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CONJUNCTIVE AND DISJUNCTIVE ITEM RESPONSE FUNCTIONS

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Abstract

Given that the examinee knows the answer to item i if and only if he knows the answer to both item g and item h, a 'conjunctive' item response model is found such that items g, h, and i all have the same mathematical form of response function. Since such items may occur in practice, it is desirable that item response models satisfy this condition. For models with two parameters per item, the most general functional form satisfying this condition is found. A third, 'guessing' parameter may be added. The corresponding disjunctive model is also derived.

Conjunctive and Disjunctive Item Response Functions*

Consider two free-response spelling items that ask the examinee to spell, respectively, the word wife and the word house. Now consider a free-response spelling item that asks the examinee to spell the word housewife. Assume that an examinee will answer the third item correctly if and only if the examinee would answer both of the first two items correctly. If so, then

$$P_3(\theta) = P_1(\theta)P_2(\theta) \tag{1}$$

where θ is the ability of the examinee and $P_i(\theta)$ is the item response function for (probability of giving a correct answer to) item i .

Kristof (1968) pointed out that it is highly desirable that P_1 , P_2 , and P_3 should all have the same mathematical form, but that this condition is not satisfied by the usual logistic or normal ogive models in item response theory (IRT). He derived one-parameter families of item response function, $P(\theta)$, satisfying (1), stating that this result "is attainable if and only if all item [response] functions are powers of each other.... The model

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rules out normal ogives as well as logistic functions as possible item [response] functions...."

More recently Yen (1984) discussed certain difficulties encountered in vertical equating using a logistic model. She "hypothesized that [these difficulties] occur because the items increase in complexity as they increase in difficulty." Such problems might be avoided if the model $P(\theta)$ satisfying (1) were used.

In Kristof's study, each item is characterized by a single item parameter. Here, Kristof's result is generalized, using a different approach, so as to allow items to differ in each of two parameters.

Necessary Condition for Solving the Functional Equation

Suppose the item response function for any item has the form $P(a^*, b^*, \theta)$: it is a function of θ characterized by two item parameters, a^* and b^*. Denote the values of a^* and b^* for item 1 by a and b ; for item 2, by A and B ; for item 3 by α and β. Then (1) becomes

$$P(a,b,\theta)P(A,B,\theta) \equiv P(\alpha,\beta,\theta) \quad . \tag{2}$$

If such an identity is to hold for all θ , a , b , A , B , α , and β , it is necessary that the item parameters α and β

be functions of a , b , A , and B (item parameters by definition do not depend on θ):

$$\alpha \equiv \alpha(a,b,A,B) \quad \text{and} \quad \beta \equiv \beta(a,b,A,B) \quad . \tag{3}$$

Define $L(a^*,b^*,\theta) \equiv \log P(a^*,b^*,\theta)$. From (2)

$$L(a,b,\theta) + L(A,B,\theta) \equiv L(\alpha,\beta,\theta) \quad . \tag{4}$$

This is a functional equation in five variables and three unknown functions, $L(\ )$, $\alpha(\ )$, and $\beta(\ )$.

Take the derivative of this with respect to a , to b , and to A :

$$L^a(a,b,\theta) \equiv L^\alpha(\alpha,\beta,\theta)\alpha^a + L^\beta(\alpha,\beta,\theta)\beta^a \quad , \tag{5}$$

$$L^b(a,b,\theta) \equiv L^\alpha(\alpha,\beta,\theta)\alpha^b + L^\beta(\alpha,\beta,\theta)\beta^b \quad ,$$

$$L^A(A,B,\theta) \equiv L^\alpha(\alpha,\beta,\theta)\alpha^A + L^\beta(\alpha,\beta,\theta)\beta^A \quad ,$$

where each superscript denotes a derivative: for example, $\alpha^a \equiv \partial\alpha(a,b,A,B)/\partial a$. Continuity and differentiability of functions are assumed as needed here and in the following derivations.

