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ABSTRACT

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Abstract

This essay clarifies what it means to know mathematics by examining ways of knowing multiplication and explores what those ways of knowing imply for the teaching and learning of mathematics in schools. The author reviews the perennial argument about whether computational skill or conceptual understanding should guide the school curriculum. A mathematical analysis of the process of multiplication, a conceptual analysis of mathematical cognition, and the authors' speculative research on classroom teaching and learning are presented to support this argument. Included are descriptions of several lessons in which children are being taught about multiplying large numbers. The descriptions focus on the connections that can be made in teaching among students' naive, concrete, computational, and conceptual knowledge.

KNOWING, DOING, AND TEACHING MULTIPLICATION[1]

Magdalene Lampert[2]

Ever since schools have existed in this country, Americans have debated about what children should be learning in them. A significant part of that debate has addressed the subject of mathematics and pits the proponents of teaching computational skill against the advocates of fostering conceptual understanding. Because computation is an aspect of mathematical knowledge that is familiar to most teachers and parents, they are likely to support its place in the school curriculum. Most mathematicians, in contrast, see computation as an almost insignificant branch of their subject and, thus, are likely to believe that it is less important to be skilled in computation than to understand how abstract mathematical principles can be used to analyze and solve problems. Two different views of what it means to know mathematics underlie this disagreement. Developments in curriculum and instruction have exacerbated the conflicts between these views and teachers are often left to figure out acceptable policies and practices for themselves.

This paper will clarify what it means to know mathematics by examining ways of knowing a particular piece of the mathematics curriculum--multidigit multiplication--and explore what those ways of knowing imply for how mathematics might be taught and learned in schools. As a foundation for this

[1]The production of this paper was partially funded by the Children's Television Workshop with support from the National Science Foundation. The opinions expressed in this paper do not necessarily reflect the position, policy, or endorsement of either the Children's Television Workshop or the National Science Foundation. The author would like to acknowledge the contribution of Jacqueline Frese to this essay. The lesson descriptions are based on her careful observations and note taking.

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analysis, the mathematical meaning of the process and structure of multiplication will be considered along with theories of mathematical knowledge being developed in the field of cognitive science. These theoretical perspectives will be complemented by descriptions of fourth-grade mathematics lessons I taught. These descriptions are intended to inform a consideration of instructional strategies that do not favor either computation or concepts but take an active approach to teaching students about the connection between these two ways of knowing mathematics. The classroom research reported here is not oriented toward conclusions about how to teach the multiplication of large numbers, however. It is intended to be conjectural, that is, to outline an approach to instruction based on a broad definition of mathematical knowledge that seems to be worth trying (cf. Noddings, 1985).

Multidigit multiplication was chosen as the focus of this paper because it has not been examined much by cognitive scientists or math educators and because it seems to be a watershed topic for learners. Whereas from a mathematical point of view the principles underlying the process of multidigit multiplication can be associated with fundamental concepts in our number system, from the school learner's point of view, it often is the place where arithmetic stops making sense. The rules for getting through the procedure make less and less intuitive sense as the numbers get larger, and little in the conventional curriculum is oriented toward helping students to understand what they are doing.

What Does It Mean to Know Multiplication?

One can interpret the question asked by an arithmetical expression like 9 x 5 as a counting question. In this sense, multiplication is an operation used for counting a total quantity when the quantity can be organized into a number of groups, each of which has the same number of members. So 9 x 5 means th number of the total of nine groups, each containing five members. The multiplicative composition of groups derives from the operation of addition; if addition is applied to two sets, one numbering *a* and the other numbering *b*, and there is one-to-one correspondence between the members of these sets (i.e., if a = b, then the total of a + b can be given by the number 2a (or 2b). So if there are nine groups, each having one-to-one correspondence with a group having five members, then the total would be given by 5 + 5 + 5 + 5 + 5 + 5 + 5 + 5 + 5 or 9 x 5.

One need not know the meaning of multiplication to know that 9 x 5 = 45, however. It has been considered appropriate for children to memorize such combinations for pairs of numbers up to 10 x 10 or 12 x 12. It is not knowledge about the multiplication of these relatively small whole numbers that will concern us here. Multiplying numbers larger than 10 or 12 (combinations that are unlikely to be memorized) is the operation under consideration. Even though the term is imprecise, because 10, 11 and 12 all have more than one digit, I shall refer to this operation as "multidigit" multiplication.

Given a facility with the basic combinations, what are the ways in which one might *know* that 8 x 76 = 608? Or be able to figure out any other multidigit multiplication without memorizing? In order to do that, one needs to have some knowledge about how to take the larger numbers apart, operate on them, and put them back together in legitimate ways. A classification of kinds of knowledge proposed by Leinhardt (1985) provides a useful outline for thinking about the different kinds of knowledge that could be used to do that. In her analysis of the process of getting to know subtraction, she uses four different categories to define kinds of mathematical knowledge: naive, concrete, computational, and principled conceptual. Each of these terms could be used to describe a way of knowing how to figure out the meaning of an expression like 8 x 76.

Naive Knowledge

Naive knowledge consists of applied, real-life knowledge that is not the result of direct instruction. Other cognitive scientists have called it "intuitive" and distinguish it from the formal knowledge taught in school (Bamberger, 1979; Posner, Strike, Hewson, & Gertzog, 1982). This kind of knowledge is derived from and bound to a context in which the knower is confronted with problems to solve that matter to him or her. It is sometimes thought of as the source of misconceptions, which are incorrrect theories constructed by a learner based on limited experiences (Anderson and Smith, 1983; Roth, 1985), the measure of correctness being derived from the abstract context of an academic discipline. The classification of knowledge as naive in this paper is not meant to imply that it is either correct or incorrect, however. In fact, the definition itself suggests that its correctness must be judged according to the context in which it is applied. The term "naive" will be used here only in its formal sense to mean that is is not the result of formal instruction; it is not meant to imply that the knower is a naive person.

In the case of multidigit multiplication, this kind of knowledge is nicely illustrated by Scribner's (1984) study of the "working intelligence" that milkmen use to pack and price various products to be delivered from a dairy. The workers Scribner observed faced several situations in the course of their workday in which they were required to do multidigit multiplications to get the job done. They invented context-specific decomposition and recomposition procedures to make their work easier. Skilled loaders of delivery cases learned the value of various configurations of containers--one layer of half-pints is 16, two rows of quarts is 8, etc.--and used this compositional structure to figure out how to fill cases with multiples of items without counting out each item. If they needed 35 half-pints to fill an order, for example, they would know to fill two layers and then add 3 more half-pints on top. Delivery men solved billing problems using a similar process of taking numbers apart and putting them back together to serve their purposes. To figure out the cost of 98 half-pints, for example, one strategy was to take two times the case price (a case held 48 half pints) and add two times the unit price. These invented algorithms for multidigit multiplication actually saved considerable time and energy on the job when compared with the procedures used by workmen who followed the more conventional rules for arithmetical computation they had learned in school.

Concrete Knowledge

Concrete mathematical knowledge requires the actual manipulation of objects to find an answer. This kind of knowledge might be used, for example, if the multiplication 8 x 76 were given by 8 equal piles of money, in which each pile contained 7 ten-dollar bills and 6 one-dollar bills. Finding 8 x 76 then would mean putting the 8 piles together to get a total of 56 ten-dollar bills and 48 one-dollar bills. The multiplication is complete at this point, but to translate the "answer" into a more conventional form, 40 of the one-dollar bills could be traded for 4 tens, making a total of 60 tens; if hundred-dollar bills were available, these could be traded for 6 hundreds, making a total of 608 dollars.

In contrast to naive procedure, concrete procedures are based on the actual manipulation of objects. No manipulation of symbols and no mental processes other than trading and counting according to given rules are required to find the answer. (Practice with this sort of grouping, trading, and counting has been advocated by Z. P. Dienes, 1960, as the most appropriate introduction for children to arithmetic procedures.) Concrete multiplication requires knowing that "multiplied by" or the symbol x indicates the physical operation of pushing a collection of equal piles of something together and that you get the "answer" by counting the total number. Getting a familiar looking answer when the number of objects involved is larger that 10 requires knowing that you can trade and knowing the comparative values of the objects involved. Being able to group and trade objects does not necessarily indicat either an ability to record those activities symbolically or an appreciation of the mathematical principles they represent.

Computational Knowledge

Another kind of mathematical knowledge about multiplication is computational. It entails doing things with numerical symbols according to a set of procedural rules. In order to compute 8 x 76 in the conventionally accepted form, for example, one would need to know it should be written as follows:

$$\begin{array}{r} 76 \\ \underline{\times\ 8} \end{array}$$

Then, beginning with the numerals on the right, one would multiply 8 x 6 to get 48 (by virtue of having memorized that combination), put the 8 down under the right hand column and "carry" the 4 up on top of the 7. Now, multiplying 8 x 7 to get 56, one adds the 7 and writes down 60 to the left of the 8, so that the answer looks like this:

$$\begin{array}{r} 76 \\ \underline{\times\ 8} \\ 608 \end{array}$$

In order to do this correctly, one need *not* know that the 7 means 70 because of its placement to the left of the 6, nor that one is actually adding 40 to 560 (the product of 8 and 70) to get the 600 in 608. What is important here is knowing which operation (multiply or add) to do to which pair of digits in what order, and where to place the answer, and to have memorized a store of associations between pairs of digits and their products (the multiplication tables). One can check on one's accuracy visually, by knowing, for example, that multiplications of two-digit numerals by one-digit numerals always produce answers with at most, three digits. So if one's work looked like this,

$$\begin{array}{r} 76 \\ \times\ 8 \\ \hline 5648 \end{array}$$

one might go back and try to figure out what went wrong. (These sorts of visual "procedural critics" have been studied extensively by Brown & Burton, 1978, in their attempts to describe the thinking that might explain mistakes children make in subtraction.) Note that it is *not* the judgment that 5,648 is too large a quantity to make a sensible answer to 8 x 76 that results from computational knowledge. It is an assessment of the way the answer *looks* that is employed here, rather than an understanding of what it *means*. Procedural critics, although superficial, can provide the basis for competent performance of arithmetic computation.

Principled Conceptual Knowledge

Using the fourth kind of knowledge in Leinhardt's scheme--principled conceptual--to do multiplication requires understanding some fundamental mathematical principles. It underlies the invention of computational procedures such as the familiar one described above. But someone who has this kind of understanding might also invent different ways to figure out 76 x 8; for example: "76 is one more than 75 which is 3 times 25; to get 8 x 75, then, 8 times 25 is 200, and 3 times 200 is 600; now we need 8 x 1 which is 8. So 76 x 8 is 608." In order to invent such a procedure, one needs to know that 8 x 75 is the same as 8 x 25 x 3 and that it is the same as 200 x 3. One needs to know that you can take 76 apart into 75 and 1, multiply each part by 8 and then put it back together. With this sort of conceptual knowledge, one would also know that there are many other ways to figure out 76 x 8 because there are many ways to "take apart" 76 and 8 (Greeno, Riley, & Gelman, 1984).

To approach the problem another way, for example, one might think: "76 is 4 less than 80; 8 times 80 is 640; 8 times 4 is 32; 76 times 8 is the difference between 640 and 32 or 608." Why can one do that? Why do both of these methods "work" in the sense that they get the same answer that was arrived at using other kinds of knowledge about multiplication? What does one need to know to be sure that these procedures are "allowed" mathematically? This invention of procedures is what Noddings (1985) calls "the search for algorithms," which calls upon an understanding of the structure of the operation coupled with computational ability and the sort of mathematical insight that comes from naive or concrete knowledge. It is, in a sense, knowledge about what one knows about numbers and how they work that is used in this domain to carry out arithmetical operations. It is not necessary to know principles to *use* any of these procedures correctly, if one is taught the steps to follow, but one does need to know the principles to *invent* the steps or to explain why they work.

