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ABSTRACT

This document reports on a project that is examining some of the difficulties encountered in teaching word problems involving multiplication, division, and intensive quantities. Some of the various uses of these operations and their structures are considered. Described are discoveries and assumptions regarding students' cognitive models of these operations, especially as they pertain to intensive quantities. The report also describes the project's computer software being developed to enrich and render more flexible student cognitive models of these operations and their quantities. (TW)

MULTIPLICATIVE WORD PROBLEMS AND INTENSIVE QUANTITIES:

AN INTEGRATED SOFTWARE RESPONSE

Technical Report

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Preface

This report will review our own work and that of others dealing with our previously identified and now more tightly defined target of difficulty, word problems involving multiplication, division, and intensive quantities. We will first outline the various uses of these operations and their structures, and describe what we know and what we assume regarding students' cognitive models of these operations, especially as they relate to intensive quantities. We will then go on to describe our planned multiple representation integrated software, which is intended to enrich and render more flexible, student cognitive models of these operations and quantities.

TABLE OF CONTENTS

1. Overview.

1.1 The Bases of Our Program

Recent Project activity has refined and focused inquiry on ever more tightly defined targets of difficulty associated with student inability to solve school word problems. We have narrowed the issues to be addressed as those having to do with building appropriately rich and flexible student cognitive models of multiplication, division, and intensive quantities. Among these we have decided to begin with intensive quantities because they are a source of difficulty in both multiplication and division problems, because they are intimately involved in multiplication and division, and because exploring an effective curricular response to student difficulty with intensive quantities will prepare the way for dealing with the operations as well. We also expect that the style and structure of software planned for intensive quantities will extend to the software for modeling the operations.

But before outlining our approach to the building of cognitive models, we will sketch the philosophical basis for both our research investigations and our technological-pedagogical initiatives.

The approach taken by the Word Problems Project stands on three philosophic foundations:

(i) A philosophy of quantity that includes explicit and systematic provision for the role of referents for numbers. This view animates our concern with the semantics of problem situations and the distinctions we consistently make between intensive and extensive quantities.

(ii) A philosophy of educational technologies that exploits their new capacity to provide multiple, interactively coordinated representations. Broader dimensions of the Center's philosophy of educational technologies are discussed in available Center Reports (e.g., "The Use of Information Technologies for Education in Science, Mathematics, and Computers," Educational Technology Center, March, 1984). The aspect of concern to this particular project is the new (to school microcomputers) capacity to support simultaneously visible representations of varying concreteness that are dynamically linked. By acting on such coordinated representations, the student will be aided in building an increasingly powerful repertoire of cognitive representations <u>that are thereby intimately coordinated.</u> We feel that the new dimension of representational power embodied in the simultaneity and depth of interfacing among representations may rival other of the microcomputer's educational gifts, such as graphics and interactive capabilities. Actions and consequences across representations that were once representable only serially in temporal sequence, and usually in clumsy fashion, will now become simultaneously visible or controllable, and so connections can be explicitly represented, inspected, utilized and internalized.

(iii) A philosophy of curriculum that puts a premium on longitudinal coherence, coherence with respect to mathematical concepts and forms for their representation. The linkages now possible in the learning environment will greatly enhance our ability to build more direct and accessible routes to more advanced mathematical ideas and representation systems. These linkages will also, when coupled with new understandings of how complex mathematical ideas develop and interrelate, help to build the cognitive coordination of the various strands of those complex ideas. The long sought-after ideal of unity and coherence in the mathematics curriculum may now become much more achievable than ever before.

1.2 Student Cognitive Models

Without rich and flexible cognitive representations of multiplication, division and intensive quantities, students fail to recognize the appropriate mathematical structures in situations calling for their application. Recent work has provided a general understanding of students' common cognitive models of multiplication (dominated by the repeated addition model), division (the partitive - "fair share" - and quotative models, dominated by the former), and intensive quantities (dominated either by familiar semantic-based structures when the quantity refers to a well-chunked "rate" such as miles per hour, or by simple ratios when less familiar referents are involved).

See, for example, the work of Fischbein, et al (1985), Greer & Managan (1984), Bell, et al (1984), Silver (1985), and Greer (in press). Furthermore, as detailed in the body of this Report, we have reanalyzed our earlier data to relate our definitions of extensive and intensive quantities to the partitive and quotative models of division, thereby helping to confirm hypotheses regarding the dominance of the partitive model among students. We also will relate our perspective and work to others' work on ratio, rational number, proportion, and more general "multiplicative structures" in the sense of Vergnaud (1983). A large body of research bears upon our work by virtue of its being at the nexus of several complex mathematical ideas.

1.3 Development of Linked-Representation Software

We plan to exploit the above-mentioned new microcomputer capacities to address directly the paucity, inflexibility, and poverty of student cognitive models for multiplication, division and intensive quantities. The manipulation of multiple representations of these concepts in a coordinated window environment appears to provide a potentially ideal vehicle for building those cognitive representations now lacking _and_ for linking these to students' existing mental models.

We now sketch briefly our initial plan for building a four-representation software learning environment to deal with intensive quantities, ranging from very concrete iconic representations of physical entities to a potent graphical representation of intensive quantity as slope of a line in a coordinate plane.

(i) The most primitive representations will provide concrete iconic representations of intensive quantities. A student can act upon these in any of several ways - for example, by replicating or deleting copies of the base set of (labeled) objects to yield a specific numeric value of the numerator or denominator. More than one form of such a representation will be available to account for the different semantic relationships among the referents in intensive quantities as well as for the differences between discrete and continuous quantities. This will also provide more than one starting point for students who may differ in their primitive conceptions of intensive quantities.

(ii) Coordinated with the previous representations will be a vertical data table (with appropriately labeled columns), so that as the number of base sets changes, the corresponding entries in the data table list change (or are appropriately highlighted). Actions on or tasks involving the concrete representation are, of course, paralleled in the data table representation.

(iii) A critically important high level "goal" representation will be a coordinate graph where the axes will be labeled to match the object labels and where points would be plotted to parallel activities in the other representations (and vice-versa, of course) - the numerical ratio associated with an intensive quantity is thus represented by the slope of the line of points.

(iv) A fourth representation will be a Semantic Calculator-like "operations pad" where one manipulates and does arithmetic with formal expressions involving the involved quantities (Schwartz, 1983).

The first, third, and fourth components would thus provide at least three very different cognitive tools for getting hold of a problem, and the second (the table) would provide a numerical bridge between any of the other two. Each of the three basic tools would engage a different portion of the student's cognitive apparatus, involving, respectively, concrete perceptual processing, visual-imagistic processing, and the formal-linguistic processing associated with the manipulation of formal expressions. We may incorporate other mediating representations into the software environment if they prove necessary or desirable - for example, a "function machine" to assist with the generation and interpretation of the coordinate graphs. The concrete representations would serve as a flexible starting point to accomodate differences in student approaches to a given task; but also, since they would all be coordinated with the same mathematical representations embodied in the other three, their mathematical commonalities would be highlighted as the one aspect that they all share despite their surface differences. By adjusting the tasks, a given student could be brought into contact with a sequence of such concrete starting points.

A major hypothesis to be investigated beyond the effectiveness of the individual components is the extent to which experience with such dynamically linked systems builds cognitive coordination of the associated cognitive representation systems. We and others (e.g., Dickson, 1985) see great promise in the technology's power to present and integrate representations simultaneously that have previously been approachable only serially in dynamic media, including the "live" classroom teaching medium. In static media, simultaneity of presentation is possible, e.g., a table of data and a graph; but only the products are available for presentation, not the processes.

As components become available (a few pieces already exist in prototype form as the result of earlier work at the Educational Development Center), we will videotape and analyze student interaction using a split screen format, half for the student and half for the computer screen, as well as record keystrokes. We expect that this stage of the project will continue for at least half a year. As coordinated components reach beta stage (essentially the final stage for our purposes), we will enlarge the testing of their impact to larger groups involving 3rd - 7th graders.

More specifically, this research/development enterprise has two direct purposes and one methodological purpose:

(i) To address the identified target of difficulty by beginning the process of appropriate curriculum design through application of state-of-the-art technology. (Note that this is by no means a curriculum development project, but rather an investigation into the nature of potentially appropriate technological components of a curriculum.)

(ii) To investigate the cognitive impact of the new capabilities of school microcomputers in a more general sense, especially their effect on the building and coordinating of cognitive models and their potential for introducing more powerful representations to younger students that lead more smoothly and coherently to the more advanced ideas of mathematics.

(iii) A methodological objective (not unique to this particular Center project) is the rigorous assessment of the above-mentioned cognitive impact. This task will be quite challenging given the lack of curricular or pedagogical parallels that would support any kind of "control" experience for comparison purposes. Not only is the experience of simultaneous and coordinated visual representations of elementary mathematical concepts quite novel, but in addition there are few if any precedents for the kinds of activities that are possible in such an environment, especially at the 3rd-7th grade levels at which we will be concentrating our work. As indicated in Section 4, we are utilizing the insights and materials of several earlier curriculum development projects from the sixties and early seventies (in addition, of course, to current research regarding cognitive representations of mathematical ideas).

We should also point out that in the much more completely studied and somewhat less mathematically complex domain of addition and subtraction (Carpenter, et al, 1982), some pilot software development joining concrete and formal representations has taken place (Moser & Carpenter, 1982 - the Math Boxes Program; Feurzeig & White, 1984 - the Summit Programs; Sybalsky, Burton & Brown, 1984 - the Arithmekit Environment; Larkin & Briars, 1982 - the CHIPS Program). While none of these efforts addresses the specific mathematical content of interest to us, they do direct attention to the goal of providing connections between concrete and more powerful abstract representations, especially the earlier Logo-based numeration system programs of Grant, Faflick, and Feurzeig (1971).

