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ABSTRACT

The Educational Technology Center has attempted to develop a series of computer based learning environments to support the learning and application of multiplicative reasoning. The work and software described in this paper, including the teaching experiment that generated the error phenomena examined, is part of a larger ongoing research project. This document looks closely at certain difficulties occurring when students use a concrete environment to model situations involving multiplication and division of discrete quantities. Student difficulties seem to be the result of incongruences between the students' visual experience and the semantic structure of the situation being modeled. The environments themselves and the context in which the phenomena of interest occurred are described. Then discussed are observed difficulty, theoretical underpinnings, and suggestions for a modified environment. (PK)

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SUPPORTING CONCRETE VISUAL THINKING IN MULTIPLICATIVE REASONING: DIFFICULTIES AND OPPORTUNITIES

Technical Report
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Supporting Concrete Visual Thinking in Multiplicative Reasoning: Difficulties and Opportunities

Technical Report
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INTRODUCTION

Over the past several years at the Educational Technology Center we have developed a series of computer based learning environments to support the learning and application of multiplicative reasoning. These comprise a kind of concrete-to-abstract software ramp, beginning with icon-based calculation environments that support multiplication and division and ending with environments that use multiple, linked representations of ratios. The overall aim of this effort is to provide sufficiently varied experiences with the conceptual field of multiplicative structures (Vergnaud, 1983) to help build the rich meanings that seem to be lacking for many students and that seem to be required to use multiplicative structures competently. Our effort is also marked by a deliberate attempt to tie visually concrete and enactive operations on objects (in this case, objects on a computer screen) with more formal and abstract representations of these operations (Kaput, Luke, Poholsky, & Sayer, 1987). The work and software described in this paper, including the teaching experiment that generated the error phenomena examined, is part of a larger ongoing research project.

Here we will look closely at certain difficulties occurring when students use one of the concrete environments to model situations involving multiplication and division of discrete quantities. The difficulties seem to be the result of incongruences between the students' visual experience and the semantic structure of the situation being modeled. But first we must describe the environments themselves and the context in which the phenomena of interest occurred. We will then turn to an observed difficulty, discuss theoretical underpinnings, and close with a resolution involving a modified environment.

ENVIRONMENTS FOR CONCRETELY ENACTING MULTIPLICATION AND DIVISION

As described in Kaput & Pattison-Gordon (1987), the larger set of environments deals first with multiplication and division in a one-icon-type calculation environment, denoted ICE-1, which is where the phenomena of interest occur. The programs all run on the Macintosh computer and attempt to capitalize on the bit-mapped graphics and mouse to manipulate screen objects in a way that preserves some of their "objectness" and hence enables us to exploit the fundamental quantitative experience with objects that most children accumulate as part of natural human development, experiences with grouping, matching and counting.

Each environment begins with an icon-choice, where the student is to choose the icon(s) to represent the items in the situation being modeled. Students are presented with a screen similar to that in Figure 1, and must point at an icon and click the mouse button to make their choice.

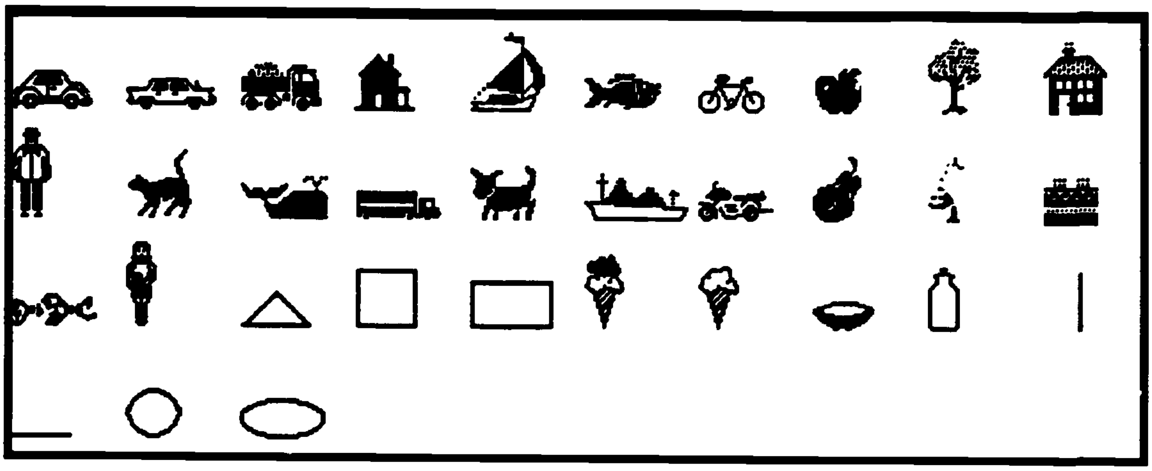

Figure 1

Suppose, for example, that the situation to be modeled involves the planting of groups of trees to provide shade for picnickers in a park. Hence the student would click on one of the tree icons in Figure 1. After confirming the choice, the student would be presented with a screen such as appears in Figure 2. Actually, the student has already begun to input numbers in the problem specification window at the lower right portion of the screen - by clicking on either the "up" or the "down" arrow specifying the quantity of trees to be used. (The user could also simply type in the appropriate number if desired.)

