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ABSTRACT

Reports from representatives in 12 countries formed the basis of this review of the role and use of applications in mathematics in elementary and secondary schools. The editor provides an overview, drawing in part upon the various reports and adding additional structure to the scope of applications in the curricula. He describes the intent of the report, defines applications, presents the results of a survey to which each country responded, discusses why mathematics teachers avoid applications, and suggests ways we can help teachers use applications. Then follow the reports from Australia, Belgium, Brazil, Great Britain, Canada (Ontario), West Germany, Finland, Israel, New Zealand, Norway, Tanzania, and the United States. Generally, in spite of good intentions and substantial efforts, applications seem to play a small part in the curriculum. Reasons for this are identified and discussed, and current initiatives aimed at changing the situation are described. (MNS)

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International Association for the Evaluation of Educational Achievement (IEA)

edited by

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An International Review of APPLICATIONS IN SCHOOL MATHEMATICS — the elusive El Dorado

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A NOTE OF THANKS . . .

We want to thank each of the persons who prepared the national reports for this publication. Their concise but explicit review of activities in their countries on the role of applications in school mathematics programs help us to attain a better understanding of this topic in relation to curricular change. We appreciate the insights presented by these authors.

We also thank Hugh Burkhardt, without whom the review would not have reached the publication stage. The questionnaire he developed helped to form the reports from the various countries, and his editorial work assured a coordinated document. We appreciate the contribution he has made to mathematics education.

Kenneth J. Travers
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An International Review of Applications in School Mathematics

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OVERVIEW **Hugh Burkhardt**

Applications Worldwide — What We Aimed to Learn

Everybody is in favour of applications as an element of the school curriculum. Applications show that mathematics can be useful, provide practice-with-a-purpose of mathematical techniques at mastery level, and provide concrete illustrations of mathematical principles that help to reinforce conceptual understanding.

How are these excellent intentions realised in practice? There is a fairly general uneasy awareness that things are not nearly as good as we should like, and that we are faced with a difficulty that has resisted energetic and ingenious efforts in the past. In such a situation it is worth sharing experiences — ideas, successes, and failures. As the mathematics curriculum differs so much in approach and emphasis from one country to another, it seemed worth trying to compile a review of what happens in various places. The Second International Mathematics Study of I.E.A. provides a suitable framework for this. This review is a consequence.

It was compiled in the following way. Correspondents in countries linked to I.E.A., chosen for their knowledge of applications, and of the local scene in mathematical education, were invited to present a picture of the position of applications in school mathematics in their country, containing three different elements:

(1) A response to a structured set of questions on various aspects of the situation; the rationale behind this survey and the analysis of responses comes later in this Overview.

(2) A descriptive review, emphasising particular aspects or projects of interest, but summarising and commenting on the overall position.

(3) A more detailed description of one or two hopeful ideas that other people in other countries might want to try.

In planning this study, we aimed to build up a picture of the achievements and difficulties, similarities and differences, the activity and the apathy, the hope and the despair.

It must be said immediately that the survey is an exceedingly rough one. To get a reasonably accurate picture of the pattern of activity in different classrooms, which differs so much even in broadly similar schools, across such a range of countries, would be an enormous task. For example, Her Majesty's Inspectors of Schools felt it necessary to observe about 10,000 hours of mathematics teaching in over 300 schools in order to get a balanced picture, when they undertook such a task in England and Wales in the 1970s. This Survey [1], the corresponding Primary Survey [2], and the Cockcroft Report [3] to the British Government are of wider interest, both for the harder evidence they bring from one country and as illustrations of the mismatch between our intentions, or even predictions, and their realisation that is no doubt found elsewhere.

Here we simply asked an author in each country, chosen for a knowledge of and work in advancing the curriculum development of the applications of mathematics, to comment on the present position and future prospects as they see it from the various sources of information, personal, anecdotal, or systematic, on which they rely. There must be some question about the reliability of the resultant picture, since observation in classrooms anywhere will reveal great differences between our intentions and our achievements; this is as true in teaching as in most fields of human endeavour. However, the main features that emerge from the reports of nearly every correspondent are too clear to doubt.

The study is limited in various other ways. The 12 countries that have in the end been able to respond is a small proportion of the membership of I.E.A., let alone of the United Nations. Happily they do cover a fair spectrum of economic development, and every continent except Antarctica. It must be recognised that in those countries with local control over the curriculum, the variety is in principle endless, while only a limited picture can be drawn. Even in those countries with centrally defined "national" curricula, there is evidence that the variations in practice are substantial. Most importantly, for the reasons I have stated, the evidence on which the picture is based is in most cases indirect — involving what is said to be done, rather than what is seen to be done.

With such crude data one is looking for two opposite things — a rich, detailed picture of great variety varying from country to country and, perhaps, a few bold qualitative features that appear to be general. Both aspects are clearly present here. The rich variety is displayed in the reports from individual countries, each of which brought, to me at least, new ideas and insights on old problems. In editing these, we have tried to keep the flavour, rhythm, and balance of each report even at the price of some repetition. An element of repetition is indeed essential in bringing home the generality, indeed the universality of the common features; apart from my attempt to channel, through the structured part of the responses, the broad picture of applications in each country, correspondent after correspondent stresses:

(a) that there is a recognised need, to which lip-service at least is paid, for mathematics to be used in tackling practical problems;

(b) that the school mathematics curriculum after the first few years has not responded to this need, being very largely formal and academic with any applications appearing only as artificial illustrations of mathematical techniques;

(c) that there are a few projects which have worked to produce material for teaching applications in school, so that lack of such materials is not the main constraint; and

(d) that there are a few teachers who have shown that applications can provide a valuable and stimulating element in the school mathematical diet.

The contributions under (c) and (d) provide the excitement and optimism of the study, and each report contains an element of this. However, the lesson (b) is surely the most important — if we are to make progress towards satisfying (a), a clear recognition of our failure so far to make any significant impact on the mainstream of schools and classrooms is an essential precursor to doing better. It suggests that we must look for change using a broader range of approaches. Several authors, notably D'Ambrosio of Brazil [4], point to the importance of cultural factors in society and in the mathematics teaching profession. I and others believe that, at a technical level, insufficient attention has been paid to the range of teaching style and the demands placed on the teacher. It is notable that other aspects of mathematics that demand a flexible and non-routine approach from teachers as well as pupils (such as problem solving, in that sense of this much-abused phrase, and open "investigation") are as rarely found in classrooms as are realistic applications. We shall return to these questions and the hope they bring after reviewing in more detail the picture that emerges from the authors' responses. However, the fundamental message is clear — we have a very difficult challenge to face, and a history of hopeful failures which should help to forge a new toughness in our future approaches.

What Are "Applications"?

The word is used in different countries, and by different people in the same country, to describe a wide range of different mathematical activities, So, for example:

- pure probability theory may be described as an application of combinatorial techniques,
- predicting coin tosses may be described as an application of probability theory,
- a study of situations in gambling involving detailed analysis of one game, perhaps Blackjack, bringing in among other things probability theory, is another kind of application, as is
- a realistic study of such a gambling situation, either from the individual gambler's or from the casino's point of view — with its inevitable emphasis on making decisions.

In going through such a sequence of alternative views, two things change in particular:

- The emphasis shifts gradually from mathematics, often just a single mathematical "tool", to a situation outside mathematics that is of interest.
- The range of skills involved becomes wider, with a greater part of the challenge arising from modelling skills such as the formulation and choice of models, the interpretation of answers, and the critical validation of the predictions with reference to the real situation, leading to further develop-

ments of the model. Even the range of mathematical skills often widens — for example, estimation and numerical evaluation, and the translation of information to and from graphs or tables often become important.

Here I shall outline a framework [5] and a language for talking about applications in school mathematics which aims to reduce this ambiguity of meaning. The following aspects seem important:

(1) An application involves the use of mathematics in describing a situation from outside mathematics, usually involving a mathematical model reflecting some aspects of that situation.

(2) In teaching applications two opposite approaches are used, each valuable in different respects:

Illustrations arise in the teaching of a particular mathematical technique, showing how it can be used — the model, usually highly idealised, is related to the new mathematical technique and a straightforward didactic teaching approach based on exposition plus exercise ("tell them and test them") is the norm, and can be reasonably effective. Several illustrations of a given technique are normal — for example, exponential growth might be illustrated by compound interest, population models, and radioactive decay.

Situations from outside mathematics can provide the central theme, linking a variety of aspects of a coherent set of phenomena, involving mathematical models of a much more varied kind — for example, a study of personal finance might look at income and expenditure, capital and interest, methods of consumer choice and borrowing, in more or less detail. The many aspects of and viewpoints on any real situation, each requiring some modelling, makes such a discussion strategically more demanding on the student, so that the technical demand of the mathematics has to be lower, the models simpler. Although the didactic approach has its place here, the absence of the single well-defined right answer is clear to all concerned; the consequent need for discussion of the assumptions of the model, the interpretation and the validation of its consequences, and how the analysis may be improved is equally clear. However, the "open" styles of thinking and, particularly, of teaching that it implies are very demanding. The temptation to narrow down to a single track, taught in the traditional didactic way, is exceeded only by the temptation to leave situations out of the curriculum altogether.

It is worth looking at the position of statistics in this regard. In many countries this is regarded as a field of applications, and has indeed displaced other areas, such as mechanics, in some countries' curricula over the last two decades. While statistics courses usually contain some illustrative applications, a situational approach is relatively rare. The subject seems more often to consist of a set of techniques and models which are extremely useful in a variety of applications, rather than a field of application itself. Similar things can be said about computing. Both techniques deserve a central place in mathematics curricula, but mastery of them is no adequate preparation for their flexible use in a range of situations from outside mathematics. However, it is true in a number of countries that statisticians, despairing of their mathematical colleagues, have taken the lead in introducing a greater element of realism into applications.

(3) Another dimension which is useful for classifying applications distinguishes the learning of standard models from the tackling of new problem situations.

Standard models of situations that occur frequently, and importantly in some sense, have formed the basis of most teaching of applications. They are important in several ways — in providing models useful in practice, in building up a tool kit of useful models and techniques, and in acquainting the student with some impressive intellectual achievements of mankind. They also provide practice in mathematical techniques and reinforce understanding by providing further "concrete" illustrations of abstract concepts in a way that is well recognised. (Illustrations and situations, noted in the previous section, both have roles to play here.) Standard models can obviously be taught didactically by exposition and imitative exercise, though more varied problems are essential if the student is to be able to adapt and use the method in practice; this brings us to

New situations. There is no possibility of covering, for students of any age, more than a small fraction of the problem situations that they will face in practice in their life and work; there is also good reason to doubt that they would remember a large number of such models well enough to be useful. Thus it is of the highest importance for every pupil to acquire skill and experience in tackling new situations, and in choosing and adapting models and techniques from his or her repertoire to help in understanding, explanation, and decision taking. This is a relatively new area of mathematical education, demanding new methods and attitudes — for example, the problems have to be much simpler than where only imitation is demanded, while teaching methods have to include a substantial element of "open teaching" where the pupil leads at his or her own level with the teacher acting as "adviser", "counsellor", and "fellow pupil" rather than as "manager", "explainer", and "corrector". (These features apply equally to the tackling of unfamiliar problems in pure mathematics; the fact that they demand a range of teaching style which only a few teachers presently command is probably the central and most difficult problem we must face.)

(4) The final dimension I should mention is the interest level of the problems to the students who tackle them. Are they:

Action problems, concerning decisions that will affect the student's own life — for example, for most of us at any age, organising our time to balance the various things we need or should like to do is an action problem.

Believable problems are action problems for your future, or for someone else whom you care about — perhaps:

"How can I borrow some money most cheaply?" or "Is it worth my studying for two more years, or should I try to get a job?"

Curious problems are simply fascinating, intellectually, aesthetically, or in some other way — perhaps:

"Why are total eclipses of the sun rarer than those of the moon?" or "How can sap be 'drawn' up a tree 100 metres high" might be curious problems for you.

Dubious problems are there just to make you practice mathematics:

"Describe the motion of two smoothly jointed rods lying at right angles in a smooth horizontal table, after receiving an impulse applied at 45° to the end of one of them", or,

"Calculate the volume of the solid produced by rotating $y = \sin 4x$ about the x axis between 0 and π".

Educational problems are dubious problems that illuminate some mathematical insight so beautifully that no one would want to lose them:

"One drachma was invested at 5% compound interest in the year BC759; to what sum has it now grown?"

This rating is, of course, partly subjective and is clearly only a partial ordering — most of us are involved with mathematics at least partly because we enjoy Curious problems. However, for many students this motivation is not enough and we should not ignore that provided by the power of mathematics in use. Applied mathematics has rarely aimed higher than the Curious and for many students it has been almost entirely Dubious; if mathematics is to be useful in practice, the curriculum must involve the tackling of Action and Believable problems on a significant scale.

The Survey

The individual reports from different countries of which this review is largely composed inevitably take somewhat different standpoints and have different emphases. It seemed to me worthwhile asking in addition each contributor to respond to a series of questions of a fairly basic nature related to the framework of this overview. This section collects these responses. In reading them, the same caution must be exercised by the reader as with the review as a whole — they represent the view of a well-informed member of the mathematical education community of each country, based on the evidence available to them. Equally, the differences in response may partly reflect different viewpoints and methods of collecting data.

Here we simply give the questions and the various responses. For quantitative answers, two different measures were encouraged — a straightforward numerical proportion and a more qualitative scale $(0, \lesssim, \checkmark, 1)$, where $\lesssim$ represents a token and $\checkmark$ a substantial amount; where two symbols such as $0\lesssim$ or $\begin{smallmatrix}.5\\.7\end{smallmatrix}$ are given, they represent a stated range.

Table A gives the response to two questions on the proportion of time spent on applications. The questions were:

Table A1: "Roughly what proportion of mathematics time is spent on applications to problems outside mathematics in the normal mathematics curriculum for the age ranges shown?"

Table A2: "What proportion of this involves applications in other school subjects for which mathematics is providing a service?"

Table B reviews the different areas of applications covered; each age range is shown separately. The question was (referring back to question A):

Table B: "Of the rest, i.e. applications to the outside world, roughly in what proportions are the following subject areas represented?"

Table C reviews the different subject areas studied from a situational point of view. The question was:

Table C: "Applications commonly appear in two ways — as illustrations of a mathematical technique, or as situations from outside mathematics studied coherently; which subject areas are studied from a situational point of view?"

In Table C, each entry refers to the age range (5 means 5-11, etc.); $\lesssim$ means only a token amount occurs. Several authors emphasized the artificiality of the treatment; conversely, the improved situation in some U.S. statistics courses and the importance of project work in Tanzania were brought out. These and other points are covered in the individual reports.

The question for the next table was:

Table D: "It is most common for students to be shown standard models of important situations and to be asked to solve problems which are minor variants on these. Are skills in the modelling of significantly new situations taught to and demanded of students?

Do the skills demanded include
formulation of models of practical situations?

interpretation of the predictions of the model?

validation, by observation or experiment?

explanation of the result in oral or written form?

Does the relevance to the students' experience enter the choice of problems?"

The following question was:

Table E: "How far are these elements included in teacher training?"

Table F probes the balance of teaching style, asking:

Table F: "How far is the style of teaching involved in the teaching of applications:

exposition and imitative exercises

structured open investigation

project work

in the normal curriculum and, separately, in some experimental projects?"

TABLE A1

PROPORTION ON APPLICATIONS

Age Range	Australia	Brazil	Britain	Canada	Germany	Hungary	Israel	New Zealand	Nigeria	Norway	Phillippines	Tanzania	USA
5-11	.1	.03	.3	.2	.2	.3	.1	.2	.2	.2	.5	1	.05
11-16	.2	.02	0	.1	.2	.2	.1	.2	.1	.2	.2	✓	.3
16-18	0 .3	.02	0 (.5)	0 .2	.1	.2	.1	.2	.1	.2	0 .2	≤	.1
18+	0 .3	.02	0 (.5)	0 .2		.1	.05	0 1	.2	.2	0 .2	✓	0 (.5)*

TABLE A2

PROPORTION IN OTHER SHCOOL SUBJECTS

Age Range	Australia	Brazil	Britain	Canada	Germany	Hungary	Israel	New Zealand	Nigeria	Norway	Phillippines	Tanzania	USA
5-11	.5	.9	0	0		0	0	.2	0	0	0	✓	0
11-16	.3	.7	-	.2		0	0	.5	.1	0	.5	✓	.3
16-18	.5	.7	.2	.5		0	.5	.7	.2	0 .5	.7	✓	.7
18+	.2	.9	0	.7		0	.7	0 .5	.2	.5 1	.2	≤	.2

* only in some statistics courses

TABLE B

SUBJECT AREAS BY AGE

Key

Economic:	Money — personal	e.g.,	wages, expenditures
	— national		gross national production, inflation exchange rates
	Time — personal		work and play scheduling
	— at work		efficiency
	Resources planning		travel to school, holiday
Physical:	Mechanical		traffic, sport, machines
	Energy		fuel resources, heating
	Pollution		litter, traffic, crowds
	Popular sciences		space, weather, disaster
Geographical:	Topographical		maps, navigation, holidays
	Physical		weather, rivers, geological features
	Social		growth of towns, distribution of goods
Human:	Domestic		chores, possessions
	Two-person interactions		friends, relations, sex
	Large group behaviour		crowd reaction, control
	Interaction at work		friends, bullies, authority
	Learning		style, efficiency
	Leisure, arts, sport		music, colour, tactics, design
Biological:	Physiological		food, sport
	Medical		epidemics, innoculation
	Reproductive		genetics, contraception
Occupations:	Various		typing, painting, carpentry, farming, nursing, etc.

TABLE B1

SUBJECT AREAS, 5-11

	Australia	Brazil	Britain	Canada	Germany	Hungary	Israel	New Zealand	Nigeria	Norway	Phillippines	Tanzania	USA
Economic													
Money													
-personal	✓	1	✓	✓		✓	ξ	✓	.5	✓	✓	1	✓
-national	0	0	0	ξ			0	0	.1			0	
Time													
-personal	ξ	ξ	0	0		✓	0	ξ	.2		ξ	1	
-at work	0	0	0	0			0	ξ	0			0	
Resources planning	0	0	0	ξ			0	ξ	0			ξ	
Physical													
Mechanical	0	0	ξ	0			0	0	0			0	
Energy	0	0	0	0			0	0	0			0	
Pollution	0	0	0	0			0	0	0			0	
Popular sciences	ξ	0	0	0			0	ξ	.04			ξ	ξ
Geographical													
Topographical	0	0	0	ξ		✓	0	0	0		ξ	ξ	
Physical	ξ	0	0	ξ			0	0	.04			ξ	ξ
Social	ξ	0	0	ξ			0	0	0			✓	
Human													
Domestic	ξ	0	ξ	0		ξ	0	ξ	0	ξ	✓	✓	
Two-person interactions	ξ	0	0	0			0	0	.04	ξ		✓	
Large-group behaviour	0	0	0	0			0	0	0			✓	ξ
Interactions at work	0	0	0	0			0	0	0			✓	
Learning	0	0	0	0		✓	0	0	.1			✓	
Leisure, arts, sports	✓	✓	0	0			0	0	.04			1	
Biological													
Physiological	0	0	0	0			0	0	0	ξ		✓	
Medical	0	ξ	0	0			0	0	0			ξ	
Reproductive	0	0	0	0			0	0	0			ξ	
Occupations	ξ	0	ξ	0			0	0	0			ξ	ξ

TABLE B2

SUBJECT AREAS, 11-16

	Australia	Brazil	Britain	Canada	Germany	Hungary	Israel	New Zealand	Nigeria	Norway	Phillippines	Tanzania	USA
Economic													
Money													
-personal	✓	1		✓	✓		✓	✓	.1	✓	✓	≤	✓
-national	≤	0		≤	✓1	✓	0	≤	0			0	
Time													
-personal	≤	≤		0	✓		0	≤	.05		≤	✓	
-at work	0	≤		≤	✓1	✓	0	≤	0			0	
Resources planning	≤	0		≤	✓		0	≤	0			✓	
Physical													
Mechanical	≤	0		0	✓		≤	0	0			0	≤
Energy	0	0		0			0	0	0			0	
Pollution	0	0		0		✓	0	0	0			0	
Popular sciences	≤	0		0			≤	≤	.05			≤	≤
Geographical													
Topographical	≤	0	≤	≤	✓	✓	≤	≤	.1		≤	≤	
Physical	≤	0		≤			0	0	.05			≤	≤
Social	≤	0					0		0				
Human													
Domestic	≤	0		0			0	≤	.05	≤	✓	✓	
Two-person interactions	≤	0		0			0	0	0	≤		✓	
Large-group behaviour	0	0		0			0	0	.1			✓	≤
Interactions at work	0	0		0			0	0	0			✓	
Learning	0	0		0			0	0	.1			✓	
Leisure, arts, sports	≤	0	≤	0		✓	0	0	.05			1	
Biological													
Physiological	0	0		0			0	0	.05	≤		✓	
Medical	0	0		0			0	0	0			≤	
Reproductive	0	0		0			0	0	.05			≤	
Occupations	0	0		0	✓	≤	0	≤	.05			≤	≤

TABLE B3

SUBJECT AREAS, 16-18

	Australia	Brazil	Britain	Canada	Germany	Hungary	Israel	New Zealand	Nigeria	Norway	Phillippines	Tanzania	USA
<u>Economic</u>													
Money													
-personal	≤	0	≤	≤			≤	✓	.1	≤	≤	≤	≤
-national	≤	0		✓			0	≤	0				
Time													
-personal	≤	0		0			0	≤	.07			≤	
-at work	0	0		≤			0	≤	.07				
Resources planning	≤	0		≤			0	≤	.07			≤	
<u>Physical</u>													
Mechanical	≤	0	1	✓		✓	≤	≤	.07			✓	≤
Energy	≤	0		✓		✓	0	≤	.07		≤	≤	
Pollution	0	0		✓			0	0	0		≤	0	
Popular sciences	≤	0		✓			0	≤	.07		≤	✓	≤
<u>Geographical</u>													
Topographical	≤	0		✓			≤	≤	.1		≤	≤	
Physical	≤	≤		✓			0	0	.07		≤	≤	≤
Social	≤	≤		✓			≤	≤	0	≤	≤	≤	
<u>Human</u>													
Domestic	≤	0		✓			0	≤	0			≤	
Two-person interactions	≤	0		✓			0	0	0			≤	
Large-group behaviour	0	0		≤			0	0	0			✓	≤
Interactions at work	0	0		≤			0	0	0			≤	
Learning	0	0		0			0	0	.1			✓	
Leisure, arts, sports	≤	0		≤		≤	0	0	.07			≤	
<u>Biological</u>													
Physiological	≤	≤		✓		≤	≤	0	.07			≤	
Medical	≤	0		✓			0	0	0		≤	✓	
Reproductive	≤0	0		✓			0	0	0		≤	≤	
<u>Occupations</u>	≤	0		≤		≤	≤	≤	.07		≤	≤	

TABLE B4

SUBJECT AREAS, 18+

	Australia	Brazil	Britain	Canada	Germany	Hungary	Israel	New Zealand	Nigeria	Norway	Phillippines	Tanzania	USA
Economic													
Money													
-personal	≤	0	≤	≤			≤	0	.1			≤	≤
-national	≤	≤		✓			0	0	0				
Time													
-personal	≤	0		0			0	0	.07			≤	
-at work	0	≤		≤			0	0	.07				
Resources planning	≤	✓	≤	≤			0	0	.07			≤✓	
Physical													
Mechanical	✓≤	0	✓	✓			0	0	.07			✓	
Energy	✓≤	≤		✓			0	0	.07			✓	
Pollution	0	≤		✓			0	0	0			≤	
Popular sciences	≤	0		✓			0	0	.1			✓	
Geographical													
Topographical	≤	0		✓			0	0	.1			≤	
Physical	≤	0		✓			0	0	.07			≤	
Social	≤	0		✓			0	0	0			≤	
Human													
Domestic	≤	0		≤			0	0	0			≤	
Two-person interactions	≤	0		≤			0	0	0			≤	
Large-group behaviour	0	0		≤			0	0	0			≤	
Interactions at work	0	0		✓			0	0	0			≤	
Learning	0	0		0			0	0	.1			✓	
Leisure, arts, sports	≤	0		≤			0	0	.07			≤	
Biological													
Physiological	≤	0		✓			0	0	.07			≤	
Medical	≤	0		✓			0	0	0		✓	✓	
Reproductive	≤	0		✓			0	0	0		✓	≤	
Occupations	≤	0		≤			0	0	.07		≤	≤	

TABLE C

SITUATIONS STUDIED

	Australia	Brazil	Britain	Canada	Germany	Hungary	Israel	New Zealand	Nigeria	Norway	Phillippines	Tanzania	USA
Money		5	5	5	11≤	5≤		5	16	5	5	5	5
Time											5		
Work					11≤								
Domestic						5≤		5	5		5	5	5
Travel										11		5	
Social										18			
Games												5	
Mechanics			16 18	16 18								16	
Traffic										11			
Optimisation			18≤										
Control			18≤										
Finance												11	
Economics			18≤	16 18					18	11		18	
Currencies												11	
Population			18≤									18	
Surveying												11	
Engineering												18	
Planning												16	

TABLE D

NEW SITUATIONS

		Australia	Brazil	Britain	Canada	Germany	Hungary	Israel	New Zealand	Nigeria	Norway	Phillippines	Tanzania	USA
In normal curriculum-	5	0	0	0	0	Σ	0	0					Σ	
	11	0	0	0	0	Σ	0	0					✓	
	16	0	0	0	0	Σ	0	0	Σ				✓1	
	18	0	0	0	0		0	0	Σ	✓			✓	
In experimental projects-	5	✓Σ	✓	✓		✓							Σ✓	✓
	11	✓	✓	✓		✓							✓	✓
	16	✓	✓	O	Σ	✓	Σ		✓		Σ	Σ	✓	✓
	18	✓	✓	✓						✓			✓	✓
Formulation		✓	Σ	✓	✓	✓	Σ	0		Σ	Σ	✓	Σ	✓
Interpretation		✓	Σ	✓	✓	✓	✓	0		✓	✓	Σ	Σ	✓
Validation		Σ	Σ	Σ	Σ	0Σ	Σ	0		✓	✓	Σ	Σ	Σ
Explanation		✓	✓	Σ✓	✓	0Σ	✓	0		✓	✓	✓	Σ	Σ
Relevance to children		Σ	Σ	✓	✓	✓	Σ	0		Σ	✓	✓	Σ✓	Σ

TABLE E

INCLUDED IN TEACHER TRAINING

	Australia	Brazil	Britain	Canada	Germany	Hungary	Israel	New Zealand	Nigeria	Norway	Phillippines	Tanzania	USA
Yes	Σ	0Σ	0Σ	Σ	Σ	Σ	0	Σ	Σ	Σ	Σ	✓	Σ

TABLE F

TEACHING STYLE

	Australia	Brazil	Britain	Canada	Germany	Hungary	Israel	New Zealand	Nigeria	Norway	Phillippines	Tanzania	USA
<u>Normal curriculum</u>													
Exposition and imitative exercises	✓	1	1	✓	1	✓	1	✓1	Σ	Σ	1	✓	✓
Structured open investigation	0	0	Σ	Σ	Σ		0	0	✓	Σ		Σ✓	0
Project work	Σ	Σ	Σ	Σ	0Σ		0	Σ	✓	Σ		✓1	Σ
<u>Some experimental projects</u>													
Exposition and imitative exercises	Σ	Σ	✓	✓	✓		1	✓	✓	✓	✓	Σ✓	✓
Structured open investigation	✓	✓1	✓	✓	✓	✓	0	Σ	✓	✓	✓	Σ	✓
Project work	✓	1	Σ✓	✓	Σ	Σ	0	Σ	✓	✓		✓	Σ

Why Do Mathematics Teachers Avoid Applications?

