1/3 becomes .3, 3/4 becomes .04 or .4. This translation recognizes that only one part of the fraction actually is shown in the decimal notation, but it mistakes the part. It is a less advanced response than the numerator-only encoding.

3. Keep both the numerator and the denominator of the fraction, and put a decimal in someplace. For example, 3/4 becomes 3.4 or .34. These are "syntactic" translations. They produce a number that has the surface structure of a decimal but has no sensible relation to the quantity expressed in the fraction.

As shown in the table, Fraction rule subjects made entirely numerator-only errors--ones in which they encoded the fraction's *numerator* in the decimal notation. Only one Whole Number rule subject ever made this type of error; instead, Whole Number rule subjects made mostly syntactic errors or encoded only the denominator.

Table 10 about here

Discussion

The study clearly replicates the earlier findings of Sackur-Grisvard and Leonard concerning the rules for comparing decimal fractions but goes beyond their study in two respects. First, it tries to understand the rationale behind the rules in terms of the children's entire conceptual framework. Second, it tries to find the relative place of these rules in decimal knowledge development. The Whole Number rule occurs frequently and early in learning. The Zero rule appears as a variant mostly among the older French children and the sixth grade Israelis. The Fraction rule occurs mostly in the Israeli and the U.S. samples. It is strongest among the Israelis, who were older than the U.S. children. This trend toward an increase in the Fraction rule as children progress in learning has been replicated subsequent to our study by Bilha Zuker from Israel, who in her M.A. thesis looked for developmental trends for these rules (Zuker, 1985). Zuker tested 74 seventh graders, 106 eighth graders. and 60 ninth graders. She found that the Whole Number rule declined from 18% in the 7th grade to 5% in the 9th grade, while the Fraction rule was more persistent and appeared in 23% of the sample in 7th and 9th grades.

The present data also demonstrate substantial internal consistency in rule application. Overall, 88% of children could be classified as consistently using one of the three rules under study. In this respect, each individual's "buggy" performances in this task are much more systematic than they are in domains such as algebra, where an individual typically alternates between two or more incorrect transformation rules, and perhaps a correct one as well, within a single problem solving session (Carry, Lewis, & Bernard, 1980; Greeno et al., 1985). There may be even more consistency in the decimal domain than in buggy subtraction, where recent work has demonstrated considerable "migration" of bugs, leading to a revision of the view that buggy performances are normally very stable (VanLehn, 1982).

There seem to be fundamental differences in the kinds of conceptual understanding that produce the Whole Number rule and the Fraction rule. Whole Number rule children appear to have a very impoverished representation of decimal numbers. Apparently their representation of the place value system does not contain the integration of the crucial information of column values, column names, and the role of zero as a placeholder. In terms of the knowledge elements shown in Table 1, these children do not seem to think about the column values of the number's decimal portion at all; rather, they simply "import" from whole numbers a comparison rule that may not itself even require that all of the elements shown in section A for whole numbers be represented fully. Their responses to column value comparisons and decimal writing tasks clearly reveal their weak or nonexistent coding of column values and their borrowing of whole number column names to apply to decimal fractions (Section B of Table 1). They also quite clearly do not have a complete representation of how zero works as a placeholder (element A3). Even more important, Whole Number rule children show no signs of recognizing that the number's decimal portion represents a fractional part of a whole, and so none of the knowledge elements in Table 2 is invoked. Their failure to represent the decimal portion of the number as a fraction means that nothing in these children's representations constrains their use of the Whole Number rule to compare decimal fractions.

Fraction rule children are most sharply distinguished from Whole Number rule children by their use--albeit incompletely coordinated--of elements of fractional knowledge. In particular, Fraction rule children appear to know and apply the principle (congruent for decimal and ordinary fractions--see element A2 of Table 2) that the more parts a whole is divided into, the smaller each part is. They also know that the number of places in a decimal fraction tells the size of the parts (element B1 of Table 2)--specifically, that if there is one place after the decimal, the parts are tenths; if there are two places, the parts are hundredths; if there are three places, the parts are thousandths. However, these children could not coordinate information about the size of parts (element B1) with information about the number of parts (element B2). Thus, when attending to size of parts (specified by the number of columns), they ignored the number of parts (specified by the digits).

