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ABSTRACT

The goal of this study was to describe characteristic stages in the development of children's understanding of place value. Sixty children (15 in each grade, 2 through 5) at 5 elementary schools in Butte County, California, were individually administered 18 tasks designed to reveal their thinking about place value and part-whole relations. On six tasks children were to identify the number of objects represented by the individual digits in a two-place numeral. Seven fifth-graders, seven fourth-graders, two third-graders, and no second-graders were successful on all six tasks; 28 children failed to convincingly demonstrate any understanding of the tens digit. Success on prerequisite tasks measuring rational counting by tens, identifying the tens and ones places, partitioning collections into a tens part and a ones part, and conserving grouped number were all found to be necessary for success on more than four digit-correspondence tasks. Understanding of part-whole relations as reflected by success on arithmetic word problems and on logical classification tasks was necessary for success on more than two digit-correspondence tasks. A five-stage model of the development of children's understanding of place value was then proposed. A three-page bibliography concludes the document. (MNS)

THE DEVELOPMENT OF CHILDREN'S PLACE-VALUE NUMERATION CONCEPTS IN GRADES TWO THROUGH FIVE

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THE DEVELOPMENT OF CHILDREN'S PLACE-VALUE NUMERATION CONCEPTS IN GRADES TWO THROUGH FIVE

Abstract. The study was designed to identify characteristic stages in the development of children's understanding of place-value numeration. Sixty children in grades two through five were individually interviewed and their performances on 18 tasks are described and compared. A five-stage model of the meanings children attribute to two-digit numerals is proposed and evidence for each stage, from previous research and the present study, is cited.

Our familiar numeration system for denoting numbers is known as a "base-ten positional" system. The value of a given digit in a multi-digit numeral is dependent not only on the face value of the digit, but also on its place, or position, in the numeral. The values of the places increase in powers of ten from right to left.

For children to understand place-value numeration requires that they coordinate numeration knowledge with number concepts. While a numeration system represents cultural information which must be socially transmitted, number concepts, according to Piagetians, represent logico-mathematical knowledge which must be constructed by the individual learner. Since the numbers represented by the individual digits in a multi-digit numeral must sum to the number represented by the whole numeral, an understanding of numerical part-whole relations is needed.

While children are introduced to the meaning of "tens and ones" as early as the first grade, to allow for their understanding of the computational algorithms for operating with large numbers, numerous studies have documented that children's understanding of place value is generally poor throughout the primary grades. Most early research, however, allowed or'v limited inferences about the reasons for children's poor performances. More recent research, utilizing interview methodologies, has been more revealing (Bednarz and Janvier, 1982; M. Kamii, 1980; and Resnick, 1983). In the present study, tasks were adapted from both the Kamii and Resnick research in order to allow for the possible integration of the limited models proposed by each into a more comprehensive model of children's conceptual development in the place-value domain.

Procedures

Scope. An understanding of place-value numeration can be held at many levels; it may include an understanding of "tens and ones," of computational algorithms, of decimal numerals, binary numerals, or of scientific notation. In the present study of children in grades two through five, the assessment of understanding was limited to the following concept: In a two-digit numeral, the whole numeral represents a whole quantity (of objects), while the individual digits represent a partitioning of the whole collection into a "tens part" and a "ones part." The whole must equal the sum of the parts. For example, the numeral "25" can represent the cardinality of a collection of 25 objects; the "2" represents 20 of them while the "5" represents the remaining 5. The study was designed to examine factors that might contribute to or inhibit children's acquisition of this concept. The present study did not assess the effects of instruction.

Tasks. During the months of October through December of 1984, 60 children were administered, in individual one-hour interviews, 18 tasks designed to reveal their thinking about place value and part-whole relations. Six of the tasks, adapted from Kamii, asked children to make correspondences between the individual digits in a two-place numeral and collections of objects. These six tasks were used to measure children's understanding of place value. Six additional tasks, some adapted from Resnick's work, assessed knowledge predicted to be prerequisite to understanding place value. These tasks assessed numeration knowledge and number concepts. The final six tasks were in the domain of part-whole relations.

The order of task presentation was partially counterbalanced. Children's responses to each task were recorded on coding sheets developed during a pilot study and were also recorded on audio tapes. Task protocols and coding sheets are available from the author.

Analysis. The responses were qualitatively analyzed and levels of performance identified for each task. A variety of quantitative methods, including prediction analysis, chi-square tests of independence, Pearson's product-moment statistic, Guttman scaling, and McNemar's test of homogeneity for correlated proportions, were then used to examine patterns of children's performances across tasks.

Sample. Three children in each grade, two through five, were randomly selected from the grade level enrollment lists of five elementary schools in Butte County, California. The resulting sample had 60 children, 15 in each grade and 12 from each school. The schools were selected to represent urban and rural communities, public and private funding, and diversity with respect to the mathematics textbook series used, school size, and social class. In all, children from 33 classrooms were interviewed, assuring a cross-section of teachers' methodologies as well.

Results and Discussion

For each of the eighteen tasks levels of performance were identified and described. New, previously unavailable descriptions of children's characteristic performance were provided for several of the tasks. The levels established by qualitative analysis were found to correlate with age for all of the tasks except the combine word problem ($p < .05$).

Digit-Correspondence Tasks

Six tasks focused on the meanings children attributed to the individual digits in a two-place numeral. These tasks asked children to match each digit with the appropriate number of objects in a collection.

Sticks. In one of the digit-correspondence tasks the child is presented with a collection of 25 sticks (tongue depressors). The interviewer asks the child to count them and "write down the number." The interviewer then circles the digit in the ones place ("5") and asks "Does this part have anything to do with how many sticks you have?" After the child's response the interviewer indicates the digit in the tens place ("2") and repeats the question. Children's responses to the Sticks task were assigned to four levels, which describe the qualitatively different meanings that children assign to two-digit numerals.

Level 1. While a whole two-digit numeral represents the whole numerical quantity of a collection of objects, the child indicates that the individual digits in a two-digit numeral have no numerical meaning.

