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INFLUENCE OF NUMBER SIZE, PROBLEM STRUCTURE AND RESPONSE MODE ON CHILDREN'S SOLUTIONS OF MULTIPLICATION WORD PROBLEMS

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ABSTRACT

One important finding from recent research on multiplication word problems is that children's performances are strongly affected by the nature of the multiplier (whether it is an integer, decimal larger than 1 or a decimal smaller than 1). On the other hand, the size of the multiplicand has little or no effect of problem difficulty. The aim of the present study was to collect empirical data concerning this type-of-multiplier effect in combination with two additional task variables which have not yet been seriously addressed in previous research, namely (1) the symmetrical/asymmetrical character of the problem structure and (2) the mode of response (choice of operation versus free response mode). While the data of the present study provide additional evidence for the type-of-multiplier effect hypothesis, they show at the same time that the two other task variables also strongly influence children's difficulties with multiplication problems.

1. INTRODUCTION

Since the late seventies a considerable amount of work has been done on young children's problem solving of simple addition and subtraction word problems (for an overview see: Briars & Larkin, 1984; De Corte & Verschaffel, 1985; Riley, Greeno & Heller, 1983). More recently, researchers have started to analyze pupils' solution skills and processes with respect to multiplication and division problems too (Bell, Fischbein & Greer, 1984; Fischbein, Deri, Nello & Marino, 1985; Greer, 1987a; Kaput, 1985; Mangan, 1986; Nesher, 1987; Vergnaud, 1983). While there is a lot of communality in both areas of research - such as the attempt to construct problem taxonomies based on the semantic structure of the problems and the efforts to unravel children's understanding of how the arithmetic operations model these semantic structures - there are also some important differences (Greer, 1987a, p. 65-66). First, while the work on addition and subtraction word problem solving has mainly been done with children in the 5-8 age-range, the subjects in the multiplication and division studies were mostly between 10 and 15 years old. Second, the research techniques were somewhat different: in the former kind of studies individual interviews and computer simulation were the prevailing methods, while in the latter one has relied heavily on collective paper-and-pencil tests. Moreover, those tests were mostly of a particular nature: pupils were not asked to carry out any computation but they only had to indicate which arithmetic operation with the two given numbers would yield the correct answer. Finally, contrary to the addition and subtraction research, the work on multiplication and division has not only paid a lot of attention to problem structure as a variable, but also to the type of numbers used.

In the present article, we first give a brief overview of the state-of-the-art in the field of multiplication and division word problems, focusing on one particular aspect, namely the influence of number type on the choice of an appropriate solution strategy. Afterwards we present the design and the main results of a recent study in which we addressed some unsolved questions related to that topic. Finally, we discuss some theoretical and practical implications of this study.

2. THEORETICAL AND EMPIRICAL BACKGROUND

2.1. Classifying multiplicative word problems

During the past decades most studies about the effects of particular task variables on children's arithmetic word problems solving have concentrated on the surface characteristics and on the mathematical structure of those problems. Some typical variables that were examined are: the number of words in the problem, the structure of the number sentence "hidden" in the problem, the place of the question, the presence of a cue or key word, etc. In more recent work the focus has shifted from mathematical and surface aspects towards the semantic structure of the problem. This new approach is based on two assumptions: (1) word problems solvable using the same arithmetic operation, can be described in terms of different networks of concepts and relationships underlying the problem, and (2) constructing an appropriate representation of such a conceptual network is a crucial aspect of expertise in word problem solving. In the late seventies this approach was applied to addition and subtraction word problems (Briars & Larkin, 1984; De Corte, Verschaffel & Verschueren, 1982; Greeno, 1978; Nesher & Katriel, 1977). At that time it was already suggested that a similar analysis could be undertaken for word problems about multiplication and division; however, extensive attempts to build a classification of situations modelled by these arithmetic operations are a rather recent development (Mangan, 1986, p. 55). Examples of such classifications have been provided by Bell, Grimison, Greer and Mangan (1987), Dou ven (1984), Greer (1987a), Nesher (1987), Kaput (1985), Schwartz (1981) and Vergnaud (1983). As an illustration, Table 1 presents the one developed by Greer (1987a).

Table 1

In Greer's (1987a) classification scheme, a basic distinction is made between symmetrical and asymmetrical problems. In the asymmetrical problem types (Table 1a) the two given numbers play different roles. For example, in the "multiple groups" problem ("3 boys had 4 marbles each; how many marbles did they have altogether?") the situation is naturally conceived as "3 lots of 4": thus 4 is the multiplicand and 3 is the multiplier, which operates on it to produce the answer. In the symmetrical problem types (Table 1b) on the other hand, the roles played by the quantities multiplied are essentially equivalent. Consequently, which of the two given numbers is the multiplier and which the multiplicand, is merely a matter of convenience.

2.2. Empirical findings concerning the difficulty of multiplicative word problems

A first well-documented finding from recent research on multiplication word problems is that problems modelled by the same arithmetic operation but differing in their semantic structure, can have substantially distinct levels of difficulty. For example, Hart (1981) asked pupils to choose the appropriate operations for a series of multiplication tasks. She found that those involving a Cartesian product, such as the "combinations problem" mentioned in Table 1b, were considerably harder to solve than, for example, the "multiple groups" problem from Table 1a. Similarly, Douwen (1984) found that problems describing a "change of scale" situation were more difficult than problems about "multiple groups".

A second robust finding is that children's difficulty in choosing the correct operation depends also strongly on the type of the given numbers in the problem. Indeed, several researchers have shown that two multiplication problems with the same mathematical, semantic and surface structure but differing in terms of the nature of the given numbers, can elicit very distinct levels of problem difficulty, even when pupils are only asked to indicate the appropriate operation (Bell, Swanson & Talyor, 1981; Bell et al., 1984, 1987; Fischbein et al., 1985; Mangan, 1986). Generally speaking, multiplication problems involving integers are solved better than problems involving decimals larger than 1, which are in turn easier than problems having decimals smaller than 1. However, in order to complete the picture of the effects of this variable on the solution of multiplication word problems, one must add the following observations (for more details see: Bell et al., 1981, 1984, 1987; Fischbein et al., 1985; Greer, 1987a; Mangan, 1986).