Given any fixed set of values of the three variables α , β , and θ , both $L^\alpha(\alpha,\beta,\theta)$ and $L^\beta(\alpha,\beta,\theta)$ are fixed. Equations (5) are then three nonhomogeneous linear equations relating L^α

and L^{β} . These three equations will be inconsistent unless the augmented matrix

$$\begin{bmatrix} \alpha^{a}(a,b,A,B) & \beta^{a}(a,b,A,B) & L^{a}(a,b,\theta) \\ \alpha^{b}(a,b,A,B) & \beta^{b}(a,b,A,B) & L^{b}(a,b,\theta) \\ \alpha^{A}(a,b,A,B) & \beta^{A}(a,b,A,B) & L^{A}(a,b,\theta) \end{bmatrix}$$

is singular. Therefore the corresponding determinant must vanish:

$$(\alpha^{a}\beta^{b} - \alpha^{b}\beta^{a})L^{A} + (\alpha^{b}\beta^{A} - \alpha^{A}\beta^{b})L^{a} + (\alpha^{A}\beta^{a} - \alpha^{a}\beta^{A})L^{b} \equiv 0 \quad .$$

Rewrite this as

$$h(a,b,A,B)L^{A} \equiv f(a,b,A,B)L^{a} + g(a,b,A,B)L^{b} \quad . \qquad (6)$$

This result accomplishes the elimination L^{α} and L^{β} from immediate consideration.

Given any fixed set of values of a , b , and θ , both L^{a} and L^{b} are fixed. Equation (6) expresses a linear relationship between L^{a} and L^{b} . This relationship must hold as (A,B) takes on different values (A_1,B_1), (A_2,B_2), (A_3,B_3), ... Thus (6) represents an (infinite) set of linear nonhomogeneous equations relating L^{a} and L^{b} . These equations will be inconsistent if the augmented matrix,

$$\begin{bmatrix} f(a,b,A_1,B_1) & g(a,b,A_1,B_1) & h(a,b,A_1,B_1)L^A(A_1,B_1,\theta) \\ f(a,b,A_2,B_2) & g(a,b,A_2,B_2) & h(a,b,A_2,B_2)L^A(A_2,B_2,\theta) \\ f(a,b,A_3,B_3) & g(a,b,A_3,B_3) & h(a,b,A_3,B_3)L^A(A_3,B_3,\theta) \\ \vdots & \vdots & \vdots \end{bmatrix}$$

is of rank 3. Thus (using an obvious notation) the determinant

$$(f_1g_2 - f_2g_1)h_3L_3^A + (f_3g_1 - f_1g_3)h_2L_2^A + (f_2g_3 - f_3g_2)h_1L_1^A \equiv 0 \quad . \tag{7}$$

This result accomplishes the elimination of L^a and L^b from immediate consideration.

Given any fixed set of values of a, b, A_1, B_1, A_2, B_2, A_3, and B_3, the f's, g's, and h's in (7) are fixed. Equation (7) must still hold for $\theta = \theta_1, \theta_2, \theta_3, \ldots$. Thus (7) represents an (infinite) set of homogeneous linear equations in the three fixed quantities $(f_1g_2 - f_2g_1)h_3$, $(f_3g_1 - f_1g_3)h_2$, and $(f_2g_3 - f_3g_2)h_1$. Such equations can be consistent only if the matrix

$$\begin{bmatrix} L^A(A_1,B_1,\theta_1) & L^A(A_2,B_2,\theta_1) & L^A(A_3,B_3,\theta_1) \\ L^A(A_1,B_1,\theta_2) & L^A(A_2,B_2,\theta_2) & L^A(A_3,B_3,\theta_2) \\ L^A(A_1,B_1,\theta_3) & L^A(A_2,B_2,\theta_3) & L^A(A_3,B_3,\theta_3) \\ \vdots & \vdots & \vdots \end{bmatrix}$$

is of less than full rank. If the equations are consistent, the third row of the matrix must be linearly dependent on the first two rows: in other words $L^A(A,B,\theta_3)$ must be a linear function of $L^A(A,B,\theta_2)$ and $L^A(A,B,\theta_1)$.

This conclusion is a contradiction, since θ_3 is arbitrary and cannot be expressed as a function of θ_1 and θ_2 . Thus the premise that the equations are 'consistent' must be false. This proves that when θ varies, (7) represents a set of 'inconsistent' equations, according to common terminology. This does not mean, however than (7) is invalid: the 'inconsistent' homogeneous linear equations will be satisfied provided

$$(f_1g_2 - f_2g_1)h_3 \equiv (f_3g_1 - f_1g_3)h_2 \equiv (f_2g_3 - f_3g_2)h_1 \equiv 0 \quad . \quad (8)$$

This proves that if differentiable functions satisfying (4) exist, then the f , g , and h must satisfy (8). According to (8) and (7), either