To achieve the necessary coordination requires a rather sophisticated representation of place value notation. The global understanding of place value that Fraction rule children seem to apply specifies only that one place signals *tenths*, two places *hundredths*, and so on. However, with this representation of place value alone, it is not possible to compare decimals that have different numbers of places. To deal with this problem, the representation of place value would have to change to a form that reflects the additive structure of decimal numbers. This representation specifies that if there are two places after the decimal point, the parts are tenths *plus* hundredths; if there are three places, the parts are tenths *plus* hundredths *plus* thousandths. Furthermore it specifies that the digit in each place tells how many parts of a given size are to be included in the addition. Such a representation would allow children to make column by column comparisons of values, thus producing correct judgments even for pairs with different numbers of digits.

We see from the different patterns of rule categorization among the three countries that different curriculum sequences produce different patterns of rule invention. This confirms our working hypothesis that errors derive from students' attempts to integrate new material that they are taught with already established knowledge. The fact that the French children by and large avoided the errors associated with the Fraction rule and instead seemed to pass directly to use of the correct decimal comparison rule might seem to suggest a superiority of the French curriculum sequence (which is shared by several other countries) in which decimal fraction instruction precedes ordinary fraction instruction by a substantial period of time. However, it is important to note that we have no evidence that any of the children classified as experts in fact understood the conceptual basis for decimal comparisons. French and other children who gave consistently correct answers may very well have arrived at their correct rules on the basis of purely surface and syntactic considerations.

A commonly taught procedure for comparing decimals illustrates how this might happen. Children are often taught an essentially syntactic rule for comparing decimals in which they first add zeroes to the shorter number in order to give both numbers the same number of digits; they can then compare the decimals as if they were whole numbers. This is a relatively simple rule to learn and gives reliably correct answers. Children applying it would have been classified as experts in our study. Yet they would have been applying the Whole Number rule, and their conceptual understanding could have remained at the level we have attributed to Whole Number rule children. Such syntactic teaching would serve to suppress errors in performance without improving children's conceptual understanding.

This brings us to a more general consideration of the status of systematic errors in mathematics learning and teaching. From a cognitive point of view, almost all instruction is incomplete in the sense that it is not possible in any single demonstration or explanation to cover all special cases or all possible implications of principles or rules that may be presented. Instruction, like all normal human communication, proceeds on the assumption that learners will use the presented material to make inferences and interpretations that complete and make sense of what the teacher or text has said. In making these inferences and interpretations, children are very likely to make at least temporary errors. Errorful rules, on this view, are intrinsic to all learning--at least as a temporary phenomenon--because they are a natural result of children's efforts to interpret what they are told and to go beyond the cases actually presented. Several analyses (e g., Resnick, 1987; VanLehn, 1986) have shown that these errorful rules are intelligent constructions based on what is more often incomplete than incorrect knowledge. Errorful rules, then, cannot be avoided in instruction. In fact, they are best regarded as useful diagnostic tools for instructors, who can often use children's systematic errors to detect the nature of children's understanding of a mathematics topic. Mathematics education researchers can support this instructional function by discovering and documenting common errors and the conceptual understanding that underlies them.

References

Brown, J. S., & Burton, R. R. (1978). Diagnostic models for procedural bugs in basic mathematical skills. *Cognitive Science, 2*(2), 155-192.

Brown, J. S., & VanLehn, K. (1982). Towards a generative theory of "bugs." In T. P. Carpenter, J. M. Moser, & T. A. Romberg (Eds.), *Addition and subtraction: A cognitive perspective* (pp. 117-135). Hillsdale, NJ: Erlbaum.

Carry, L. R., Lewis, C., & Bernard, J. E. (1980). *Psychology of equation solving: An information-processing study.* Austin, TX: University of Texas.

Greeno, J. G., Magone, M. E., Rabinowitz, M., Ranney, M., Strauch, C., & Vitolo, T. M. (1985). *A cognitive skill early in its acquisition.* Unpublished manuscript, University of Pittsburgh.

Grossman, A. S. (1983). Decimal notation: An important research finding. *Arithmetic Teacher*, 32-33.

Hiebert, J., & Wearne, D. (1985). A model of students' decimal computation procedures. *Cognition and Instruction, 2*, 175-205.

Kieren, T. E. (1975). On the mathematical, cognitive, and instructional foundations of rational numbers. In R. A. Lesh (Ed.), *Numbers and measurement* (pp. 101-144). Athens, GA: University of Georgia, The Georgia Center for the Study of Learning and Teaching Mathematics.

Matz, M. (1982). Towards a process model for high school algebra errors. In D. Sleeman & J. S. Brown (Eds.), *Intelligent tutoring systems* (pp. 25-50). New York: Academic Press.

