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ABSTRACT

Since Polya, Wertheimer, and Hadamard's descriptions of qualitative reasoning strategies used by scientists and mathematicians, very little data have been collected on whether these strategies are actually used by experts. This study used video-taped thinking-aloud interviews to examine the problem solving strategies of professors and advanced graduate students in technical fields. Evidence from these interviews documents the use of analogies, visual transformations, extreme cases, creative partitioning, and other plausible reasoning strategies used by experts. (Author)

NON-FORMAL REASONING IN EXPERTS' SOLUTIONS TO MATHEMATICS PROBLEMS

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March 25, 1984

Abstract: Since Polya, Wertheimer, and Hadamard's descriptions of qualitative reasoning strategies used by scientists and mathematicians, very little data has been collected on whether these strategies are act. ly used by experts This study used video-taped thinking-aloud interviews to examine the problem solving strategies of professors and advanced graduate students in technical fields. Evidence from these interviews documents the use of analogies, visual transformations, extreme cases, creative partitioning, and other plausible reasoning strategies used by experts.

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TABLE OF CONTENTS

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Considering helpful analogous and extreme cases, breaking problems into analyzable parts, and performing simplifying spatial transformations are key reasoning processes in solving non-trivial problems. These processes allow talented scientists to attack problems outside the domain of familiar problems for which they have established algorithmic procedures. They allow them to attack problems they have never seen before, giving them a degree of problem solving power and scope that is truly impressive. In previous reports, (1,2,3,4), I have documented the fact that these qualitative reasoning processes are used by expert scientists in solving physics problems. This paper shows that empirical evidence for these processes can also be collected in the case of experts' solutions to mathematics problems. I will first briefly describe results from the earlier physics problem study, and then describe those from the current mathematics problem study.

EXPERT REASONING ON A PHYSICS PROBLEM

In the previous study, ten expert subjects were asked to solve the spring problem shown in Fig. 1. All subjects were advanced doctoral candidates or professors in technical fields. The study concentrated most on documenting and analyzing the use of analogies.

Some examples of analogies generated for this problem are as follows: One subject thought about a horizontal saw blade held fixed at one end and

loaded with a weight at the other end. He felt that a long blade would bend more easily than a short one, and this indicated to him that the wider spring might stretch more. Other examples of proposed analogies were that a longer horizontal "hairpin" shaped wire would extend more than a shorter one (see Fig 2), and that a larger single "square coil" would stretch more than a smaller one. Another subject examined the relationship between coil diameter, coiling angle, and wire length by thinking about mountain roads winding up narrow and wide mountains.

The correct answer to the problem is that the wide spring will stretch farther (the stretch in fact increases with the cube of the diameter). This seems to correspond to most peoples' initial intuition about the problem. However, explaining why the wide spring stretches more (and explaining exactly where the stretch of the spring comes from), is a much more difficult task when taken seriously.

Some of the findings from this study were as follows:

(1) Spontaneously generated analogies were observed to play a significant role in problem solutions of scientifically trained subjects. Seven of the ten subjects generated at least one salient analogy.

(2) The subjects generated a large <u>variety</u> of analogous cases. Not all of the analogies were to situations familiar to the subject. Some were novel cases in the form of Gedanken experiments that appeared to be invented by the subject.

(3) In addition to the initial process of <u>generating</u> an analogy, there is a second process that is just as important in expert problem solving, that of <u>critically evaluating</u> the validity of the analogy.

(4) Analysis of the transcripts indicated that there was more than one type of analogy generation method used. Two of these methods are the associative leap, and the generative transformation.

The subject using an <u>associative leap</u> jumps to an analogous situation that differs in many ways from the original problem.

For example, one subject compared the wide and narrow springs to two blocks of foam rubber, one made with large air bubbles, and one made with small air bubbles in the foam. He had a strong intuition that the foam with large air bubbles would be easier to compress.

However, the associative leap was not the only analogy generation method observed. Another pattern was observed in which a subject generates an analogy via a transformation which "warps" or changes the original situation A to produce the analogous situation B. Such a generative transformation occurs when a subject modifies an aspect of the original situation A that was previously assumed to be fixed. For example, some subjects "unrolled" the spring into a horizontal wire and thought about how the wire would bend if the wire were held at one end and the weight were placed on the other end.