(Case 1) $h(a,b,A,B) \equiv 0$, or

(Case 2) $h(a,b,A,B)$ is nonzero but $g(a,b,A,B) \equiv 0$ and/or $f(a,b,A,B) \equiv 0$, or

(Case 3) f/g is independent of A,B :

$$\frac{f(a,b,A_1,B_1)}{g(a,b,A_1,B_1)} \equiv \frac{f(a,b,A_2,B_2)}{g(a,b,A_2,B_2)} \equiv F_1(a,b) \quad , \qquad (9)$$

where $F_1(a,b)$ denotes a (more or less) arbitrary function of a and b only.

If Case 1 were to hold, then, from the definition of h in (6),

$$\begin{vmatrix} \alpha^a & \beta^a \\ \alpha^b & \beta^b \end{vmatrix} \equiv 0 \quad .$$

But for fixed A , B , this is the Jacobian of the transformation $\alpha \equiv \alpha(a,b)$, $\beta \equiv \beta(a,b)$. When the Jacobian is zero, α and β do not vary independently. In the present problem, this would mean that for fixed A , B , $L(\alpha,\beta,\theta)$ is only a one-parameter family, and thus that $P(\alpha,\beta,\theta)$ is only a one-parameter family of item response functions. This is contrary to the original requirements, so Case 1 is inappropriate.

Case 2 will be treated later. The remaining possibility is given by (9). Substitute (9) into (6), obtaining

$$h(a,b,A,B)L^A(A,B,\theta) \equiv g(a,b,A,B)[F_1(a,b)L^a(a,b,\theta) + L^b(a,b,\theta)] \quad .$$

Denote the quantity in brackets by $F_2(a,b,\theta)$. Take the logarithm of the absolute value of both sides:

$$\log|h| + \ell_A \equiv \log|g| + f_2 \qquad (10)$$

where $\ell_A \equiv \log|L^A|$ and $f_2 \equiv \log|F_2|$. Differentiate (10) on θ :

$$\ell_A^\theta(A,B,\theta) \equiv f_2^\theta(a,b,\theta) \quad . \qquad (11)$$

Now the left side of (11) is not a function of a or b , the right side is not a function of A or B . It follows that each side must be independent of all four variables a , b , A , B . Thus

$$\ell_A^\theta(A,B,\theta) \equiv \phi_1(\theta) \quad , \qquad (12)$$

a (more or less) arbitrary function of θ only.

Integrate (12) on θ :

$$\ell_A(A,B,\theta) \equiv \phi_2(\theta) + F_3(A,B) \quad ,$$

where $\phi_2(\theta) \equiv \int\phi_1(\theta)d\theta$ and $F_3(A,B)$ is the 'constant of integration.' Exponentiate, obtaining

$$L^A(A,B,\theta) \equiv \exp[\phi_2(\theta) + F_3(A,B)] \quad ,$$

$$L^A(A,B,\theta) \equiv F_4(A,B)\phi_3(\theta)$$

where F_4 and ϕ_3 are (more or less) arbitrary nonnegative functions. Integrate on A to find, finally

$$L(A,B,\theta) \equiv F_5(A,B)\phi_3(\theta) + G_1(B,\theta) \quad , \qquad (13)$$

where $F_5(A,B) \equiv \int F_4(A,B)\, dA$ and $G_1(B,\theta)$ is the constant of integration, both (more or less) arbitrary functions.

The only remaining case, Case 2, also leads to (13), as will now be shown. If both $g(a,b,A,B)$ and $g(a,b,A,B)$ were zero in Case 2, (6) would become

$$h(a,b,A,B)L^A(A,B,\theta) \equiv 0 \quad .$$

Since h( ) is nonzero in Case 2, this would imply $L^A = 0$. But $L^A \equiv 0$ means that the log of the item repsonse function does not vary with A , a special limiting situation of no general interest here. The only interesting alternatives under Case 2 is that either f = 0 or else g = 0 , but not both.

When $g \equiv 0$, $h \neq 0$, and $f \neq 0$, then (16) becomes

$$hL^A \equiv fL^a \quad .$$

Take the logarithm of the absolute value of both sides and then differentiate on θ to find

$$\ell_A^\theta \equiv \ell_a^\theta \quad .$$

This is equivalent to (11). Thus Case 2 with $g \equiv 0$ also leads to (13).