First, let me review the evidence that they do. Everywhere the importance of applications is recognised — I have yet to meet a teacher who does not believe that using mathematics is important; yet it does not happen in classrooms. In some countries (for example, the USA), very substantial efforts in the provision of workable, indeed often exciting materials, and in the advocacy of their use, have produced no sign of general change; on the other hand, the successes of a few teachers suggest that pupils can cope with these activities — indeed, the evaluation results of some of the projects with the most realistic approach (e.g., USMES in the USA) show strong positive pupil attitudes.

This evidence is indirect — strong but uninformative. Direct comments from teachers show a pattern of concerns which will surely have to be reconciled if applications are ever to arrive in most classrooms. What are these concerns?

"No decent mathematics"

The mathematics curriculum in every country is based on an exposition-exercise approach, where the tasks children are asked to perform involve close imitation of processes demonstrated by the teacher. It is not surprising that they can operate at a much higher level at such imitative exercises than at tasks where they have to select and deploy mathematics from their own "tool kit" of techniques, but teachers find the scale of the difference unwelcome and, indeed, hard to accept. (Our experiments show that this difference is enormous when pupils are faced with situations new to the mathematics classroom — 11-year-olds who are learning the transformation geometry of the plane could scarcely organise the tabulation and arithmetic for shopping lists of Christmas presents, while very able 17-year-olds with five years of success at algebra used only arithmetic in tackling realistic problems.) The performance gap can be reduced by teaching problem-solving skills explicitly, but it is likely to remain large.

"Not really mathematics"

The apparently clean, well-specified nature of pure mathematics, with the security that "right answers" provides, is part of the attraction of mathematics teaching to many of us. It confers on the teacher membership an exclusive, almost esoteric priesthood with admired, difficult mathematical skills which, once mastered, are actually undemanding — a piece of mathematics, as we learn it, moves from the impossible to the easy much more dramatically than in most other subjects.

Realistic applications, on the other hand, are intrinsically messy and uncertain, making a wide range of open demands, and it should be no surprise that they are uncongenial to many mathematicians. Of course, the tackling of unfamiliar open problems in pure mathematics has many of these features — "Which whole numbers can be represented as a sum of successive whole numbers, and how, and why?"; however, that kind of activity, too, is largely excluded from the curriculum.

"Too hard for children"

Realistic problems require a broader view than imitative exercises, some wider experience and maturity, and since we see no sign of these qualities in the mathematics classroom, it is clearly unreasonable to set tasks that demand them. This view that applications are too hard often accompanies the view (see above) that they are trivial.

"They should do these things later — when they know some mathematics"

This theme is to be found at every stage beyond the early years (where practical applications of counting and measuring are to be found in most classrooms in most countries), despite the fact that all children "know" far more mathematics than they can use.

"Nobody wants it"

The mathematics curriculum has "got on" without applications, let alone realistic ones, for many years. The social pressures are for better technical skills, particularly in arithmetic, not for crude practical uses, which are "all fairly obvious once you see them".

What Can We Do to Help the Teacher?

This is a difficult challenge to which there is no established answer, but many hopeful ideas. Over the last 20 years or so, a great deal of the time and effort of able and ingenious people has been devoted to tackling these problems, which have been recognised in various forms for at least a hundred (and probably 5,000) years. In exploring new possibilities, two things are crucial:

(a) we should recognise that changing teaching style is very difficult — individually for the teacher concerned as well as collectively for society.

(b) our methods of working in curriculum development and teacher support should recognise the high probability of failure and should include ways of diagnosing it swiftly and with a detailed understanding of its symptoms that may guide the next attempt. This requires an integrated research-development approach based on detailed study of the human situations involved, in the classroom and elsewhere.

Now for some suggestions. All these elements are currently being pursued in various places, but the need for much more high quality work is manifest in the reports from the various countries, and in this overview. If you find an emphasis on teachers, their individual teaching styles, and how they may broaden to encompass open teaching which promotes the full range of classroom activities discussed above, then this reflects the weight of evidence as to where the real difficulties lie.

(1) Teaching materials. Most teachers in their teaching follow closely some published teaching material such as a textbook.

There is a clear need in every country at each age and ability level for well-developed materials incorporating useful applications in an interesting way. There is around the world a considerable fund of such material; Max Bell's review [6] for the USA makes this point in some detail. The process of adaption and extension in every country is a substantial development exercise, which is likely to succeed only if it is based on detailed study of the trial material in use by a representative group of teachers and pupils. There is still a shortage of material in some areas, particularly in the provision of easy interesting problems relevant to everyday life.

(2) Teaching style development. Observational evidence suggests that printed material has very little influence on teaching style — an "explainer" will not long resist explaining "the solution" (his or her solution) to the most open problem, while it is equally possible to base an open investigation on a page of multiplication "sums". Since applications, and indeed all problem solving, demand a substantial component of "open" activity with the teacher in a variety of counselling roles of a nondirective kind, and since few teachers have this within their style repertoire, there is a central difficulty here. Many attempts over many decades (at least) in various countries have not produced a major change, though small-scale successes suggest hopeful elements:

(a) activity based approaches offer the only real hope of modifying behaviour as well-established as a teacher's style. Equally, the new activities involved have to seem credible and achievable in the context of the teacher's own classroom. Thus, "taught courses" are ineffective, though they may be a useful adjunct to "school-based" work. What stimuli can help?

(b) personal example — team teaching with another whose style is broader is one way that has succeeded on a small scale, though such ways of working only occur where there is a willingness to look at change, so this evidence may not be unbiased. The spread of this approach has surely been slow.

(c) workshop activities with other teachers and groups of pupils at out-of-your-own-school meetings can provide a flavour of (b), though the demand of translating it back to your own classroom context is considerable -- for example, the pupils have to see and accept the new relationships implied.

(d) microcomputers [7] have recently been shown to have considerable potential in this area; when programmed to act as a "teaching assistant" the micro can temporarily take over some of the usual authoritarian roles of the teacher. This happens in such a way that the teacher naturally assumes the elusive "counselling" roles while the computer is "task setting" and "managing" the activity. These style shifts were noticed with teachers with a wide range of natural styles, and persisted to some extent beyond the microcomputer's use.

(e) other media can help, though they do not seem to assume in the pupils' eyes the independent "personality" that is an essential feature of (b) and, surprisingly, of (d). Video can help by showing open teaching in action, but the viewer is passive. Practical work with apparatus can be invaluable, but places greater demands on the teacher.

(3) Organisation. A variety of organisational possibilities may help. For example, teachers of young children who cover the whole curriculum seem naturally to work in more open ways — it may be that such arenas, including "general studies" teaching in secondary schools, may be an easy place to introduce realistic problem-solving activities — perhaps with some support from the Mathematics Department. Conversely, style specialisation has something to commend it — physics is taught by a physics teacher, rather than a mathematics teacher, but different syllabus content is much more easily adopted than new style elements. If only 10 percent of the school mathematics team can teach open problem solving, they might concentrate on that, teaching all pupils for part of their mathematics curriculum time. Many open learning activities, such as project work, also have strong organisational implications.

(4) Pressures. Pressures on teachers of many kinds crucially affect the pattern of classroom activities [4]. These may be changed to make their influence more benign. The pressures include professional expectations and public examinations (where the urge for fairness and consensus tend to exclude any non-routine tasks) and social pressures of various kinds, expressed directly by children, parents, colleagues, and supervisors, and indirectly through television and other media. Any change in the curriculum is only likely to come with a change in the pattern of pressures.

The educational system in any country has enormous inertia. Experience suggests that it is unlikely to move unless all the efforts are pulling coherently in much the same direction. This implies coherent contributions from central and local government, examination authorities, teacher trainers, professional associations, and research and development centres. In addition, an effective, supportive, two-way communication system between these different elements and with all teachers must be created; in many countries this is perhaps the most difficult task.

There is much to be done. There is much being done. We shall all watch its progress with a great deal of interest.

References

[1] Aspects of Secondary Education in England (Report of the National Secondary Survey by H. M. Inspectorate). London, HMSO, 1979; also Supplementary Information on Mathematics.

[2] Primary Education in England (Report of the National Primary Survey by H.M. Inspectorate. London, HMSO, 1978.

[3] Mathematics Counts (Report of the Committee of Inquiry into the Teaching of Mathematics in Schools, under the Chairmanship of Dr. W. H. Cockcroft). London, HMSO, 1982.

[4] D'Ambrosio, Ubiratan. Report on Brazil and Latin America. This volume, pages 52-55.

[5] Burkhardt, Hugh. The Real World and Mathematics. Glasgow, Blackie, 1981.

[6] Bell, Max. Report from the USA. This volume, pages 142-155.

[7] See, e.g., Fraser, Rosemary; Burkhardt, Hugh; Coupland, Jon; Phillips, Richard; Pimm, David; and Ridgway, Jim. Learning Activities and Classroom Roles — With and Without the Microcomputer. Nottingham, Shell Centre for Mathematical Education, 1983.

AUSTRALIA

Vern Treilibs

Education in Australia is the responsibility of the individual States and hence there is a real potential for significant differences in their curricular offerings. The reality of the situation, however, is that there is remarkable uniformity between the corresponding syllabuses of the States and differences are confined largely to newly developed areas of the curriculum (e.g., "social education") or to relatively small-scale innovatory projects.

The situation in Mathematics Education is that most courses are very "pure" and "academic" in nature — a legacy of the 1960s emphasis on fundamental and abstract mathematics. Applications of mathematics appear to arise only incidentally in the large majority of courses. In most States, the curricula are still dominated by public exminations, despite the fact that the number of these is decreasing (at most two per student in a given State), and that the retention rate (to Year 12) is increasing. In the typical classroom, mathematics teaching still centres on the textbook — and usually a textbook in the "modern" style (viz., cheap, quickly produced, and in an example-exercises format). In the current economic climate, it is hard to see much change occurring, in the short term, in this methodology.

Initiatives

On the positive side, there is a healthy communications network amongst curriculum developers across the country, and the Australian Association of Mathematics Teachers and the embryonic Australian Mathematics Education Program continue to make an increasing contribution to the dissemination of good curriculum practice. There is considerable interest and acceptance of the "new" emphases on problem solving and applications, some small-scale experimentation, but few major implementation projects.

Where projects emphasising applications are initiated, the usual target group is the less-academic student. For example, in New South Wales the Mathematics 2 — Unit A course contains optional units on:

- personal finance
- mathematics in construction
- land and time measurement
- mathematics of chance and gambling
- computing
- elementary coastal navigation
- space mathematics

Approaches aimed more specifically at vocational training are also being considered. For example, in Victoria the "Careers and Mathematics" (C.A.M.) project has recently been established. It is sponsored by the Australian Institute of engineers and its brief includes the production, during 1982-83, of student materials which present in context the mathematics used in industry and commerce. It is believed that this thematic approach based on situations will present a better picture of the applications of mathematics in those areas than does the current practice of using applications to illustrate the mathematics being studied.

Three projects, differing greatly in style, scope, and purpose, are making interesting contributions to the teaching of applications of mathematics in Australia. They are:

- the School Mathematics Curriculum Project of the Australian Academy of Science
- the Realities in Mathematics Education project of the Education Department of Victoria
- the "Mathematical Modelling" topic in the Year 12 Secondary School Certificate course of the Education Department of South Australia

These are discussed in more detail in the following sections.

School Mathematics Curriculum Project

This project was established by the Australian Academy of Science in the mid-1970s with the aim of producing student materials which "are enjoyable and useful to students, relevant to the mathematical needs of their everyday life and future employment . . .". The project materials meet these aims by placing strong emphasis on applications of mathematics. The project went into publication with a six-book series entitled "Mathematics at Work". The books, together with an indication of the applications they cover are:

Making the Best of Things — a comparison of the applications of Euclidean and "taxicab" geometries, some introductory work on transport problems.

Taking Your Chances — modelling using basic probability.

People Count — applications of the basic statistical concepts (various averages, deviation, correlation measures) together with an introduction to the use of "statistical argument".

Shape, Size and Place — applied geometry, particularly the application of ratio and trigonometry.

Understanding Change — functional dependence studied by examining its application to simple growth situations.

Maths and Your Money — application of mathematics to personal money management.

Although these books were originally intended for the "non-academic" Year 11 (17-year-old) student, teachers have found the materials valuable for enriching the work presented to most secondary classes.

Some appreciation of the approach adopted in this series can be obtained by examining the sample pages in Attachments A and B.

Reality in Mathematics Education (RIME)

An attempt to increase the "applications" component of the common (for all students) junior secondary curriculum is being made by the Victorian Education Department's Mathematics Curriculum Committee, through its Realities in Mathematics Education project (RIME). The junior secondary (Years 7-10) mathematics syllabus guidelines in Victoria classify content into three categories:

- concepts
- skills
- applications

It is considered that adequate textbook resources are available to support the skills development, but teachers need additional materials for the development of concepts and applications. The RIME writing team is producing units of lesson plans (typically two or three lessons in length) which are keyed to sections of the syllabus. The main emphases of these materials are problem-solving methods and applications, as illustrated by the example in Attachment C. Naturally, the writers take the opportunity presented by the writing of lesson plans to promote good methodology — e.g., the use of group work and of computers as a learning aid.

The RIME project has the most potential of any in Australia for producing significant curriculum reform, and the extent of its uptake by teachers will be keenly watched by curriculum developers in other States. A commendable feature of this project, which has been in existence since 1979, is that it was allowed two or three years to "find its feet". Had it been forced, by pressure of time, to follow its initial directions, it would certainly have had much less potential for acceptance in schools.

Mathematical Modelling Unit

The Education Department of South Australia offers the Year 12 Secondary School Certificate (S.S.C.) courses as alternatives to the public examinations. The S.S.C. Mathematics course contains a six-week unit on "Mathematical Modelling". This unit aims to develop some skills in solving real problems. The main focus is on the formulation stage of the modelling cycle with the emphasis being on the process rather than the result. Naturally, the problems solved by the students tend to be fairly simple. Examples of more complex modelling in industry, commerce, etc., are given as case studies in other parts of the course (e.g., queuing, C.P.A.). The approach in the modelling unit is empathetic in that there is no explicit teaching of the various steps in the modelling cycle; rather, the students learn by example and practice. Typically, the teacher and class work together through two or three "teaching models" (e.g., designing the in-ground sprinkler system for the school oval) for about seven lessons each, and students solve assignment problems of increasing difficulty as the course progresses. A discussion about the nature of mathematics and its applications in the real world is an ongoing feature of the unit; another is the opportunity for students to nominate a range of problems from which the teacher is expected to choose and solve one. The problems in Attachment D illustrate the style of the course.

Student reaction to the mathematical modelling unit is generally initial apprehension followed by a gradual increase in enthusiasm and confidence. Almost all teachers offering the unit report some doubts about whether or not "it's mathematics", but none has doubts about its educational value.

Overview

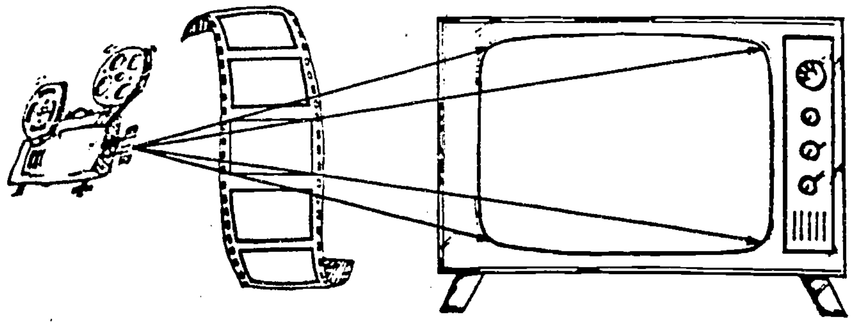
In conclusion, the Australian Mathematics Education scene is somewhat lacking of examples of mathematics curricula which give due emphasis to applications, but current curriculum developments do give some cause for optimism.

Attachment A

School Mathematics Curriculum Project

Sample from

<u>Shape, Size and Place</u>

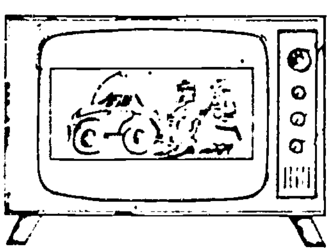


For films made for theatre-going audiences, however. the situation differs. Before 1953, most motion pictures were also made using a film which created a picture with aspect ratio 1.3:1. In 1953 Cinemascope was introduced, and since then many other wide-screen processes have been added. Some of them are listed here.

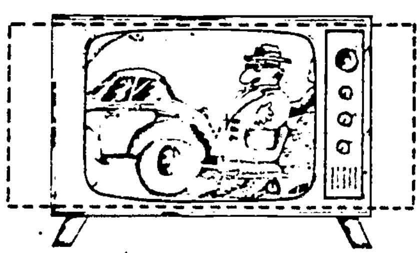
Process	Aspect ratio
Cinemascope (stereo sound)	2.55:1
Cinemascope (mono sound)	2.35:1
Techniscope	2.35:1
Todd-AO	2.20:1
Ultra Panavision	2.70:1
Vista Vision	1.85:1

Consider a Cinemascope film with an aspect ratio of 2.35:1. There is a problem in enlarging it to fit the dimensions of any television screen. There are four possibilities.

A Both width and height dimensions too small.

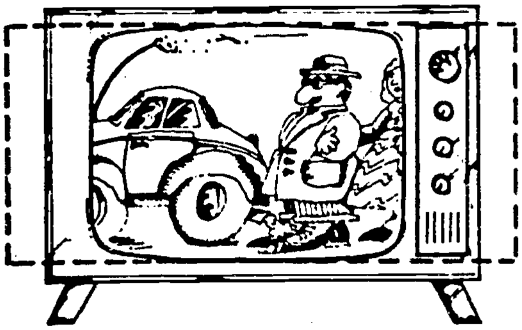


B Both width and height dimensions too large.

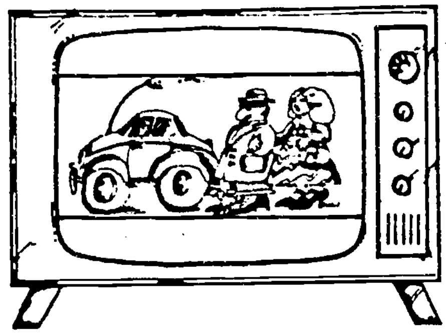


From: Shape, Size and Place, p. 24.

C Suitable height, but unsuitable width.



D Suitable width, but unsuitable height.



1 Which of these have you seen used to show Cinemascope movies on television?

2 Sometimes when the credits (names of the stars and the film crew) are being shown, figures in the background seem very tall and thin. After the credits finish, the figures change into their usual proportions and look 'normal'.

What (non-similar) process is being used to fit the credits onto the screen? Why would this be used only for the credits, and then one of the four possibilities mentioned previously takes over for the rest of the film?

3 If the television screen represented above measures 44 × 33 cm

a find the height of the image in case *D* above, and

b find the (theoretical) width of the image in case *C* above.

(Remember the aspect ratio of this image is 2.35:1.)

4 How would the problems of differing aspect ratios be overcome in picture theatres?

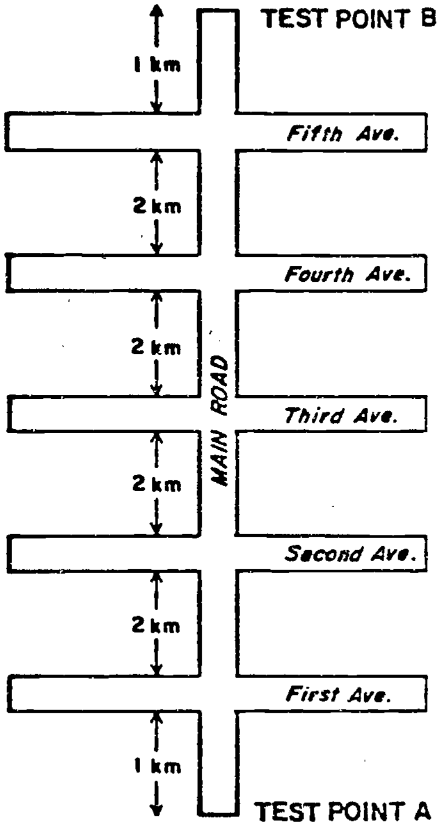
From: Shape, Size and Place, p. 25.

Attachment B

School Mathematics Curriculum Project

Sample from

Taking Your Chances



The situation

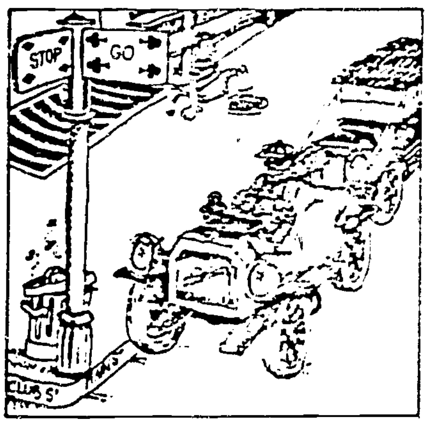
A section of main road passes through five major intersections. Each is controlled by a set of traffic lights. The roads, and the distances between them, are shown in the accompanying diagram.

On this section of road cars can average 40 km/h between intersections. A car travelling from A to B travels 10 km. If there were no hold-ups due to the lights, it would take 15 minutes for the trip.

In this section a mathematical model will be set up to test the effect of the lights. Some changes can be made and tested. Here is a case where it is very convenient to simulate the real world. Drivers do not enjoy waiting in long queues until the best system is found.

Model 1: Fixed assumptions

1 There is only one lane of traffic at each intersection.
2 The stage of the light cycle when the test car reaches an intersection is a random event.
3 The number of cars waiting at an intersection is a random number between 0 and 15.
4 Cars can cross intersections at the rate of 12 per minute when the lights are green.
5 The lights are green for 60 seconds, then red for 60 seconds.
6 The time the lights are amber is counted as zero.



Model 1: Simulation

Playing cards can be used to simulate the working of the lights. A suitable spinner can simulate the length of queues. Only the assumptions listed above will be used. A test car will be driven from A to B. It will travel at 40 km/h, except when stopped by lights or queues.

The lights: Make a pack of A, 2, 3, 4, 5, 6 of red cards and A, 2, 3, 4, 5, 6 of black cards. The cards are shuffled together and one is drawn. The tables then show the stage of the lights.

Thus, drawing 2 ◊ means the lights are red, and will be red for 20 seconds.

Red card: Red light	A	2	3	4	5	6
Time before light changes (seconds)	10	20	30	40	50	60

Black card: Green light	A	2	3	4	5	6
Time before light changes (seconds)	10	20	30	40	50	60

From: Taking Your Chances, p. 36.

The queue: Obtain Spinner 2. Spin it twice, recording the two numbers obtained.

Queue length = (first number) × 4
+ (second number)

Thus, the length of the queue can be any number from 0 to 15.
Are all the numbers equally likely to occur?
Write down the answers to the following exercises. They check whether you understand how the lights and the queue models work.

Exercises

5 a You have drawn a black two. What is the stage of the lights?
b You have drawn diamond four. What is the stage of the lights?
c Spin 3, then 1. What length of queue does this represent?
d Spin 0, then 2. What length of queue does this represent?

6 The test car reaches an intersection. The queue length is 4, and the diamond 4 is drawn.
a How long before the lights turn green?
b The test car is fifth in line. Once the lights turn green, how many seconds will it take to pass through the intersection? (Remember it takes 12 cars 1 minute to cross.)
c How long altogether did the test car spend at the intersection?

7 Worksheet 1 contains a table. Fill in the first two columns for all intersections, using the card and spinner. Then complete the entries across. Here is a sample:

Intersection	Card	Spins		Queue Length	Light Colour	Time To Change	Total Time
		1	2				
1	2D	2	3	11	R	20 sec.	*

The queue length (= 2 × 4 + 3) is 11, so the test car is 12th in line. It will be 20 seconds before the lights turn green, and then it will take 60 seconds for the test car to cross. 12 cars per minute are all that can cross.
The entry * would be (20 + 60) seconds = 1 minute 20 seconds.

8 By adding up the times in the 'total time' column calculate the total time that the test car spends waiting at intersections.

From: Taking Your Chances, p. 37.

9 To your answer in Exercise 4 add the 15 minutes 'travelling time' to get from point A to point B. What is the total time that the test car takes to make the journey?

10 Compare your results with those of

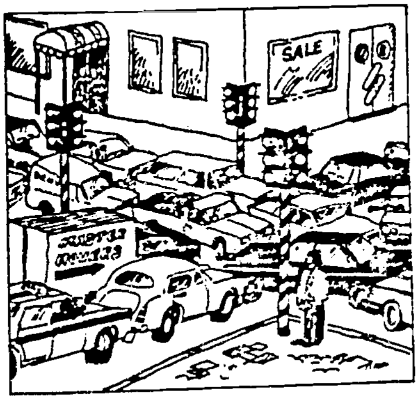
a your neighbour

b the rest of the class.

The model being used gives a feeling for traffic flow, but it could be more realistic.

The queue length was assumed to be independent of the state of the lights. At what stage would you expect to find the longest queue?

Can the simulation be changed to match this? Let us look at a model of the situation which does take this factor into account.



Model 2: Fixed assumptions

1 There is only one lane of traffic at each intersection.

2 The stage of the light cycle when the test car reaches an intersection is a random event.

3 The number of cars in a queue is partly determined by the light cycle stage.

4 Cars can cross intersections at the rate of 12 per minute when the lights are green.

5 The lights change every 60 seconds.

6 The time the lights are amber is counted as zero.

Model 2: Simulation

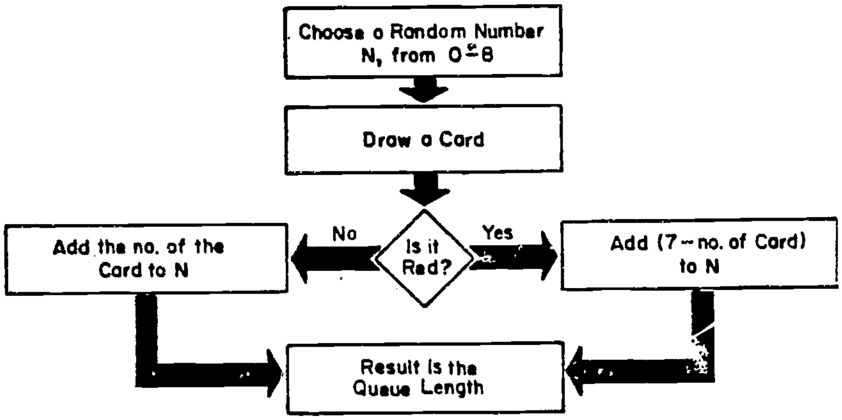
The problem is to make the queue length partly random and partly light-affected. Here is one way:

- Choose a random number from 0 to 8 (*you* find a way to do it!).
- Draw a card to find the light cycle stage (as in Model 1).
- If the card is black, add the number of the card to your random number.
- If the card is red, calculate: 7 – (the number of the card). Add this result to your random number.

For example, if you drew the two of hearts, calculate $(7 - 2) = 5$. and add 5 to your random number.

This flow chart summarises the steps more neatly:

From: Taking Your Chances, p. 38.



11 Use the flowchart to calculate the queue length in these cases.

a random number 5, card 3D
b random number 7, card 2C
c random number 0, card AD
d random number 8, card AC

12 Use the above method to produce five new trials—one simulated queue length for each intersection.
13 Obtain another copy of Worksheet 1 and enter your results from Exercise 8.
14 Now complete the simulation as in Model 1. What is the total time that the test car takes to travel from point A to point B?
15 Compare your results with those of

a your neighbour
b the rest of the class.

