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SIMILARITY: EXPLORING THE UNDERSTANDING OF A GEOMETRIC CONCEPT

Technical Report
January 1988

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Technical Report
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ABSTRACT

Previous work has identified four areas of difficulty that students seem to have with the topic of similarity: understanding the definition of similarity, proportional reasoning, dimensional growth relationships, and correspondences in right triangle similarity. This paper reports the results of an investigation into high school students' understanding of similarity. A unit addressing three of these difficulties was constructed for use with the GEOMETRIC SUPPOSER. Students were observed as they learned similarity with this unit and were given pretests and posttests on fractions, ratio and proportion, and similarity. From the observations and tests, clarification of these three difficulties will be sought. The resulting greater understanding of student difficulties with similarity will be of use to practitioners and of interest to the mathematics education research community.

The use of technology, specifically the GEOMETRIC SUPPOSER, provides two benefits. First, it supports a pedagogy which seeks to attack directly the students difficulties in understanding similarity. Second, the lab setting allows researchers as well as teachers to examine directly student thought processes in the classroom.

TABLE OF CONTENTS

B. Suggested classroom activities

1. Activities to combat the incorrect additive strategy

2. Helping students recognize and relate different growth relationships

3. Preparing to solve right triangle proportion problems

FIGURES

A3-1. The shapes used in the interview task

A3-2. The construction tasks posed in the interviews

B3-1. A figure like those created by students for computer task #8

B3-2. Two non-similar rectangles that have sides of the same length

TABLES

I. INTRODUCTION

Since the work of Piaget, it has become commonplace to note that children have theories about phenomena before they receive instruction about these phenomena. In recent years the influence of cognitive science has led to much research into children's "naive" theories. Some researchers in the field view these misconceptions as the result of conflicting partial theories that students hold (For an introduction to some of these phenomena in physics, see Clement, 1982, and McCloskey, 1983.).

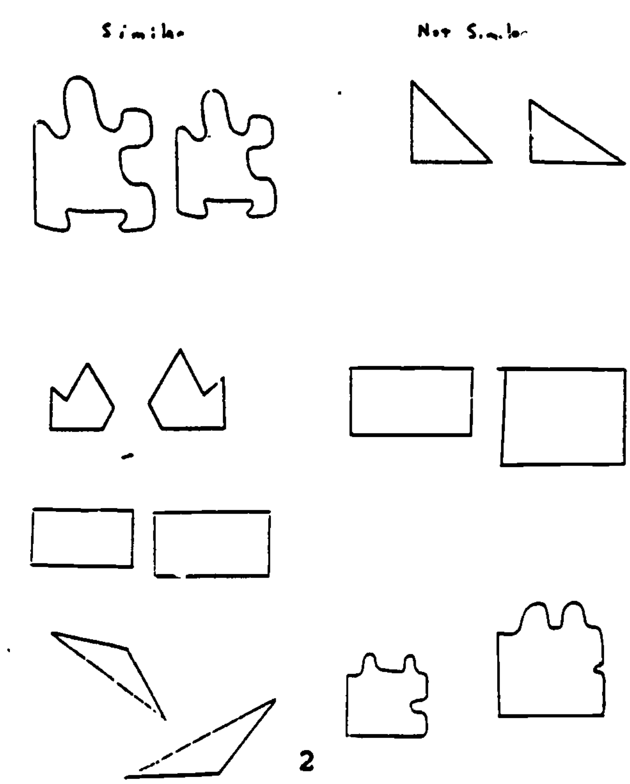
This paper reports the results of an empirical study predicated on the assumption that it is important to investigate students' notions about mathematical concepts. It examines students' difficulties with similarity, a central concept in the high school geometry curriculum, by analyzing students' understanding of proportion and similarity before, during, and after instruction on similarity. Students observed during their instruction on similarity were members of classes that studied the topic by exploring problems using the GEOMETRIC SUPPOSERS. The SUPPOSERS are a set of microcomputer programs designed as a tool to aid students in constructing and exploring geometric figures.

Some of students' difficulties with similarity arise when they bring their everyday experience to high school geometry. Their misconceptions are related to common sense

notions and prior perceptual experience. For example, in geometry, the words "similar" and "similarity" are used differently than in everyday life. In everyday usage, all triangles are similar; they all have three angles and three sides and therefore are not that different one from the other. In geometry, all triangles are not similar.

An additional difficulty is that the standard high school textbook definition of similarity is vague. Similarity is defined as a relationship between two shapes, where the two shapes have the "same shape," yet are not the same size. The ambiguity in the definition resides in the words "same shape." After all, what does it mean for shapes to be the same? Are all right triangles the "same shape," and thus "similar?"

Figure 1
Some similar and non-similar shapes



Other judgments about similarity between figures are even more complicated. For example, some of the pairs of shapes in Figure 1 are similar; others are not, though one might plausibly describe them as being the "same shape." In fact, the definition of "same shape" changes according to the kind of shape one is examining. In high school courses, convex, polygonal shapes are emphasized. A sufficient definition for such cases, and the one which will be used throughout most of this paper, is that, "Two polygons made of line segments are similar if their corresponding angles are congruent and corresponding sides are proportionate."

II. RATIONALE AND PURPOSE

A. Similarity and the Geometric Supposers

Similarity is a key concept taught in high school Euclidean geometry courses. Indeed, similarity relationships between figures may be second only to congruence relationships in their centrality to a typical geometry course.

When teaching with the GEOMETRIC SUPPOSERS, the importance of the concept of similarity rivals that of congruence. Whereas, in a typical course, students' proof exercises mainly involve deducing the results of congruence relationships between figures, in a SUPPOSER course, many of the quadrilateral and circles conjectures that students develop require a knowledge of similarity for their proof.

The impetus for the present study came from the Educational Technology Center's 1985-86 study of the effectiveness of teaching geometry combining both empirical and deductive approaches to the subject using the GEOMETRIC SUPPOSER (Yerushalmy et al 1987; See Yerushalmy, et al., 1986, for a preliminary report.). One focus of the study was students' understanding of geometric concepts.

Teachers reported that students seemed to have little difficulty with the concepts presented in most parts of the course. Of the difficult topics, similarity was the most

difficult for their students to understand. Moreover, students' empirical exploration with the GEOMETRIC SUPPOSER seemed to do little to build their understanding of similarity (Yerushalmy, et al., 1987). Thus, in keeping with the Educational Technology Center's (ETC's) orientation towards "Targets of Difficulty" as "Windows of Opportunity," students' understanding of similarity was chosen for exploration.

Experiences in the 1985-86 classrooms led to a characterization of students' initial conceptions about similarity as well as the difficulties they experienced in learning about the topic. With no presumption of completeness, four areas of difficulty with similarity were identified:

(1) understanding the definition of similarity,
(2) proportions in enlargement,
(3) dimensional growth relationships, and
(4) proportions in right triangles.

All four of these issues are addressed during a typical high school geometry similarity unit. Each of them has been observed in previous research to be an obstacle for high school students. Therefore, it appeared likely that observing students as they study similarity in their high school classes would offer an opportunity to gain new insights into these difficulties.

This belief provided the impetus for the creation of a unit of similarity tasks for use with the GEOMETRIC SUPPOSER. The unit asks students to explore constructions involving

similarity and, where possible, tries to force a confrontation between student conceptions and contrary evidence.

This unit capitalizes on opportunities provided by a method of instruction which the GEOMETRIC SUPPOSER supports. First, the microcomputer tool supports conflict teaching. This kind of pedagogy attacks students' difficulties in understanding by confronting them with tasks that bring forth these difficulties. Second, the fact that students work in pairs in the lab setting allows researchers as well as teachers to observe students at work doing mathematics, revealing their thought processes through discussion.

The primary purpose of the present study is to investigate the three student difficulties with similarity that are not directly related to the definition of similarity (proportions in enlargement, dimensional growth relationships, and proportions in right triangles). The fourth issue, students' recognition of similar figures or their understanding of the definition of similarity, is well-suited for comparison of regular and SUPPOSER teaching interventions. It will be examined by another Educational Technology Center study, which compares SUPPOSER and non-SUPPOSER classes.

A secondary purpose of this study is to provide support for the belief that observations in a classroom computer lab setting allow for a rich understanding of what students are

thinking. If this is true, then given the appropriate microcomputer tools, some research which is now done in clinical interview settings may be accomplished <u>in situ</u>, during regular classroom computer lab sessions. Results of such research may be more practical for teachers and may more easily traverse the chasm between theoretical research and everyday practice.

The following section outlines in greater depth the three areas of difficulty that emerged from the 1985-86 study. Specific questions are then posed for study in each of these areas of difficulty. In the following "Methods" section, the instructional intervention and the data collection methods are described. Specific descriptions of the data analysis methodology for each area of difficulty are presented in the "Results and Discussion" section.

B. Three difficulties in learning about similarity

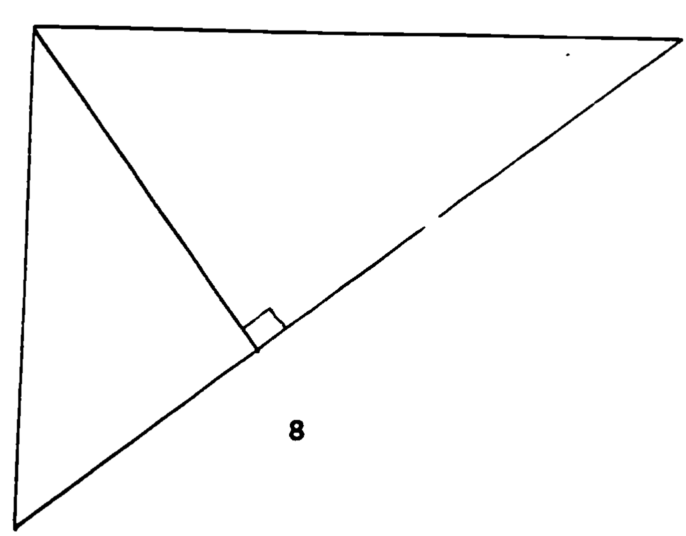
The most basic difficulty that students experience in studying similarity involves a crucial aspect of similarity, ratio and proportion. For example, in the 1985-86 ETC study, students were given a production task--produce similar triangles by extending two of the sides. They seemed to think that by extending the sides of a triangle by equal lengths they would always get a similar triangle. This conception was very resistant to change and appears to be

related to similar additive strategies exhibited by students on ratio and enlargement tasks (Piaget and Inhelder, 1967; Karplus, et al., 1975; Hart 1984).

A second difficulty with the application of the concept of similarity indicated in the literature involves an understanding of the relationship of area growth in similar plane figures and the generalized problem of dimensional growth in higher dimensional similar objects. This difficulty is not with the definition of similarity itself, rather it is with a consequence which can be derived from that definition. Students confound similarity with dimension; they are surprised to see that the enclosed area does not grow in the same ratio as the lengths of the sides. Extant research on this difficulty focuses on junior high school students' understanding of the relationship between linear and area growth (Friedlander et al., unpublished).

Finally, students have difficulties with the mean proportional relationships found in right triangles with an altitude drawn from the right angle vertex (See Figure 2).

Figure 2
The difficult altitude in a right triangle configuration



Students experience difficulties solving problems that demand the correct identification of the correspondences and proportions among the segments involved. Such problems involve two steps. First one must understand which figures are similar, then one must identify the correct correspondence in order to choose the correct proportions.

C. Research questions

Specifically, this paper will describe findings on the following questions about the origins and instructional remediation of the three above difficulties.

1. Is students' preference for incorrect additive versus multiplicative strategies in a production task which asks them to creace similar figures a replicable phenomenon? If so, does instruction directed at this misconception improve the situation?

   Is this incorrect strategy a misunderstanding of ratios or is it a geometric difficulty? If the misunderstanding is a geometrical one, are there geometrical explanations for the types of mistakes students make?

2. When working on their own, without the aid of a teacher, are high school students able to formalize the relationship between the rates at which areas and sides grow in similar triangles? If not, how do teachers cope with this situation?

   At the end of their studies of similarity, can students recognize linear, area, and volume growth relationships in similar solids? If so, how do they explain the difference between the numerical values of these relationships? If not, are there any clues as to why students miss some of the descriptions?

3. Is students' ability to recognize similar triangles dependent on the fact that the similar

triangles are rotated, flipped, or overlapping? Do students understand that flipping two-dimensional figures does not change the similarity relationship between the figures?

In the case of the right triangle problem, do students' difficulties stem from the fact that the same segment, the altitude, is simultaneously the small leg of one triangle, the large leg of another triangle, and the altitude of a third, that in mean proportions the length of the same segment appears in the numerator of one fraction and in the denominator of another?

Might other aspects of the problem also explain its difficulty?

III. METHODS

In order to explore these questions and other questions about the effectiveness of GEOMETRIC SUPPOSER use, members of the Educational Technology Center's Geometry Research Group constructed a similarity unit for use with the GEOMETRIC SUPPOSER and observed two classes in each of two schools as students studied similarity using these lessons. One school was in an urban setting and the other was in a suburban community. In the urban school, the two classes were untracked classes which were taught by the same teacher. In the suburban school, the classes were for juniors in the average college-bound track. Members of the research group also identified a comparable class in each school (as judged by the school administration) which did not use the GEOMETRIC SUPPOSER. These students were also observed as they studied similarity.

It is important to note that this study does not compare students who used the GEOMETRIC SUPPOSER with those who did not or students in the urban school with students in the suburban school. Instead this study focusses on these students' understanding of three difficulties aspects of similarity.

A. The Unit

The similarity unit used in these schools consisted of

eight tasks for use with the GEOMETRIC SUPPOSER in a computer lab during class time. Over a three to four week period, students did the lab tasks while also learning about similarity in their regular classroom through discussions and teacher presentations. The computer tasks were added to the structure of the teacher's existing similarity unit (See Appendix A1 for a copy of the computer tasks.).

The curriculum in the suburban school system where this study was conducted integrates the study of algebra and geometry. Thus, the classes in this setting spent more time working on similarity and proportions than the urban students. The experimental classes in the suburban location had no prior experience with the SUPPOSER. They covered all eight tasks in a five and one-half week similarity unit (including some introductory work to familiarize them with the SUPPOSER). The suburban comparison class's similarity unit was four weeks long.

The experimental classes in the urban location had used the SUPPOSER from September until March prior to beginning the similarity unit. They did seven of the tasks in three and one-half weeks, while the comparison class studied similarity for two and one-half weeks. The length of these units represents the amount of time spent introducing similarity. Similarity is then a concept which is used throughout the remainder of the course.

B. Data Collection

1. Observations

Observers from the research team visited all six classes while the topic of similarity was taught. Observations of the experimental classes included sessions that took place both in the regular classrooms and in the lab setting. After each classroom observation, observers organized their comments by outlining the objective of the lesson (after consultation with the teacher), the flow of the session, issues of student understanding that arose during the session, and pedagogical or classroom interaction issues that arose.

2. Pencil and paper sources

Students' computer assignments were collected from the experimental classes. Since three of the tasks were explicitly designed to investigate the three misconceptions, these papers were especially valuable.

In addition to collecting students' computer assignments, students were given pre- and post-tests. The tests were the same except for questions added to the posttest (numbers 7, 8, 9, and 10) on students understanding of similarity (See Appendix A2). The pre-test provided a baseline of the students' ability to manipulate fractions and indicated whether they used an additive strategy on ratio tasks. The post-test used the same questions to assess

students' fraction ability and ratio and proportion skills. It was examined for changes in the use of additive strategies, improvement in ratio and proportions skills, as well as understanding of different aspects of similarity.