Type A. The child thinks that the individual digits have no meaning at all.

Type B. The child assigns a whimsical or graphic meaning to the individual digits.

Level 2. While the whole numeral represents the whole quantity, the child invents numerical meanings for the individual digits; the invented meanings are not related to place-value notions of groupings into tens and/or ones.

Type A. The individual digits represent groupings of objects.

Examples: In 25 sticks the "5" means groups containing five sticks, "2" means groups containing two sticks.

Type B. The individual digits represent other invented numerical meanings.

Level 3. While the whole numeral represents the whole quantity, the individual digits have meanings related to groups of tens or ones but the child has only a partial or confused idea of how this all works. The sum of the parts need not equal the whole.

Type A. The place-value meanings assigned to the individual digits are incomplete or inconsistent.

Type B. Both individual digits mean ones.

Type C. The child reverses the meanings of the the digits; the right digit means groups of ten and the left digit means ones.

Level 4. The whole numeral in a two-digit numeral represents a whole quantity of objects. The individual digits represent the partitioning of the whole quantity into groups of ten units (the tens digit) and a part composed of units (the ones digit). The whole must equal the sum of the parts.

While every child in the study was able to determine the number of sticks and write the appropriate numeral, not until grade 4 did half the children demonstrate that they knew that the "5" represented five sticks and the "2" represented 20 sticks. The number of children performing at each level, by grade in school, is reported in Table 1.

TABLE 1

GRADE IN SCHOOL BY PERFORMANCE ON TASK H:

STICKS DIGIT CORRESPONDENCE

	Level of Performance			
Grade[a]	1	2	3	4
2	5	2	5	3
3	7	1	2	5
4	0	7	0	8
5	1	4	0	10
Total	12	14	8	26

[a] n = 15 for each grade.

chi-square = 30.1 df = 9

level of significance = .0004

Other digit-correspondence tasks. In the Sticks task described above, the child was presented with an ungrouped collection of 25 sticks and asked to make correspondences between digits and objects. In the other five digit-correspondence tasks, the collections of objects were chosen and grouped by the child as part of a prerequisite task. The groupings and size of the collections for all six tasks are shown in Table 2. Typically the child did not arrange the objects in the left-to-right order shown in the drawings. Descriptions of the levels for each of the tasks were the same as those given above for the Sticks task.

Differential results for the digit-correspondence tasks. As expected, children did not find all six digit-correspondence tasks of equal difficulty. The number of children performing successfully (level 4) on each of the six tasks are reported, by grade level, in Table 3.

TABLE 2
DIGIT-CORRESPONDENCE TASK MATERIALS

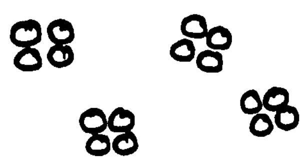
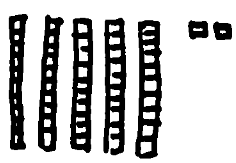
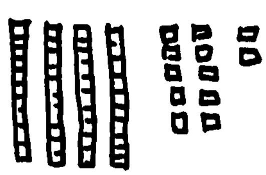
Task	Objects	Number
I-1 Beans		48
I-2 Regrouped Beans		48
H Sticks		25
G Wheels		16
J-1 Dienes Blocks		52
J-2 Regrouped Dienes Blocks		52

TABLE 3

NUMBER OF SUBJECTS PERFORMING AT LEVEL 4

ON DIGIT-CORRESPONDENCE TASKS BY GRADE IN SCHOOL

Task	Grade[a]				Total
	2	3	4	5	
J-1: Dienes Blocks	7	12	12	13	44
I-1: Beans	5	10	12	12	39
H : Sticks	3	5	8	10	26
I-2: Regrouped Beans	0	8	8	9	25
G : Wheels	2	5	7	8	22
J-2: Regrouped Blocks	0	4	8	8	20

[a]n = 15 for each grade.

When the tasks were originally designed for the present study, it was expected that those tasks which used canonical representations would be easier that those that used non-canonical representations. Canonical representations are those that follow the convention that no more than nine objects may be used in any position. Non-canonical representations allow more than nine and are necessary when representing multi-digit computational algorithms (for example, "carrying" and "borrowing"). For example, for a child to correctly link the individual digits in "48" with the correct number of beans in a representation which used three cups of ten beans and 18 loose beans was expected to be more difficult than making the same links when the representation contained four cups of ten beans and eight loose beans.

The data supported this conjecture--to a point. No child who performed successfully (level 4) on the regrouped beans task (non-canonical) failed to perform successfully on the canonical beans task; however, of the 39 children who performed at level 4 on the canonical task, only 25 were successful on the non-canonical task. The remaining 14 children performed at lower levels. Similar results held for the two Dienes-block tasks. When the differences between the numbers of children performing successfully on each of the six digit-correspondence tasks were tested for statistical significance, it was found that significantly more children were successful (performed at level 4) on the two canonical tasks, Beans and Dienes

Blocks, than on the remaining four tasks. (See statistical note 1.)

The surprise, however, was that none of the other contrasts produced statistically significant differences. Children found the Sticks task, in which the 25 objects were not grouped at all, to be as difficult as the non-canonical tasks which required regrouping. The pairwise comparisons of the proportions of children succeeding on each task are reported in Table 4. Clearly canonicity did not account for the observed differences. An alternative explanation was sought.

It was concluded that children could succeed on the canonical Beans and Dienes blocks tasks using an interpretation of individual digits that did not include the concept of tens and ones. In these tasks, the representations of the groups of ten are quite salient. For example, with the Dienes-block representation, white centimeter cubes are used to represent units, "longs" are 1 x 1 x 10 centimeter purple blocks which resemble ten unit-blocks glued together. In a canonical representation of a two-digit number, no more than nine unit-blocks are allowed; ten is always represented by a long-block.