The remaining possibility, that $f \equiv 0$, $h \neq 0$, and $g \neq 0$ (in Case 2) also leads to (13), except that a and b , also A and B , and also α and β are interchanged. Since the original problem is invariant under this interchange, (13) will still apply, given an appropriate initial choice of parameter assignment. Thus with suitable parameter assignment, (13) determines a specific form of L(, ,) that is necessary (but not yet sufficient) for satisfying (4).

Necessary and Sufficient Condition for a Solution

Using (13), dropping numerical subscripts and rearranging, (4) can now be written

$$[F_5(a,b) + F_5(A,B) - F_5(\alpha,\beta)]\phi_3(\theta) \equiv G_1(\beta,\theta) - G_1(b,\theta) - G_1(B,\theta) \quad . \qquad (14)$$

Now, $\phi_3(\theta)$ is not identically zero, since by (13) this would make make $L(A,B,\theta)$ and hence $L(\alpha,\beta,\theta)$ independent of A . Divide by (14) by $\phi_3(\theta)$:

$$F_5(a,b) + F_5(A,B) - F_5(\alpha,\beta) \equiv G_2(\beta,\theta) - G_2(b,\theta) - G_2(B,\theta) \quad , \qquad (15)$$

where $G_2(\ ) \equiv G_1(\ )/\phi_3(\theta)$. Differentiate on θ :

$$G_2^{\theta}(\beta,\theta) \equiv G_2^{\theta}(b,\theta) + G_2^{\theta}(B,\theta) \quad . \qquad (16)$$

Since the right side is independent of a and A , $\beta(a,b,A,B)$ must be also. Hereafter the notation $\beta(b,B)$ will be used.

Differentiate (16) on b to find:

$$G_2^{\theta\beta}(\beta,\theta)\beta^{b} \equiv G_2^{\theta b}(b,\theta) \quad .$$

Take the logarithm of the absolute value of both sides:

$$\log|\beta^{b}| + g_{\theta\beta}(\beta,\theta) \equiv g_{\theta b}(b,\theta) \quad .$$

where $g_{\theta b} \equiv \log|G_2^{\theta b}|$, and $g_{\theta\beta}$ likewise. Differentiate on θ :

$$g^{\theta}_{\theta\beta}(\beta,\theta) \equiv g^{\theta}_{\theta b}(b,\theta) \quad .$$

Repeat the foregoing operations on (15), switching the roles of b and B to find

$$g^{\theta}_{\theta\beta}(\beta,\theta) \equiv g^{\theta}_{\theta B}(B,\theta) \quad .$$

Eliminate $g^{\theta}_{\theta\beta}$ from the last two equations, obtaining

$$g^{\theta}_{\theta b}(b,\theta) \equiv g^{\theta}_{\theta B}(B,\theta) \quad .$$

Since the left side is not a function of B and the right side is not a function of b , it is seen that each side is a function of θ only; so

$$g_{\theta b}^{\theta}(b,\theta) \equiv \psi_1(\theta) \quad ,$$

say.

Integrate this on θ :

$$g_{\theta b}(b,\theta) \equiv \psi_2(\theta) + k_1(b) \quad ,$$

where $\psi_2(\theta) \equiv \int\psi_1(\theta)d\theta$ and $k_1(b)$ is the constant of integration. Exponentiate:

$$|G_2^{\theta b}(b,\theta)| \equiv \exp[\psi_2(\theta) + k_1(b)]$$

$$G_2^{\theta b}(b,\theta) \equiv k_2(b)\psi_3(\theta) \quad .$$

Integrate on b and then on θ :

$$G_2^{\theta}(b,\theta) \equiv k_3(b)\psi_3(\theta) + \chi_1(\theta) \quad ,$$

$$G_2(b,\theta) \equiv k_3(b)\psi_4(\theta) + \chi_2(\theta) + k_4(b) \quad .$$

Thus, finally, by the relation of G_2 to G_1 ,

$$G_1(b,\theta) \equiv k_3(b)\psi_5(\theta) + k_4(b)\phi_3(\theta) + \psi_6(\theta) \quad . \qquad (17)$$

From (13) and (17),

$$L(A,B,\theta) \equiv F_5(A,B)\phi_3(\theta) + k_3(B)\psi_5(\theta) + k_4(B)\phi_3(\theta) + \psi_6(\theta)$$

$$\equiv F_6(A,B)\phi_4(\theta) + k_3(B)\psi_5(\theta) + \psi_6(\theta) \quad .$$