(5) Some subjects gave evidence of using spatial reasoning by referring spontaneously to imagining or picturing situations they were thinking about.

(6) Extreme cases such as considering a very narrow or very wide spring were observed as well. These were effective in adding confidence to a subject's prediction for the problem.

ANALOGICAL REASONING IN A MATHEMATICS PROBLEM

A set of eight subjects were also asked to solve the "Doughnut" problem shown in Figure 3.

All subjects were advanced graduate students or professors in technical fields. This paper reports on results from the eight solutions to this problem and looks in detail at one of the solutions. Some behaviors parallel to those in the solutions of the spring problem have been identified, as well as some completely new behaviors.

A common analogy generated for this problem was to consider the case of the "straightened out" torus in the shape of a cylinder. Subjects conjectured that the volume of these two objects might be the same. The condensed transcript excerpt from subject SS below gives one example of this approach.

001 S: .. Ok. So here's a doughnut. Now the question is how to get its volume. Uhh, the first thing that comes to mind is that it's probably pretty close to a worm, er,

002 S: I mean a cylinder. Where you know, if you laid out the doughnut on the ground, uh, if you cut it open and laid it out, it would basically be the area of the base times the length around the middle. So let's see, I'll put down here number 1 is my first approximation which in fact may turn out to be the exact thing. Uh, I'll just turn it into a cylinder.

003 I: Mmm

004 S: In other words here is this volume of doughnut---and so that would--Pi r_2 squared would be the bottom of the cylinder and then, uh, you know, I think the relevant length of the cylinder would be not r_1, but the distance, uh, uh, to the middle there--namely r_1 oops, r_1 minus r_2--uh, right, that gets us to the middle, and then times 2 Pi...

005 I: .. When you thought about the cylinder, do you know how that arose? Did it just sort of flash in your head?

006 S: Well I mean, in fact, er, this little r_2 was drawn here very nicely so I just imagine the knife cutting it open and you know, laying it out...

008 S: ...I mean to say I thought of cutting it at one edge and it sort of flopping down band then the uh, the doughnut becomes a cylinder.

009 I. OK

010 S: And so I guess, uh, you know with, er, I don't think I've made any real algebraic mistakes here. I mean I think this er, probably I mean I feel pretty confident about that answer because what happens when you open the doughnut is that the, the top of the doughnut kind of expands a little.

011 S: I mean er, the part of the doughnut which was the inside stretches out a bit but then on the other hand, the part of the doughnut which was the outer perimeter gets crunched a little bit. And so probably those two things cancel. And you know, if the world is made correctly, but now I guess...

012 I: Just put a confidence number on it at this point and then you can go on.

013 S. Ok. I would say that on that I'm probably like 80% confident.

The cylinder idea fits the definition of a spontaneous analogy as used here because it is a case which differs from the doughnut with respect to a feature (the shape of the doughnut) that is a fixed feature in the original problem. The observational definition of a spontaneous analogy used in both studies was the following:

(1) The subject, without provocation, considers another situation B where one or more features ordinarily assumed fixed in the original problem situation A are different; (2) the subject indicates that certain structural or functional relationships (as opposed to surface features alone) may be equivalent in A and B; and (3) the related case B is described at approximately the same level of abstraction as A.

The act of violating a feature previously assumed to be fixed is the creative aspect of producing an analogy. The difficulty of such acts is presumably the underlying source of Wertheimer's finding (5) that many students do not think to modify the shape of a parallelogram in order to compute its area.

RESULTS: ANALOGIES

As shown in Table 1, all eight subjects wrote an equation for their answer that was correct in principle, with one subject making an algebra mistake.

Analogies. Six of the subjects spontaneously considered the analogy of the cylinder and used it in their solution. The two other subjects used Pappus' theorem which states that if one has a surface of arbitrary shape and one rotates it around a line in the same plane as the surface (but not cutting the surface), the volume o the solid generated will be the area of the shape times the distance that its centroid covers in the circuit. The more interesting solutions were from subjects who did not think about Pappus'

theorem, and of those six, four used the cylinder as the main route to their solution. In a fifth solution, the cylinder appeared to be of equal importance to another approach, and in a sixth solution, it played a confirming role. Thus, spontaneous analogies were observed to occur in a number of these expert solutions to a mathematics problem.