(1) The difficulty of choice of operation seems to be affected by the nature of only <u>one</u> of the two given numbers, namely the multiplier. For example, in a study with several types of asymmetrical problems from Table 1a, Mangan (1986), found that children performed significantly better on problems with an integer as multiplier than when the multiplier was a decimal larger than 1; problems with a multiplier smaller than 1 were still much more difficult. (The most common error on the latter problem type was dividing instead of multiplying the two given numbers.) On the other hand, the size of the multiplicand had <u>no</u> significant effect on problem difficulty.

(2) Another typical finding is that a lot of pupils who indicate the wrong operation, seem to be able to reason correctly about the size of the answer, but cannot relate their correct reasoning to the appropriate

formal mathematical operation with the numbers in the problem. For example, in a study with 12-13-year olds, Bell et al. (1984) observed a lot of wrong-operation errors on problems like "How much does it cost for 0.53 gallons at 1.33 pound per gallon?"; the most common errors was 1.33 : 0.53. Afterwards a subgroup of selected pupils were individually interviewed using the same problems. When asked whether the answer on the "gallons" problem would be more or less than 1.33, most of them gave promptly and confidently the correct answer. Moreover when instructed to estimate the answer, they quickly said between 0.60 and 0.70 pound. But when asked again to indicate the operation that would yield the exact answer, they chose division, because "multiplying by 0.53 would make the answer larger than 1.33".

(3) A final remarkable observation is that pupils generally seem to be unaware that for a given problem structure the mathematical relationship between the known and the unknown quantities remains invariant, regardless of the sizes of the numbers involved. For example, when interviewed about the "gallon" problem ("How much does it cost for 0.53 gallons at 1.33 pound per gallon?"), several pupils did not perceive any conflict in choosing division for 0.53 gallons, but multiplication for 2 gallons (Bell et al., 1984). Greer (1987b) introduced the term "non-conservation of operation" to refer to children's propensity to base their choice of operation on the type of numbers involved. Furthermore he discovered that this misconception is very resistant. Indeed, pupils who were unaware of the invariance of operation with respect to a particular multiplication or division problem type, did not easily acquire such an awareness during a clinical interview.

2.3. Theoretical account for the observed effects of number type on children's solutions of multiplication word problems

Several explanations have been put forward for the experimental findings and observational data concerning the effects of number type summarized above. At this moment, Fischbein et al.'s (1985) concept of an intervening intuitive model probably provides the most plausible theoretical account of children's difficulties in solving multiplication or division word problems with decimals.

According to this theory, each arithmetical operation remains linked to a primitive "intuitive model", even long after that operation has acquired a formal status. Those intuitive models are basically behavioral in nature; that is "when trying to discover the intuitive model that a person associates with a certain operation, one has to consider some practical behavior that would be the enactive, effectively performable counterpart of the operation" (Fischbein et al., 1985, p. 5). The same authors further assume that the process of identification of the operation needed to solve a word problem is mediated by these intuitive models. When the constraints of the underlying model are incongruent with the numerical data given in the problem, the choice of a wrong arithmetic operation may be the result. Interestingly, these models appear to act unconsciously to a great extent: "They manipulate a person's problem-solving efforts 'from behind the scene', and thus their impact can hardly be controlled by the solver" (Fischbein et al., 1985, p. 6).

The theory specifies that the primitive model associated with addition consists of putting together two (or more) disjoint sets of objects to obtain a new one that is their union. The model affecting the meaning and use of multiplication is "repeated addition", in which a number of collections of the same size are put together. Under the "repeated addition" interpretation, one number (i.e. the number of equivalent collections) is taken as the multiplier, the other (i.e. the magnitude of each collection) as the multiplicand. A first logical consequence of the "repeated addition" model is that, while the multiplicand can be any positive number, the multiplier must be an integer. For example, one can intuitively easily conceive of 3 times 0.63 (namely 0.63 + 0.63 + 0.63), but not of taking a certain quantity - e.g. 3 - 0.63 times. Second, because under the "repeated addition" interpretation the multiplier is always a whole number, multiplication necessarily results in a number that is bigger than the multiplicand.

As said before, the theory states that if there is an incongruity between the specific numerical data given in the problem and the constraints of the underlying intuitive model, the pupil may get into difficulties when deciding upon the appropriate arithmetical operation. In multiplication problems with an integer as the multiplier, there are no such incongruities; consequently, they will elicit the lowest amount of wrong-operation errors. In problems with a decimal multiplier larger than 1, only the first constraint of the "repeated addition" model - namely that the multiplier must be an integer - is violated; therefore, they are solved better than problems in which the role of multiplier is played by a decimal smaller than 1, and the second constraint - namely that the result must be bigger than the multiplicand - is violated too.

3. HYPOTHESES OF THE PRESENT STUDY

While the experimental and observational data concerning the effect of number type presented in paragraph 2.2 are consistent with Fischbein et al.'s (1985) theory as elaborated above, there still remain a lot of questions requiring further investigation.

First - with the exception of Mangan's recent study (1986) - the available evidence on the effects of the type of multiplier on the choice of operation (regardless the nature of the multiplicand) is not convincing, because it is based on comparisons between problems that differ also in several aspects other than the nature of the numbers (Bell et al., 1987, p. 5). Consequently, a first objective of the present study was to collect additional data about the effects of type of multiplier and type of multiplicand in a more carefully designed way.