Each of the tests was designed to be given during a classroom period. The fractions and ratio and proportion sections were taken from the Chelsea Math Series tests and were graded on a four-level scale designed by Kathleen Hart (See Hart, 1980). The similarity section was designed for this study. The scoring scheme for this section will be described in an upcoming ETC paper (Chazan, in preparation). It is an incidence scoring scheme which tallies how many different relationships between similar figures appear in students' response to a set of figures. Such relationships include: explicit mention of similarity, congruent angles, proportionate sides, correct correspondence between vertices, parallelism of sides (where applicable), and statement of area relationships (all the figures were two-dimensional).

For the current study, the key results of the pre- and post-test are the identification of the strategies that students use to solve problems and the differences between student responses to different questions. Chi-square and odds ratio statistics are used to compare students' work on different problems.

3. <u>Interviews</u>

Members of the research group also conducted taped

interviews with four students from each of the six classes after they completed their study of similarity. With the advice of the teachers, group members chose the students to represent a range of student ability: an able student, a student who was having difficulties, a quiet student and a student whose comments in class suggested that he/she would be interesting to interview. The interviews were one hour long and included five tasks which focused on the three areas of difficulty outlined above. The interview tasks are not typical school problems, yet the knowledge of similarity taught in school is helpful in devising solutions (See Appendix A3 for a description of the interview tasks.).

IV. RESULTS AND DISCUSSION

This section will address each area of difficulty separately. The discussion of each difficulty begins with a quick review of previous research followed by the findings of the study. These findings are reported by data source, however the order of the reporting is different for each difficulty.

A. Additive versus multiplicative strategies

1. Prior research

Hart's (1984) investigation of additive strategies in enlargement problems begins with an observation from the "Concepts in Secondary Mathematics and Science" investigation of ratio. According to that study, about one-third of the interviewed fifteen-year-olds (10th graders) used incorrect additive strategies on enlargement problems (p.74).

Table 1
Incidence of additive strategy in 1976 CSMS data

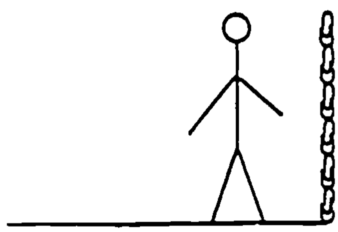
Age (year group)	*Incidence of error answer Questions*[a]				
	1	2	3a	3b	
13(2)	51.4 (28.1)[b]	47.6 (7.9)	43.6 (13.7)	32.3 (15.4)	per cent
14(3)	50.6 (29.6)	39.4 (11.0)	40.3 (19.7)	29.3 (20.5)	
15(4)	39.1 (42.0)	39.7 (19.7)	34.2 (28.7)	25.0 (28.8)	

[a] Number 3 has two parts, hence there are four questions altogether.

[b] The percentage frequency of the correct answer is shown in brackets.

Figure 3
Enlargement questions on Hart's test

1) You can see the height of Mr Short measured with paperclips.



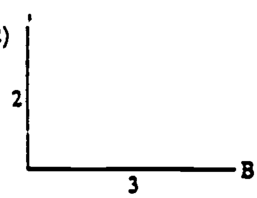
Mr Short has a friend Mr Tall. When we measure their heights with matchsticks

Mr Short's height is four matchsticks
Mr Tall's height is six matchsticks.

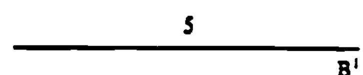
How many paperclips are needed for Mr Tall's height? ....

Error Answer: 8.

2)



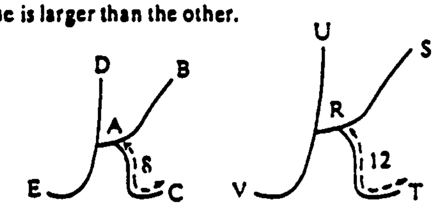
Work out how long the missing line should be if this diagram is to be the same shape but bigger than the one on the left.



Error Answer: 4.

3a) These 2 letters are the same shape, one is larger than the other.

AC is 8 units. RT is 12 units.



a) The curve AB is 9 units. How long is the curve RS?

Error Answer: 13.

b) The curve UV is 18 units. How long is the curve DE?

Error Answer: 14.

Her study focuses specifically on enlargement problems that require computation, because "it had been felt that enlargement items which could be solved by the use of a drawing technique using centers of enlargement did not sufficiently test understanding so ιl these set items required some computation" (p.20).

Hart's interview work with enlargement problems led her to isolate a group which she calls the "adders," ₋hose who use incorrect additive strategies on enlargement tasks. "The adders:

> 1) Replace multiplication by repeated addition on 'easy questions'.
> 2) Add a fixed difference on enlargement questions which have a greater complexity or involve ratios other than n:1 or n:2.
> 3) Never multiply by a fraction.
> 4) Can be made aware of their [illegible] e., are aware of distortion in figures, [illegible] little idea how to replace the strategy [illegible]at led to the error." (Hart, 1984, p.33)

After interviewing students, Hart constructed a teaching module to alleviate this particular student misconception. The teaching module made a direct assault on the misconception.

> The steps to be taken...were designated as follows:
>
> 1) The incorrect addition strategy should be seen by the child to be incorrect...(p.34)

This is a "Diagnostic Teaching" approach as advocated by Bell (1986).

The study reported in this paper follows in Hart's footsteps by attempting to replicate Hart's findings and to

determine whether the incorrect additive strategy is a ratio or geometric difficulty.

2. Test data

Insights into students strategies on enlargement tasks can be gleaned from the ratio and proportion part of the pre- and post-tests (See items 1-6 and 11-17 in Appendix A2.) which was taken from Hart's SESM test. The ratio portion of the tests was given to all six classes and was scored using a four-level system designed by Hart (1984). This system includes a procedure for coding incorrect student responses. Certain incorrect responses, such as adding the same amount to the numerator and denominator of a fraction to get an equivalent fraction, indicate an incorrect additive strategy. Although one cannot ascertain from these responses what a child was thinking, they suggest that the child used an incorrect additive strategy. Most students who give such a response on one enlargement question give analogous responses on other enlargement problems.

The test data provide a record of the number of students who seem to use incorrect additive strategies. These data also show whether any students change from additive strategies on the pretest to correct strategies on the post-test. Such changes would suggest that the instruction was effective at combatting this misconception.

On the ratio pretest, the two urban experimental groups

had a similar number of "adders"[1]; the urban comparison class had slightly fewer (See Table 2.).

In the two suburban experimental classes, fewer students added than in the corresponding urban classrooms. In this case, the stronger SUPPOSER class had fewer "adders" than the weaker class (see Table 2). The number of "adders" in the comparison class was comparable to the number in the weak experimental class. Thus, each of the six classes had students who used an incorrect additive strategy on the enlargement problems.

Table 2
Pre- and post-test number of adders by class

Class	Urban			Suburban		
		"adders"			"adders"	
	N	pre	post	N	pre	post
Strong experimental	14	10	2	22	2	0
Weak experimental	18	12	4	21	7	2
Comparison	10	5	1	24	6	1

The post-test results corroborate Hart's findings (p.62). Instruction in both the experimental and comparison classes produced an immediate change in students' strategies. It is unclear, of course, whether these changes would persist

1 An "adder" is a student who provides a response to any question on the ratio and proportion problems that indicates an additive strategy. None of the students in the sample gave only one such response. So in fact all of the "adders" gave responses of this type on more than one problem. See page 22 for an indication of which problems elicited additive strategies.

if students were interviewed at some later time.

Hart's work does not address the issue of the relative effectiveness of her intervention. Her teaching module provides instruction only on enlargement problems (p.34). Its efficacy was not compared to other approaches. Bell (1986), on the other hand, has data on the effectiveness of major features of his diagnostic teaching methodology. He did three experiments.

> The first of these showed the superiority of a 'conflict' as against a 'positive-only' approach to the teaching of decimal place value... The second experiment showed that, of seven classes using similar teaching material but with varying degrees of conflict discussion, the more vigorous and intensive discussions were associated with greater progress. The third experiment showed greater learning in seven diagnostically taught classes compared with two taught by 'exposition for understanding.'"(p.332)

The similarity unit used in the study reported here also mounts a frontal attack on students' additive misconceptions. The post-test data suggests that this strategy was not significantly more effective than regular instruction. In both urban and suburban settings, a comparison of the number of students using an additive strategy on the pretest and on the post-test indicates no significant difference between the experimental and comparison group. However, in the urban setting, there was a significant difference between the SUPPOSER and comparison class "adders" on the posttest. Eighteen of the twenty three SUPPOSER classes adders used the additive strategy on fewer questions on the posttest than on

the pretest, while none of the comparison class adders used the additive strategy any less frequently on the posttest than the pretest. Since the number of "adders" was so small, further study will be necessary to corroborate this observation.

The test data from the current study also indicate which problems elicited an additive strategy, which may indicate whether the additive strategy represents a geometric or numerical misunderstanding. All of the additive strategy errors were either in Hart's enlargement problems, Piaget's sprats and eels problem or Karplus's Mr. Short and Mr. Tall (see Karplus and Karplus, 1975; Piaget, 1967). All of these problems are enlargement problems with a geometric context.

3. Classroom observations and students' papers

In earlier research by our group, we observed that students used incorrect additive sxtrategies on certain production tasks, consistent with the experience of other researchers cited on pp____above. In early 1986 we observed these errors on a task which asked students to construct similar triangles. A group of students in one of the classes using the SUPPOSER which was studied by the Educational Technology Center developed the notion of "rescaled" triangles, triangles that have the same angles but are different sizes. The name seemed to derive from the "Scale Change" option in the SUPPOSER. The students were convinced that in order to get the rescaled version from the smaller

version, one should add a set amount to the length of each side. They stuck to this opinion in the face of contradictory evidence (See Yerushalmy, et al., 1987.). Based on this experience, we expected that students would have difficulty with construction tasks, that they would use inappropriate additive methods. Our teaching activities were designed to illuminate further the nature and sources of studoents' difficulties in this area.

To our surprise, the particular phenomenon described above was not recreated in any of the experimental classes in the study here described. When asked to explain how one could create a rescaled version of a triangle, students used proportions in their explanations. However, evidence from one of the computer tasks in the unit suggests that the students preference for multiplicative strategies was not strongly rooted. Students had little difficulty recognizing similarity in the figures that they constructed, however they did have a hard time figuring out appropriate methods of construction. They did sometimes use additive strategies. We now turn to that problem (See Figure 4), students' responses, and our analysis of these.

One of the principles behind Hart's teaching module was that "The child (adder) needs to: realize that the incorrect addition strategy produces a distorted figure..." (p.32). The unit in which this problem occurs uses a similar strategy: students had a computer assignment whose point was

that an incorrect addition strategy does not create a similar figure.

Figure 4

Computer Task #4 from the similarity unit-The choice version.

Draw a scalene triangle ABC. Extend sides AB and AC, from B and C respectively, in such a way that the triangle formed by connecting points D and E is similar to the original triangle. (Editor's note: The two triangles should have angle A in common.) Make a sketch of your first attempt..

Diagram:

Determine whether or not your triangles are similar.

If your triangles are not similar, describe how you attempted to do it and then try again.

In this task, students were asked to create a triangle similar to a given triangle, by extending two of its sides. Students were observed as they carried out the lab assignment. The observers looked for incorrect additive strategies as opposed to multiplicative strategies. Did a significant number of students use additive strategies? When they tried an additive strategy were they surprised that it did not work? Did they change to a multiplicative strategy, or were they perplexed?

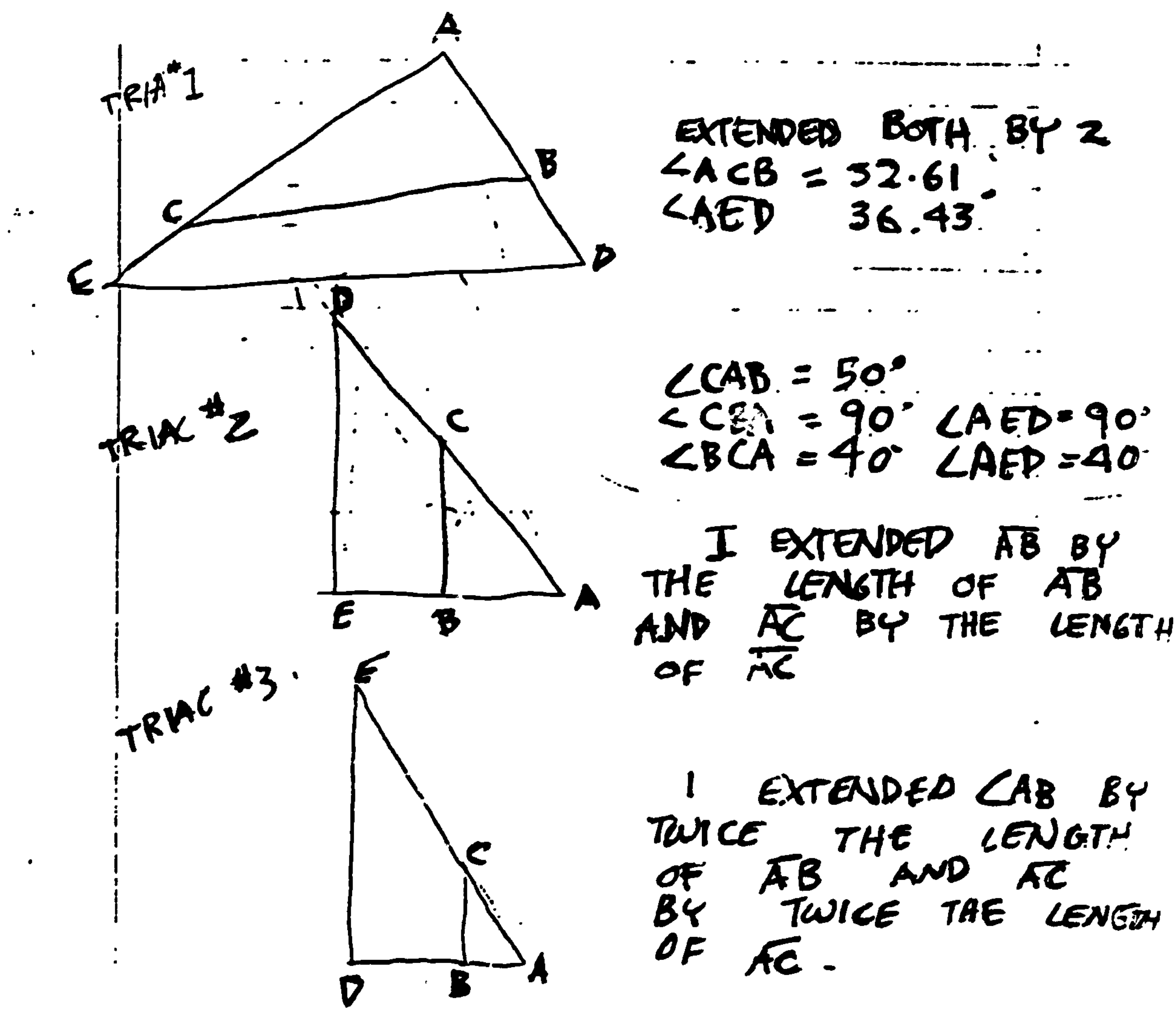
Students' lab papers were collected. From these papers, it was possible to reconstruct the strategies used by students on this task. Along with the direct observations described above, these papers helped to identify whether

students were using incorrect additive strategies and whether such usage was common.