Projects

Now that you have some practice at using a simulation, try developing some models of your own. Choose one or more of the following projects and develop a suitable model. You will need to develop different random number generators. You will also have to decide whether the events are independent or not.
Design your own recording sheets as you need them.

Project A

The local council wants to cut down on the delays along Main Road between A and B, and asks two town planners to make recommendations.
One town planner recommends altering the light cycle to 30 seconds green, 30 seconds red.

From: Taking Your Chances, p. 39.

Attachment C

Reality in Mathematics Education (RIME)

Sample Lesson

HANGING WIRE

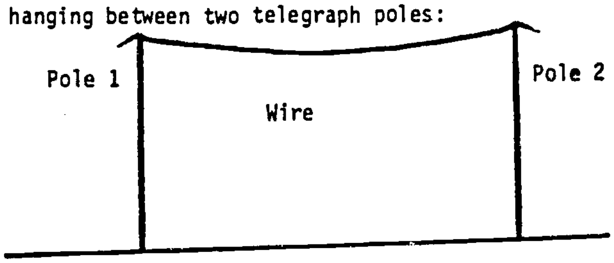
A10.2 Quadratic Graphs

Features of this lesson

1. A practical activity using mathematics to model a real situation.
2. Group work.
3. Finding a quadratic rule from a table of values and graph.

Overview

The wire hanging between two telegraph poles:



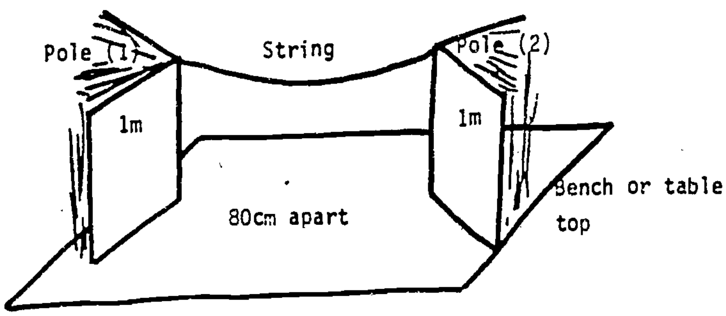
hangs in a shape very close to an extremely flattened out parabola. In this lesson we will make an actual physical model for this situation and try to find the corresponding mathematical model (in the form of a rule).

Preparation

For each group, two books, 1 metre of thin non-stretching string, sticky tape. For each pupil, graph paper and a ruler.

Plan

1. Demonstration
   (a) Set up the apparatus as follows



From: RIME.

(b) Talk about telegraph or SEC wires hanging in approximate parabola shapes. Point out how the apparatus now set up is a physical model of the real situation.

Explain that by making measurements and graphing we will be able to make a mathematical model of the situation. When we also find a rule for the graph we will have another, even more useful, mathematical model.

Hand out the worksheets. Go over the procedure briefly.

2. Investigation

Organize pupils into groups, supply equipment and graph paper, and set them to work.

3. Conclusion

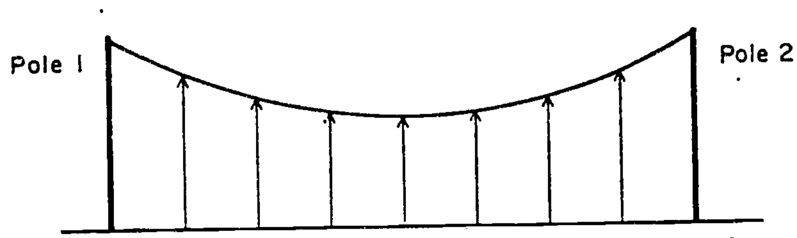
At a suitable time, go over the mathematics of the formula $y = ax^2 + G$ and discuss the range of answers for the problem. How could they find the rule for a real SEC poles and wire?

From: RIME.

HANGING WIRE - WORKSHEET

(a) Set up your own apparatus, with poles 80 cm apart exactly.

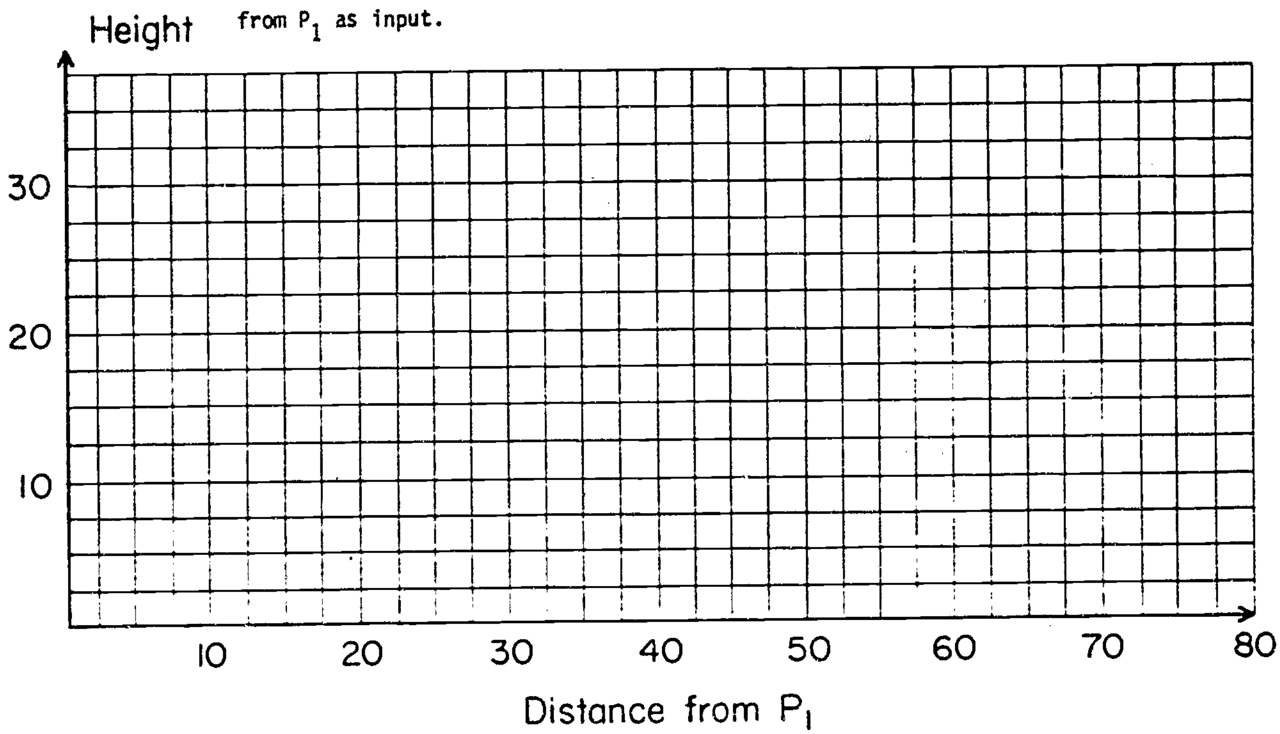
(b) Measure the height (to nearest .5 cm) of the string above the table top at 10 cm intervals along the table top starting at pole 1 until you reach pole 2. Put the results into the table of values below.



Distance	0	10	20	30	40	50	60	70	80
Height									

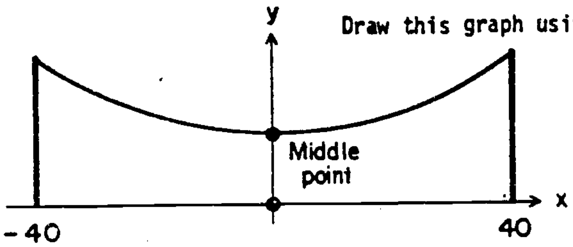
Is it symmetrical?

(c) Draw a graph of height in cm as output against distance in cm from P_1 as input.

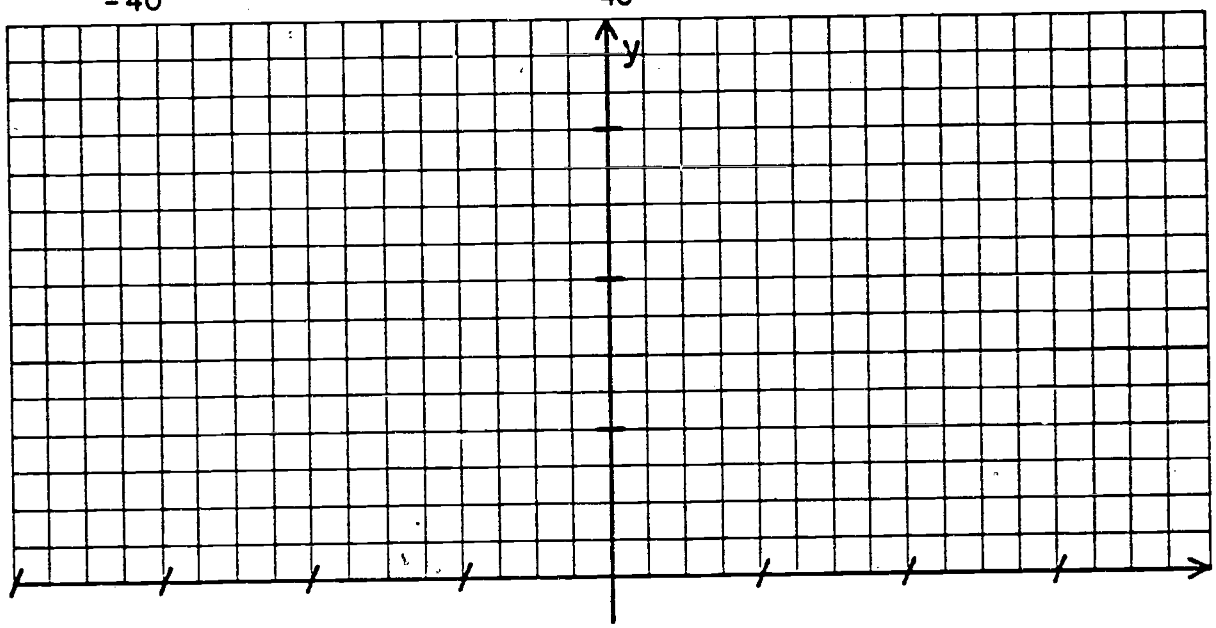


From: RIME.

(d) Mathematical Model: We will try to find a suitable rule, and draw its graph as a mathematical model for the situation. To do this we look at the picture in a slightly different way. We set the lowest point of our graph along the Y axis; and place P_1 and P_2 an equal distance either side.



Draw this graph using your previous table of values.



(e) Try to find a rule of the form $y = ax^2 + c$

where y = height
x = distance from 0, to fit your table and graph.

Give a reasoned argument on why you chose the values in your rule, and compare the values you get from your rule with the actual values.
Comment on any differences.

From: RIME.

Attachment D

Mathematical Modelling Unit in South Australia

Sample Lesson

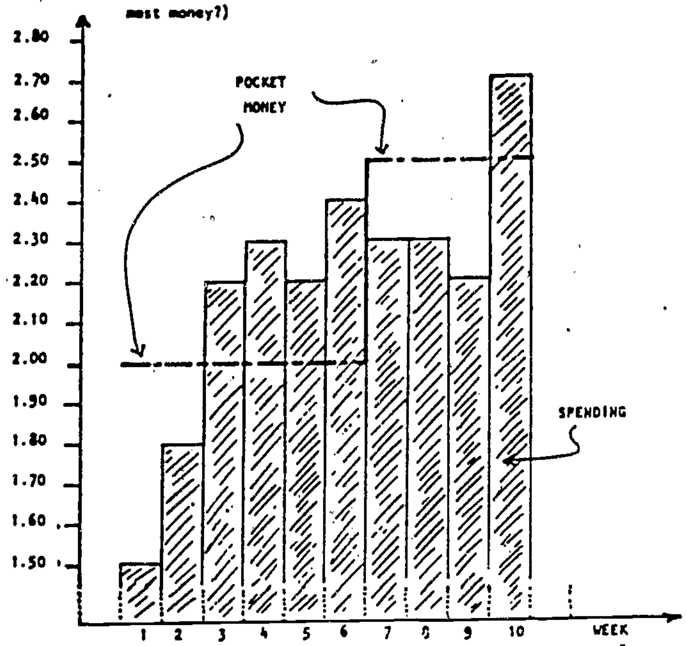
1. The graph shows young Jody's income (i.e. pocket money) and spending during a 10 week period.

   (a) In which week did Jody first receive a "rise"?

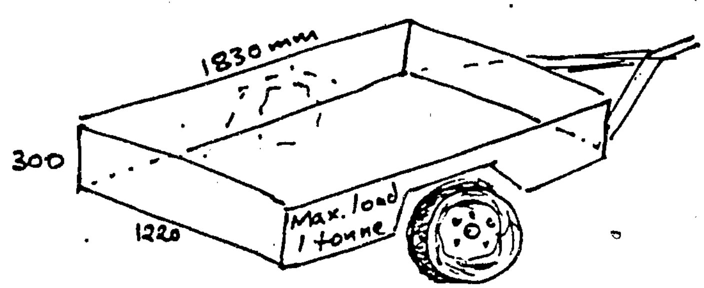
   (b) What was Jody's total income and total spending during the 10 week period?

   (c) During which weeks did Jody receive an "advance"?

   (d) When was Jody richest? (i.e. at the end of which week did Jody have the most money?)



2. I plan to carry 500 bricks from Littlehampton to Balhannah (a 30 minute "hilly" trip) using my trailer (pictured).



Naturally I don't want to balance bricks on top of my load (and have them fall off), nor do I want to risk overloading my trailer (and breaking the axle!).

How many trips should I take?

From: South Australia.

3. The section of railway timetable (right) shows that it is possible to travel from Adelaide to Brighton by:
- "express train" (no intermediate stops), or
- "slow train" (stops at each station along the way)

Sections		a.m.	a.m.	a.m.	a.m.
—	ADELAIDE	7 05	7 11	7 11	8 37
1	MILE END PSGR.	—	—	7 14	8 40
1	MILE END GOODS	—	—	7 16	8 42
2	KESWICK	—	—	7 18	8 44
2	GOODWOOD	—	—	7 20	8 46
3	CLARENCE PARK	—	—	7 22	8 48
4	EMERSON	—	—	7 23	8 49
4	EDWARDSTOWN	—	—	7 25	8 51
5	WOODLANDS PARK	—	—	7 27	8 53
5	ASCOT PARK	—	—	...	8 55
6	MARION	—	—	...	8 57
6	OAKLANDS	—	—	...	8 59
7	WARRADALE	—	—	...	9 00
7	HOVE	—	—	...	9 02
8	BRIGHTON‡	7 22	7 28	...	9 04

(a) How long does the express take to make the trip from Adelaide to Brighton?
(b) How long does the slow train take to make the same trip?
(c) How many more stations does the slow train stop at?
(d) What is the average time allowance given to the slow train for stopping at an intermediate station?
(e) How long would you expect the slow train to stop at an intermediate station — Goodwood for example. (Give reasons for your answer)
(f) If there was an express from Adelaide to Woodlands Park, how long would you expect the trip to take? (show your working)

From: South Australia.

I have four wooden boards which I plan to use to build a sandpit in the backyard for my little girl to play in. I have measured the boards; their sizes are:

2510 x 300 x 25

2000 x 302 x 25

1790 x 301 x 25

1500 x 300 x 25 (all measurements in mm)

(a) How should I design the sandpit?

The sand for this sandpit will have to be delivered to the front of my house, a distance of 50 metres from the place where I want to put the sandpit.

(b) How much sand should I order?

(c) How long will it take me to "wheelbarrow" the sand for the sandpit from the front of the house?

NOTE: Some background information that you might have tried to discover is supplied below. NOT ALL OF THE INFORMATION GIVEN IS USEFUL - CHOOSE SENSIBLY!

My little girl's toys include:

2 sandbuckets (small)

1 sandbucket (large)

2 plastic shovels (small)

1 wheelbarrow (small)

a plastic tiptruck

One tonne of sand occupies approximately 1.3

CUBIC FEET TO CUBIC METRES

To convert equivalent to [illegible] decimal point 2 places [illegible]

1	cubic	foot	=	0.028 316 85	cubic	metre
2	"	feet	=	0.056 633 70	"	"
3	"	"	=	0.084 950 54	"	"
4	"	"	=	0.113 267 41	"	"
5	"	"	=	0.141 584 26	"	"
6	"	"	=	0.169 901 11	"	"
7	"	"	=	0.198 217 96	"	"
8	"	"	=	0.226 534 82	"	"
9	"	"	=	0.254 851 67	"	"
10	"	"	=	0.283 168 5	"	"
20	"	"	=	0.566 337 0	"	"
30	"	"	=	0.849 505 6	"	"
40	"	"	=	1.132 674 1	"	metres
50	"	"	=	1.415 842 6	"	"
60	"	"	=	1.699 011 1	"	"
70	"	"	=	1.982 179 6	"	"
80	"	"	=	2.265 348 2	"	"
90	"	"	=	2.548 516 7	"	"
100	"	"	=	2.831 685	"	"

CUBIC YARDS TO CUBIC METRES

1	cubic	yard	=	0.764 555	cubic	metre
2	cubic	yards	=	1.529 110	"	metres
3	"	"	=	2.293 665	"	"
4	"	"	=	3.058 220	"	"
5	"	"	=	3.822 775	"	"
6	"	"	=	4.587 330	"	"
7	"	"	=	5.351 885	"	"
8	"	"	=	6.116 440	"	"
9	"	"	=	6.880 995	"	"
10	"	"	=	7.645 55	"	"

(Also supplied: a page on woodwork joints and on selection and use of woodscrews).

From: South Australia.

BELGIUM

C. Brusselmans-Dehairs
George Henry

In this paper we will discuss the application domain for both parts of Belgium. Our framework for this discussion will correspond in general terms with the Howson [1] four-dimensional curriculum model. We will consider in a first step the elementary education level and in a second step we will describe the situation at the secondary education level (academic, technical, and vocational tracks).

Elementary Education

The introduction since 1968 of the so-called renewed Mathematics in Secondary Education in Belgium has had a subsequent effect on the mathematics curricula at the elementary school level. Georges and Frederique Papy [2] of the Belgian Center for Didactics in Mathematics have had a great influence upon the further development of these curricula. The ideas of Dienes [3], Cuisenaire [4], Picard [5], and of the Nuffield Project also played an important role. These ideas were therefore integrated in the elementary school curricula. The renewal of the curricula was also affected by the idea that children can be involved in other activities than those normally included in the traditional curriculum, which mainly consists of arithmetic, measurement, elementary geometry, etc.

It is, however, worthwhile to note that the implementation of the so-called renewed mathematics was not the same in different districts and different parts of Belgium.

Aims

The inclusion of application aims in the elementary school curriculum must be considered as an important part of the general framework of the mathematical aims. Mathematics does indeed prepare pupils for social life as it will be at the end of this century. The ability to solve "real-life problems" should be developed at school. Also, other social aims can be identified, such as increasing cooperative skills (e.g., students can conduct problem-solving activities in a team).

New ideas concerning applications specifically aim to avoid students' considering mathematics as a difficult subject. One should hope that students will be more able to understand the nature of mathematics through a more diversified approach of mathematical applications.

At the elementary school level, the mathematics teacher is an all-around teacher. Mathematics is therefore considered as a suitable tool for environmental studies in different fields (biology, geography, etc.).

We are grateful to the I.E.A. Belgian Flemish Mathematics Committee for their important suggestions and comments.

Methods

Didacticians [6] have influenced teaching methods. In an increasing number of schools, real-life problems are presented, either by teachers or by pupils. More attention is given to team work which aims primarily at the development of social attitudes related to the cognitive areas (e.g., perseverance, critical thinking, ability to be involved in project work including other school subjects).

Content

The aims and methods have a decisive influence on the selection of the content that will be taught. The following topics are so far being taught: sets, operations on sets, relations, numbers (whole numbers and rational numbers), operations on numbers, structure, terminology, problems, measurement, discovery of space, geometry, ... Again, important differences occur in the different parts of Belgium in the emphasis given to these different topics.

Applications are used at the beginning and at the end of the teaching process to give concrete form to the theory. A typical lesson:

- begins with a well-defined real-life problem (example: computing a benefit expressed in percentage);
- points out the theory which allows one to compute this benefit;
- applies the theory of percentages to other application domains.

Rather rarely, a wider problem is defined (example: how to prepare a school trip) which is used as a source of multiple mathematical applications.

Evaluation

The results of the evaluation of the outcomes of the renewed mathematical curriculum are or will be available in both parts of Belgium. In the French part, a survey was carried out three years ago [7, 8]; in the Flemish part, a survey is organized every year [9]. Some efforts are made for developing pre- and in-service training from a didactical and from a mathematical point of view [10]. New evaluation techniques should be developed for evaluating high-level activities such as problem solving.

Summary

The Belgian mathematics curricula at the elementary school level (grades 1-6) have been deeply influenced at the intended level by the psychological approach (e.g., Papy, Dienes), but this is not so much the case at the attained level. In this prospect, learning strategies are strongly linked to the mathematical content in order to attempt to unify the intellectual structures and the mathematical structures [11].

The mathematics curricula are also influenced by a more general educational approach. Mathematics is only a part of the content that contributes to more general educational aims such as development of the personality and development of cognitive and affective structures [12].

Secondary Education

Educational trends differ drastically at this level (grades 7-12) when the educational tracks of the school system are taken into account:

- academic education (including classical, general, and transitional technical tracks);
- final technical and vocational education.

1. Academic Education

Aims

Mathematical educational aims are conceived from a hierarchical perspective by taking into account the prerequisites defined by the universities for those students who will follow mathematical or scientific tracks at the university level. Mathematics is, moreover, conceived as a progressive building, based on general structures deductively embedded across the whole curriculum.

The approach is typical for "new maths". The mathematics curriculum aims at training future mathematicians able to think in a highly oriented mathematical way.

The applications mostly aim at the illustration of the theoretical aspects. The integration with the other school subjects (except in some transitional technical tracks) is absent in nearly all cases. Also, there is a lack of coordination as far as the school programs, the language, and the terminology are concerned.

Method

The applications primarily aim at the illustration of theoretical concepts. Closed and open types of methods are simultaneously used. It can happen (but it is rather exceptional) that the maths teacher works together with another teacher (science, geography, history, ...) and illustrates an application from a mathematical point of view.

Content

Mathematical concepts are selected according to the hierarchical properties of the "new maths"; the organization of the content is a linear one. Mathematics is taught as a highly hierarchical subject based on some general structures.

Evaluation

During the past 15 years, educational innovations in mathematics have led to the following problems:

- an over-emphasis on mathematical thinking based on sets;
- an over-emphasis on pure abstractions, allowing no applications and no real concrete use;
- a pseudo-scientific jargon emphasizing symbols and terminology;
- an over-emphasis on pure reasoning as a source of mathematical ideas;
- a lack of a global view based on spatial perceptions and an emphasis on algorithmic thought based on formal algebra;
- the isolation of mathematics from the other school subjects and therefore from possible application areas.

The new maths curricula are very ambitious with respect to the content that the students have to master every year. This is a real danger because of the need to cover all of the content from a theoretical point of view: there is not enough time for applications.

Two trends exist within the mathematics curriculum committees. The first trend does not sufficiently emphasize the educational aspects of maths teaching. The efforts made by didacticians are often perceived as an intrusion.

The second trend is a reaction against the over-abstraction in the last 20 years. More and more people (curriculum developers, but also teachers and students) are convinced that a better integration of mathematical didactics into the maths curriculum must be attained [13].

A survey* is being carried out in both parts of Belgium in grades 8 and 12 for measuring the achievement of the Belgian students both on internationally based and nationally derived items. This survey will allow us to compare Belgian mathematical achievement with the achievement observed in other developed countries. It will give valuable information to the curriculum developers.

Curriculum Developer's Level

The mathematics curriculum does not achieve its aim if the student has not been involved in real mathematical applications.

- A greater individualization is necessary within the school system (especially in the comprehensive sub-system).
- It becomes, moreover, more and more clear that students achieve better on the specific topic when different approaches are used; this frequently can be done through the diversity of applications.

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* The I.E.A. survey is being carried out in all tracks of the school system (academic, technical, and vocational).

Teacher's Level

- The problem of the students' motivation becomes more and more crucial. By the use of applications, an increase in motivational level could possibly be expected.
- Many teachers of other school subjects are not sufficiently informed of the possible opportunities to use mathematics in their own domain. In the opposite case, the mathematicians do not feel the need to be informed of the school curricula in other school subjects because applications are not emphasized within the mathematics curriculum.

Student's Level

Students do not always perceive the potential use of mathematics because mathematics is taught in an abstract way and because of the lack of applications in real-life situations.

Actual Trends — Experimental Projects

In the Flemish part of Belgium, a Mathematics Committee has been set up since the beginning of the Second I.E.A. Mathematics Study. In this committee are curriculum developers from the different educational tracks. Using an action research methodology, the National Research Coordinator is trying to make the Committee members more aware of the problems related to the applications domain. A similar action should be developed in the French part of Belgium.

Some efforts have already been made to:

- inform the mathematicians of international trends concerning the mathematics curricula in the field of applications;
- organize in-service training with the help of people working within international organizations;
- stimulate coordination between mathematicians and didacticians.

This work will undoubtedly influence the Belgian maths curricula. One may hope that curriculum developers will invest more efforts in the field of applications.

From the interest growing towards applications in the international scientific literature, it is possible to outline the future trends:

- constructive cooperation between mathematicians and didacticians;
- the development of problem situations reflecting real-life problems;
- an analysis of the different levels of processes involved in problem solving within a class context reflecting the heterogeneity existing in mathematical abilities;

- an increasing importance of mathematics viewed as a game;
- the development of mathematical modeling.

2. Final Technical and Vocational Tracks

It is very difficult to give a clear outline of maths curricula in these tracks because there are a lot of differences between districts, schools, and types of job preparation. It is, however, possible to outline some general trends concerning the mathematics curricula.

Aims

The curriculum generally focuses on the needs of the students. Mathematics is considered by students as a tool for their real life. They must therefore be confronted with problems similar to those they will encounter in the future.

Differences do exist, however, between the expectations of the maths teachers and the expectations of the teachers of vocational subjects. Vocational subject teachers expect the maths teachers to train the students only for those skills which are prerequisites for vocational practice.

Mathematics teachers, on the other hand, think that the maths curriculum must also contribute to general education by leading to a development of understanding and reasoning. Therefore, the mathematics curricula must not be limited to those skills which are relevant for real life (vocational or daily).

Method

As previously mentioned, one aims here at a better integration of mathematics and of the other school subjects than is the case in the academic track. Depending on the districts and the schools, this integration is more or less effective. For instance, in the Flemish part of Belgium, the curriculum is developed following a "project unit" approach emphasizing "team work" [14]. In that case, content is of less importance; the main focus is on the method. Such a project unit approach is less usual in the French part of Belgium.

The integration of school subjects raises some serious problems (especially when a project unit approach is not used). The maths teachers often face difficulty in coordinating the teaching of the prerequisites (to the practical learning) and their own specific planning for teaching mathematical objectives. Moreover, mathematics teachers consider the development of a mathematical concept within a vocational framework as too complex for students in these vocational or technical tracks. Therefore, there is a tendency to build simplified situations allowing applications of skills. For these teachers, real-life problems are not a source for the learning process.

Content

Curricula are very often concerned with the fundamental prerequisites necessary to solve practical problems. The selection of the subject matter is therefore made by taking into account the real future needs of the student in his or her professional and private life. The content can differ strongly according to the characteristics of the schools. Real emphasis is often given to arithmetic (real numbers, decimals, rationals; operations; proportions and percentages; scales; and so on), metric system and units, geometric forms, etc.