Analogy Generation Methods. A striking feature of S5's protocol above is the explicit evidence for generating the analogy via an associative leap. The most explicit criterion used to code for a generative transformation is the subject referring to changing a fixed feature of the problem. Here the subject makes statements like: If you cut it open and laid it out...", and "I'll just turn it into a cylinder" (Line 2), referring explicitly to changing the shape. This method contrasts to an associative leap, where the subject is simply reminded of a familiar situation via a direct association (For example, if the subject were reminded of another problem he had seen about a torus). This protocol provides fairly explicit evidence for the possibility of generating an analogy via a transformation. As shown in Table 2, evidence for a generative transformation of this type was observed in four of the six cylinder analogies In a fifth case, the cylinder idea grew out of considering the extreme case of a very wide thin doughnut with r_1 much greater than r_2. In a sixth sense, the generation method idea was unclear.

Evaluating the cylinder conjecture. Some subjects, such as S5, critically evaluated the analogy relation they had constructed by questioning whether the volume of the cylinder they constructed was really the same as that of the torus, and by seeking out alternative paths to the solution. For example, S5's transcript continues as follows:

013 S: Uhh, now what would happen if you did various things to the doughnut? Certainly you could argue that... this answer [the formula for a cylinder] is closer and closer to the correct one if uh, you know, if r_1 is much, much greater than r_2, then in that limiting case, you've [illegible] get this. Because that's just... going to approach being a cylinder more and more. So whatever the correct answer is, it's got to have that [formula] as a limiting case if r_1 is much greater than r_2...

026 S: ...I suppose the other way you could imagine doing it if you wanted to break it up would be to break it up into little wedges of doughnuts. So that if you were looking at it that way then you say OK, here's er, here's another

027 S: infinitesimal element which is a wedge like that, both faces of which have an area of uh Pi r_2 squared. Ha, ha. Of course, the thing is what's the volume of a little wedge like that? Um, well, if it's small enough, then it's just the thickness of it.

031 S: ...Volume of wedge would be Pi r_2 squared, and we'll call th[illegible]w dZ and again, you know, it would boil down to that same equ. again, if you added them all up into...

032 I: yeah

033 S: ..into equation 2, right? And dZ essentially would be the midpoint, distance there [the thickness of the middle of the wedge]. OK... I've, I mean, I think my confidence level at this point would be-- like 95%.

Earlier, it was stated that the process of criticizing and evaluating an analogy is just as important as the process of generating it in solving science problems. This appears to be true in the case of mathematics problems as well. Subjects who think about an equivalent cylinder must choose a cylinder of the right length, and they often take pains to critically evaluate their choice of length. For example, S5 above chooses the central or "average" circumference of the torus, $2Pi(r_1 - r_2)$, as the length of the cylinder. But he then evaluates the plausibility of this choice in lines 10 and 11 by giving a qualitative compensation argument about the inside stretching and the outer part getting "crunched". He also evaluates his prediction further by using an extreme case in line 13.

OTHER "INSIGHTFUL REASONING" PROCESSES

Other strategies observed in the doughnut problem solutions are shown in Table 3. For example, S5 cutting the 'wedges' out of the doughnut above is an example of a partitioning process. Each of the processes are discussed in turn below.

<u>Extreme cases.</u> Five of the subjects generated an extreme case in the problem and there were six extreme cases generated altogether. For example, several subjects thought about the extreme case where r_1 is much greater than r_2. Typically, they reasoned that if r_1 is very large, a small section would look locally very much like a cylinder since it would have very little curvature. Thus, they felt that the formula derived from the case of the cylinder would be correct at least in that extreme case. Other subjects thought about the case where r_2 goes to zero and checked whether the formula they had derived was correct in that situation.