Observers also attended class discussions of this problem in which students tried to justify their construction strategies. In addition to documenting these justifications, the observers looked both at conflicts between students who used different strategies and at students' reasoning as they attempted to resolve these conflicts. Finally, students' comments and rationales on their lab papers were also examined to understand how the students viewed the problem.

In the urban experimental classes, students were first given the problem as stated above. They had the option of choosing additive or multiplicative strategies. The two classes had different experiences and made different choices. In the strong class, where this task was given the same day that students investigated figures whose sides were proportionate, every student but one chose a multiplicative strategy. In the weak class, where a weekend intervened between the Side-Side-Side activity and the task of extending the two sides, eight out of eleven pairs tried incorrect additive strategies. Thus, on their lab papers, they first recorded incorrect additive attempts and then multiplicative attempts.

Figure 5
The additive strategy
in the work of one student from the weak, urban class.



On the next day, the teacher followed up the two-option task with a version of the same task that did not allow choice (See Figure 6). In this version, a student is asked to investigate an incorrect additive strategy and then a multiplicative strategy.

Figure 6
Computer task #4--The no-choice version

I Construct triangle ABC such that AB = 3, AC = 3 and BC = 3. Extend side BC from C two units. Similarly, extend BA from A two units. Connect points D and E to get a new diagram. Copy the diagram below and state any observations that you have.

II Use the REPEAT key to repeat the above constructions on a new triangle ABC in which AB = 3, AC = 5, and BC = 4. Copy the diagram and state observations.

III Use the repeat key to again copy those constructions on another new triangle ABC. This time AB = 3,3AC = 40° and BC = 3.2. Copy the diagram and state observations.

IV Use the REPEAT key one more time and construct triangle ABC with AB = 2, AC = 4, and BC = 3. Copy the diagram and state observations.

V Using the same diagram that you used in IV, extend BC from C the length of BC. Likewise, extend BA from A the length of BA. Connect the points F and G to get yet a new triangle. Copy the diagram and state your observations.

VI Use the REPEAT key on a previous triangle. Copy the diagram and state observations.

VII Use the REPEAT key on another previous triangle. Copy the diagram and state observations.

VIII Use the REPEAT traingle on the last of the previous triangles. Copy the diagram and state observations.

What was this project about?

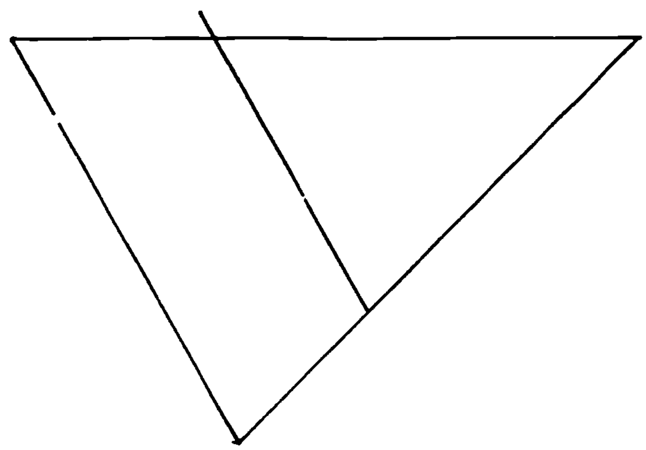
The strong class, the class that previously had used multiplicative strategies, now expressed surprise that an incorrect additive strategy would not work. It took half the period for the students to connect this phenomenon to their previous assignment. Their performance on the previous version had been buoyed by the Side-Side-Side activity. In the weak or "additive" class, the students immediately explained that the activity was the same as the activity that they had done earlier.

In the suburban classes, students were not given choices; they were told first to construct additively and then multiplicatively. On the additive portion, in both the strong and weak experimental classes, many students didn't bother measuring to check that the triangles were similar or that the lines were parallel. They were positive that an additive strategy would yield similar figures and parallel lines. Only when they tried a multiplicative strategy on the same figure did they feel the need to go back and make measurements. They expressed consternation when they found that extending by equal lengths did not necessarily yield parallel lines.

At this point it is appropriate to make a short analytic excursion and examine the role of the geometrical configuration in this problem before returning to classroom observations to examine this issue empirically. It is interesting to note that the configuration of the shapes in

this construction task (See Figure 7.) seems to play an important role in students' behavior. It suggested incorrect additive strategies even to those students who showed an understanding of multiplicative ratio. Even students who had used multiplication one day were stymied when asked to add the next day.

Figure 7
The configuration which suggested additive strategies

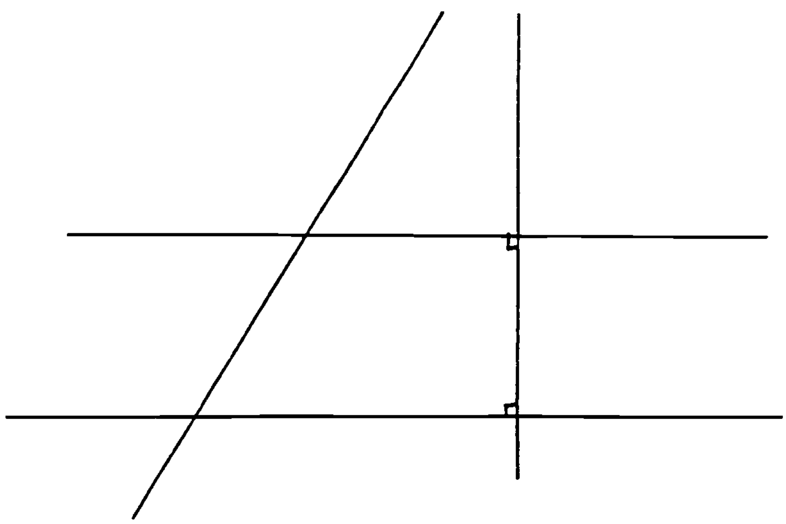


What about this configuration suggested an incorrect additive understanding? From later observations in the urban classes it was clear that students in these classes did not seem to have a strong notion of distance. They were not aware that the distance between two parallel lines is measured by the length of the shortest segment between the two lines, the perpendicular. In fact, it was as if they believed that "since parallel lines are equidistant, any segments drawn between two parallel lines are the same length." According to this view, in order to construct a line parallel to one side of a triangle, one extends the

other two sides by equal lengths.

The connection between the difficulty with this configuration and students' difficulties with the concept of distance was supported by an observation in one of the suburban classes. Students were hotly debating whether additive or multiplicative strategies would work. Neither camp had convinced the whole class that their strategy was correct. One student then asked to come to the board and drew the following:

Figure 8
A visual argument against the incorrect additive strategy



This student's argument was that the perpendicular distance is the shortest distance and that other distances are longer. Therefore, he was showing a case where the incorrect additive strategy would not work at all. If one used an equal length, one wouldn't get a triangle. The two segments of equal length would have to be both perpendicular to the parallel lines and thus would be parallel to each other and would not form a triangle.

4. <u>Interview tasks</u>

A generalization of the incorrect additive strategy came to light during an interview task on recognizing similar shapes[2] where some of the shapes were concave shapes (See Appendix A3 for a description of the task).[3] The two shapes in Figure 9a were created on a xerox machine and are similar, however one of the students interviewed in the suburban setting declared that they were not.

His argument was that since the smaller shape didn't fit inside the larger shape (it overlapped), then the shapes could not possibly be similar (See Figure 9b). Further probing elicited his idea of similarity. The whole shape should be moved in an equal distance. "See if I take a pencil and start drawing the same basic outline getting smaller and smaller, I would try to fit this in and see if it would fit the basic area around where I would draw (See Figure 10a)." This definition works in convex shapes, but not in concave shapes.

---

2 Additive strategies were also exhibited by some students on another interview task when they were asked to produce a triangle similar to a given triangle (See Appendix 1C for complete details of the task). Four of the twenty four interviewed students used additive strategies, two recognized their error and two did not.

3 With such shapes, the definition of similarity is more complicated. Without exploring this definition, one way to produce similar shapes of this type is to use a xerox machine to enlarge or reduce a shape.

Figure 9a
Two similar figures which were judged not similar

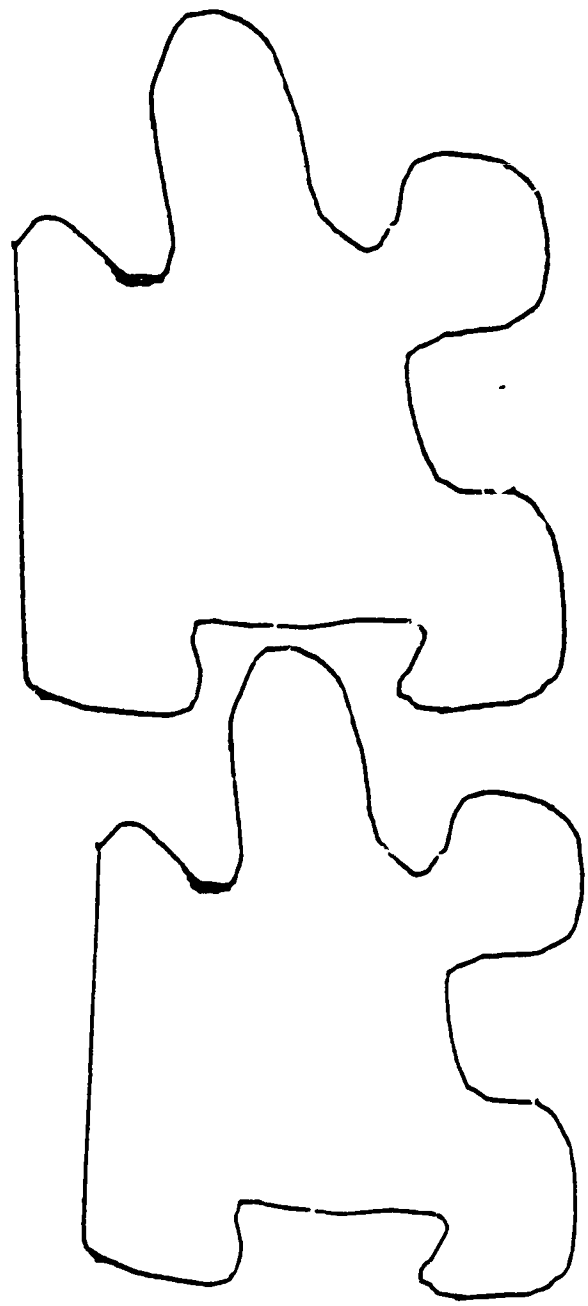


Figure 9b
The two similar figures from 9a superimposed

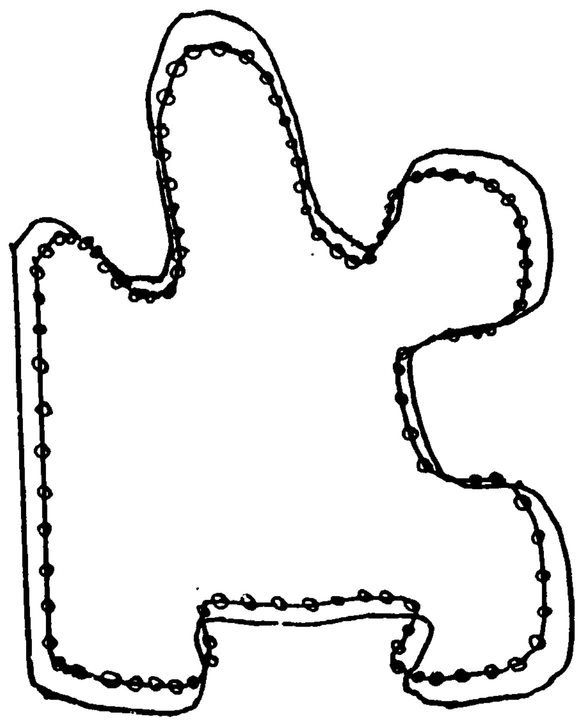


Figure 10a
One student's idea of how to create similar figures

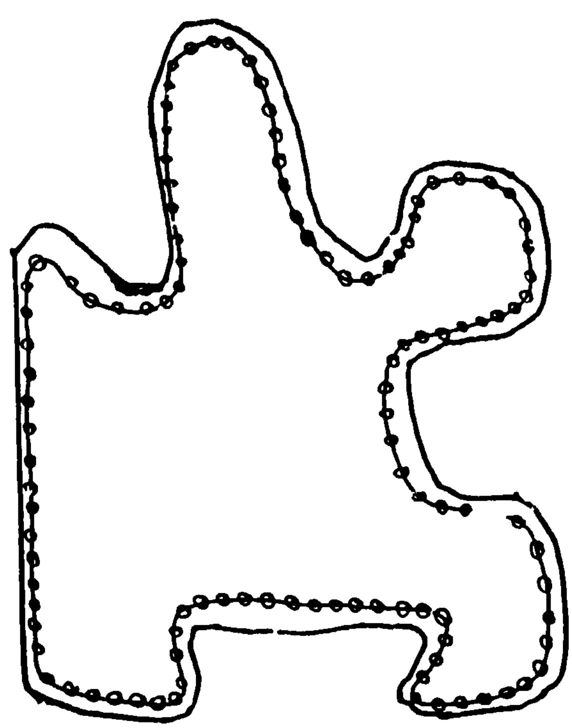
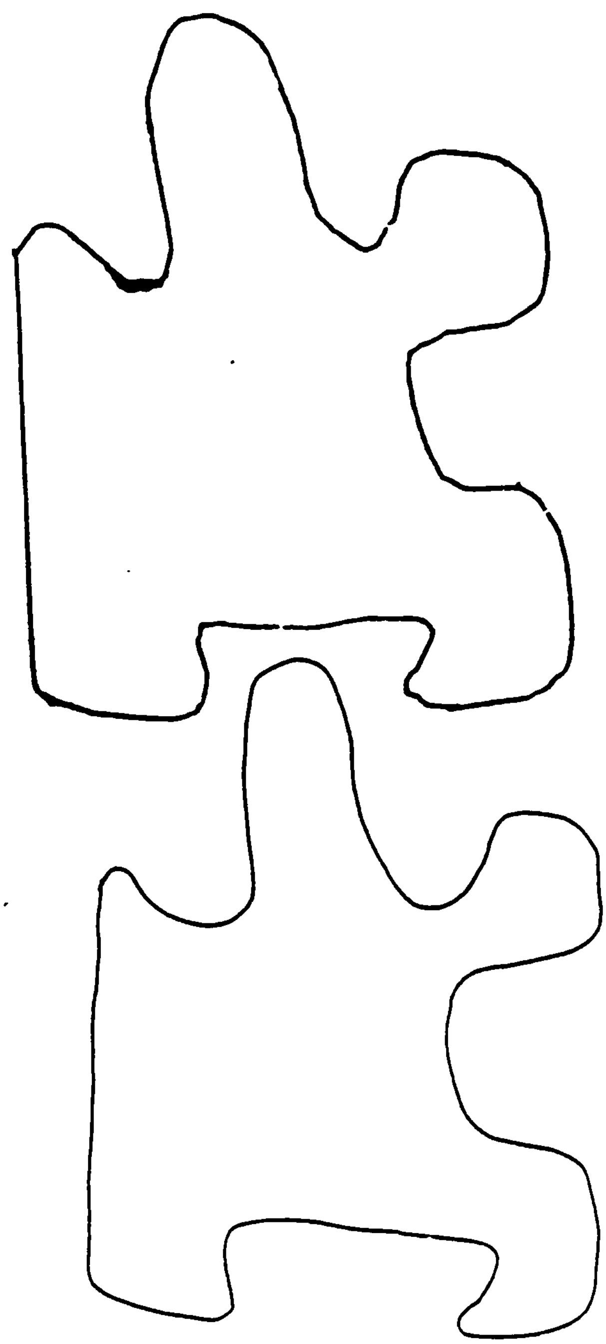


Figure 10b
The two shapes in Figure 10a drawn separately



A student from the suburban school who was interviewed earlier had argued that the two shapes in Figure 9 were similar. He argued that as a shape grows proportionally smaller, the holes have to get smaller, which causes the parts that jut out to be drawn in. Thus, he concluded correctly that the shapes were similar even though they overlapped.