Second, the word problems included in previous investigations always had asymmetrical structures. This means that the two quantities multiplied play psychologically a different role in the problem situation, and are therefore non-interchangeable. This raises the question whether the type of the given numbers affects the solution of symmetrical problems too. In this respect, one could argue that it is less likely that the choice of operation for a symmetrical problem is mediated by the misconceptions inherent in the "repeated addition" model; other primitive models (with other constraints imposed on the numbers that can be used), such as the "rectangular pattern" model, might be more influential (Freudenthal, 1973, p. 248-249). But even if the choice of operation for a symmetrical problem is mediated by the "repeated addition" model, its constraints do not always have to divert or block the solution process; indeed, the symmetrical structure does not force the solver to attribute the role of multiplicand and multiplier to a particular given number. For both reasons, one can hypothesize that the type of multiplier will have no significant influence on the solution of symmetrical problems. In view of testing that hypothesis, the present study involves both symmetrical and asymmetrical problems. According to Mangan (1986, p. 289-290), such a systematic comparison of problems from both structures is necessary in order to get a better understanding of the influence of the primitive models on children's difficulties with solving multiplication and division word problems.

Third, with the exception of some clinical data from specially selected children, in most previous studies pupils were not asked to answer the problems, but to indicate which formal arithmetic operation would yield the correct answer. As argued in paragraph 2.2, errors on such a choice-of-operation task may be attributable to pupils' failure to match their otherwise accurate reasoning about the size of the answer to an explicit statement of the operation needed to obtain the correct answer. However, as several researchers have suggested (see for example, Anghileri, 1986; Bell et al., 1981; Dekker, Ter Heege & Treffers, 1982), selecting and executing a formal arithmetic operation with the two given numbers, is not the only way in which a one-step word problem can be solved. Besides, there are a lot of informal solution strategies that may lead to a correct solution. In such cases, the processes by which this goal is obtained often remain unclear to the solver; in particular, he is unable to describe them in terms of an formal arithmetic operation or a number sentence with the two given numbers (Bell et al., 1981; De Corte & Verschaffel, 1985b). Therefore, one could hypothesize that when children are not forced to choose a formal arithmetic operation, but are allowed to rely on other, more informal solution strategies, they will make less wrong-operation errors, especially for those problems in which the constraints of the intuitive model underlying the corresponding formal operation, are violated. To test that hypothesis, all problems in our study were presented in two different response modes: choice-of-operation and free-response. According to Bell et al. (1984, p. 146), such a systematic comparison of different response modes would be of both great practical and theoretical interest.

Before discussing the design and the results of the present study, we summarize the general research questions in the form of three hypotheses:

1. Multiplier effect hypothesis: While children's difficulties with choosing a correct operation for multiplication problems will be strongly affected by the type of the multiplier (whether integer, decimal larger or smaller than 1), the type of number which is the multiplicand will have little or no effect. More specifically, problems with the multiplier being an integer will be solved significantly better than problems with the multiplier being a decimal smaller than 1, which in turn will be solved much better than when the role of multiplier is played by a decimal smaller than 1. For problems with the multiplicand being an integer, a decimal larger than 1, and a decimal smaller than 1, no significant differences will be found.
2. Multiplier-problem structure interaction hypothesis: The negative effect of the multiplier being a decimal larger or smaller than 1 on the choice of operation, will only be found in the context of asymmetrical, and not for symmmetrical problems. More specifically, while asymmetrical problems with the multiplier being an integer, a decimal larger than 1 and a decimal smaller than 1 will have significantly different levels of problem

difficulty as described in hypothesis 1, for symmetrical problems no significant differences between the three problem types will be found.

3. Multiplier-response mode interaction hypothesis: The negative effect of the multiplier being a decimal larger or smaller than 1 on the choice of operation, will be much stronger when pupils are forced to choose between a fixed set of operations, than when they are simply asked to solve the problem. In other words, the difference between problems with the multiplier being an integer, a decimal larger than 1 and a decimal smaller than 1 will be be much greater in the choice-of-operation than in the free-response mode.

4. METHOD

4.1 Subjects

The subjects were 116 pupils from four sixth-grade classes (12 year-olds) in two Flemish schools. The experiment took place in the middle of the school year. According to the math curriculum followed in these classes, the notions of multiplying and dividing are imported already in the first grades. In Grades 3 and 4 the pupils learn the algorithms for multiplication and division. In Grade 5 numerical as well as verbal problems involving multiplication and division with decimals, are introduced.

4.2. Instrument and procedure

A paper-and-pencil test consisting of 24 one-step problems was constructed. The test contained 16 multiplication problems. The remaining eight problems - four division, two addition and two subtraction problems - were included only to reduce the likelihood of stereotyped, mindless response strategies on the 16 target problems. Therefore, the data for these "filler items" will not be reported here.

Half of the multiplication problems had an asymmetrical structure; the other half were symmetrical. For the asymmetrical structure the rate problem type was chosen ("One litre of milk costs x francs; someone buys y litres; how much does he have to pay?"). For the symmetrical structure we chose the area problem ("If the length is x meters and the breadth is y meters, what is the area?"). All eight symmetrical and asymmetrical problems differed only with respect to the type of the multiplier or the multiplicand (either an integer, a decimal larger than 1, or a decimal smaller than 1). Table 2 gives an overview of all multiplication problems.

Table 2

The word problems were presented in a way that allowed us to control for task variables that were not central to the present investigation, such as: familiarity with the context, the presence of so-called cue words, the place of the question, etc. The addition, subtraction and division "filler" items were interspersed among the 16 multiplication problems. Finally, in order to reduce any order effect, two different sequences of presentation were used.

The pupils solved this 24-items test twice: once in a choice-of-operation form and once in a free-response form. In the first form, the pupils had to indicate the operation they would use to solve the problem in a list of six alternatives; in the latter the pupils were simply asked to solve the problem, and - if possible - to explain their solution path by writing down either their calculations, a diagram, or the numbers that went through their mind while solving the problem. To control for learning effects, half of the children got the choice-of-operation test first and the free-response test one week later; for the other half the sequence was reversed.

4.3 Data analysis

The answers on the multiplication problems were submitted to a detailed qualitative and quantitative analysis. Children's answers on the free-response mode were classified into one of the following categories:

(1) Correct answer (CA): the pupil gives the correct answer;
(2) Technical error (TE): the pupil chooses an appropriate solution strategy, but fails in carrying out the computations;
(3) Wrong operation (WO): the pupil chooses in inappropriate solution strategy, which can formally be described as either adding, subtracting or dividing the two given numbers;
(4) Given number error (GN): the pupil answers with one of the given numbers in the problem;
(5) No answer (NA); the pupil gives no answer to the problem;
(6) Rest category (R): errors for which we have no ready explanation.