The reader will notice that any two of the three quantities specifiable in that part of the screen determine the third. Thus, providing three different problem types amounts to providing numbers in two of the three positions of that window and requesting the third. In particular, the student is now in a position to model any of the three situation types abstractly represented by positions of the unknown in the following equation:

$$E' = E*I$$

We have used the letters E and E' to denote extensive quantities (which denote the amount of something) and I to denote an intensive quantity (the amount of one thing "per" the amount of another) . The quantity E' is instantiated as the number of objects - trees in this case - E as the number of boxes, and I as the number of objects per box.

After the given (known) numbers are input, the user determines the third by an appropriate grabbing and dragging action, depositing the objects into the rectangular cells. Clearly, each of the situation types has a distinct set of icon-object manipulations constituting its solution:

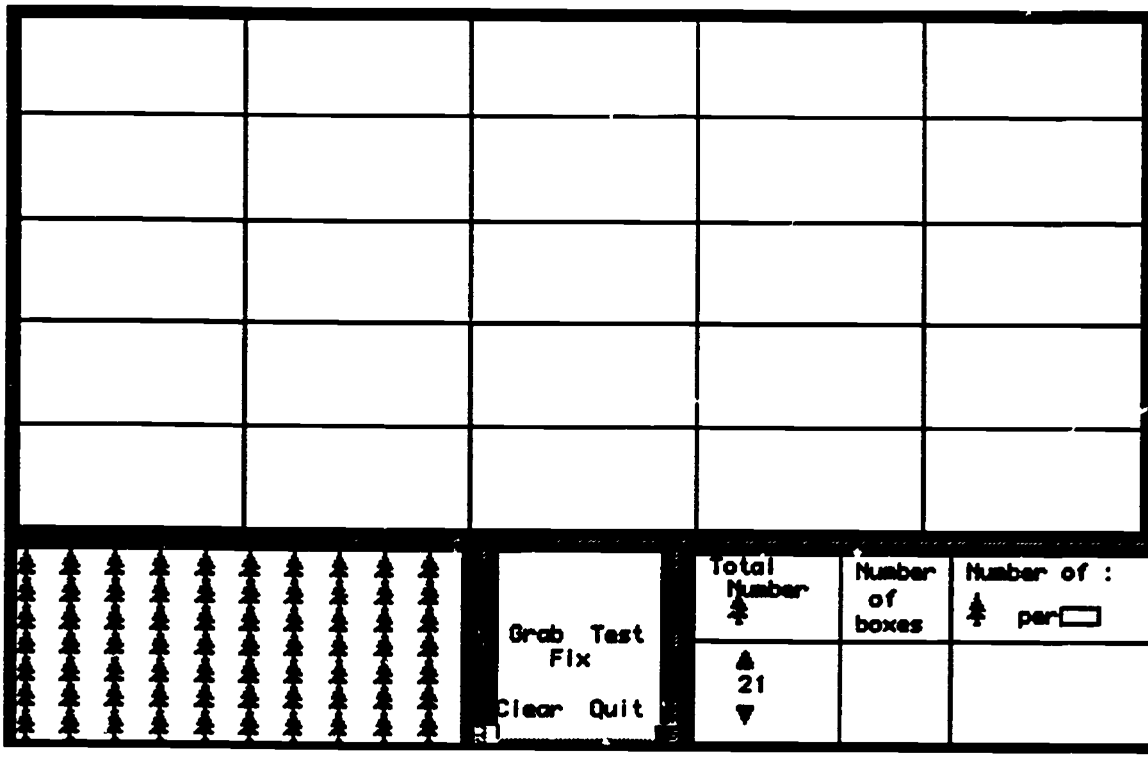

Figure 2

(1) providing the number of boxes and the number of icons/box yields a "rate" type of multiplication problom - find E*I: *the total number of icons;*

(2) providing the number of boxes and the total number of icons yields a partitive (fair share) division problem - find E'/E: *the number of icons/box;*

(3) providing the number of icons/box and the total number of icons yields a quotative division problem - find E'/I: *the total number of boxes.*

Since they are not at the heart of the issues of interest, we will not describe details regarding various options and scaffolding that are available except to note that the user can choose the two quantities to be entered in the appropriate quantity label-area in the lower right side of the problem window.

In Figure 3, for example, the user is determining how many apples will be needed altogther if four children are to get three apples apiece. The user is grabbing and dragging three apples at a time into the cells (grouping three at a time is not necessary unless computer help is asked for, in which case the computer constrains the grabbing process so three are grabbed automatically). Here the user is using the respective boxes (cells) to represent the children, so that the semantic relation of "giving" apples is modeled directly by the act of depositing the apple icons in a box.