The principles underlying a conceptual knowledge of multidigit multiplication include knowing that

1. the way digits are lined up in a number has meaning, that is, the 7 in 76 means 70 because it is to the left of the 6 (place value);

2. numbers are composed by addition and can be decomposed in many different ways without changing the total quantity, that is, 76 can be thought of as 70 + 6 or 75 + 1 or 25 + 25 + 25 + 1 or 38 + 38 etc. (additive composition);

3. the elements of these additive compositions can be grouped and added in different ways without affecting the total quantity, that is, 25 + 25 + 25 + 1 can be recomposed as 50 + 26 or 75 + 1 or 25 + 51, etc. (associativity of addition);

4. the order in which additions are done does not affect the final sum, that is, 70 + 6 gives the same total as 6 + 70 (commutativity of addition);

5. numbers are also composed by multiplication in the sense that they can be decomposed into equal groups, that is, 76 can be thought of as 38 + 38 which is 2 x 38, or it can be thought of as 19 + 19 + 19 + 19 which is 4 x 19 or even as 25 + 25 + 25 + 1 which is (3 x 25) + 1 (multiplicative composition);

6. the elements of those multiplicative compositions can be grouped and multiplied in different ways without affecting the total quantity, that is,
76 = (2 x 2) x 19 = 4 x 19 or 76 = 2 x (2 x 19) = 2 x 38 (associativity of multiplication);

7. the order in which multiplications are done does not affect the final product, that is, 4 x 19 gives the same product as 19 x 4 (commutativity of multiplication); and

8. numbers to be multiplied can be decomposed additively, each of the elements operated on separately, and the product obtained from recomposing the "partial products", that is,

76 = 70 + 6
70 x 8 = 560
6 x 8 = 48
so
76 x 8 = 560 + 48 = 608
or
76 = 25 + 25 + 25 + 1
25 x 8 = 200
1 x 8 = 8
so
76 x 8 = 200 + 200 + 200 + 8 = 608
(distributive property of multiplication over addition)

These principles are basic building blocks, not only of processes used to multiply large numbers but of much of pure mathematics. They represent the implicit conceptual knowledge of anyone who understands multiplication procedures in our notational system, but they also underlie the construction of algebraic tools such as factoring and the derivation of sophisticated theoretical systems such as symbolic logic. To the general reader, they may seem so obvious as to be trivial, but they are important considerations here because the errors children make in computation suggest that they are often *not* operating on the basis of these principles.

For the teacher, these ways of knowing multiplication raise important questions. Can children learn to manipulate concrete objects or abstract symbols in ways that enable them to arrive at correct answers without having principled understanding? Does the naive invention of procedures that make computational processes more efficient mean that the inventor does possess understanding? What if a learner can do multiplication in some situations and not in others? In what order should a child be presented with ways of thinking about multiplication?

Doing Multiplication

The arithmetic of doing the multiplication below:

$$\begin{array}{r} 76 \\ \underline{\times\ 8} \\ 608 \end{array}$$

can be described in terms of principles for operating on quantities symbolized by numbers as well as in terms of mechanical procedures for operating on those symbols. If we think of 8 x 76 as asking us to figure out how many things are in a total of 8 groups, each of which contains 76 objects, we can see the mechanical procedures as representations of concrete operations on those groups. The principles tell us how we can take the groups apart, find out how many are in each subgroup, and then put the groups back together to find the total. Working along conventional "place-value" lines, we can take each of the 8 groups of 76 apart so we have 8 groups of 70 and 8 groups of 6. We can do this because of the additive composition of 76 from 70 and 6 and because of the distributive law, which tells us that 8 x 76 = (8 x 70) + (8 x 6). The language of groups enables us to see that these principles "make sense" in terms of actions done to concrete objects, linking principled conceptual knowledge to concrete knowledge. We need not call the principles "decomposition" or "distributivity" to understand that they work.

In order to figure out how many are in 8 groups of 70, we need to do some more taking apart in order to enable us to use our knowledge of the multiplication tables or combinations. The principle of multiplicative composition tells us that each group of 70 can be taken apart into 7 groups of 10, and since we had 8 groups of 70, there will be 56 groups of 10 all together. The eight groups of 6 will have a total of 48 objects in them. At this point, a diagram might help us keep track of what we have done:

	8 groups of 6 =	48	
8 x 76		70 = 7 groups of 10	
		70 = 7 groups of 10	
		70 = 7 groups of 10	
	8 groups of 70 =	70 = 7 groups of 10	= 56 groups of 10 = 560
		70 = 7 groups of 10	
		70 = 7 groups of 10	
		70 = 7 groups of 10	

Now we add the 48 to the 560 to get a total of 608 objects.

$$\text{total:}\quad \begin{array}{r} 560 \\ \underline{+48} \\ 608 \end{array}$$

By talking about multiplication in terms of groups and groups of groups, we have illustrated abstract mathematical principles in a familiar concrete language. At this concrete level, one's appreciation of the truth of mathematical principles might be said to be intuitive. There remain very large questions about how one's knowledge of these principles is obtained and, in particular, about the nature of the relationship of intuitive to principled knowledge. Questions about the relationship between worldly common sense and the uncommon sense of the hypothetical realm of mathematics are at the very heart of mathematical epistemology and have been debated by philosophers throughout the 19th and 20th centuries. (Davis & Hersch, 1981; Guillen, 1983; Hawkins, 1985), if not since the time of Plato and Aristotle.

I shall sidestep these ponderous issues for the moment by taking the perspective of a mathematics teacher. The mistakes I have observed children making when they multiply large numbers suggest that they have little sense that what has been described above is what they are doing when they do "8 times 6 is 48, put down the 8 carry the 4; 8 times 7 is 56, add the 4 put down the 60." When we add the 48 and the 560 in the conventional procedure, we put the 4 of 48 up on top of the 7 of 76 to indicate that it means 4 tens or 40. After we multiply 8 x 7, which is really 8 x 70, we add the 40 to it and write down 60 which means 600 because it is to the left of the 8. These procedures are more transparent in the "grouping" diagram used above than they are in the symbolism of the conventional algorithm. The groups of 10 that are meant by numbers of more than one digit are unpacked in the diagram in a way that makes the procedures used to operate on the numbers more obvious than the symbolic "carrying" that is done in computation.

Many of the common mistakes in multiplying multidigit numbers have to do with the digits that are "carried" (Buswell, 1926). For example, we might find the following in different children's work:

```
A.   86        B.   86        C.   86        D.   86
    x 3            x 3            x 3            x 3
   2438            278           2418            222
```

In each of these cases, the child knew that 3 x 6 = 18 and 3 x 8 = 24 and that these two multiplications needed to be done to find the answer. These are not errors of fact or the simple careless placement of numbers. They are errors that seem to indicate that the students reached an impasse in their knowledge of what to do when they got to the carrying part of the procedure and they invented a way to cope with it (cf. Brown & Van Lehn, 1980).

In Example A the child multiplies 3 x 6, writes down the 8 and carries the 1, then multiplies 3 x 1 and writes down the answer, then multiplies 3 x 8 and writes down the answer. His or her sense of the procedure might be that all the numbers not on the same line as the 3 need to be multiplied by the 3. This rule works as long as there are no carries; typically multidigit multiplication is first practiced with these kinds of numbers and this practice gives the child a context for constructing naive computational rules.

In Example B, the child multiplies 3 x 6, writes down the 8 and carries the 1, then adds the one to the 8 to get 9 and multiplies 3 x 9 to get 27. The rule that was applied here might be "add the carries and then multiply," a rule which could not be based on understanding what the "carries" symbolize, but might be derived from a learned pattern for doing *addition* of columns for numbers in which you add the "carry" to the top number before adding it to the rest of the numbers in the column.

In Example C, the child multiplies 3 x 6, writes down 18, then multiplies 3 x 8 and writes down 24. Again, this process would work if there were no carries, but here indicates that the child does not take account of the fact that the 24 resulting from the multiplication of 3 x 8 symbolizes 240 because it is the product of 3 and *80*. To symbolize the number 240 in relation to the other numbers in the problem, the 4 should be written in the same column as the 8 (the "tens" column) and the 2 should be written in the column to the left of the 8 and the 4 (the "hundreds" column). Because there is already a 1 in the "tens" column from 18, there seems to be "no room" for the 4, and so it is moved over to the left, with the result that 240 becomes 2400.

In Example D, the child has also carried and added incorrectly, indicating a lack of attention to the meaning of place value. The errors in this example may derive from the fact that he or she starts on the left instead of on the right, first multiplying 3 x 8, then multiplying 3 x 6. Starting with the 8 in 86 might be explained by the fact that when we *say* 86, we say the 8 first and then the 6. This child might have even learned that 3 x 86 means the sum of 3 x 80 and 3 x 6. But lacking an understanding of the importance of place value in representing quantities numerically, this child multiplies 3 x 8 to get 24, puts down the 2 under the 8 and carries the 4, then multiplies 3 x 6, adds the carried 4 and obtains 22, which is put down to the right of the 2. Here the "2 hundred" of the product of 3 x 80 is maintained when the answer is read even though it is under the 8 tens because the answer line is moved over to the right. But the 4 that is carried, which was supposed to be 4 tens becomes 4 ones as it is added to the product of 3 x 6.

Most of these errors result from bits and pieces of procedural knowledge incorrectly applied (Brown & Burton, 1978). Procedures are brought from other arithmetic domains or done in the wrong order, perhaps indicating naive knowledge which is not coupled with a knowledge of contextual constraints. Each of the erroneous procedures can be explained in a way that suggests they might make sense to the child who uses them, especially if "making sense" is based on the assumption that arithmetic is a procedural system rather than a conceptual one. They are not "misconceptions" about place value; rather they suggest that place value (i.e., understanding of the value of the quantities represented by the numbers being operated upon) is simply not a relevant consideration to these children in the process of doing computation.[3]

[3]Resnick and Omanson (in press) have speculated similarly with regard to the errors children make in subtraction with borrowing.

As more digits are added to the multiplier or the multiplicand, these errors based on place-value are compounded since the child is now being asked to multiply by numbers composed of groups of tens, hundreds, or even thousands, as well as groups of units.[4] The errors described here suggest that whether or not children understand the concept of place value, they have not learned that it is a relevant consideration in doing multidigit multiplication.

The most common errors that children make in doing multidigit multiplication can be seen as a general indication of how they view the doing of arithmetic. The faults in the procedures they use seem to express a separation between their sense of what numbers mean *as numbers representing quantities* (46 is 40 + 6) and their sense of how numbers function in computational processes. It also would not be unreasonable to conclude that the children who made these computational errors did not "understand" multiplication, and that is the sort of conclusion typically drawn in an instructional situation.

Teaching Multiplication

A great divide in opinion exists about what children who make these kinds of mistakes need to know, how they will come to learn it, and how they should be taught. One side believes that what needs to be known are correct procedures. Learning these procedures is most often a matter of watching, listening, practicing, and remembering, whereas teaching is showing, telling,

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[4]The principles that give meaning to the procedures used to multiply large numbers are the same principles as those used to form large numbers in the first place. In the number "536," for example, the "3" is a symbol for 3 groups of 10, whereas the "5" is a symbol for 5 groups of 100 or 5 groups of 10 groups of 10. The additive composition of groups of groups of 10 (5 groups of 100 + 3 groups of 10 + 6) is the conceptual basis from which symbols for numbers are derived. Additive composition is coupled with multiplicative composition in multidigit numbers because they are based on sums of groups of 10 and powers of 10.

providing multiple opportunities for practice, and testing students' ability to carry out the procedure correctly. This kind of teaching can be done to a whole-class group by a teacher or it can be "individualized" by having students read and work the exercises in a textbook. Occasionally a diagram or a demonstration with concrete materials might be used to explain why the procedure is what it is, but the ultimate purpose of these demonstrations is generally to have learners come to know *the procedure* rather than to acquire any concrete knowledge of the operation.

The computational way of knowing mathematics is the easiest to teach and learn in a traditional school setting because it lends itself well to the social organization of the classroom: large groups of learners under the direction of one teacher, a curriculum that is sequentially organized and standard within grade levels, and a system of assessment that clearly distinguishes between right and wrong answers. It is not necessarily the easiest kind of mathematical knowledge to learn, but it is easier to teach than naive, concrete, or conceptual knowledge because of these organizational features of schools. Traditional instructional systems consider computational knowledge as "basic," perhaps implying that if principled knowledge is to be acquired, it will be built on a base of computational competence. This progression from computational skill to conceptual understanding might be thought of as a responsibility of the teacher or the curriculum or as a function of student talent. In the latter case, computation is the only kind of knowledge that is directly taught and it is assumed that the brighter students will ascertain the principles that underlie the procedures without ever having those principles be the focus of instruction. Students who do ascertain something of the meaning of what they are doing would probably not make the sorts or errors in doing multiplication described above. But for others, a more direct approach to connecting *doing* with *meaning* might need to be taken.