When presented with, for example, five long purple blocks and two small white cubes, and the interviewer posed the questions about whether the "2" and the "5" in the numeral "52" had anything to do with how many blocks there were, some children may have readily responded that the "5" represented the five purple blocks and the "2" represented the two white blocks, without having any thought of tens and ones in mind. The meaning they assigned the digits was that of

TABLE 4

SUMMARY OF MCNEMAR TEST RESULTS FOR ALL PAIRS OF DIGIT-CORRESPONDENCE TASKS

Contrast	z
Dienes Blocks (J1) x Beans (I1)	1.51
Dienes Blocks (J1) x Sticks (H)	4.02*
Dienes Blocks (J1) x Regrouped Beans (I2)	4.36*
Dienes Blocks (J1) x Wheels (G)	4.69*
Dienes Blocks (J1) x Regrouped Dienes Blocks (J2)	4.90*
Beans (I1) x Sticks (H)	3.15*
Beans (I1) x Regrouped Beans (I2)	3.74*
Beans (I1) x Wheels (G)	4.12*
Beans (I1) x Regrouped Dienes Blocks (J2)	4.36*
Sticks (H) x Regrouped Beans (I2)	-0.33
Sticks (H) x Wheels (G)	1.41
Sticks (H) x Regrouped Dienes Blocks (J2)	1.73
Regrouped Beans (I2) x Wheels (G)	1.73
Regrouped Beans (I2) x Regrouped Dienes Blocks (J2)	1.89
Wheels (H) x Regrouped Dienes Blocks (J2)	0.71

* $p < .05$

their respective face values; to respond, they searched for two of something and then five of something else.

It was difficult to distinguish, in these tasks, between those children operating with this less competent quality of thinking and those with a more fully operational place-value concept which included groupings of tens. Verbalizations sometimes suggested the different qualities of thinking; if a child used the word "fifty," it was clear that s/he was thinking of the five purple blocks as representing fifty units. However, if the child said "five tens," the meaning was ambiguous; one is unsure whether the child is thinking of five sets of ten units or simply using a verbal name s/he had learned for the purple blocks. Even with additional probes the tasks were insufficiently sensitive to distinguish between the qualitatively different modes of thought.

Similar ambiguity was found in the canonical Beans task. Some children who pointed to the four cups of beans when asked if the "4" (in "48") had anything to do with how many beans there were, actually meant the four cups, not the beans in the cups. Again the interviewer was unable to clearly distinguish between the two modes of thought. Ultimately all children who responded "correctly"--identifying which blocks were represented by the "5" and which by the "2," were coded as level 4. This decision, though unsatisfying, led to the result that the proportion of children performing successfully on the two canonical tasks was significantly higher than the proportions successful on the other digit-correspondence tasks.

Composite digit-correspondence score. Each child was assigned a composite score on the digit-correspondence tasks equal to the number of tasks on which the child was successful (0-6). Twenty of the 60 children were found to understand place value, operationally defined as performing competently on at least five of the digit-correspondence tasks. Table 5 shows a cross-tabulation of the composite digit-correspondence score by grade level.

Understanding of place value as measured by the digit-correspondence tasks was found to be related to gender, with boys significantly outperforming girls. The Chi-square value was 13.24 ($6, N = 60, p < .05$).

Prerequisite Tasks

SIx tasks assessed children's ability to 1) determine by an efficient method the number of objects in a collection that has been grouped into sets of ten and a remaining part, 2) conserve grouped number, 3) represent a given number with base-ten Dienes blocks using no more than nine unit blocks, 4) represent a given two-digit numeral with Dienes blocks using more than nine unit blocks, 5) identify the tens and ones places in a two-digit numeral and 6) count orally (rote) by tens.

For the first two prerequisite tasks, materials included 48 lima beans and nine one-ounce plastic cups. Children were asked to put ten beans in each cup. Spontaneous discussion of the remaining eight beans and five cups usually followed. The unused five cups were set aside so that the result set was composed of four cups, each

TABLE 5

COMPOSITE SCORE ON DIGIT-CORRESPONDENCE TASKS (G - J2)

BY GRADE IN SCHOOL

Grade[a]	Score on Digit-Correspondence Tasks						
	0	1	2	3	4	5	6
2	8	1	3	2	1	0	0
3	2	4	1	1	2	3	2
4	1	4	2	0	0	1	7
5	1	2	1	2	2	0	7
Total	12	11	7	5	5	4	16

[a] n = 15 for each grade

chi-square = 35.26   df = 18

level of significance = 0.0088

containing ten beans, and eight loose beans.

Counts 48 Beans. The interviewer then asked the child to tell how many beans were there altogether. If necessary the interviewer prompted the child to include the loose beans as well as those in the cups. Observable behaviors and verbalizations were noted, and children were asked to tell "how they knew." Simple counting errors were ignored; counting strategies were the focus of the task.

Three qualitatively different levels of performance were identified. Three children were unable to quantify the grouped collection and were designated as level 1. Nine children who performed at level 2 could quantify the whole collection but depended heavily on counting by ones rather than using any of the more efficient methods characteristic of level 3 performance.

Forty-eight children were able to quantify the set in an efficient manner; their performances were designated as successful and assigned to level 3. These children, for example, counted by tens and then mentally added or counted on the remaining eight beans. Some of them used implicit ("four tens is forty") or explicit (four times ten is forty") multiplication. One child used pencil and paper (to add 20 + 20). No child spontaneously counted "8, 18, 28, 38, 48." The grade level results are reported in Table 6.

Conservation of Grouped Number. An adaptation of Piaget's familiar conservation of number task was used to assess whether a

TABLE 6

GRADE IN SCHOOL BY PERFORMANCE ON TASK C:

COUNTS 48 BEANS

	Level of Performance		
Grade[a]	1	2	3
2	2	4	9
3	0	4	11
4	1	1	13
5	0	0	15
Total	3	9	48

[a] $\underline{n}$ = 15 for each grade.

chi-square = 11.0 df = 6

level of significance = .0884

child knows that the whole quantity is conserved when a collection is transformed from a canonical ten-and-ones partitioning into a non-canonical partitioning. After completing the Counts 48 Beans task and the canonical Beans digit-correspondence task, the interviewer then spilled one of the four cups of beans onto the table, so that there were on the table ten beans in each of three cups and 18 loose beans. The child was asked: "Do you think there are now more beans or fewer than there were before?" After the child responded, the interviewer would ask "how do you know?" Non-conserving children, designated as level 1, were those who were quite convinced that the total quantity was altered to either more or less when one cup of beans was spilled. There were 10 such children.