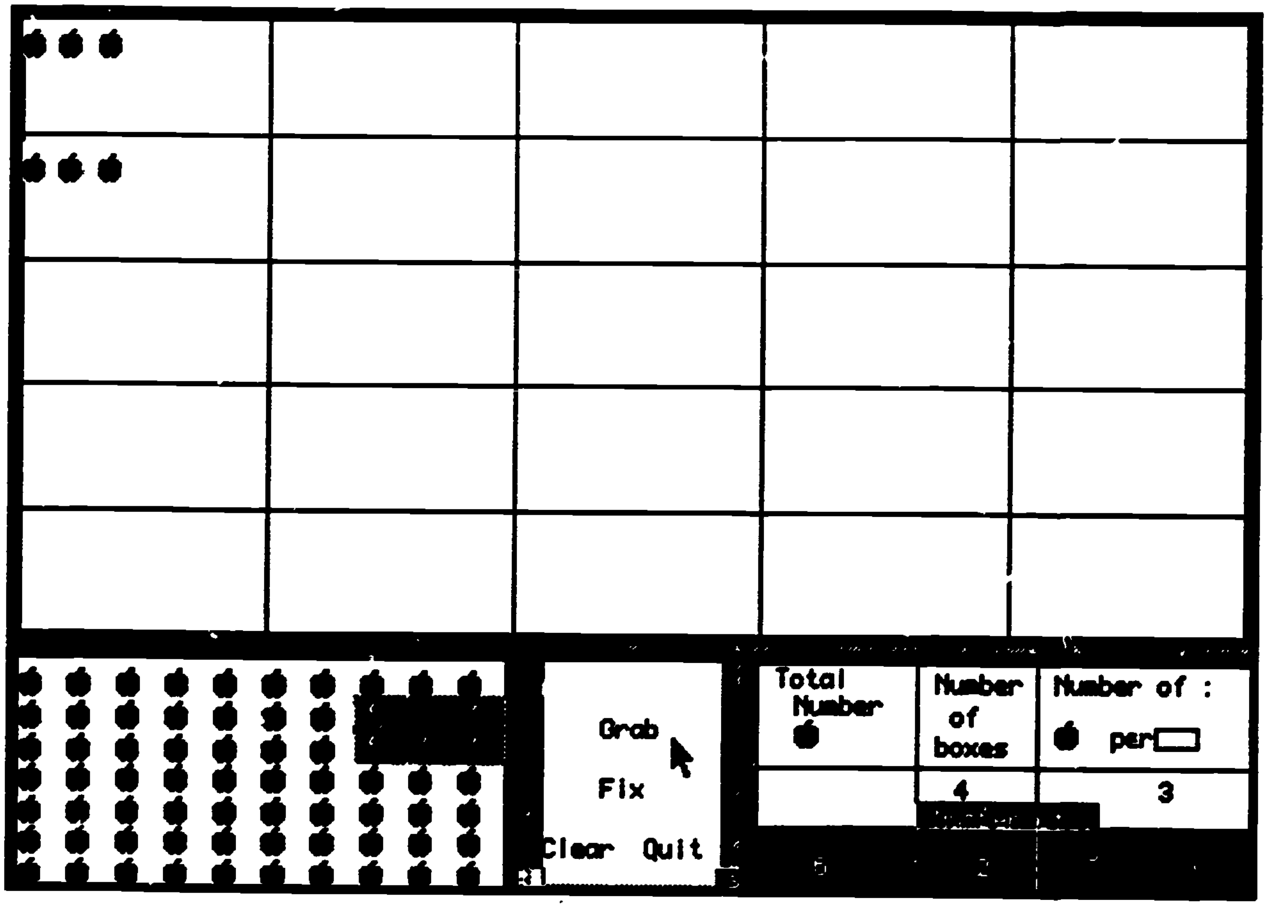

Figure 3

The computer keeps track of how many objects and boxes have been used, and, if the number of items deposited per box is the same for all boxes, it displays that number as well - this information is continuously updated in inverse video as indicated in Figure 3.

In Figure 4, we can take the problem situation to be a partitive division - 20 apples are to be given equally to 5 children, how many will each child get? Here, as is often the case among young students, the apples are distributed in a "round-robin" one-at-a-time style, a very primitive action familiar to many pre-school children (Hunting & Sharpley, 1988).

However, if the problem were its numerical <u>quotative</u> equivalent - twenty apples are to be distributed so that each child gets five, how many children will get apples? - the more likely distribution pattern would be to grab and distribute five at a time (again supportable by a computer scaffolding option). Indeed, the two forms of division, formally identical, have clearly different concrete enactments in ICE-1 (Bechtel & Weaver, 1976; Kratzer & Willoughby, 1973).