If this work proves successful, we have the choice of either a horizontal expansion to include the other identified aspects of word problem targets of difficulty (multiplication and division problems not covered in the work with intensive quantities, such as those involving combinatoric models, etc.) or a vertical expansion to extend the earlier representations into algebra, to the general function concept, and to non-linear quantitative relationships. Our preliminary plans are to expand horizontally with similar-genre software, but a firm decision awaits the results of the planned development and testing.

2. Project Starting Points

2.1 Our Earlier Work and Data.

Early deliberations of the groups defining the research agendas for the Educational Technology Center found a strong sense among practitioners that classic word problems are a source of special difficulty for students at all grade levels. Further discussion narrowed the types of problems to be addressed to single step word problems, i.e., those requiring the application of a single operation. Subsequent work (Schwartz, 1984) took the form of a pair of studies designed to fix empirically the semantics of problems (i) that students generate in response to a request for problems requiring a single operation, and (ii) that students have difficulty in solving. The first study collected and classified student-formulated problems according to certain taxonomic schemes devised by the researchers and appearing elsewhere in the literature (e.g., Riley, Greeno, & Heller, 1983; Carpenter, Moser, & Romberg, 1982). The second study confirmed, at least at the aggregate level, that those problem types not appearing among student-formulated problems were also those that students were least able to solve.

This work thus helped identify more specifically the target of difficulty by identifying certain categories of problems as especially difficult while confirming the assertion that a student's ability to recognize the contexts in which a particular operation applies (and thus solve word problems requiring that operation) is tied to his or her cognitive repertoire of situations calling for that operation.

We followed upon these discoveries with a tentative decision to create a multidimensional "problem web" -- a software environment in which a student, when confronted with a word problem that he or she could not solve, could systematically move along certain dimensions ordered by problem difficulty to a solvable problem. The student would then take the understanding developed in that problem situation (perhaps augmented by further exploration in the problem web) to help solve the original problem. The web's dimensions would be determined by (i) the semantic categories discovered and confirmed in the empirical work, and (ii) certain numerical characteristics of the problems (e.g., computational complexity, familiarity of numerical relationships among the referents, etc.) However, the number and irregularity of the actual task variables in the problem domain refused to accomodate a sufficiently simple and clean taxonomy which could then be mapped onto the dimensions of a problem web.

Nonetheless, the attempt to sort out the problem domain has yielded valuable insights and partial taxonomies (some of which appear below) that will serve us well in the next phase of our Project. Indeed, most of the web work was prerequisite to our current work.

2.2 Our Philosophy of Quantity.

During the past decade a consensus has developed among the mathematics education community that calls for more attention to the applicable aspects of mathematics, a consensus based mainly on ubiquitous reports of student inability to apply the meager mathematics that they do learn to their wider world of experience. Our view of this situation goes a step beyond the assertion that mathematics is best learned in context, to the assertion that THE ELEMENTARY MATHEMATICS OF SCHOOL SHOULD NOT BE, AS TACITLY ASSUMED, EXCLUSIVELY THE MATHEMATICS OF NUMBER WITH APPLICATIONS REGARDED AS SEPARATE, BUT RATHER SHOULD BEGIN WITH THE MATHEMATICS OF QUANTITY, SO THAT THE MATHEMATICS AND ITS "APPLICATIONS" ARE OF A PIECE FROM THE VERY BEGINNING.

The formal outline of our notion of quantity was sketched in the November, 1984, Technical Report, particularly Appendix A (Schwartz, 1984), and in (Schwartz, 1976). We shall not repeat its description here, but instead simply will note that the concept of quantity has a distinguished lineage, tracing back to Gauss and Bolzano (cited in Janke, 1980), Lebesque (1933-36), and Whitney (1968a, b), and that its use has been advocated in educational proposals as far back as the turn of the century (Speer, 1897). Note also that Freudenthal (1973) had also developed a formal treatment of quantity paralleling that of Schwartz. In fact Freudenthal (1973, p.207) is cited by Usiskin and Bell (1983b) as follows:

The argument of rigor against computations with concrete

> numbers (what we have called quantities) is completely mistaken. Concrete numbers are absolutely rigorous, and the resistance of some mathematicians to them is sheer dogmatism.

In fact, as argued by Janke (1980), Gauss' definition of quantity - the result of more than thirty years' effort - comprised a systematic response to a crisis in the foundations of mathematics and its relationship with science that rivaled in depth the much more heralded crisis a century later. Whereas the later crisis generated an abstract and highly formal set-theoretic and logicist response setting mathematics free from its experiential origins, the earlier crisis led to an attempted union of mathematics with its origins in the understanding of the natural world that was philosophically quite sophisticated in its accomodation of the new abstract mathematics of the time (e.g., Gauss' invention of the geometric representation of complex numbers). The logical and metamathematical contents of either approach are much less important to us than their cognitive-developmental, curricular and pedagogical implications, which in our view greatly favor the earlier as a starting point, particularly given the widespread student inability to apply mathematics.

2.3 Our Philosophy and Application of Educational Technology.

While the approach of the Center as a whole is deliberately eclectic, based on the breadth of application arenas it is addressing, we have chosen a particular approach based on the needs of our particular domain of inquiry, multiplication and division word problems involving intensive quantities. We see enormous, as yet untapped, potential in the next generation of school microcomputers' ability to provide multiple and simultaneous representations of important mathematical ideas. Even more important is their ability to provide explicit, easily controllable coordination of such representations, an ability not possessed by earlier generations of school microcomputers.

Discussions with industry decision makers strongly suggest that the new microcomputer software environments will be distinctly Macintosh-like, including new Apple computers compatible with the educational standard Apple II family. Hence our prototype development will take place on a Macintosh computer in a new structured BASIC that utilizes the Macintosh windowing and mouse interface routines, versions of which are necessary for the applications that we have planned. Although immediate school application of our development efforts is clearly not intended or expected, it is comforting to know that the next generation of school microcomputers is likely to have the multiple representation capability that we regard as critical to our approach.

Since the software environment that we envision for our mathematical representations is, by virtue of the Macintosh, not new, it is perhaps wise to emphasize the <u>educational</u> and <u>cognitive</u> novelty of its features. Except for very modest educational examples of linked representations such as the simultaneous presentation of small amounts of data and their graph, or small numbers of iconic screen objects and their simple additive combinations (in the software cited in the Overview), no significant interaction of several simultaneous representations on a single screen has been possible. On the other hand, recent research has shown that the cognitive coordination of several representations is critical to success in physics (Chi, et al, 1980; Heller & Reif, 1984), and in complex mathematical domains such as the domain of rational numbers, for example, where several subconstructs and their different representations must be learned and coordinated (Post, et al, 1985; Behr, et al, 1983; Lesh, et al, 1985; Lesh, Landau, & Hamilton, 1983). As indicated below, the operations of multiplication and division may likewise be decomposable into subconstructs based on usage-types (Usiskin & Bell, 1983a, b, c) and other more structural features (Vergnaud, 1982, 1983). Again, the different cognitive structures associated with these subconstructs and their different representations need to be learned <u>and</u> coordinated.

Furthermore, this parsing of mathematical concepts into distinct subconstructs with partially separate developmental trajectories, examining their individual features, applications and representations, and then devising cybernetically coordinated representations to tie them cognitively togther, may, in fact, be repeated all across mathematics. This would render the new microcomputer capacity that we will be exploring and exploiting all the more important to understand. See Section (4) for more discussion of these issues.

2.4 Coherence in the Mathematics Curriculum.

Common wisdom on the subject of curricular coherence calls for logical vertical coordination of topics, spiralling where possible, ease of access, and an absolute minimum of dead ends or abrupt notational/representational shifts. Implementation of this wisdom is codetermined by the perceived utility and learnability of the mathematics under consideration. Historically, this implementation has been little informed by an understanding of the underlying cognitive issues and has been only marginally affected by information technologies (e.g., calculators). A radical change in both aspects of this pattern is now possible if not likely, at least in the domain of interest to the Word Problem Project.

Since we have taken as our base task the deliberate enrichment and coordination of cognitive models of intensive quantity, the question arises <u>which</u> external representations of this mathematical concept to use? THE PRINCIPLE OF MAXIMAL LONGITUDINAL COHERENCE ARGUES FOR THOSE REPRESENTATIONS THAT SUPPORT WIDE APPLICATION OUTSIDE OF MATHEMATICS AND CONTINUING GROWTH WITHIN MATHEMATICS. This is our primary curricular justification of the four representation types listed in the Overview.

Our choice of the multiple-window microcomputer environment as the chief pedagogical tool in this endeavor then aids in achieving the longitudinal coherence in several ways:

(i) We can create a range of representations differing in concreteness and complexity that can connect to those the students already possess.

(ii) We can explicitly link these representations so that a student can "ramp up" from his or her more primitive representations to more abstract and flexible ones. Facilitating the ability to move to more powerful representations should make the latter more accessible at an earlier age, enabling us to introduce potently general models far earlier than commonly attempted, models such as the graphical model of intensive quantities. (In this representation, the intensive quantity "10 mi/hr" can be viewed as the slope of a straight line in the first quadrant of the coordinate plane whose horizontal axis is measured in hours and whose vertical axis is measured in miles.)

(iii) In domains such as the rational number domain where several subconstructs are involved, each with its own "best" or most natural representation, these different representations can be more dynamically coordinated, particularly through the use of simultaneous representations.

(iv) In some cases, where operations with a particular idea is computationally messy and as a result postponed, we may be able to introduce it and tie it to its natural conceptual relatives by leaving the computations to the software.

In all the above examples, the technology may provide the vehicle for curricular coherence that has been historically out of reach.