To set the scene for the digit-correspondence task, children were then asked how many beans the collection now contained. This led some of the children who initially gave a non-conserving response to establish by counting that there were, in fact, the same number of beans as initially. These children were designated as level 2. Also designated as level 2 were those children who chose not to respond to the conservation question ("Are there now more or fewer?") until they counted to find out. In all, there were nine children at level 2.

Level 3 was assigned to children who conserved number in this task by responding, without counting, that the number of beans remained the same. Some gave logical rationales; others gave empirical (numerical) rationales. Only 5 of the 15 second-grade children were at level 3; of the total sample of 60 children, 41 were found to be at level 3. The results are reported in Table 7.

TABLE 7

GRADE IN SCHOOL BY PERFORMANCE ON TASK D:

CONSERVATION OF GROUPED NUMBER

	Level of Performance		
Grade[a]	1	2	3
2	6	4	5
3	3	1	11
4	1	3	11
5	0	1	14
Total	10	9	41

[a] n = 15 for each grade.

chi-square = 15.6 df = 6

level of significance = .0163

Builds 52 with Dienes Blocks (canonical). The goal of this task was to assess children's strategies for constructing a whole quantity from concrete base-ten materials. The quantity of unit-blocks available (40) was insufficient to construct the whole quantity without using ten-blocks.

From a set of Dienes block representations of ones tens and hundreds, the child was asked to "use these counting blocks to build 52." If the child did not understand the task, various restatements of the problem were provided, especially "so they will count up the 52." After the child indicated that s/he was finished, the interviewer asked "how do you know that is 52?"

Three levels of performance were identified. For children at levels 1 and 2 this task was a non-routine problem solving task. Children at level 1 were unsuccessful in solving the problem. Children at level 2 were successful; generally they solved the problem after an initial unsuccessful attempt to represent 52 using only the unit-blocks. After discovering that the quantity available was insufficient, children at level 2 eventually, some only after a prompt from the interviewer, solved the problem using some of the ten-blocks. Six children were at level 1, seven were at level 2.

The 47 children who performed at level 3 behaved as though this were a routine problem. There was no uncertainty in their behavior; they quickly chose five ten-blocks and two unit blocks to represent 52. The results are reported in Table 8.

TABLE 8

GRADE IN SCHOOL BY PERFORMANCE ON TASK E:

BUILDS 52 WITH DIENES BLOCKS

	Level of Performance		
Grade[a]	1	2	3
2	5	1	9
3	0	3	12
4	1	1	13
5	0	1	14
Total	6	7	47

[a] $\underline{n} = 15$ for each grade.

chi-square = 14.5 df = 6

level of significance = .0245

Builds 52 multiple ways (non-canonical). The children who succeeded in producing an initial representation of 52 using Dienes blocks were asked if they could find another way to represent 52. The sequence of levels was the same as for the canonical task above. Eight children were unsuccessful and their performances were designated as level 1. Eighteen children first attempted to use soley unit-blocks; upon discovering the insufficient quantity they eventually, with or without a prompt from the interviewer ("Could you use some of these blocks?) were successful. The performances of these children were designated as level 2.

Thirty children were successful; their performances were designated as level 3. They transformed the initial collection of five ten-blocks and two unit-blocks into a "different way" without initially counting the 40 unit-blocks. Some of these children counted up from 40 (40, 41, 42....) and others used a trade--one ten-block for ten unit-blocks. The results are reported in Table 9. Children who were successful at building 52 the first (canonical) way were sometimes quick to respond that there was no other way it could be done; when prompted however, they became engaged in the task. Most children were much slower to construct the second representation than they had been in the first and many more children were operating in a problem-solving mode.

TABLE 9

GRADE IN SCHOOL BY PERFORMANCE ON TASK F:

BUILDS 52 TWO WAYS

	Level of Performance			
Grade[a]	0	1	2	3
2	4	5	4	2
3	0	2	5	8
4	0	1	5	9
5	0	0	4	11
Total	4	8	18	30

[a] n = 15 for each grade.

chi-square = 25.2 df = 9

level of significance = .0027

Positional Knowledge. The previous four tasks were viewed as assessing children's understanding of number concepts that might be necessary for children to understand place-value numeration. The goal of the final prerequisite task was to assess the subjects' knowledge of the positional feature of our numeration system for designating numbers. Children were first asked to read the numerals "37" and "84." They were then asked to point first to the "tens place" and then to the "ones place" of the 37 and tell "how many tens are in 84."

Children's responses to this task reflected the following sequence of ideas: In level 1 the children demonstrate no knowledge of the fact that individual digits in a two-place numeral are distinguished as "tens" and "ones." In level 2 the child is aware of the distinction but confuses which digit is which. In level 3 the child can distinguish correctly between the tens and ones digits but doesn't know that the face value of the tens digit represents how many tens are contained in the whole quantity. In level 4 the child demonstrates immediate and strong knowledge of the positional feature of our numeration system; s/he answers all questions correctly.

One child made a reversal error when asked to read "37." All other children read both numerals correctly. Thirty-seven children performed at level 4. Only three of these were in grade 2. The formal tabulation of results, by grade level, is found in Table 10.

TABLE 10

GRADE IN SCHOOL BY PERFORMANCE ON TASK B:

POSITIONAL KNOWLEDGE

	Level of Performance			
Grade[a]	1	2	3	4
2	5	5	2	3
3	0	4	1	10
4	0	2	1	12
5	0	1	2	12
Total	5	12	6	37

[a] n = 15 for each grade.

chi-square = 24.9 df = 9

level of significance = .0031

Prerequisite tasks x digit-correspondence score. The knowledge assessed in each of the six prerequisite tasks described above was predicted to underlie an understanding of two-digit numeration. To evaluate the support that the data from this study provide for such predictions, digit-correspondence scores were cross-tabulated with prerequisite task results and the prediction analysis statistic ∇_P was used. The prediction analysis statistic was developed by Hildebrand, Laing, and Rosenthal (1977) specifically for use in analyses where a priori predictions have been made that certain of the cells in a cross-classification are equal to (or near to) zero. (See statistical note 2.)