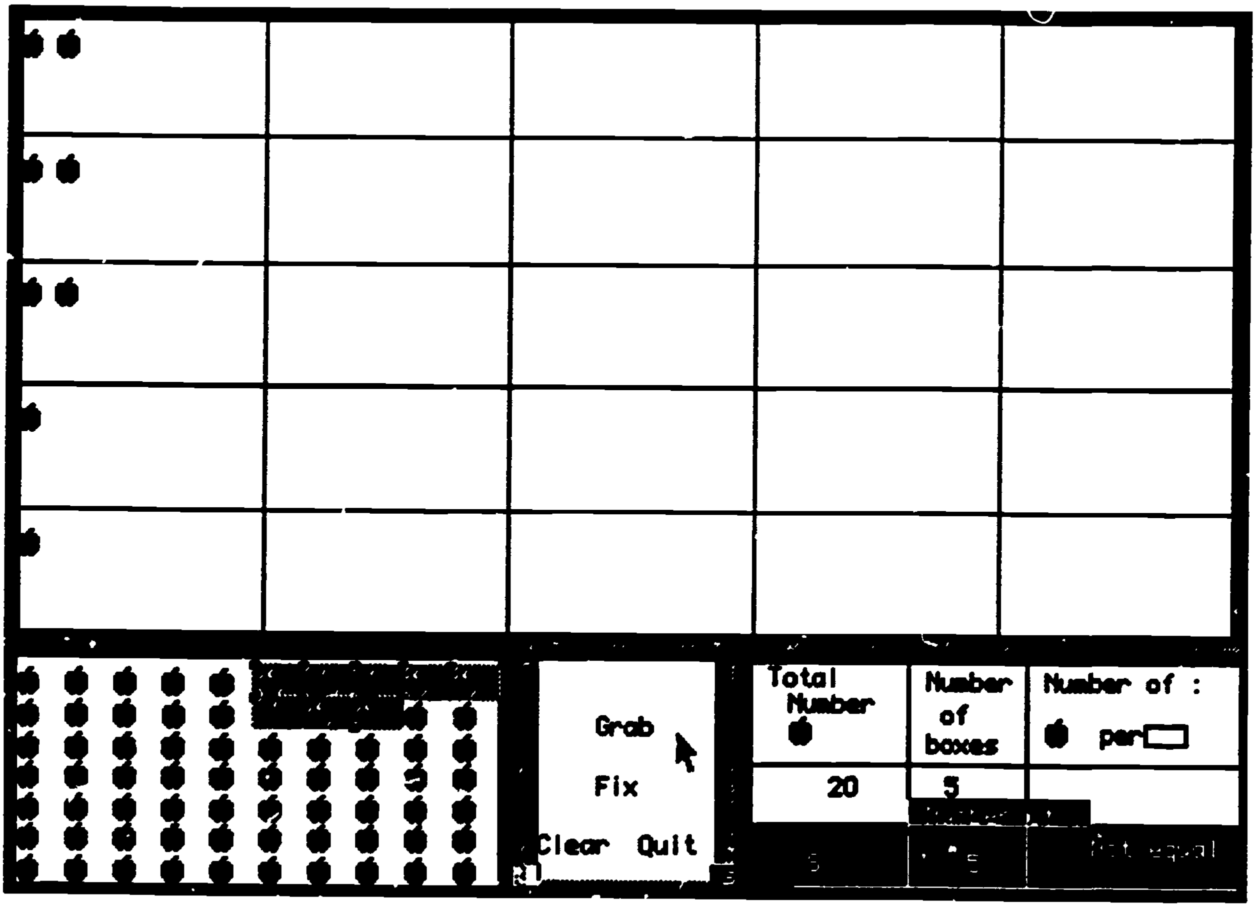

Figure 4

DIFFICULTIES - INTERFERENCE BETWEEN VISUAL AND SEMANTIC RELATIONS

We used the above environment as well as extensions of it that included tables of data and coordinate graphs that are action-linked to the icon calculation environments in a ten session teaching experiment spread over a three week period as part of a summer program for children in an urban school setting. The group of interest here consisted of seven 6th and 7th graders classified as average or below average in school mathematical performance. With one exception, none had prior competence in ratio reasoning as measured by an individually administered pretest, and most had difficulty with division at some level of complexity. Thus instruction began with activities using ICE-1 as indicated above.

Three of the group had difficulty using ICE-1 to solve problems represented by the following problem (which is the one that caused the most difficulty):

> The Boston Sunday Globe has 7 sections. If you deliver 5 newspapers, how many sections do you deliver?

One student, "Antwon" illustrates the difficulty. He picked the rectangle icon and said it stood for a newspaper. He then began to fill out the table as shown in Figure 5 by putting the 5 in the place indicated.