<u>Partitioning and symmetry arguments.</u> As mentioned earlier, S5 partitioned the torus into wedges in order to help confirm his solution. Altogether there were thirteen attempts to partition the problem generated by five of the subjects. Subject S5 generated a second interesting partition by breaking up the doughnuts into smaller doughnuts as described below:

018 S: ..let's see? Is there any other limiting case we can look at? (15 second pause) I suppose another way to you know, uh, increase my confidence on that is to say well suppose if I really believe--which I do--that this limiting case [r 1 >> r 2] is correct, then why not imagine the doughnut being made up of a lot of other little doughnuts you know, which are

019 S: tightly packed in there In other words, a whole series of thin doughnut rings that are all packed together in just the right way

to give the slightly bigger, fatter doughnut. And then, you know, again, that would indicate that this [equation] is the correct answer. Uh--those are not really space filling though. There's

020 S: little interstices between those doughnuts... In the final analysis, I think that I feel very confident about that because you would--if you were to do the integral, you would break it up into doughnuts that have a square cross section. And then you would just add 'l those up.

Figure 4 shows a cross-section of the doughnut with tiny doughnuts which can be thought of as wires passing through the cross section. Although S5 does not complete the argument here, we can use his imaginative way of partitioning the doughnut to show that the length of the equivalent cylinder should be the same as the length running through the center or midline of the doughnut. To do this, we imagine the doughnut being filled with a multitude of tiny thin doughnuts or "wires." We consider a cross section of the doughnut and notice that the average circumference of the thin doughnuts in the cross section should be the same as the length of the conjectured cylinder. We can prove this to ourselves by drawing a vertical line down the center of the cross section in Figure 4. One then notices that for every wire on the left side of the line, there is a symmetrically placed wire on the right side of the line. The wire on the left will be of length $2Pi(r_1 - r_2)-d$, and the wire on the right will be of length $2Pi(r_1 - r_2)+d$. Thus, each long wire on the left has a short counterpart on the right which cancels its extra contribution to the volume, and the average length of a strand is the same as the length of the conjectured cylinder. This argument is interesting because of its use of a creative partitioning strategy. It is also interesting because of the use of symmetry. The key insight seems to occur when one recognizes that one can cancel differences by creating a one-to-one matching between wires on the left and right of the cross section. This

symmetry argument allows one to cancel an infinite number of contributions to the volume in one stroke, even though each contribution has a different value.

Reassembly of a partition. Another observed strategy is to partition an object in an attempt to rearrange the pieces into a more "congenial" (simpler or more familiar) object. Figure 5 shows a partition of the torus into what another subject, S2, called "apple rings". He convinced himself that the volume of each ring would be equivalent to a rectangular solid whose length is that of the mid-circumference of the annulus. This allows one to "restack" the slices in the shape of a cylinder. Five attempts to generate and reconstruct a partition were observed in the solutions, but these were all generated by a single subject. The classical example of creative partitioning and reconstruction of the problem is found in Wertheimer's discussion of the parallelogram, whose area can be found and understood by partitioning the parallelogram and reconstructing it into a rectangle.

Embedding. Six attempts to embed the problem in a larger problem were observed in three of the solutions. For example, one subject embedded the torus in a "washer" shown in Figure 6 which snugly wraps around the torus. The washer is a cylinder with a hole in it, and its volume is easy to calculate by subtraction. He then noticed that the ratio of the area of the torus' cross-section to that of the washer could be calculated and that the volume of the torus and the volume of the washer should have the same ratio. So he determined the volume of the torus by embedding it in a larger object.

Spatial reasoning. The protocol also provides evidence for the role of spatial reasoning. First and most obviously, there are references to spatial relations between objects that are primarily qualitative and often dynamic in nature, such as: (line 2) "You know, if you laid out the doughnut on the ground;" and (line 11) "the part of the doughnut which was inside stretches out a little bit." Passages of this kind suggest that the subject may be. (1) imagining manipulating concrete or idealized objects; and (2) experiencing the anticipated outcomes of his manipulations via imagery.

Secondly, there are more explicit references to imager. An imagery report is defined as occurring when the subject refers to imagining, picturing, "remembering a diagram for", hearing, or "feeling what it's like to manipulate" a situation. We refer to a dynamic imagery report if the reference is to imagining a situation which does not remain fixed, but changes with time. In this study, we are concerned only with spontaneous imagery reports where the interviewer does not ask the subject whether an image was used. Examples of dynamic imagery reports in the protocol are: (Line 6) "I just imagine the knife cutting it open;" and (Line 26) "You could imagine...if you wanted to...break it up into little wedges of doughnuts." Thus, it is possible to point to some evidence in protocols which supports the hypothesis that spatial reasoning involving imagery of a qualitative and dynamic nature is involved in expert problem solving.