The digit-correspondence scores were collapsed into "successful" and "unsuccessful" categories: scores of 0, 1 and 2 were designated "unsuccessful" and scores of 5 and 6 were designated "successful." On the prerequisite tasks children performing at the highest level were designated as "successful," children performing at all lower levels were designated as "unsuccessful." The theoretical predictions maintained that no child who was unsuccessful on the prerequisite task would understand place value (digit-correspondence scores of 5 or 6); in the 2 by 2 cross-tabulation shown in Table 11, the upper right-hand cell was predicted to contain no entries.

TABLE 11

PREDICTED CELL ENTRIES

FOR SELECTED PREREQUISITE TASKS BY DIGIT-CORRESPONDENCE SCORE

	Digit-Correspondence Score	
Prerequisite Task	0, 1, 2	5, 6
Unsuccessful	+[a]	0[b]
Successful	+	+

[a]"+" indicates the non-error cells. [b]"0" indicates the error cell.

Table 12 shows, for each prerequisite task, A-F, a summary of the results of computing the prediction analysis statistic for the cross-tabulation with the digit-correspondence score and testing it for significance. The tasks for which the prediction that the prerequisite task represents knowledge which is necessary for understanding place value is supported for all the prerequisite tasks except Task A, counting orally by tens.

Part-Whole Tasks

The six part-whole tasks included two addition/subtraction word problems identified in previous studies as requiring part-whole knowledge (c.f. Riley, Greeno & Heller, 1983), and four logical

TABLE 12

SUMMARY OF PREDICTION ANALYSIS FOR PREREQUISITE KNOWLEDGE BY DIGIT-CORRESPONDENCE SCORE

Prerequisite Task	∇_p	z
Task A: Counts Orally by Tens	0.38	0.71
Task B: Positional Knowledge	0.86	6.54**
Task C: Counts 48 Beans	0.79	4.07**
Task D: Conservation of Grouped Number	0.68	3.44**
Task E: Builds 52 with Dienes Blocks	0.79	4.07**
Task F: Builds 52 Two Ways	0.33	1.77*
Prerequisite Knowledge Composite Measure	0.60	3.76**

* $p < .05$. ** $p < .01$.

classification tasks: Markman's (1973) "family" relation inclusion task, Piaget's "flowers" class inclusion task (Inhelder & Piaget, 1969), and multiple class membership and horizontal reclassification tasks adapted from Lowery (1981).

Each part-whole task was cross-tabulated with composite digit-correspondence scores. Prediction analysis was used to test whether success on each part-whole measure was necessary for place-value understanding. Each of the six part-whole tasks was found to represent knowledge necessary (though not sufficient) for understanding place value.

The digit-correspondence scores were collapsed as in Table 13 for the prediction analysis. To examine the suggestion that more understanding of part-whole relations might be required to understanding regrouping than is required for understanding place value in canonical contexts, Tasks K, M1, and M2 were predicted to be necessary for digit-correspondence scores of 3 and above. Tasks L, N, and O were predicted to be necessary for digit-correspondence scores of 5 or 6, which would include success on at least one of the regrouping digit-correspondence tasks. Table 13 shows which cells in the contingency table are predicted to be zero.

TABLE 13

PREDICTED CELL ENTRIES FOR

PART-WHOLE TASKS BY DIGIT-CORRESPONDENCE SCORES

	Digit-Correspondence Score		
Part-whole task levels	0,1,2	3,4	5,6
1	+[a]	0[c]	0[b]
2	+	+	+

[a]"+" indicates the non-error cells. [b]"0" indicates the error cell. [c] The prediction is 0 for tasks K, M1 and M2, + for tasks L, N, and O.

The results of the prediction analysis for the part-whole tasks are summarized in Table 14.

As predicted, success on Task K (Ladybugs Class Inclusion) and on each of the arithmetic word problems (M1 and M2) was necessary for demonstrating at least a partial understanding of place value (scores of 3 or more). In addition, the results of formal analysis suggest that in order to demonstrate even a partial understanding of place value (digit-correspondence scores of 3 or more), it is necessary that children demonstrate an understanding of part-whole relations as measured by all the part-whole tasks used in this study.

The predictions that success on Task L (Flowers Class Inclusion) and Task O (Horizontal Reclassification) were necessary for digit-correspondence scores of 5 and 6 were supported. The data do

TABLE 14

SUMMARY OF PREDICTION ANALYSIS

PART-WHOLE TASKS BY DIGIT-CORRESPONDENCE SCORE

(See Appendix F.)

Part-Whole Task	∇_P	z
Contrasting Digit-Correspondence Score of 0,1,or 2 versus 3, 4, 5 or 6		
Task K: Ladybugs Class Inclusion	0.40	2.24*
Task L: Flowers Class Inclusion	0.55	3.44**
Task M-1: Combine Word Problem	1.00	**
Task M-2: Change Word Problem	1.00	**
Task N: Multiple Class Membership	0.54	2.46**
Task O: Horizontal Reclassification	0.41	2.09*
Contrasting Digit-Correspondence Score of 0, 1, 2 versus 5 or 6		
Task K: Ladybugs Class Inclusion	0.56	2.66**
Task L: Flowers Class Inclusion	0.74	4.45**
Task M-1: Combine Word Problem	1.00	**
Task M-2: Change Word Problem	1.00	**
Task N: Multiple Class Membership	0.42	1.58
Task O: Horizontal Classification	0.50	2.13*

* $p < .05$. ** $p < .01$.

not support the prediction that success on Task N (Multiple Class Membership) is also necessary.