On the other side of the instructional divide is teachi directly aimed at the acquisition of principled conceptual knowledge. In the extreme, this kind of teaching begins with instruction in the structure of mathematics, assuming that if one understands the structure of the subject, the ability to solve contextual, concrete, or computational problems will follow. Students would not be taught to do computations, although they might be expected to invent their own procedures. In teaching multidigit multiplication, then, one would first teach the distributive law and the concept of place value. Such teaching would range over the structure and composition of numbers in all base systems, and our familiar base 10 numerals would only be considered as a particular and not especially important example of how the concept might be applied. Instruction would be carried out in an abstract and symbolic language as this is most appropriate for capturing mathematical concepts. This is the sort of instruction that was intended by the "new math" of the 1960s.

As with instruction that has as its goals the acquisition of computational knowledge, teaching toward the acquisition of concepts might be supplemented by reference to concrete materials or intuitively derived understanding. Particularly where educators have adopted the notion that theories of cognitive development should serve as guidelines for instructional design, the four kinds of knowledge described above are viewed as existing in a necessary progression, from naive to concrete, from concrete to computational, and from computational to principled conceptual, with principled conceptual knowledge being the ultimate goal. One or more of these kinds of knowledge might be ignored along the way to the acquisition of concepts, depending on what one believes about how abstract knowledge develops.

If, for example, one takes the view that naive knowledge can develop into principled knowledge, learners would need to be provided with many opportunities to engage in activities from which they might develop their own intuitive theories about mathematical relationships. The teacher would need to monitor classroom safety and perhaps social relationships, but the only teaching that would occur would be encouraging learners to test their theories in different contexts and to define the conditions under which they do and do not apply. Both mathematical philosophers (e.g., Brouwer, 1913) and developmental psychologists (e.g., Kamii, 1985) have argued that intuitive theories of number will develop into principled mathematical knowledge through this process: The principles themselves need not be taught if learners are in a position to actively construct them and ascertain their limitations from experience.

Close by this approach to mathematics instruction is the use of carefully designed concrete materials to help learners arrive at a knowledge of mathematical concepts. Guidebooks for such manipulable materials as Cuisenaire rods, attribute blocks, pattern blocks, and unifix cubes suggest structured activities that are intended to produce a knowledge of the principles of arithmetic, geometry, or algebra. (See, for example, Trivett, 1976; Pasternack & Silvey, 1975.) The process by which concrete knowledge becomes connected to conceptual knowledge is rarely made explicit in these activities, and connections between concrete manipulations and computational procedures remain equally vague.

Psychologists are currently at work on the explanation of how those connections might occur, and Leinhardt (1985), for one, has argued that we can "come close" to a definition of mathematical understanding if we think of it as a collection of *all four* ways of knowing mathematics as well as an appreciation of the connections among them. In this view, no one kind of knowledge is linked automatically with any other, but by definition, understanding can be increased by increasing competence in any one of the four areas. This way of thinking about "understanding" is broader than many; by including procedural skill as one element of understanding, it circumvents the perennial debate about whether or not procedural skill is related to understanding. At the same time, procedural skill is not equated with mathematical knowledge as it might be in many classrooms.

Noddings (1985) argues for a similarly broad view of mathematical knowledge. She links the "informal knowing" of the naive and concrete categories and the "domain of formal procedures" with a "metadomain" of conceptual principles used to critique and discuss why things work in both informal systems and in formal computational procedures. In her view, *doing* mathematics involves making connections among activities in all three domains, whether one is a new learner or a practicing mathematician. She supports this view with descriptions of the work of both learners and experts as they turn to the domain of informal experience to interpret or bring commonsense meaning to formal procedures and to the metadomain of mathematical principles to explain the legality of their procedural systems. Davis (1984) also attributes the acquisition of "meaningful" mathematical knowledge to making explicit connections among different ways of knowing, as when written procedures for borrowing in subtraction are associated with the trading of base-ten blocks to be able to "take away" a given quantity.

The positions taken by these scholars are confirmed by my observations of fourth- and fifth-grade students learning mathematics. When they make errors (as described above), it seems as if they are not connecting one kind of mathematical knowledge with another. When they approach a problem to be solved, it is often the bringing together of the naive with the computational or the principled with the concrete that produces a solution. There is, therefore, a reasonable basis in both theory and practice from which to experiment with methods of instruction that both enhance these connections and build competencies *in all areas*. Research on the relationship among concrete experience, procedural skill, and understanding has been taken up by Resnick and Omanson (1985) and Carpenter (1984), among others. What I wish to focus on here, however, is the potential role of classroom instruction in increasing competencies in each of these categories of knowledge or strengthening the connections among them.

A consideration of some of the social features of school learning can also be used to support this approach. Because school learning is organized to occur with large groups of children in the charge of one teacher, the teacher will necessarily face learners who have all different kinds and levels of mathematical competence. Some children will have quite a usable naive knowledge of multiplication bound to familiar contexts like buying toys; others will have learned to move objects (like fingers) around effectively to get their answers; and still others will be efficient at numerical computation but unable to explain the processes they have learned. One way to approach this diversity in an instructional setting with the intention of developing each of the students' mathematical competence and understanding is to recognize the legitimacy of *all* of these kinds of knowledge and work to build the connections among them. Since those connections rest on conceptual principles, such an approach to instruction has the potential to increase understanding at many different levels. Like Leinhardt and Noddings, I would not neglect the computational in my attempts to increase mathematical understanding. Whether or not it can be conceptually linked to naive, concrete, and principled knowledge, it is a form of knowing mathematics that children, teachers, and the general public recognize as useful. Since classrooms exist within a larger society, it seems appropriate to make connections with what that society values.

Most fourth-grade children have probably been introduced to the idea that 7 x 6 means seven groups of six, and they can probably speak this "grouping" language about multiplications up to 10 x 10, expressing the connection between the concrete and the computational. When larger numbers come into play, however, it is less likely that children will see multiplication as simply a way of counting the total number of objects in situations in which those objects are arranged in groups and all of the groups contain the same amount of objects. Children can easily picture in their minds, (and the teacher can easily draw on the board) seven groups of six objects, and they can count the total number to get the answer to 7 x 6. When they consider the question asked by 82 x 152, however, it becomes much more difficult to produce a concrete image that will lead to an efficient counting procedure foi finding "the answer." Teachers usually stop using objects to represent mathematical processes in second or third grade. That is when the numbers get bigger, making the objects more unwieldly; it is also when the connection between concrete and computational processes becomes more complex.

Concrete activities with older children become more difficult to manage for social reasons. Even if one were to find the answer to 82 x 152 by physically counting out the objects in 82 groups of 152, the kinds of connections between that concrete activity and computing the total by the steps in the conventional multiplication algorithm that could give the algorithm meaning would not be immediately obvious. Children are likely to see the counting and the computing as separate and unrelated activities, and it is therefore very difficult for them to assign any meaning to the numerical process for computing 82 x 152 even though they might be successful in finding the answer using the conventional algorithm.

It is possible, without actually having students count out groups of blocks or sticks or dots, to teach in a way that makes connections among naive, concrete, and computational processes for finding products of large numbers by making explicit the operatings on quantities and relating those operations and quantities to symbolic procedures.

Consider, for example, the mental imagery that results from the following story: "Eighty-two new apartment houses have just been built on the north side of the city and each building has exactly 152 apartments in it." Such a story enables children who are familiar with apartment houses to get an idea of what 82 x 152 might mean in terms of quantity without counting out 82 groups of 152 or even seeing a picture of 82 groups of 152. It makes available a semiconcrete representation of the structure of numbers that is easily imaginable mentally by children and provides an intermediate step between the concrete and the abstract (cf. Hatano, 1980). It builds on their naive knowledge of a situation in which there are "groups of equal sized groups" to give meaning to multidigit multiplication.

Such an image can be connected with the processes of decomposing and recomposing numbers to help students to invent a way to compute 82 x 152. Each of the 15 floors in each building might have 10 apartments on it, with 2 more in the penthouse. How many apartments are on all the first floors? the fifth floors? all the penthouses? How many apartments will be in these buildings all together? If such problems are posed regularly using different kinds of familiar equal groupings, like cartons of soda pop bottles or rows of chairs in the auditorium, children can gain experience with applying multiplication principles to different contexts.

What follows are descriptions of several ways of teaching children about the familiar arithmetical process of multidigit multiplication that are designed to increase and connect students' naive, concrete, computational, and conceptual knowledge. I do not claim that these methods are new or unique to me, but simply that we need more of such descriptions of practice in the literature. The descriptions of instruction are all drawn from my own experience teaching a heterogeneous group of 28 fourth graders. Several weeks of lessons are described which include telling and illustrating multiplication stories, writing out the place-value groupings explicitly in numerical computational procedures, and working on problems with coins that require inventing different configurations for grouping and counting. All of these activities are designed to make children familiar with the principles of additive and multiplicative composition, associativity, and commutivity, and the distributive property of multiplication over addition. These principles were never named or taught directly in my class, but they underlie the naive, concrete, and computational proc dures that were introduced and practiced.

When these lessons began, the children ranged in computational skill from beginning to learn the single-digit multiplication combinations (times tables) to being able to calculate accurately n-digit by n-digit multiplications. The activities were designed so that all of the children could participate in and learn something from whole-class instruction. My purpose here is not to evaluate what was learned from these lessons, but to describe classroom teaching practices that are congruent with an appropriately complex view of mathematical knowledge. I did not consider these as lessons in "finding the answer" to multiplications like 4 x 86. Some members of the class already were able to do that; others were not. My concern was to teach all of the students something about what multiplication means while at the same time enhancing their computational competence.

Telling and Illustrating Multiplication Stories

Stories and drawings are modes of representation that are more familiar to most children than configurations of numbers arranged on a page. Many of my lessons on multidigit multiplication used stories and drawings like the apartment building story described above as a vehicle for giving meaning to the decomposing and recomposing processes that are at the center of the computational procedures used to find the products of large numbers. Conventionally, numbers are taken apart and put back together along place-value or "groups-of-10" lines. It is important for learners to become familiar with that convention, but it is also not always the most obvious or efficient basis on which to decompose numbers for multiplication; hence some of the stories and drawings I used represented base-10 decompositions and others did not. [24 x 9, for example, might be more easily thought of as (12 x 9) + (12 x 9) than as (20 x 9) + (4 x 9).] The essential principle I was attempting to teach with these lessons was that a multiplier or multiplicand can be taken apart, each of the parts multiplied, and then the total product could be found by adding the partial products, or more formally, the distributive law of multiplication over addition: $a(c + d) = ac + ad$.

We started with stories that would represent what happens to quantities when you multiply a one-digit number by a one-digit number, for example, "There were six parties last week and seven children attended each party," as a story for 6 x 7. These came easily; the children I was teaching had done them in third grade. We quickly progressed to stories that would represent the multiplication of a two-digit number by a one-digit number, and then to stories for two digits times two digits.

All of these multiplications could be represented by the same sorts of stories as the one-digit by one-digit multiplications, and the progression was meant to help the children see that the meaning of the operation was the same, no matter how big the quantities. It was just as sensible (from a mathematical point of view!) to have 43 parties with 26 children at each as to have 6 parties with 7 children at each. Once the routine of telling stories for multiplication problems was established, I began to use the stories in lessons with pictures and numerical symbols to accompany them to represent the process of taking apart bigger numbers so as to make them easier to multiply and count. Throughout, multiplication was always discussed as a procedure for counting the total quantity in a collection of groups in which each group has the same number of members.

What follows is a sample dialogue from one of these lessons (T means teacher). In the right-hand column is what I wrote or drew on the board during the lesson.

T:	Can anyone give me a story that could go with this multiplication?	12 x 4
Jessica[5]:	There were 12 jars and each had four butterflies in it.	
T:	And if I did this multiplication and found the answer, what I know about those jars and butterflies?	
Jessica:	You'd know you had that many butterflies all together.	

[5]All students' names are pseudonyms.

Jessica has constructed a way of giving meaning to the operation 12 x 4. The *meaning*, that is, "what I would know if I did it" is not the same as the *answer*, "48."