The data were submitted to two different analyses of variance (with a randomized block factorial design 3x3x2x2), each time with the following four task characteristics as independent variables: (1) size of multiplicand: an integer, a decimal larger than 1 or a decimal smaller than 1; (2) size of multiplier: an integer, a decimal larger than 1 or a decimal smaller than 1; (3) problem structure: symmetrical or asymmetrical; (4) response mode: choice of operation or free response. In the first analysis of variance the proportion of correct answers was choosen as the dependent variable. In the second analysis the dependent variable was the sum of the correct answers and the technical errors; the underlying idea here is that answers resulting in technical errors nevertheless reflect than an appropriate solution strategy has been chosen. ... and interaction effects significant at the 5 % level were further analyzed using Duncan's multiple range test ($p < .05$). In this paper we restrict ourselves to the results of the second analysis of variance, in which the dependent variable is the sum of the number of appropriate solution strategies. Although the outcomes of the first analysis are also interesting, they are not directly relevant to the main theme of this contribution, namely the effects of four different task characteristics - number type of the multiplicand, number type of the multiplier, problem structure and mode of response - on children's difficulties with choosing an appropriate solution strategy.

5. RESULTS

5.1 Main effects

Multiplier effect hypothesis

Although the existing evidence so far is inconclusive, previous research nevertheless suggests the multiplier effect hypothesis, involving that the nature of the multiplier (either an integer, a decimal larger than 1 or a decimal smaller than 1) strongly influences children's performance on choice-of-operation tasks, while the type of multiplicand has only a marginal effect on problem difficulty (see paragraph 2.2). The data from the present study provide additional support for this hypothesis.

First, the results of an analysis of variance with the proportion of appropriate solution strategies as the dependent variable, revealed a main effect for the independent variable type of multiplier (see Table 3).

Table 3

A supplemental analysis (using Duncan's multiple range test) showed that the "integer" problems were significantly easier than the "decimal larger than 1" problems, and that the latter were easier than the "decimal smaller than 1" problems (see Table 4). Although both differences were significant at the 5 % level, it is obvious from this table that the "decimal smaller than 1" problems were substantially more difficult than both other types. Mangan (1986) found also smaller differences between multipliers which are integers and decimals greater than 1, than between multipliers which are decimals larger and smaller than 1. This suggests that violating the first constraint of the "repeated addition" model (i.e. the multiplier must be an integer) is much less harmful than violating the second one (i.e. the multiplier must be larger than 1).

Table 4

No main effect was found for the independent variable type of multiplicand (see Table 3). The proportion of correct strategy choices for multiplicand as integer, decimal larger than 1 and decimal smaller than 1 was .86, .83 and .82 respectively. This finding is also in accordance with our first hypothesis.

Other main effect results

Although we did not state any other main effect hypothesis, it is nevertheless interesting to mention the results for the other two independent variables.

Table 3 also shows a main effect for the third independent variable, namely problem structure. The supplemental test revealed that the symmetrical problems elicited a larger proportion of correct strategies than the asymmetrical ones (see Table 5).

Table 5

In this study the symmetrical as well as the asymmetrical problems were represented by one single subtype, respectively "rate" and "area". It therefore would be premature to conclude that in general symmetrical problems are easier than asymmetrical ones. In this respect we remind that other researchers have found that "combination" problems - another symmetrical type - are much more difficult than asymmetrical "multiple group" problems (see paragraph 2.2).

Finally, there was no significant difference between the proportion of correct operations for the problems presented in the two response modes, namely multiple choice and free response (see Table 3); the proportion of correct strategies was .83 and .85 respectively.

To summarize, our results confirm the multiplier effect hypothesis: the type of multiplier strongly influences children's choice of an appropriate solution strategy, while the nature of the multiplicand has no significant influence on their choices. Furthermore, it was found that the "area" problems were significantly easier than those about "rate", and that the free-response mode did not elicit significantly more wrong operations than the multiple-choice mode. Our two remaining hypotheses concern interaction effects between type of multiplier on the one hand and problem structure or response mode on the other. The next section gives an overview of our findings with respect to both hypotheses.

5.2 Interaction effects

A main goal of the present study was to analyze how two additional task characteristics, namely problem structure and response mode, affect the influence of the type of multiplier on the proportion of correct strategy choices. More specifically, it was expected (1) that the negative effect of the multiplier being a decimal larger or smaller than 1 would only be found in asymmetrical problems and not in symmmetrical ones, and (2) that the negative effect of the multiplier being a decimal larger or smaller than 1 would be much stronger when children are forced to choose between a fixed set of operations, than when they can choose their own problem solving strategies.

Multiplier-problem structure interaction hypothesis

Table 3 shows a significant type of multiplier by problem structure interaction, which seemed to be disordinal (see Figure 1). A supplemental Duncan multiple range test, based on p .05, revealed that for the asymmetrical structure, problems with an integer as multiplier were significantly easier than those with a decimal larger than 1 as multiplier, and that the latter were easier than those in which the multiplier was a decimal smaller than 1 (see Figure 1); this is entirely in line with the overall results reported in the previous section. For the symmetrical structure, on the other hand, there was much less difference between the proportions of correct strategy choices for the three distinct problem types. Moreover, although here too significant differences were found, they were not in the same direction: "decimal smaller than 1" and "integer" problems were both significantly easier than "decimal larger than 1" problems, but did not differ mutually.

Furthermore, a comparison between the proportion of correct operations in the context of a symmetrical and asymmetrical structure for each of the three types of multiplier revealed that "integer" and "decimal larger than 1" problems were easier when embedded in an asymmetrical structure; for problems with a decimal smaller than 1, on the other hand, the symmetrical structure was the easiest (see Figure 1). All three differences were significant.