To investigate similarity in concave shapes further, two new shapes were added to the set of shapes to be classified by the students from the urban school. The two shapes were created from the two figures in Figure 10a. Geometrically, these two shapes are not similar (Notice that these shapes do not look similar when separated. See Figure 10b).

These shapes were created after 12 students from the suburban setting had already been interviewed. Twelve students from the urban setting received these two new pieces along with the other pieces previously used in this interview task (including those in Figure 9a). Six of the twelve students classified the shapes correctly. One explicitly said that the concave shapes show that placing one shape in the center of the other is not a good strategy for demonstrating similarity. He said, "What I said (put one shape in the center of the other) was wrong or it is only true in certain situations." Four of these six students explicitly used proportional reasoning to explain that the ratio between the size of the cavities and the knobs had to

be kept the same. Of the six remaining students, four kept all of the curvilinear, concave shapes together arguing that they had the same number of cavities and knobs. The other two grouped these shapes into two piles by size. Thus, most students did not seem to be attracted to the generalized additive strategy in a recognition task.

5. Summary

The phenomenon of using incorrect additive strategies to construct similar triangles noted last year was replicated, though in a different task. When asked if the shapes which result from the incorrect additive strategy are similar, students do not think that these shapes are similar; however, they expect these strategies to work and are surprised to see them fail. Thus, this misconception seems to be linked to production tasks, not recognition tasks.

Both direct teaching and indirect teaching about similarity are effective in changing some students' strategies, although some students stick with an incorrect strategy even after direct teaching.

The data from this study supports Hart's suggestion that this incorrect additive misconception is restricted to enlargement contexts (p.75). However, in contrast to Hart's study, even those students who show an understanding of multiplicative ratio may exhibit that understanding with certain geometric configurations and not with others. For example, the no-choice variant of the problem of extending

two sides of a triangle to make a similar triangle encouraged additive understandings of proportions even among those students who had already used multiplicative understandings of proportions on another variant of the same task. Therefore, it is important to look for aspects of the geometrical configuration in a problem that lead students to use an additive strategy. In this particular construction, a misunderstanding about parallelism may be partially responsible for the reversion to an additive strategy.

B. Side, area, and volume growth relationships in similar solids

1. Prior research

Many phenomena in the world around us are determined by the interplay between the growth relationships that obtain in different dimensions. This interplay explains why the largest animals in the world live in the ocean and why trees need leaves. Understanding the way lengths, areas, and volumes vary as a shape grows helps one understand why Gulliver's travels could never have happened (See PSSC, 1960) and why sculptors must be careful when making stone enlargements of their clay models.

Researchers at Michigan State University (Friedlander, Lappan. and Fitzgerald, unpublished) have studied the difficulties that students have in acquiring an understanding of this growth concept. Most of their work was with middle

school students. Following the suggestions of Lunzer (1968, 1973), they suggest that isolating area growth relationships from perimeter growth relationships may require formal operational capabilities.

They taught similarity concepts using what would later become two separate units. One of the units, The Mouse and the Elephant, investigates what would happen to a mouse that grows to be the size of an elephant. The second unit focuses on Similarity and Equivalent Fractions.

After instruction, these researchers found no gain in understanding area growth relationships among sixth graders and some gain for seventh and eighth graders (Friedlander et al, unpublished, p. 17). The results for sixth graders correspond
to previous results reported by Fitzgerald and Shroyer (1979).

The current study does not use a Piagetian framework, and no claims are made about students stages of reasoning. Instead, failure to distinguish the growth relationships for attributes of different dimensionality, such as area and length, is treated as a misconception unto itself. Since this study focuses on older students, in addition to the area growth relationship, this study investigates a generalization of this relationship, the volume growth relationship.

2. Classroom observations and student papers

The experimental classes were given one lab exercise involving area growth in triangles which asked about the

ratios between sides, angles, perimeters, altitudes, and areas of two similar triangles. From observations of the classes' work in the lab and in associated class discussions and from students' papers, it is possible to determine whether some students arrive at an understanding of dimensional growth in triangles without the aid of the teacher. If some do, what portion of the students are able to do so? Observations of class discussions indicate how teachers dealt with students who did not discover this relationship on their own.

In the four experimental classes, very few students discovered the relationship on their own prior to instruction. None of the students constructed a proof for the relationship on their own.

In the suburban setting, one pair in the strong class concluded that "since area is two-dimensional, the ratio of the areas would have to be squared." After a question from an observer, they added, "Three dimensional would be cubed." Most students remarked that area behaves differently than the rest of the ratios, but could go no further.

When this problem was discussed in the strong class, the teacher referred to the pair's conjecture. The other students were intrigued. The student responsible for the conjecture suggested that the idea came from "squares." The teacher then ended the class session by providing students with the formula for the area of a triangle and then proving

that the ratio of the areas of similar triangles is the zoom factor (ratio of similitude) squared.

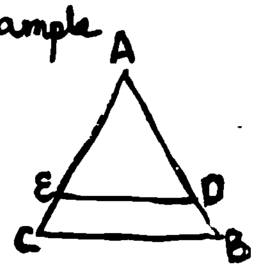
In the weaker class, no student could explain the area ratio, but in the class discussion one student suggested that there was a difference between area and perimeter relationships because "area involves multiplication of sides, perimeter is adding of segments." Again, the teacher reminded students of the area formula for triangles. Then by working with an example, the teacher helped students realize the exact relationship. This was an "aha!" experience for the class.

Thus in each case, the teacher in the suburban class realized that the key piece of information that students needed in order to make a conjecture about the area growth relationship in triangles was the area formula and it was she who introduced this formula to the class.

Figure 11
An attempt to formalize the triangle area growth relationship

I think this is true because the area is dealing with a whole region, where as the others deal with lines.

Example



AB is app. 1/4 the size of AD.
However △ABC is greater than 1/4 the size of △ADE

In the urban setting, only one pair of students in the strong class came close to formalizing the triangle area growth relationship in the lab. They came to an incorrect formalization which, however, clearly indicates that they had some inkling of what the relationship might be (figure 11).

The remaining students were not able to generate this relationship on their own in the lab. They all observed that the area ratio was different from the ratio between the sides, but had no description of what the relationship was. In class the next day, one student from the weak class noted that though he didn't know the relationship in triangles, he had tried it with squares. When the ratio of sides was 1 to 5, the ratio of areas was 1 to 25. He concluded that it had something to do with "the power of a side."

The teacher in the urban setting took a different approach to leading his students to the triangle area relationship. He gave the students another lab assignment. This time students were asked to surround a triangle with the rectangle of smallest area that would still contain the triangle. From working on this problem, students were able to generate the traditional area formula. After this lab problem, one student suggested that if you divide the ratio of the areas by the ratio of the sides you get the ratio of the sides. Another student formulated the conjecture in a more typical manner. The teacher was satisfied with an informal proof of these statements.

3. Interview Tasks

At the end of the unit, one of the interview tasks focussed on students' ability to generalize the area growth relationship to higher dimensions. The task involves two similar rectangular solids. The dimensions of one of the solids were twice the dimensions of the other. Students were asked to describe how much larger the large solid was than the small solid.

Table 3
Number of students giving different responses to blocks task

Response		(N=24) Number of students
Additive strategy		1
Incorrect ratios only		1
One correct ratio-volume 8:1		5
One correct ratio-volume 8:1, but aware of sides 2:1		2
Two correct ratios	sides and volume	7
	surface area and volume	3
Three correct ratios		5

The results of this interview task in Table 3 document the number of relationships that students observe between the two blocks. They indicate students' explanations for these relationships, whether any of the relationships are additive, how students explain the simultaneous existence of more than one relationship, and whether the interviewed students mention that the shapes are similar and are able to

generalize the area relationship to other dimensional relationships.

Only two of the fifteen students who suggested more than one strategy expressed surprise that there was more than one relationship. One student thought that the area relationship should be the same as the sides relationship. He counted, expressed surprise, then revised his opinion to 2:1 for sides and 4:1 for area.

When asked to explain the possibility of different answers or when asked to choose between two answers they had given, twelve of the fifteen students who had given multiple answers settled on the volume relationship as the truest description of the situation and the best answer to the question. Only one of these students was disconcerted by the fact that two different descriptions seemed to hold.

Four of these fifteen students decided that the other relationships, though they existed, were either simply a part of the volume relationship or were incorrect descriptions of the phenomenon.

Only three students explicitly responded that two or all three descriptions were true. These students suggested that "the answer depends on the question" or that "the quantities being measured are different."

> "It could be 2, 4 or 8. If you're looking at all of these two objects as having three different axes, it's 8. If you're looking at 'em as having 2 different axes, then it's 4, If you're looking at the two as having one axis, then it's 2. Zero axis, this is just as big as this, sort of...You

can only look at them one way at a time."

"So I guess they're dependent on how you specify what you want. It depends on where you want the answer. Taking and converting something and doing its different units. Like the speed of light has a certain amount of units. There will be a different value in different units."

Eight of the students explicitly mentioned that the s s were similar. Five students had explanations for the volume relationship, 8:1, that seemed like generalizations. One student explained that since it is a cube, one cubes the ratio of the sides. No student was moved to make any generalizations for dimensions greater than three.

4. Summary

Most students were not able to understand the relationship between area and side growth in triangles without the direction of a teacher. Specifically, they needed teachers to remind them of the area formula. The two teachers used different strategies to provide the necessary direction. One strategy was to provide students with the formula, the other was to have students generate this formula by asking them to do a computer task. It is not clear at the end of these exercises whether all of the students would claim that this area relationship is true for all two dimensional figures. Some may think that it is only true for triangles.

Students were able to recognize the different growth relationships that hold for solids. They tended not to experience conflict between the different descriptions, even

when they agreed that different descriptions existed and were valid. However, as on the area problem, they were not able to coordinate the descriptions for a pair of solids explicitly. With one exception, they did not understand the relationships between sides, areas, and volumes in similar solids as representing one underlying relationship. Students did not construe the phenomena in solids as generalizations of the relationships in plane figures.

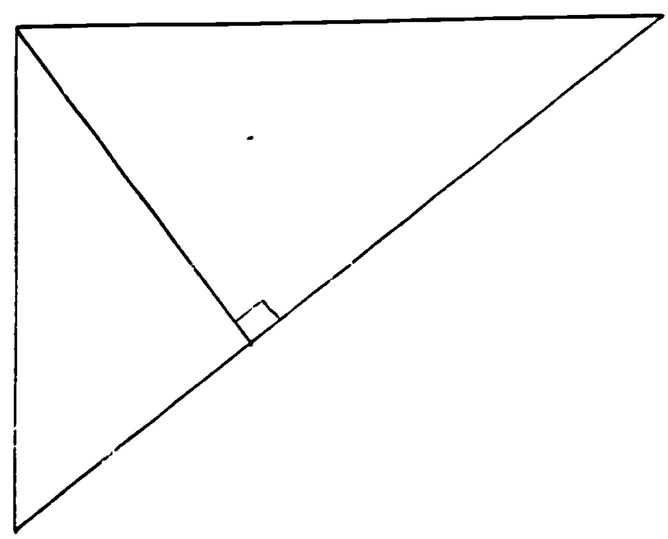
It is not clear why some of the students were not able to recognize all of the descriptions. Some students provided two descriptions; clearly, they believed that more than one description actually could hold. When students missed one of the dimensional answers, the problem seemed to be perceptual in nature. It was as if they could not reorganize their perception of the whole to highlight the missing dimension. Those who saw 4:1, missed the edges; those who saw 2:1, missed the faces. Frequently, these students misused geometrical vocabulary about solids. Volume was used for area and vice versa. Edges were used to describe faces. In contrast, only two students were unable to discover the volume relationship. One added and the other could not come up with any relationship whatsoever.

C. Corresponding ratios in right triangles

1. Prior research

In the 1985-86 study, one teacher spent a considerable amount of time on relationships in right triangles when an altitude is dropped from the right angle (See Figure 12).

Figure 12
The altitude from the right angle vertex in a right triangle



She was perturbed by two phenomena that she observed in her students. First, her students did not grasp the notion of corresponding sides in similar figures, though they had done well with this idea for congruent triangles. Second, she commented that most of the work with similar triangles in the textbook did not involve triangles that were rotated and flipped with respect to each other. In a right triangle with an altitude from the right angle vertex, suddenly students

are confronted with triangles that are rotated and flipped (See Figure 12--In order to align Triangles ABD or ADC with triangle ABC, they must be rotated and flipped). Possibly as a result, her students were not able to solve the simple "Find the missing sides..." in problems with an altitude drawn from the right angle vertex of a right triangle.

This difficulty has not been documented as such in the geometry literature, though in conversations with geometry teachers this topic is mentioned as difficult to teach. Related research has explored students' judgements of when triangles are similar and how rotation of triangles effects students judgements of triangle classification (Vollrath, 1977, and Hershkovitz, 1987).

Thinking more about this difficulty, the problem splits conveniently into two separate parts. In order to succeed at the "Find a missing side..." problem, one must first recognize that the shapes are similar. One must know that similarity involves proportions among sides and then one must be able to set up the correct proportion. This study focusses on the first part of the process, identifying similar triangles.

Two hypotheses are tested. The first hypothesis is that students have difficulty recognizing the similarity of triangles which must be flipped in order to be superimposed with their corresponding sides aligned. The second hypothesis is that students find the similarity relationships

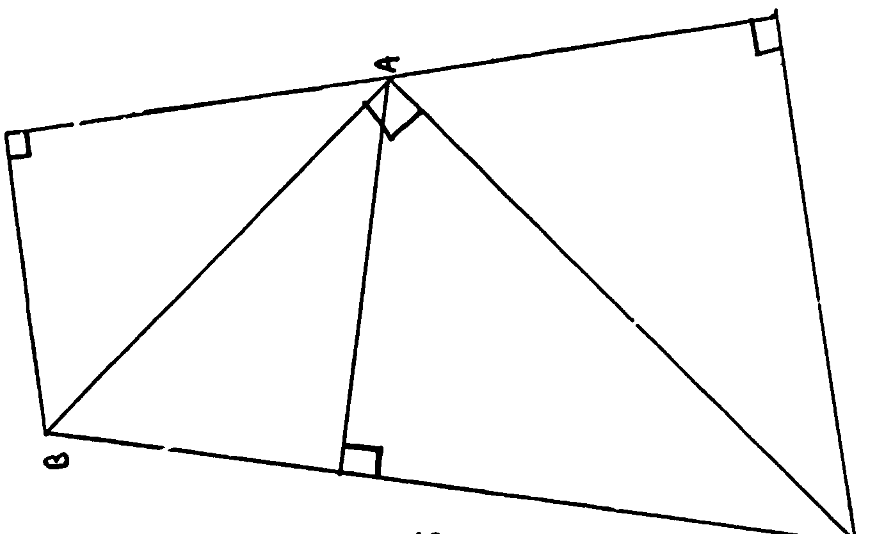
in right triangles difficult because in the typical configuration the altitude plays a different role in each of the three triangles (In Figure 12, AD is the altitude in triangle ABC, it is the longer leg in triangle DBA, and the shorter leg in triangle DAC).