Evaluation

Evaluation focuses mainly on basic skills and less on mathematical processes.

Actual Trends — Experimental Projects

Different trends exist within the experimental projects. On the one hand, there is a tendency for promoting a half-open curriculum; this kind of reform is based on project-unit and team work. So far, projects have been developed on the following topics: house, leisure, sport, traffic, and food.

On the other hand, there is a tendency to develop a more rigid curriculum (including evaluation units) where all of the mathematical skills needed for a specific job are carefully defined (as in the UNICAP project).

Summary

As previously mentioned, it is very difficult to get a clear understanding of the general trends of applications in the mathematics curricula. Differences do indeed exist between districts, schools, and tracks. Moreover, it is important in the educational system to distinguish between the intended curricula and the attained curricula, the last one being more difficult to grasp.

It is, however, clear that in Belgium applications are not considered as very important where the academic tracks are concerned. It is our hope that the contemporary trends in mathematics curriculum reform will lead to corrective measures which will have a considerable influence and will increase the role of applications in the near future.

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BRAZIL

Ubiratan D'Ambrosio

It seems unquestionable that mathematics related to real situations is an educational trend all over the world. Many educators even use, to my view in a distorted way, the concepts of "back to basics" and "problem solving" with this in mind to imply such ideas as "useful mathematics", mathematics for everyday life, and similar ideas. This has not been at all the way "back to basics" and "problem solving" have been practiced. In developing countries, traditionally uncritical followers of trends of the developed countries, the danger is even greater. I want to discuss a few such dangers, with some illustrative examples.

Let me first elaborate on my concept of applications of mathematics. There is no doubt that I think of real situations as a starting point — reality in its full content, i.e., socio-cultural, physical, phenomenological (e.g., weather), and even psycho-emotional. This means that an individual is immersed in a reality of his or her own; feelings, interest, perceptions of reality are very personal. The complexity of situations that are real, i.e., belong to his or her reality, cannot be denied. The way he or she depicts reality is very personal, and the parameters one distinguishes in this depiction are privileged according to values, intentions, and even contingencies. This does not deny the objective reality of the real world, but the perception of this reality is the key issue in education. At the same time, the natural non-submissiveness of man to this reality is undeniable. It is proper to our species to act upon its environment. The most striking characteristics of our species is action, specifically intelligent action. This action, even when purely cognitive, directly modifies reality. The real world changes as the result of continuing action, the perception of the real world changes as the result of cognitive action, and action is generated by the perception of reality. Now, the crux! How directed can this action be? What degree of direction is acceptable in guiding the perception of reality, which will generate the actions? School systems provide direct mechanisms (see [1]).

Mathematics Education is a particularly strong directing mechanism. Mathematics, as with every codified discipline, has its rules, signs, and basics, and its effectiveness in explaining real situations or solving problems derives from the appropriate use of accumulated accepted knowledge and the handling of necessary techniques — of course preceded by ingenuity, creativity, and insight. The first, i.e., the codes, rules and signs, results, and techniques, are necessary to "speak" mathematics, and can be taught and learned to some extent by just about any human being. The second set, i.e., ingenuity, creativity, and insight, must be cultivated as a plant needs seeding in a proper ground and nourishment. Here, fertility is determined by motivation, interest, curiosity, and initiative. We cannot prevent ourselves from recognizing that formal school systems, particularly in the developing world, fail to develop such qualities, indeed work in the opposite direction. Excessive authority, identified with language and even posture of the teacher, reinforced by culturally absurd evaluation mechanisms, kill the natural components of insight, creativity, and ingenuity, with which children arrive in school (see [2]).

Unfortunately, we are gathering results which show that mathematics as a school subject has been responsible for much of this sad state of affairs. Particularly striking evidence has been obtained in the case of Latin America in the framework of the Second International Study in Mathematics (see [3]). It is shown that the ability to perform the basic operations of arithmetic, the component most energetically and intemperately sought for in the back-to-basics movement, is completely unrelated to functioning intelligently in new situations. This is also shown, for example, in the research conducted by Ruggiero [4]. "Problem solving" does not improve the situation. The way it is so often done, it is more of a drill in solving type-problems, artificially mounted to allow for immediate applications of the basic operations and usually to present artificial results.

Initiatives

If we look into what is being done to improve the situation in Latin America, particularly in Brazil, it will lead us to an overall analysis of the complexity of factors which Education faces in the region — lack of public support, lay teachers (sometimes with no understanding of what they are required to teach), lack of adequate literature, and pressure resulting from the school environment which causes an excessive conservatism. This is well-explained by applying the recent psychoanalytic approach to classroom practice in changing societies, as in the case of developing countries (see [5]).

Many factors can be identified. Indeed, a project is underway in defining socio-cultural variables in Mathematics Education. This is particularly important when we realize that there is an ethnomathematics going on, with all the characteristics of what we would call applicable mathematics, which is practiced by children and adults up to entering school. Then a conflict arises which disables children from functioning mathematically in the real world. This has been studied elsewhere (see [6]). It is remarkable how the introduction of calculators produces fewer of these problems, i.e., current everyday practice allows for an easy transition to the use of machines, while traditional school practice (paper-and-pencil operations) practically kills any previously known mathematical skills. This is now under analysis from the cultural anthropology viewpoint in several projects going on in Brazil.

Unlike most Latin American countries, Brazil does not have a national curriculum of studies. Each school system, indeed each individual teacher, has the legal right to organize his or her own program. In practice, this does not occur and programs are shaped more or less homogeneously all over the country, the major cities serving as models. There, largely urban populations have expectations of schools either as a step toward upward social mobility (hence aiming at college educatiom), or as a need to fulfill basic legal requirements for lower middle class employment in the public services, commerce, banks, etc. These latter rarely require more than primary education, and the standards are minimal. On the other hand, the former group look for an education that allows them to pass the highly competitive entrance examination to the universities. The program is strongly dictated by what is required in these exams, called "vestibular". Secondary education is dominated by these requirements, which are classical and based on training and drilling for multiple-choice testing.

Hopeful Signs

A few examples of attempts to introduce more lively and creative programs can be found, both in the primary (8 years of compulsory schooling, beginning at the age of 7 years) and secondary (three years of schooling) levels. These are isolated cases, and cover interesting applications of the most varied nature. Two examples can be mentioned: the Project developed under the sponsorship of the Ministry of Education (PREMEN) from 1973 through 1982 on Statistics and Probability and Computer Science for Secondary Schools, and Geometry, Functions, and Equations as applied to a large variety of problems for upper primary (ages 13-15), as well as Models for early primary school (ages 7-10).

This last project brings the concept of modelling into the very early years of schooling, dealing with real-life situations. Problems dealt with are, for example, the construction of a kite and the construction of scale models of houses, cars, etc. The novelty, in the Brazilian school scene, is that "theory" goes together with "doing". Measurements are taken, analysis and planning take place, material is bought, and an object is the final product. This approach is that of a "project", with "theoretical" reflections on every step of the process.

As we have made clear, these innovations are restricted to a few research groups and special training centers and are hardly known of in most of the country. From the point of view of national visibility more hope was put, at the secondary level, in a series of books on applications [7]. In trying to reconcile what is required in the entrance examination to the universities with more creative and interesting mathematics, the authors introduced applications. For example, some problems are taken from newspapers, analysing financial news perhaps. These books did not displace the traditional textbooks and have been used mostly by teachers who are themselves curious and eager to learn a little more. Best sellers, on the other hand, have trivial and non-challenging applications of the drill kind.

The most effective efforts now are concentrated at the graduate level, to prepare trainers of teachers. A considerable number of seminars and workshops are being held on applied mathematics for college teachers who are in charge of training prospective primary and secondary school teachers. This was motivated by observing that, because of the highly traditional and theoretical curricula in the teacher training courses, it is unlikely that teachers will be able to stimulate other than routine and trivial applications in their classrooms. Calculator use is generally forbidden, even in university calculus (and numerical analysis!) classes. Statistics rarely appears in teacher training courses.

Overview

This is the general picture in developing countries: the less developed the country, the more formal and theoretical is mathematics teaching. The examples from Brazil can be easily extrapolated to other countries.

As mentioned elsewhere, in spite of protests from conservative teachers and parents, calculators are a thriving business, and children manage them with ease. Last week, my ten-year-old nephew beat me easily at his Christmas gift, a calculator game, although he has recently failed badly in mathematics! There is now no mystery in the core of "basics", so that teachers are afraid of being replaced by calculators!

Summing up, except for artificially formulated drill problems that hope to give an impression of usable or applicable mathematics, little on applications of mathematics is going on in the formal school systems of developing countries. As yet unrecognized, ethno-mathematics is being practiced by uneducated people, sometimes illiterates. Some effort at the research level is going into identifying those ethno-mathematical practices and to incorporating them into the curriculum. But there is much to be done before the pedagogization of ethno-mathematics takes place, and even more before ethno-mathematics becomes recognized as valid mathematics.

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BRITAIN

Hugh Burkhardt

In writing about the position of applications in school mathematics in Britain, I shall not go over the general features which have been noted in the introductory Overview. Despite the proper identifications of this country with the pragmatic approach to mathematics, based on an emphasis on applied mathematics at all levels stretching back through Dirac and Maxwell to Newton and beyond, the picture is much the same here as elsewhere — very little in the way of applications in the main-stream curriculum after the early years of school, together with some interesting projects that have as yet no large-scale impact. The one established exception to this tradition, a 50 percent component on theoretical mechanics in the 16- to 18-year-old range for those pupils who choose mathematics as one of only three Advanced Level subjects, has been substantially eroded over the last two decades by the offering of statistics as an alternative. In this report, I shall concentrate on current trends and ideas, but with a backward look as well.

In heading this report "Britain", I have chosen an informal title to reflect the informality of the review of several more or less independent education systems. England and Wales, to which the bulk of my comments apply, have separate but closely linked and similar education systems. In each, as far as anything is the same in this land of unwritten constitutions and varied traditions, the curriculum is formally the responsibility of the Headteacher of the individual school. The relative uniformity of what actually happens in classrooms suggests that these distinctions may not be as important as they appear, but nonetheless they do allow and encourage a great deal of experiment and innovation (computing, for example, has spread slowly but steadily from a beginning in a few schools in the 1960s, when universal provision would have been regarded as prohibitively expensive, as it was in other countries). Scotland, on the other hand, perhaps partly due to its traditional links with France, has a curriculum that is in principle centralised. Textbook provision, public examinations, and many other matters reflect this. Here, too, there are a number of interesting projects that should be mentioned. Northern Ireland has again a separate but linked approach, more like the English. Eire is not discussed explicitly here, though that too belongs to the same historic educational tradition.

Background

As already mentioned, applied mathematics in Britain has meant mechanics. Starting at the age of 16, able students have successively studied the mechanics of particles, systems of particles, rigid bodies, fluids, and so on. Apart from this specialised pursuit, only in primary school, from ages 5-11, have applications played a significant part — with counting and measuring and "reckoning" money important classroom activities. Even here, schooling from the age of 7 or 8 is increasingly dominated by the search for maximum proficiency in imitative exercises on arithmetic operations on whole numbers, fractions, and decimals. Not surprisingly, this has resulted in an unbalanced curriculum for all but the most able children who could take these demands "in their stride". For most children an almost total concentration on arithmetic was accompanied by repeated failure at it — a recipe for educational disaster, well authenticated by the research of the Concepts in Secondary Mathematics and Science Project [1], which shows the total mismatch between targets and achievements. The normal secondary education for the 11-16 age range contains virtually no applications, apart from artificial illustrations:

> "Milk bottles have either gold or silver tops. If there are 3 gold and 2 silver bottles in a box and I choose one at random, what is the chance I pick a silver top bottle? Picking 2 bottles at random, what is the chance that they are both silver topped?".

It is worth considering how the present situation has arisen from recent history. Over the last two decades, secondary school mathematics classrooms in England have been dominated by the books of the School Mathematics Project — over 60 percent of them used it as the standard text at one time. The Project began in the early 1960s, a collaboration of able teachers, teaching able children in selected (and selective) schools. They saw a need for a more lively, more interesting mathematics curriculum related to applications of mathematics in everyday life and other fields. The original draft text was stimulating and contained interesting applications. Revised and published as a five-year (11-16 year-olds) course for the top 20 percent of ability, these Books 1-5 included a small proportion of tasks such as the design of a home heating system and scheduling problems including usually critical path analysis. These were found too challenging even for their target group. The publishers saw a need for a course for average-ability children, and Books A-H are the outcome. The mathematical techniques covered are much the same, but the more difficult examples, including all serious applications based on situations, are omitted.

Other projects with similar ambitions, such as Mathematics for Education and Industry, have had similar motivation, found similar difficulties, and produced similar results. The history of this period has been nicely reviewed by Howson [2].

The second special circumstance to be mentioned is the "Cockcroft Report" [3]. It is the report of a Government Committee of Inquiry into the Teaching of Mathematics in Primary and Secondary Schools, which over three years collected "evidence" from 900 contributors on which its report was based. The very warm welcome it has received reflects the skill of the Committee in identifying a consensus on what the major problems are and in which directions we should try to move; they are understandably less clear about mechanisms for progress, though they make some interesting suggestions. Applications are central to their thinking. They say, for example, in paragraph 34:

> Most important of all is the need to have a sufficient confidence to make effective use of whatever mathematical skill and understanding is possessed, whether this be little or much.

Implementation of the Report's recommendations will be a slow and incomplete process, but it is beginning. The limited outcomes in practice from the good intentions in principle in the past, however, serve as a useful warning of the challenges involved.

Examinations

Public examinations play a very important role in determining the curriculum in mathematics in Britain. In England there are examinations taken at age 16 by over 80 percent of the pupils, with further examinations at 18 for those who stay on at school. These have a dominant effect on the curriculum from 14 years onwards, and a strong influence at least from the age of 11. The selection exam at 11 for secondary education, which has largely disappeared over the last 20 years, had a similar effect on the curriculum from age 7 which has largely persisted in practice. What are these effects? They reflect a number of pressures, all of which tend to narrow the range of tasks set in the examination, and thus to narrow the curricula in practice; these include a concern that questions:

(a) should be clear and unambiguous, so pupils know what is expected,

(b) should be straightforward, so that pupils who have worked hard and have been properly taught should always be able to do them,

(c) should cover the syllabus topics,

(d) should discriminate clearly over the whole ability range.

The result is an overwhelming predominance of content-based imitative exercises in mathematical techniques at a level of difficulty indicated by a pass mark of 40 percent. As for applications, any but the most stereotyped illustrations tend to be excluded by (a), while most of problem solving is precluded by (b). It is (d) that, by concentrating on discrimination at the top of the ability range, condemns many pupils to an examination experience which feels like failure, even though they may pass. Those components, particularly at Advanced Level, which partly break these general rules, represent extremely difficult tasks, because the concept of lowering the technical demand to match an increased strategic problem-solving demand is not sufficiently recognised.

The influence of the examination system is widely recognised and there have been attempts to use it to support the curriculum, usually through alternative examinations tied to particular curriculum development projects. These, at least when they have spread beyond the pioneering teachers who were involved with the project, have been subject to similar pressures with similar results — leaving differences largely related to syllabus content. There are exceptions, including particularly some statistics examinations which have a significant realistic component, but they tend to be taken by few schools, presumably for the same reasons.

Initiatives

The initiatives which aim to get more applications into the mathematics curriculum are many. In the new courses that are being developed in Britain, illustrations are more frequent and more realistic than in the previous generation of courses. The first calculator-based course, by Peter Kaner and others, which aims to cover the 11-16 age range, is an example, as is the new SMP 11-16 course currently being developed and trialled. A study of situations, however, still forms a very small part. This approach is found more commonly in courses for low achievers, particularly towards the end of compulsory schooling at age 16; free from examination constraints, courses on "money maths", for example, take a down-to-earth situational point of view. None of these initiatives face the teaching style difficulties inherent in handling real problems. Thus all are more concerned with exposition and practice of standard models — a useful but insufficient preparation for using mathematics in the world.

An effort to go beyond these limitations is the Open University course for teachers, "Maths Across the Curriculum" [4]. Based loosely on the U.S. project, Unified Science and Mathematics in the Elementary School (USMES), it aims to develop open methods of classroom work in the tackling of realistic problems. It gives teachers experience in realistic problem solving and suggestions on how to stimulate it in the classroom. It has had some success with some teachers, but it is not clear how far it will have any substantial impact.

Of earlier effort along rather different lines is the Mathematics Applicable [5] project, led by Christopher Ormell and funded by the Schools Council. It aimed to teach mathematics to able non-specialist 17-year-old students through a modelling approach. It introduced some interesting new examination techniques, including clues-at-a-price on the written paper, and included a written project on a modelling topic. This work again continues on a small scale, and has tended to shrink rather than to grow.

The Shell Centre for Mathematical Education has made a number of initiatives in this area. After an early review [6] of possibilities, it has produced a "starter pack" [7] which aims to give the interested but uncommitted teacher a flavour of what is involved in teaching real problem solving in the classroom while minimising the demands for style change this imposes. Most of the material, though open in its demands on the pupils, presents closed demands on the teacher, involving mainly explanation and the grading of student responses according to explicit marking schemes. By the end of the 5-10 hours of teaching involved, it has reached the stage of involving complete realistic open problems. Teachers have found this material useful and interesting, but it doesn't pretend to be an answer to all the difficulties. Work on more technical skills with a strong illustrative link to real situations is quite successful — for example, material on the interpretation and sketching of graphs [7] and on understanding decimals works well within the conventional spectrum of classroom activities, and yet extends it a considerable way in the direction of realism and usefulness. A current initiative aims to use the examination system to enhance, rather than as at present to narrow, the curriculum. Studies of the use of the microcomputer as a class teaching aid show real evidence, and perhaps the best hope currently available, of giving teachers the support they need to widen their range of teaching style [8].

In Scotland, the Munn and Dunning Reports have led to a serious attempt by the Scottish Curriculum Development Service to provide examinations and materials for lower-achieving pupils that are closely linked to practical applications — travel distances, working periods, poster layouts, and measuring problems of many kinds are included. However, the tasks are all neatly posed elements of relevance to a realistic situation rather than credible realistic problems. Impetus towards a curriculum with realistic applications remains strong.

The Future

Despite the difficulties that the study so far clearly illustrates, a number of these initiatives do give hope for the future, though clearly careless optimism is out of place. We have already mentioned some developments going along traditional lines, and a few that are more innovative.

Another in this category has a pleasant irony. The Assessment of Performance Unit of the Department of Education and Science was set up by the Government to monitor standards nationally — the impetus related to concerns about basic skills, particularly in arithmetic, and many people were worried that it would produce yet another pressure for narrowing the curriculum. In the outcome, the designers of the mathematics tests (and to an even greater extent those of the science tests) chose to include items relating to applications and to include a mathematics practical test. Though these tests are not as "visible" as those of the public examinations, their content is known and does exert some influence. The reports make interesting reading [9].

A feasibility study is in progress to see if it is possible to use the main Board examinations to enhance the curriculum in a similar way. A large examination board and the Shell Centre are working on the design of modules, each containing examination tasks, teaching materials, and teacher support material related to a particular curriculum activity — consumer decisions, interpreting graphs, everyday problems, number investigations, and a realistic approach to one-dimensional mechanics are among the early developments. Great care is being taken not to make too-great demands on teachers, and all elements are being developed and tested on a representative sample of the target group, which includes most teachers of secondary mathematics. Two new elements give some hope of success. First, the modular nature of the change means that teachers have only a small amount of new material to take on board at any time, allowing some concentration on it. Secondly, the teacher support package, a multimedia one, is based on the evidence of success in promoting style change by using a microcomputer as a teaching aid [8]. Of course, the key element in this initiative is the dominance of the curriculum by public examinations; though this is not a factor in all countries, there are others which might find the same approach useful, if it should prove acceptable and successful.

The balance between ambition and acceptibility in such initiatives is a crucial one. The middle ground between our ideals and the status quo has perhaps not been sufficiently explored.

Finally, the most recent Government initiatives for low attainers, arising out of the Cockcroft Report [3], also aim to produce tests of achievement in curriculum material aimed at curriculum enhancement. This is merely the most explicit example of the impetus which the Report has given to initiatives of this kind in England and Wales. It has created a climate of thought in which applications are once again at the centre of attention. However, as we have seen so often, intention is not enough. When the realistic targets for performance by pupils and teachers will really be recognised and accepted remains, at best, an open question.

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CANADA (ONTARIO)

H. Howard Russell

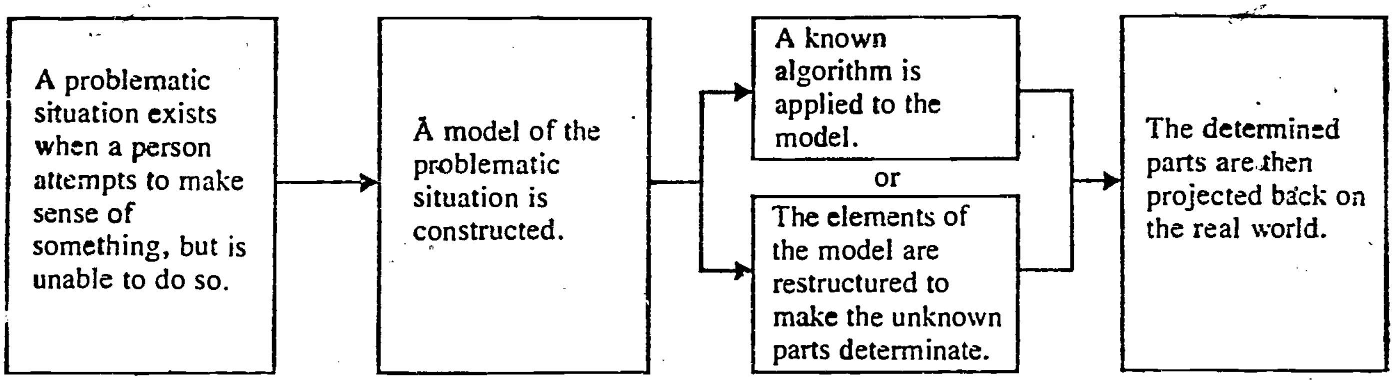
There is no shortage of mathematics topics which can be proposed for inclusion in the school programme of a nation on the grounds of applicability to "real life". Rather, there is a shortage of ways of using mathematics in practical situations. There is a shortage of people who know how to use mathematics in real life; there is a shortage of people who know how to teach mathematics so that it can be used and will be used in practical situations; and there is a shortage of effective, documented means by which either of the first two shortages can be overcome. Ontario seems to be similar to other countries around the world with respect to these particular shortages.

The precise nature and extent of the alleged shortages may be determined in the near future through the Second International Mathematics Study of the I.E.A., which has introduced the concepts "Intended Curriculum", "Implemented Curriculum", and "Attained Curriculum" and which will report summary data about the various curricula. If the Intended Curricula show serious attention to mathematical applications and problem solving whereas the Implemented Curricula or the Attained Curricula show no such thing, then the truth of the allegation of shortages will be more likely to be widely understood.

There are two documents already in existence which deal with the Intended Curriculum and the Implemented Curriculum, at least insofar as they apply to Ontario. These documents are especially revealing on the issue of problem solving or applications. The document which outlines the Intended Curriculum is the official guideline produced by the government [5], and it states "the guideline places a major emphasis on application of mathematics". This same guideline contains Figure 1; its presence in the official document gives it special importance. It is a clear indication that application is an important part of Ontario's Intended Curriculum.

The second document is a research report that describes the Ontario programme as observed in the classrooms in 1974. This relatively recent study shows some clear gaps between the Intended and the Implemented Curricula, and the areas of greatest concern were geometry, measurement, and problem solving. If the situation is basically unchanged since 1974, and if other countries have problems similar to Ontario, then the SIMS reports are almost certain to show that problem solving and mathematics applications are really in need of concentrated attention.

The editors of mathematics education journals seem to have made the judgment that attention to problem solving is needed and they have devoted considerable space to the topic in recent years. The <u>Ontario Mathematics Gazette</u>, a regional journal of 20 years' standing, has presented articles by Smith [9], Kulm [3], Nicholls [6], and Woods [10], and there are other publications by researchers such as Carlow [1] and Robinson [7] which give some specific attention to how to proceed with problem solving. One of the features that these publications have in common is the recognition that the problem is of a very fundamental nature. They suggested that a firm



Although this progression applies to problem-solving situations in general, it is being considered here in relation to mathematics. In learning to solve problems in a systematic manner, students should:

- develop their ability to identify and formulate problems of a logical-quantitative nature;
- develop their model-building skills;
- develop a widening awareness of the genuine applications of mathematical ideas in both school subjects and real life, as well as an understanding of the uniqueness and limitations of logical thought;
- use increasingly sophisticated models and procedures to deal with problems arising from real-life experiences.

In order to place appropriate stress on problem solving, teachers will need to introduce the above emphases in their mathematics courses. When students are involved in activities of this kind over an extended period of time, they will improve their ability to solve problems in a systematic manner.

Figure 1. Excerpt from Ontario Ministry of Education Guidelines.

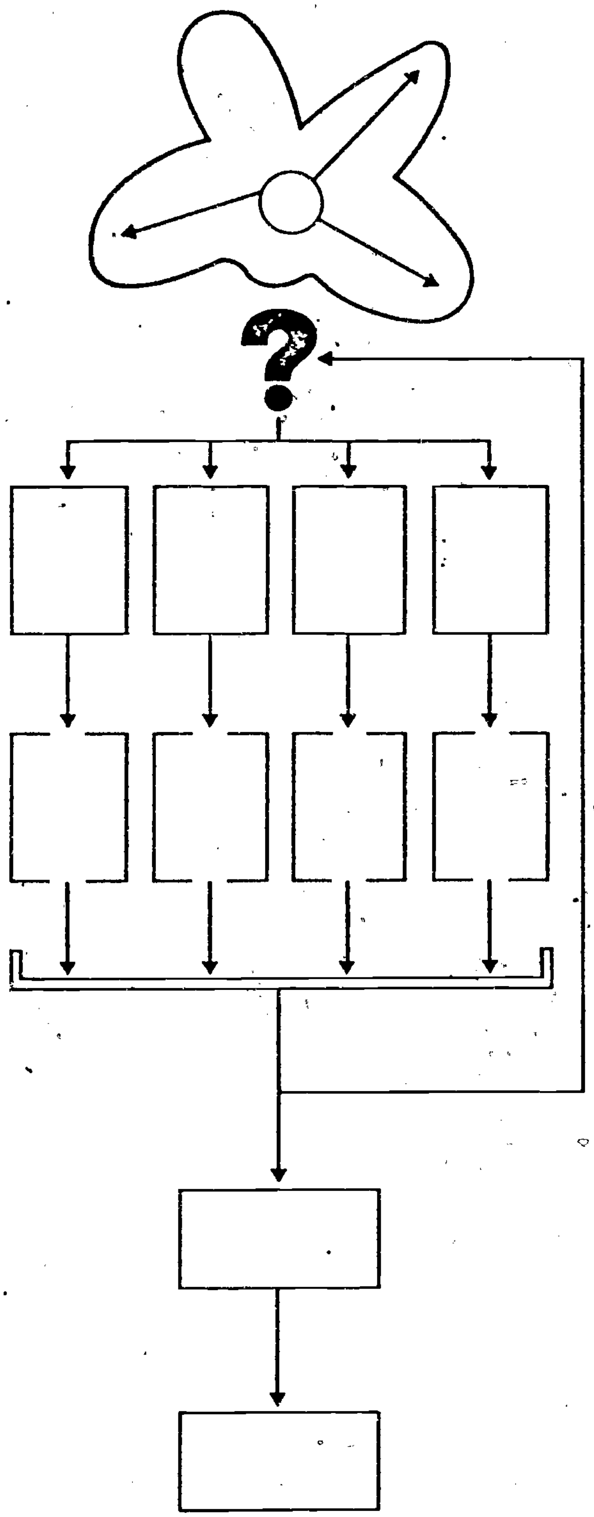
theoretical foundation is essential to any proposed plan to rectify the situation. Also, there seems to be a recognition that the theory base should involve not only mathematics but also other relevant disciplines such as psychology and sociology. Furthermore, there is a need for something more than the strong theoretical base, and that is a "down-to-earth", practical way of developing and carrying out the plan.