Figure 1

As Figure 1 convincingly shows, the observed type of multiplier by problem structure interaction effect is mainly due to the very small proportion of correct operations for asymmetrical problems with a multiplier being a decimal smaller than 1, as compared to the parallel problems in the symmetrical context. Taking into account the unexpectedly good overall performances of the pupils on the problems with a decimal larger than 1 as the multiplier (see section 5.1), this is not surprising.

Overall, these findings provide a relatively good support for our second hypothesis. There is obviously a multiplier-problem structure interaction effect, but it is not entirely in line with the expectations. The larger and smaller than 1 multipliers had indeed respectively a significant and a very significant negative effect in asymmetrical problems. However, some small but significant differences between the distinct types of multiplier problems were found in the symmetrical structure too. Interestingly, these differences were not in the same direction as in the asymmetrical structure.

Multiplier-mode of response interaction hypothesis

A significant type of multiplier by response mode interaction was also found (see Table 3); in this case too the interaction was disordinal (see Figure 2). A supplemental analysis revealed that in both response modes, problems with an integer as multiplier were significantly easier than those with a decimal multiplier larger than 1, and that the latter were in turn significantly easier than those having a decimal smaller than 1 as multiplier. However, when we compared the proportion of correct operations in both response modes for each of these three types of multiplier, it was observed that "integer" and "decimal larger than 1" problems were easier in the choice-of-operation than in the free-response condition, whilst the reverse was true for problems having a decimal smaller than 1 as the multiplier (see Figure 2). All three differences were significant.

Figure 2

As Figure 2 suggests, the observed "type of multiplier" by "response mode" interaction effect is mainly due to the high level of difficulty for problems with a multiplier smaller than 1 in the multiple-choice form as compared to the parallel problems in the free-reponse mode.

Again the data confirm only partially our third hypothesis, in the sense that the observed interaction is not entirely in accordance with the predictions. The greater negative impact of the type of multiplier in the choice-of-operation than in the free-response mode, is only found for the multiplier being a decimal smaller than 1; contrary to our expectations, the negative effect of the larger than 1 multiplier is similar in both response modes. This result can again be related to the more general finding that problems with a decimal larger than 1 as multiplier, were solved remarkably well in the present study (see section 5.1).

Interaction between type of multiplier, problem structure and response mode

Taking into account that the independent variable "type of multiplier" interacts significantly with both problem structure and response mode, it was interesting to explore the triple interaction between those variables, which is also significant (see Table 3). The results are represented in Figure 3. No supplemental statistical analysis have been carried out on these data.

Figure 3

Figure 3 strongly suggests that the observed main effect of the type of multiplier on children's choice of an appropriate solution strategy (the multiplier effect hypothesis) is almost totally due to the negative influence of the multiplier as a decimal smaller than 1 in asymmetrical problems; in the symmetric context multiplier smaller than 1 problems were even solved slightly better than both other types. Furthermore, the effect of the decimal smaller than 1 in an asymmetrical context seems to be especially disastrous when the pupils have to choose the correct operation out of a fixed set of alternatives: while in the free-answer mode, the proportion of correct strategies was .72, this number dropped to .43 in the multiple-choice form.

6. DISCUSSION

A major finding of recent research on multiplication word problems is that pupils' difficulties with selecting the appropriate operation to solve a problem, are strongly affected by the nature of one of the given numbers, namely the multiplier: problems with an integer as multiplier were found to be much easier than those where the multiplier is a decimal larger than 1, and problems with a multiplier smaller than 1 are still more difficult. By contrast, the type of number of the multiplicand seemed to have only a marginal effect on problem difficulty.

However, so far these empirical findings were not very robust, because of confounding effects. Indeed, the results of most prior studies were based on comparisons between groups of problems differing also in other aspects than the type of multiplier and/or multiplicand. Furthermore, those findings were derived from studies using only asymmetrical problem types presented moreover in only one particular format, namely a choice-of-operation response mode.

The results of the present study support the multiplier effect hypothesis. Integers as multipliers were coped with better than decimals greater than 1, which elicited in turn much less difficulties than those smaller than 1; the most frequently observed type of error on multiplicative problems with a decimal smaller than 1 multiplier, was dividing instead of multiplying the two given numbers. The number type of the multiplicand had no significant effect.

These results are consistent with Fischbein et al.'s (1985) theory that pupils' performances on choice-of-operation multiplicative word problems are strongly affected by their primitive, intuive conceptions about that operation (i.e. the "repeated addition" model) and by the numerical constraints involved in it (i.e. the multiplier must be an integer, and multiplication always makes bigger). However, our results enable us to specify the multiplier effect hypothesis in two respects: (1) the above-mentioned differential effect of number type for the multiplier is only found in asymmetrical problems, not in symmetrical ones, and (2) this differential effect is much weaker when pupils are not forced to choose between a fixed set of formal arithmetical operations, but are simply asked to solve the problem.

The multiplier-problem structure interaction raises an important question, namely what mechanisms might account for the observed absence of a "type of multiplier" effect in our symmetric problems. Two possible explanations were already suggested earlier when we presented our hypothesis. On the one hand, one could - in line with Fischbein et al.'s (1985) theory - argue that the constraints of the "repeated addition" model do not affect negatively the solution process of symmetrical problems with decimals, because their symmetry does not require the problem solver to attribute the role of multiplicand and multiplier to particular numbers. Another explanation might be that the "repeated addition" model does not at all influence the conceptualization of multiplicative problem situations involving "area"; other primitive models, such as the "rectangular pattern" model (with other constraints imposed on the numbers that can be used) may be more influential. However, there is even a third plausible account for the absence of the multiplier effect in our symmetrical problems, namely that pupils' selection of the operation is not the result of a "deep" understanding of the problem structure and a mindful matching of that understanding with a formal arithmetical operation (mediated by a primitive model), but is simply based on the direct and rather mindless application of a well-known formula (area = length x breadth), associated with the key word "area" in the problem text. We were already aware of that possibility when designing our study. However, we could not replace "area" by another problem type, because the remaining two symmetrical problem types in Table 1b do not allow the use of decimals.