A Proposed Model

The agenda for research on children's concept development in the place-value domain includes several related components. First, a viable model of place-value knowledge must be constructed. We have known little about which concepts children find especially difficult, what misconceptions they hold, or which concepts are prerequisite to the development of other concepts. Within a model, the relationships of various concepts that comprise the domain must be defined; those concepts that are prerequisite to the learner's understanding of other concepts must be identified. Valid and reliable means of measuring a child's level of concept development within the model must be designed.

In an effort to trace the development of place-value understanding, several studies in recent years have provided descriptions of children's thinking about multi-digit numeration (Bednarz and Janvier, 1983; M. Kamii, 1980, 1982; Resnick, 1982, 1983). In each of these studies, children were asked to perform certain tasks using concrete materials and to explain their reasons for their actions. Results led the researchers to propose various theories of how understanding of place value develops, but we do not yet have a unified, comprehensive model.

The descriptive model proposed below synthesizes the results of this study and previous research. Five stages of development are proposed. In the first stage children interpret a two-digit numeral as the whole number it represents. In Stage II, children demonstrate knowledge of the positional property of the individual digits. In Stage III the digits are interpreted by their face values. In Stage IV the tens digit is interpreted as representing groups of ten, though this understanding is limited and performance is unreliable. In Stage V understanding is easily demonstrated and performances are reliable. Evidence from the present study as well as prior research is cited for each stage.

The Model: Description and Evidence

Stage I: The child is able to read and write two-digit numerals and associate the whole numeral with the number it represents. The child assigns no meaning, however, to the individual digits which comprise the two-digit numeral.

The child can read and write "48, for example," and knows that "48" represents 48 objects. Neither the "4" nor the "8", however, have any meaning except as parts of the whole numeral. Children know that when reading the numeral "48," the "4" part of the numeral is read "forty." Most children, by the beginning of second grade, have reached this stage.

All the children in this study were at least at Stage I. They were able to count collections of as many as 52 objects and write the two-digit numeral that corresponded to the count. Only two children made slips in writing two-digit numerals; each reversed the digits in a single trial. One child made a reversal error in reading a two-digit numeral. Most children, by the fall of second grade, though they do not understand the meanings of the individual digits, understand the number indicated by the whole (two-digit) numeral.

Nine children were unable to demonstrate any understaning beyond stage I.

Results of previous research support the conclusion that understanding of the whole numeral precedes understanding of the individual digits (Baroody, et al., 1983; Barr, 1978; M. Kamii, 1982; National Assessment of Educational Progress, 1983; Rathmell, 1969; Resnick, 1982, 1983; R. Smith, 1972, 1973).

Stage II: The child knows that in a two-digit numeral the digit on the right is in the "ones place" and the digit on the left is in the "tens place." The child's knowledge of place value is limited, however, to the position of the digits and does not encompass the quantities indicated by each.

Many children in second and third grade are still working to sort out the left-right distinction of "tens and ones." They often make reversal errors. When a child in Stage II says the "4" in "48" means "four tens," s/he is demonstrating only verbal knowledge based on the left and right positional labels; the child does not recognize that the "4" represents 40 objects.

As summarized in Table 4, twelve children in this study made reversal errors in identifying the positions of the tens and ones digits (Task B). Although 43 were successful on Task B, only 20 of the 43 were able to demonstrate on the digit-correspondence tasks an understanding of the numbers represented by the individual digits. Three children demonstrated Stage II knowledge but no understanding beyond Stage II.

With Stage II knowledge, children are able to succeed on a variety of tasks typically found in their textbooks and standardized tests, such as the following:

In 27, which digit is in the ones place?

How many tens are in 84?

35 = ____ tens and ____ ones.

7 tens + 5 ones = ____.

Previous assessments have shown children can succeed on such items at a relatively early age (Baroody, et al., 1983; Flournoy, 1967; Heibert & Wearne, 1983; National Assessment of Educational Progress, 1983; Scrivens, 1968; R. Smith, 1972, 1973). Among Kamii's (1982) 16 seven-year-old subjects, six "mentioned tens and ones" but only two of these six were able to demonstrate understanding of the quantities indicated by the individual digits. Based on children's responses to interview tasks, Bednarz and Janvier (1982) concluded that order concepts preceded grouping concepts in children's development of understanding place value.

Stage III: Children interpret each digit as representing the number indicated by its face value; the set of objects represented by the tens digit, however, are different objects than the objects represented by the ones digit. They may verbally label as "tens" the objects which correspond to the tens digit, but these objects do not truly represent groups of ten units to children in stage III; the child does not recognize that the number represented by the tens digit is a multiple of ten.

The whole does not equal the sum of the parts in the Stage III interpretation. For example, for the numeral "48," 4 (long blocks) + 8 (unit-blocks) does not equal 48 since the long blocks are not interpreted as representing 40. A child need not understand part-whole relations to hold a Stage III mental model.

When presented with a digit-correspondence task using base-ten embodiments, the child in Stage III seeks, for a given two-digit numeral such as "48", four units of one kind and eight units of another kind. When confronted with a new embodiment, the child adapts by identifying which of the new objects corresponds to the tens digit and which corresponds to the ones. For both digits, however, the child is actually counting the objects as units. Even when containers or drawings of sets of ten objects are used, the child ignores the ten objects when counting the containers or "circles" around the collections of ten.

Sixteen children in this study operated with a Stage III interpretation of digits. These children were initially identified during the data collection phase of the present study. In the Beans task (I-1) and the Dienes Blocks task (J-1) children were able to succeed on these tasks by counting the cups and long purple blocks as units, even though the materials were intended to represent sets of ten.

The results of the McNemar tests of homogeneity for correlated proportions supported the observation; significantly more children were successful on these two tasks than on the other four digit-correspondence tasks. The Guttman scaling data showed that of the 18 children who succeeded on only one or two of the six digit-correspondence tasks, 16 of them did so on the canonical Beans and Dienes blocks tasks. To succeed on these tasks, prediction analysis results showed that success on neither the part-whole tasks nor prerequisite tasks was necessary for success.