DISCUSSION

As summarized in Tables 1, 2, and 3, it is possible to document several kinds of creative reasoning strategies in expert solutions to mathematics problems, including analogy generation, extreme cases generation, partitioning, the reconstruction of the problem into a different shape,

embedding the problem in a larger context, and the use of spatial transformations

S5's reference to cutting the doughnut into wedge-shaped pieces and computing their volumes documents the strategy of breaking a problem into parts--in this case, the subject partitions the problem symmetrically into a number of equivalent parts. One could treat this as the simple application of a heuristic, but the trouble with the heuristic "break the problem into simpler parts" is that it does not tell you which parts to form. Such an act can require considerable creativity and ingenuity.

By speaking of an infinitesimal slice, S5 is in danger of breaching the request in the instructions to refrain from taking an integral. Indeed, creative cutting and partitioning of just this kind is an essential skill for applying the integral calculus to non-trivial situations. As in the case of analogies, the breaking into parts process is in effect an attempt to find a conserving transformation which leaves one with one or more simpler problems. It is interesting to note that the wedges can be stacked alternately as shown in Figure . In the limit, this can provide an elegant argument for the validity of the original analogy to a cylinder of length $2Pi(r_1 - r_2)$ (An analogous argument for calculating the area of a circle is shown in Figure 8.) The discovery of such conserving spatial transformations and equivalence can be a great source of satisfaction and appreciation for the interconnectedness of mathematical ideas. It was physical-spatial transformations of this type that apparently allowed Archimedes to develop many fundamental ideas underlying the integral calculus over two thousand years ago. (See description in [6].)

CONCLUSION

The ability to perform relevant spatial transformations, the ability to consider and evaluate analogous cases and extreme cases, and the ability to break problems into parts intelligently are crucial skills for solving non-trivial problems. The process of partitioning is an essential skill underlying the concepts of subtraction, division, fractions and integration. The fact that experts use these processes can be documented in problem solving protocols, and the nature of the processes can be analyzed. In saying that these strategies played an important role in a number of the problem solutions, we mean that they were involved in a serious attempt to understand or solve the problem and were not just proposed by the subjects as an ornamental side comment or as a check on a firm answer. We are currently investigating the abilities of students as well as experts to use these processes, and as we understand more about their nature, we should be able to design more effective instructional experiences which foster them.

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SPRING PROBLEM

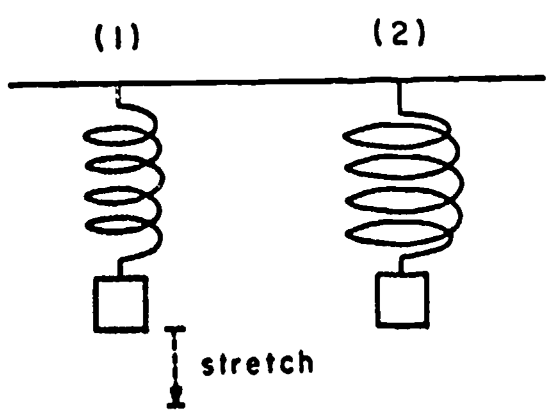
A WEIGHT IS HUNG ON A SPRING. THE ORIGINAL SPRING IS REPLACED WITH A SPRING

--MADE OF THE SAME KIND OF WIRE,

--WITH THE SAME NUMBER OF COILS,

--BUT WITH COILS THAT ARE TWICE AS WIDE IN DIAMETER.

WILL THE SPRING STRETCH FROM ITS NATURAL LENGTH, MORE, LESS, OR THE SAME AMOUNT UNDER THE SAME WEIGHT? (ASSUME THE MASS OF THE SPRING IS NEGLIGIBLE COMPARED TO THE MASS OF THE WEIGHT.) WHY DO YOU THINK SO?



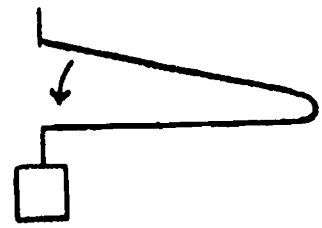


Figure 2

DOUGHNUT PROBLEM

Compute the volume of the torus (doughnut) below without taking an integral. Give an approximate answer if you cannot determine an exact one.