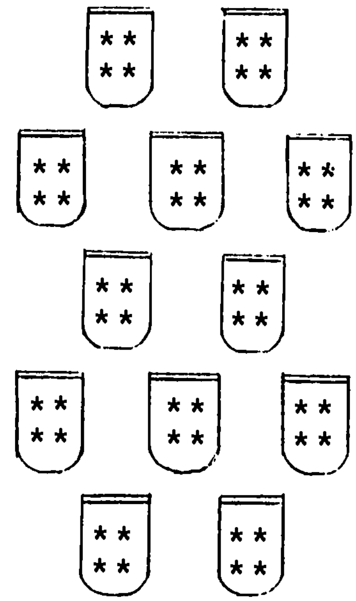
From this point on in the lesson, I am going to be structuring a concrete representation of Jessica's story that focuses on the intuitive procedure for counting large numbers of objects arranged in groups by taking the groups apart and putting them together. In my dialogues with students, I almost always follow an unexplained "answer" with a question about how it was derived, partly to see what the child will say, partly to give him or her the opportunity to verbalize the reasoning, and partly to establish the culture of "sense making" as part of our relationship in the classroom. I assume that what they say can both teach them something and tell me something about what they know.

By having the students "capture" their procedures in their own language, this kind of lesson makes the concepts underlying multiplication more explicit and accessible to learners and enables us to learn a great deal about how children can use principles to explain actions.[6] Here, the work of the teacher and the work of the researcher are parallel. If one is going to make connections in the classroom between children's naive knowledge of arithmetic and the procedures and concepts they are to learn, it is necessary to include inquiry into their ways of thinking as part of the instructional strategy.

As a teacher I diverge from the researcher, however, in that my job goes beyond finding out what students know or how they learn. Thus, I do not judge the value of "thinking aloud" only in terms of its legitimacy as a data

[6]Resnick and Omanson (1985) conclude from their research on subtraction that the amount of verbalization by a student during instruction designed to link concrete and procedural performance was important in transferring understanding from one realm to the other.

gathering device; for me it is a teaching technique as well. In order to carry on this sort of lesson, I must teach children how to explain their thinking and that it is an appropriate thing to do because I am committed to their acquiring a particular kind of knowledge.



Returning to the lesson:

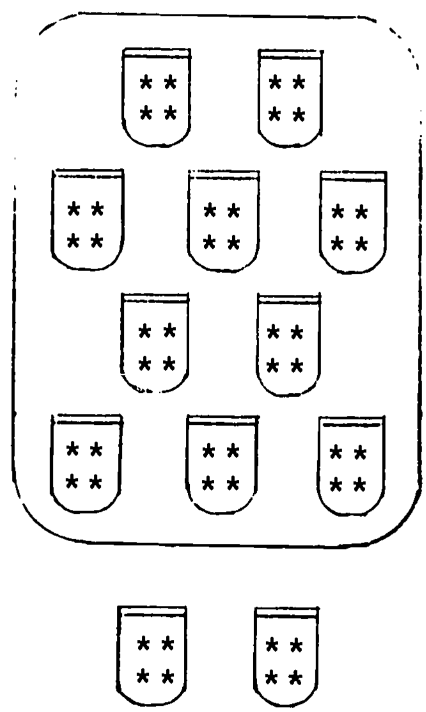
T: Okay, here are the jars. The stars in them will stand for the butterflies. Now, it will be easier for us to count how many butterflies there are all together, if we think of the jars in groups. And as usual, the mathematician's favorite number for thinking about groups is? (Draw a loop around 10 jars.)

Sally: 10

T: Each of these 10 jars has four butterflies in it, so how many butterflies are inside this circle?

John: 40

T: How'd you figure that out?



John: It's 4 x 10. That's easy, you just add a zero.

T: I put the jars in groups of 10 because I knew it would be easy for you. How many more butterflies are there outside the circle?

Jim: Eight. (He does not know his "tables" but he can count them easily now that there are only a few.)

T: I add 10 jars and 2 jars and I get 12 jars. Each jar has four butterflies in it. (Point to the two fours in 4 x 10 and 4 x 2.) So how many butterflies are there all together?

Chorus: 48

Even though we have arrived at "the answer" at this point, I continue on with the lesson, analyzing the procedure we used and verbalizing its structure.

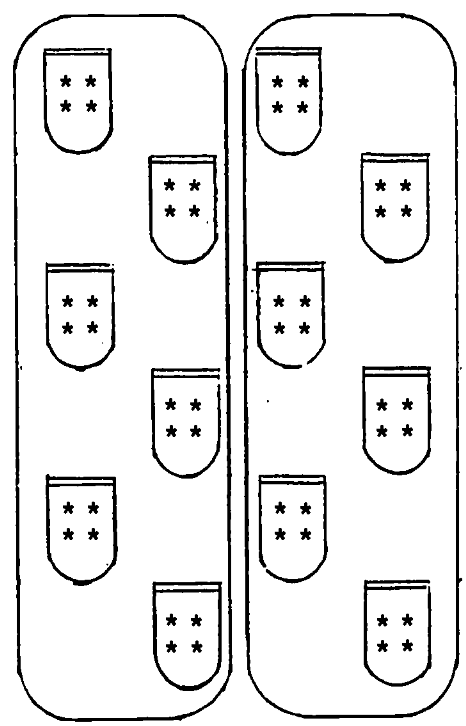
One message that I hope to convey with this kind of lesson is that even children who do not have computational competence can "figure out" the answer.

T: I added the 10 and the 2 to get 12 jars. Should I also add the 4 and the 4 to get 8 butterflies?

Shawn: No. There are just four butterflies in each jar. That will never change.

This is usually the glitch for children in what mathematicians call the "distributive law"; it's an easy principle to see when it is attached to quantities in stories like this one, but very confusing when presented in the abstract. When students see only (4 x 10) + (4 x 2) it is very hard to explain why the answer is *not* obtained by doing 8 x 20. Yet math books often ask children to do such figuring without referring to any representation of quantities as a way of introducing the partial products used in multidigit multiplication.

The next part of the lesson is intended to get at the idea of finding a grouping that makes the figuring easier. Usually that means grouping by "tens" and "ones," but not always.



T: Suppose I erase my circle and go back to looking at the 12 jars again all together. Is there any other way I could group them to make it easier for us to count all the butterflies?

Jean: You could do 6 and 6.

T: Now, how many do I have in this group?

Steve: 24

T: How did you figure that out?

Steve: 8 and 8 and 8. (He put the 6 jars together into 3 pairs, intuitively finding a grouping that made the figuring easier for him.)

T: That's 3 x 8. It's also 6 x 4. Now, how many are in this group?

Jean: 24. It's the same. They both have 6 jars.

T: And now how many are there all together?

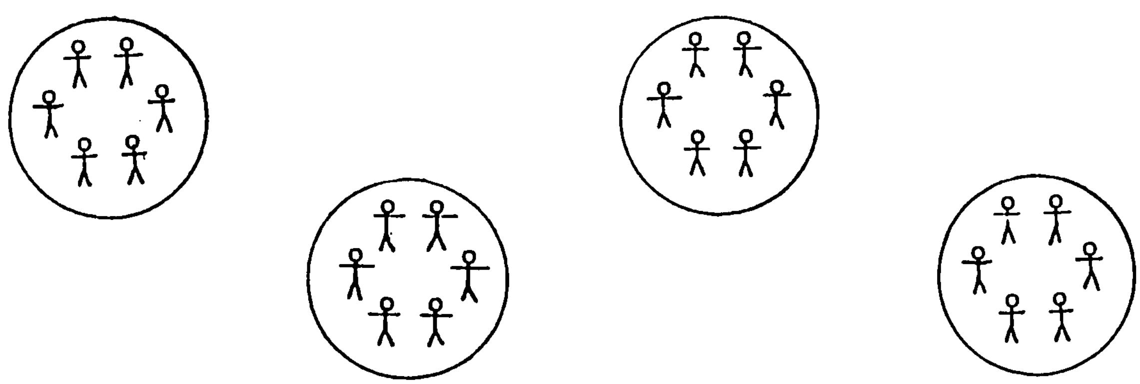
Patty: 24 and 24 is 48.

T: Do we get the same number of butterflies as before? Why?

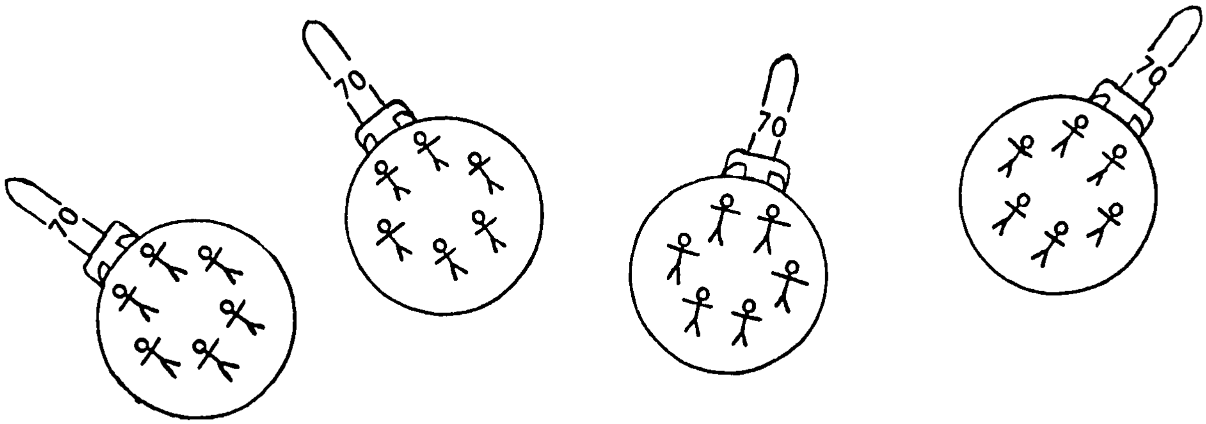
Patty: Yeah, because we have the same number of jars and they still have 4 butterflies in each.

Jean's and Patty's answers are what I would call "explained." I had a feeling of what made sense to them here without having to ask. I asked several other children to explain in their own words why there were the same total number of butterflies each time. It was clear from watching them that some of the kids were surprised that it came out the same, which was a cue to me to do lots more of these different kinds of groupings.

In subsequent lessons, I made the numbers in the multiplication bigger, for example, 76 x 4, and I became even more didactic. I said that I was going to tell a story to help us think about what this multiplication means. I chose a theme from some stories the students told in another context: "There were four planets and each had six astronauts exploring on it." I drew the following on the board:



Then I said, "Then a big spaceship with 70 more crew members landed on each planet," and I drew a spaceship with the number 70 on each "planet."



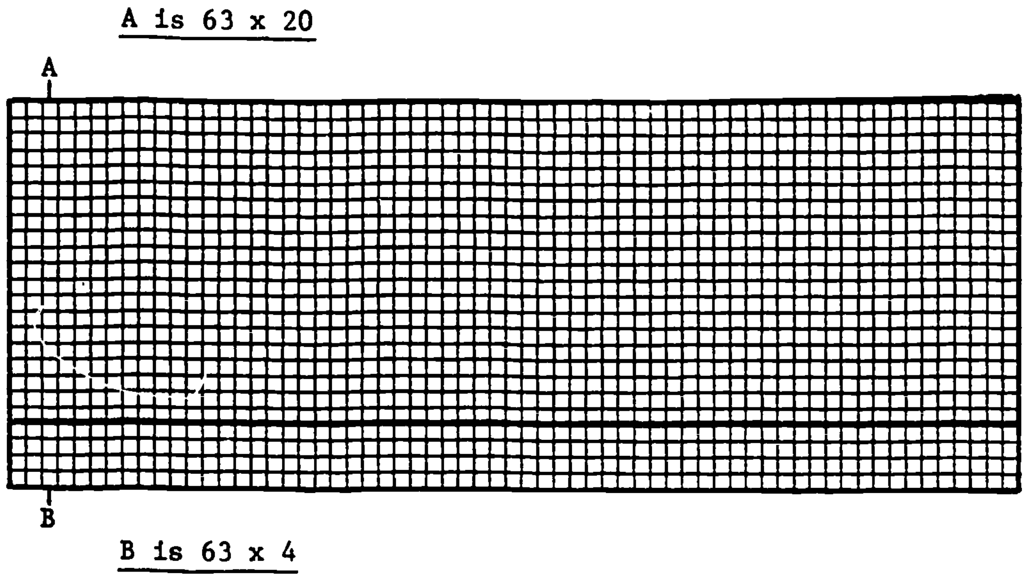
I asked how many space men were on *each planet* now, and then I asked them to figure out how many space men there were *all together*, on all four planets. Some children added 76 + 76 + 76 + 76. Another doubled 70, then doubled 140 and got 280, and then added 24 to get it to a total of 304. Another multiplied 70 x 4 and then multiplied 6 x 4 and added the products together.