From this and (4)

$$[F_6(a,b) + F_6(A,B)]\phi_4(\theta) + [k_3(b) + k_3(B)]\psi_5(\theta)$$

$$\equiv F_6(\alpha,\beta)\phi_4(\theta) + k_3(\beta)\psi_5(\theta) - \psi_6(\theta) \quad .$$

This shows that $\psi_6(\theta)$ is a linear function of ϕ_4 and $\psi_5(\theta)$:

$$\psi_6(\theta) \equiv C\phi_4(\theta) + K\psi_5(\theta) \quad ,$$

where C and K are constants (since $\psi_6(\theta)$ does not depend on a , b , A , B). Substitute this into each of the two preceding equations, replace $\phi_4(\theta)$ by $\Phi(\theta)$, $\psi_5(\theta)$ by $\Psi(\theta)$, and $k_3(\ )$ by $G(\ )$, dr ,ping numerical subscripts to obtain finally

$$L(A,B,\theta) \equiv F(A,B)\Phi(\theta) + G(B)\Psi(\theta) \quad , \tag{18}$$

$$[F(a,b) + F(A,B)]\Phi(\theta) + [G(b) + G(B)]\Psi(\theta)$$

$$\equiv F(\alpha,\beta)\Phi(\theta) + G(\beta)\Psi(\theta) \quad . \tag{19}$$

For equality to hold in (19), it is necessary that $\beta(b,B) \equiv G^{-1}[G(b) + G(B)]$ where $G^{-1}(\ )$ is the inverse function of $G(\ )$. Also that

$$\alpha(a,b,A,B) \equiv F^{-1}_{\beta(b,B)}[F(a,b) + F(A,B)]$$

where $F^{-1}_{\beta(b,B)}$ is the inverse function defined for each given b by $F^{-1}_{b}[F(a,b)] \equiv a$.

The solution to (2) is found by exponentiating (18):

$$P(A,B,\theta) \equiv \exp[F(A,B)\Phi(\theta) + G(B)\Psi(\theta)]$$

$$\equiv [f(A,B)]^{\Phi(\theta)}[g(B)]^{\Psi(\theta)}$$

$$\equiv [\phi(\theta)]^{F(A,B)}[\psi(\theta)]^{G(B)} \quad , \tag{20}$$

where $\Phi(\theta) \equiv \log \phi(\theta)$, $\Psi(\theta) \equiv \log \psi(\theta)$, $F(A,B) \equiv \log f(A,B)$, and $G(B) \equiv \log g(B)$.

The Conjunctive Item Response Function

A reparameterization of the IRT model will simplify (20) for present purposes. Define new item parameters by $b' \equiv G(B)$, $a' \equiv F(A,B)$. The general conjunctive item response function (20) is now simply

$$P(a',b',\theta) \equiv [\phi(\theta)]^{a'}[\psi(\theta)]^{b'} \quad . \qquad (21)$$

Without loss of generality, since $\phi(\theta)$ and $\psi(\theta)$ remain to be chosen, take $a' > 0$ and $b' > 0$. To be an item response function (IRF), it is necessary that $P(a',b',\theta)$ be a monotonic increasing function of θ and that $P(a',b',-\infty) = 0$, $P(a',b',\infty) = 1$. If it is possible to have $a' \to 0$, then $\psi(\theta)$ must satisfy corresponding conditions. If it is possible to have $b' \to 0$, then $\phi(\theta)$ must satisfy corresponding condition. It will be assumed hereafter that both $\psi(\theta)$ and $\phi(\theta)$ satisfy such conditions.

For convenience, the primes in (21) will now be dropped. If $\psi(\theta) \propto \phi(\theta)$, it would follow that $\psi(\theta) \equiv \phi(\theta)$, since otherwise $\psi(\infty) = \phi(\infty) = 1$ could not hold. If $\psi(\theta) \equiv \phi(\theta)$, then $P(a,b,\theta) \equiv [\phi(\theta)]^{a+b}$. In this case, the reparameterization $a^{*} \equiv a + b$ would reduce $P(\ )$ to a one-parameter family. Thus it is essential that $\psi(\theta)$ and $\phi(\theta)$ should not be proportional.