An Initiative

One illustration of an approach to mathematics applications cannot reveal all of the many details of the various approaches under consideration, but it can provide some insights into the nature of the approaches. Robinson [7] is responsible for the particular illustration which follows, and as an introduction to it, a significant anecdote from notes dated 1973 may be helpful. Robinson was essentially a mathematician in the 1950s, a psychologist in the 1960s, and a researcher in the 1970s, and beyond that he was responsible for a considerable amount of "field service" work with schools. The most revealing episode in this anecdote took place in the office of a senior education official who was at that time, and still is, in charge of programme planning for a school system. This official had a long history of support for educational research in general and for the work of Robinson in particular. At the time of the meeting in question, Robinson had been working with the official's staff for about two years in an effort to improve the problem-solving capability of students by constructing a new part of the mathematics programme especially focussed on problem solving and applications of mathematics. The plan which emerged at that time was based on the elements of Figure 1, and it included an algorithm with the essential steps found in most problem-solving schemes (see Figure 2). The school official reviewed the programme with Robinson and commented that it seemed to be the same scheme that he had seen Robinson present several years earlier. He was very annoyed when he found that Robinson agreed that the "new" material which was created by the official's staff was firmly based on existing schemes. The official was quite emphatic in his comment that he and his staff did not need to be "tricked" into going along with a plan for improving the programme. He was also quite confident that they would have gladly saved two years' time and effort, and implemented the scheme on the basis of the earlier documentation.

The non-trivial point of this story is that Robinson was able to reach to the bookshelf behind the official and take out a report that had arrived in 1969, and which contained the plan and a considerable amount of documentation. It was then clear to the official that Robinson could not change a school system's programme merely by handing them an empirically validated programme with the features they needed. He could not do this even in a school system which had a positive view of his work. He had to begin by working with the staff of the school system, and he needed to help them attack the issues that they perceived to be most critical. The fact that they came up with a programme that looked like "Robinson" is not really so much a duplication of effort as it may seem to be. It is a testimony to the fact that Robinson had done his homework. He had devoted many man-years of effort to the task and that effort represented academic expertise in mathematics, psychology, and some other disciplines as well. This programme would work now because the school system staff had built it piece by piece so that it would respond to their problems. It was

The Basic Inquiry Model



Initial Experience:
Exploratory activities are introduced.

Question:
The student poses a suitable question around which the study will develop.

Alternatives:
The student suggests a range of reasonable alternatives to answer the question. (Additional alternatives may arise in the subsequent data-collection stage.)

Data:
The student collects information on each alternative.

Synthesis:
The student arrives at a conclusion by deciding, on the basis of the accumulated information, which of the alternatives give(s) the best answer to the question.

Assessing the Conclusion:
The student ascertains whether the conclusion adequately answers the original question.

Expressing the Conclusion:
The student organizes a clear expression and presentation of the conclusion.

Evaluation:
The student assesses the appropriateness of the conclusion and its expression in the light of the original question.

Figure 2. The Basic Inquiry Model.

now their programme. There is more to the problem of implementing innovations than this one anecdote reveals. The work of Fullan [2], as well as that of Leithwood [4], deals with the issues in more detail and yet there is no suggestion that they have found the cure.

One instance of an application of mathematics in the manner proposed by Robinson will clarify in some ways just how he works. Typically, students are presented with a "problem" of the type, "In order to buy a motor bike that costs $470, Sam saves his allowance for 50 weeks. This was enough when added to the $120 Sam already had saved. How much was Sam's weekly allowance?" Such a problem is presented so that the students can use the three-stage algorithm that they have been rehearsing. However, such a problem is not a "real" problem because Sam is almost certain to know what his allowance is, and furthermore he would not think of such a cumbersome scheme for finding out his allowance anyway.

A more realistic problem comes from the same setting. Sam really wants a motor bike. One of his friends now has one, so the need is evident. Sam has $320 in the bank, which his father has encouraged him to save toward his post-secondary education. Sam does not really have an allowance, but he earns $2.00 per hour for doing odd jobs for his father. He averages about $12.00 per week revenue and spends about half that on adolescent luxuries. Sam's father will probably lend him the money he needs, at the going rate of interest. Now Sam knows he has a problem, but what is it? Not knowing what the problem is, precisely, is a characteristic of real-life problems.

Here are some of Sam's thoughts: "Will my dad let me take some of my savings, say $100, to put down on the motor bike? How long will it take to pay off the debt if I get the motor bike like my friend's and if my earning and spending stay the same as the past few months? What additional costs will I have, such as license, maintenance, and required equipment? If I am to pay off the debt within a year, as Dad wishes, how many extra hours of 'paid work' per week would I have to do?" The problem has emerged more clearly now and models can be contemplated as in Figure 1 and Figure 2.

Sam can continue to refine his problem until it is sufficiently well-stated to permit the use of an algorithmic approach. When he moves to the second box in Figure 1, he can choose the "Basic Inquiry Model" shown in Figure 2. Sam had to choose among alternative questions in the first instance. How he must choose among alternative ways to proceed. He needs more information, however, and he can see this readily. He needs to know the full cost of the bike including all the extras. He needs to know about insurance and when he must pay it. He needs to know about operating costs and how he will pay them. He needs to distinguish between weekly and monthly revenues and expenditures and he needs to accommodate all the information.

When the computations are completed, Sam must be able to interpret the result. He must know from his own intuition whether or not h s answer is likely to be correct. He must know whether or not he thinks it is worth it. The next step is a discussion with his father, and at this stage he should be able to explain all the data, the steps, the computations, the assumptions, and the result.

The Robinson scheme for handling real-life problems is well-documented now, and there are other similar schemes available for school systems and teachers, wherever they may be. Those who will not be content until today's youth are provided with improved opportunities to learn to apply mathematics have their work cut out for them. They must find a theoretically sound approach to the issue and then they must confront the more difficult problem of ensuring that their solution works in all types of classrooms. If and when they are successful, there will be one less gap between the Intended Curriculum and the Implemented Curriculum.

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FEDERAL REPUBLIC OF GERMANY

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1. The School System in the Federal Republic of Germany

In order to facilitate understanding of the following review, we shall first give a brief description of the school system in the Federal Republic. Longitudinally, the school system is subdivided into Primarstufe (ages 6 to 10), Skundarstufe I (ages 11 to 16), and Sekundarstufe II (ages 16 to 19). Various school types are assigned to these levels.

School level	School type				
Primarstufe age 6-10	Sonderschule	Grundschule			
Sekundarstufe I age 11-16		Hauptschule low level	Realschule middle level	Gesamtschule all levels	Gymnasium. high level
Sekundarstufe II age 17-19	Berufsbildende Schulen				

Graduation from Skundarstufe II at Gymnasien and Gesamtschulen means that graduates can enter the university. Students may "in principle" change from one school type to another according to performance. The Federal Republic of Germany is politically organized into eleven Federal States which have different syllabi. Our reports says nothing about the Sonderschule (special schools) or the Berufsbildende Schulen (vocational schools), which have evolved into a multitude of specialized types.

2. On the Conceptual Discussion about Applications in Mathematics Education

Since about the mid-1970s, the topic of how to treat applications of mathematics in the classroom, and why, has played an important role in the discussion among mathematics educators. A host of application examples and a number of theoretical conceptions have been published in the ten or more West German periodicals dealing with mathematics education, and there have been several conferences exclusively devoted to applications. Notable among these are the conferences "Anwendungsorientierte Mathematik in der Sekundarstufe II" in Klagenfurt (Austria) in 1976 [6], "Anwendungsbezüge im Mathematikunterricht" in Oberwolfach in 1979, and the international conference on "Co-operation between Science Teachers and Mathematics Teachers" in Bielefeld in 1978 [14].

In the many published articles containing examples with applications, the standpoint adopted is for the most part a pragmatical one. There is a quest for problem situations which are realistic, relevant, acceptable in terms of extramathematical knowledge required, mathematically accessible, compatible with syllabi, and possibly even suited to take over a decisive function in teaching some part of mathematics. The fact that there are much too few examples of this kind in literature which, moreover, are not even accessible to the teacher, has led to various activities of documentation and of developing application examples in the frame of projects, and to the publication of several volumes containing application examples (see sections 4 and 5). Besides these examples, a theoretical discussion arose which aims at integrating the role of applications into an overall conception of mathematics education. This increased discussion about applications in mathematics education is partly due to the failure of the attempt to reform the mathematics curriculum in the sense of "new math", which has rather increased the distance between curricula and practical life. One of the characteristics of this reform was alignment of the curricula to the "structure of the discipline" as demanded, for instance, by Bruner. Criticism of this position first led to formulating an extreme counter-position, i.e., the demand for interdisciplinary project-teaching. In part, the debate saw strong contradictions between aligning the curriculum to mathematics on the one hand, and aligning it to the pupils' interests and to the real-life situations for which school should prepare them on the other. Meanwhile, there are a number of German educators whose concern is to develop positions mediating between those two, even if this problem has not yet been solved to satisfaction. In doing so, they do not primarily criticize that the "new math" curriculum reform was aligned to mathematics as a discipline, but rather, first, that it was based on a somewhat one-sided idea of mathematics (Bourbaki's interpretation) and, second, that there was a too-direct transposition into the classroom without prior reflection on the question of pedagogically justifiable general educational goals.

One of the fairly early approaches which has, moreover, met with great acceptance for Sekundarstufe I is that of the general educational goals developed by Winter [39], which correspond to general aspects of mathematics and of man. Mathematics is seen as a creative science, as a proving/deductive science, as an applied science, and as a formal science, and is related to the general goals of learning heuristic strategies, learning to prove and argue, learning to mathematize, and learning to formalize. These goals correspond, more generally, to general human activities and abilities such as creativity, reasoning, shaping technology and the economic and social environment, and communicating and representing information. This serves to derive perspectives and concrete proposals for treating individual applications in the classroom within the frame of this overall conception. In the following, we shall attempt to present some of the ideas about the role of applications in mathematics education which have been developed in the course of the last ten years and which could also be of interest to the reader abroad.

Just as in other countries, the concept of mathematical modelling or of mathematization is central for the theoretical debate. This amounts to taking a common perspective with regard to the question of how independent of the type of mathematics and of the kind of problem situation mathematics is applied. There are two more substantial doctoral theses with regard to educational theory (Beck [20], Weber [18]) which treat principles for the so-called "application-oriented mathematics education" globally and on several levels. The central point of reference, for both doctoral theses, is a general differentiated scheme of the process of mathematization which serves as foundation for considerations pertaining to classroom activities, learning goals, and concrete application examples. The authors demand — just as others do — that unabbreviated (or at least consciously abbreviated) modelling examples be realized in the classroom. They postulate that there is something like general abilities to mathematize. Mathematics education is to develop these in order to equip students with transferable general abilities which enable them to apply mathematics in new situations as well. Often, less demanding objectives are formulated: the student is to experience modelling processes in an examplary way and think about them in order to get an idea of how mathematics is used in real life. Here, the development of abilities of one's own to mathematize is referred to the background.

A more detailed analysis of various conceptions which are all presented under the designation of mathematization or modelling, however, shows that these conceal in part quite different accentuations and purposes which are also revealed in different process descriptions, and differ quite significantly from approaches pursued in other countries. We shall briefly treat some of these approaches in an exemplary fashion.

In some approaches, the principle of mathematization is tied in with ideas on "genetical teaching" which are of long standing in Germany. Wittmann [42] defines the "genetical principle" in his well-known volume on mathematics education, saying that teaching mathematics should set out from the students' previous understanding and should be aligned, on the basis of intra- and extramathematical problem situations, to the "natural" processes of creating mathematics. The concept of mathematization is connected with processual ideas about how mathematical concepts and methods develop from application contexts. This necessarily implies consciously changing the point of view: from the creation of new mathematical instruments across the conceptual and structural analysis and specification of these instruments to the further application to new situations. Such global processual ideas are to provide the guideline for the method of teaching; the student is to be made aware of the changes of point of view in order to achieve a motivated transition from coping with the situation to dealing with more mathematical questions. Beck [20] has included this conception in his doctoral thesis, specifying it by means of teaching examples. In this context, mathematization is mainly also a general principle of teaching, which, as compared to the simple transfer of process descriptions given by applied mathematicians to the classroom, has already been "educationally reflected".

Steiner [38] has developed a process schema for processes of mathematization which differs from the usual descriptions of modelling processes as well. Its emphasis is both on processes of genesis of new mathematics from real problem situations and on their subsequent organization in logical-deductive or axiomatic form. Then follows an application and reinterpretation of the theories developed for new situations, which may lead to an extension or modification of the mathematical theory. Steiner stresses processes such as defining intuitive concepts, idealizing and abstracting, local and global logical organizing, proving theorems and axiomatizing. Students, so to say, are to assume the role of scientists in the classroom, experiencing the process of creating and re-applying mathematics while actively participating in it. The processes which have taken place are to be reflected upon in the classroom, in order to have the students acquire exemplary insights about how mathematics can be generated from coping with reality, but then gets very distant from the original situation, organizing its knowledge in an abstract and logical-deductive way. What is to be made clear is the usefulness of this method, particularly the connected, general applicability of abstract mathematics to so many different situations. Here, mathematization is not so much understood as a general teaching principle for introducing mathematical subject matter, in relation to problems outside mathematics. General abilities to mathematize in the sense of general strategies of coping with the environment are not to be taught either. Rather, knowledge about mathematics and its reference to reality is to be taught, and this in the sense of philosophical and methodological reflexion.

Fischer and Malle [8] have taken up the processual schema developed by Steiner, formulating further general goals in connection with the treatment of examples of mathematization: the insight that mathematics develops formal models for areas of reality; the insight into the difference between model and reality, i.e., that various mathematical models can be developed for one situation, and that one mathematical model can be used to describe and analyze different real situations. Besides, the expectation is that more general cognitive stragegies and intellectual skills can be developed by an appropriate treatment of application examples. These more general goals are also taken up in the two doctoral theses by Beck [20] and Weber [18], and partly specified by further teaching proposals.

The goal of teaching knowledge about mathematics and its reference to reality were originally very strongly linked with the educational conceptions for the upper grades of the German Gymnasium, which were mainly concerned with preparing for scientific studies and which strove to teach a critical view of the importance of individual sciences and their applications. It was later, however, transposed in partly modified form to other school forms of Sekundarstufe I, though without adopting the philosophical intensions. The assumption was that this education, too, is not meant to teach simply "technical knowledge" in the sense of "modelling skills". Hence, there is often the demand that mathematics education in school should teach a balanced image of mathematics as a comprehensive cultural and social phenomenon (see also Beck [20]; Blum [23]; Fischer and Malle [8]).

We have tried to show that the concept of mathematization is in part understood differently in the German discussion, and has given rise to educational intentions which are different from, say, the conception of "mathematical modelling" in the English-speaking countries. With regard to the application examples published, the following can be noted:

on the one hand, there are a number of mathematics educators working on the "program of mathematization", for instance by developing additional specific teaching proposals in this direction, and by providing in-service training for teachers. On the other hand, many examples are published which neither explicitly nor implicitly adopt this concept. Their authors do not share the partly high transfer expectations with regard to "abilities to mathematize". There is confinement to examples which are mathematically rich and interesting and relevant in terms of extramathematical content (cf. the "pragmatical point of view" characterized at the outset).

In the following, we shall briefly deal with some developments which we should like to interpret as critical of the universalist/normative conception of mathematization even if this criticism is not explicitly stated. Conceptions which intend to develop a general ability to mathematize often normatively assume that pupils should themselves adopt the role of an active 'mini-modeller" in present or future situations of life which mathematics education shall help them to cope with.

There are some studies in the German Federal Republic which have approached the question of which qualifications are necessary to cope with situations in life not only normatively, but also empirically — and this means in a concrete-historical way as well. We shall describe one of these studies in more detail as it excels in scope and methodological sophistication while advocating an approach part of which seems to us to be of more general importance (Damerow et al., [4]).

The study's point of reference is the future vocational role of most of the students of Sekundarstufe I as wage-earners in the present society. This role is treated from the standpoint of an educational theory critical towards society: it is not only a matter of identifying qualifications in the working environment which are necessary to cope with these situations in a technical sense. Rather, the goal is to identify qualifications which are required to behave competently and autonomously, too, in such situations. The postulate of self-reliance is derived from the requirement of a critical educational theory which says that school should qualify the pupil to act independently. The accounting and bookkeeping of a large industrial enterprise serves as an example which is analyzed from this perspective, identifying more than 20 qualifications related to mathematics and to applying mathematics. Particular emphasis is on so-called translation skills which are required to transform knowledge partly formulated in everyday language into mathematical-formal formulations and vice versa. In addition, the ability to discern between the mathematical model and its empirical content is stressed.

The accent placed on a critical attitude towards society, and the emphasis on emancipation as an educational goal, are objectives presently pursued by the MUED project (see section 4.2), in which the student is not only seen as a wage-earner, but also as a participant in public debate. The approach sketched would seem to be of more general importance, in our opinion, in the following respect: the student is not seen in the role of a "mini-modeller", but as a person objectively affected by the application of mathematical models and methods. The goal perspective adopted is his or her ability to act and orient himself or herself in a partly mathematized reality. In our opinion, a number of teaching proposals have adopted this perspective, while a theoretical superstructure comparable to that of the conception of mathematization has not yet been developed in mathematics education.

Also to be criticized is the fact that principles with regard to methods of teaching are normatively derived from a general notion of how mathematizing processes take place. We consider it to be a deficit of the current research into mathematics education that there are practically no systematical empirical studies about teaching-learning processes which refer to treating applications in the classroom and from which new perspectives for teaching could be obtained.

In the Federal Republic, there are, besides, some studies which approach the problem of integrating applications into teaching from a theoretical perspective other than that of mathematization. Some educators consider it necessary to analyze specific parts of mathematics and their reference to applications from an epistemological and historical perspective in order to attain a differentiated understanding of mathematics which includes the reference to reality and does not consider the latter something additional, as do the "classical" philosophical interpretations of mathematics in the sense of formalism, intuitionism, logicism, and Bourbakism. In this perspective, there are some studies on stochastics (probability and stochastics, Steinbring [37]) and on geometry (Bender [21], Schreiber [36]). For them, the concept of mathematization is secondary for two reasons: first, they do not adopt the perspective of the process in an application situation, but rather aim at developing an application-oriented mathematical knowledge. Second, the problems of applying, say, the probability concept and statistics are raised in a specific form. Relevant for this is the philosophical and methodological discussion in the "philosophy of science", the concept of mathematization providing only a few extremely unspecific orientations.

Finally, there are efforts to approach the problems raised by the application of mathematics not from epistemology and philosophy, but rather from cognitive psychology. Notable among these studies is that of Scholz [35] which deals with psychological modelling of empirically established thinking processes in solving application tasks from stochastics.

This trend toward specializing on part disciplines of mathematics in theoretical research is matched, in the more pragmatically oriented studies, by an evident focus on the question of how individual curricular topics can be taught in a way referring to applications. Many studies in this field are concerned with stochastics, which was introduced as a regular feature of most curricula in recent years. In 1980, there was a conference on this topic attended by many mathematics educators from the German Federal Republic in Klagenfurt (Austria). The proceedings (Dörfler and Fischer [7]) inform about the manifold activities with regard to application-oriented stochastics teaching.

3. On the Role of Applications in the Curricula of Various School Levels

In this section we shall give an overview about the role applications assume in the various curricula for the school levels. The materials referred to are syllabi, textbooks, and impressions concerning practice in school.

Systematical empirical studies, however, exist only for Sekundarstufe II in the shape of a representative survey among teachers. Before that, section 3.1 gives a brief description of the evolution of an educational tradition concerning the integration of applications, which, under the designation of "Sachrechnen", has had a great influence on the curricula of the Primarstufe and the Sekundarstufe I.

3.1 On the Development of Sachrechnen

Until the beginning of the 1970s, the so-called "traditional Sachrechnen" prevailed. Since then, this conception has been transformed into a broader conception. In his comprehensive analysis, Strehl [15] gives the following characterization of "traditional Sachrechnen":

1. "With regard to subject matter, Sachrechnen is calculating with measures and weights" (Strehl [15], p. 10). The following fields were considered to be the nucleus of traditional Sachrechnen: computing interest, computing percentages, alligation, profit and loss, computing proportions, rule of three.

2. Sachrechnen should be treated on all levels of school, as the objective of mathematics teaching was conceived to be the preparation for the so-called "computing situations of everyday life".

3. Sachrechnen is also a certain method for teaching arithmetic operations. Four stages are distinguished. Teaching should start with more or less real situations, then the mathematical content should be made explicit. After a stage of practicing the new skills, they should be applied to new situations. In practice, however, no great importance was attached to real-life situations. The situations concerned were "presented almost exclusively linguistically in the shape of word problems, which means that "Sachrechnen" and solving word problems can be used almost synonymously" (Strehl [15], p. 11).

4. The comprehensive claim of Sachrechnen was often connected with separating mathematics education in the Grund-, Haupt-, and Realschule from that in the Gymnasium. This means that the pretended realistic character of Sachrechnen in Grundschule and Hauptschule is placed in opposition to the exact, abstract mathematics of the Gymnasium. Sachrechnen was also treated in the Gymnasium, but only until grade 7.

At the beginning of the 1970s, the small success of Sachrechnen gave rise to criticism. The following drawbacks of traditional Sachrechnen were noted:

- it was confined to such problem situations of everyday life which could be solved by using the four basic arithmetical operations, a fact which led to unrealistic examples;

- the subject matter of Sachrechnen was organized according to intra-arithmetical aspects;

- questions of correct notation and pronunciation, of text design and computation were dominant;

- it was hostile towards mathematical methods and concepts (cf Winter [40]).

From this criticism, broader conceptions were developed which aim at extending the traditional understanding of Sachrechnen. Thus, the mathematical core of Sachrechnen was stressed, the repertoire was extended by including modern mathematical subject matter such as statistics, probability, calculus, and functions, and new methods of teaching were developed, including ideas of mathematization (see section 2).

Winter proposes the following understanding of "modern Sachrechnen": "I should like to understand, by the term of 'Sachrechnen', rather a guiding idea of mathematics education, which is realized by developing mathematical concepts in close correspondence with relevant applications (in everyday life, nature, society, ...), and this for mutual clarification" (Winter [41], p. 82).

3.2 The Curricula of the Primarstufe

3.2.1 Practice in School

There are no systematic studies as to school practice. This is why we must depend on individual impressions which result from classroom observations, and from interviews with teachers, educators, and representatives of publishing houses. These yield the following impression: a great number of small word problems taken from multiple fields of reality are used. Frequently, they are an occasion to exercise mathematical ideas — predominantly arithmetical ones. The teacher's attention is on correct computations, while the discussion of the situation seems marginal.

Small situations are often used as a motivational and intuitive basis for the introduction of new mathematical ideas. As the textbook has a guiding function in primary school instruction, situations are presented in the shape of word problems and pictures. Projects starting out from the children's immediate experience are less widespread.

With regard to subject matter, applications of arithmetic prevail. Geometry and stochastics are underrepresented in the classroom as compared to syllabi and textbooks.

In the educational debate about applications in primary school, interdisciplinary teaching is of considerable importance. In school practice, however, it is hardly implemented. Part of the teachers shun the effort it implies and think that it is too demanding for the pupils. Mathematical ideas from the topics of magnitudes and statistics are copiously used in Sachunterricht (science). Art teaching uses ideas taken from geometry.

3.2.2 Textbooks

In the German Federal Republic, there are about 20 different textbook series which are licensed to be used in schools. The textbooks (see Rohrbeck [33]) show the author's efforts to use application situations referring to various fields (school, leisure, the working environment, home, traffic) and to communicate information and data in different ways (tables, graphs, statistics, texts, toy money, sketches, diagrams, ...).

Realistic problems prevail, but there are also tasks containing unrealistic questions and data, or even false information: a dealer's "profit", for instance, is calculated from the difference between buying and selling price.

Besides, the social role of men and boys on the one hand, and women and girls on the other, as represented in the textbook problems, is aligned to stereotypes. These stereotypes do not result from an individual problem's lack of realism, but rather from the frequency of problems which mainly use men or women in certain situations (women, for instance, are always shopping and hardly active at all in other situations). Where women appear in a working context, they are most frequently badly paid. While the lower salaries for women are a fact of real social life, there are no hints in tasks of that kind which lead towards critical discussion.

In order to get a more precise impression of primary school textbooks, the role of Sachrechnen in two textbooks will be presented in more detail. The first of these volumes is widely used in school, while the second is new on the market and gives wide scope to interdisciplinary teaching.

- In the textbook Mathematik, Denken und Rechnen (Schmidt et al. [34]), about 50 percent of all pages are arranged according to the following pattern: the upper third of the page shows a picture (usually a photo, sometimes a drawing) which represents situations of everyday life (for instance, a snack bar with price boards; children playing on a see-saw; a post office with mail rates; photos of the speedometer of a bicycle, a motorcycle, a car; a network of trains; etc.). The pictures represent genuine or construed situations. The rest of the page often contains problems in which the pictures are analysed. The pictures are meant to provide a surrogate for genuine experiences, to demonstrate the course of activities, and to provide data.

- The new textbook Spielen, Rechnen, Selber Denken (Guderian [28]) makes considerable effort to establish relationships to the teaching of arts, science, and German language. Besides, it attempts to encourage the pupils to project-like work. The solutions of the problems usually require about two lessons.

  To give an example: a double page provides information about the automatized production, the packing, and the sales path of nails to the consumer by means of pictures, tables, and text. Explicit questions are not asked. The project's goals are to teach realistic information about production processes and buying and selling, to place pupils in an open situation in which they are encouraged to ask questions. In this situation, they are intended to apply and confirm their knowledge about the value of money, weights, and numbers. Throughout the textbook, there are reproductions of paintings and statues by modern artists (e.g., Le Witt, R. Indiana, E. Higelman, Jung and Back, F. Stelle, M. C. Escher) which are studied as to their geometrical and arithmetical principles of construction. The children also are encouraged to draw pictures of their own according to the principles just discovered. The intention is to ensure that allowance is made for the concerns of both subjects (mathematics and arts), but also for the children's expectations.

3.2.3 Syllabi

In those parts of the syllabi concerned with goals, the role of Sachrechnen is traditionally important. The parts referring to subject matter mainly distribute the magnitudes to be treated among the respective school years under the topic of Sachrechnen. Suggestions with regard to how to teach real problems are rare. The 1973 syllabus still valid in the Federal State of Nordrhein-Westfalen goes beyond this frequently unbalanced presentation of Sachrechnen, and this is why we are going to describe it.

Several aspects of Sachrechnen are contained in the syllabus:

(a) The subject matter aspect: Sachrechnen comprises teaching magnitudes (money values, lengths, weights, numbers, periods of time, and areas and spaces in a propaedeutic way), and an introduction of stochastical ideas (collecting, representing, and classifying data from situations experienced; propaedeutics on the concept of probability).

(b) The psychological aspect: Sachrechnen may serve to develop and confirm mathematical concepts by making use of real situations, as these offer opportunities to motivate pupils, to illustrate facts, to exercise, and to link intramathematical concepts to those of the environment.