Resnick, L. B. (1982). Syntax and semantics in learning to subtract. In T. P. Carpenter, J. M. Moser, & T. A. Romberg (Eds.), *Addition and subtraction: A cognitive* *perspective* (pp. 136-155). Hillsdale, NJ: Erlbaum.

Resnick, L. B. (1987). Constructing knowledge in school. In L. S. Liben (Ed.), *Development and learning: Conflict or congruence?* (pp. 19-50). Hillsdale, NJ: Erlbaum.

Resnick, L. B., & Omanson, S. F. (1987). Learning to understand arithmetic. In R. Glaser (Ed.), *Advances in instructional psychology, Vol. 3* (pp. 41-95). Hillsdale, NJ: Erlbaum.

Sackur-Grisvard, C., & Leonard, F. (1985). Intermediate cognitive organization in the process of learning a mathematical concept: The order of positive decimal numbers. *Cognition and Instruction, 2*, 157-174.

Siegler, R. S., & Vago, S. (1978). The development of a proportionality concept: Judging relative fullness. *Journal of Experimental Child Psychology, 25*, 371-395.

Sleeman, D. (1982). Assessing aspects of competence in basic algebra. In D. Sleeman & J. S. Brown (Eds.), *Intelligent tutoring systems* (pp. 185-199). New York: Academic Press.

VanLehn, K. (1982). Bugs are not enough: Empirical studies of bugs in passes and repairs in procedural skills. *Journal of Mathematical Behavior, 3*(2), 3-71.

VanLehn, K. (1986). Arithmetic procedures are induced from examples. In J. Hiebert (Ed.), *Conceptual procedural knowledge: The case of mathematics* (pp. 133-180). Hillsdale, NJ: Erlbaum.

Zuker, B. (1985). *Algorithmic knowledge vs. understanding decimal numbers*. Master's thesis, University of Haifa, Israel.

Table 1

Comparison of Decimal Fraction and Whole Number Knowledge

Elements of Decimal Fraction Knowledge	Corresponding Elements of Whole Number Knowledge	Similar (+) or Different (-)
A. Column Values:	A. Column Values:	
1. Values decrease as move left to right	1. Values decrease as move left to right	+
2. Each column is 10 times greater than column to right	2. Each column is 10 times greater than column to right	+
3. Zero serves as a place holder	3. Zero serves as a place holder	+
4. Zero added to rightmost column does not change total value	4. Zero added to leftmost column does not change total value	-
5. Values decrease as move away from decimal point	5. Values increase as move away from decimal point	-
B. Column Names:	B. Column Names:	
1. End in <ths>	1. End in <s>	-
2. Start with tenths	2. Start with units	-
3. Naming sequence (tenths, hundredths...) moves left to right	3. Naming sequence (tens, hundreds...) moves right to left	-
4. Reading sequence is tenths, hundredths, thousandths	4. Reading sequence is thousands, hundreds, tens, ones	- -
C. Reading Rules:	C. Reading Rules:	
1. The units must be explicitly specified and they vary	1. The ones implicitly serve as the units in all cases	-

Table 2

Comparison of Decimal Fraction and Ordinary Fraction Knowledge

Elements of Decimal Fraction Knowledge	Corresponding Elements of Ordinary Fraction Knowledge	Similar (+) or Different (-)
A. Fraction Values:	A. Fraction Values:	
1 Expresses a value between 0 and 1	1. Expresses a value between 0 and 1	+
2. The more parts a whole is divided into, the smaller is each part	2. The more parts a whole is divided into, the smaller is each part	+
3. There are infinite decimal fractions between 0 and 1	3. There are infinite ordinary fractions between 0 and 1	+
B. Fraction Notation:	B. Fraction Notation:	
1. The number of parts a unit is divided into is given implicitly by the column position	1. The number of parts a unit is divided into is given explicitly by the denominator	-
2. The number of parts included in the fractional quantity are the only visible numerals	2. The number of parts included in the fractional quantity is the numerator of the fraction	-
3. The whole is divided only into powers of 10 parts	3. The whole is divided into any number of parts	-
4. The ending <th> ("tenth") is typical for a fractional part	4. The ending <th> ("fourth") is typical for a fractional part	+

Table 3

Relations Among Decimal Pairs and the Underlying Rules for Comparing Them

Relation	Example A = $0.a_1a_2..a_n$	Example B = $0.b_1b_2..b_k$	Child's Choice	The Assumed Underlying Rule
$n<k$			A	Correct or Fraction Rule
	0.8	0.64		
$a_1>b_1$			B	Whole Number
$n<k$			A	Fraction Rule
	0.2	0.64		
$a_1<b_1$			B	Correct or Whole Number Rule

Table 4

Comparison Items That Detect Rules

Question: For each pair, circle the number that is bigger.