3. Multiplication, Division and Intensive Quantitities: A Selected Theoretical Review.

3.1 An Overview of the Pedagogical Problem.

The one clear consensus of earlier and current work on multiplication and division word problems is that many students, grades four and onward, have inadequate cognitive models of multiplication, division, and intensive quantities - these operations and quantities have limited conceptual content for them. The recent work of Fischbein, et al (1985), Bell, Fischbein and Greer (1984), and Greer and Managan (1984) shows that the primitive models that students do possess constrain their ability to solve word problems. More particularly, their research has shown that students' primitive models of multiplication are based on repeated addition, and that their models for division are mainly based on the partitive interpretatation. Fischbein and Greer each used problems involving choice of numbers that violated the assumptions of those primitive models (e.g., fractions or decimals) to expose the weaknesses of existing student cognitive models. In addition, our work has shown (1) that multiplication or division word problems involving intensive quantities are especially difficult, even for above-average 12th graders (Schwartz, 1984), and (2) student-formulated problems contain relatively few non-partitive division problems and a very limited variety of multiplication problems. This last result dovetails well with the Fischbein-Greer results.

The consensus view is that the building of appropriately flexible cognitive models of multiplication, division, and intensive quantity is not easy, because at an early age the materials and situations leading to the building of cognitive models must necessarily be simple. Yet the early models seem to control what kinds of models are built or applied later. The problem is well drawn by Fischbein, et al (1985, p.15):

> What are the sources of the primitive models? Two explanations seem plausible. The most direct explanation is that the model reflects the way in which the corresponding concept or operation was initially taught in school. As the first interpretation learned by the child, it tends to be strongly rooted in his or her mental behavior... A second explanation is that these primitive models are so resistant to change and so influential because they correspond to features of human mental behavior that are primary, natural and basic. People naturally tend to interpret facts and ideas in terms that are behaviorally and enactively meaningful. This tendency may maintain the primitive models above and beyond any formal rules one may have learned. In our view, both explanations are correct.

They go on to point out that as first choices for the operations, these models make perfect sense, especially from a Piagetian perspective of mental operations as internalized external actions. But then, as they put it (p.15) "teachers of arithmetic face a fundamental didactical dilemma." Finally, after noting that later instruction and cognitive development do not undo the influence of the primitive models, they conclude the following (p.16):

> Our findings show that the dilemma is much more profound than it might appear at first glance. <u>The initial didactical models seem to become so deeply rooted in the learner's mind that they continue to exert an unconscious control over mental behavior even after the learner has acquired formal mathematical notions that are solid and correct.</u> [Their emphasis]

Our development of new forms of linked-representation software is directed at this most difficult problem. Indeed, we feel that two critical aspects of the problem are vulnerable to a software assault: insufficiently linked and coordinated cognitive representations, and numerically limited exemplification of each model. But before detailing our proposed attack on the problem, we offer a more detailed analysis of the relations between certain cognitive models and certain of our previously developed word problem task-types. Although our first cognitive model building work is directed at intensive quantities, the operations of multiplication and division are intimately related to intensive quantities and help comprise a natural family of issues based on a set of interrelated concepts developing within a student over a period of perhaps ten years. As a matter of fact, Vergnaud's unifying notion of conceptual field applies - in particular, we are dealing with the conceptual field of multiplicative structures as defined in Vergnaud (1983).

3.2 Intensive and Extensive Quantities: Semantic Issues

To date we have given serious attention to two semantic factors:
(i) the type of referent - is it an extensive or intensive quantity;
(ii) the relation between the referents in the problem.
Since a quantity is in fact a pair of things, a magnitude M and a referent R, the referent for an intensive quantity involves two entities, say N and D (Schwartz, 1976, 1984). In fact there is also a third entity to deal with in the intensive case, the relationship between N and D. Given that there are more things to keep track of when dealing with intensive vs. extensive quantities, and these other features of the quantities may have a significant influence on how a student approaches a problem, it is not surprising on these grounds alone that problems involving intensive quantities are generally harder - except in those multiplication problems where the intensive quantity is familiar enough to be well "chunked" into a single familiar "rate" entity such as price or speed. (There are several other factors relating to problem difficulty such as imageability and whether the problem calls upon the concepts of ratio, proportion and division. We will discuss these below.)

A student's cognitive model must be somewhat more elaborated to handle unchunked intensive quantities. This is what was predicted and observed regarding both measures of problem difficulty in both earlier studies (Schwartz, 1984). Moreover, the necessary cognitive model elaboration would be expected to evolve with age and mathematical experience, and this likewise agrees with the data, especially with the "most intensive" of problem types, those involving the quotient of one intensive quantity by another ("I/I" problems). However, even in the 11th and 12th grades, performance on such problems with congenial numbers was far less than satisfactory. Students obviously lack functional models of intensive quantities to apply in such situations and have difficulty in coordinating their models of division with those of intensive quantity.

Given the two referents N and D for a given intensive quantity, we have already observed (Schwartz, 1984, p.18) that they can be related in several ways at differing levels of "strength." For example, one referent may physically contain the other (candies per bag); one may contain the other more abstractly (children per family); one may be ordinarily related to the other in a familiar way (tires per car) or in an unfamiliar way (cars per tire); one may not normally be related to the other (trees per heartbeat). The two referents may also be connected together in a pre-constructed linguistic entity such as speed or price (Bell, Fischbein & Greer, 1984). Moreover, the individual referents can be of varying familiarity, abstractness, and imageability. Given these semantic dimensions of referent relatedness in an intensive quantity (derived from the "web" work mentioned earlier), we have the basis for dimensions of relatedness for the two referents of a given multiplication or division word problem. Such orderings of problems according to semantic relatedness dimensions guide current empirical work associated with the task development for use in software, especially regarding choices of concrete representations.

3.3 Multiplication.

3.3.1 Size change, conversion rates, and "scalars."

The operation of multiplication has been analyzed according to a variety of principles over the years: applications, mathematical structure, place in the curriculum, cognitive or developmental features, etc. We shall briefly recount some of the taxonomies in order to provide a context for our current investigations involving intensive quantity and for extensions of these investigations that focus more on the operations.

Two of the most ambitious and complete taxonomies based on application-types are those of Usiskin & Bell (1983b) and a similar, but less explicitly taxonomic analysis by Freudenthal (1984). Usiskin and Bell describe three basic types of multiplication useage: size change, acting across, and rate factor. We shall review their taxonomy in the light of the nature of the quantities involved. The major difference between their application-based account and ours has to do with the idea of "scalar."

SIZE CHANGE, according to Usiskin and Bell, involves multiplying a quantity by a (referent-free) scalar, with no resulting unit change. It changes only the number of the quantity and not the referent. While in such contexts Usiskin and Bell (as well as many others) invoke the notion of scalar as pure quantity - with no referent associated with it - our framework consistently associates a referent with _every_ number in an applied context: a consistent arithmetic of quantities avoids the ambiguities alluded to by Usiskin and Bell when the notion of a referent-free scalar is introduced, especially in the context of multiplication.

To be more specific consider the following two size change occasions to apply multiplication:

(i) Charles is 3 times as heavy as his son, whose weight is 60 lb.

(ii) Charles is 3 times as tall as his son, whose height is 24 in.

What is Charles' weight (height, resp.)?

The _arithmetic_ role of the "3" in each case is as a numerical factor and is the same in both cases; but its _quantitative_ role is different in each case. In (i) the "3" is acting on units of weight (but not changing the unit) and has a referent of lb/lb. The role of the seemingly superfluous referent is entirely consistent with the unit "bag" in the multiplication 3 bags times 5 candies/bag, which yields 15 candies. In the weight case, the referent structure is 3 lb (in Charles' weight) per lb (in his son's weight). This is entirely different from the quantitative role of "3" in the height case, where the referent is in/in (3 in. in Charles' height per in. in his son's height):

3 lb(in Charles' wt)/lb(in son's wt) x 60 lb(in son's wt)
= 180 lb(in Charles' wt)

3 in.(in Charles' ht)/in.(in son's ht) x 24 in.(in son's ht)
= 72 in.(in Charles' ht)

The parenthetical extensions of the referents are included to clarify the real meaning of the "scalar" as an intensive quantity. Notice also from this perspective that the size change in weight, say, is being described as "intensive" growth - that is, each lb in the son's weight is being changed to 3 lb. (Visually, we could think of this as a single entity growing to 3 times its previous size.) This is in distinction to "extensive" growth, which would be described as a triple replication of the entire 60 lb extent of the son's weight (visually, a single entity being replaced by 3 copies of itself).

Not only is attachment of referent to the "3" consistent with the larger theory of quantity in which this case is embedded, it also clarifies the conversion rate case that Usiskin and Bell find ambiguous with respect their taxonomy. Suppose we want to convert Charles' height measure from inches to feet. We would multiply 72 in by 1/12 ft/in to get 6 ft. This multiplication statement has exactly the same intensive quantity structure as the previous statements. The difference is that a change in units for a fixed height results whereas in the previous (height) case, the height changed and the units were fixed. In both cases we have the product of an intensive quantity with an extensive quantity. The conversion factor case is distinguished as an intensive quantity by the common attribute of the two parts of the referent (length in this case).

Similar analyses apply to size change expressed through percentages. We regard Usiskin and Bell's "scalar" percent as an intensive quantity in diguise.

Size change can also apply to intensive quantities: If a certain job pays 4 dollars/hr and I get time and a half for overtime, what is my overtime rate? Answer: 1.5 (dollars/hr) / (dollars/hr) x 4 dollars/hr = 6 dollars/hr.

3.3.2 Acting across and rate factor uses.

ACTING ACROSS involves the product of two quantities, whose referents may be entirely arbitrary and whose product referent is the product of the referents. It corresponds to an E x E multiplication (the product of two extensive quantities). If the referents have spatial dimensions, then the product referent has, of course, a spatial dimension equal to the sum of the dimensions of the factors. Some forms of acting across involve quantifying the set of all possible (ordered) pairs of elements from two sets, especially in combinatoric situations - see the "outfits" problem below. Other examples of acting across abound in the sciences, often with special units designating the product referent: 110 volts x 8 amperes = 880 watts (of power); 10 pounds (of force) acting across 5 feet equals 50 foot-pounds (of work). As is well known, the different types of acting across are associated with widely different cognitive structures and processes and vary considerably in their learnability and their involvement with other specialized knowledge.