While previous research has also identified children who interpret digits by their face value (Kamii, 1982), the additional requirement that each digit be associated with <u>different</u> objects has not been identified in earlier models. Ashlock (1978) maintains that children can count single objects and can group objects but they cannot count groups of objects, a limitation which could explain Stage III performance, but other evidence shows that children can count groups (Bednarz & Janvier, 1982; M. Kamii, 1982; R. Smith, 1973).

The Stage III interpretation of digits may be fairly persistent, since it is sufficient for success on many tasks. When engaging in activities designed to help them understand regrouping in the addition and subtraction algorithms, children may encounter tasks which cannot be accommodated by a Stage III interpretation. Without the conflict that such tasks provide, children may not experience sufficient disequilibrium to motivate them to seek a more accommodating interpretation.

Teachers who limit instructional activities to those using base-ten embodiments for canonical partitionings may unwittingly reinforce a Stage III interpretation. With these materials the teacher and manufacturer may have "embodied the ten," but the child need not.

Stage IV: The child knows that the left digit in a two-digit numeral represents sets of ten objects and that the right digit represents the remaining single objects but this knowledge is tentative and characterized by unreliable task performances.

Children in Stage IV, like those in Stage V, consider both the face value and the place value of the tens digit; they require that the "objects" that correspond to the tens digit represent sets of ten units. The whole and the sum of the parts are equal; an understanding of part-whole relationships is necessary for the development of a Stage IV model. However, Stage IV is a transitional stage where task performances are uneven.

Sixteen children in this study were able to make digit-correspondences on at least one, but not all, of the tasks that required an understanding that the tens digit represented groups of ten. Children in this study did not find the digit-correspondence tasks in which collections were presented in non-canonical partitionings significantly more difficult than the Sticks task, which presented the collection without any partitioning. In fact, six children who were unsuccessful on the Sticks task were successful on one or more of the non-canonical digit-correspondence tasks. Based on the evidence provided by the present study, it was concluded that the unreliable performances which characterize Stage IV can best be explained by viewing it as a transitional stage, when children's understanding of the base-ten property is weak and unstable.

While a Stage III interpretation of digits does not require an understanding of part-whole relations, in Stage IV the numbers represented by the individual digits sum to the number represented by the whole numeral. In the present study, prediction analysis showed that understanding part-whole relations as measured by arithmetic word problems and logical classification tasks was necessary for success on more than two digit-correspondence tasks; to succeed on more than two, children had to be using a stage IV or stage V interpretation of the individual digits.

Evidence from earlier studies suggested that the unreliable performances typical of Stage IV might be because the understanding of these children is limited to canonical partitionings. Resnick's (1983) model of place-value understanding distinguished between children who recognize only the unique canonical partitioning of objects into a tens part and a ones part and those children who recognize the possibility of multiple partitionings of collections while maintaining a tens part and a ones part. The results of the prerequisite task performances in the present study supported her finding. Fifteen children who were easily able to build canonical representations of a number with Dienes blocks resorted to empirical methods to build the non-canonical representation, without using what Resnick calls a "1C for-1 trade schema." One child who easily built the canonical representation was unsuccessful at the non-canonical construction.

Other ear'ier place-value studies have shown that regrouping and renaming numerals are especially difficult (Flournoy, 1967; Heibert & Wearne, 1983; Rickman, 1983; Scrivens, 1968; C. Smith, 1969; R. Smith, 1973). The present study, however, did not produce strong evidence for extending from the concrete to the symbolic level Resnick's distinction between understanding canonical and understanding non-canonical partitioning. Children who were characterized in previous research as understanding place value for simple canonical partitionings but not for regrouping may have been operating with what has been described here as a stage III interpretation of digits.

Stage V: The child knows that the individual digits in a two-digit numeral represent a partitioning of the whole quantity into a tens part and a ones part. The quantity of objects corresponding to each digit can be determined even for collections which have been non-canonically partitioned.

To understand two-digit numeration at this highest level, children need to coordinate numeration knowledge, which can be socially transmitted, with number concepts, which must be individually constructed. In addition, an understanding of part-whole relations is necessary.

Sixteen children in this study demonstrated understanding of the numbers represented by the individual digits in two-place numerals by succeeding on all six of the digit-correspondence tasks. Two of these tasks required children to make correspondences between digits and non-canonical base-ten representations. Many of these children were able to express themselves verbally as they made the appropriate correspondences--for example, "tens ones is the same as a ten," "these three tens (gesturing to three sets of ten beans) and these (gesturing to ten loose beans) are the same as four tens," "the '4' means (gesturing to two sets of ten beans and twenty loose beans) forty beans."

Success on the prerequisite numeration and number tasks used in this study were shown by prediction analysis to be necessary for success on more than four digit-correspondence tasks. Success on the part-whole tasks was shown to be necessary for success on more than two digit-correspondence tasks.

Kamii did not use digit-correspondence tasks to assess children's interpretation of digits for collections of base-ten embodiments that had been non-canonically partitioned. Her Wheels task, however, did require children to regroup from sets of four to a base-ten partitioning. She reported no differences between children who could make correspondences between digits and quantities of objects in the Sticks task and those who could do so in the Wheels task.

To understand addition and subtraction algorithms for numbers up to 99, children probably need to be capable of a Stage V understanding of place value.

Applying the Model to the Data in the Present Study. Each of the stages of the model has been operationally defined post hoc in terms of performances on the tasks used in this study. The operational definitions are described in the following section. Table 15 shows the number of children in this study, by grade level, who performed at each stage of this proposed model. The chi-square test of independence showed that the relationship of grade in school to the stages in the model was statistically significant.

Stage I. Children who did not perform at the highest level on the positional knowledge task (Task B), and who succeeded on none of the digit-correspondence tasks were defined as holding Stage I knowledge of two-digit numeration. There were nine such children.

Stage II. Three children performed at the highest level on Task B, Positional Knowledge, but were successful on none of the digit-correspondence tasks. These children were characterized as holding a Stage II understanding.