2. Classroom observations and student papers

The geometry curricula in the urban and suburban sites were very different when it came to proportions. In the suburban setting, geometry is integrated with algebra and therefore right triangle mean proportions are emphasized. In the urban setting, these relationships are not stressed. Thus, the classroom observations described below took place only in the suburban setting and not in the urban setting.

In the suburban classes, one of the problems in the unit asked students to reflect the altitude from the right angle in a right triangle over each of the legs. This construction creates a figure like the one below.

Figure 13
A figure like those created by students for computer task #8



Students were asked to explore this figure and come up with conjectures. Students first noticed that there were two pairs of congruent triangles. They then noticed that there were four similar triangles in the figure. A small number of students also noticed that these four triangles were similar to the fifth, original triangle (triangle ABC).

In the suburban setting this problem was followed by an activity to help students identify the correct proportions. The classes were broken up into small groups and a class session was devoted to discovering the three mean proportional relationships. Each group was given two copies of a right triangle with an altitude (As in Figure 12). They were to cut one copy into two triangles and rotate and flip these triangles to discover the correct mean proportions.

In one group, the mean proportion involving the altitude was discovered very quickly. Students then focussed on writing the correct correspondences for the similarity. They were able to align the two small triangles that they had cut, but had a hard time with the third triangle, the original uncut one. They called it the "weird one." They explained that it was weird because it needed to be flipped over. Even after aligning the three triangles, this group had a hard time finding the two other correct mean proportions. Only one of the small groups discovered all three correct proportions. The other groups each found only the altitude mean proportion.

3. Interview tasks

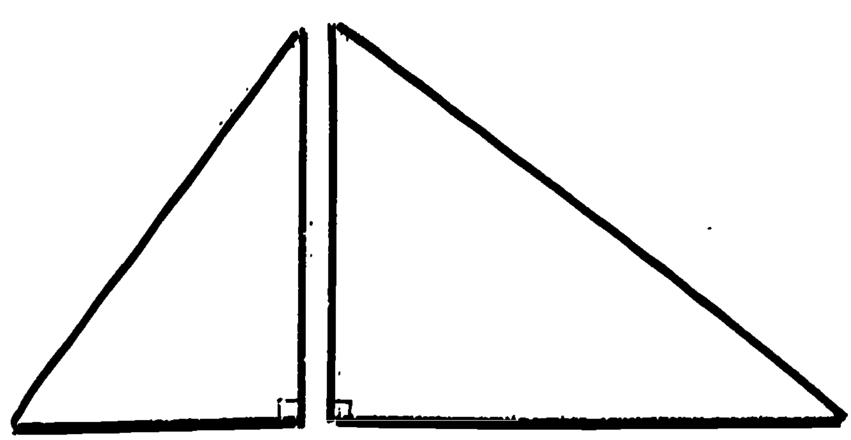
Two of the interview tasks were structured to test the hypotheses about students' difficulties with proportions in right triangles. Each of these tasks focuses on whether students recognize similarity relationships; they did not ask students for correct correspondences, or proportions.

a. Metal pieces

One task involves six metal pieces (two of which are of equal length) that can be joined end to end to create two similar right triangles. In the interviews, after students succeed at this part of the task, one of the equal-sized pieces is removed. Students are asked to make two or three similar right triangles with the remaining five pieces.

If students successfully complete the second part of the task, they are asked how many similar triangles they recognize in the finished shape. This question was designed to discover whether students were able to see all three similar triangles or just the two smaller similar triangles.

Figure 14
One solution to the six rods task

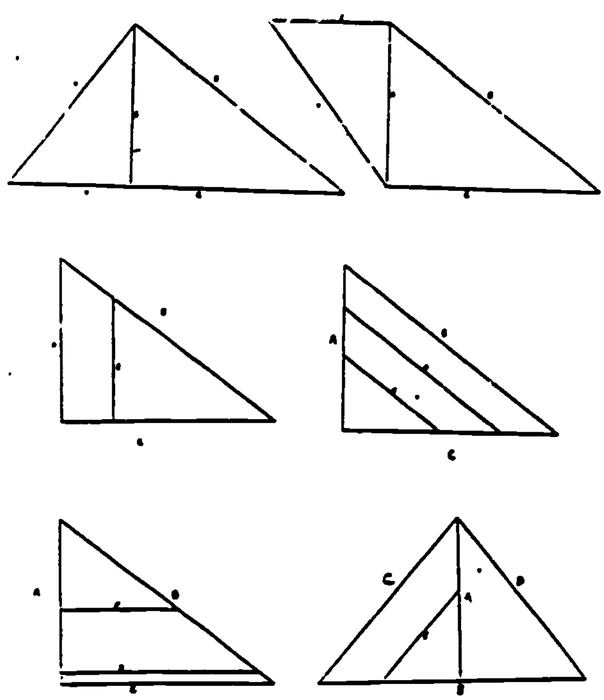


In general students did not seem to have a problem with the notion that one metal piece would have to function as a different element of two or three triangles. However, one student did. As she explained her reasoning she realized that it was mistaken.

> "We have two triangles here and they are not similar and you (the interviewer) asked me how I know they are not similar. Because I have one altitude for both triangles, so if one side is the same, in order for them to be similar, they would have to be the same triangle..."

Only one of the interviewed students thought that the second part of the task was not possible. She was rushed; she might have changed her mind had she had more time.

Figure 15
Alternative solutions to the five rod task



Students created many configurations which satisfied the second part of the task (See Figure 15.). The first one is the altitude in a right triangle configuration. Five of the six students who created that configuration recognized all three similar right triangles. Of these five, two saw the large triangle after some delay, a third recognized it only when pressed by the interviewer and a fourth identified all three triangles, but was not able to write correct proportions. One student did not recognize the third larger triangle at all.

b. Classifying shapes

The second relevant task involves a series of shapes, some of which are similar and some of which are not. Students were asked to group the shapes into piles of similar shapes. Five of the similar shapes had to be flipped in order to align their corresponding sides (See Appendix A3 for more complete details). The task tests the notion that students do not identify the similarity of triangles which must be flipped in order to have their corresponding sides aligned. In addition, a tally was kept of the number of flips that students made. This tally is an indication of the degree to which students feel that flipping the shapes is a legal operation that does not change the similarity relationships.

Nineteen out of twenty three interviewed students had no difficulties performing the necessary flips and recognizing

the similar figures. One student classified the shapes in a different way, not according to the geometrical definition of similarity, and did not use flips at all. Four students missed one or two of a total of five necessary flips. Two of these students named the similar triangles on a paper and pencil post-test question involving flipped triangles and two did not (problem 9 on the post-test).

4. Test data

The third opportunity to check the two hypotheses and to investigate students' difficulties with right triangle mean proportions involved three written problems that were given as part of the post-test. Would student performance be different in a paper and pencil setting versus a manipulative setting?

On the post-test, there are three sets of comparisons between questions which can be used to assess students' difficulties with flipped and rotated similar triangles. Specifically, are similarity relationships between flipped triangles harder to recognize? Unfortunately, none of the problems can be used to investigate the "two uses for one segment" issue.

All of these comparisons will be done by examining frequencies of student responses to each question. Chi-square analyses will determine if the patterns of response are significantly different.

a. Problem seven versus problem nine

In problem 7, students did not need to flip or rotate the triangles in order to align their vertices. In problem 9, students need to rotate the smaller triangle to align it with the medium triangle and flip and rotate each of these triangles in order to align them with the large triangle (See page 10 in Appendix A2 for a copy of these problems.).

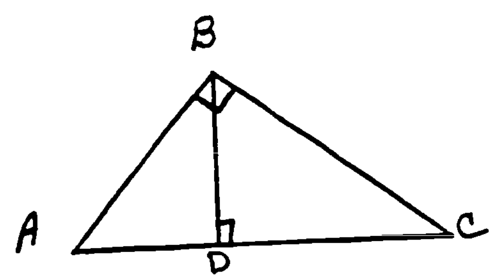
As our criterion of comparison in this case, we looked at the number of students who mentioned that there were similar triangles. We did not compare the number of correct correspondences or proportions in this case (The correct correspondence was given in problem 7.). Seventy percent of the 108 students mentioned similarity in problem 7, while only 49% of the students mentioned similarity in problem 9. A comparison of the number of students who mentioned similarity in these two problems versus those who did not yields a strong, significant difference in favor of problem 7 (chi-sq=8.4567, Fisher's exact $p<0.0000$, phi=0.2073). Thus, a problem that involved flipping and rotating created a context where students were less likely to recognize similarity relationships.

b. Problem nine--which relationships are noted?

Do students write relationships between the small triangle and the medium triangle (rotated), the medium triangle and the large triangle (flipped and rotated), or the small triangle and the large triangle (flipped and rotated)?

Problem nine inadvertently favored the relationship between the medium and large triangles. The right angle in each of these triangles was marked, while in the small triangle the right angle was not marked though it could be deduced that it was a right angle.

Figure 16
The diagram from problem nine



Large-triangle ABC, Medium-triangle ADB, Small-triangle BDC

Interestingly, even though the problem favored the relationship between the medium and large triangles, 83% of the 53 students who wrote a similarity relationship recognized the small to medium relationship, while 72% recognized the medium to large relationship and 62% wrote that the small and large triangles were similar.

Table 4
Similarity Relationships Written by Students for Problem 9

Relationship	Freq.	%age of those writing similarity relationships	%age of those who took the test
Small to medium (rotated)	13	24	12
Medium to large (flipped)	7	13	6.5
Small to large (flipped)	3	5.5	3
S to M and M to L, but not S to L	1	2	1
All three	30	56	28
Any	54	100	50

Table 5
Correct Proportions Written by Students for Problem 9

Relationship	Freq.	%age of those writing similarity relationships	%age of those who took the test
Small to medium (rotated)	7	23	6
Medium to large (flipped)	6	19	6
Small to large (flipped)	0	0	0
All three	18	58	17
Any	31	100	29

Eighty one percent of the 31 students who wrote a correct proportion wrote a proportion involving the small and medium triangles, while 77% wrote a proportion involving the sides of the medium and large triangles and 58% wrote a proportion using the sides of the small and large triangles.

Problem 9 seems to be a difficult problem, few students recognize the similarity relationships in this problem and even fewer write proportions when presented with this configuration. There is some evidence that the rotation relationship was more recognizable than the relationships that required flipping and rotating, even though the problem militated against recognizing the relationship between the small and medium triangles.

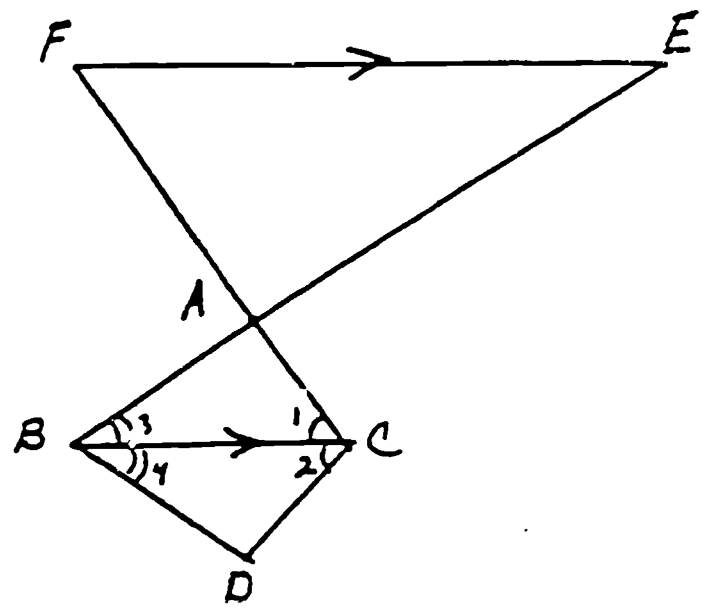
c. Responses to 10a versus 10b

In comparing student responses to questions 10a and 10b. our criterion was the identification of correct correspondences and not simply recognition of similarity.

In 10a (See Figure 17.), the two similar triangles share one point. There are two simple ways to move these shapes mentally in order to align them. The easiest way is to rotate triangle ABC 180 degrees around point A. An alternative is first to flip ABC over a line parallel to BC and then over a line perpendicular to BC. 96% of the students attempted the problem, 68% used the correct correspondence, and 28% used an incorrect correspondence.

Figure 17
Diagram for problems 10a and 10b

EF is parallel to BC,
angles one and two and three and four are congruent



10a compares triangles ABC and AEF, 10b compares BCD and EFA.

In 10b, both of the easiest manipulations involve flips. One possible course of action is to rotate triangle BCD 180 degrees and then flip it over a line parallel to BC. An alternative is to flip triangle BCD over a line perpendicular to BC and then to translate it up to triangle EFA in order to align the vertices. 91% of the students attempted the problem, 44% of the students used a correct correspondence, and 47% used an incorrect correspondence.

A comparison of the number of correct versus incorrect proportions on these two problems yields a significant difference in favor of 10a (chi sq= 9.6232, Fisher's <0.0015,

phi=0.2284). Thus, it does seem that some similarity relationships are easier to recognize than others. In this case, the distinguishing factor between the two problems may be the necessity for using flips in 10b.

5. Summary

The results reported for this student difficulty are not conclusive. The two hypotheses about students' difficulties with right triangle similarity relationships have been investigated, however much work remains to be done.

The evidence from the two interview tasks involving manipulatives does not strongly support either of the two hypotheses. Few students had difficulty recognizing the similarity of shapes which needed to be flipped. Students did not see flipping as an inappropriate action which does not preserve similarity relationships. Students also did not have difficulties with the notion that one rod could be a part of two or three triangles. Only one student was stymied by the task of making similar right triangles when limited to five pieces.

In contrast, flipping does seem to be an issue when students work in paper and pencil contexts where one must pick up and flip the figure mentally. Students were less likely to recognize similarity and to write correct proportions when the figures drawn on paper needed to be flipped to have their corresponding sides aligned.

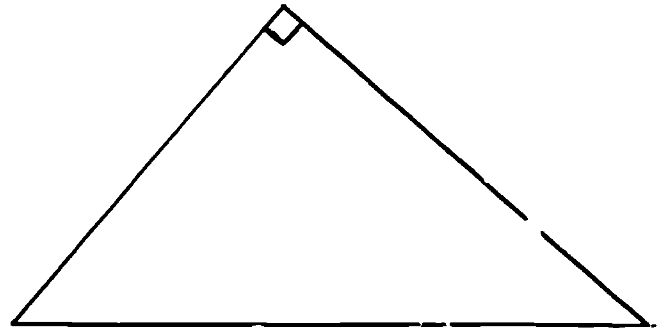
There are two other possible elements of an explanation

for the particular difficulty which students seem to have with the right triangle with an altitude configuration. These hypotheses seem plausible in light of the data reported in this study.