The RATE FACTOR use class identified by Usiskin and Bell includes the product of an intensive quantity by an extensive quantity where the denominator referent of the intensive quantity matches the referent of the extensive quantity, thereby yielding an extensive quantity for the product (E x I = E). This use class also includes IxI applications of the following sort: If a team sells an average of 6000 tickets/game at 5 dollars/ticket, what is their average revenue per game?

Of special interest to Usiskin and Bell is the conversion rate case, where I is a unit conversion factor, e.g., 3 ft/yd (see the size-change discussion). They regard this as the quotient of two quantities, 3 ft by 1 yd. From this point of view, since numerator and denominator are equal as quantities in this case, their quantity quotient is the scalar 1. But the intended effect of multiplying a given extensive quantity by such a factor is to change the referent (unit) of the original quantity to that of the numerator of the conversion factor - from yards to feet in the given example. Hence they suggest that conversion rates share some of the properties of scalars and some of intensive quantities, so their taxonomy is not disjunctive.

Our point of view is that both size change and rate factor uses of multiplication, including the conversion rate special case of the latter, are simply products usually involving an intensive and an extensive quantity, although on occasion both may be intensive. This is not to deny, however, that important cognitive differences exist among these uses. In fact, when isolating the important cognitive differences in the applications of multiplication (and the other operations for that matter), the discriminations must be much finer that any we have discussed thus far in the Report. In the longer run the term "scalar multiplication" is likely to survive our theoretical subsumption of its meaning, so we can only hope that it will come to be used within this more consistent and economical framework - as a special kind of intensive quantity whose referent consists of a "numerator" and "denominator" that are equal.

3.3.3 Other multiplication use issues.

We would note that Vergnaud (1983) and others (notably Bell, Fischbein and Greer, 1984), have separated out area and volume situations for special consideration and study because of their special features. As Vergnaud's work shows, these concepts develop and differentiate only gradually among most students.

Another special feature of the spatially related models of multiplication are their extremely wide utility in modeling other forms of acting across - serving a dual modeling role: one as models for multiplication, and second as models for other multiplicative phenomena. Area as the product of two linear dimensions, for example, can be used to model a variety of products, especially when at least one factor happens to be a function of time and where one linear dimension is chosen to represent time: distance traveled as a function of time, force as a function of acceleration, work as a function of force over time. The power of this model is its extendability to non-constant and nonlinear functions via integral calculus.

Others have categorized types of multiplication more by semantic features of the referents as found in school word problems and their place in the curriculum (Kansky, 1969; Kennedy, 1970; Swenson, 1973; Vest, 1968). Generally, these analyses do not clarify underlying issues beyond the analyses already given, but do expose other assumptions regarding the authors' curricular preferences. For example, they almost all give repeated addition as a primary category, frequently identify distance-speed problems or unit price problems as categories, and taxonomize on the basis of number size of the factors. Sutherland (1947) analyzed textbook multiplicative word problems in grades 3-6 and found the great majority involved distance-speed, item pricing and repeated addition. No examples of acting across were found. We would suspect a similar pattern holds today.

3.3.4 Multiplication as repeated addition vs ExE multiplication.

Usiskin and Bell deliberately avoided making repeated addition a use class (although an earlier analysis by Usiskin (1976) did separate out repeated addition) because (1) it cuts across the other use classes if small whole numbers are involved, (2) it is only a computational feature of the problems in which it applies, and (3) all the repeated addition situations are subsumed in the other categories. However, as is emphatically clear in the results of Fischbein, et al, (1985) and Greer and Managan (1984), when multiplication is introduced as repeated addition or perhaps even identified with repeated addition (Grossnickel and Reckzeh, 1973 - cited by Usiskin and Bell, 1983b), the implicit numerical constraints of the repeated addition model are then built into student cognitive models of multiplication. As a result the students' ability to recognize the multiplication inherent in problem situations is likewise constrained. Since so many situations do not fit the repeated addition model, this is a serious handicap in problem solving, a handicap camouflaged by the predominance of repeated addition or formulaic approaches to school word problems involving multiplication. Software development involving models for multiplication should certainly take this into account.

Our data from student generated problems (Schwartz, 1984) indicate that the acting across, or ExE type of multiplication, is not a ready part of students' repertoire, a fact fitting the poor representation of ExE problems in the curriculum. Our data suggesting that the IxE type of multiplication is easier and more commonly a part of student multiplication models may at least in part be a consequence of our (since revised) decision to go with purely integral coefficients, as the Fischbein and Greer results imply. (See also Ekenstam & Greger, 1983, for dramatic evidence of the constraining effect of the repeated addition model.) Thus the students' assumptions of the multiplication-as-repeated-addition model were never challenged - EXCEPT in the ExE case, where a different model needs to be invoked, the combinatorial version of the acting across model.

There is another issue connected with the combinatorial ExE case relating to cognitive development in the classic Piaget-Inhelder sense. They found that the ability to deal with combinatorial possibilities was developmentally sensitive, and could be associated with the transition to "formal thought." We have some tentative clinical evidence from interviews involving a small sample of above average 3rd, 4th and 6th graders that students may be able to generate and apply tree diagram representations to solve such problems as:

Betty has 5 blouses and 3 skirts. How many different outfits can she make?

The students not only spontaneously generated tree diagrams, but were able to generalize from them to solve parallel problems involving numbers too large for convenient visual description.

We wish to make one last point regarding the dominance of the repeated addition model and the understanding of coefficients of variables in algebra. Strong evidence supports the hypothesis that the repeated addition model of multiplication in arithmetic, when coupled with an introduction to algebra that likewise presents "7X" as "seven X's," yields a deep misunderstanding of coefficient multiplication that identifies it with the adjectival use of number (Kaput & Sims-Knight, 1983). In turn the adjectival use of number is deeply embedded in natural language syntax (Schwartz, 1976; Nesher & Schwartz, 1982). The result of building coefficient multiplication on a natural language-based schema, given other weaknesses in the understanding of variables, is the widespread and well documented "Students-Professors problem reversal" (Clement, 1982). This reversal results when students are asked to write an algebraic equation that expresses the following statement: "For every 6 students at a certain university, there is 1 professor." They are told to use S for the number of students and P for the number of professors. The common error is to write "6S = P" and thereby treat 6 as an adjective and S as a label for either a student or a set of students. Use of the letters "x" and "y" did not significantly change the outcome.

We would suggest, therefore, that enriching student models of multiplication may be one half of solving the "Student-Professors Problem" problem. The other half is enriching and strengthening student concepts of variable.

3.4 Division and Intensive Quantities.

3.4.1 The partitive and quotative models.

Before relating division to intensive quantity, we shall first review the two models normally associated with division, the partitive and the quotative (also known as the "measurement" model). The partitive is thought by some to be the strongest among children, perhaps because it is based on the presumably easier to understand "fair-share" idea. Later we shall further subdivide the partitive model into two types, the "share" and the "cut" models.

Partitive interpretation of p/q:
Here an entity or collection is to be divided into a number of equal parts, and p/q represents the amount in each part - divide by the number of parts. For example: (A) A mother has 15 candies to give equally to her 3 children. How many should each child get? Or: (B) 12 people want to make 3 teams. How many people should be put on each team?
The numerical assumptions of this model are:

(i) $p > q$

(ii) q is a whole number

(iii) $p/q < p$

Quotative (or "measure") interpretation of p/q:

Here one wants to know how many entities of size q are in an entity of size p. If discrete objects are involved, one divides by the number of objects per group. For example: (A) A mother has 15 candies to put into bags that will hold 5 candies per bag. How many bags will she need? Or (B): If there are 12 people are to divided into teams of 3 people, how many teams will there be?

The numerical assumptions of the quotative model seem to be simply that p > q, although if p/q is a whole number (i.e., no remainder), then the division can be interpreted as repeated subtraction.

3.4.2 Intensive quantities and the partitive model: reanalysis of earlier data.

Note first that the basic ideas of partitive and quotative models assume division of quantities rather than of numbers. We now interpret them in terms of our explicit theory of quantity.

First of all, the partitive case involves an E/E situation leading to an I quotient:

(A) [15 candies/3 children] = [5 candies/child]

(B) [12 people/3 teams] = [4 people/team]

Both are partitive E/E=I situations. Does the partitive model extend to I/E situations? The answer appears to be yes, although a more abstract version of partition is involved. For example, consider the following problem, #7 from Study B (Schwartz, 1984), slightly modified:

A ball is dropped from the roof. After 3 seconds of constantly increasing speed it is falling at a rate of 96 feet per second. How fast is the speed of the ball changing?

A reasonable partitive interpretation of this situation regards the question as asking how the increase in speed is distributed across the 3 seconds - or how it is partitioned into 3 pieces.

A similar abstract partitive interpretation can be applied to I/I problems where each intensive quantity has the same referent in its "denominator." The following I/I problem, # 10(a) from Study B, illustrates the analysis:

Stephen drives 200 miles per week, and spends $16 per week on gas. How many miles does Stephen drive on $1 worth of gas?

Here the "per week" "cancels" from the quotient, formally as well as semantically, leaving an E/E question - He drives 200 miles on $16. How many miles can he drive on $1? The solution can be thought of as determining of the 200 miles driven "how much does each dollar get." The relatively low percentage of correct solutions to this problem indicates that students do not easily see this as a partitive problem. Either the semantic "cancelling" of the "per week" information (the discarding of extraneous information) causes cognitive overload difficulties, or the abstract and complex nature of each intensive quantity obscures the partitive nature of the situation. More recent data indicate that high ability 12th graders regard this situation in partitive terms (See below.)