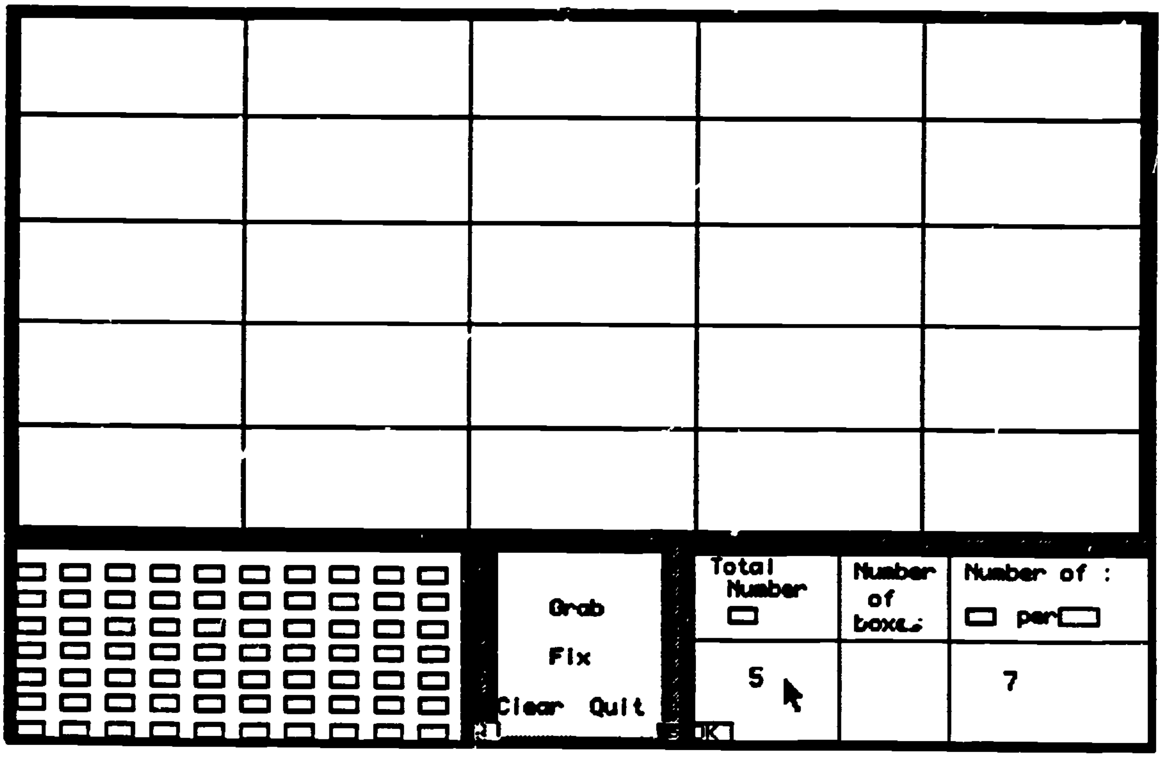

Figure 5

When probed, he said that the boxes stood for the houses to which the newspapers are delivered. Hence the sections were not represented. When asked "what is going into what?" he could not answer. He could not even recall how many sections went into a newspaper when asked. He guessed 3, and the interviewer pointed out the 7 in the problem statement. Antwon then entered the 7 as indicated in Figure 5 and began dragging rectangle icons into the boxes as shown in Figure 6.

He was lost, unable to model a simple containment relation and its multiplicative consequences. In response to a later question about what the filled in column of boxes stood for -- after the left column of boxes was filled with 5 rectangle-icons each -- he said that a whole column of boxes stood for a newspaper.

It seems apparent that the interference between, on one hand, the visual impact of his choice of icon and the associated placement of the sets of icons and, on the other, the actual semantic membership relation between sections and newspapers, led to an incoherent conceptualization of the given situation.

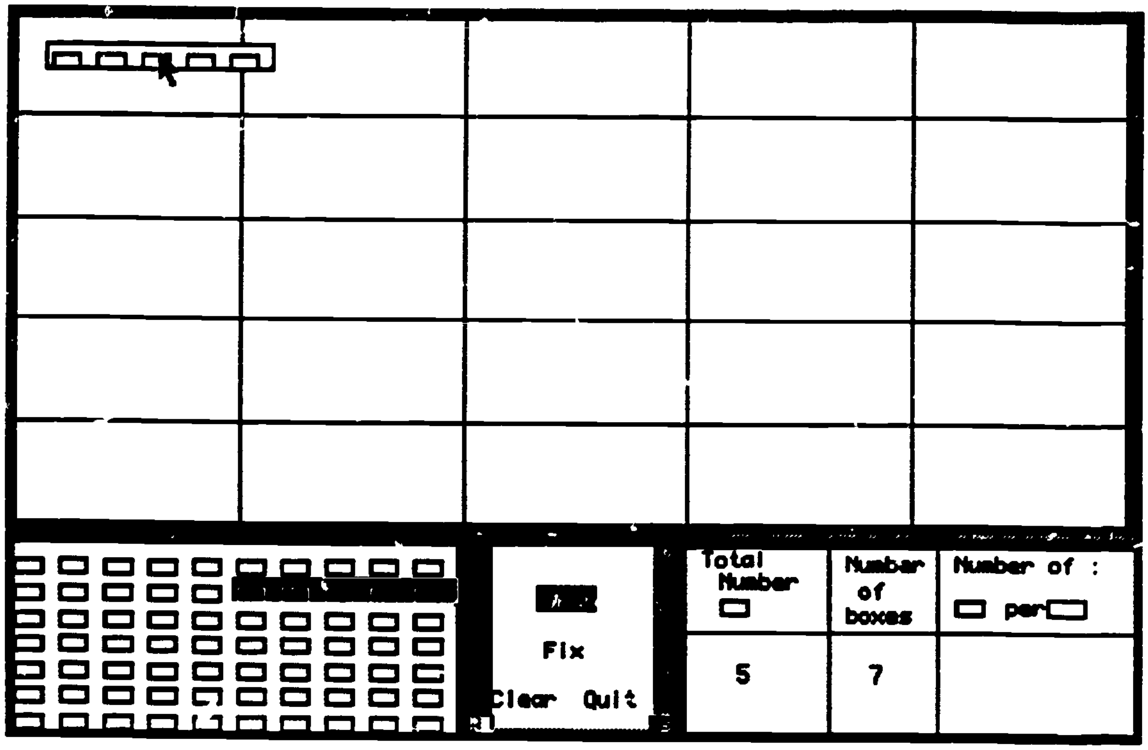

Figure 6

After some prompting, he resolved the situation without re-choosing icons by letting the houses to which papers were to be delivered be represented by boxes on the assumption that each house gets only 1 newspaper. Thus the boxes also represent (or were at least in 1-1 correspondence with) newspapers, and then the rectangle-icons could stand for sections.