(c) The pragmatical aspect: Sachrechnen serves to qualify pupils for coping with everyday life situations. It requires the pupil to do the following: to understand situations, to ask questions, to obtain and relate data, to establish mathematical relationships, to solve the problems within a mathematical model, to interpret the solution, to vary data, to modify elements of the situations, and to enquire whether the problem solution will fit other situations as well.

The above syllabus justifies treating the guiding structural concepts (set, relation, operation) not only on the basis of their intramathematical importance, but on the very basis of their significance for situations of the environment, in arts, language, and science.

The syllabus does not point out the connection between the subjects in detail. It still remains the teacher's task to establish them. If we consider, however, the syllabi of other subjects, the possibilities of cross-disciplinary teaching can be guessed. For instance, the 1977 syllabus for science teaching in the Federal State of Baden Wurtemberg alludes, among other things, to the following abilities: to organize information and to obtain it oneself; to read tables, graphs, maps, and other forms of representation; to grasp connections by means of guesses, rules, and regularities. The arts syllabus of the Federal State of Nordrhein-Westfalen of 1973 contains topics which could be taken from a mathematics syllabus: problem-solving techniques like constructing, combining, analyzing; the picture elements point, line, area; circle, spiral, zigzag, symmetries; getting familiar with body and space; getting acquainted with ornaments, grasping their regularities.

The chances for a kind of Sachrechnen which does not merely jump from one small Sachrechnen task to the next are favorable:

- In the Grundschule, the class teacher teaches several subjects to one class. He or she is familiar with the subject matter of all, a fact which ought to facilitate interdisciplinary teaching.

- In the Grundschule, the class teacher may arrange lessons more freely, a fact which enables him or her to organize projects without being obliged to prove that these are devoted to mathematics, or German language, or arts.

Sachrechnen in textbooks, nevertheless, is confined to word problems. This certainly indicates both a deficit in teacher education and in teacher attitudes.

3.3 The Curricula of Sekundarstufe I

The worldwide reform of mathematics education carried through in the sense of "new math" in the 1960s was made compulsory by a decision made by the Conference of the Federal State Secretaries for Cultural Affairs (Kultusministerkonferenz) on 3 October 1968. According to an analysis of this reform (Damerow [27]), the decisions of this conference led to the following modifications of the syllabi as compared to the traditional ones which had been developed after World War II: early introduction of concepts of set theory; teaching algebraic structures, at least in the Gymnasium; elimination of applications; early introduction of certain subject matter like algebra; and emphasis on structural concepts (cf Damerow [27], p. 230). The elimination of applications concerned especially the Hauptschule and the Realschule, in which the treatment of various applications had usually been required. They were generally treated — as described in section 3.1 — in the frame of "Sachrechnen", which was reduced, in the guidelines published by the Kultusministerkonferenz, to its formal-mathematical aspects.

There are no published analyses concerned with the present state of the orientation towards applications in the syllabi for Sekundarstufe I. A survey of the presently valid syllabi, especially of the draft syllabi of the 11 Federal States, yields the following results.

In the Hauptschule, the focus of applications is again, just as before the decisions of the Kultusministerkonferenz, on Sachrechnen, the latter having undergone the change of meaning described in section 3.1. Thus, standard equations, elementary functions, and descriptive statistics are to be treated as new topics, besides the classical fields of Sachrechnen like percentages, interest, rule of three, etc. Additionally, some syllabi continue certain tendencies of "Reformpädagogik", which was of great influence in Germany before World War II: the treatment of extramathematically structured fields, which serve a more extramathematical purpose, is prescribed for the graduation classes. The topics of these fields are usually taken from business and trade. Some syllabi developed in recent years require that the relation between reality and model be reflected, and the students be enabled to mathematize.

The syllabi of the Realschule remain diverse — just as they were before the decisions of the Kultusministerkonferenz. As in the case of the Hauptschule, applications tend to gain in importance by taking recourse to parts of traditional Sachrechnen and by including modern mathematical fields such as stochastics, elementary functions, and standard equations. Extramathematically structured teaching units are prescribed but in a few syllabi, and often in the shape of additions, especially optional subject matter. As a whole, these syllabi are less application-oriented than the syllabi of the Hauptschule, and less than the syllabi of the Realschule were before the decisions of the Kultusministerkonferenz.

The syllabi of the Gymnasium are even less application-oriented than those of the Realschule. There are hardly any applications, and Sachrechnen, too, remains mostly confined to its formal-mathematical aspects. Some syllabi developed for the Gesamtschule tend to counteract this; thus, some syllabi strive to assign a bigger status to Sachrechnen in its extended meaning, and to introduce, similar to Hauptschule and Realschule, extramathematically structured teaching units toward the close of Sekundarstufe I.

A survey of the most widespread textbooks (the first five per school type) yields similar results, which are, among other things, due to the fact that textbooks, in the Federal Republic, must be authorized by the Kultusminister before they can be introduced in the classroom. Authorization depends on their congruence with the syllabi of the Federal State concerned.

In the textbooks for the Hauptschule, the focus of applications is on parts of traditional Sachrechnen, supplemented by modern mathematical subject matter taken from the fields of stochastics and functions. Some of the textbooks published during the last two years go beyond the syllabi by providing an extramathematically oriented chapter for every grade, not only for the graduation classes. Besides newer fields such as problems of (economic) growth and energy resources, they make allowance for more traditional fields which have come to be forgotten, e.g., surveying.

The textbooks for the Realschule are just as diverse as the syllabi with regard to applications. A tendency evident in more recent textbooks, especially in new editions of textbooks developed in the 1960s, is de-emphasis on structural elements and allowance for the extended understanding of Sachrechnen. As in textbooks used in the Hauptschule, only rarer, there are extramathematically oriented teaching units; textbooks of the Realschule often exceed the requirements of the syllabi with regard to applications. More than other school types, the Gymnasium still works with older textbooks which were written towards the end of the 1960s and at the beginning of the 1970s. In these textbooks, there are almost no applications at all: structural elements prevail. More recent textbooks which are often conceived for Realshcule and Gesamtschule as well show characteristics similar to those of the Hauptschule and the Realschule.

For the more recent textbooks for the various school levels which can be expected to determine the development of the years to come, the following can be noted with respect to reality content and function of applications: for the fifth and sixth grades, applications mostly serve intramathematical purposes, i.e., they serve to develop conceptions which are to be used in application situations presented at higher grades, or to guide pupils towards intramathematically oriented problems. From the seventh grade on, applications' purpose of contributing to an understanding of extramathematical situations becomes increasingly important. Especially the extramathematically oriented teaching units aim at teaching extramathematical knowledge and hence give more scope to representing the extramathematical context. Besides these units, grades 5 and 6, in particular, are dominated by short application tasks with little extramathematical context. This corresponds to the fact that applications in grades 5 and 6 tend to be more removed from reality. From grade 7 on, there is a shift of emphasis towards tasks closer to reality. Some textbooks — very few — attempt to reflect upon the connection between model and reality, partly by discussing the adequacy of solutions, especially mathematical models, partly by pointing out how application problems can be solved.

3.4 The Curricula of Sekundarstufe II

The following considerations refer exclusively to the generally educating part of Sekundarstufe II, i.e., mainly to the last three grades of the German Gymnasium. It contains no statements about the field of vocational school which has the largest proportion of pupils by far.

3.4.1 Some Results of a Representative Survey among Teachers

In Fall 1982, the DIMGO project, organized at the Institut für Didaktik der Mathematik (IDM) at Bielefeld University, which is empirically concerned with mathematics education in the senior classes of the Gymnasium (ages 17 to 19), carried out a representative survey among teachers. Each (!) German school of that type received a comprehensive questionnaire. Since the response was 85 percent (about 2000 teachers), the results can be considered to be highly representative.

Table 1 presents the answers to those items of the questionnaire which refer to applications. For each item, six possibilities of response were given, ranging from 1 = always to 6 = never.

TABLE 1

	(always) 1	2	3	4	5	(never) 6
1. How often did you start out from one or several extramathematical problems when introducing a new topic?	1%	16%	21%	22%	34%	5%
2. How often did you, in order to exercise, pose:						
a) problems which were worded in an extramathematical context and for which the emphasis is on solving the mathematical problem?	1	13	28	30	25	4
b) problems taken from mathematics fields of application for which the emphasis was on solving the extramathematical application problem?	1	9	14	23	40	14
3. How frequently did you use "project teaching" in your course?	0.1	0.3	1	3	9	87
4. How often did you present the following problem types in written tests:						
a) "word problems" which require that the facts stated in the problem are transformed into a formal language?	7	28	32	24	8	2
b) problems which require that mathemathical models be found in order to solve extramathematical problems?	0.1	1	3	8	29	59
5. When introducing a new concept, how often did you have the students work first on an extramathematical problem which made it necessary to introduce the new concept, and to define it?	1	9	14	19	39	19
6. How often was there a discussion in your course about several different mathematical models for one extra-mathematical problem situation?	0.4	5	11	20	45	19
7. How often was there a discussion in your course about the limits of the applicability of mathematics to extramathematical problem situations?	1	4	9	17	43	27

Table 1 (continued)

	1	2	3	4	5	6
8. How important were the following aspects for you in your course:						
a) applying mathematical methods to solve extramathematical problems?	8	21	26	23	19	4
b) social significance of mathematics?	2	8	14	18	35	23

[For item 8, 1 equals "very important" and 6 equals "not important at all".]

9. What part of the teaching time (in percent) did you devote to the treatment of extramathematical applications in your course?

≤ 5%	≤ 10%	≤ 15%	≤ 20%	≤ 30%	≤ 40%
18	58	62	83	94	98

[The mean time for applications — in the broadest sense — is approximately 15 percent of the whole teaching time.]

Among the results whose detailed interpretation is left to the reader, it must be noted that word problems are relatively more important than "genuine applications" (items 2 and 4). The extremely rare use of application tasks in written tests seems to be significant (item 4b). Broad conceptions for teaching applications (modelling and model discussion, items 3, 6, 7) are less frequent. Besides, it seems remarkable that insight into the social significance of mathematics has a very low status or none at all for almost 60 percent of the teachers (item 8b). The difference as compared to the answers to item 8a might perhaps be interpreted as showing that the applicability of mathematics is emphasized — if at all — rather more in a technical sense without a broader intention of teaching general insights about the importance of mathematics.

3.4.2 Some Results of an Analysis of Syllabi

In the senior classes of the Gymnasium, the dominant feature with regard to the mathematical subject matter to be taught are the so-called "three pillars": calculus, linear algebra/analytical geometry, stochastics; time allotted and binding character decrease in that order. In a roundabout way, the following trend can be noted: the number of applications in the three subject matter fields decreases in the following order: stochastics, calculus, linear algebra/analytical geometry.

In the Germany Federal Republic, there are 11 Federal States which have syllabi differing considerably as to type and scope. Some of them treat general educational goals of mathematics education and general questions of teaching methods very extensively; others confine themselves essentially to listing mathematical subject matter. With regard to applications, there are considerable differences as well. In the following, we shall briefly report some results obtained from a detailed analysis of syllabi (see Biehler [22]), in particular of those parts of the syllabi which formulate general principles for mathematics education, and of those which refer to the subject matter of calculus. We shall also consider four syllabi which, in some sense, represent the extremes.

The syllabus for the Federal State of the Saar does not contain any principles for mathematics education and is essentially limited to subject matter catalogues and detailed learning goals with regard to mathematical contents. Dozens of learning goals are stated with regard to calculus. Only one of them (!) refers to extramathematical applications. As this syllabus is intended to be of "intentional completeness", in particular as there are central graduation examinations in the Saar as opposed to most of the other Federal States, it can be said that teaching applications is basically incompatible with this syllabus.

Contrasting with all others, the syllabus of Nordrhein-Westfalen excels by its specificity with regard to stating general principles for mathematics education. Among other things, it requires teachers to teach

- the ability to develop mathematical models of real situations in simple cases and to use reality to check statements obtained by deducing within the model;
- knowledge of the part mathematics has had in the development of science, technology, and economy, and insights into the social significance of the subject;
- knowledge of important fields of application of mathematics, of relations with other subjects, etc. ...

In addition, there is an entire chapter in the syllabus on the topic of "integrating applications of mathematics in the natural and the social sciences", which expressly provides encouragement to carry through projects in the classroom "if there is sufficient time". With that, the syllabus, in principle, seems to provide unlimited opportunities for application-oriented mathematics education. In the field of calculus, for instance, it can be noted, however, that the prescriptions for the basic course do not contain treatment of applications at all. For the advanced course, only one single application example is required. Besides, the amount of compulsory mathematical subject matter may be an additional obstacle to including more extensive applications in the curriculum. In addition to that, the syllabus presents some optional ideas for courses; but even these proposals do not show how the high general learning goals of the syllabus can be attained.

A third extreme is formed by the syllabi of the Federal States of Bremen and Berlin. These are relatively brief with regard to general learning goals. They contain, however, (partly optional) course proposals which are application-oriented in a way we consider exemplary. The said proposals attempt to link calculus closely to application problems, departing from the usual conception of adding a sprinkling of applications problems to an otherwise mathematically structured course. Beyond that, the courses proposed are the only ones which suggest analysing empirical data when treating functions in calculus, and describing them by means of the latter. In spite of all proclaimed reference to applications, the concept of "data" occurs in no other syllabus on calculus.

4. Curriculum Projects and Teacher Education

4.1 Teacher Education

In the Federal Republic, teacher education occurs in two stages:

- First ("theoretical") stage (3 to 5 years): studies at a university or a Pädagogische Hochschule (teacher training college).

- Second ("practical") stage (1½ years): the teacher-to-be gives several lessons at a school, advised by several teacher educators. Simultaneously, he or she attends seminars, requiring several hours per week, which aim at integrating theory and practice.

After the second stage, the official training ends. Several institutions offer courses for in-service training, which is voluntary.

A survey of the catalogues of some universities would seem to indicate that applications play a subordinate role or are neglected altogether in the first stage. No courses are offered which place the focus on knowledge about applications (e.g., significance of mathematics for society, of the role of applications in the genesis of mathematics, etc.), or on the problem of qualifying teachers for project-teaching or teaching mathematical modelling. This, however, does not say that such considerations are not included in other courses on mathematics education of the first stage.

As there is no sophisticated system of courses on applications at German universities, we shall describe some courses as they are offered at the Pädagogische Hochschule Flensburg (and, in similar fashion, at Dortmund University and at the Gesamthochschule Kassel). These courses are based on the following simple idea: before the teacher-to-be enters school life, he or she is to have experienced how university teachers, he or she, and other students relate mathematical ideas to phenomena of science or of the environment.

- In the introductory lectures on calculus, algebra, and geometry, "short modelling problems" of historical or general significance are mathematized. Special lectures devoted exclusively to applications are offered for advanced students.

- In seminars, students themselves are to mathematize situations taken from everyday life, from the working environment, or from technology. Important efforts consist of collecting information about the situation from statistics, newspapers, catalogues, interviews with workers, etc. The main concern of these seminars is that the students carry out mathematizations, with their manifold activities, on their own. Questions concerning the social importance of mathematics, of the significance of applications for the genesis of mathematics, etc., can be treated in these seminars. The hope is that the student will acquire a certain competency in coping with application situations.

- In a school traineeship preceded by a preparatory seminar, the student may give some lessons in the sense of an application-oriented mathematics education. It would seem important that prospective teachers do not have their first teaching experience in an application-oriented mathematics education under the stress of a full-time teaching job, but while they are still able to concentrate on a short course, and with the assistance of university teachers in the planning stage. The objective is that students have first positive experience and become certain that they, too, will be able to teach mathematics in an application-oriented way.

How much space is allotted to applications during the second stage of teacher education is little known. We shall briefly sketch a project which concentrates its innovatory efforts on the second stage, developing materials for the latter.

The project "Stochastik im Unterricht der Sekundarstufe I" has the objective of developing a conception for application-oriented mathematics education (using probability calculus as an example). It is part of the EPAS (Entwicklung praxisorientierter Ausbildungsmaterialien für Mathematiklehrer der Sekundarstufe I) project at the Institut für Didaktik der Mathematik (IDM) at Bielefeld University. The central concern is to establish, in cooperation with teachers and educators, materials for the second stage of teacher education with regard to curricular, educational, and pedagogical topics.

The theoretical background for this project was developed in Steinbring's [37] study. Application-oriented teaching cannot be confined to treating problems which refer to reality in mathematics education. Besides references to reality, it also has to develop appropriate mathematical concepts and specific means to cope with real situations. This means, in particular, that experiments must be made, models constructed, decisions made, estimators chosen, etc., in many ways in the classroom. For mathematics education, this leads to a conception in which there is a stronger emphasis on the experimental and empirical aspects of mathematics than in traditional conceptions.

For its work with teachers and teacher educators, the project first developed extensive materials which contain an educational conception on probability in Sekundarstufe I and describe contexts of application in an exemplary manner. In a double way, this material served as a basis for joint work with teachers and teacher educators: on the one hand, improvements of and supplements to texts were discussed from the perspective of the teachers; on the other hand, this text provided the educational frame for establishing empirical case studies on the teaching of probability calculus in the various grades of Sekundarstufe I.

With regard to in-service training, there are a number of activities. Among these, we could also count the projects described under section 4.2, which are activities of teacher education at the same time. Further material for this stage is also to be found in the literature presented in section 5.

4.2 Curriculum Projects

The following will describe two projects which have attempted to implement some of the ideas for application-oriented mathematics education in the classroom. These two projects are not the only ones working in this direction; there are other projects in experimental schools and in teacher education which, however, often only publish "unofficially". The two projects selected are in a certain sense exceptional.

Mathematikunterrichtseinheiten-Datei (MUED)

The Mathematikunterrichtseinheiten-Datei (MUED, documentation of mathematics teaching units) is a project organized, financed, and supported by teachers themselves. It is described by its authors as "producers' association, communication network, and advisory center for decentral work on alternatives to present math teaching" (MUED [32], p. 1). The project was initiated in 1977 and has since extended to about 500 mathematics teachers belonging to different school types and school levels. Their common starting point is radical criticism of mathematics education: "The current mathematics education has almost nothing to do with the world and with the life of its students ... Where it comes into contact with our world, the world of mathematics textbooks is characterized by a piecemeal attitude towards technology, an uncritical one towards progress, by hostility towards women and towards ecology" (MUED [32], p. 3).

On the basis of these objections, MUED developed an educational conception which contains elements of various critical approaches such as mathematics education "oriented towards practice", "referring to situations", "related to reality", or "project teaching", which have been introduced, since 1974, to the educational debate, among others by Damerow et al. [4], Münzinger [11]. and Volk [16]. The pedagogical program of MUED is to realize, with emancipatory intentions, a mathematics education which is problem-oriented:

> Students and teachers should be motivated to develop abilities to act in an emancipatory way. Emancipation is understood as establishing a self-determined practice in solidarity, as establishing a reasonably founded practice ... This basic goal implies direct requirements for classroom practice ... An ability to act independently can be taught only if there is an opportunity for efforts to act independently and if this effort is encouraged. (Böer and Volk [25], p. 35[

This leads to the following conclusion for the selection of the subjects to be taught: they should refer to the present or expected situations where pupils must act, and pupils should be aware of their relevancy for themselves. Mathematical methods and abilities to act should be situation-specific and appropriate means to cope with the situation. As mathematical problems are mostly part problems belonging to more complex situations, problem-oriented mathematics education in the sense sketched above is usually organized in a interdisciplinary way. To treat problem situations adquately requires that the mathematical treatment be integrated into a network of overall treatment.

In order to implement this program, regionally organized groups and regularly organized MUED conferences develop teaching materials and report experiences made with attempts at implementation. These activities serve, on the one hand, to develop teaching materials for the most important mathematical topics prescribed by the syllabi which try to meet the objectives sketched above. A comprehensive package on "Calculus Used for Realistic and Relevant Applications" has been developed, from which a teaching series on turnpike construction has already been published (Böer and Volk [26]). On the other hand, teaching materials are being developed for relevant extramathematical topics such as the ecology and energy resources discussion, wages, raises, women's vocational activity and salaries, weaponry and peace, land surveying, traffic. A series on the problems of energy resources for Sekundarstufe I has been published (Böer and Maass [24]).

The Curriculum Project of "Stochastik in der Hauptschule"

At the University of Saarbrücken, there was from 1975 on a research project organized by mathematics educators and mathematics teachers, which in 1983 concluded with presenting a curriculum proposal for stochastics education in grades 7 to 9 of the Hauptschule. The project pursued goals on two levels:

- With regard to content: "The intention was to develop a curriculum which enables Hauptschule students, by providing an appropriate minimum of concepts and methods, to act adequately and to decide reasonably in simple stochastic situations, e.e., in situations characterized by chance or mass phenomena" (Jäger and Schupp [30], p. 6).

- And methodologically: "The intention was to develop strategies for planning, testing, and evaluating a curriculum new for the Hauptschule which is oriented towards practical teaching, i.e., close to practice, feasible, and open" (Jäger and Schupp [30], p. 6).

Principles guiding the curriculum development were, among other things, demands to close the gap between intensive use of stochastic methods in many fields and ignorance about these means in large proportions of the population, to take up situations relevant for practice, to provide links between statistical and probabilistic ideas, and to present stochastical phenomena and methods according to "genetical teaching" (see section 2). The curriculum proposed contains the following contents: in grade 7, pupils are to be sensitized for stochastical situations by analyzing the frequency distributions, e.g., in experiments. In grade 8, the distribution concept is to be mathematically specified. In grade 9, the knowledge acquired is to be applied in complex real situations; for this purpose, interdisciplinary projects are to be carried out. The topics intended are:

- Packaging: stochastical analysis of the packaging process; developing and analysing various packaging regulations to protect consumers;

- Heredity of body characteristics: types of heredity, Mendelian Laws, exemplifying blood group heredity and family trees, etc.;

- Unemployment: description of the occurrence of unemployment, local and age-group dependencies;

- Chances of winning in public games of chance like lottery, football bets, roulette, slot machines.

The curriculum was developed in the following stages: first, raw sequences organized according to stages were developed for courses in grades 7 and 8 and for projects in grade 9; these materials were educationally founded and contained learning goals, potential subject matter, and field-specific background knowledge. Subsequently, the curriculum proposals were tested in many classes; evaluation was effected by classroom observation, establishing pupil attitudes and performance by means of tests and questionnaires; establishing teacher attitudes by means of questionnaires and interviews. After the results had been used to modify the curricula, further testing ensued. Parallel to curriculum development, a research project "Stochastics in the Working Environment" was carried out which, by means of an empirical survey of jobs relevant for Hauptschule graduates, determined which abilities with regard to stochastics are present in the worker, and which are necessary for his or her work (for details, see Jäger and Schupp [29]).

5. Literature on Application-Oriented Mathematics Education

Since the beginning of the 1970s, there has been a host of literature on application-oriented mathematics education, which is dispersed in many periodicals and books which, being "unofficial" literature, is hardly accessible. In order to provide an overview of more important literature on application-oriented mathematics education both German and international, Kaiser, Blum, and Schober [31] have published a documentation of selected literature on this topic. The volume provides detailed summaries, evaluates the literature presented according to the relation to reality, type of mathematics used, etc. Diverse extra- and intramathematical indexes make it useful as a source for authors looking for literature on application-oriented mathematics education. The references below were taken from this documentation. For the most part, they concern books (not articles) published recently which contain comprehensive theoretical conceptions, especially more substantial proposals for teaching.

[1] Beck, U. Mathematikunterricht zwischen Anwendung und reiner Mathematik. Frankfurt, 1982. (Partial publication of Beck's doctoral thesis, 1979.)

The theoretical part starts by critizing two diametically opposed understandings of mathematics: mathematics as a (pure) science independent of reality and mathematics as an auxiliary science. A "balanced picture" of mathematics is sketched which integrates the two extremes and which is meant to provide the foundation for designing application-oriented curricula. A certain view of the modelling process is a central component of Beck's approach. Also, a scope of general educational objectives related to applications is developed and justified. Exemplary teaching examples corresponding to various educational intentions are developed, such as the growth of a population of field mice as an example for project teaching, draining processes in water tanks as an example for cooperation between the sciences and mathematics, and optimal use of a group ticket as an introduction to the problems of mathematizing (see also section 2).

[2] Becker, G. et al. Anwendungsorientierter Mathematikunterricht in der Sekundarstufe. Bad Heilbrunn, 1979.

The theoretical part formulates an integrative position with regard to application-oriented mathematics education which considers applications to be both a help in coping with life situations and for teaching a more profound understanding of mathematics. A detailed schema to classify applications is presented. The practical part provides examples which are closely aligned to practical teaching; among other things, geographical maps to introduce and use basic geometrical concepts, queuing problems to introduce and use simulations, elections as project-oriented exercising of percentage, and proportion calculus.

[3] Christmann, N. et al. Anwendungsorientierter Mathematikunterricht — unter besonderer Berücksichtigung der Möglichkeiten von Rechnern. Paderborn, 1981.

Teaching examples suited mainly for the upper grades of the Gymnasium concerning, among other things, income tax, investment decisions, traffic flow, etc.; proposal of a course containing an application-oriented introduction to linear algebra.

[4] Damerow, P. et al. Elementarmathematik: Lernen für die Praxis? Stuttgart, 1974.

Qualifications are determined and justified on the basis of a structural concept for the reform of the curriculum, which intends to relate goals and subject matter of mathematics education to the situations for which school is meant to prepare, thus determining the goals and subject matter on the basis of those situations (for details, see section 2).

[5] Deutsches Institüt für Fernstudien — Abteilung Mathematik (Hrsg.). Sachrechnen in der Wirtschaft, Wachstum und Zerfall. Tübingen, 1980.

The materials for teacher education contains knowledge and teaching ideas with regard to payment by installments, economic problems, growth and decay processes in economics, physics, and biology. Teaching units for these topics are developed.

[6] Dörfler, W. and Fischer, R. (Hrsg.). Anwendungorientierte Mathematik in der Sekundarstufe II. Klagenfurt, 1977.

The volume contains the proceedings of a symposium on application-oriented mathematics in Sekundarstufe II. There are various contributions on the relationship between pure and applied mathematics which, after having analyzed this relationship, develop educational

conceptions, e.g., a process-oriented methodology of mathematizing. The volume also contains didactical analyses of linear optimization; numerical analysis/computing and their applications; and examples taken from physics, information theory, and ecology.

[7] Dörfler, W. and Fischer, R. (Hrsg.) Stochastik im Schulunterricht. Stuttgart, 1981.

The volume contains the proceedings of a symposium on stochastics in school. The contributions are, among other things, about the development of conceptions for a stochastics curriculum meeting the requirements of developing the ability to mathematize and of teaching how to think stochastically. These ideas are implemented in teaching units containing examples taken from the pupils' everyday experience, traffic, and life expectancy.

[8] Fischer, R. and Malle, G. Fachdidaktik Mathematik. Lehrbrief für das Fernstudium "Pädagogik für Lehrer an Höheren Schulen". Klagenfurt, 1976.

The material for teacher education develops a position with regard to application-oriented mathematics education, in which the modelling process and modelling skills are central. Also, the philosophical reflection about the modelling process is presented and exemplified.

[9] Hessisches Institüt für Lehrerfortbildung (Hrsg.). Materialien zur Unterrichtsgestaltung: Mathematik, Sekundarstufe I; Ergebnisse regionaler Lehrerfortbildung. Fuldatal/Kassel, 1978-1982.

This material for in-service training of teachers contains issues concerning working environment and salary, decorating a room for an adolescent, and income distribution.

[10] Meyer, K. Algebra und Geometrie. Frankfurt, 1980.

A comprehensive collection of problems taken from economics, everyday life, geography, the natural sciences, and technology, which can be solved by geometry and algebra; for the most part for Sekundarstufe I.

[11] Münzinger, W. (Hrsg.). Projektorientierter Mathematikunterricht. München, 1977.