Item on Questionnaire	Number Pair*		Answer by Subject Group			
	A	B	Whole Number Rule	Zero Rule	Fraction Rule	Experts
Whole Number Rule:	4.8	4.63	B	B	A	A
	0.5	0.36	B	B	A	A
	0.25	0.100	B	B	A	A
Zero Rule:	4.7	4.08	B	A	A	A
	2.621	2.0687986	B	A	A	A
	4/100	0.038	B	A	A	A
Fraction Rule:	4.4502	4.45	A	B	A	A
	0.457	4/10	A	B	A	A

* The numbers in Column A are the larger numbers of the corresponding pairs, therefore, an "A" answer is a correct answer

Table 5

Items Detecting Underlying Rules

A. U.S. Sample

Item		Student Responses (Grouped According to Rule Used)												
		Whole Number						Fraction			Experts		
A	B	12	13	14	15	16	17	5	7	11	1	2	3
4.8	4.63	B	B	B	B	B	B	A	A	A	A	A	A
0.5	0.36	B	B	B	B	B	B	A	A	A	A	A	A
0.25	0.100	B	B	B	B	B	B	A	A	A	A	A	A
13/100	0.125	B	A	B	B	B	B	A		B	A	A	B
4.7	4.08	B	B	B	B	B	b	A	A	A	A	A	A
2.621	2.068796	B	B	B	B	B	B	A	A	A	A	A	A
4/100	0.038	A	A	B	B	B	B		A	B	A	B	B
4.4502	4.45	A	A	A	A	A	A	B	B	B	A	A	A
0.457	4/10	ʌ	A	A	A	A	A	B	B	B	A	A	A

5 children could not be classified

Table 5, continued
Items Detecting Underlying Rules

B. Israeli Sample

Item		Student Responses (Grouped According to Rule Used)																	
		Whole Number				Zero			Fraction							Experts			
A	B	4	10	12	16	6	17	21	3	5	8	11	13	18	20	1	2	15	19
4.8	4.63	B	B	B	B	B	B	B	A	A	B	A	A	A	A	A	A	A	A
0.5	0.36	B	B	B	A	A	A	B	A	A	B	A	A	A	A	A	A	A	A
0.25	0.100	B	B	B	B	B	A	B	A	A	A	A	B	A	A	A	A	A	A
13/100	0.125	B	B	B	B	B	A	B	A	B	A	A	A	A	B	A	A	A	A
7/10	0.678	A	B	B	A	A	A	B	A	A	A	A	A	A	A	A	A	A	A
4.7	4.08	B	B	B	B	A	A	A	A	A	B	A	A	A	A	A	A	A	A
2.621	2.0687986	B	B	B	B	A	A	A	A	A	A	A	A	A	A	A	A	A	A
4/100	0.038	A	B	A	A	B	A	B	A	B	A	A	A	A	B	A	A	A	A
4.4502	4.45	A	A	A	A	A		A	B	B	B	A	B	B	B	A	A	A	A
0.457	4/10		A	A		A		A	B	B	B	B	B	B	B	A	A	A	A

3 children could not be classified

Table 5, continued

Items Detecting Malrules

C French Fourth Grade Sample

Item		Student Responses (Grouped According to Rule Used)																																
		Whole Number															Zero				Fraction			Experts										
A	B	1	7	8	10	11	13	16	17	22	30	31	33	35	36	37	5	15	26	29	2	9	34	3	4	12	14	19	20	23	24	27	28	32
4 8	4 63	B	B	B	B	B	B	B	B	B	B	B	B	B	B	B	B	B	B	B	A	A	A	A	A	A	A	A	A	A	A	A	A	A
0 5	0 36	B	B	B	B	B	B	B	B	B	B	B	B	B	B	B	B	B	B	B	A	B	A	A	A	A	A	A	A	A	A	A	A	A
0 25	0 100	B	B	B	B	B	B	B	B	B	B	B	B	B	B	B	B	B	B	B	A	B	A	A	A	A	A	A	A	A	A	A	A	A
4 7	4 08	B	B	B	B	B	B	A	B	B	B	B	B	B	B	B	A	A	A	A	A	A	A	A	B	A	A	A	A	A	A	A	A	A
2 621	2 0687986	B	B	B	B	B	B	B	B	B	B	B	B	B	B	B	A	B	A	A	A	A	A	A	A	A	A	A	A	A	A	A	A	B
4 4502	4 45	A	A	A	A	A	A	A	A	A	A	A	A	A	A	A	A	A	A	A	B	B	B	A	A	A	A	A		A	A	A	A	A

4 children could not be classified.