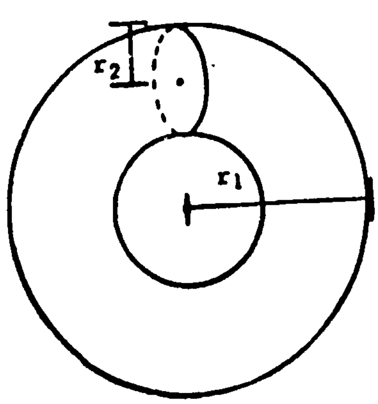


Figure 3

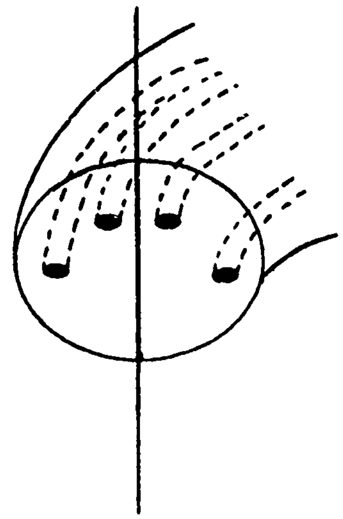


Figure 4

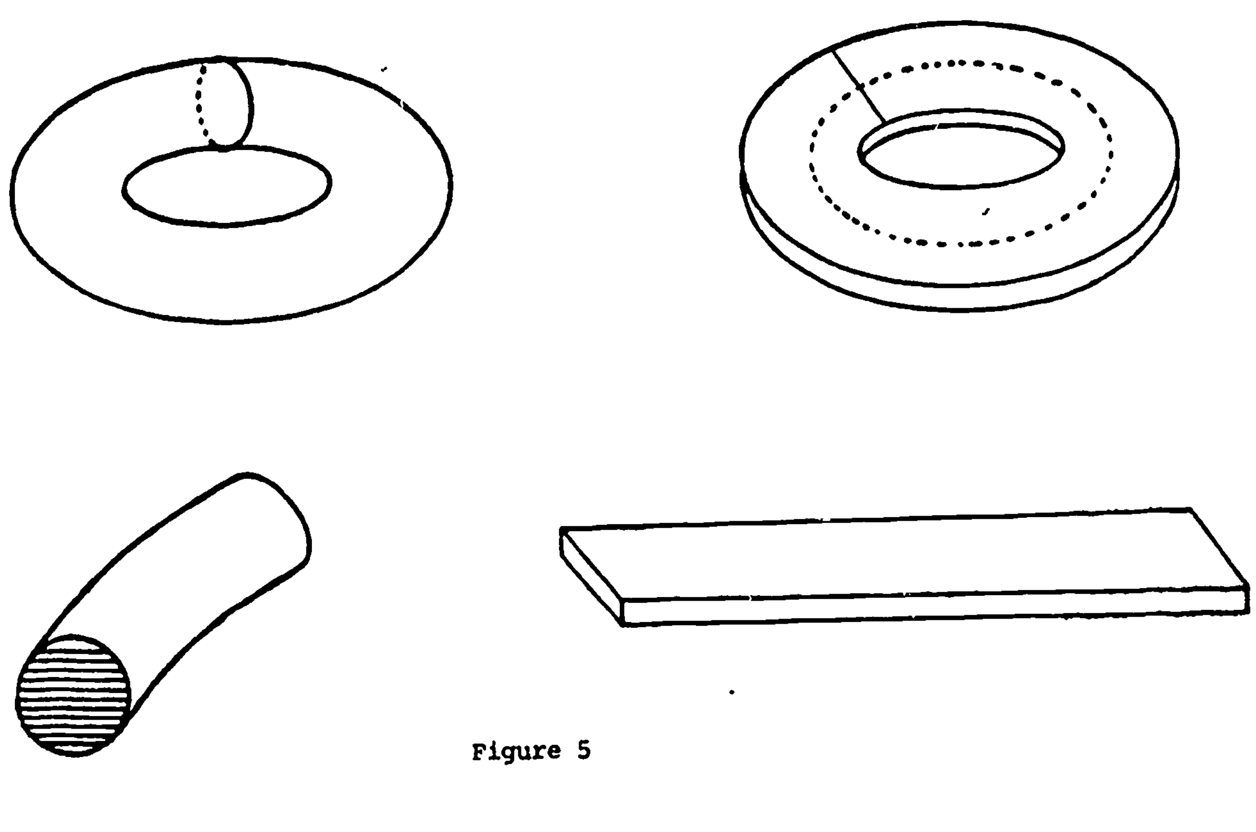


Figure 5

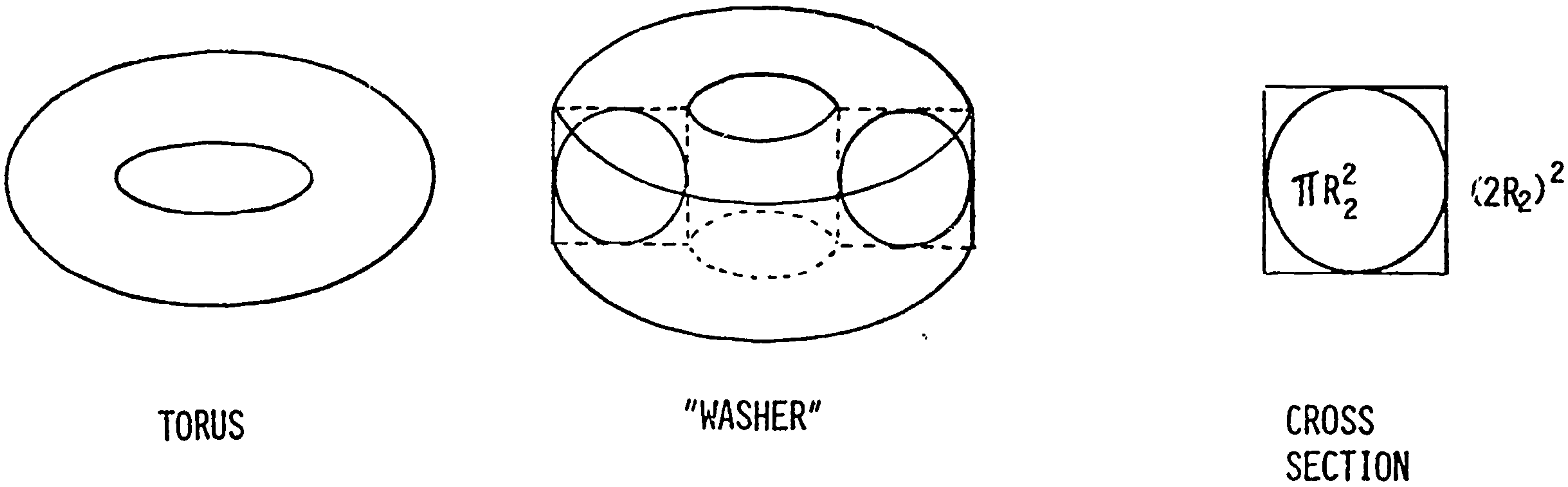


Figure 6

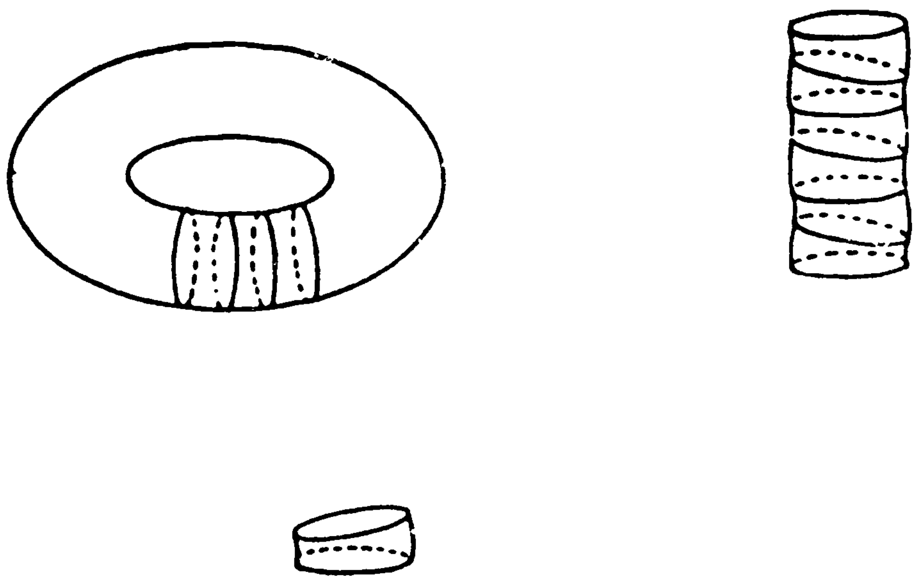


Figure 7