A simpler I/I example was provided by a student who generated the following problem in Study A (Schwartz, 1984):

A boy produces 24 paper airplanes each day. If he works 6 hours a day, and at a constant rate, how many planes does he make each hour?

The answer can be obtained by asking how "the" (after cancellation of the "per day") 24 planes are distributed across the 6 hours.

The I/I situation where the intensive quantity denominators are not the same is much more problematical and was not represented in either study.

Bell, Fischbein and Greer (1984) have examined the concept of "rate" and division in a manner close to our analysis. Thus they distinguish speed = distance/time as partitive and time = distance/rate as quotative.

3.4.3 Intensive quantities and the quotative model: reanalysis of earlier data.

The quotative model involves an E/I situation leading to an E quotient:

(A) [15 candies/5 candies per bag] = [3 bags]

(B) [12 people/3 people per team] = [4 teams].

Both of these E/I examples, and all the others appearing in our earlier two studies generated either by us or by students (Schwartz, 1984) can be regarded as requiring the quotative model of division - with the following possible exception:

Jim has $20 dollars (sic). Sarah has $5.00, how many times as much does Jim have?

It seems reasonable to regard this question as asking "How many five dollars are there in 20 dollars?" The answer is best thought of in terms of an intensive quantity:

20 dollars/5 dollars = 4 dollars (that Jim has)/dollar (that Sarah has)

It is worth noting that the pure number quotative version is the model most commonly used in school when discussing "division facts" and teaching the most common division algorithms. This fact, coupled with the apparent predominance of the partitive model as students' primitive model of division, may lead to a failure of communication between students and the division curriculum. Considerable work has been done in earlier decades relating the two models to appropriate algorithms (Van Engen & Gibb, 1956; Dilley, 1970; Kratzer, 1971). Failures of fit between student models and division algorithm teaching may also affect the interpretation of remainders in applied problems (Silver, 1985).

3.4.4 The partitive vs quotative distribution of student generated problems.

We reexamined the student-generated division problems from Study A (Schwartz, 1984) to determine whether they confirm tentative suggestions in the literature that the partitive model is the more primitive than the quotative in the sense of developing earlier and therefore being more primary in students' conceptualization of division (e.g., Fischbein, et al, 1985; Silver, 1985). Our data strongly support this assertion. Of 81 problems generated by students from grades 4 to 13, 81% were clearly partitive, 17% were clearly quotative, and the other 2% were the two problems discussed above. There was no obvious decline in percentage of partitive problems with increase in age. See Appendix A for numerical details.

This conclusion is at some variance with the results of Bechtel and Weaver (1976) who gave 2nd graders division problems based on the two models in search of order and interference effects that might provide the basis for a curricular decision on which model should be introduced first in schools. They found that students tended to get slightly higher mean scores on quotative problems. However, their results are likely to be strongly affected by their use of concrete manipulatives and other task variables and may not generalize to less concretely presented problem situations and solution modes - see below.

3.5 Two Versions of the Partitive Model: Share and Cut

In attempting to determine more clearly how the the models for division might interact with the semantic features of different kinds of intensive quantities to affect problem difficulty, let us examine the kinds of actions that might be the basis for a primitive cognitive model associated with partition.

Suppose a student is using a physical action-based cognitive model to carry out a division problem. If it is a quotative problem (How many groups or entities of size q are there in a set or entity of size p?), most researchers (e.g., Bechtel & Weaver, 1976; Moser, 1952) agree that the action will be repeated subtraction (of the groups of size q from the group of size p). On the other hand, if it is a partitive problem (partitioning a set or entity of size p into q groups entities of size to be determined), it appears that there may be two types of actions.

The first, and perhaps the more primitive (we have no first hand data supporting this conjecture) involves counting down by units of one, distributing the elements into the q groups. We refer to this as "share partition."

If there is a good marker or signal when a given stage in the sharing is completed (when each of the q groups has received one additional element), then keeping track of the process is facilitated. Such occurs, for example, when the problem context involves a true distribution schema with discrete elements to be distributed into explicitly defined recipients (e.g., candies per bag or candies per child). By contrast, if the problem context violates any of these givens, the execution of the distribution requires more cognitive processing, or more structure to be provided by the person performing the action.

The second model of partitive division, and perhaps a cognitive consolidation of share partition, is "cut partition." In cut partition the set or substance being divided is cognitively "marked" at its "cut points" and the amount in each subsection is counted or otherwise measured. We have found clinical evidence for cut partition in interviews with third and fourth graders as well as in a task involving twenty advanced placement calculus twelfth graders.

The 12th graders were requested to show on paper how they thought of an I/E division problem: almost all drew a representation of the I quantity that they then subdivided using a visual partition, even when they were forced by the size of the numbers involved to draw a representative sample of the entire quantity before partitioning it to indicate a representative element of the partition. Followup questions helped confirm that their representations underlay their solution not only of the problem at hand, but other division problems as well.

Several grade school students gave clinical evidence of thinking of a cut partition in working out a sharing problem, which seems to indicate that the cut partition may develop early from its apparently more concretely enactive relative, share partition. In particular, a third grader actually drew a cut partition while solving a sharing problem.

The above kind of analysis and clinical work should provide reasonable clues regarding which semantic or contextual features of a division problem might bear upon its difficulty. For example, if the entity to be partitioned is not "mentally divisible" into distributable elements, or, as suggested earlier, if there is no natural or familiar means for receiving such elements in a distribution partition, or if there are no ready cues as to which group is which, or if the quantity is not easily imaged or "cut," then cognitive actions are required to compensate for the absence of support for either partition model. Such is more likely the case, as seen earlier in the I/E and I/I cases, when intensive quantities are involved in the dividend of the division problem. Such clues regarding features bearing upon problem difficulty are also clues to the construction of appropriate software learning environments.

Similarly, the repeated subtraction version of the quotative model requires the creation and maintenence of a temporary unit to be subtracted off the original quantity and an intermediate recording act to keep track of how many of such units have already been subtracted. Further, there must be some indication of when the process is complete – or whether a remainder is involved. The cognitive load here appears greater than in either partitive case, although again, we must identify those semantic or contextual problem features which might aid in the process, e.g., how well held together is the unit, how imageable is the quantity being subtracted from, are they continuous or discrete quantities, are there remainders, etc. (See Silver, 1985, regarding the vexing question of remainders and their interpretation in differing problem contexts.)

3.6 Other Classifications of Division Processes and Problems.

The classifications by Usiskin and Bell (1983b) mainly parallel their classifications of multiplication, although they distinguish "ratio" and "rate" uses. They regard the former as a comparison of quantities having the same referent, hence results in a referent-free scalar (our view is that an intensive quantity results). The latter (rate) involves two different referents and hence yields an intensive quantity – unless the division has an intensive quantity in either the numerator or denominator, in which case the result may be an extensive quantity.

Classifications by Sutherland (1947) break down into "measurement" and "partition" and in fact amount to quotition and partition as outlined previously, although her subcategories allude to many of the semantic distinctions that we have examined regarding the types of quantities involved in a division problem and the relationship between the two quantities.

Zweng (1963) likewise used the same fundamental categories and, in working with second graders, found that students' primitive models of division were sensitive to the semantic factors we have identified. For example, she found that of the following two quotative problems, the second was easier:

(1) If I have 8 pencils and separate them into sets of two pencils, how many sets will I obtain?

(2) If I have 8 pencils and put them into boxes, placing two pencils in each box, how many boxes will be used?

As we have noted, the existence of a concrete "recipient" for the pencils lightens the cognitive load. She found similar results in partitive problems.

Zweng (1972) also points out that since the curriculum is not often clear about the two interpretations of division, the students also have dificulty. The difficulty is compounded by the fact that the formal representation of the division in the two types of problems is identical. Consider, for example, the problem of cutting a 30 foot ribbon into pieces 8 feet long -

how many pieces? - and the problem of cutting a 30 foot ribbon into 8 pieces. - how long is each piece? The formal description of the computation in each case is "30 divided by 8 equals n" but the referents of the elements of the formal statement are very different in the two cases. The differences show up dramatically when students are asked to interpret the results, confusing feet with pieces or vice versa. Zweng gathered data on 2nd grade students who solve such problems (not involving remainders) before they have formal instruction in division, and who therefore use missing factor reasoning. She found that they interpreted their solutions correctly. We would interpret this finding in terms of the cognitive models that students at that age have for multiplication. In addition to the simple repeated addition model, they have an E x I model (number of sets times the number per set). Thus by applying the missing factor reasoning, the cognitive model for multiplication carries the student to a correct interpretation - the student sets up the appropriate product equation, each of whose parts has an I or E interpretation, and then computes and interprets the missing factor. Our clinical data on a 3rd grader who correctly solved both a partitive and a quotative division word problem using the missing factor method agrees with Zweng's data. Our earlier data (Schwartz, 1984) also provides evidence of a stable E x I model for multiplication, at least when familiar intensive quantities are involved.

Zweng suggests, since division statements hide the two very different meanings of division, that the teaching of division facts be dropped from the elementary curriculum and that more time be spent with multiplication. Perhaps a better alternative now would be to offer concrete representations of division that do make the distinction clear from the beginning. In fact, we expect our multiple representation software to assist in this directly (see Section 4), in part by building systematically on existing student models of intensive quantities.

Many other cognitive issues relating multiplication to division have yet to be systematically explored, and we expect that this will be a very active and fruitful area of research in the coming years. See (Greer, in press) for an overview.