ANALYSIS

Observations of Antwon and the others helped convince us of the following. First of all, his choice of icon was governed by a quick, perhaps unconscious attempt to symbolize the most salient aspect of the problem, the newspapers. He saw it as a "newspaper problem" and referred to it as such. (Interestingly, so do we when discussing it among ourselves.) Second, the icon choice menu (see Figure 1) offered an object that, again, without much conscious deliberation, supported this initial choice of item to be symbolized. Third once this choice has been made, the environment leaves no option regarding how the containment relationship can be modeled. Once the object icon is chosen in ICE-1, the boxes automatically act to contain it as they are distributed - there is no choice remaining here regarding what contains what. Fourth, the dissonance between the visual experience and the semantic relationship embodied in the text disoriented the student so that he was unable to be coherent, and even lost a grasp on what was given vs what was asked for.

Indeed, we would speculate that, just as the visual experience is a powerful organizer of appropriate cognitive structure in some cases, when it acts in other cases to produce a structure contradictory to an existing one, it leads to debilitating confusion. In the case at hand, the existing cognitive structure was associated with the containment relation that holds between sections of a newspaper and the whole newspaper. This was violated by the visual representation. This type of phenomenon occurred with two other problems, each of which embodied a containment relation that was to be mapped to an intensive quantity, an "x per y" statement. For example, another problem involved putting ice cubes into glasses, and the students chose a bottle icon for the glasses.

We take the heart of the difficulty to be the fact that the modeling act requires two student choices, one for the containing thing and the other for the contained thing, while the software asks only for one choice, with the other being made by default. The choice, by the nature of the rectangular array and the fact that one puts items into the arrayed boxes, requires that the contained thing be represented by an icon (which is the thing chosen) while the containing thing be represented by the boxes themselves (which does not require an active choice). If the most salient item in the situation description is the containing thing and that thing has an icon-equivalent, then the observed difficulty occurs. Otherwise it does not occur. For example, in another problem, we asked how many basketballs would go equally into each of 6 bins if there were 18 basketballs. All students chose the ball icon, the default representation of the bins as boxes was an ideal match, and no difficulty arose, even among another group of students who were judged very weak mathematically.

OPPORTUNITIES - EXPLICIT MODELING CHOICES

The above-described difficulties are avoided if one needs to make two explicit choices regarding the two parts of the intensive quantity being modeled, and indeed we have never observed such difficulties among several dozen users of the two-icon calculation environment, ICE-2. This environment was designed to support modeling the following kind of problem situation: We are making applesauce in such a way that 3 apples are needed for each 2 bowls of sauce. How many apples are needed to make 10 bowls worth of sauce. (Note that this is not a "pure" containment relation.)

In ICE-2, the student selects two icon-objects from the Figure 1 menu, say apples and bowls. This leads to the kind of screen shown in Figure 7, where there are two icon-reservoirs from which to grab and deposit objects.

The student can now select and deposit apples and bowls into the boxes in a way that reflects the numerical correspondence between apples and bowls, by putting 3 apples and 2 bowls into a box and then repeating this action until 10 bowls are used - which, of course, requires 5 boxes via a quotative division of the type described earlier.