A plausible model is obtained by choosing $\psi(\theta)$ and $\phi(\theta)$ to be some familiar two-parameter IRF. If the two-parameter logistic function is used, for example, the IRF for item i is

$$P(a_i,b_i,\theta) = (1 + e^{-\sigma\theta-\mu})^{-a_i}(1 + e^{-\theta})^{-b_i} \tag{22}$$

where σ and μ are arbitrary constants. These constants must be the same for all items in a given test, but they can vary from one test to another. Ideally σ and μ should be constant for all items of a single type: they are test or item-type parameters, not item parameters.

Up to this point, the argument has dealt with the probability that a given examinee knows the correct answer to an item or to a component part of an item. If the final item is presented in multiple-choice form, however, the examinee may answer correctly simply by random guessing. To deal with this situation, a 'guessing' parameter c_i can plausibly be introduced into (22), so that now

$$P(a_i,b_i,c_i,\theta) = c_i + (1 - c_i)(1 + e^{-\sigma\theta-\mu})^{-a_i}(1 + e^{-\theta})^{-b_i} \quad . \tag{23}$$

This IRF does not satisfy the functional equation (2). Nevertheless, (23) may be quite appropriate; for example, when guessing does not contribute to the probability that an examinee knows the correct answer to the component parts of the item.

The Disjunctive Item Response Function

The original reasoning dealt with situations where an examinee would know the correct answer to the actual item only if he knew the correct answer to two separate subproblems in the actual item. One can also imagine situations where the examinee will know the answer to the actual item whenever he knows the answer to either one of two separate subproblems: In other words, there are two independent routes to knowing the answer to the actual item. This is the disjunctive case.

The corresponding functional equation is the same as (2) except that $Q(a,b,\theta) \equiv 1 - P(a,b,\theta)$ is substituted for $P(a,b,\theta)$:

$$Q(a,b,\theta)Q(A,B,\theta) \equiv Q(\alpha,\beta,\theta) \quad . \tag{24}$$

The corresponding IRF is thus

$$Q(a,b,\theta) \equiv [1 - \phi(\theta)]^{a_1}[1 - \psi(\theta)]^{b_1} \quad , \tag{25}$$

where $\phi(\theta)$ and $\psi(\theta)$ satisfy the same conditions as before. In the logistic case with guessing,

$$Q(a_i,b_i,c_i,\theta) \equiv (1 - c_i)(1 + e^{\sigma\theta+\mu})^{-a_i}(1 + e^{\theta})^{-b_i} \quad . \tag{26}$$

Practical Implementation and Conclusion

The conjunctive (disjunctive) IRF's have one convenient feature: When there is no guessing ($c_i = 0$) , the probability of answering all of a set of m items correctly (incorrectly) has a much simpler mathematical form than is usually the case. For conjunctive items,

$$\text{Prob}(u_1 = 1,\ u_2 = 1,\ldots,u_m = 1) = [\phi(\theta)]^{\sum^m a_i}[\psi(\theta)]^{\sum^m b_i} \quad . \tag{27}$$

The parameters a_i and b_i of conjunctive or disjunctive items are not at all readily interpreted in terms of item difficulty and item discriminating power. It is hard to find even a complicated function, let alone a simple function, of a_i and and b_i that can be comfortably interpreted as a satisfactory measure of item difficulty or of item discriminating power.

Figure 1 shows six plots of IRF's from (22) illustrating various degrees of skewness. One can, of course, obtain a pure logistic curve from (22) by setting either a_i or b_i equal to zero. It is also possible to obtain double ogives with three points of inflexion (not illustrated here).

A new computer program has been written for simultaneous maximum likelihood estimation of the item parameters a_i and b_i, the test parameters σ and μ, and the ability parameters θ for the conjunctive model (23). The new program uses fixed c_i values determined in advance by a run of the data on the computer program LOGIST (Wingersky, 1983). Although the estimates converge in any one computer run from a given set of trial values, the result is apparently not yet useful because from different sets of trial values the computer reaches different local maxima corresponding to different assignments of zero values to a_i or to b_i for different subsets of items.