3.7 Multiplicative Structures, Ratio, and Proportion.

Vergnaud (1983, p. 127) defines a conceptual field as "a set of problems and situations for the treatment of which concepts, procedures, and representations of different, but narrowly connected types are necessary." He defines the conceptual field of multiplicative structures as that which involves multiplication, division, fraction, ratio, rational number, linear and n-linear function, dimensional analysis, and vector space. Obviously, most of these are in some way or another connected to the issues before us. Although we will not review all of Vergnaud's work on this conceptual field, we will examine more closely in the language already developed the first of the three subfields he has identified. The three are isomorphism of measures, product of measures (very closely related to the

Usiskin-Bell "acting across" concept), and multiple proportion other than product. In each case where the word "scalar" appears below, Vergnaud regards it as a pure number, whereas we regard it as an intensive quantity the two parts of whose referent are the same - see 3.3.1.

This analysis provides the foundation for our choice of representations of intensive quantities to be employed in our planned software environment - those representations that lead directly to and support the evolution and application of a powerful <u>function</u> approach to comparing quantities.

3.7.1 Isomorphism of measures: Multiplication.

Vergnaud offers the following four-celled array as an economic means of theoretically framing a large class of multiplication and division word problems.

M1	M2
1	a
b	x

The M1 and M2 very nearly refer to the referents of the numbers on the respective sides of the vertical line. Vergnaud's example of a multiplicative problem follows:

Richard buys 4 cakes at 15 cents each. How much does he have to pay?

a = 15 b = 4 M1 = # of cakes M2 = # of cents (cost)

Vergnaud proposes that most students approach this problem in one of two ways, either by extracting a scalar operator or by extracting a function operator, with the conception of the scalar operator developing ahead of the function operator. (Again, we regard the scalar as an intensive quantity.) Each can be viewed in terms of the diagram.

$$\begin{array}{ccc} \underline{M1} & : & \underline{M2} \\ 1 & : & a \\ b & : & x \end{array} \qquad \text{"times } b\text{"}$$

Scalar Approach

$$\begin{array}{ccc} \underline{M1} & : & \underline{M2} \\ 1 & \longrightarrow & a \\ b & \longrightarrow & x \end{array} \qquad \text{"times } a\text{"}$$

Function Approach

In the scalar approach the student extracts the multiplicative relationship that exists within M1 between 1 and b and transposes it to M2, establishing a corresponding (isomorphic) multiplicative relationship between a and x based on the (scalar) operator "times b." The assumed reasoning is "b cakes is b times as much as 1 cake, so the cost of b cakes is b times the cost of 1 cake." Here b is a scalar because it is the ratio of two quantities with the same referent (cakes): [b cakes / 1 cakes].

In the function approach, the student extracts the multiplicative relationship between 1 and a and transposes it to create a multiplicative relationship between b and x. Each relationship in the function approach crosses referents, from M1 to M2, relating two different quantities. Here "times a" is a function operator in the sense of being the numerical slope of the numerical linear function from M1 to M2. As a quantity, a is intensive, with referent cents/cake.

3.7.2 Isomorphism of structures: division.

The same scheme can be used to characterize division problems, which amount to traversing the diagrams in opposite directions.

Partitive types of problems have the following representation, where the problem is to find the unit value x – the amount that "each gets," or the amount that corresponds to the unit of the <u>opposite</u> measure. Thinking in terms of a function operator, the problem is to determine f(1) given 1, a, and b = f(a).

<u>M1</u>	:	<u>M2</u>
1	:	x = f(1)
a	:	b = f(a)

Partitive Problems Scheme

Two problems illustrate the discrete and continuous cases:

Mom wants to give 12 candies equally to her 3 children. How many will each child get?

$a = 3 \quad b = 12 \quad M1 = \#\ \text{children} \quad M2 = \#\ \text{candies}$

Nine large peaches weigh 4 pounds. How much does each one weigh?

$a = 9 \quad b = 4 \quad M1 = \#\ \text{peaches} \quad M2 = \#\ \text{pounds}$

This class of problems can be solved by applying the "scalar" operator $1/a$ to the number b, that is, finding what on the right side of the diagram corresponds to the unit on the left. (Note that a child may not actually be thinking in terms of the $1/a$ notation - this is merely used to describe the mathematical structure of the situation.) We also recognize this to be an E/E division as described earlier. Although this is the structural form of the solution, Vergnaud observes that younger children are likely to use more primitive calculation strategies than actual division, e.g., missing factor approaches or trial and error. In fact, as we have seen earlier in Zweng's work (Zweng, 1963, 1972), fewer errors occur in such circumstances.

Quotative Problems.

Here the problem is to find x given $f(x)$, $a = f(1)$, and $b = f(x)$.

M1		M2
1	:	$a = f(1)$
x	:	$b = f(x)$

Quotative Problems Scheme

Peter has $15 to spend on miniature cars. If they cost $3 each, how many cars can he buy?

a = 3 b = 15 M1 = # cars M2 = # dollars

Dad drives 55 miles per hour on the highway. How long will it take him to travel 410 miles.

a = 55 b = 410 M1 = # hours M2 = # miles

The usual approach in these problems is to invert the function operator "times a" to get "divide by a," which we shall denote by 1/a, and then apply this "divide by" operator 1/a to b. Vergnaud suggests that because of the inversion associated with the referents in such problems (cars/dollar in the first problem and hours/mile in the second), younger children find these problems hard. Perhaps this is another way of accounting for the fact that students' primary model of division may be partitive.

3.7.3 Isomorphism of structures: The general case.

The entire previous analysis can be generalized by replacing the unit in each scheme by an arbitrary number. Consider the following example.

<u>M1</u>	:	<u>M2</u>
a	:	b
c	:	x

The General Isomorphism of Measures

My car burns 10 gallons of gas every 250 miles. How much gas will I use on a 3000 mile vacation trip?

a = 250 b = 10 c = 3000 M1 = # miles M2 = # gallons

When she makes strawberry jam, grandma uses 3.5 pounds of sugar for 5 pounds of strawberries. How much sugar does she need for 8 pounds of strawberries?

a = 5 b = 3.5 c = 8 M1 = lbs of strawberries
M2 = lbs of sugar

Just as in the earlier unit-ratio multiplication problems, these problems can be approached either via a scalar or a function approach. Our perspective on such problems is to emphasize their functionality and their linearity, because it is this feature that is generalizable and that uses the more powerful representations.

Vergnaud (1983) did detailed empirical work showing the scalar approach to be easier and preferred by students aged 11 - 15, but with slow evolution toward the functional approach with increase in age and mathematical experience. Most researchers agree that in a typical missing-value proportion reasoning problem, the relative divisibility of the numbers usually determines which approach will be used.

Similar findings have appeared in the proportional reasoning and ratio literature, with the terms such as "internal comparison" (scalar) and "external comparison" (functional) appearing frequently (see, for example, Karplus, et al, 1983; Noelting, 1980 a,b; Tourniaire & Pulos, 1985). Note that "within" and "between" are also used.

Lybeck (1978) has also made a parallel analysis based on clinical observations of students doing classic proportional reasoning tasks. He ties the distinction to the historical development of the function concept as a means of formalizing proportional relationships, noting how, for example, Gallileo described lengths of inclined planes of a fixed height and the times needed by a ball to roll their length in scalar proportional language (comparing distance to distance ratios with time to time ratios), whereas Newton and Leibniz moved toward functional descriptions (comparing distance to time ratios - actually, comparing vertical changes in graphs with horizontal changes).

Freudenthal (1978) likewise distinguishes between the two ways of comparing a pair of ratios. In his terms, using s for distance and t for time, the two ways are written as

SCALAR $s1 : s2 = t1 : t2$

FUNCTIONAL $s1 : t1 = s2 : t2$

he notes that for the mathematically sophisticated, interchanging the middle terms to get from one to the other is a trivial matter, but is a large step for the learner because of the differing cognitive content of the two representations. However, Lybeck points out (1978, p. 32):

> It is the quantification of the relation between qualitatively different variables [i.e., the functional approach -- JJK] that produces the fertile relationship. Also, it gives the desired simplicity in the abstract description of the world around us, which is characteristic of modern scientific perspectives.

We agree that the function view is to be emphasized. Since proportions are nothing more than linear functions with only a small sample of data from the domain of the function under consideration, they should be viewed in the way that generalizes to larger sets and which emphasizes the underlying quantititative invariant, the "m" in the "y = mx + b."

4. Software Development Plans.

4.1 Goals.

We should emphasize at the outset that we by no means expect to build a curriculum in our Project. Such is beyond both the scope of our mandate and the reach of our resources. Rather, our primary goal is to build potential technological components of a desirable future curriculum in line with a particular organizing theme: the application of multiple linked representations, simultaneously available and visible. Such a software environment is intended to support the learning and integration of the different representations and different aspects of a given complex mathematical concept, in this case the concept of intensive quantity.

We sketch briefly below a sample set of coordinated representations and potential actions on them. Most details, and perhaps important innovations not yet considered, await empirical guidance to be derived from clinical testing of alternatives currently underway. The bulk of this clinical work takes the form of clinical probe sessions followed by brief teaching interventions involving the use of some potential component of the software manifested as well as possible in non-computerized form. As pieces of the software environment become available, these will be used in a clinical environment and video recorded using a split screen - half for the child and half for the computer screen.

Of considerable interest beyond the learning of the mathematics in these situations will be the cognitive impact of various software features, such as the timing of the microcomputer's reaction to a particular input in one representation and its consequences in another: Should they be nearly simultaneous in some real time sense? Should there be a specifiable delay? Should the timing be under the control of the student? Which options should be made available to the student? How many representations should be present simultaneously on the screen? How much "history" of the student's actions should be made available and, if so, in what form? Such design issues will form important side questions that will be the subject of considerable input from other ETC experts.

4.2 Major Representational Issues and Concrete Illustrations.

4.2.1 Forms of representations.

As suggested earlier, student models for intensive quantities often lack the inherent "typicality" of an intensive quantity: to know that there are 5 red cars produced for every 3 blue cars produced by a factory is to know something about <u>all</u> samples of production and not merely the color of a particular sample of eight or sixteen cars. However, many students seem able only to think of one or two small samples so, while their representation may not be structurally incorrect, it is an inadequate, overly concrete representation of the intensive quantity involved and needs to be "ramped upward" to a more general and potent one.