Contains various ideas and examples on project-oriented mathematics education. Examples on soccer, agriculture, and speed limits, all on the level of Sekundarstufe I, are included.

[12] Schick, K. Mathematik und Wirtschaftswissenschaft: Bd. 1: Probleme aus der Preistheorie. Bd. 2: Probleme aus der Produktionstheorie. Bd. 3: Probleme aus der Steuertheorie. Bd. 4: Probleme aus dem Bereich der Linearen Optimierung. Frankfurt, 1980, 1981.

The volumes are an introduction to the problems of price theory, production theory, tax theory, linear optimization. Description of economic problems and presentation of potential shares indexes; suitable for the upper grades of the Gymnasium.

[13] Schmidt, G. (Hrsg.). Methoden des Mathematikunterrichts in Stichwörtern und Beispielen. Bd. 2: Jahrgangsstufe 7/8. Bd. 3: Jahrgangsstufe 9/10. Braunschweig, 1981, 1982.

The theoretical part argues that pupils should be enabled to mathematize and that this is the central concern of application-oriented mathematics education. The various functions of applications in the curriculum are analysed. The practical part presents several teaching sequences conceived for Sekundarstufe I which intend to teach basic concepts of stochastics, geometry, functions, and equations, using examples taken from the pupils' everyday life, like sport scores, car expenses, and land surveying.

[14] Steiner, H.-G. (Hrsg.). Cooperation between Science Teachers and Mathematics Teachers. Materialien und Studien des Institut für Didaktik der Mathematik, Bd. 16. Bielefeld, 1979.

The volume contains the proceedings of a symposium on cooperation between mathematics teachers and science teachers; there are contributions on the fundamental relationship between mathematics, the sciences and reality, problems and possibilities of cooperation with regard to curricula, teacher knowledge, and teacher attitude. In addition to that, there are contributions to problems such as teacher education, development of the probability concept, and problems and strategies in connection with innovating school practice by integrating applications of mathematics.

[15] Strehl, R. Grundprobleme des Sachrechnens. Freiburg, 1979.

The volume develops an extended conception of Sachrechnen as the mathematization of situations. It contains a comprehensive analysis of Sachrechnen dealing with the latter's traditional subject matter, with more recent extra- and intramathematical fields, and with educational problems of Sachrechnen (for details, see section 3.1).

[16] Volk, D. Handlungsorientierende Unterrichtslehre am Beispiel Mathematikunterricht. Bd. A. Bd. B. Doctoral thesis. Bensheim, 1979, 1980.

The study develops a position which intends to modify teaching according to emancipatory ideas. It provides detailed arguments with regard to educational theory and mathematical epistemology. It is of importance as pedagogical-theoretical background for the MUED project (see section 4.2).

[17] Vollrath, H.-J. (Hrsg.). Sachrechnen. Stuttgart, 1980.

Educational analyses concerning various fields of Sachrechnen such as teaching the concept of function on the basis of pupils' experience and of experiments and teaching algebra in the context of Sachrechnen. Teaching examples which aim at enabling pupils to mathematize are included.

[18] Weber, H. Grundlagen einer Didaktik des Mathematisierens. Doctoral thesis. Frankfurt, 1980

Development of a position which considers enabling pupils to mathematize a central guiding idea of mathematics education, thus showing

a possibility for overcoming both purely structural and purely pragmatical standpoints (see section 2).

[19] Winkelmann, B. (Hrsg.). Mathematische Modelle in der Biologie. Materialien und Studien des Instituts für Didaktik der Mathematik, Bd. 14. Bielefeld, 1979.

A book which gives a survey of deterministic mathematical models in biology relevant for the upper grades of the Gymnasium and which contains an extensive bibliography on the topic of mathematics and biology.

References

(not included in section 5)

[20] Beck, U. Zur Didaktik der Anwendungen innerhalb eines ausgewogenen Mathematikunterrichts. Dissertation. PH Ruhr, Dortmund, 1979.

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FINLAND

Paavo Malinen

Changes in the Curriculum

Until the end of the 1960s, we had a parallel school system in elementary education in Finland. For all pupils, the first four school years (age range 7-11) were common primary school where practical arithmetic was taught. After this, part of the pupils attended lower secondary school (for five years, age range 11-16) where arithmetic was still taught for two years and after that algebra and geometry. The rest of the pupils continued their studies for four years in primary school (age range 11-15) and they had practical arithmetic teaching all the time.

In traditional primary education, arithmetic has been applied mathematics. New processes of calculation were learnt in practical situations. Textbooks for primary schools in the countryside and towns had been compiled separately so that examples could be taken from the pupil's own environment. Especially in grade levels 5 and 6 in both types of school there were a lot of series of calculations on the same area so that they were like small study projects. In the lower secondary school, calculations of percentage and interest were very common. As applications for these there were tasks like calculation of solution levels, profit and loss in commerce, how profitable is it to save money, etc. Positive rational numbers (decimals and fractions) were used; the process of calculation had first been learnt in simple applications.

The real teaching of mathematics started in grade 3 in the lower secondary school (7th school year). Fairly soon the characteristics of rational numbers and the solution of equations and pairs of equations were treated in algebra. In geometry the systematics according to Euclid's Fundamentals formed the basis for teaching, completed with practical exercises. Calculations of percentage and interest were included as applications in the way that equations were used at the same time. Elementary applications in trigonometry appeared, too, and calculation of surface area and volume.

The transition to the comprehensive school system gradually took place in Finland from 1970 onwards. The primary and lower secondary schools were unified into the nine-year comprehensive-type compulsory school. At the same time we started to realize the modernization of mathematics teaching in the lowest grade levels. The idea for this had come from a plan drawn up by "The Nordic Committee for the Modernization of School Mathematics" on the basis of the seminars of the OECD. The report of this committee, "New School Mathematics in the Nordic Countries", was published in 1967.

In the Comprehensive school, the curriculum was changed to that mathematics is now the name of the school subject from the first-grade level. According to the modernization plans, set theory was widely used; the concept of function was introduced at an early stage like the mathematical characteristics of rational numbers. In higher grade levels algebra, geometry, and trigonometry were no longer separated but unified into one subject called mathematics. Different domains were taken into consideration in the textbooks according to the spiralling principle, which means that a certain task (like geometrical descriptions) was dealt with more comprehensively year after year.

In the mathematics curriculum in the comprehensive school, applications are not used as a basis of teaching as they are in the primary school. Instead, there are set models and function models. Before pupils learnt mechanical rules for calculation, but now the aim is to learn to understand the process of calculation on the basis of models and deduction tasks. So, there are considerably fewer applications than before, especially as we now try to teach at least as much to the whole age group as was earlier taught in the lower secondary school. The curriculum has proven to be very heavy, especially in grades 6-9. However, there are some applications in the calculation of percentage, for example, in situations connected with commerce, but, for example, calculations of solution levels and interest have been left out. In the lower secondary stage there are, however, some new types of applications, for example, of function theory and statistics.

In the upper secondary school (grade levels 10-12), the role of applications has always been of little value. Earlier there were applications in geometry, trigonometry, and mathematical analysis to some extent, but their role has now diminished. Instead, there are some applications in probability theory and statistics. At the equivalent stage in vocational education, there have been plenty of applications in pupils' own vocational branch. On standardizing education in school years 10-12, the aim has been to increase the teaching of basics in vocational education also. This results in applications in mathematics being more and more connected with general technology and life in society.

Research on Teaching Applications

An international I.E.A. study (Project for the Evaluation of Educational Achievement) in mathematics was pursued in Finland in 1964 and the same tasks were presented in a new study in 1981. During the 15 years between those studies, the school system had undergone essential changes and also the structure of the mathematics curriculum had changed as described above. Some results of the retesting are available for the 7th school year. Both studies concerning this level of education included ten tasks. They were mainly word problems on practical situations; some included solving algebraical sentences or equations.

Almost 1000 pupils were involved in the sample in both tests for the 7th school year so that the sample covered the whole age group. Let's compare the results of the applications in the whole data without separating different types of schools. In 1964, the solving percentage was higher in seven tasks and in 1981 in three tasks. Differences in achievement in the years 1965 and 1981 were relatively small. Earlier, pupils were better able to solve word problems, whereas in retesting, problem-type applications in geometry were better done. Differences were thus clearly connected with change in the curriculum. Although the ability to apply mathematical knowledge has apparently decreased along with the "modernization" of mathematics teaching, the change is not significant. The amount of practice on applications is considerably smaller in 1981 than in 1964. At the moment we are checking the new curriculum in the comprehensive school and more applications will again be introduced, while pupils' ability and willingness to calculate will be confirmed.

The role of applications in the new curricula has been analyzed in research by the Association of Teachers of Mathematical Subjects, the executants of which were Associate Professor Jarkko Leino and Dr. Pekka Norlamo. They published a report, "The Contents of School Mathematics from the Point of View of Technology and Economics" [1].

The main object of the study was to find out what the contents of school mathematics should be like to meet the needs of technology and economics. For this purpose, mathematics courses of different school types were analyzed and vocational teachers' opinions of what should be taught in school mathematics were asked. In the teachers' opinion, the least significant changes in the reforming of mathematics were areas like set theory, probability theory, and treatment of the concept of function. Teachers would like to have more application in the examination of quantities and the treatment of ratio and proportion. Vocational teachers thought that pupils should be better taught to use fractions, to solve first-order equivalents, and to solve word problems in the comprehensive school.

Mathematical textbooks used in the comprehensive and vocational schools were also compared. In new textbooks for vocational education, there are more and more applications that could be taught in the comprehensive school. Seventy to ninety percent of the tasks in the textbooks in the comprehensive school are purely mathematical, whereas these comprise 30% to 60% in vocational schools. Integration between mathematics teaching in the comprehensive and vocational schools is still poor because of the changes in the school system. No total schemes for teaching applications were presented. But the authors presented plenty of practical tasks suitable for teaching in the comprehensive school, to develop pupils' problem-solving skills.

Assessment of the Present Situation

In the 1970s, quite a considerable modernization of mathematics teaching was carried out in all grade levels (1-12) in Finland. That reform in the curricula was only partly accomplished. Teachers did not have enough in-service education, and teachers in the lowest grade levels especially changed their way of teaching slowly. At the same time, efforts were made to clarify basic subject matter and to concentrate on teaching it. However, there has been no "back to basics" trend in Finland — the aim is to develop teaching to emphasize essential subject matter — "forwards to fundamentals". Professor Tamas Varga from Hungary put forward the latter idea when he visited Finland several years ago.

On moving to the comprehensive school system, new methods have been developed to give support to low achievers. They can have special tuition by special school teachers outside normal lessons or another teacher is present in the classroom and gives simultaneous teaching. The aim has been to provide all pupils with adequate skills to continue their studies after the comprehensive school. Remedial teaching has contributed to the mastery of fundamentals, whereas it has not been found to have much influence on learning applications. Low achievers in mathematics have continual difficulties in perceiving the mathematical techniques and concepts needed in practical situations.

Because of rapid changes in society, the mastery of fundamentals has been emphasized in school teaching. Therefore, the requirements have been quite high for the whole age group in the 1970s. It is, however, difficult to motivate low achievers to study more extensive courses. About 15% to 20% of an age group have serious difficulties in mathematics in the lower secondary stage (school years 7-9). Their achievement can be best improved by trying to make their attitude towards mathematics more positive and by giving them mathematical tasks connected with their practical life. According to the study mentioned above and to teachers' experience, pupils' skills for applications in mathematics have decreased to some extent. As we have tried to raise the level of teaching as far as basics are concerned, we should have the same objective with applications. All these factors have contributed to the reassessment of teaching applications in mathematics.

Goals for Modern Applications

In the study by Leino and Norlamo mentioned above [1], three main functions were attributed to the teaching of applications:

(1) They provide concreteness needed for learning mathematical concepts — in other words, observations and mental images which the learning of concepts (abstraction) is based on.

(2) They view mathematics as an instrument which can be used for any kind of quantitative and qualitative information-processing.

(3) They motivate pupils to study mathematics and help retention.

In addition, application tasks offer good opportunities for developing general problem-solving skills.

These main functions have been poorly accomplished in teaching mathematical applications. However, the psychological basis for applications has been examined to such an extent that the chance to develop pupils' problem-solving skills by means of teaching applications in mathematics has increased.

It is also important to practise the use of variables occurring in different practical stituations in life — such as changes in direction (positive-negative, forwards-back, profit-loss), change taking place from a 'starting point', measuring of time intervals with different methods, use of derived quantities (e.g., km/h), and so on. It is possible to practice the use of geometrical figures and concepts in many other lessons than mathematics even in lower grade levels. In higher grade levels, one must carefully consider the applications, keeping in mind pupils' interests and their possible future careers.

Pupils often make practical rules to solve mechanically a problem without understanding the whole idea. Erroneous associations and performance algorithms should be corrected at once. In many situations pupils must be made to use a heuristic method which is not easily generalized. It is also important to present situations where algorithmic thinking is possible. Studying the flow diagram for problem solving serves as a basis for later ADP teaching. In connection with applications at an elementary level, it is also possible to start to prepare pupils' concepts of what kind of tasks can be put into a preprogrammed form.

Earlier, simple examples were used in teaching applications so that calculations did not become too long. At the moment pupils are allowed to use electronic pocket calculators from the seventh-grade level onwards, and this makes it possible to make use of real practical situations as pupils can quickly do mechanical calculations. It is no longer necessary, either, to use tables, so that in trigonometry, for example, more time can be spent on applications.

No study projects where applications would play an essential role have been planned in Finland. Coordination of the teaching of mathematics with physics and chemistry is possible, especially in the comprehensive school: the three subjects are taught by the same teacher. A clear general plan of the areas of applications in different grade levels has been made for natural sciences. However, there is no general plan of how the aims of applications mentioned above could be systematically realized. But interest in teaching applications has increased so much that this domain will be most central in the development of mathematics teaching in the 1980s.

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ISRAEL

Shlomo Vinner

In the preface of the 1979 NCTM yearbook, Applications in School Mathematics [1], were mentioned four major questions used as guidelines to the book: (1) What are applications? (2) Why include applications in school mathematics? (3) How can applications be brought to the classroom? (4) What issues are related to applications? (p. vii).

These are very relevant questions for a book meant to advertise applications. This paper, however, is supposed to describe the applications aspect of mathematics education in one particular country, and therefore, the four questions above are not suitable as guidelines for it. Nevertheless, we would like to use them as a starting point. Question (3) should be modified to: How are applications brought to the classroom?

Questions (2) and (4) seem to us irrelevant unless one wishes to report about people's ideas. But ideas are general and we want to be specific. We also think that our task is to report about actions and not about intentions. So we won't discuss (2) and (4) in any form. The answer to (1) in such a survey paper should be practical and not theoretical: applications of mathematics are all those things people call applications of mathematics. Since there is no chance to obtain a consensus, the only thing one can do is to give examples. It seems to us that a theoretical discussion about the nature of applications belongs to the methodology of science and mathematics. But it turns out that theoretical answers to (1) had certain impact on the field of mathematics education, an impact which we would like to discuss later on. Hence, in order to be clear we will first have a short theoretical look at applications of mathematics.

The Idea of Mathematical Model

We do not intend to explicate the notion of mathematical model here. It has been done many times in various publications (for instance, see [1]). The concept of mathematical model is based on the idea that mathematics is an abstract theory which deals with abstract notions such as numbers, numerical operations, relations between numbers, equations, functions, derivatives, differential equations, and so on. If one of those is chosen to represent or to describe a real-world situation, then it becomes a mathematical model for this situation. This distinction is the basis of almost all diagrams describing mathematical models and their relation to the real world. Such a diagram is given in Figure 1, taken from Kerr and Maki [2].

Mathematical Models and the University Complex as a Modern Version of the Bourgeois Gentilhomme Complex

The above conception of mathematical models has led some people to the recognition that if in the morning I had five apples and I ate two of them for lunch, then $5 - 2 = 3$ is a mathematical model for this particular

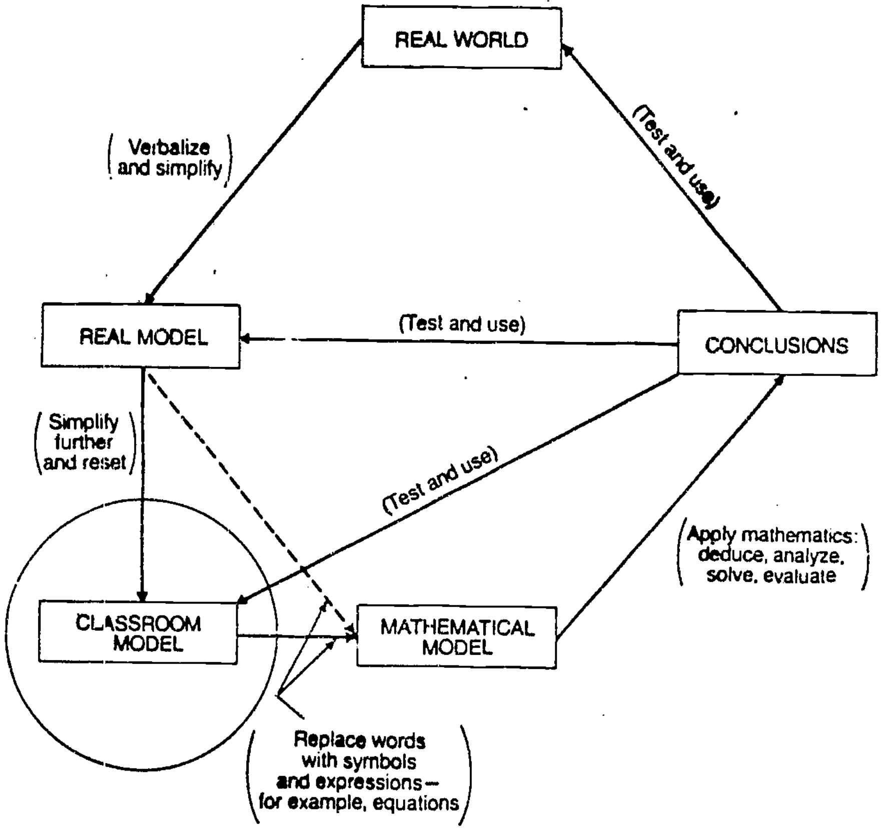


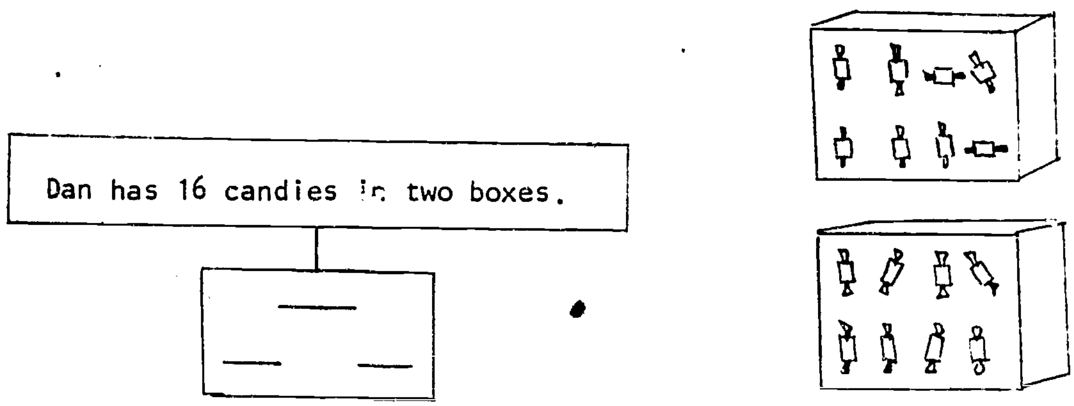
Figure 1. Mathematical models to provide applications in the classroom. [2, p. 3]

part of my biography. Some people also believed that this fact should be brought somehow to the knowledge of the students, starting at the first grade. It is clear that since the beginning of culture and agriculture, children did manipulations on apples. However, nobody told them that they invented mathematical models. Now at least we try to tell them.

This is quite similar to the discovery of Molière's Bourgeois Gentilhomme. After becoming rich, he decided to learn a few things about culture. One of them was "prose". To his great surprise he discovered [3] that he had spoken prose for over 40 years without knowing about it. Our question is whether this was fortunate or not. If we correctly understood Molière, then this discovery did not contribute a lot to Mr. Jourdain's cognitive development. We do not want to argue this point here since this is a survey paper. We only want to point at some facts. Since the beginning of "new math", it is quite typical that observations, distinctions, and ideas which belong to the university level were offered in a simplified version to high school, junior high school, and elementary school students. We would like to call it the university complex of mathematics education. (Another example of this complex is related to the notion of number. In the foundations of mathematics, the concept of whole numbers is based on the concept of sets and one-to-one correspondences. This brought "set theory" to kindergarten and first grade, together with plenty of "finite set theory" activities. Also, the distinction between numbers, as abstract objects, to their representations, as concrete signs, found its way to elementary and junior high levels, expressed by the distinction between numerals or number names and numbers.)

We will illustrate the university complex in the domain of applications by three typical examples from the elementary level. Of course, the idea of mathematical models, if introduced at the elementary level, should find an appropriate terminology. Such terminology was indeed invented. Terms such as "numerical sentence", "addition form", and "multiplication form" now decorate many word problems (considered as applications). These terms make some old ladies look quite different in their new makeup, but if you look for the ladies only they remain the same. For instance:

(1) Write a multiplication form for the following picture which has a multiplication line.



This lady in her original makeup probably looked like this:

> Dan has 16 candies in two boxes. Each box has the same number of candies. How many candies are there in each box?

Another type of word problem is the result of the understanding that equality and inequality are relations. This is expressed in questions like the following: Write a numerical sentence and answer the question.

(2) Dan ate six candies. Mira ate four candies. Dorith ate the same number of candies as Mira. (____________) How many candies did Dorith eat? ____________

(3) Dina has two cats and Yoram has three bears. Who has more animals?

(____ > ____) ____________

If the reader is not sure about the answers, they are as follows:

(1)

	16	
8		2

(2) $\underline{4} = \underline{4}$ $\underline{4}$

(3) $\underline{3} > \underline{2}$ Yoram

Note that $16 = 8 \times 2$, $4 = 4$, and $3 > 2$ are mathematical models for (1), (2), and (3) respectively.

Similar examples can be found at the junior high and the senior high levels.

Two Ways in Which Applications Appear

The questionnaire distributed to the contributors of this volume mentions two ways in which applications commonly appear: (1) illustrations of a mathematical technique (taught in connection with that technique), and (2) situations from outside mathematics studied coherently, using whatever mathematics may be useful and with understanding of the situations as a central aim.

This distinction (taken from [4]) is quite helpful to describe the domain of applications. We would like to advise that the outside-mathematics approach won't be considered as a complete opposite to the illustration approach. The two should be regarded as two end points of one continuous scale. The difference between them is similar to the difference between routine and non-routine problems. If you wish to teach non-routine problems, you need much more time and/or much better students than you usually have.* Therefore, the outside-mathematics approach is an appropriate approach for special university courses or perhaps also for honors high school students. Thus, most applications in the common curriculum are illustrations of mathematical techniques. Moreover, the illustrations should belong to certain archetypes. This is a result of the expectation that students should be able to solve additional application problems on their own. There is no chance that a student can solve on his or her own a non-routine application problem like the traffic light problem (see [4]). Every problem like this should be taught to the students by a teacher and cannot be given as a homework assignment. Hence, every problem like this is only an <u>illustration</u> to a non-routine application problem, and there is no time for more than two or three illustrations like it in a regular course (which is supposed to teach a list of concepts, theorems, and techniques). As a result of that, the decisive majority of application problems look like:

* Or much easier problems! [Ed.]

(1) 63 tons of cement arrived at a railway station. A contractor has 2 trucks. One can load 3.5 tons and the other one can load 4.5 tons. How many trips are needed in order to transport the cement to the warehouse if the contractor uses only the first truck? only the second truck? (a junior high problem)

(2) Pipe A and pipe B together fill a pool in 2 hours. Pipe B and pipe C together fill the same pool in 3 hours and 45 minutes. All three pipes together fill the pool in 1 hour and 40 minutes. How long does it take for each pipe to fill the pool? (a senior high problem)

A Third Way — Applications as a Trigger for New Topics

The suggested two ways in which applications commonly appear assume as prerequisites certain mathematical techniques. But there are some cases where an outside-mathematics situation can be a trigger for new concepts and techniques. These cases are quite few and quite hard to find and, therefore, their discovery should not be considered as a revolution in the curriculum. On the other hand, they might be an important contribution and worthwhile to mention. Such a case is the case of a radioactive element which is used in Israel to establish the exponential functions and the exponential laws. (It was first introduced in Israel by Professor A. Amitsur from the Mathematics Department of the Hebrew University, Jerusalem. It is taught mainly to eleventh graders who learn mathematics with emphasis.) We will introduce it briefly here.

Assume that we observe 1 gram of a radioactive element starting at the time $x = 0$. Let $f(x)$ be the mass of the element after x hours. Namely, f is the function which tells us how much of the 1 gram is left after x hours. Note that if from 1 gram, after x hours, is left b gram $(0 < b < 1)$, then from k gram, after x hours, is left kb gram. Now observe the value $f(x_1 + x_2)$, where $x_1, x_2 > 0$. This is the mass left from the 1 gram after $x_1 + x_2$ hours. After x_1 hours, the mass left from the 1 gram will be $f(x_1)$. If we observe this mass — the $f(x_1)$ gram — for an additional x_2 hours, we will get $f(x_1)f(x_2)$ gram. This is because from 1 gram after x_2 hours we get $f(x_2)$ gram and instead of 1 gram we took $f(x_1)$ gram. (We are using here a previous remark taking $k = f(x_1)$, $b = f(x_2)$, and $x = x_2$.) Thus we obtain the equation: (1) $f(x_1 + x_2) = f(x_1)f(x_2)$. It is easy to see that this equation holds for any two real numbers x_1, x_2, not only for $x_1, x_2 > 0$ (of course, for $x < 0$, $f(x)$ needs a suitable interpretation). Remember also that: (2) $f(0) = 1$. Denote: (3) $a = f(1)$. Since $f(2) = f(1 + 1)$, it follows from (1) that: $f(2) = f(1)\cdot f(1) = a^2$. In the same way $f(n) = a^n$ for every whole number n. Moreover, since

$$f(1) = f\left(\frac{n}{n}\right) = f\Big(\underbrace{\frac{1}{n} + \frac{1}{n} + \ldots + \frac{1}{n}}_{n \text{ times}}\Big)$$

it follows from (1) that $f(1) = \left(f\left(\frac{1}{n}\right)\right)^n$ Hence,

$f\left(\frac{1}{n}\right) = \sqrt[n]{f(1)} = \sqrt[n]{a}$. As an immediate result of that, since

$$f\left(\frac{m}{n}\right) = f\left(\underbrace{\frac{1}{n} + \ldots + \frac{1}{n}}_{m \text{ times}}\right)$$

we obtain:

$$f\left(\frac{m}{n}\right) = \left(f\left(\frac{1}{n}\right)\right)^m = \sqrt[n]{a}^{\,m}.$$

Finally, for every positive number x, since $f(0) = f((-x) + x)$, it follows from (1) that $f(0) = 1 = f(-x)\cdot f(x)$. Therefore

$$f(-x) = \frac{1}{f(x)}$$

Let us denote $f(x)$ by a^x for every x. This notation is consistent with the previous meanings of z^x for any rational number x (including negative numbers) if these meanings were introduced to the students previously. If not, they get it from the above notation.

Namely, $a^n = \underbrace{a \cdot \ldots \cdot a}_{n \text{ times}}$, $a^{\frac{m}{n}} = (\sqrt[n]{a})^m$, $a^{-x} = \frac{1}{a^x}$

Moreover, because of physical consideration, $f(x)$ is a continuous function and therefore we can estimate a^s for any irrational number s (using a rational approximation for s). This without the necessity to mathematically define a^s for irrational s (which is quite a hard task).