First, the fact that in the altitude in a right triangle configuration the large triangle shares its sides with the other two triangles seems to add a level of difficulty to this problem. The large triangle is obscured; many students do not easily recognize that there is a third, larger triangle. They neglect the third triangle, just as in the blocks task with similar solids some students neglected either sides or faces.

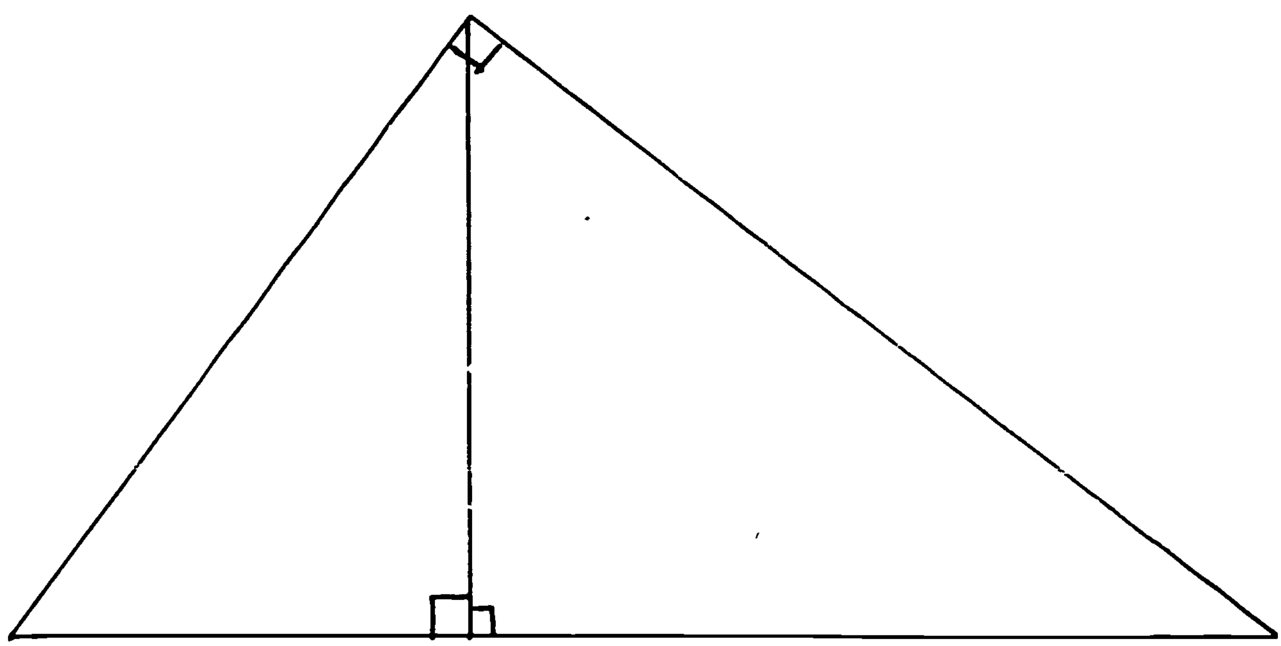
An alternative suggestion for students' difficulty in recognizing the third triangle comes from the work of Rina Hershkovitz (1987). She indicates that right triangles in the following position are hardest to recognize as right triangles.

Figure 18
A position in which right triangles are hard to recognize



In most textbooks, the right triangle configuration is presented in the following position so that the altitude will be in a comfortable vertical position.

Figure 19
The typical textbook orientation
of the altitude in a right triangle configuration



In such a case, the third triangle may not be recognizable as a right triangle. It would be interesting to know if students recognize that there is a third triangle, but simply do not think that it is a similar right triangle. If this were the case, the GEOMETRIC SUPPOSER's random orientation of right triangles should help combat this issue. However, this hypothesis was not studied.

V. CONCLUSIONS AND RECOMMENDATIONS

The primary goal of this study was to explore students' understanding of similarity. Specifically, the study focussed on three areas of student difficulty with similarity:

- incorrect additive strategies,
- dimensional growth relationships, and
- right triangle proportions.

Each of these three areas of difficulty was indeed difficult for students in the observed classes. In each case, this study described students' difficulties and attempted to understand their misunderstandings. The results reported in Section III shed light on each of these difficulties and suggest directions for further research which are described below.

In addition earlier it was argued that, beyond illumination of student difficulties, this research would be beneficial in providing practical suggestions for teachers. To be "practice-oriented", this study should produce recommendations for teaching. While this study did not attempt to evaluate systematically the effectiveness of the activities used, our experience indicates that they can be recommended to teachers. The activities embody two tactics to combat students' misconceptions. We describe these tactics, and then discuss how the teaching tactics relate specifically to each of those areas of difficulty. The

activities themselves are outlined in Appendices B1 - B3. A review of the results for each area of difficulty and an explanation of how the teaching tactics relate specifically to each difficulty will follow.

The goal of the activities described in Appendices B1 - B3 is to make students' difficulties with similarity explicit and then to address them in the context of existing curricular materials on similarity. The activities used in this study were successful in bringing student misconceptions to light and to the attention of teachers, indeed both of the teachers who taught the similarity unit were surprised by some of the misconceptions demonstrated by their students. These activities also underscore the difficulty of similarity and make it clear that much time can profitably be devoted to similarity and scale during a high school geometry course. For these reasons, we recommend these activities. We are not suggesting that these activities will eliminate students' difficulties with similarity. In addition, these activities are but one way of addressing the three student difficulties, we are not suggesting that they are the only way, or even the best way.

A. Student difficulties: teaching tactics

These tactics are based on the premise that it is important for students to explore mathematics and to

construct their own understanding. The first tactic is simply to acknowledge students' misconceptions and argue against them deductively and empirically (See Bell, 1986 for an explanation of conflict teaching). This is an unusual approach in mathematics classes where students' misconceptions are seldom recognized or explicitl confronted.

The second tactic is analogous to those used in physics education, particularly by John Clement, who uses analogies between understood situations and ambiguous situations in order to transfer or extend understanding. Understanding is built by starting with situations where students apply knowledge successfully a working to situations where they do not. The teacher must make an explicit argument that connects the two situations and that highlights the analogy between the two. Discussions play a key role in such a pedagogy.

1. Incorrect additive strategies

The phenomenon of using incorrect additive strategies to construct similar triangles was replicated. The data from this study support Hart's suggestion that this additive misconception is restricted to enlargement contexts (p.75). However, in contrast to Hart's study, in this study even students who showed an understanding of multiplicative ratio in some geometric contexts failed to do so in others. Therefore, it is important to look for aspects of the

geometrical configuration in a problem that lead students to use an incorrect additive strategy. In the extending two sides of a triangle construction, a misconception about parallelism may be partially responsible for the reversion to an incorrect additive strategy.

In this teaching experiment, both direct teaching and indirect teaching about similarity were effective in changing some students' strategies, though some students stayed with this incorrect strategy even after direct teaching.

The activities in Appendix B1 add a progression of tasks to the direct teaching done during the study. The main idea of the sequence of activities is to emphasize in the more accessible problems that similarity involves multiplication so students will see the relevance of multiplication to more difficult similarity problems (as suggested by the second tactic), such as the extending two sides of a triangle problem (computer task #4 in Appendix A1). When implementing these exercises using the two strategies outlined above, students who exhibit incorrect additive strategies are challenged and asked why they are changing their strategy from strategies used on previous problems. The teacher or other students act out the first strategy by making empirical and deductive arguments against the incorrect additive strategy.

2. <u>Recognizing and relating different growth relationships</u>

In this study, most students did not understand the

relationship between area and side growth in similar triangles without the direction of their teacher. It is not clear whether students generalized this relationship to all two dimensional shapes.

Students were able to recognize the different growth relationships in solids; for example, if the sides double, the area quadruples, and the volume grows by a factor of eight. They tended not to experience conflict between the different descriptions of growth, even when they agreed that different descriptions existed and were valid. However, as in the plane figures, they were not able to explicitly coordinate the descriptions. With one exception, they did not understand the relationships between sides, areas, volumes, and other dimensional relationships as the consequence of one underlying relationship. They needed knowledge of area and volume formulae in order to make explicit the connection between the growth relationships.

Some of the students did not recognize all of the descriptions. It was not that they could not believe that more than one description actually would hold, rather the problem seemed to be perceptual in nature. It was as if students could not reorder their view of the whole to focus on one of the dimensions.

Based on these results, the activities in Appendix B2 were designed to help student integrate area and volume formulas into their understanding of how scale change affects

similar figures. They also make use of a progression from triangles, to other two dimensional figures, and finally to three dimensional figures.

3. Right triangle proportions

Though some aspects of students' difficulties with right triangle proportions have been investigated, the perspectives gained in this study do not present a complete explanation for these difficulties.

The results of this study suggest that the dual or treble function of the altitude in the right triangle, does not seem to a source of much confusion. This result is surprising and must be considered provisional until further data are collected. The data come from an interview task that involves hands-on manipulation of metal rods. It may be that this dual or treble function is still difficult in pencil and paper tasks that require the identification of correct proportions. The flipping issue does not seem to be an issue with manipulatives, though it still is a difficulty in a paper and pencil context where mental, imagined flipping is necessary.

These results also suggest that the right triangle configuration is especially hard because the third triangle shares all of its sides with the other two triangles. Students must recognize that there is a third larger triangle. They neglect the third triangle, just as in the blocks they neglect either sides or faces. Alternatively,

students' difficulty in recognizing the third right triangle may be because of its typical, hypotenuse-horizontal orientation.

Keeping all of these sources of misunderstanding in mind, the activities in Appendix B3 are suggested. Again, an important strategy is the "ramping upwards" from contexts where students succeed in identifying similar triangles and writing correct proportions to more difficult contexts. Since students exhibit a better understanding of issues of "flipping" when they work with concrete, three-dimensional shapes, the activities begin with manipulatives before moving to pencil and paper. When students move to two-dimensional shapes, it is possible to order configurations of similar figures by students' ability to recognize the similarity of the shapes. Based on the second tactic, the activities build from the clearest configuration to more difficult ones.

B. School computer labs as research settings

Finally, in addition to insights into students' understanding of similarity, this study has also provided some evidence for a claim of a more general nature. The observations made in the classroom lab setting were a crucial and valuable source of data which allowed researchers to look at students' understandings. For example, much of the evidence for the existence of the incorrect additive strategy

came from students' work in the lab. This experience supports the claim that when microcomputer tools are used in a guided exploration approach, the computer lab classroom session can be an important and valuable setting for research on childrens' understanding. Hopefully, in the future, more researchers will come to make use of this opportunity to do research on understanding in classrooms.

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APPENDIX A1: THE COMPUTER TASKS

Project #2 10/10/86

Use the Supposer to draw any triangle that you are drawn to today.

a) Copy it as carefully as you can below.

b) Find the ratios of all the pairs of sides (please leave your results as the ratios of two integers).

c) Find the measures of all three angles.

Diagram I:

Ratios of sides:

Names	Values

Sizes of angles:

Names	Values

Use the "Scale" option and repeat steps a), b) and c) above.

Diagram II:

Ratios of sides:

Names	Values

Sizes of angles:

Names	Values

What statements can you make, if any, about how the two triangles look in relation to each other?

What statements about ratios of sides?

What statements about sizes of angles?

Repeat this whole procedure on the back of this paper with a new pair of triangles. (Make them very different from those on the [illegible]

Project #3 10/17/86

Construct a rectangle.
Construct a second rectangle inside the first which is similar to the first. Describe how you constructed it and draw a diagram to accompany your description.

Construct a parallelogram which is not a rectangle.
Will your method work in this situation? (Use a diagram and describe why or why not.)

Construct a quadrilateral which is not a parallelogram.
Will your method work in this situation? (Use a diagram and describe why or why not.)

Extra Credit Project 10/17/86
(after you finish projects 2 and 3)

Draw a scalene triangle (do you know why I keep on using scalene triangles?).

Draw two triangles inside the original which are similar to the original (they may have points in common).

Diagram your results.

Why do you think they are similar?

Draw three triangles inside the original which are similar to the original.

Diagram your results.

Draw four triangles inside the original which are similar to the original.

How did you do it?

How could you draw n triangles inside the original similar to the original?

PROJECT #4A

Draw a scalene triangle ABC. Extend sides AB and AC, from B and C respectively, in such a way that the triangle formed by connecting points D and E is similar to the original triangle. (Editor's note: The two triangles should have angle A in common.) Make a sketch of your first attempt.

Diagram:

Determine whether or not your triangles are similar.

If your triangles are not similar, describe how you attempted to do it and then try again.

We have discussed the AA postulate and SSS theorem for similarity. This lab attempts to shed some light on yet another method for showing similarity of triangles. In conjecture form, can you state what the "hidden agenda" was for this lab.

Lab Project #4

10/20/86

I Construct t.iangle ABC such that AB = 3, AC = 3 and BC = 3. Extend side BC from C two units. Similarly, extend BA from A two units. Connect points D and E to get a new diagram. Copy the diagram below and state any observations that you have.

II "se the REPEAT key to repeat the above constructions on a new tria..gle ABC in which AB = 3, AC = 5, and BC = 4. Copy the diagram and state observations.

III Use the repeat key to again copy those constructions on another new triangle ABC. This time AB = 3,3AC = 40° and BC = 3.2. Copy the diagram and state o..ervations.

IV Use the REPEAT key one more time and construct triangle ABC with AB = 2, AC = 4, and BC = 3. Copy the diagram and state observations.

V Using the same diagram that you used in IV, extend BC from C the length of BC. Likewise, extend BA from A the length of BA. Connect the points F and G to get yet a new triangle. Copy the diagram and state your observations.

VI Use the REPEAT key on a previous triangle. Copy the diagram and state observations.

VII Use the REPEAT key on another previous triangle. Copy the diagram and state observations.

VIII Use the REPEAT traingle on the last of the previous triangles. Copy the diagram and state observations.

What was this project about?

Project #5 10/22/86

Construct an acute triangle. Measure its sides, perimeter and area. Use the rescale option. Measure sides, perimeter and area of the new triangle. Determine the ratios of a) corresponding sides, b) perimeters, c) areas.

	1st triangle	2nd triangle	Ratio
AB			
AC			
BC			
Perimeter			
Area			
Altitude			

Repeat the above on an obtuse triangle and its rescale.

	1st triangle	2nd triangle	Ratio
AB			
AC			
BC			
Perimeter			
Area			
Altitude			

Observations?

Project #6

Construct a triangle whose sides are 3, 4, and 5. Measure its angles.

Construct a triangle whose sides are 6, 8, and 10. Measure its angles.

Observations?

Construct a triangle whose sides are 3.1, 1.7 and 3.1. Measure its angles.

Construct a triangle whose sides are 9.3, 5.1 and 9.3. Measure its angles.

Observations?

Construct a triangle whose sides are 4, 5, and 6. Measure its angles.

Construct a triangle whose sides are 6, 7½ and 9. Measure its angles.