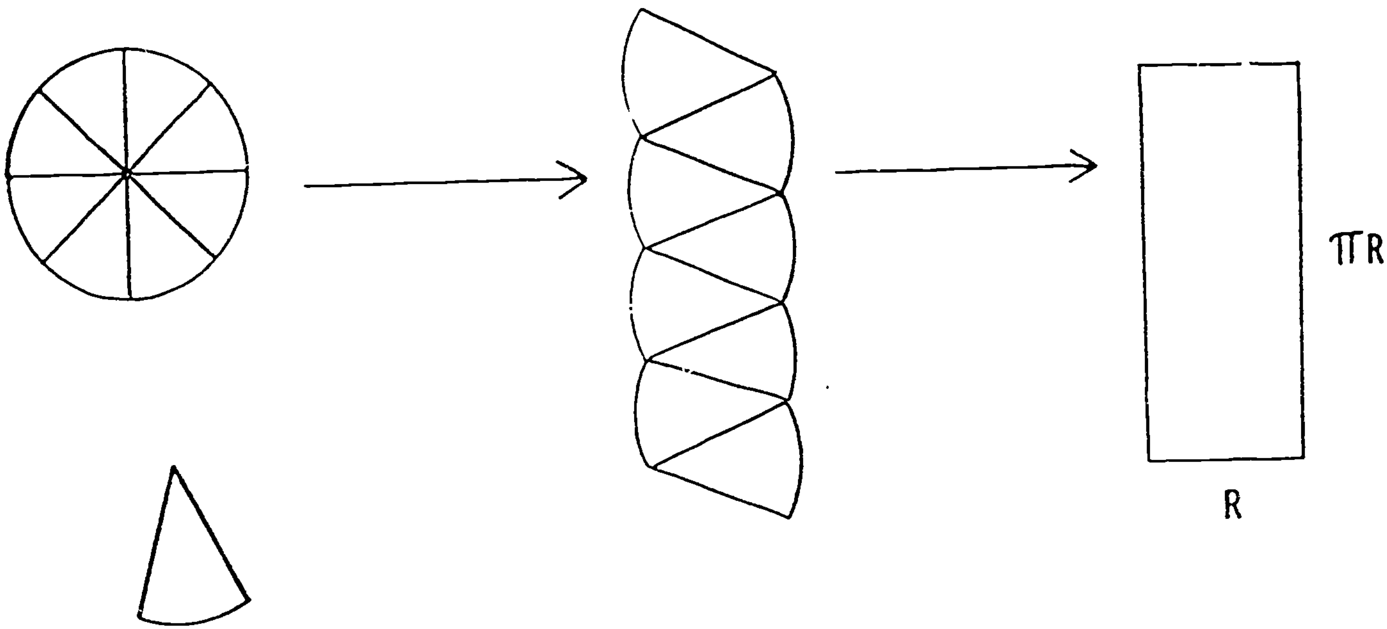


Figure 8

SOLUTIONS TO DOUGHNUT PROBLEM

N = 8

CORRECT	7
CORRECT IN PRINCIPLE	8
PAPPUS' THEOREM	2
CYLINDER:	
PLAYED A ROLE	6
MAJOR ROLE	5
MAIN ROUTE	4

Table 1

GENERATION METHODS FOR CYLINDER ANALOGY

N = 6 SALIENT CYLINDER ANALOGIES

EVIDENCE FOR GENERATIVE TRANSFORMATION (UNBENDING)	4
EXTREME CASE TRANSFORMATION ($R_1 >> R_2$)	1
UNCLEAR	1

Table 2

OTHER STRATEGIES USED IN DOUGHNUT PROBLEM

	NUMBER OF SUBJECTS	TOTAL CASES
EXTREME CASES	5	6
PARTITIONING	5	13
ATTEMPTED REASSEMBLY OF A PARTITION	1	5
EMBEDDING ATTEMPTS	3	6

Table 3