Table 5, continued

Items Detecting Underlying Rules

D French Fifth Grade Sample

Student Responses (Grouped According to Rule Used)

Item		Whole Number							Zero									Fraction
A	B	2	6	7	12	13	14	36	1	4	8	10	18	21	26	31	34	32
4 8	4 63	B	B	B	B	B	B	B	B	B	B	B	B	B	B	B	A	A
0 5	0 36	B	B	B	B	B	B	B	B	B	B	B	B	B	B	B	B	A
0 25.	0.100	B	B	B	A	B	B	A	A	B	B	B	A	B	B	B	B	A
4 7	4 08	B	B	B	B	B	B	B	A	A	A	A	A	A	A	A	A	A
2 621	2 0687986	B	B	B	B	B	B	B	B	A	B	A	B	A	B	A	A	A
4 4502	4 45	A	A	A	A	A	A	A	A	A	A	A	A	A	A	A	A	

Item		Experts																			
A	B	3	5	9	11	15	16	17	19	20	22	23	24	25	27	28	29	30	33	35	38
4 8	4 63	A	A	A	A	A	A	A	A	A	A	A	A	A	A	A	A	A	A	A	A
0 5	0 36	A	A	A	A	A	A	A	A	A	A	A	A	A	A	A	A	A	A	A	A
0 25.	0.100	A	A	A	A	A	A	A	A	A	A	A	A	A	A	A	A	A	A	A	A
4 7	4 08	A	A	A	A	A	A	A	A	A	B	A	A	A	A	A	A	A	A	A	A
2 621	2 0687986	A	A	A	A	A	A	A	A	A	A	A	A	A	A	A	A	A	B	A	A
4 4502	4 45	A	A	A	A	A	A	A	A	A	A	A	A	A	A	A	A	A	A	A	A

One child could not be classified.

Table 6

Proportion of Underlying Rules Used in the 3 Countries

Sample	Number of Students	Rule Use Category: Whole Number	Zero	Fraction	Experts	Not Classified
United States	17	.35	.00	.18	.18	.29
Israel	21	.19	.14	.33	.19	.14
France.						
4th graders	37	.41	.11	.08	30	.11
5th graders	38	18	24	.03	.53	.03

Table 7

Hidden Numbers Comparison Task · Number of Children in Each Rule Use Category Giving Each Type of Answer (Combined U.S. annd Israeli Samples)

	Rule Use Category		
Answer	Whole Number	Fraction	Experts
Choosing the long string	10	0	0
Choosing the short string	1	18	0
I don't know	13	2	14

Table 8

Zero Insertion Task: Number of Children in Each Rule Use Category Giving Each Response (Combined U.S. and Israeli Samples)

		Rule Use Category			
		Whole Number	Zero	Fraction	Experts
	2.35 < 2.305	8	3	0	0
2.35 compared	2.35 > 2.305	1	0	9	6
with 2.305	2 35 = 2.305	1	0	1	0
	unclear	0	0	0	1
	2.35 < 2.035	6	1	0	0
2.35 compared	2.35 > 2.035	1	2	9	6
with 2.305	2.35 = 2.035	3	0	0	0
	unclear	0	0	1	1

This sample only of U.S and Israel.
The French formulated a different question on this item and, therefore, did not yield information for this table.

Table 9

Ordinary Fraction Notation Frequency of Correct and Incorrect Responses Comparing 1/333 and 1/334; 1/3 and 1/4 (Combined U S. and Israeli Samples)

	Rule Use Category			
	Whole Number	Zero	Fraction	Experts
Correct	5	3	11	9
Incorrect	14	3	7	5

Table 10

Fraction to Decimal Translation for 3/4, 1/5, 3/2, 2/3 (U.S. Sample)

	Rule Use Category	
	Whole Number	Fraction
Numerator encoded	1	6
Denominator encoded	3	0
Syntactic translation	10	0