Whilst our data about the "area" problems are thus not necessarily inconsistent with Fischbein et al.'s (1985) theory, they suggest nevertheless that we may have to search for a more comprehensive theory, based on the principle that the selection of an appropriate solution strategy is affected by a large number of factors interacting in complex ways. The type of number involved is only one of these factors. Among the other candidates we mention: the particular structure to be modelled (e.g. "rate" versus "area"), the specific context described in the problem (e.g. "price" versus "speed" in "rate" problems), and the presence of key words (such as the words "times" and "area" in the problem text) (Bell et al., 1987, p. 38).

The multiplier-response mode interaction is the second additional finding of our study: the negative influence of the multiplier being a decimal smaller than 1, is much weaker in the free-response than in the multiple-choice format.

Effects of different modes of response have also been signaled by Bell et al. (1981, 1987) and Luke (1987), who found that pupils can make correct estimates about the relative size of the answer (i.e. greater than, smaller than, or the same as the multiplicand), while being unable to choose the appropriate formal arithmetic operation. The results of the present study are in line with this finding, but at the same time go beyond it, by showing that many children who make predictable errors on a choice-of-operation task, are able to choose a correct solution strategy when simply asked to solve it (<u>and not just to make an estimation of the relative size</u>).

Although Fischbein et al.'s (1985) theory cannot <u>totally</u> account for this finding, it is in line with that theory. Indeed, Fischbein et al. (1985)

attribute pupils' difficulties with choice-of-operation tasks to the incongruity between the numerical data given in the problem on the one hand and the specific constraints of the intuitive model underlying the necessary formal arithmetic operation on the other. Consequently, one could argue that when pupils are not forced to choose between a fixed set of alternative formal operations, but are allowed to rely on indirect and informal solution methods, less wrong-operation errors will occur, especially on those items that violate the specific constraints of the intuitive model underlying the correct operation.

As the finding concerning the multiplier-response mode interaction was obtained using collective paper-and-pencil tests, we have little or no information about the precise nature of the cognitive processes in the free-response mode that led to the correct strategy choices on problems with a multiplier smaller than 1. Of course, there is already a relatively large body of knowledge on the informal strategies youngsters use to solve simple multiplication problems with small integers (Anghileri, 1987; Dekker, Ter Heege & Treffers, 1982; Bell et al., 1981), indicating that pupils are able to give the correct answer - sometimes almost immediately - without apparently being aware that the solution could be obtained by multiplying the two given numbers. However, the specific question raised by our data is: which solution paths - other than multiplying the two given numbers - can lead to the solution of problems <u>in which the multiplier is a decimal smaller than 1?</u> Although the written notes on the response sheets in the free-response mode suggest that most correct answers on these problems were indeed <u>not</u> the result of simply multiplying the two given numbers but of other, more informal or indirect solution strategies, it is impossible to trace the exact reasoning processes that have led to these correct answers. Therefore, we plan to collect more systematically data on children's solution processes while solving different types of multiplication problems, using the individual interview and eye-movement registration as the main data-gathering techniques.

Another aspect of Fischbein et al.'s (1985) theory concerns the origins of children's primitive operation models, such as the "repeated addition" model for multiplication. These authors discuss two plausible explanations. On the one hand, these models might reflect the way in which the corresponding operation was initially taught at school. Another possibility is that these models correspond to "features of human mental behavior that are primary, natural, and basic" (p. 15). Although the authors think both explanations are correct, it is necessary to analyze both sources more thoroughly. A first step might be a detailed analysis of the range of multiplicative situations children are confronted with inside as well as outside the classroom.

Finally, we have restricted ourselves in this article to an analysis and interpretation of the difficulties encountered by pupils when solving simple multiplication problems involving decimals. As Bell et al. (1987) and Fischbein et al. (1985) have pointed out earlier, it is necessary - from an instructional point of view - to go beyond that kind of ascertaining studies, and to construct and evaluate appropriate teaching materials and methods aimed at preventing and/or remediating the observed errors, misconceptions and difficulties. Valuable steps towards that goal have already been made by Bell et al. (1981) and Greer (1987b). In these studies, the effectiveness of several general and specific strategies has been tested, such as drawing diagrams, using easier numbers and estimating and checking one's answer (Bell et al., 1981) or provoking a cognitive conflict within the pupil's mind (Greer, 1987b). These studies have shown that children's misconceptions leading to wrong operation choices for multiplicative word problems are very resistant to remedial instruction. Preventing the occurrence of these misconceptions probably will be equally difficult, because one has to deal with what Fischbein et al. (1985) refer to as the fundamental didactical dilemma:

> "On the one hand, if one continues to introduce the operations of multipication and division through the models described above, one will create strong, resistant, and, at the same time, incomplete models that soon will come in conflict with the formal concepts of multipication and division. On the other hand, if one tries to avoid building the ideas related to arithmetical operations on a foundation that is behaviorally and intuitively meaningful, one certainly will violate the most elementary principles of psychology and didactics" (p. 15).

The need for more intervention studies in view of a solution for this dilemma is apparent. The major question to be answered is: how can we help children to construct appropriate formal notions of multiplication and division starting from their informal and intuitive knowledge and skills?

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Table 'a. Types of multiplication and division problems (asymmetrical cases) (Greer, 1987a)

Category	Multiplication	Division
Multiple groups	3 boys had 4 marbles each. How many marbles did they have altogether?	12 marbles were divided equally among 3 boys. How many marbles did they get each?
Iteration of measure	4 pieces of wood are each 3.2 m long. What is the total length of wood?	A piece of wood 12.8 m long is cut into 4 equal pieces. How long is each piece?
Change of scale	In a photograph, the length of a car is 3.2 cm. If the photograph is enlarged by a factor of 4.5, how long will the car be in the enlarged photograph?	A photograph is enlarged by a factor of 4.5. In the enlarged photograph a car is 14.4 cm long. How long is the car in the original photograph?
Rate	A man walks for 4.5 hours at a steady speed of 3.2 m.p.h. How far does he walk?	A man walks 14.4 miles in 4.5 hours. What is the speed in m.p.h.?
Measure conversion	If the rate of exchange is 1.5 dollars per pound, how many dollars will you get for £ 3.20?	If you get 4.80 dollars for £ 3.20, what is the exchange rate in dollars per pound?