In each case, I had the child explain to the whole class "how he/she figured it out" and gave equal praise to each solution. Each of their explanations is an expression of naive, concrete, or computational knowledge of the principles that underlie multidigit multiplication. By having the children explain their different procedures to one another and by legitimating them all in a whole-class discussion, my intention was to provide experiences that have the potential to expose concepts. This kind of mathematics teaching contrasts with the prevalent tendency to have students learn *one* way to find the answer in which they are meant to remember procedures.

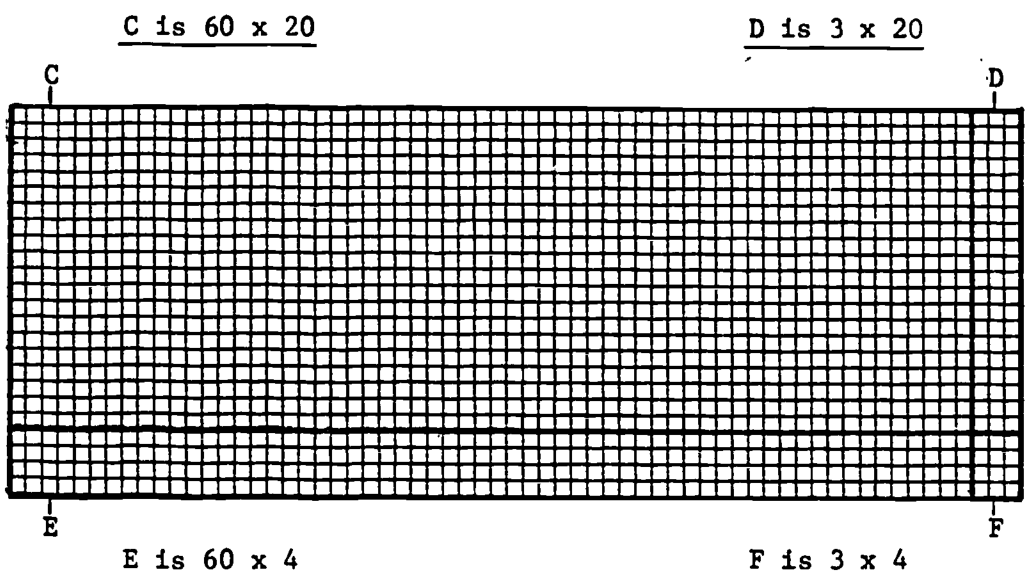
After everyone had done some experimenting with both base-10 and less conventional decompositions as ways to find the product of a one-digit and a two-digit number, we moved on to two-digit numbers multiplied by two-digit numbers. These are more complex because there are two numbers to decompose and, even if one uses only base-10 decompositions, there are several different ways to go about finding the total product. For example, 63 x 24 can be interpreted as 63 groups of 24. Those 63 groups can be separated into two sets of groups: 60 groups of 24 and 3 groups of 24. If we could figure out the total quantity in both of these sets of groups and add them together, we could know how many are in 63 groups of 24. But it's not very easy to see how many are in 60 groups of 24 or how many are in 3 groups of 24, so we need to treat each of these as a separate multiplication to be represented using the same sort of taking apart and putting together as I described above to find how many are in each of the two sets of groups.

We can start with either one. How might we break down 60 groups of 24 to better be able to count what is there? We could think of it as 10 large groups each containing 6 groups of 24, and work from there, first finding 6 groups of 24 [(2 groups of 24) + (2 groups of 24) + (2 groups of 24)] and then multiplying that by 10. Or we could think of 60 groups of 24 as 60 groups of 20 and 60 groups of 4. Sixty groups of 20 is 120 groups of 10, and that's 1,200; 60 groups of 4 is 240. The total is 1,440. Now we need to figure out how many are in 3 groups of 24. One of my fourth graders who had played around a great deal with groups and numbers might say, "Well that's the same as 6 groups of 12, and that's 72." You could also think of it as 3 groups of 20 (60) plus 3 groups of 4 (12) for a total of 72.

All of this becomes more clear if the different kinds of taking apart and putting together can be shown in a picture. The most *convenient* way for books and teachers to illustrate this sort of decomposition is with squared-off areas, using the rows and columns to help with the counting. The multiplication of 63 x 24 would be represented by a rectangle 63 blocks high and 24 blocks wide. The decomposition would break the rectangle up, first into two rectangles, then into four rectangles as follows:



The second breakdown splits A into C and D, and splits B into E and F:



This is meant to show that in order to multiply 63 x 24, we need to decompose both numbers: 63 is 60 + 3 and 24 is 20 + 4. The rectangle shows that the total area is given by (60 x 20) + (3 x 20) + (60 x 4) + (3 x 4), but this is still quite an abstract representation for kids, unless their parents work in the tile business. It gets all mixed up with ideas about length of sides and area. If you talk about illustrating 2 x 3, for example, by  , you can say this figure is 2 blocks across and three blocks down, 2 columns of 3 *blocks*, or 3 rows of 2 *blocks*, and 6 *blocks* all toegther. But many books and teachers do not talk about such an illustration in terms of rows and columns of blocks. Instead, they switch into measures of length and area, saying that such a figure has a width of 2 *inches* and a height of 3 *inches* and its area is 6 *square inches*. Now the difference between inches and square inches is a significant one; one measures length whereas the other measures area. To try to connect these measures to the ideas about grouping associated with multiplication is difficult: Are we talking about 2 groups of 3 inches? or 3 groups of 2 inches? and if so, in either case, how do we wind up with a total of 6 *square inches* instead of just 6 *inches*? These complications made me reluctant to use blocks built up into rectangles as an illustration for large number multiplications.[7] One would need to be very careful with the language and checking children's sense of what it all means in order for this to make sense to them as an illustration of the multiplication process.

When I asked my fourth graders to come up with a story about a two-digit by two-digit multiplication, we worked out a rather nice pictoral representation together. Given the multiplication 28 x 65, Coleen suggested that we could think of it as 28 glasses with 65 drops of water in each glass.

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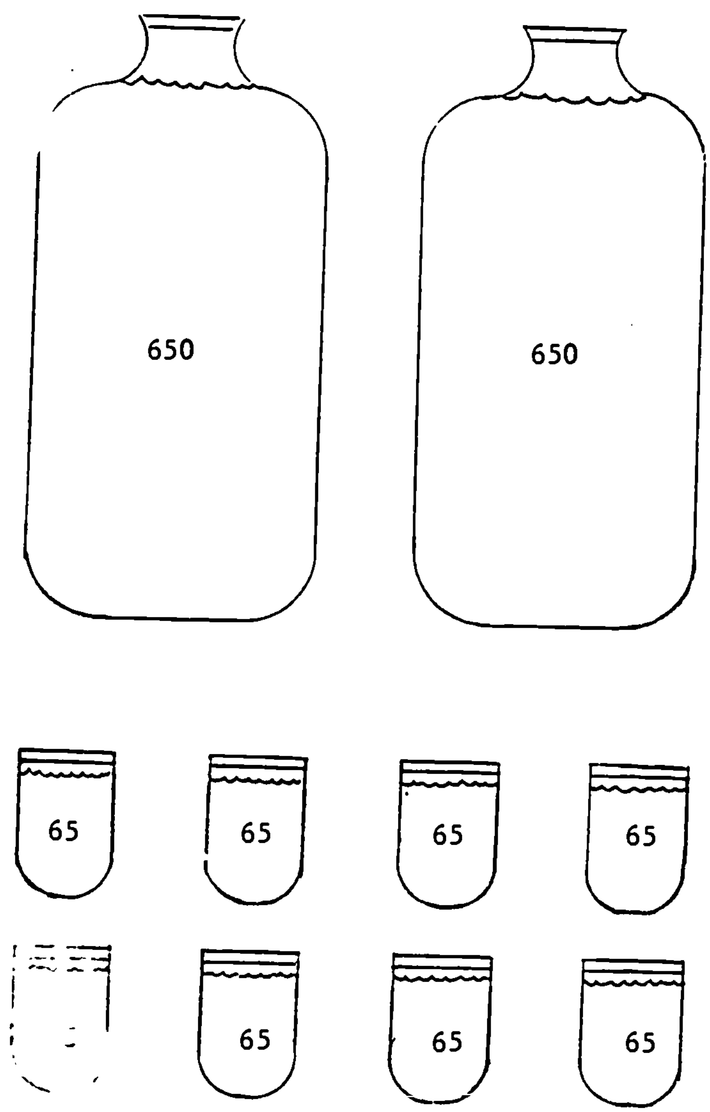
[7] Some textbooks avoid the issue by illustrating two-digit by two-digit multiplications using a rectangular array of a large number of dots, and show that it is easier to count dots by grouping them. This is an improvement, but the activity remains for the most part passive and abstract, since one is unlikely to actually count the dots, even in the smaller groups.

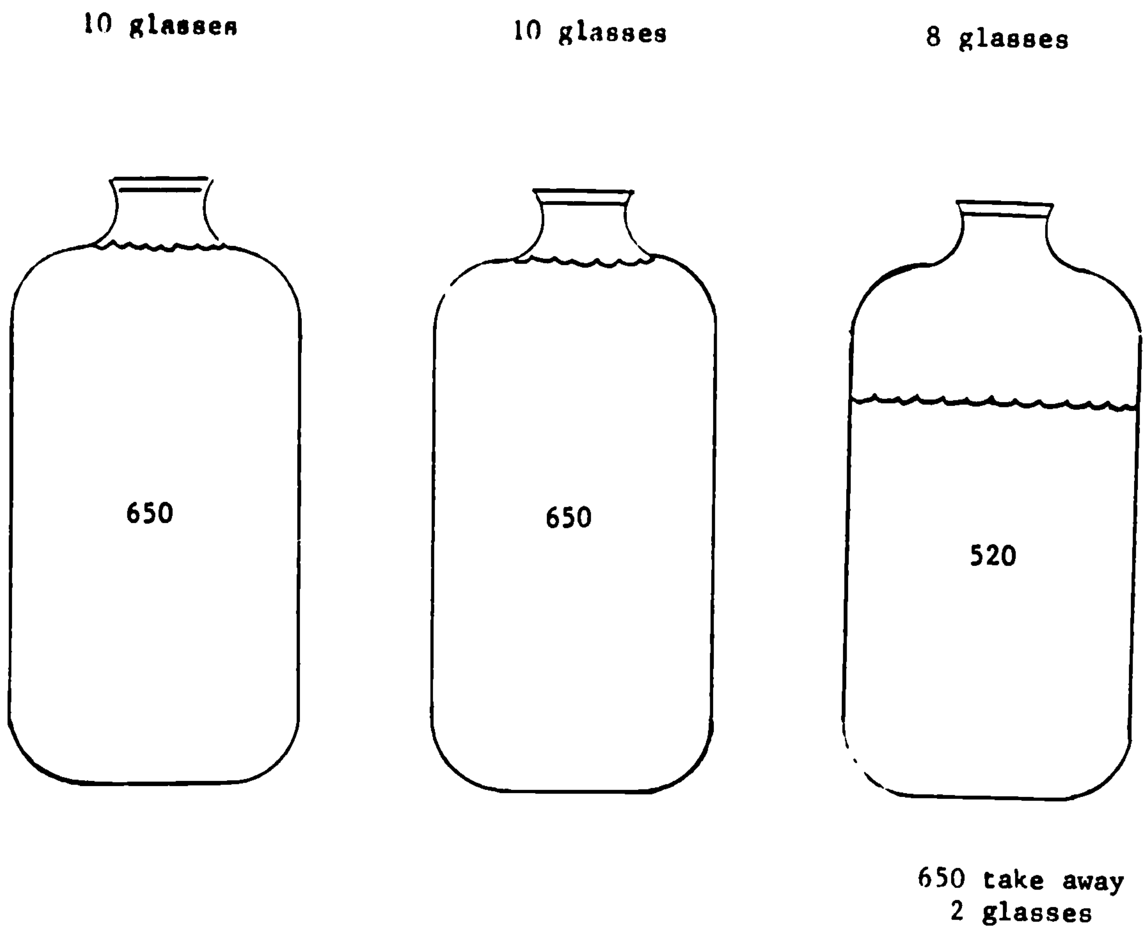
I said I didn't want to draw 28 glasses on the blackboard, so instead I would draw two big jugs that would each hold the same number of drops as 10 glasses. I questioned the students as I drew: How many jugs would I need? Two. And how many more glasses? Eight. How many drops in each glass? Sixty-five. And how much in each jug? Six hundred fifty.