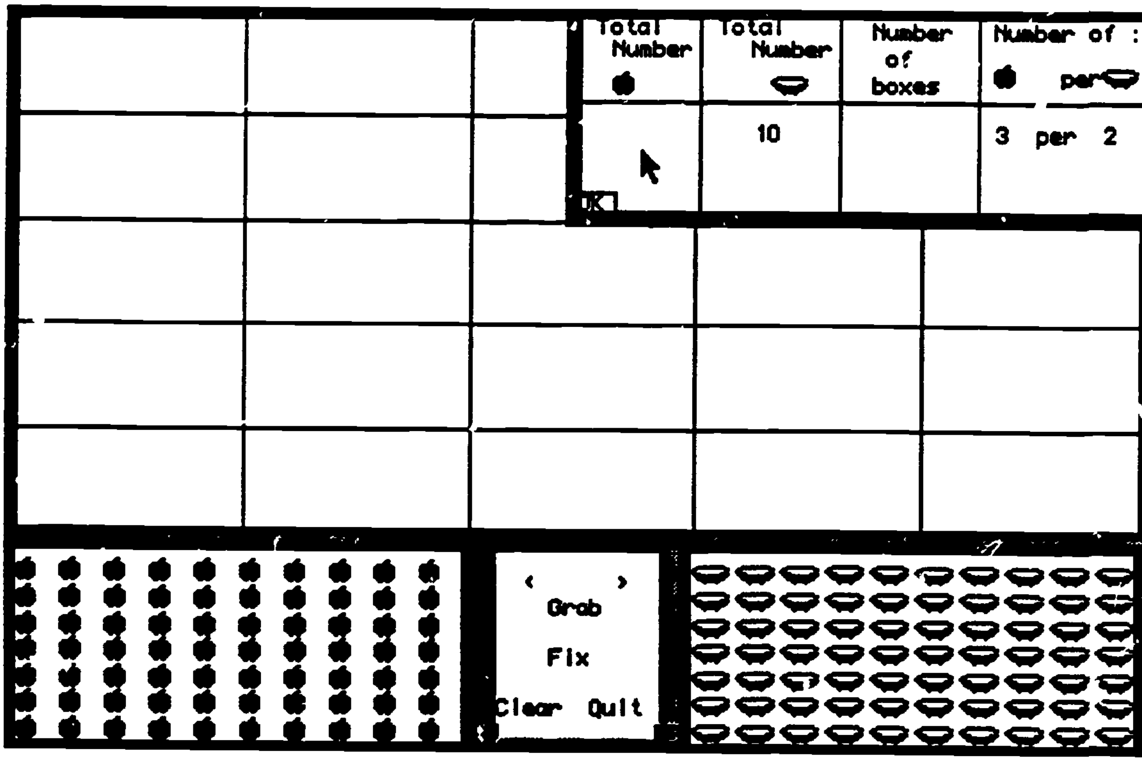

Figure 7

As with multiplication and the two forms of division discussed earlier in the context of ICE-' the particular actions on the objects carried out by the student amount to a concrete, visually explicit reasoning strategy. For example, the student can deposit 10 bowls two at a time into separate cells, and then deposit 3 apples in each of those cells matching each pair of bowls. Then the number of apples can be counted by counting either the "grayed out" apples in the reservoir or the apples in the boxes (which don't show very well in this reproduction). In Figure 8 we have asked for a computer report on the number of objects and boxes used. This information appears in inverse video in the upper right side of the screen.

Note that the full semantic relationship itself between the apples and the bowls is not being directly or literally represented - it is indicated in a more abstract way by using juxtaposition of the two icon-types in the same box. But of course it is in the nature of models that they leave out certain features of the situations that they are modeling and maintain only those essential to the purposes of the model - which in this case is to determine the appropriate number of apples needed. The essential feature here is the correspondence between the numbers of bowls and the numbers of apples, which is explicitly provided by juxtaposition within cells and the organized collection of identical cells. See Figure 8. (Of course, one could envision more explicit visual models, such as drawings of bowls containing one and a half apples each. However. it may not always be the case that the explicit visual detail of a model will contribute to its utility in quantitative reasoning.)

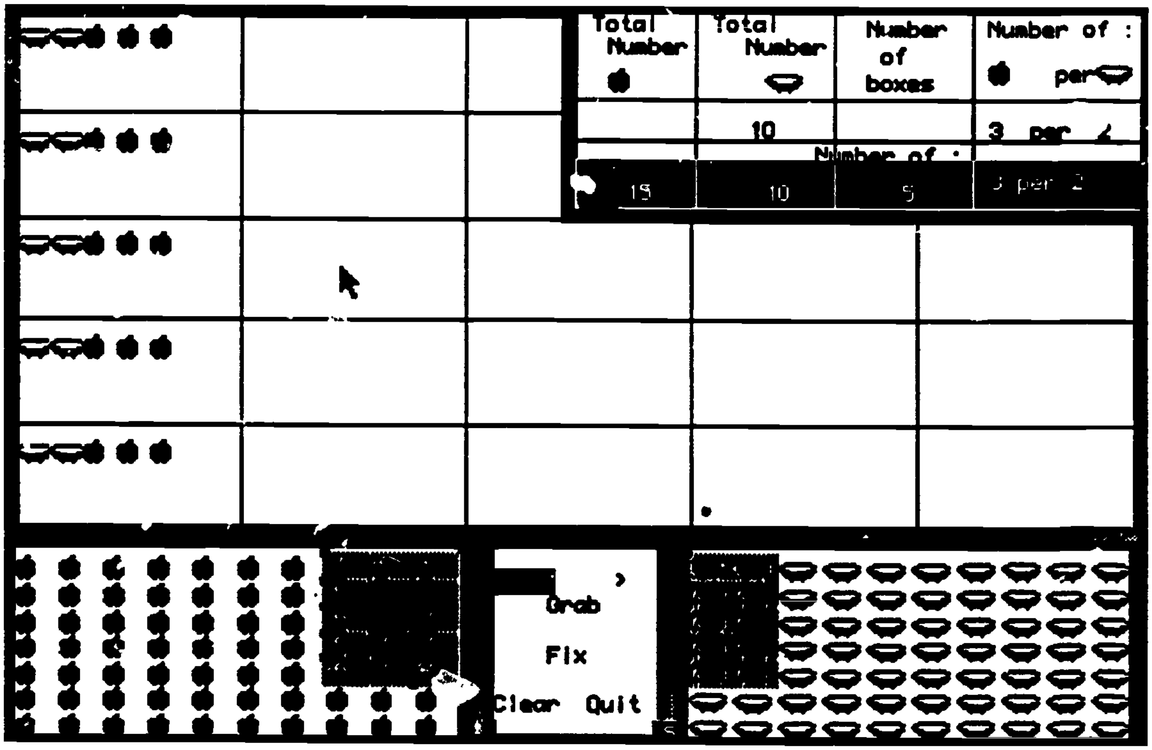

Figure 8

The particular concrete reasoning strategy for solving missing value problems supported by ICE-2 has been referred to as the "boxes strategy" in Kaput, et al (1987). In Kaput (1988a) the underlying cognitions are discussed in much more detail, especially as they relate to the <u>forms</u> of the representations involved.