As indicated above, a much more flexible and potent representation of intensive quantity is provided by the slope of a straight line on an a pair of coordinate axes labeled by the referents involved. In the case at hand the axes would be labeled "number of blues" and "number of reds," for example, and the line would have slope 5/3 or 3/5 depending on the choice of labels for the horizontal and vertical axes, respectively. This visual model has two immediate virtues:

(1) Its simultaneous representation of "all" samples, although to be more precise, the discreteness of the quantities involved would need to be taken into account perhaps via the use of a dashed line with markers at points with strictly integral coordinates.

(2) The visually explicit embodiment of the ratio's numeric constancy in the straightness of the line.

The longer term virtues of this kind of representation in terms of mathematical power were argued in Section 3.7 in the context of the functional approach to proportion and ratio, and in Section 2.4 it was proposed as a vehicle for curricular coherence and unity. It and the ideas in which it participates provide entree' to some of the greatest intellectual achievements of western culture. However, this representation must be learned - its syntax and style of reference require explication. Even more importantly, it must be cognitively linked to other, perhaps more concrete, understandings that the student may have of ratio and intensive quantity.

To this end we propose the creation of an iconic representation of an intensive quantity - a concrete "sampling" device that provides a direct visual representation of the two quantities involved in the composite, intensive quantity. One such that has been developed in prototype form provides a screen window tesselated with "boxes" in which are put sets of icons in a given ratio. Thus, to model the 5 red cars per 3 blue cars production ratio mentioned above, the student inputs the ratio and the computer responds with a window tesselated with boxes, each of which contains 5 icons of one type and 3 of another directly below in the same box. The student can then highlight boxes as he or she chooses and the computer responds with the total number of reds and blues in highlighted boxes. We would expect to organize this data in vertical tabular form with the two columns of the table appropriately labeled. As the student highlights or "de-highlights" boxes, the table grows or shrinks in length (or perhaps has the appropriate ordered pair highlighted in a table with a long list of pairs already provided).

In a multiple representation environment the concrete and the tabular representations would appear in adjacent windows. In another window the appropriate ordered pairs would be either plotted or highlighted to correspond with the actions taking place in the other windows.

More than one type of such a concrete sampler may be necessary to accomodate discrete vs continuous quantities. Consider, for example, the modeling of the ratio of a pair of line segment lengths - the above form of concrete representatic (using "boxes" containing icons) would not be appropriate. Other factors, such as the different kinds of semantic relationships between the referents, also call for more than a single style of concrete iconic representation. Consider, for example the intensive quantity associated with 3 candies per bag - it might be useful to provide a representation that models a containment relationship rather than a simple association. Again, some of the dimensions of the previously mentioned Problem Web may prove to be relevant, although instead of providing sets of problems organized along the various dimensions bearing upon problem difficulty, we take a more pedagogically direct approach modeling more directly the features that associate with problem difficulty. Thus, instead of moving around in a collection of problems in order to find one that is sufficiently understandable to solve and generalize from, the student will have the freedom to choose (from an available menu of representative types) a sufficiently concrete and sufficiently iconic representation of the quantities involved so as to facilitate understanding and choice of appropriate actions.

An important consequence of having multiple choices of iconic representations all having the same tabular and graphical associates is to force to the foreground the common mathematical structure that inheres in the several superficially different concrete models. We regard this as a kind of educational payoff from the technology that has enormous potential, yet has had very little, if any, systematic pursuit. Dickson (1985) likewise argues for juxtaposed symbol systems in software environments that emphasize and render more learnable important translation skills, although his examples involve significantly less sophisticated interactions among representations than we have envisioned.

A fourth representational form will be provided for computing with intensive quantities. Our current plans call for a Semantic Calculator type of work pad that helps organize and record computations with the quantities involved in a particular situation by keeping track of the referents as quantities are operated upon. Again, the operations on the pad will reflect and be reflected by actions in the other representations.

Shavelson and Salomon (1985, p4) also argue explicitly for multiple representations (using the language of "symbol systems"):

> The power of representing information in more than one symbol system lies in the ability to: (a) provide a more complete picture of a phenomenon than any single symbol system can; (b) increase the chances of linking new information to the learner's preferred mode of learning

(i.e., to the learner's preferred symbolic representation); and (c) cultivate cognitive skills in translating or shifting among symbolic representations. Our expectation is that this symbo ic flexibility will increase the learner's knowledge and understanding.

4.2.2 Actions on representations: reasoning and computing.

A representation is a tool to think with, to perform cognitive actions with or to be acted upon - in Olson's and Goody's terms, a "tool of the intellect" (Olson, 1976, 1985; Goody, 1977). It is sometimes convenient to speak in terms of internal cognitive representations and external or, in Vygotsky's terms (Vygotsky, 1978), extracortical, representations. Of course, these exist in fundamental partnership - although some would argue that the distinction is philosophically indefensible. This is not the place for an airing of the matter - fuller discussions of the distinctions and philosophical issues can be found in Salomon (1979), Kaput (1985, 1985a), Lesh (1985) and Olson (1985).

Hence the key to the software environment is the kinds of actions its representations support. We shall list some likely possibilities based mainly in our experience with non-cybernetic intensive quantity tasks. Undoubtedly, experience in the environment will suggest some new possibilities, including tasks with no paper-pencil or physical analogs. However, work by Schultz (1984) suggests that even when limited to use of physical manipulatives, the problem solving ability of 7th graders improves considerably. She used a microcomputer to generate word problems in the presence of physical materials that students used to model the relationships in the problems as part of an extended teaching experiment. The microcomputer also provided menus of potentially useful physical materials as well as animated sequences describing the problem situation. As Post (1980) points out, such physical materials are in fact abstractions from the given problem situation. Thus an iconic microcomputer representation is but another step in the same direction taken by physical manipulatives, without all the practical drawbacks of physical materials.

Note also that we are thinking in terms of two broad classes of activities:
(1) those centered within a particular representation and traditionally associated with the domain at hand, and (2) those that deliberately exercise the linked multiple representation nature of the software in acts of translation. The former class would include, for example, the standard proportional reasoning tasks involving the comparison of two ratios and the various missing-value tasks. The latter would include such tasks as performing an action in one representation and predicting the consequences in another representation.

In the concrete, iconic representation one will be able to perform actions beyond the highlighting of samples. For example, one base activity for younger students is to predict a numerator given a denominator, and vice versa. In concrete terms, this may take the form of having a section of the tesselation's icons deleted (again, refer to the red and blue cars, with, say, a certain set of boxes highlighted having the red car icons deleted) and then requesting the student to input the number of red cars to match the blue ones in the affected boxes. A corresponding stack of red car icons appears and is then "loaded" by the student into the highlighted boxes. Success or failure is measured by whether the stack of red car icons is exactly consumed or not. In a continuous representation, the stack may be replaced by an appropriately long line segment (perhaps with unit markers) with pieces snipped off and moved to the appropriate places to complete a match. Interesting possibilities present themselves in the continuous case with the use of non-integral values. Earlier work by Quintero (1981, 1983) and Schwartz (1981) provides guidance, as does continuing clinical work, on appropriate forms of iconic representations.

It is important to realize that the above actions on the concrete representation have analogs in the other representations which "follow along" the actions generated in the "active" or "input" representation. In particular, it is easy to imagine the tabular and work pad analogs of these activities. The work pad version gets particularly interesting when non-integral numbers are involved and a choice for a unit of comparison is needed (the finding of an appropriate unit fraction). [Unit factor methods for solving these and related problems have been strongly advocated from time to time (e.g., Herron & Wheatley, 1978) and are likely to be easily implementable in this environment.]

4.2.3 Representational Detail and Research Issues.

We have already mentioned how the various semantic features of intensive quantities will affe t the style of concrete representations used. However, even within a particular semantic context, e.g., the discrete red car - blue car situation involving the icons within boxes, serious issues arise at a finer level of detail. Consider, for example, the extensive work uncovering the role of perceptual distractors in student interpretation of fractions in concrete representations (Behr & Post, 1981; Behr, et al 1983) and the difficulties that some students have with the idea of equivalence of fractions (Behr, et al 1984; Post, et, al 1985).

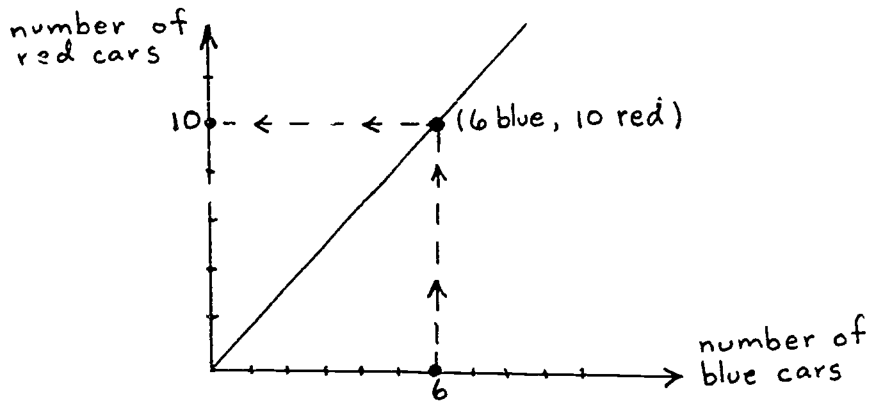
This work raises, but does not resolve the issue of what kinds of actions to make available when a student highlights a set of boxes, say two (each containing 5 icons of one type and 3 of another). Should the boundaries between the two boxes be dissolvable and should the icons be rearrangeable producing a box with 10 and 6 icons? If so, how might this action be presented in the other representations? More importantly, how would students who had been previously identified as having difficulty with equivalence of fractions react to different such possibilities? And what kinds of additional activities and discussion are needed to surround this cybernetic action in order to make it pedagogically meaningful? We expect that our exploratory work in this new context to shed further light on the nature of the underlying cognitions associated with equivalence of fractions and the role of perceptual distractors.