For each of these answers, I requested an explanation, and the student verbalized how he or she "figured it out." I drew the following illustration on the blackboard:



Instead of drawing the drops in each glass, I put numbers in each glass and on the jugs. I asked the class what we should do to find the total number of drops? Add the jugs together. That's 1,300 drops. Then I said I would show them a trick that would make it easy to add the drops in all the eight glasses together. This procedure illustrates the base-10 decomposition of 65 that occurs in the conventional multiplication procedure: Suppose I take five drops out of each (circle each 5 with colored chalk) and put them in a jar: How many are in the jar? 40. How many are left in all the glasses? 60. And how many glasses all together? 8. How many drops are left in each glass? 480. So, now I have 1,300 drops in jugs, 480 drops in glasses and 40 drops in the jar. How many drops all together? 480 + 40 is 520 and 520 + 1,300 is 1820. One of the students pointed out that we could have also made it easy by pairing off the glasses, making four pairs with 130 drops in each. Pairing these pairs, we have two groups with 260 drops in each, and adding those together, 520 drops all together. The children figured all of this out without any paper and pencil, and I pointed that out to them. By clever groupings, we had reduced the calculations to those that could be done "mentally" by most of the class. Some members of the class are quite capable of doing 28 x 65 using the conventional algorithm, paper, and pencil; for them, this was a lesson about thinking rather than about doing.

After we finished this illustration of 28 x 65, one of the girls in the class (Ko) came up with "another way of thinking about it" (and that's just what she called what she was about to tell me). I followed along and also put Ko's explanation on the board so as to give it equal weight in the eyes of the class. She said, "I thought you could have three jugs. Two would have 650 drops in them, just like you said there. But if you put 650 drops in the third one, you'd have too much. You'd have to take out two glasses, because there are not 30 glasses in the story but only 28. And each glass has 65 drops, so you'd have to take out 65 plus 65 or 130 drops. Then in that third jug, you'd have 650 - 130 which is 520. You'd have to add that to the other two, 1,300 + 520, and you get 1,820." I also illustrated Ko's thinking to relate it to what we had done before:



Although I did not use these symbols or terms with the class, we could explain what Ko did, using the mathematical principles of multidigit multiplication, as follows:

$30 \times 65 = (28 + 2) \times 65$	Additive composition
$(30 \times 65) = (28 \times 65) + (2 \times 65)$	Distributive law
$(30 \times 65) - (2 \times 65) = 28 \times 65$	Subtracting the same quantity from both sides of the equation
$(20 + 10) \times 65 = 30 \times 65$	Additive composition
$[(20 + 10) \times 65] - (2 \times 65) = 28 \times 65$	Substitution of an equal quantity
$(20 \times 65) + (10 \times 65) - (2 \times 65) = 28 \times 65$	Distributive law
$1{,}300 + 650 - 130 = 28 \times 65$	Computation
$1{,}300 + (650 - 130) = 28 \times 65$	Associativity
$1{,}300 + 520 = 28 \times 65$	Computation
$1{,}820 = 28 \times 65$	Computation

(I have used "computation" here as a shorthand to avoid writing out the conceptual foundation of multiplication by multiples of 10 and addition so that it would be possible to focus on the principles of multidigit multiplication.) If I wanted to know whether Ko had a principled conceptual understanding of mathematics, I might ask her to try to invent another legitimate procedure for the same problem, and another. But I include her story here to provide evidence that kids *do* invent computational procedures and to support the argument that what they really need to be competent at school mathematics is some help connecting what they can do with the conventional symbols and procedures.

After three or four lessons in which I used students' stories to do drawings and numerical symbolization representing the decomposition process on the blackboard, I constructed assignments in which the children would do their own stories, numerical representations, and drawings on paper with decreasing amounts of teacher direction. In some of those assignments, they were directed to "find the total" according to whatever decomposition and recomposition method they chose and then to find it again using a *different method.* Some of the children became quite interested in showing me how many different ways they could decompose one of the factors to find the partial products. At this point, they were using the concrete processes we had practiced to make sense of a procedural strategy, or in Nodding's terms, using informal "commonsense" knowledge to inform their competence in handling formal procedures, and in the process, ranging into the metadomain of reflecting on the nature of what they know.

This "game" of multiple decompositions was related to a set of lessons I had done earlier in the year on coin problems. Although these earlier lessons were not directly meant to teach multidigit multiplication, they provided a context for making connections with children's naive knowledge of the distributive law and the multiplication and additive composition of numbers.

Coin Problems

"Using only two kinds of coins, make $1.00 with 19 coins." Early in the school year, I gave my fourth graders the challenge of solving this problem with two goals in mind: I wanted to demonstrate that there were many possible routes to the solution of a math problem, and I also wanted to show them that math problems could have more than one right answer. At first, the students did not know where to begin. I was challenged to help them find a "way in" to this problem that would leave most of the thinking in producing a solution up to them. The activities that I designed turned out to involve several weeks worth of practice with many of the principles underlying multidigit multiplication. The coin problems I gave them to do were simplified versions of the problems stated above. They required the same process of taking numbers apart and putting them back together again that involved in using decomposition and the distributive law to do multidigit multiplication.

I did not tell the class much about the connections between what we were doing and the more conventional multiplication procedure, although such connections could certainly be made. Instead, I capitalized on the fact that the coin problems did not look much like the arithmetic they had been learning from other teachers. These problems therefore had the potential to free them up to experiment and think mathematically rather than relying on mechanical routes to right answers. They also were less likely to trigger the sorts of thoughtless procedures that are routinely applied to arithmetic problems (Carpenter, 1984). The procedures the students used to puzzle out the different combinations of coins that satisfy given conditions are isomorphic to the computational procedures involved in multiplication; they represent the concepts of multiplication in a context (i.e., money) that has its own internal equivalence relationships.

The relationships among different kinds of coins are concrete in contrast to the more abstract trading represented by place value in numbers. *One* dime is worth *ten* cents. One dime is in some sense "the same as" ten pennies, but a dime is only *one* coin, while ten pennies is *ten* coins. A dime could also be worth the same as *six* coins if one were a nickel and the others were pennies. So the total number of coins and the total amount of money are related, but they are related in complex ways. One coin might be worth one cent, but it also might be worth five, ten, twenty-five, or fifty cents. These relationships are similar to those that give place value meaning, but placement is a more abstract way to represent comparative value: A "6" can be understood not only as 6 ones, it can also mean 6 groups of 10 ones or 6 groups of 100 ones, depending on where it is in a number.

I began these lessons with a review of all the coins we use in the United States, making a chart on the board which listed each coin's name and how much it was worth in cents. I also identified each coin with the first letter of its name so that I would not have to write the name of the board each time I wanted to refer to it. I was careful to use the word "cent" when I was talking about the value of a particular coin and the word "penny" when I was talking about the coin that is worth one cent, even though these words are often used interchangeably.

Coins	Amounts of Money
penny	p = 1 cent
nickel	n = 5 cents
dime	d = 10 cents
quarter	q = 25 cents
half-dollar	h = 50 cents
silver dollar	s = 100 cents

The more advanced students liked the chart because they thought it had to do with algebra.

We began working with simple multiples: How much money is 3 dimes worth? 5 dimes? 7 nickels? 2 quarters? And then we did combinations: What are 3 dimes plus 2 quarters worth? 7 dimes plus 3 pennies? 2 quarters plus 1 dime? These were sometimes written on the board as 3d + 2q, 7d + 3p + 2q + 1d, and the use of these letters did not seem at all problematic to the children. I did not need to explain that 7d meant 7 x 10 cents; their familiarity with coins gave them this knowledge. These exercises represented the concept of additive composition which is a central idea in the process of multidigit multiplication. Finding the monetary value of combinations of coins was also an important exercise in working in a context where the "order of operations" matters; everyone knows that you multiply first and then add to get the value of the coins because of their familiarity with how money works (i.e., to find the value of 3d + 2q, first you multiply 3 x 10, then you multiply 2 x 25, then you add the results of these two multiplications.)

This familiarity gave the students the opportunity to do mathematics confidently in an area where they would later be introduced to more abstract forms. Because they knew how coins worked, they would be unlikely to add first and then multiply or make other procedural errors. (Adding first would be like adding the "carry" before multiplying. See pages 11-14). The symbols used to indicate the "order of operations" in a mathematical expression are parentheses. Taught in the abstract, they constitute just another rule to be remembered: when children see (3 x 10) + (2 x 25) they are supposed to *remember* to do the multiplications first and then add.

Another activity we did was figuring out how many of which kinds of coins would add up to a certain total. At first we used only pennies or only nickels or only dimes or only quarters or only half dollars, and we usually began by figuring out how many would make a dollar. The multiplications and divisions here all involve fairly "easy" combinations of 5 and 10, but there are a many variations. Children can work confidently with the processes of decomposition and regrouping in these activities because they do not have to strain to remember the "number facts" involved in the computations. The "problems" are ones they can solve and yet there is enough of a challenge to make it satisfying when they find the solutions.

When we switched to using *two* kinds of coins to add up to a certain amount of money (e.g., nickels and/or pennies to make 82 cents) the problems became more interesting and fun because there are multiple solutions. The children were most enthusiastic about figuring out many *different* ways to make an amount "82 cents" using nickels and pennies, and their enthusiasms gave me many opportunities to help them generate their solutions using strategic rather than random thinking. The nature of these problems is such that once they come up with one solution (like 16 nickels and 2 pennies for making 82 cents), they can make more solutions by trading: for example, everytime they take five away from the pennies pile, they need to add one to the nickels pile. The process gets children involved in using mathematics in a way that enable them to challenge *themselves* to find solutions, constructing mathematical hypotheses along the way about whether or not they have found them all yet. Students who are not yet able to use multiplication and division can find solutions by adding, and so a lack of computational "skills" does not hinder their participation. On the contrary, they develop their own more sophisticated adding strategies which come to look more and more like multiplication, moving from a naive knowledge of the process toward understanding its principles.

As my class worked on these problems, I encouraged them to "guess and check": that is, to pick numbers of coins that they thought might add up to the desired total, and then to check to see if they actually would. Once they got going on this procedure, they were able to think mathematically and modify their guesses strategically. But first I had to find a way to help them overcome a habit that inhibited their use of this way of getting a solution: Once children in school recognize that something they have put down on paper is wrong, they are intent on erasing it and forgetting it.

They don't want to leave any evidence around of their "failures," especially once they have figured out how to "succeed." They are used to being judged on the basis of answers produced rather than on the basis of strategies for arriving at answers. This way of approaching mathematics is not only conceptually problematic; it causes logistical difficulties as well. I needed to find a way to reward students for all of their guesses so that they would record their "wrong" guesses and think about *why* they did not work, so I gave them a framework for writing down their thinking about the coin problems. For example, for a problem like "How many ways can you make a dollar using dimes and/or nickels?" I directed them to put the following down on paper:

d	n	total amount of money

I suggested that they list *all* of their attempts, not just the ones that "come out right," and I directed them to put a line through the ones that did not satisfy all the conditions of the problem. I praised them for their number of tries, not just for the number of solutions they could find that met all the conditions. This not only gave them a record of their work to analyze; it gave me some sense of how their thinking progressed and it provided a transition to using the placement of symbols on a page to represent mathematical operations. After several minutes of working on the problem: "Make $1.00 using nickels and dimes," one student's paper looked like this:

d	n	total amount of money
5	10	$1.00
~~6~~	~~9~~	~~$1.05~~
6	8	$1.00
~~7~~	~~7~~	~~$1.05~~
8	4	$1.00
9	2	$1.00

I would assume, from looking at such a paper that the student first thought he should decrease the number of nickels by one if he increased the dimes by one. It took him two trials to figure out that the nickels had to be decreased by *two* for the total to stay the same.