Table 1b. Types of multiplication and division problems (symmetrical cases) (Greer, 1987a)

Category	Multiplication	Division
Rectangular array	If there are 3 rows and 4 columns, what is the total number?	If the total is 12 and there are 3 rows, how many columns are there?
Combinations	If there is a choice of 3 colours and 4 styles, how many combinations of colour and style are there?	If there are 12 combinations of colour and style and there are 3 choices of colour, how many choices of style are there?
Area	If the length is 3.2 cm and the breadth is 4.5 cm, what is the area?	If the area is 14.4 cm and the length is 3.2 cm what is the breadth?

Table 2. Overview of the sixteen multiplication problems used in the present study (asymmetrical and symmetrical cases)

Problem	Multiplicand	Multiplier
Asymmetrical problems		
One pencil costs 12 Bfr. Ann buys 4 pencils. How much does she have to pay?	12	4
One package of chewing gum costs 13.5 Bfr. How much money would one need to buy 15 packages?	13.7	15
If one piece of sugar costs 0.3 Bfr, how much would it cost for 11 pieces?	0.3	11
Long ago, flour was priced only 11 Bfr. How much did then cost 6.3 kilogram of flour?	11	6.3
Pete buys a rope of 5.7 metres. One metre of rope costs 14.5 Bfr. How much does he have to pay?	14.7	5.7
Motor oil is priced at 75 Bfr a litre. How much does it cost for 0.7 litre?	75	0.7
Ann is going to make a cake. Therefore she needs 0.7 litres of milk. Milk is priced at 18.5 Bfr a litre. How much does she have to pay?	18.5	0.7
Several years ago, salt was priced only at 0.8 Bfr per kilogram. If one bought 0.6 kilogram of salt, how much did this cost?	0.8	0.6

Table 2. Overview of the sixteen multiplication problems used in the present study (asymmetrical and symmetrical cases) (Continued)

Symmetrical problems (1)

A hen-house has a length of 9 metres and a breadth of 4 metres. What is the area of that hen-house?

A garage has a length of 5.9 metres and a breadth of 3 metres. How much is the area?

The dimensions of a tennis net are 0.8 and 7 metres. What is the area of that tennis net?

The dimensions of a rectangular room are 8 and 5.1 metres. How much is the area of that room?

A floor has a length of 11.4 metres and a breadth of 6.2 metres. How much is the area of that floor?

A gymnastic bank has a breadth of 0.2 metres and a length of 3 metres. How much is the area of that bank?

A carpet has a length of 5.4 metres and a breadth of 0.8 metres. What is the area of that carpet?

The dimensions of a rectangular tile are 0.2 and 0.4 metres. How much is the area of that tile?

(1) As a matter of convenience, the length and the breadth have been chosen as the multiplicand and the multiplier respectively.

Table 3. Results of the analysis of variance with a randomized block factorial design (3x3x2x2) and the proportion of appropriate solution strategies as the dependent variable (n=116)

Source of variance	DF	SS	F value	PR>F
Multiplicand	2	0.4835	2.86	n.s.
Multiplier	2	40.1929	237.75	0.001
Structure	1	4.7069	55.69	0.001
Response mode	1	0.1443	1.71	n.s.
Multiplicand x Structure	2	0.2543	1.50	n.s.
Multiplier x Structure	2	49.9924	295.72	0.001
Multiplicand x Response mode	2	0.1954	1.16	n.s.
Multiplier x Response mode	2	9.0068	53.28	0.001
Multiplicand x Structure x Response mode	2	0.2952	1.75	n.s.
Multiplier x Structure x Response mode	2	9.4946	56.16	0.001

Table 4. Effect of type of multiplier on problem difficulty

Multiplier	Total number of responses	Proportion of correct strategies	Outcomes of Duncan's test (1)
Integer	1392	.94	A
Decimal larger than 1	928	.89	B
Decimal smaller than 1	1392	.71	C

(1) Means with the same letter are not significantly different ($p < .05$)

Table 5. Effect of problem structure on problem difficulty

Problem structure	Total number of responses	Proportion of correct strategies	Outcomes of Duncan's test
Symmetrical	1856	.88	A
Asymmetrical	1856	.80	B

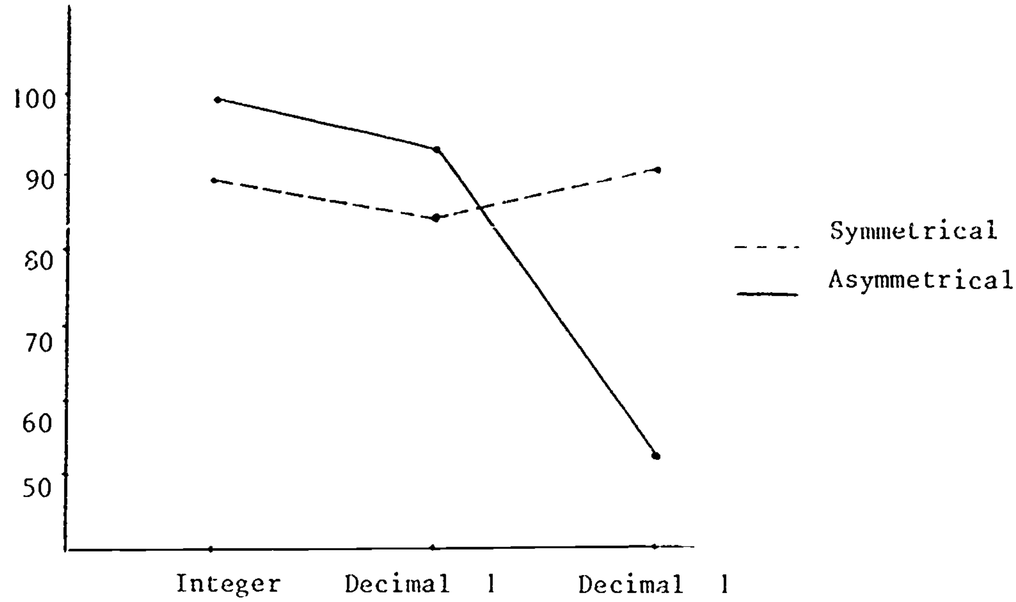

Figure 1. Proportion of appropriate solution strategies for the distinct "type of multiplier" problems in the two problem structures