Observations?

Conjectures?

Project #7

Construct a triangle, ABC. Choose a random point on BC. Through that point D, draw a line parallel to AC and intersecting AB. Draw a diagram of your results.
Determine the ratios of the pieces of the sides (i.e. BD to DC and BE to EA).

Observations? (There are a few.)

Take a different triangle. Choose a random point on one of the sides. Through it, draw a line parallel to a second side and intersecting the third.
Draw a diagram and take the ratios of the pieces of the sides.

Observations?

Now for the conjecture (be careful to say what you mean):

On the back, prove that your conjecture will be true for all triangles.

Project #8 11/5/86

Draw a right triangle with an altitude from the vertex of the right angle.
Reflect the altitude over each of the legs.
Connect the end point of each image to the vertex of the adjacent acute angle.
There are all kinds of observations that one can make about this diagram. Make as many as you can. (Sketch the diagram first.)

Use the repeat key to test the generality of your observations.
On the back, give explanations for as many of your observations as you are able.

Project # 9 10/17/86

Construct a rectangle.
Construct a second rectangle inside the first which is similar to the first. Describe how you constructed it and draw a diagram to accompany your description.

Construct a parallelogram which is not a rectangle.
Will your method work in this situation? (Use a diagram and describe why or why not.)

Construct a quadrilateral which is not a parallelogram.
Will your method work in this situation? (Use a diagram and describe why or why not.)

TASK: In this problem the goal is to state relationships that occur in a triangle when an angle bisector is drawn from one vertex.

Construction:
Draw an angle bisector in a scalene triangle from vertex A.
Through vertex B draw a segment parallel to the angle bisector until it intersects AC.

Observations about this construction.

Conjectures about this construction.

Conjectures about relationships among parts of a triangle when you draw an angle bisector--(ignore all of the construction except points A-D)

What happens if the angle bisector is drawn from a different vertex?

Can you find an argument to support any of your conjectures?

APPENDIX A2:
THE POSTTEST

NAME ____________________ BLOCK __________ DATE __________

1. Three small statues X, Y, Z come with stands, the height of the stand depending on the height of the statue.

X 10

Y 15

Z 25

(a) If Statue X has a stand 2 units high, how high should the stand for Z be?

Z .. ..

(b) If Status Y has a Stand 9 units high, how high is the Stand for Z?

Z ....

(c) If Statue Z has a Stand 10 units high, how high are the stands for X and Y?

(i) X .... (ii) Y ....

2. In a particular metal alloy there are:

8 parts zinc to 15 parts copper.
3 parts tin to 10 parts copper.
1 part mercury to 5 parts copper.

You would need how many parts zinc to how many parts tin?

.... parts zinc to .... parts tin.

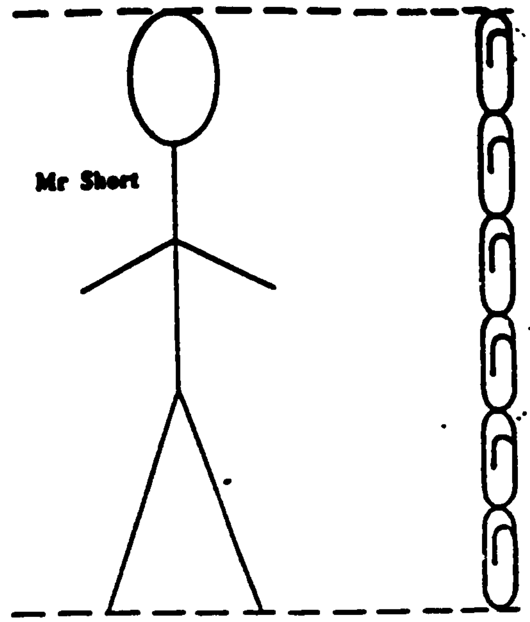
3. % means per cent or per 100. So 7% = 7 out of every 100.

(a) The price of a coat is $20. In the sale it is reduced by 5%.

How much does it cost? ....

4. $\frac{10}{\square} = \frac{2}{7}$ What number goes in $\square$? ....

5. You can see the height of Mr. Short measured with Paper-Clips



Mr. Short has a friend Mr. Tall.
When we measure their heights with matchsticks:

Mr. Short's height is Four matchsticks.

Mr. Tall's height is Six matchsticks.

How many Paper-Clips are needed for Mr. Tall's height ? ....

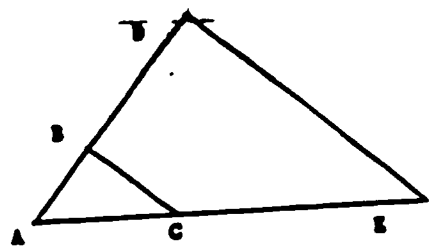
6. AVGOLEMONO Soup Recipe (Greek Chicken-Lemon Soup) for 8

2 Quart Chicken Broth
1/2 Cup Uncooked Rice
6 Egg Yolks
Juice of 2 Lemmons
Salt
Peper

I am cooking AVGOLEMONO Soup for 6 people.

How much Rice do I need ? ....

7. The sides AB and AC of $\triangle ABC$ have been tripled.

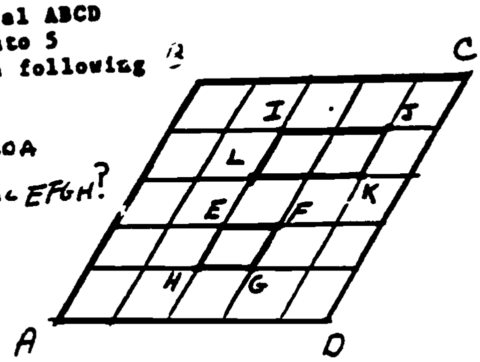


a) State relationships between parts of $\triangle$ ABC and $\triangle$ ADE?

b) State the relationships between $\triangle$ ABC and $\triangle$ ADE?

c) State relationships between $\triangle$ ABC and Quadrilateral BCDE?

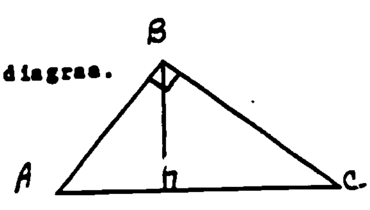
8. The sides AB, BC, CD, AD of the quadrilateral ABCD are all congruent. Each side is divided into 5 equal parts. Using the diagram, answer the following questions.



a) Stae the relationships between the BCDA sides and angles of Quadrilateral and Quadrilateral IJKL? & QUADRILATERAL EFGH?

b) What are the relationships between the Quadrilaterals themselves?

9. State relationships from this diagram.



10. Given the diagram. EF $\parallel$ BC, $\angle 1 = \angle 2$, $\angle 3 = \angle 4$.

a) If AC = 3, AB = 5 and AE = 12.5.

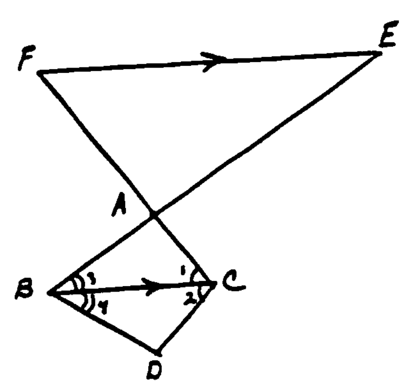
Find

AF = ____

b) If CD = 12, BD = 13, AF = 18

Find

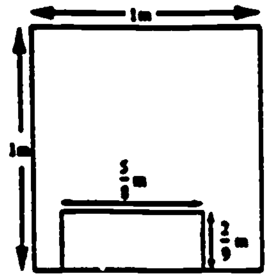
AE = ____



11. x, y, and z are positive whole numbers.

$\frac{x}{y}$ is less than $\frac{x}{z}$ when ....

12. What fraction of the square is shaded ?



....

13. A man is driving in France. He knows that 1 km is the same length as $\frac{5}{8}$ mile. His hotel is $\frac{2}{3}$ km from the Petrol(Gas) station.

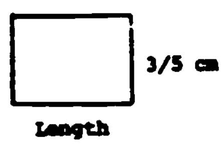
What is this distance in miles ? ....

14. An American in London converts pounds to dollars in his head when he wishes to buy something. £1 is equivalent to $1\frac{3}{5}$ dollars.

If he pays £45.50 for a coat, how much is this in dollars ?

....

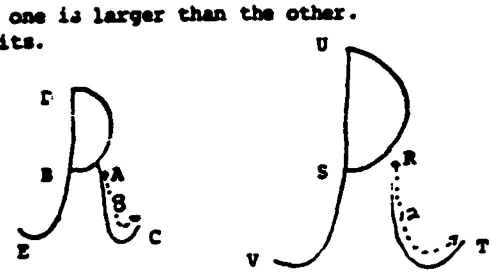
NAME ______________________ BLOCK ______________

15. 

Area = 1/3 square centimeters

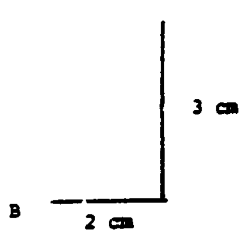
Length = ______________

16. These two letters are the same shape, one is larger than the other. The curve AC is 8 units. RT is 12 units.

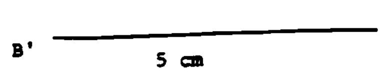


(a) The curve EB is 9 units. How long is the curve VS? ____________

(b) The curve UV is 18 units. How long is the curve DE? ____________



17. Work out how long the missing line should be if this diagram is to be the same shape but bigger than the one above.



____________ cm

APPENDIX A3:
INTERVIEW TASKS

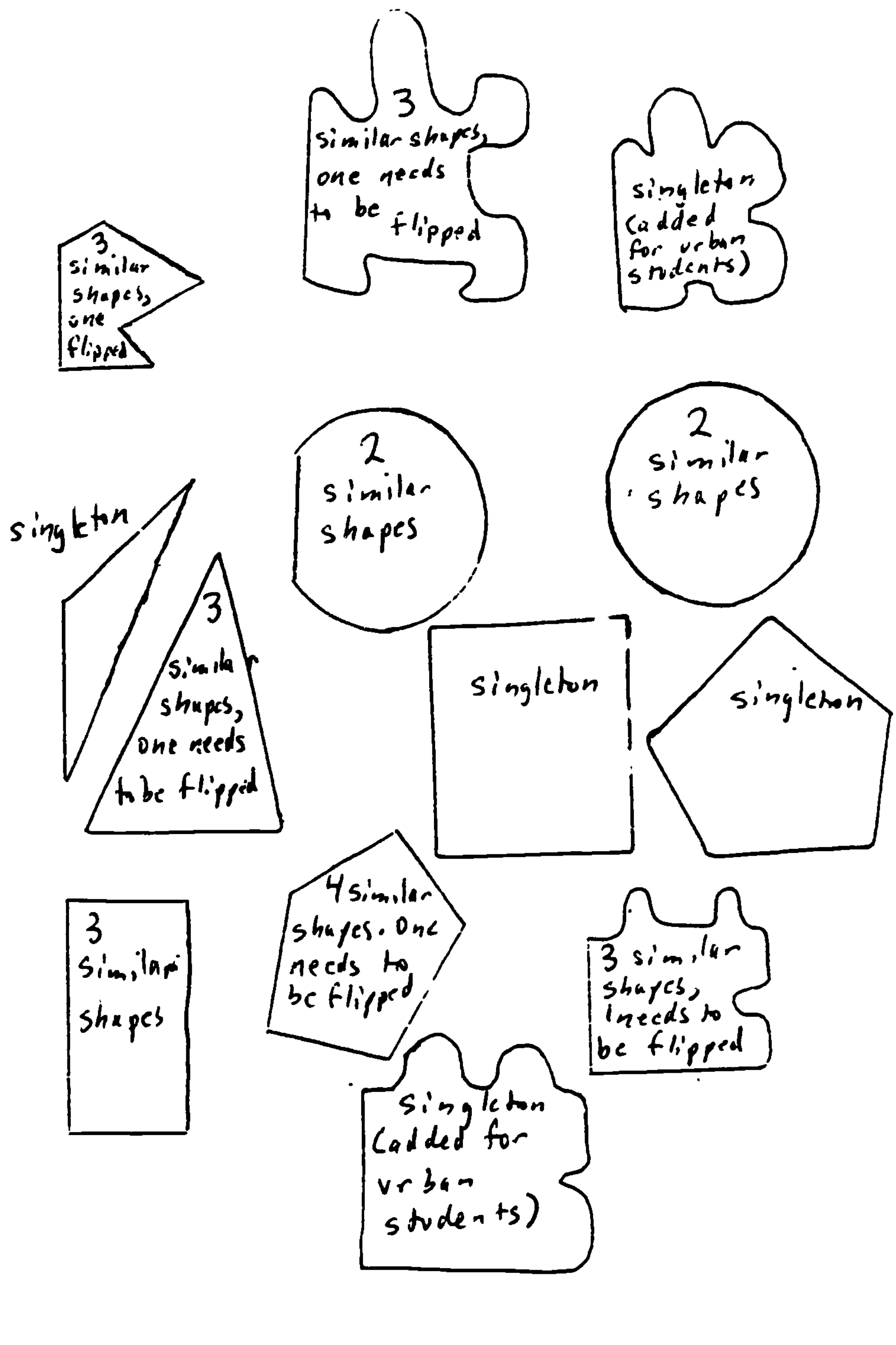
BLOCKS

Students were given two arrays made up of unit cubes. One array was a rectangular solid 2 by 3 by 4. The second array was a rectangular solid that was 4 by 6 by 8. Students were asked to specify how much bigger the larger one was than the smaller one. After a student gives one answer, the interviewer asks if there are other ways to look at the problem. We want to see whether students use multiplicative or additive strategies. If they use multiplicative strategies the object is to see which of the three relationships (linear, area, volume) they recognize and whether they see a connection between the three types of relationships. If they recognize more than one relationship, the interviewer asks how it can be that there are different descriptions.

SHAPES

The second task was a recognition task not a production task. The shapes in Figure 20 are a sample of the shapes presented to students (The shapes are not in the scale given to students). The students are asked to group these shapes into piles of similar figures. Some of the shapes must be flipped in order to observe that they are similar. The interviewer explicitly says that the piles may include congruent shapes since congruent shapes are also similar.