Twelve children in Stage III were able to perform successfully on the task requiring them to construct a given number canonically with Dienes blocks (Task E); only five were able to easily build the number non-canonically (Task F).

Stage IV. In this study, the 16 children who performed successfully on at least one but not all of the digit-correspondence tasks other than the canonical Beans and Dienes Blocks tasks were characterized as Stage IV. The digit-correspondence scores of Stage IV children ranged fom 1 to 5; their mean digit-correspondence score was 3.6.

Stage V. Sixteen children in this study performed

TABLE 15

STAGES IN THE DEVELOPMENT OF CHILDREN'S UNDERSTANDING OF TWO-DIGIT NUMERATION BY GRADE IN SCHOOL

	Stage of Development				
Grade	I	II	III	IV	V
2	8	0	3	4	0
3	0	2	5	6	2
4	1	0	6	1	7
5	0	1	2	5	7
Total	9	3	16	16	16

$\underline{n}$ = 15 for each grade level

chi-square = 41.18 df = 12, $\underline{p} < .01$.

competently on all six digit-correspondence tasks. They were defined as having Stage V understanding of two-digit numeration. As reported in Table 5, two were in grade 3, seven in grade 4 and seven in grade 5. Thirteen of these sixteen children were boys.

Conclusion

Children learn about numbers in the earliest years of elementary school. They learn that a set of three objects is different in number from a set of four objects and that sets of five objects all share the concept of "fiveness." They also learn about numeration--the word-names and symbols, or numerals, used to write numbers. In about second grade, present mathematics curricula introduce the place-value properties of our numeration system. Children are provided instruction to help them understand that in a two-digit numeral, for example, the digit on the left represents sets of ten, the digit on the right represents the remaining units, and together the two digits represent the whole quantity.

It has been repeatedly documented, however, that children throughout the elementary-school years generally understand place value poorly, if at all, and the value and nature of the early instruction is often questioned. While most educators agree that understanding the principles underlying our numeration system is a reasonable instructional goal, they often disagree about when such understanding should be expected and appropriate instruction provided.

Those who advocate early place-value instruction generally do so in order that children may understand the conventional algorithms for operating with large numbers, which are taught beginning in the second grade. Many of these believe that children's poor understanding is due to inadequate instruction.

Others feel that the early introduction to our positional, base-ten numeration system is either unnecessary or unreasonable (or both). Those who believe it is unnecessary point out that algorithmic procedures can be learned by rote, without reliance on place-value principles, and that most children always have and still do learn to compute by rote methods.

Those who believe it is unreasonable feel that children's poor performance on place-value assessments is due to the fact that the concepts are too demanding for the cognitive capabilities of the young child. They prefer to delay place-value instruction until children are older and more capable. Within this group some advocate teaching algorithms by rote, others advocate delaying algorithmic instruction until place-value can be understood.

Unfortunately, place-value research has been sparse and none of the viewpoints can be supported by robust empirical findings. This study clearly supports the finding that children's understanding of place value is poor. While older children tended to perform more competently on the place-value tasks, children's understanding throughout grades 2 through 5 was limited.

The study provided new, previously unavailable descriptions of children's ideas about place-value. Levels of performance were identified and described for each of the tasks; the levels for all tasks but one were significantly correlated to age.

The digit-correspondence tasks developed by Kamii (1982) were found to be useful measures of children's understanding of two-digit numeration. When, however, the tasks were modified by representing numbers canonically with base-ten embodiments, children were able to succeed by interpreting the tens digit only by its face value. The digit-correspondence tasks were not found useful in identifying children who understood place value except for regrouping.

To understand place value, it was found that children needed knowledge of numeration, number concepts and part-whole relations. Four of the part-whole tasks measured logical classification concepts which are usually not taught in school; these tasks are often considered measures of general cognitive development. Previous research has established that children typically do not succeed on these tasks until as late as age eight or ten. The study provides some support for those who argue that the level of cognitive development of most children in the primary grades makes them unable to understand place value.

Most of the children in this study, while they knew that the whole numeral "25" could represent a collection of 25 sticks, did not know that the "2" represents 20 of them and the "5" represents the remaining five sticks. It may be relatively easy to teach the meaning of the digits to older children who lack this knowledge. But children in the primary grades who lack prerequisite knowledge of numeration, number concepts, or part-whole relations will probably require many additional experiences before they can be said to understand place value.

Statistical Notes

Note 1: Significance of differences was tested using McNemar's test for homogeneity of correlated proportions. Each of the six tasks was contrasted with each of the others, resulting in fifteen pairwise contrasts. The McNemar z-values were considered significant if, using a one-tailed α = .05, they exceeded the Dunn critical value (2.69) for 15 planned comparisons. The only significant differences were between the canonical Beans and Dienes Blocks tasks and the remaining four.

Note 2: The prediction analysis statistic $\nabla_{\mathcal{P}}$ was developed by Hildebrand, Laing, and Rosenthal (1977) specifically for use in analyses where a priori predictions have been made that certain of the cells in a cross-classification are equal to (or near to) zero.

$$\nabla_{\mathcal{P}} = 1 - \frac{\sum_i \sum_j w_{ij} p_{ij}}{\sum_i \sum_j w_{ij} p_{i.} p_{.j}}$$

The numerator in the ratio is the proportion of the population observed in the set of "error cells," those cells predicted to equal zero. The denominator represents the proportion expected, given the marginals and an assumption of statistical independence. Assuming a non-zero denominator for the ratio, the statistic may vary from $-\infty$ to 1.00, with the upper bound obtained if no observations fall into any of the error cells. The statistic equals zero if the two variables are statistically independent, and may be negative if the total probability in the error cells is greater than would be expected under statistical independence.

The sample statistic, ∇, can be tested for its statistical significance against the null hypothesis that the two variables are statistically independent (H : ∇ = 0). The test results in a z-value. When using a one-tailed test, α = .05, the critical value is 1.64. A discussion of the uses of this statistic for developmental research can be found in Froman and Hubert (1980).

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