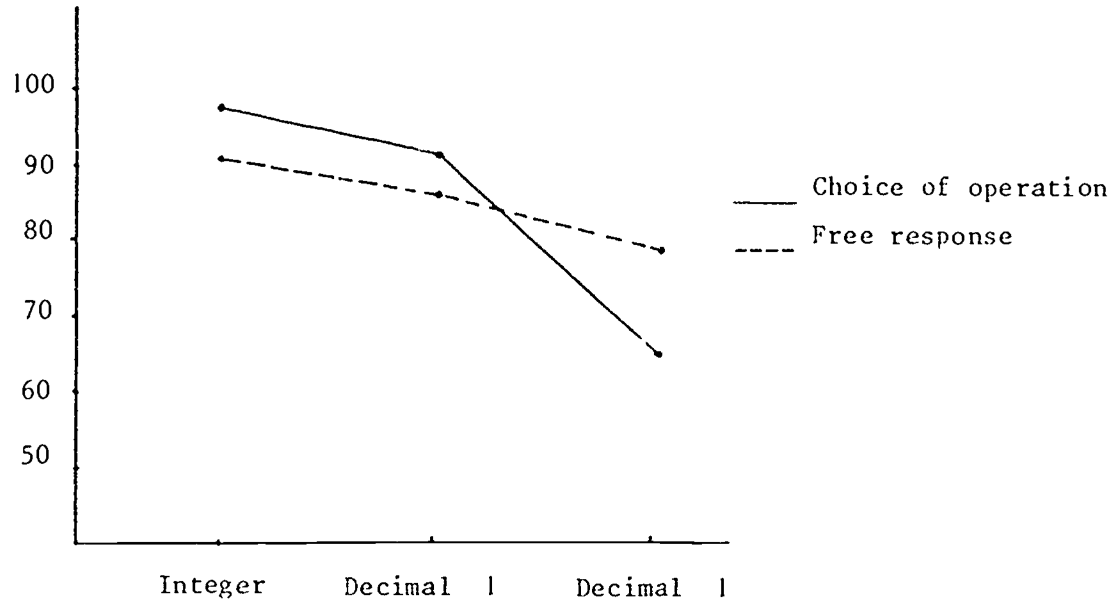

Figure 2. Proportion of appropriate solution strategies for the distinct "type of multiplier" problems in the two response modes

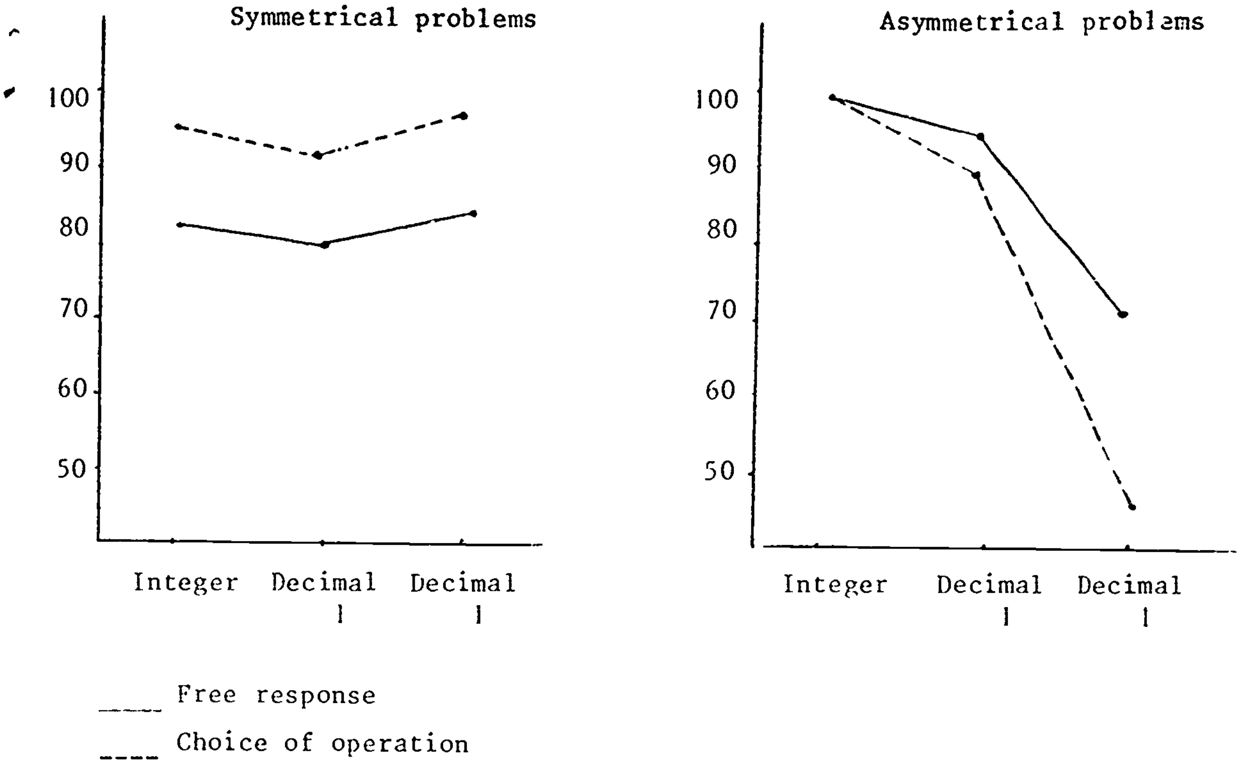

Figure 3. Proportion of correct solution strategy choices in both response modes as a function of type of multiplier and problem structure