Figure A3-1
The shapes used in the interview task

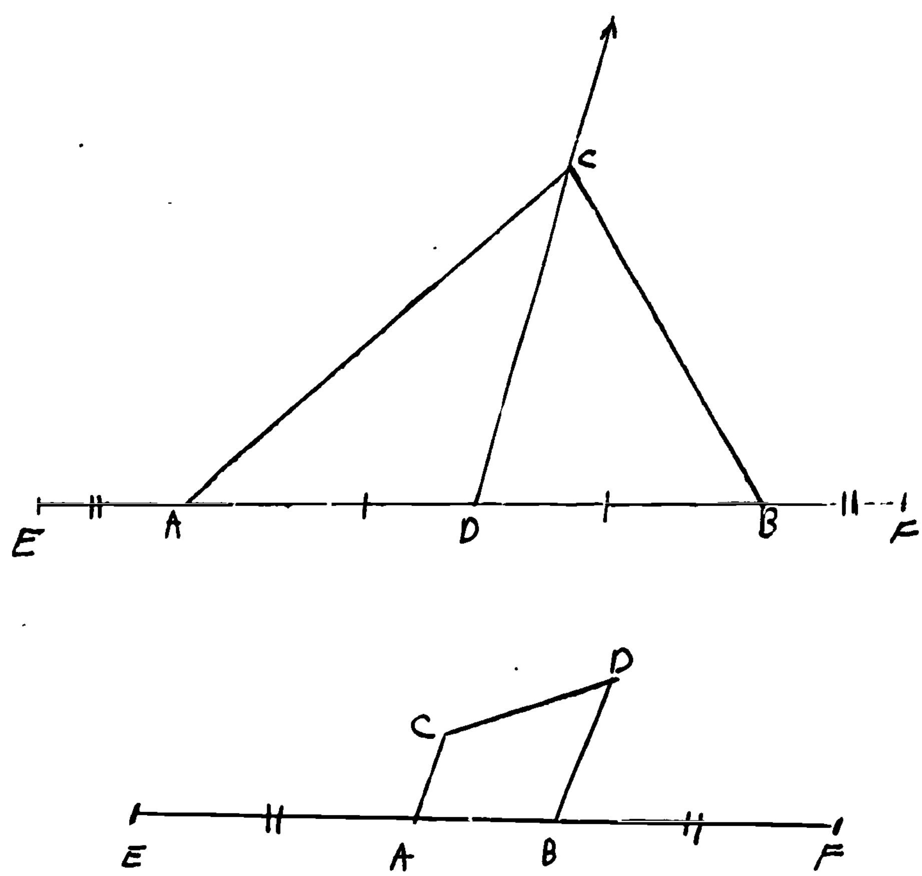


Students are encouraged to talk as they group the shapes. The interviewer then probes the students' understanding by asking why the piles are the way they are.

CONSTRUCTIONS

In contrast to the previous task, the next two tasks are productions tasks where students are asked to create similar figures, not just judge whether shapes are similar. The first part of this task is taken directly from the work of Piaget. Students are given a triangle with each side of the base extended by the same amount. They are asked to construct a similar triangle on the base. We included a median extended out to the end of the page.

Figure A3-2
The construction tasks posed in the interviews



Pencil, ruler, compass and protractor are given to the students. They are encouraged to talk out loud and describe their plan for constructing the triangle. Inferences are made about the definitions that they are using to construct the triangle. If their work exhibits no proportional reasoning, and if time permits, they are given a second construction of a similar nature with a quadrilateral (see Figure 21).

RIGHT TRIANGLES

Six lengths of hangers were cut out. The shapes are cut to be the lengths of a right triangle with an altitude dropped from the right angle vertex. There are two pieces with the length of the altitude. Students are given these six pieces and asked to arrange the pieces into two similar right triangles. After they do so, the interviewer mixes up the pieces and removes one of the two that have the same length as the altitude. Students are then asked to make two or three right triangles with the remaining five lengths. They are encouraged to talk out loud and describe their strategies. The interviewer probes constantly during this task to determine what the student is thinking.

APPENDIX A4:
THE RATIO EXERCISES

#1

Use your calculator to divide where you see the '÷' sign.

18	/	5	=
29	.	13	=
22	/	11	=
17	/	17	=
15	/	22	=
13	/	26	=

Conjectures-LOOK FOR PATTERNS:

#2

Use your calculator to divide where you see the '/' sign.

5	/	18	=
15	/	29	=
11	/	22	=
17	/	17	=
22	/	15	=
26	/	13	=

Conjectures-Compare with page #1

#3

Take the answer from sheet #1 problem #1 and take its inverse, that means if 'x' stands for that answer find 1 / x.
Then do the same thing to the answer from sheet #2, problem #1.

	sheet #1	sheet #2
Prob. #1		
Prob. #2		
Prob. #3		
Prob. #4		
Prob. #5		

Conjectures:

14

'x' and 'y' stand for two numbers. Not knowing what numbers 'x' and 'y' actually are, what can you say about the relationship between the two of these numbers.

x	/	y	=	1
x	/	y	=	2
x	/	y	=	.5
x	/	y	=	2.5

#5

This time 'x' stands for some number. In each case what can you say about 'x'.

x	/	4	=	2
x	/	3.5	=	3
x	/	3	=	.5
x	/	3	=	1.5
x	/	3.2	=	2.3

#6

'x' again stands for some number. What can you say about 'x'?

6	/	x	=	2
6	/	x	=	.5
5	/	x	=	1.3
7	/	x	=	1.4

#7

All of the sheets so far have a hidden number, a hidden '1'. The numbers that have appeared on the right hand side of the equality can be written as ratios.

For example, 1.4 = 1.4 / 1.

Sometimes it is easier to understand a ratio when instead of a denominator of '1' we chose a different denominator in order to get whole numbers in our ratios.

For example, 2.5 = 2.5 / 1 = 5 / 2 .

In general, a simple way to change a decimal into a ratio of whole numbers (if you don't see an easier way) is to:
1. Express the decimal as a fraction over '1'.
2. Multiply the numberator and denominator by a power of ten large enough to remove the decimal point.
3. Put the resulting fraction in lowest terms.

To practice, put the following decimals into whole number ratios.

APPENDIX B1:
ACTIVITIES TO COMBAT THE INCORRECT ADDITIVE STRATEGY

This sequence of activities is not an introduction to similarity. It is designed to fit into an existing similarity unit. It assumes that students have already had experience at classifying and recognizing similar shapes, especially similar triangles.

1. Arithmetic Ratio Problems

This sequence of activities begins with arithmetic work on ratios (See Appendix 1D.). These activities also provide the students with ratio experience that is helpful in understanding data as presented by the GEOMETRIC SUPPOSER. This work should also include typical missing element ratio problems that are not simple N:1 relationships and require multiplicative computation.

2. Enlargement Problems in a Geometric Context

After working on these purely numerical problems, the next step is to work on enlargement problems (See the four problems that are presented on page 17, along with problem one from the post-test in Appendix 1B). Based on Hart's experience and on this study, some students will exhibit an incorrect additive strategy on some of these problems. A discussion of students' reasons for switching strategies for these problems is appropriate, as well as empirical and deductive arguments against an incorrect additive strategy.

3. Computer Similarity/Proportion Activities

These class activities are followed by two computer tasks related directly to similarity. Task #2 in Appendix 1A

highlights the role that ratio and proportion play in the definition of similarity. After this task, it is appropriate to discuss the definition of similar shapes. Task #6 in Appendix 1A focusses on the sufficiency of SSS relationship to create similar triangles. It again highlights ratios.

4. Extending Two Sides of a Triangle

After these two activities, it is time to introduce the task of creating similar triangles by extending two sides. The teacher has more options if he/she starts with the choice version (#4a in Appendix 1A). If students correctly choose multiplicative strategies, one can come back with the first part of the no-choice version (#4 in Appendix 1A). If students choose an incorrect additive strategy and do not discover the correct strategy, then the no-choice version presents the students with a correct strategy.

Students should then be asked to make arguments against the additive strategy. Both empirical ("it doesn't work") and geometric arguments (like the one given by the student on p.30) are valuable.

**APPENDIX B2:
HELPING STUDENTS RECOGNIZE AND RELATE
DIFFERENT GROWTH RELATIONSHIPS**

This set of activities begins in triangles, moves to other two dimensional shapes and then to three dimensional shapes. It asks students to integrate formulas into their understanding of how scale change affects similar figures. It begins with a problem drawn from one of the ratios that is recorded on computer task #2.

1. Why is the area ratio different?

Based on experience in the four classrooms studied this year and on experience in other classrooms over the last two years, the expectation is that most students will notice that the area ratio is different from the sides, altitude and perimeter ratio in computer task #2. This observation can be explored by having students do a computer task on area that will focus their attention on the formula for the area of a triangle as was done by the teacher in the urban setting.

2. Understanding "Base * Height / 2 = Area"

The computer task is to create the smallest rectangle that will enclose a given triangle and then to compare the ratio of the areas of the rectangle and the triangle. Discussion of this problem can lead to the formula for the area of a triangle. The teacher can then guide the discussion towards the observation made on task #2. Students can now be asked to work more carefully on their observation and to look for the relationship between the different ratios that they observed. Finally, they can be asked to justify the relationships that they conjecture are true.

3. Generalizing to other two dimensional shapes.

The purpose of this activity is to have students conclude that the area growth relationship in similar figures works analogously in all two dimensional figures, though they will not be able to prove this result. From this year's experience, it is not clear that students would have agreed to such a generalization. Instead, they might have expected that the area growth relationship in squares to be different from the one they explored in triangles. Using the Quadrilaterals and Circles GEOMETRIC SUPPOSER disks, students should be asked to investigate the area and perimeter growth relationships in other pairs of similar two dimensional figures. For each type of figure, they should try and justify their conjecture using area formulas that they know. Such experience indicates that the relevant variable is the dimensionality of the figure and not the number of sides it possesses.

4. Area and Volume in Three Dimensions

These two computer tasks can be followed by a short unit on area and volume in different solids. Then, groups of students are presented with different solids made of unit cubes and asked the same question that students were asked on the blocks interview task, that is, "How much bigger is this one than that one?" Later, the groups present their conclusions to the whole class. Discussion of this exercise would focus on justification of the observed relationships in

terms of the formulas learned in class and a listing of the largest number of possible growth relationships. It is helpful to make sure that students use the correct terminology for area, volume, faces and edges and to mention explicitly that this exercise is a generalization of the previous computer tasks.

5. Applications

It is important that students realize from this exercise that the relationship between the two figures depends on what is being measured. If this idea is clearly understood, then students are ready to look at its applications in the sciences. Pages 40-48 in PSSC Physics which discuss growth relationships in nature are appropriate reading. Math teachers might also invite science teachers into their classes to demonstrate the importance of this concept in the sciences.

Other activities on growth relationships can be found in a recent National Council of Teachers of Mathematics publication How to Teach Perimeter, Area and Volume (Beaumont et al, 1986).

APPENDIX B3:
PREPARING TO SOLVE RIGHT TRIANGLE PROPORTION PROBLEMS

1. Sorting Similar Triangles

The first activities focus on students' ability to recognize triangles as similar. It is important to have tasks that ask students to sort similar triangles from triangles that are not similar in order to verify that they understand the definition of similarity (See Appendix A3 for shapes other than triangles used in this study.). Manipulatives are especially appropriate for such activities. With manipulatives, students can recognize that triangles which do not seem to be similar may be easily identifiable as similar when one is flipped.

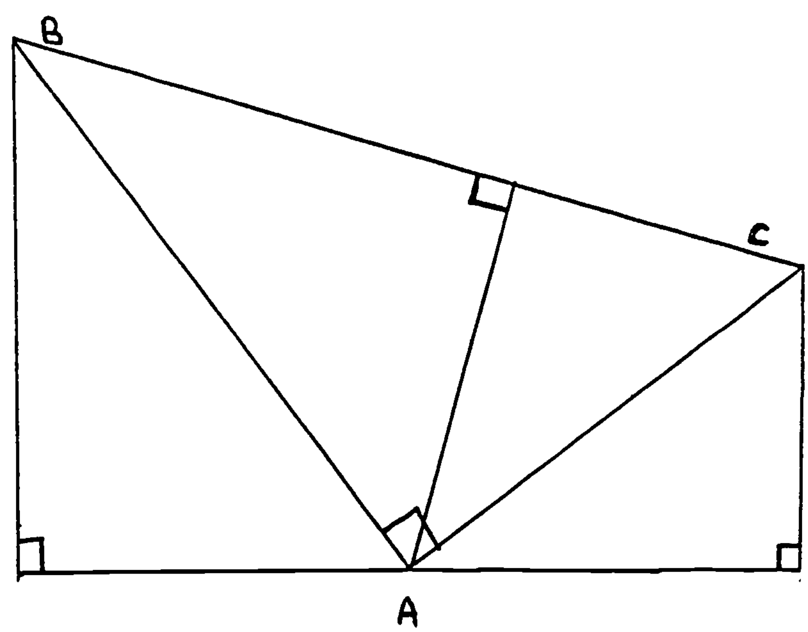
2. Pencil and Paper Similar Triangle Judgements

From similarity recognition tasks with manipulatives where students seem to make correct judgements most of the time, work proceeds to paper and pencil similarity recognition tasks (see posttest problems 7, 8, 9, and 10 in Appendix A2). Further empirical research work with students may help isolate a hierarchy of problems that moves from those that are easiest for students to those that are hardest. Based on this study, it makes sense to begin with configurations like problem 7 on the post-test and then move to rotations like 10a before doing flips, such as 10b. After making sure that students recognize similar triangles, work can be done on the correct recording of ratios between their sides.

3. A Right Triangle Computer Task

The work with triangles seems to be a necessary preliminary to work with the special right triangle relationships. This class of problems may first be addressed by using computer task #8.

Figure B3-1
A figure like those created by students for computer task #8



This task is helpful for two reasons, first because the task is posed for use with the GEOMETRIC SUPPOSER, the right triangles appear in random orientations. Thus, the larger triangle may be more recognizable to more students. Second, this construction generates two pairs of congruent triangles and five similar triangles (see figure 22). Students seem to have little difficulty recognizing the two congruent pairs of triangles. They may have less difficulty recognizing the similarity relationships between the reflected triangles and the large triangle since the only required mental movement is

a rotation. From the recognition of the similarity between these three triangles students may be able to extend the similarity to the complete set of five triangles.

4. Writing Right Triangle Proportions

A second activity, described in the study, leads students to write proportions for the right triangle configuration. Students are given two copies of a right triangle with the altitude drawn in and are asked to cut one copy into two triangles and to manipulate the three triangles to discover the similarity relationships. They are then asked to record the correct ratio relationships between the triangles. One teacher in the 1985-86 study used the following matrix to help students record the information:

	Hypotenuse	Long Leg	Short Leg
Small Triangle			
Medium Triangle			
Large Triangle			

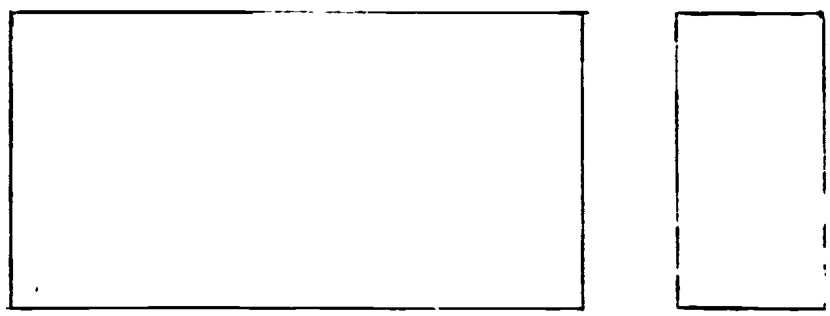
5. Two or Three Uses for One Segment

Some teachers may want to address the two or three uses of the altitude issue, though it did not appear problematic in our study. The activity with the rods that was part of this study's interview is appropriate.

Another activity presents students with similar rectangles where the ratio between the sides of one rectangle

is x:y and the ratio in the other is y:z.

Figure B3-2
Two non-similar rectangles that have sides of the same length



Some students may argue that the rectangles are not similar since one side is the same size and the other is not.

6. Textbook Problems

After all of this preparation, students should be ready to do the problems that appear currently in their textbooks. They should be encouraged to use the matrix in activity 4 above to help organize their work.