I asked the students to talk about the "patterns" they noticed in their charts. On a problem such as the one illustrated above, they would say things like, "If the dimes go up by one, the nickels have to go down by two," or "Everytime you put in another dime, you have to take away two nickels." I tried to use their language myself in noting trading patterns and to make telling about patterns a central part of our work on problem solving. Each time we began a new problem, however, the discovery that there would be a pattern and the description of patterns was left up to the students. If they were working away randomly, I did not interfere by giving them a suggestion for an "easy way" to get a lot of solutions. If there was a strategic pattern obvious in their work, I took note of it and encouraged them to verbalize it. They were, in a sense "inventing algorithms" which would help them to count large numbers of cents quickly, using the same principle of distributivity that underlies the conventional multiplication procedure.

Another kind of problem we did that focused on grouping and recognition of strategic patterns was to find a number of ways you could make $1.00 using only two kinds of coins, for example, nickels and quarters, and then to find all the ways to make $2.00 using the same two kinds of coins. Many students observed patterns; for example, they noticed that they could take the solution to the first problem and make solutions to the second problem simply by doubling the total coins in each category or by adding 20 to all the nickel amounts or by adding 4 to all the quarter amounts. These are all concrete exercises in composing a total quantity using groups of smaller quantities, an operation that is isomorphic to the procedures used in multidigit multiplication. They require students to use their naive and concrete knowledge of those procedures, and the activity of "finding patterns" provides opportunities to link that knowledge to procedures and principles.

After a few days of working with two kinds of coins to get a given total amount of money, I introduced problems using 3, 4 ,5, and 6 different kinds of coins. These problems offered many more possibilities for "guesses" and it was harder to see the patterns. It also required thinking about relationships among three variables: the number of *coins*; the number of *kinds* of coins; and the value of the total. If we had begun with this sort of problem, the students would not have had a repetoire of strategies and principles with which to begin their decompositions. In all of the problems I've described so far, no condition was placed on the total number of coins that could be used. To make a dollar, one could use 20 coins, or 2, as long as their values added up to 100 cents. We also did some problems in which the *number* of coins that could be used was specified, but the *kind* of coin was not: e.g., "Make $1.00 using 3 coins."

Considering each of these kinds of problems separately and practicing each one on several different examples helped the students to develop the confidence, the skill, and the organization to approach a more complex coin problem like the one with which I began this section: "Using only two kinds of coins, make $1.00 with 19 coins." This problem sets limits on the total number of coins to be used, the amount of money they must add up to, *and* the number of different kinds of coins that can be used. It does not specify *what kind* of coins to use, but only that there should be two kinds.

Students' solutions to these more complex problems were organized using the same chart described above, with extra columns for the total number of coins and the numbers of kinds of coins when these conditions were part of the problem. With the preliminary work we had done, they had both concepts and strategies to bring to these problems; and both the concepts and the strategies could also serve as a foundation for other mathematical work. They ranged among naive, concrete, computational, and conceptual knowledge to produce a solution; the process of connecting these different kinds of knowledge was facilitated by the routines for recording ideas that had been established while working on the less complex problems.

Using Numbers in a Meaningful Way to Illustrate the Principles of Multiplication

Another way to teach the processes involved in multiplying pairs of large numbers in a way that relates concepts or principles to other ways of knowing mathematics is to use configurations of numbers themselves to indicate the operations that are occurring. These symbolic representations of the process are "alternative algorithms" that record each step of the operation more explicitly than the conventional procedure. They are usually less efficient as a procedure for getting an answer; they are also less familiar to students and therefore often a source of distress when they are encountered in the classroom. Once students have learned a "quick and easy" method for calculating the answer and have come to believe that the answer is what really matters, they are understandably less tolerant of more cumbersome alternatives. Alternative algorithms also cause difficulties when parents try to help a child with his or her math homework. "That's not the way *I* learned it" is a comment that can carry a confusing message to a child who is already skeptical about what is going on in school.

Several pedagogical techniques can be used to avert this difficulty. Within a traditional school situation, one of the most effective might be to test children on their ability to demonstrate proficiency in using the more explicit alternative procedures, giving "points" for the clarity of process as well as for the accuracy of the answer. This strategy uses the institutional structure of the classroom to convey the message that explicitness and clarity in the procedure are at least as valuable as the right answer. I do not mean to indicate here that the capacity to reproduce alternative algorithms that have been taught should replace answers as the basis for judging students' understanding; rather, as in the other representations of the multiplication process that have been described here, the learning of explicit procedures in which the steps of an operation are clearly symbolized by the way numbers are placed on the page is meant to be an occasion for learners to make connections among different kinds of knowledge about particular mathematical principles.

As with stories and pictures, I began my lessons on numerical symbolization with one-digit by two-digit multiplications and focussed on the decomposition of the two-digit number into tens and ones. The total in 3 groups of 86, for example, could be figured out in four steps and represented as follows. (The arrows are meant to show the order in which each step is completed.) Step 1 illustrates decomposition. In Steps 2 and 3, each part of the number is multiplied and then in Step 4 the product is composed using the distributive law.

```
  86   ----->  80 + 6      Step 1
  x3
  ——
  18   <-----  3 x 6       Step 2
+240   <-----  3 x 80      Step 3
————
 258   <-----  18 + 240    Step 4
```

We did several multiplications in this way together as a class, with me writing on the board and the children doing more and more of the work on their papers.

After several such examples, I raised a question about Step 3: "Why is there always a zero on the end of the answer?" The students' responses expressed the observation that one of the factors in Step 3 always had a zero on it. My "explanation" at this point again referred to money as a concrete representation of place value[8] and a realm in which students are likely to have considerable intuitive knowledge about place value and trading. I posed

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[8]This continuing reference to money may seem overly mercenary for classroom work, but money is one of the few trading systems in our culture that uses base 10. Other trading systems with which children are familiar are dozens (base 12); minutes, seconds, and hours (base 60); and days and weeks (base 7) but these do not easily map on to the conventions we have adopted for writing large numbers. Many text books use metrics to illustrate these conventions, but metric units of measure are not yet familiar enough to most American children to provide a familiar representation of a concept.

several problems of the following sort: What is 3 times 8 one-dollar bills? What is 3 times 8 ten-dollar bills? 5 times 7 ones? 5 times 7 tens? I represented their answers in two forms on the blackboard as follows:

5 x 7 ones = 35 ones	5 x 7 tens = 35 tens
5 x \$7 = \$35	5 x \$70 = \$350

I extended the problems to hundreds and thousands which we would get to later in our work on multiplication.

5 x 7 hundreds = 35 hundreds	5 x 7 thousands = 35 thousands
5 x \$700 = \$3,500	5 x \$7,000 = \$35,000

From this work, we evolved "the zeroes rule" which stated that the product had to have at least as many zeroes at the end of it as the factors. This rule was a very useful tool in getting through meaningful representations of the multidigit multiplication process using numbers; it enabled me to teach my students to do computations using numbers in a way that directly represented the quantities being operated upon.

The next step in this set of lessons focused on the principle of commutativity of addition. Emphasizing this principle was intended to highlight the meaning of the numbers and the operations and deemphasize the generation of a mechanical procedure for computing the answer. I demonstrated several problems in which Step 2 and Step 3 were reversed to show that the answer came out the same when the order of multiplication was changed, for example:

```
  97  --->  90 + 7          97  --->  90 + 7
  x4                        x4
 ---                       ----
 360  <---   4 x 90         28  <---  4 x 7
 +28  <---   4 x 7        +360  <---  4 x 90
 ---                       ----
 388  <---  360 + 28       388  <---  28 + 360
```

That Steps 2 and 3 could be done in either order did not seem to confuse anyone, and several children were able to say why it made sense to them to be able to switch them around by referring to the decomposition in concrete terms. In fact, the version in which the "tens" part of the number was multiplied first made *more* sense to them when they were able to put the convention of "starting with the ones" aside. When you "take apart" a two-digit number in order to read it, you think first of the tens and then of the ones (i.e., 73 is *seventy*-three not three and seventy). I wanted them to be able to explain that some parts of the procedure could be "switched around" and it wouldn't make any difference to the answer, rather than just believing me when I said it. I sometimes referred to a story or some drawings so that they had a picture of what the "switching around" looked like and to give them an additional intuitive or concrete referrent for explaining why what we were doing with the numbers makes sense. (cf. Davis, 1984, pp. 8-14.)

This explicit numerical representation for multiplying a two-digit number by a one-digit number extended easily to one digit times three digit multiplication and beyond: to one digit times four, five, and six digits. So, for example, 3,652 x 8 would be:

```
  3652    --->   3000 + 600 + 50 + 2
    x8
 24000    <---   8 x 3000
  4800    <---   8 x 600
   400    <---   8 x 50
+   16    <---   8 x 2
 29216    <---   24000 + 4800 + 400 + 16
```

After we did several of these together as a class and they did several individually with success, I showed them the "short-cut" notation in which the "answer" appears all on one line; that is, instead of writing:

```
  38
  x3
  24   <---  3 x 8
 +90   <---  3 x 30
 114
```

we usually write:

```
  38
  x3
 114
```

in the short cut, 90 does not get written down anywhere and 20 is symbolized by the little 2 up on top of the 3 (which is really 30). Making the transition from writing down the 2 and the 9 in the tens column *under* the line to writing down the little 2 in the tens column *above* the problem is difficult for several reasons. First of all, the children's knowledge of the multiplication combinations is still shaky and trying to remember and add before writing anything down can be confusing. Second, and more closely related to the issue of representing the operation that is occurring, is the difference between what that "little 2" up above the given problem means in a multiplication problem and what it means in an addition or subtraction problem. "Working across the columns," that is, doing something with the three in the units column to the three in the tens column, creates a very different "traffic pattern" than those used in either addition or subtraction where work in one column is completed entirely before moving on to the left. The "little number" at the top of the tens column now means something different than it did in addition or subtraction.

If, as Brown and Burton (1978) have claimed, children use visual cues to figure out how to proceed and whether they are proceeding correctly through a computation, it would not be surprising if errors like the following occurred at this point by analogy with the procedures for addition:

$$\text{A.}\quad \begin{array}{r} 38 \\ \underline{\times 3} \\ 94 \end{array} \qquad\qquad \text{B.}\quad \begin{array}{r} 38 \\ \underline{\times 3} \\ 154 \end{array}$$

In the first example, the child has used the three units to operate on the carried "two," perhaps because it is the uppermost number in the configuration, and then by multiplying 3 x 2, obtains six and adds three to get the nine tens. In the second example, the child adds the carried two and the three tens to get five and then multiplies by the three units to get 15 tens. Either of these errors may be "corrected" by having children do a great deal of practice until a new "traffic pattern" is established, but this sort of teaching never contains an explanation for why one multiplies the three units by the three tens and *then* adds the carried two to get the correct answer. The learner is not given a referent in the concrete or naive realm of knowing that can act as a "check" on the mechanical procedures.

After teaching my students the form:

$$\begin{array}{rll} 38 & & \\ \underline{\times 3} & & \\ 24 & \leftarrow\!\!- - & 3 \times 8 \\ \underline{+90} & \leftarrow\!\!- - & 3 \times 30 \\ 114 & & \end{array}$$

and then showing them how this could be "shortened" into the conventional form:

$$\begin{array}{r} 38 \\ \underline{\times 3} \\ 114 \end{array}$$

the most common error I encountered was:

$$\begin{array}{r} 38 \\ \underline{\times 3} \\ 9024 \end{array}$$

This is quite a different sort of error than the carrying confusions described above. This student was anxious to "get it all on one line" and missed an essential aspect of conventional symbolization. The *operation* is carried out correctly but the *answer* is incorrectly symbolized. What has happened here is that 3 x 30 became 9,000 because there was not enough "room" in the tens place of the answer for both the *9* of 90 (3 x 30) and the *2* of 24 (3 x 8). In order to write all the "answers" on one line, one thing you can do is move over to the left, that is, to put the 90 to the left of the 24. In order to *read* these two answers, however, we apply the framework of place value, assigning the name "thousands" to the fourth place over from the right. What gets written down as 90 thus becomes 9,000 when it is read. I could discern from my conversations with the children who made this "mistake" that they understood the principle of decomposition and were able to use it as a way to multiply, but they wrote the results of their calculations in a way that did not have the quantitative meaning they intended (cf. Greeno, Riley, & Gelman, 1984). They did not take account of the fact that in the hierarchy of mathematical symbols, the way place value is represented by lining digits up from right to left is fundamental. Hence, it *does* make a significant difference whether products are written one on top of the other like this: $\begin{matrix}90\\24\end{matrix}$ or next to the other like this: 9024. With or without a plus sign, the first configuration suggests the addition of 90 and 24, the second does not. It is important to note that the children who wrote down 9024 did not have the "misconception" that 3 x 30 = 9,000. They simply did not know how to symbolize *the sum of* 90 and 24. With a referent to quantity made explicit, learners are able to decide for themselves whether it makes sense to write 9024.