CONCLUDING REMARKS

We have seen that the use of a visually explicit representation of the quantities involved in a problem, especially the intensive quantities, allow a concrete reasoning strategv to be used to solve classical missing value problems that are normally not solved competently until significantly later in the curriculum. We have planned a series of extended teaching experiments that introduce these representations in the 4th grade and will link actions on these to tables of data, coordinate graphs and algebraic equations in grade 5. Such representations also help attack certain well-known trouble spots, e.g., how to interpret remainders in both types of division problems.

While there is much potential in the use of concrete visual representations to build on "natural" actions on objects in the student's world, there are pitfalls as well, an example of which we described above. On the assumption that there is enormous potential in connecting actions on such concrete representations with actions on the more abstract ones of higher mathematics, we are also planning a much more complete set of object based computation environments.

However, as Fischbein, et al, (1985) and others, e.g., (Bell, Swan, & Taylor, 1981; Greer, & Mangan, 1984) have pointed out, competence in the arithmetic of discrete quantity does not necessarily translate to competence in the arithmetic of continuous quantity. In fact, habits of mind developed in the former may yield serious difficulties in the latter. Hence we have also built extensions of our environments to the continuous world, where the object-icons are replaced by continuous line segments. Such are used to model a situation such as "A summer camp has found that on most days, every 3 children drink 2 liters of soda. On a given day, how much soda will be drunk by 10 children?"

Easily executed "click-and-drag" scaling options enable a user to align parallel lines representing the quantities involved in such a way that their labeled tic marks match up. To answer the indicated question the user can then sweep a line across the parallel lines from left to right as indicated, stopping at the 10 children mark. The system maintains and can make available in what is equivalent to a dynamic odometer, the updated values of the (children, liters) ordered pair as the line is dragged along.

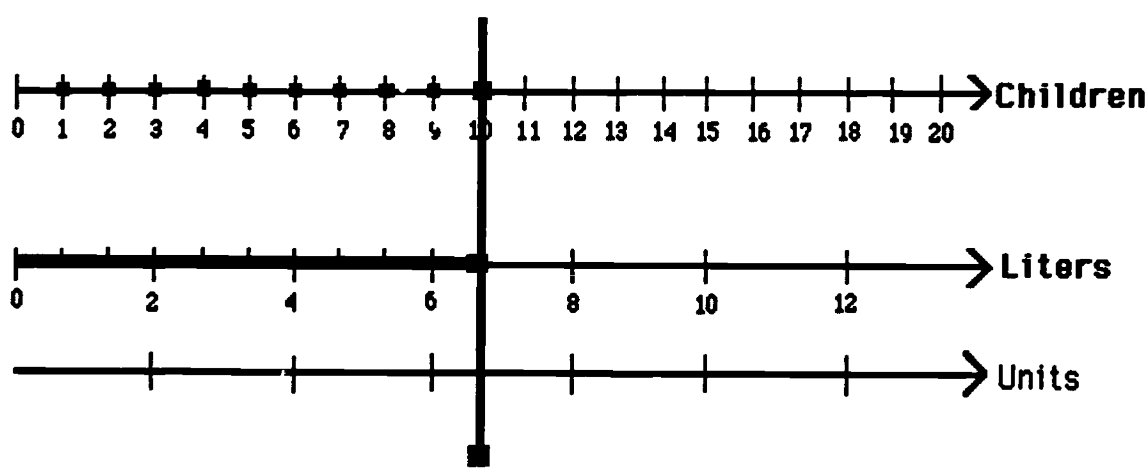

Figure 9

We are testing this type of environment with 5th and 6th graders. In addition to determining the best forms of the continuous environments and the various actions to be supported, we are attempting to determine to what extent does the discrete strategy transfer to the continuous case and how might it be extended to deal with the above kind of non-whole number producing situations? And what additional features of the continuous representation might be needed to extend the transferred strategy, i.e., is the third "unit line" given in Figure 9 needed to match the organizing cells of the discrete representation, or is the alignment of the scales sufficient?

Clearly, much more work needs to be done. In fact, we are merely at the starting point of exploiting the representational power of the computer medium in this as well as most mathematical domains (Kaput, 1988b).

One last point seems worth emphasizing - the fact that the medium we are exploiting here is a dynamic one. The key ingredient is the ability to support real time actions on visual representations. We distinguish this ability from the ability to represent actions or procedures visually, which may not necessarily require a dynamic medium. For example, the static paper-pencil medium can represent a procedure in terms of a flow chart. Or more subtly perhaps, it may represent an arithmetic procedure as a static arithmetic expression, e.g., 7*2 + 9. For much more detail on the theoretical underpinnings of distinctions between procedure representing objects and actions on objects see (Kaput, in press).

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