The fit of the model to the responses of a group of examinees could probably be much improved by substituting (26) for (23) whenever an item is estimated to have a zero value of a_i or b_i, thus assuming that such items are disjunctive items. Subsequent iterations of the estimation process might then lead to a

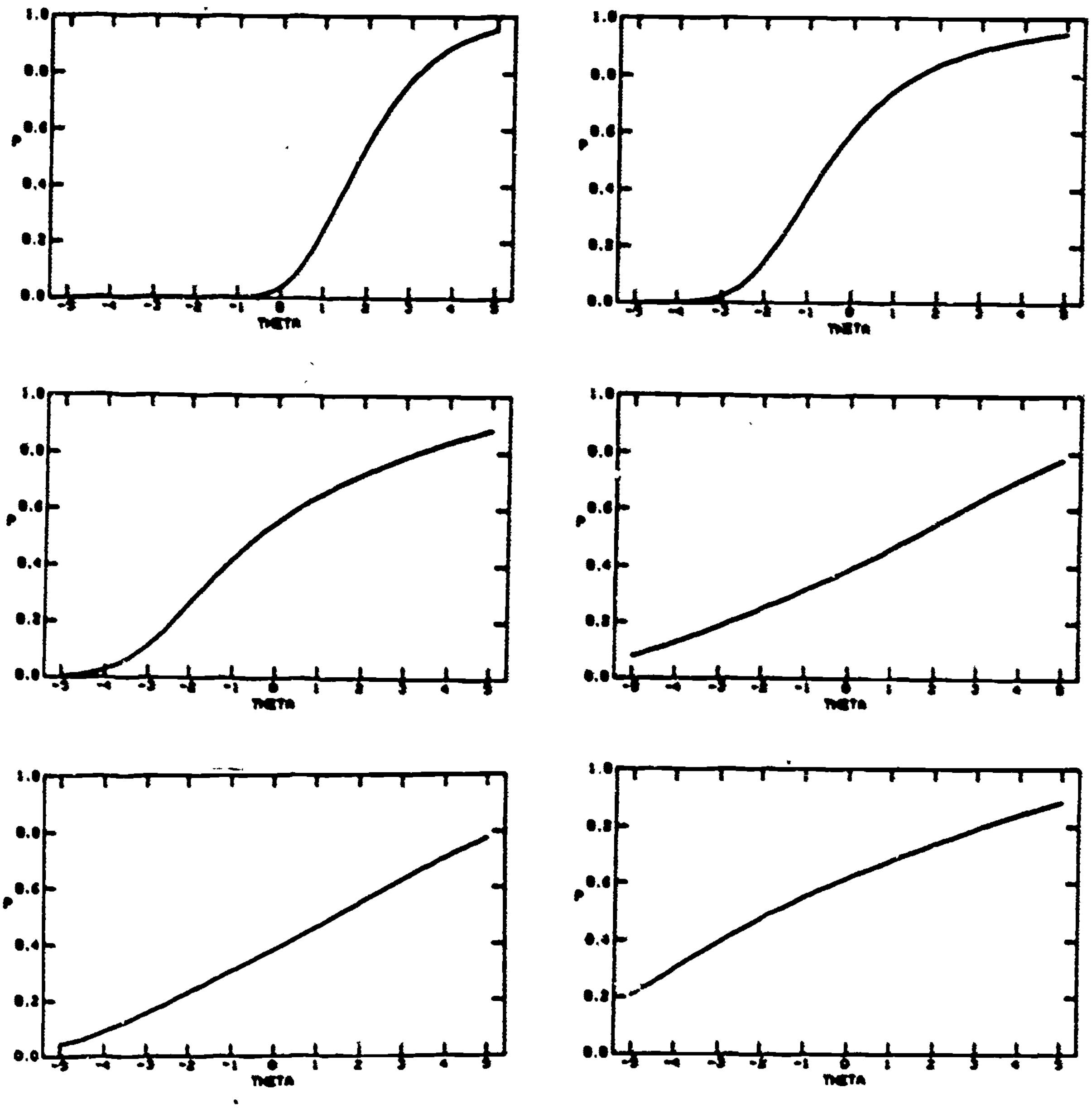

Figure 1. Illustrative conjunctive item response functions with $c = 0$ (22).

reassignment of some of these items to the conjunctive model while other items might be transferred to the disjunctive model. Ultimately, such a process might successfully classify all items as conjunctive or disjunctive in such a way as to find a global maximum of the likelihood function and an optimal fit of the models to the data. It is not at all clear, however, how much computer time such a process might require.

It is possible that the conjunctive and disjunctive models, based as they are on relevant psychological considerations, may provide a better fit to real data than the usual logistic or normal ogive models.

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