One of the questions that often gets raised at this point in math education circles is whether this particular error in symbolization would have developed if I had not taught my class to do these kinds of problems by putting the two products down separately rather than by "carrying" from the beginning. I don't know the answer to that, but I do know that I've seen children invent all sorts of similarly sensible but erroneous symbolizations in classrooms where they were *only* taught the steps in the conventional algorithm and directed to practice them over and over without any sense of why they work. Children who have a well-developed sense that numbers stand for quantities (and that multiplication is a matter of counting groups) know that counting three groups of 38 will not get you anywhere near 9024. But this sort of number sense takes a long time to develop, and its development is not helped by teaching computational procedures in isolation from the meaning of operations. The child who sees 9,024 on his or her paper after completing the two multiplications 3 x 8 and 3 x 30 faces a conflict in his or her own understanding. It does make some sense, and yet three groups of 8 and 3 groups of 30 would not add up to 9,024. It makes sense if they see 9024 as a row of answers to 3 x 30 and 3 x 8 with no spaces in between. But it *does not* make sense if they read 9024 as nine thousand twenty-four.

How is such a conflict to be resolved? We could *tell* such a child how to resolve it, but the telling should respect the fact that *both* ways of looking at 9024 do make *some* sense. If the *teacher* acknowledges that the answer 9024 does make some sort of sense, then the child who came up with it can see him or her self as a sense-maker rather than as a mistake-maker. But one cannot stop there and neglect to point out that this number can be interpreted in two different and conflicting ways. Because of the kind of subject mathematics is, the conflict between the two interpretations must be resolved. In the process, the child can be challenged to build a new "sense-making" schema that will turn up an answer without ambiguity. (Non-Euclidean geometries as well as other powerful mathematical ideas got invented this way!) Recognizing the conflict may even give learners a better appreciation for *why* we have adopted the practice of writing the little "2" (from 24) up on top of the 3 in the conventional algorithm instead of writing it underneath the line and thus give them a framework for remembering to do it.

Acknowledging the child's conflict in understanding rather than seeing this kind of situation as a right-wrong answer dichotomy is a way of looking at mathematical knowledge that is very unusual in classrooms and textbooks. It goes strongly against the grain of the way knowledge is usually organized for teaching and learning in schools, but it also has incredible power for giving the child a different picture of him or her self in relation to the process of learning mathematics. If children can learn to say "that does not make sense to me" and to tell their teachers what *does* make sense, and if teachers can respond by acknowledging that there is some sense to what the child says, it seems much less likely that so many children will "drop out" of the study of mathematics because they perceive it as something they could never really understand. It is true that some children can succeed for a while in the subject just by learning to *do* computation even if they don't understand what they are doing or why, but they probably won't get very far beyond high school algebra. They probably also will not see mathematical knowledge as useful or usable.

The use of numbers arranged in such a way as to make the multiplication process more explicit becomes, of course, more cumbersome as the numbers get larger. Yet it still seems worthwhile to illustrate the process that connects the conventional procedures used on numbers with what one is actually doing to the quantities involved. Numbers of more than one digit in both the multiplier and the multiplicand result in interesting alternatives for decomposition. If I were to multiply 352 x 82, for example, following the same process outlined above for one-digit by two-digit multiplications, I could get the following:

```
   352   --->   300 + 50 + 2    Step 1
   x82
 24600   <---   300 x 82        Step 2
  4100   <---    50 x 82        Step 3
+  164   <---     2 x 82        Step 4
 28864                          Step 5
```

Steps 2, 3, and 4 require another kind of decomposition:

```
300 x 82   -->   (300 x 80) + (300 x 2)
 50 x 82   -->   (50 x 80) + (50 x 2)
  2 x 82   -->   (2 x 80) + (2 x 2)
```

Now each multiplication involves numbers that are simple multiples of 10 or 100. One need only attend to the leading digits and be sure that the product includes the appropriate number of zeroes. Written in a somewhat more conventional way, the procedure would look like this:

352	
x82	
4	(2 x 2)
100	(2 x 50)
600	(2 x 300)
160	(80 x 2)
4000	(80 x 50)
+24000	(80 x 300)
28864	

My students call this the "no-carry way" to do multiplication and many of them favor it over the "carry way," which they tend to find confusing because it is difficult to keep track of all the "little numbers" that accumulate at the top of their work when they are multiplying by a number with two or more digits.

For several days, I presented my class with a multidigit multiplication at the beginning of class which they could do in either "the carry way" or "the no-carry way." Several did both and I encouraged them to compare their work with classmates (which resulted in the correction of many times-tables errors!) and I put both "solutions" up on the board. This activity evolved into whole class discussions of the two methods and how they were or were not related. For example, some students observed that

$$\begin{array}{r} 96 \\ \underline{\times 84} \end{array} \quad \text{and} \quad \begin{array}{r} 84 \\ \underline{\times 96} \end{array}$$

resulted in the same set of partial products if they did them the no-carry way and *different partial products* if they did them the carry way. Some students were eager and able to explain why, whereas others wondered "what would happen if" they tried mixing the digits up to make other problems like:

$$\begin{array}{r} 69 \\ \underline{\times 48} \end{array} \quad \text{or} \quad \begin{array}{r} 86 \\ \underline{\times 49} \end{array}$$

A lively discussion ensued about why these combinations turned up *different final answers* than

```
 96      84
x84     x96
```

In these discussions, students used principled knowledge to explain what they were seeing. They were able to talk meaningfully about place value and order of operations, even though they did not use technical terms to do so. I took their experimentations and arguments as evidence that they had come to see mathematics as more than a set of procedures for finding answers.

Conclusion

Throughout the course of all of the lessons described here, there were several opportunities to observe students demonstrating mathematical understanding. Greeno, Riley, and Gelman (1984) have argued that the case for understanding is strongest if a child is required to generate new procedures and the procedures are consistent with mathematical principles. My students had the opportunity to do that with stories, pictures, coins, and numbers and on several occasions they invented procedures that indicated an awareness of principles even though the principles themselves were never articulated.

In their stories and pictures, they decomposed quantities into groups of groups in both conventional and unconventional ways and after multiplying with groups, added partial products together appropriately. In solving the various kinds of coin problems, they created multiple decompositions and recompositions, applying the distributive law correctly to calculate the total amounts of money indicated by their groups of coins. In generating the explicit numerical products indicated by the place-value decomposition of multidigit multipliers and multicands, they were able to move the partial products around and recombine them in ways that indicated the appropriate application of the commutative and associative laws.

Moving around among the concrete, naive, and computational realms and making explicit connections among them, they indicated their "metaknowledge" of the process of multiplication by producing performances that were systematically consistent with the principles that underlie this operation in a wide variety of different contexts. Leinhardt (1985) observed in describing the relationship among different kinds of knowledge of subtraction, "as competence increases, naive knowledge and principled knowledge should converge and form a basis upon which unique generative solutions can be formed and into which computation procedures can be nested and legitimized."

The lessons I have described here provide some evidence that this process can occur within the context of classroom instruction. It is possible for children to learn concrete and symbolic procedures to arrive at answers without having understanding. By putting the emphasis in my teaching on the invention of procedures rather than on the production of answers, this outcome was less likely to occur.

Very few mathematics teachers or textbooks go through the process of representing multiplication to children at the various levels I have described. Most often, they proceed directly from concretely representing single-digit multiplication like 7 x 6 as seven groups of six objects or seven rows of six dots on the board to teaching children the computational procedures for multiplying 82 x 152 using the conventional algorithm. Sometimes the grouping process is pictured in textbooks, but very little is said about "groups" once children are multiplying with two numbers that are both bigger than 10. *After* they have learned to use the conventional procedures to do these multiplications, students are usually asked to "apply" their skills to solving word problems like: "If baseball shirts cost $8.95 each, what would be the total cost of 73 shirts?"

But doing such problems does not necessarily indicate an understanding of the algorithm. Children can learn to recognize problems like this as members of a set of "word problems" to which the procedure for doing multiplication should be applied (usually because they come at the end of the chapter on multiplication), but they probably could not tell the teacher *why* multiplying gets them the correct answer. If they understood that such a problem was asking them to find the amount of money in 73 groups of $8.95, they would probably also be able to answer more practical questions like: "If there were 900 dollars in the school budget for 73 baseball shirts, would we have enough money to buy them?" Understanding multiplication as a process of counting by grouping has at least as much practical value as being able to compute the exact answer to 73 x 895 precisely using the conventional algorithm; a child with only procedural knowledge of multiplication would be unlikely even to recognize the school budget question as a "multiplication problem."

More important than its potential practical value, however, is the possibility that such understanding could give children the confidence to make the transition from figuring out why arithmetic makes sense to figuring out that they can do and understand higher mathematics. When mathematics begins to be nothing more than a set of rules for manipulating symbols that "magically" turn up the right answer, children stop seeing themselves as capable of figuring out why one procedure for moving numbers around makes any more sense than another. Once this happens the only successful learning strategy available to them is to *remember* what the teacher or the book said you were supposed to do. What gets into their memory then, is a list of rules that are not tied together in any way that would help them to use the right rule on the right occasion.

The more mathematics one is taught, the longer the list of rules to remember grows. It would not be surprising if there were a point where it just gets to be too much. For some children, learning the conventional procedure for the multiplication of large numbers is the beginning of the end; others can remember everything up to the process for doing long division. Some make it through remembering the rules for the addition and subtraction of fractions, but the "invert and multiply" rule for dividing fractions is the last straw. "Turning numbers upside down???" "How could that possibly make sense?" It is not necessary for children to be able to explain every procedure that is learned or used; what I am arguing for here is that students learn that such explanations are possible.

Usually by the time they get through high school, many students (including some who go on to become teachers) are confirmed in the belief that mathematical rules generally *do not* make sense and that mathematics is an esoteric body of memorized knowledge enjoyed by a small but strange minority. Because there is hardly time in most teacher education programs to seriously challenge these beliefs, many children are taught mathematics by teachers who see mathematical knowledge as something much more limited than the complex relationships among the four kinds of knowing described above. The mathematics textbook is treated by these teachers like a cryptic document: Teacher and student together engage in puzzling out "what IT wants you to do."

The dichotomy between familiar arithmetic computation and the principled mathematics that mathematicians know and use becomes firmly set in this process of student and teacher alienation from the subject matter. If, however, teachers could be enabled to see the computational processes that are familiar to them in relation to more sophisticated mathematical principles and in relation to their own naive ways of understanding mathematical structures, they might be more likely to become the allies rather than the enemies of mathematicians in the campaign to make school mathematics more principled. They could use what they (and many children's parents) recognize as "mathematics" to teach the concepts that students will need to know if they are going to continue to be successful in the subject beyond arithmetical procedures.

Fourth graders can do and think mathematically. They have the capacity to gather information, to organize it strategically, to generate and test hypotheses, and to produce and evaluate solutions. They can talk about what they are thinking, they can listen to and appreciate another student's procedural way of understanding something, and they can invent problem-solving procedures that are both useful and sensible. What sort of help do children need from adults in order to be able to do these things and to be confident in their ability to do them? They need to be asked questions whose answers can be "figured out" not by relying on memorized rules for moving numbers around but by thinking about what numbers and symbols mean. They need to be treated like sense-makers rather than like rememberers and forgetters. They need to see connections between what they are supposed to be learning in school and things they care about understanding outside of school, and these connections need to be related to the substance of what they are supposed to be learning. They need to learn to do computation competently and efficiently without losing sight of the meaning of what they are doing and its relation to solving real problems. The lessons I have taught and described here suggest that it is possible to do these things in conventional school classrooms.

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