Similar questions arise with respect to virtually all aspects of the proposed software development. Other questions arise regarding the learnability of certain of the representations - what level of intellectual development is necessary to use the graphical representation meaningfully? And, of course, perhaps the largest and most novel question:

TO WHAT EXTENT DOES THE LINKED, MULTIPLE REPRESENTATION SOFTWARE SUPPORT THE <u>COGNITIVE</u> LINKING OF THE MULTIPLE REPRESENTATIONS?

4.2.4 Sample Activities in the Graphs Environment.

Recall the concrete version of the "name the numerator given the denominator task." In the graphical representation, this takes the form of predicting on the vertical axis (red car axis) where one would "land" if starting on the horizontal axis at the point representing the number of blue cars in the incomplete boxes. To test a prediction one travels vertically from this point to hit the "5 red per 3 blue line" and then horizontally to the vertical (red car) axis. See the figure below. (p.66)



In general, the choice of which representation to act upon is clearly dependent on both student and task variables in the specific sense of Kulm (1979). We will now illustrate some tasks that are executed in the graphics representation, but which, of course, have analogs in the other representations. We will now deal with continuous intensive quantities which are chunked into familiar unit rates, rather than the arbitrary numerical relationship embodied in the red car blue car illustration. The goal, of course, is that the student will eventually recognize the common mathematical structure that underlies both situations. Furthermore, as argued earlier, the graphical representation of the intensive quantity as a function operator extends to that grand corpus of mathematics including calculus - which was invented to deal with nonlinear change, but which uses linear change as the means of local measure of change (the tangent line). As a result, we have the potential of building at least one aspect of the bridge from student primitive models to the more sophisticated and potent models of higher mathematics.

Suppose now that the constituents of a particular intensive quantity are more explicitly understood as a rate, as with the miles and hours. In this case the two axes of the graph serve as starting points for calculations and reasoning involving particular situations. The intensive quantity 10 mi/hr is represented as a line of slope 10 on a coordinate plane with horizontal axis labeled "hr" and vertical axis labeled "mi." One can then answer questions such as "If I go 10 mi/hr for 2.5 hr how far have I gone?" One finds 2.5 on the "hr" axis, moves vertically to hit the line and then horizontally over to the "mi" axis to read off the answer, 25 mi. (See Figure A below.) One uses the same representation to answer the reverse question (involving an extensive by intensive quantity division) "How long would it take to go 35 mi at 10 mi/hr?" In this case one moves horizontally from the 35 mi position on the vertical axis to the 10 mi/hr line and then down to the horizontal hr axis where the answer can be read off, 3.5 hr. Mathematically, of course, one is tracing the inverse of the linear function $f(x) = 10x$.

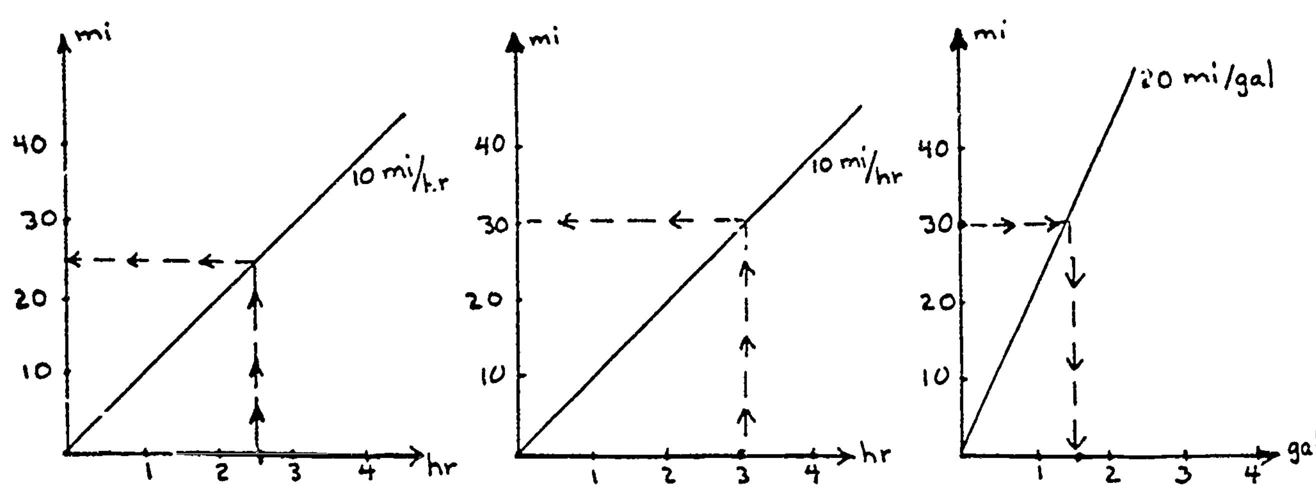


Fig. A Fig. B Fig. C

Such graphical representations can be concatenated to answer questions and deal with situations involving additional intensive quantities, such as mi/gal, for example. Thus if one knew the mileage of a certain vehicle were 20 mi/gal and were asked "How much gas do I burn in 3 hr in driving the tractor that averages 10 mi/hr?" then one would set up a second coordinate axis system with gal on the horizontal and mi on the vertical axes, respectively. (See Figures B and C.) One then uses the first representation as before to find that in 3 hr one travels 30 mi. Next, one finds 30 mi on the second system's vertical axis, travels over to the 20 mi/gal line and then down to the horizontal gal axis to read off the answer, 1.5 gal. Clearly, such representations can be replicated to deal with an enormous variety of multiplication and division situations. In so doing, the student is also being concretely introduced to the composition of operators and functions.

With the representations understood at a first level, one can then go on to tasks that build upon earlier work, such as comparing different intensive quantities in various representations, e.g., with more than one intensive quantity on a given coordinate axis, asking questions about the effect of doubling the production of blue cars and graphically comparing the previous ratio with the new one. And, of course. one can move to more sophisticated multiple proportion problems in the conceptual field of multiplicative structures described in Section 3 (g) (Vergnaud, 1983).

4.3 Earlier Curriculum Development Work.

The elusive dream of unity for the mathematics curriculum is not new. Much earlier in this century there were repeated calls for unification based on the widespread application of the function concept (Moore, 1902; National Committee on Mathematical Requirements of the MAA, 1923). Later, in the spirit of Bourbaki, unification was to be had on the basis of the foundational role played by set theory utilizing the logical structure of the subject. However, the reader will note an important shift in the organizing principles called upon in our approach. While the various abstract structural elements of the domain play an important constraining role in determining what can and should be done, other features of the domain are being addressed here as well, features having to do with the <u>forms</u> used to represent the structural content. Perhaps even more important, we (and other contemporaries) are negotiating much more explicitly and knowledgeably than ever before with the cognitive constraints, the learnability of the subject matter.

But again, this is not new. Others, especially in the ill-fated but often remarkably imaginative curriculum development projects of the past two decades, attempted coherence in the mathematics curriculum and paid close attention to the ways that the important ideas can be learned. Frequently their work foundered on the hard rocks of implementation - for one reason or another, they were difficult to implement on a large scale. New information technologies should help eliminate some of the earlier implementation barriers considerably.

We are searching out and reviewing much of this work, frequently with the assistance of earlier project directors or participants. Work being examined includes:

The Madison Project, Elementary Science Studies, Developing Mathematical Processes, The Arithmetic Project, The Minnemast Project, The SciMath Project (including software extensions of the dimensional analysis techniques being independently developed).

While isolated elements of several of these curricula will be helpful, the most sustained effort paralleling ours, especially regarding the attempt to use graphical representations and slopes at early grade levels, is that of Paul Rosenbloom formerly materials designer for the Minnemast Project. He is making available all of his materials, including a significant portion that was never published.

We are also in the process of reviewing textbook series for potentially useful materials and task possibilities, and examining the work of contemporary development projects, including:

The University of Chicago School Mathematics Project, The School Mathematics Project from England's Shell Centre and Cambridge University Press, The Ohio State SOHIO-NSF Middle School Mathematics Project, the software (in LISP) being developed by James Greeno and colleagues at Berkeley, the Ontario Institute Classroom Materials, and perhaps most important, the Rational Number Project, which has researched, developed and tested on a large scale and under a tight theoretical framework a variety of diagnostic and teaching materials which are now being extended to proportional reasoning situations.

Lastly, we are reviewing the considerable practical literature aimed at teachers over the years. This material contains much useful advice on the teaching of rate, proportion, multiplication, and division, some of which will be pertinent to the construction of activities in the planned software environment.

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Appendix A

Student Generated Single Step Division Problems
Study A
Fall 1984

We asked 290 students in the public schools of Cambridge, Newton and Watertown, MA., as well as some freshmen at Southeastern Massachusetts University, to

> Make up a problem that requires one division, and no other operation, to solve it.

The data were classified by grade, age, sex, and mathematical ability as assessed by the classroom teacher.

Of the problems generated in response to this division prompt, 90% were single step division word problems. Most of the inappropriate responses were clustered in grades 4-6. They were mainly multiple step problems involving subtraction. Children of higher ability tended to generate a higher percentage of single step division word problems. There were no differences in any respect due to sex.

Problem Distribution
per cent

		E/E	E/I	I/I	I/E
Grades 4-6	(n=25)	84	16	0	0
Grades 7-8	(n=13)	77	23	0	0
Grades 9-10	(n=19)	79	21	0	0
Grades 11-12	(n=13)	85	0	15	0
Grade 13	(n=11)	73	27	0	0
TOTAL	(n=81)	81	17	2	0