Now, it follows immediately from (1) that $a^{x_1 + x_2} = a^{x_1}a^{x_2}$

for every x_1, x_2. (Usually, this is proved only when x_1, x_2 are integers.) It is not hard to show also that $\left(a^{x_1}\right)^{x_2} = a^{x_1 x_2}$

for every s_1, x_2 and that $(\sqrt[n]{a})^m = \sqrt[n]{a^m}$

Thus, a whole technical, messy, and relatively boring chapter in the traditional curriculum is introduced to the (able) student in a short, elegant, and accurate way, tied to applications.

Summary

The applications movement is like other movements in mathematics education, such as new mathematics, back to basics, problem solving, calculators, etc. It draws our attention to a very important aspect of mathematics. It convinces us that it is most desirable to have a strong emphasis on applications when teaching mathematics.

Several years have passed since applications started to be massively advertized. Summing up these years, the results are not very impressive. Even in new programs with innovations, the applications problems are more or less the same as in traditional programs. This is mainly because of two reasons: (1) the majority of the new programs came to introduce mathematics as a theoretical discipline (a body of knowledge); (2) it is very hard to invent new meaningful applications problems.

Thus, the main applications are still illustrations to equations (linear or quadratic) and to calculus (maximum and minimum problems). Of course, one can consider trigonometry or analytic geometry as applications, but these are applications of one mathematical domain in another one.

Certain new programs include linear programming or statistics. One can find some (routine) applications problems in these topics, too.

The most frustrating situation concerning applications is at the university level. In mathematics courses for mathematics majors there are no applications whatsoever. But also in mathematics courses which are service courses, it is very hard to find applications. These courses are usually taught by members of the mathematics departments. Most of them are not familiar with the applications of mathematics in economics, management, chemistry, or biology. On the other hand, these departments do not define exactly their needs. In addition to that, the syllabuses of the service courses are so overloaded with techniques and concepts, and the time is so short, that there is almost no time for applications. The "great moment" of applications is supposed to come, if at all, in future courses in chemistry, biology, economics, or management, a long time after the mathematics courses take place. By this time the student usually has forgotten what he or she learned.

Thus, we have to conclude by saying that the applications approach is much more a wishful thinking than an everyday practice. Knowing the limitations of the educational system as a whole and the intrinsic difficulties of the subject itself, it is hard to expect a drastic change in the future.

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NEW ZEALAND

Robert A. Garden

The phrase "applications in mathematics" means different things to different people. To some it implies the topic usually included in elementary mathematics courses for which there is an obvious utilitarian purpose — shopping, measurement, or percentages, for example. To another group it means the kind of problem studied in senior-level "applied mathematics" — mechanics, numerical methods, statistics. Others see the significance of applications ". . . seeking verifiable answers to questions inspired by thought-provoking situations" [1].

In this description of the "applications" scene in New Zealand, distinctions between these interpretations are blurred. At different levels of the school system and for different ability groups of pupils, the relative emphasis between the interpretations shift in a way that cannot readily be documented.

Table 1 presents a synopsis of some aspects of the New Zealand school system which are relevant to mathematics education and which will assist in providing a context for the discussion to follow.

Compulsory schooling ends at the age of 15 years, but most students remain in school at least until the end of the fifth form (F5) year. Secondary schools are non-selective and pupils proceed through the system by age promotion until they reach this level. At each of the F5, F6, F7 years, external examinations are conducted — School Certificate (F5), University Entrance (F6), and University Bursaries (F7) examinations. Performance criteria must be met in each of the first two to ensure promotion to the next grade.

Pupils follow a common curriculum in mathematics until the end of the F4 year. In the F5 year, pupils choose the subjects (up to six) they intend to offer for School Certificate. Almost all pupils enter for the School Certificate and of these some 80 percent enter in mathematics.

At sixth form level most pupils enter for University Entrance mathematics in which there is only one course. A small proportion, however, follow alternative school-based curricula.

Two courses are available at seventh form level, Pure Mathematics and Applied Mathematics. For the latter, at least two of Statistics, Computing and Numerical Methods, and Mechanics must be studied. A very small proportion of pupils now study mechanics. Most pupils still study five subjects for the University Bursaries examination.

In general, the mathematical preparation of teachers in the primary and intermediate schools is limited to secondary school mathematics (commonly F6 level), with courses largely devoted to mathematics pedagogy during their three-year preservice training in Teachers' Colleges. The majority of teachers of mathematics in secondary schools complete a one-year preservice course in a Teachers' College following graduation

TABLE 1

SYNOPSIS OF NEW ZEALAND SCHOOL SYSTEM

School Type (for majority)	Modal Age	Grade	Hours Mathematics per week (modal)	% of cohort in school (approx.)	% of those in school taking mathematics (approx.)
Primary	5	J1	3.5	100	100
	6	J2	3.5	100	100
	7	S1	3.5	100	100
	8	S2	3.5	100	100
	9	S3	3.5	100	100
	10	S4	3.5	100	100
Intermediate	11	F1	5	100	100
	12	F2	5	100	100
Secondary	13	F3	3	100	100
	14	F4	3	97	100
	15	F5	4	88	80
	16+	F6	4	50	85
	17+	F7 Pure	4	15	67
		F8 Applied	4	15	44

from a university. It should be noted, however, that only about 35 percent of teachers of mathematics will have had three or more years of university mathematics, and that approximately 20 percent may have had no university level mathematics at all [2]. Teachers in New Zealand secondary schools commonly teach more than one subject, and thus many of those teaching mathematics are graduates in subjects other than mathematics or have a major teaching subject for which the required qualifications are not university-based.

Curriculum Aims and Applications

The place of applications in New Zealand mathematics curricula is intimated in the curriculum guides published by the Department of Education. In the guide for J1-S4, among a list of aims for primary school mathematics occurs the following:

> ". . . to apply this knowledge of basic facts and principles to developing accuracy and efficiency in computation." [3]

Here the emphasis is on applying learned skills in the development of other mathematical skills, but in discussion of how a balanced programme should be developed from the aims, the same publication recommends:

> ". . . the explanation of situations devised by the teacher or suggested by a text, sharing of discoveries, recording observations, making generalisations, refining ideas and appreciating their application in other situations."

Furthermore, in discussing the evaluation of children's work, the guide suggests that it is appropriate to establish ". . . the versatility with which facts, skills and understanding are applied". Topics in the syllabus which provide opportunity for establishing the relationship of mathematics with the students' environment include money, measurement, and graphs (to give a picture of observations and events).

As is to be expected, applications do not receive much emphasis at this level. The main thrust of instruction is to provide skills and knowledge and to begin to establish mathematical principles.

Nevertheless, it is important for the development of student attitudes which allow them to feel comfortable about applying their mathematical skills and knowledge and which encourage them to look for mathematical "possibilities" in given situations that the advice given in the guide and referred to above be understood and followed by teachers.

In the curriculum guide for F1-F4 ([4], the references to applications are sharpened considerably. Among the aims listed for mathematics at this level are included:

> To develop the basic mathematical knowledge, skills and understanding necessary for effective living and for everyday citizenship.
>
> To help pupils appreciate the importance of mathematics in their future studies and vocations.

Discussion of the aims includes the comments

> Increasing demands will be made on the mathematical competence of all pupils in their future careers. . . . New applications of mathematics are being found in all fields of thought, and

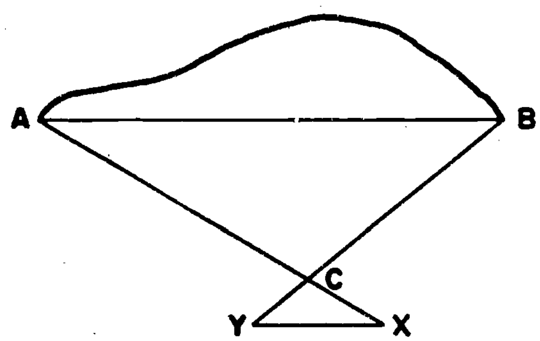
new areas of mathematical study are being developed to meet new needs.

Notes to the syllabus point out to teachers areas in which applications can and should be made.

It might be imagined then, that in New Zealand mathematics classrooms at the Forms 1-4 levels, pupils were gaining practice in applying mathematics in a creative manner to a wide range of contexts. This, alas, is not the case. Among the reasons for this are:

(1) A paucity of appropriate published resource material for teachers. In many of the textbooks in use, the "applications" problems are what Pollak [5] describes as "spurious applied". He quotes the following as an example:

To find how far a tunnel goes through a hill, a surveyor lays out AX = 100yd, BY = 80yd, CX = 20yd and CY = 16yd. The surveyor finds by measurement that YX = 30yd. How long is the tunnel AB?



Pollak points out that we cannot solve this problem unless A, B, X, Y are assumed coplanar. But if these points are on a plane, what need is there for a tunnel?

(2) Discouraging results experienced by many teachers in trying to have pupils apply mathematics to real situations. This is hypothesised to result from a set of teacher assumptions and other factors which will be discussed later.

(3) Inadequate preparation of pupils in the skills needed in the process of applying mathematics.

(4) Backwash effects of public examinations. In this respect, the School Certificate examination (F5) provides a good example. In an introduction to the prescription for this examination, aims for the curriculum implied by the prescription are stated. These include aims relating to the application of principles to unfamiliar situations, everyday living and effective citizenship, appreciation of mathematics in society and with respect to vocations. The list of aims is followed by a listing of what the examination will measure and in this there is markedly less emphasis on applications. It should be noted, however, that in

more recent years successive examiners have striven to produce examination questions intended to assess the ability of pupils to apply principles, techniques, and facts. The following are examples from recent examination papers. Question format and the size of the diagrams has been altered to reduce the space needed. Figure 28 is thus no longer to scale.

School Certificate
1980/12

There are two small radio stations, Radio A and Radio B as shown in figure 28, which is drawn to scale. 1 mm represents 1 km.

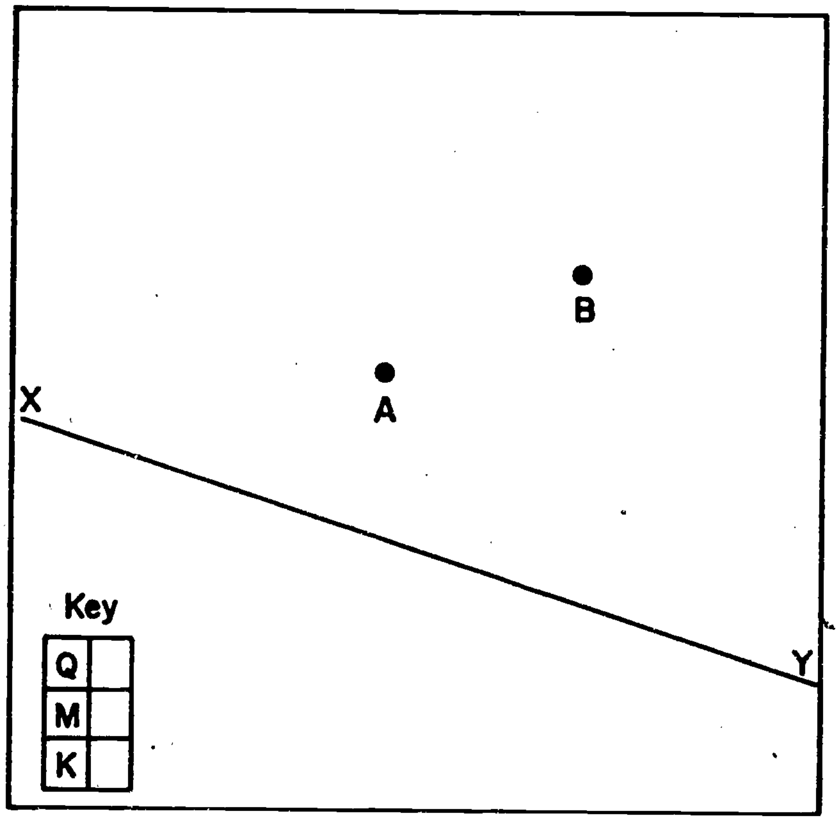


Figure 28

a What is the distance between A and B?

Each station can be received up to 50 km away. Using the given scale, draw the following sets on figure 28:

b Q, the set of points where Radio B can be received

c M, the set of points where both stations can be received.

XY is a mountain range.
Neither station can be received for a distance of 10 km from XY on the side further away from A and B.

d Draw K, the set of points where a station cannot be received because of the mountain range.

Show on the key in figure 28 the shading you used for the sets Q, M and K.

School Certificate
1981/2

Transistor radio batteries are shaped like the one in figure 12.

Batteries are 3 cm wide and 6 cm high.

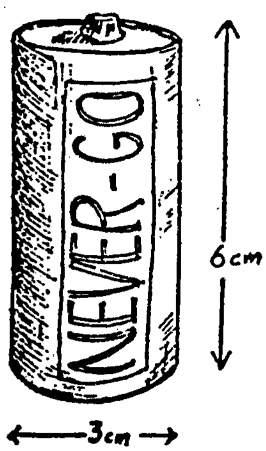


Figure 12

The batteries are packed on their sides in boxes which are 27 cm wide and 12 cm deep; see figure 13.

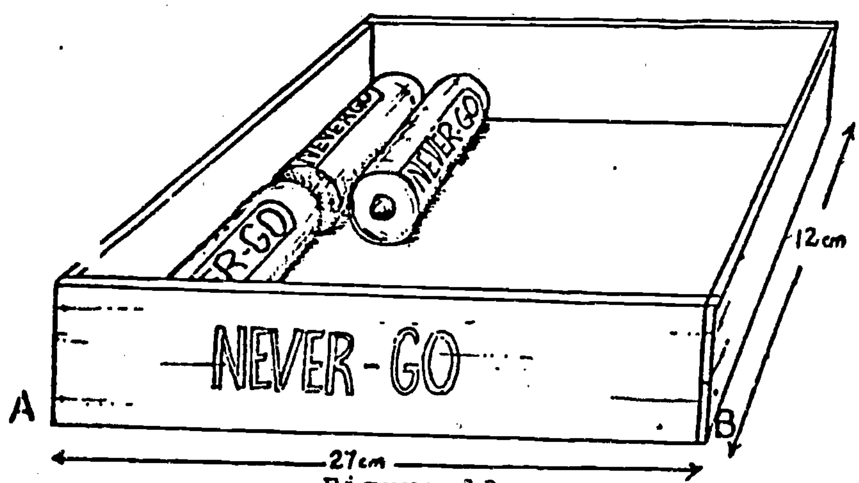


Figure 13

a How many batteries will fit along the front of the bottom layer between A and B?

b The batteries are packed in two layers; see figure 14.

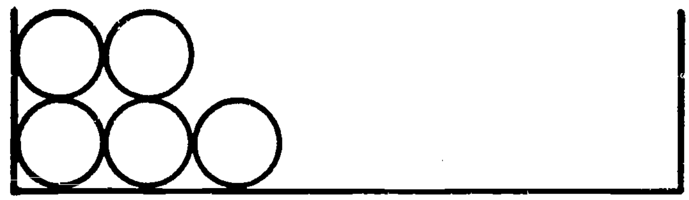


Figure 14

How high must the box be made?

c How many batteries will there be in a full, two-layer box?

d The batteries are sold for 45 cents each. What is the cost of six batteries?

e A shop gives 10% discount on batteries. How much will the six batteries cost at this shop?

f On each box is written "Average life 84 hours".

The lives of the batteries have a normal distribution. The standard deviation is 13 hours.

Complete this statement:

"About two-thirds of all the batteries have lives which lie between 71 hours and ..... hours".

In the preamble to the University Entrance syllabus, teachers of sixth forms are advised that "In teaching the syllabus, applications from the physical, social, and commercial environments should be sought". However, in the prescription itself, the only reference to applications is in the application of the derivative and/or anti-derivative to rates of change as applied to "velocity, acceleration and problems from other disciplines".

On the other hand, statistics and probability topics are well-represented in the prescription and it is noteworthy that in mathematics syllabuses throughout the school system considerable importance is accorded statistics.

In the terminal year of schooling the existence of two courses introduced a sharp distinction between pure mathematics and applied mathematics. The examination prescription upon which the Applied Mathematics course is based consists of three parts: Statistics (and Probability), Computing and Numerical Methods, and Mechanics. The content of these parts is similar to that in specialist courses in a number of other countries.

Applications of Mathematics in Other Subjects

Primary and Intermediate

The extent to which mathematics is used in other subjects at these levels of the school system varies with the interests and confidence of the teacher. Resources recently developed encourage teachers to use mathematics widely in other subjects and to undertake integrated studies with mathematics as a component. These practices are reported to be increasing. In the past, there were occasional examples of integrated studies being undertaken, particularly in intermediate schools and, more frequently, examples of the construction and use of statistical graphs, sometimes based on surveys carried out by the class. Also at Intermediate School level some of the craftwork (e.g., woodwork and metalwork) offers the opportunity for application of skills associated with measures. At lower levels activities such as counting, ordering, and simple computation with whole numbers are not uncommon in other subjects. In general, however, the application of mathematics to solve problems or illuminate results in other subjects is not widespread.

Secondary

The extent and nature of applications of mathematics to other subjects is reflected in the applications which students are expected to have encountered before taking the School Certificate examination at the end of their third year (age approximately 16) of secondary school. All secondary school students follow a curriculum which includes the subjects English, mathematics, science, and social studies for the first two years of secondary school before taking the 4, 5, or 6 School Certificate subjects of their choice in the third year. Thus, all students are likely to have encountered many of the types of applications which are tested in the School Certificate examinations.

Table 2 indicates the mathematics topic areas to be found in 16 of the 1981 School Certificate papers. In the other School Certificate papers (languages, music, and so on), no real application of mathematics can be expected.

The list for each subject is only for those topics tested in the examinations in 1981, and clearly other related topics are likely to be taught in classwork. Questions asked in the Home Economics paper, for example, implied that calculations with money (for budgeting) and calculations with measure are part of normal classwork in that subject.

In most subjects, the applications are very straightforward in topics other than statistical graphs, where considerable perception is needed in questions asked in several subjects. For applications in other areas, pupils are commonly assisted by diagrams, instructions, and hints, so that they are expected to do almost no modelling for themselves. In both physics and science, however, there is evidence that students are expected to be able to propose and test mathematical models to describe given data or given physical situations.

TABLE 2

MATHEMATICS TOPIC AREA IN SCHOOL CERTIFICATE PAPERS

Subject	Mathematics Used	Approximate % of all SC entrants
Applied Mechanics	Wide range of skills from each of Arithmetic, Algebra, Geometry, Trigonometry, Measurement	0.1
Physics	Wide range of skills from each of Arithmetic, Algebra, Geometry, Measurement	4.6
Chemistry	Whole number calculations Addition and subtraction of decimal fractions Addition and subtraction of measures Calculation of percentage, ratio Scale reading Reading and interpretation of line graphs	2.6
Electricity	Range of skills from Arithmetic, Algebra, Measurement	0.1
Science	Four rules for whole numbers, formulae, plotting and interpreting graphs of functions, reading and interpretation of line graphs, vectors	60.0
Biology	Four rules for whole numbers and decimals, scale reading, rate, measurement, percentages, graph plotting and sketching, reading and interpretation of variety of statistical graphs, deducing trends from tables	17.7
Human Biology	Plot, read and interpret graphs of statistics	2.6
Horticulture	Ratio, percentage, scale drawing	0.2

TABLE 2 - continued

Subject	Mathematics Used	Approximate % of all SC entrants
History	Four rules for whole numbers and decimal fractions, ordering, ratio, percentage, reading and interpreting charts and tables, reading and interpreting a variety of graphs and statistics	20.1
Geography	Percentages, rate, scale reading, reading and interpretion of a variety of statistical graphs	47.2
Economic Studies	Understanding of percentage and ratio, reading, interpreting and drawing of graphs, calculation with money	20.2
Accounting	Calculations with money, percentages, profit and loss, ratio	0.1
Bookkeeping	Calculations with money, discount, profit and loss, depreciation, percentage	8.1
Engineering Shopwork	Calculations with whole numbers and decimal fractions, scale reading, ratio, reading tables, formulae, measurement including angle measure, spatial visualization	8.8
Technical Drawing	Scale drawing, measurement including angle measure, spatial visualization	20.3
Woodwork	Calculation with money (costing), measurement, spatial visualization	7.6

Clearly, the most widely applied branch of mathematics at fifth form level is statistics, and this continues to be the case in the sixth and seventh forms. At seventh form level, the statistics option of the separate subject Applied Mathematics is well-supported and use is made of inferential statistics in at least one subject, biology. Whether the extent and variety of use made of mathematical methods in other subjects is adequately taken account of by the system is an open question. As mentioned earlier, many teachers of mathematics teach physical or social sciences or technical subjects in addition to mathematics, so it can be expected that, at least in some cases, teaching programmes will be structured so that what is taught in mathematics supports the mathematics needed elsewhere. On the other hand, examination of textbooks indicates that some techniques are used in science and social studies some time before these techniques are taught to mathematics classes. (Variation in science and some statistical graphs in social studies are examples.) It is also claimed by mathematics teachers that teachers in other subjects sometimes teach the mathematical skills and knowledge needed for specific applications in a way which is out of tune with methods used in the mathematics classroom. Each of these situations will give rise to "interference" in the learning process itself and, in addition, is likely to induce negative attitudes in students to applying mathematics.

Innovations

During the early 1970s, and within the space of a few years, dramatic changes took place in the nature of the Form 5 mathematics population. This resulted from, first, a significant increase in the number of pupils remaining at school until at least the end of the Form 5 year; second, a considerable increase in the proportion of the Form 5 cohort choosing to take mathematics for School Certificate; and, third, a rapid increase in the proportion of girls choosing to take mathematics for School Certificate. Classroom teachers, many of whom were still coping with the changes to newer syllabus content and with the new approaches to the teaching of mathematics being advocated by curriculum developers, were not always able to adapt their teaching rapidly enough to take account of the much wider range of ability and motivations of their pupils. This led to a growing number of Form 5 pupils for whom the instruction they were receiving in the classroom was too abstract in nature and who were gaining little from their study of the subject.

In response to these concerns, leaders among mathematics teachers in two regions of New Zealand established "Local Certificate" courses for lower ability Form 5 pupils. The courses, while including most aspects of Form 5 mathematics, placed less emphasis on abstract topics (e.g., group theory, mathematical proof) and more on the practical use of mathematics. Such was the success of these schemes that within a few years similar schemes were developed, by local teachers, in most regions of the country. The success of the schemes was measured by the greatly improved attitudes and achievement reported by teachers of the pupils for whom they were designed. Many pupils regularly experienced success in mathematics classes for the first time in many years and were able to relate what they were learning to the world of their experience.

Procedures for between-school moderation for the awarding of grades within local regions are designed to enable teachers to adapt their courses to the abilities and interests of their pupils without the constraint of one-shot end-of-year examinations. Furthermore, where reference tests are used, the nature of some of the questions in the tests encourages teachers to ensure that classroom instruction in mathematics provides pupils with the techniques and the approaches to a problem necessary to apply mathematics successfully.

For example, a typical reference test from the Canterbury certificate scheme includes sets of questions which explore the mathematics of such activities as buying a corner shop (calculations based on components of purchase price, percentages, commission, interest on borrowed money, bank withdrawals), taxi charges (formulae, substitution, simplification), and a golf game (measurement, scale drawing). On the other hand, questions are structured in such a way that pupils are rarely required to develop a mathematical model, or even complete a step in the development of a model, and it is likely that this is true of most class instruction.

Concurrently with the developments at local levels, a Working Party on Syllabuses in Mathematics: Forms 5 and 6, set up in 1972 by the Minister of Education, was considering the mathematical needs of pupils at this level in relation to their general education and future careers, and suggesting guidelines for the development of syllabuses to meet a wider range of needs than those met by existing examination prescriptions. The most pressing need at that time was seen as suitable resources for lower ability Form 5 students and, in addition to fostering the "Local Certificate" schemes referred to above, the Working Party sponsored several in-service courses at which teachers produced teaching units. Some 40 units were produced for Form 5 pupils and issued to schools under the title of "Units of Related Mathematics". The aim was not only to provide resources, but to encourage teachers to produce similar materials themselves. The units produced included such topics as those indicated on Table 3.

Many topics are treated in a similar manner to these and deal with a wide range of applications. In addition, several units are games designed to give pupils facility with, and insight into, aspects of mathematics.

In recent years, a great many schools have started to use Outdoor Education Centres in the forest and mountains for group and communal activities. One or more classes at a time spend a week camping at these centres and receive instruction relating to the physical and biological environments and gain social experience. One of the units (a series of six topics giving a total of 16 activities) of "Related Mathematics" is entitled Mountain Mathematics. The topics range over activities such as estimating and calculating heights of trees on slopes, widths of rivers, volume and speed of river flow, water content of snow, and the like. An example of one activity which can be adapted to suit the level of knowledge of the pupil group follows:

TABLE 3

UNITS OF RELATED MATHEMATICS

Topic	Major Mathematical Content
A method for finding the length of a roll of paper (without unrolling it)	Measurement, decimals, area of circle, area of trapezium
Post Office centred activities	Computational skills, data collection, graphing
Measurement of reaction time	Measurement, recording and graphing data, calculating means
The rotary clothesline	Lines, angles, polygons, measurement, symmetry, loci, calculations of length and area, costing, rotational speed
Designing an open box with maximum volume	Spatial visualisation, measurement, nets of cuboids, calculation of volumes, graphing functions
Geometry and early architecture	Common geometrical constructions (and to awaken pupils to the geometrical basis of architectural design)

RL 1 RELATIONSHIP BETWEEN HEIGHTS OF TREES

AND ALTITUDE UP A MOUNTAIN

Target Pupils: Form 5 - Form 7.

Aims: To investigate whether any relationship exists between the height of beech trees and the altitude at which they grow, by drawing a scatter diagram and a curve (line) which best fits the results.

Introduction:

The emphasis is to be placed on ways of tackling the problem rather than accuracy of measurement, e.g.,

1 Ways of obtaining equally spaced heights up a hill if no altimeter is available

2 How to select trees at random

3 How to obtain heights of trees in a forest.

Requirements:

Clinometer, metric tape (altimeter), pencil, notebook.

Method:

1 Select a suitable bush-clad face of a mountain

2 Measure the heights of five randomly selected trees at the bush edge

3 Repeat this at equal intervals (at least four) up the mountain

4 Draw a scatter diagram of the results

5 Fit a curve of best fit to the scatter diagram.

Extension:

Investigate whether trees

a On a ridge

b On a north face

c On a south face

fit the curve.

In terms of general acceptance, the Local Certificate schemes and the introduction of mathematics directly and perceivably related to the pupil environment for less mathematically able pupils proved to be very successful developments. This was, in large measure, due to the fact that a felt need of mathematics teachers was seen to be responded to by mathematics teachers producing schemes and materials for mathematics teachers.

Innovation at Form 6 level has been largely in the hand of individual teachers. During the mid-1970s, and as a result of a movement initiated by the Working Party referred to above, efforts were made to introduce a course in applicable mathematics at this level. While there was a large measure of support for the idea, and for proposed curricula amongst teachers, various political considerations have combined to delay introduction of such an option.

The trend for pupils to remain at school longer and to study mathematics in greater numbers resulted in pressures in Form 6 mathematics classrooms similar to those felt in Form 5 classrooms of the early 1970s. Opportunity exists for schools to develop alternative programmes for less able pupils, but these pupils commonly choose to attempt the competitive University Entrance course. Most alternative programmes are remedial in nature, but there are examples of imaginative courses having been developed in individual schools. One such course, for example, deals with the establishment of a supermarket from site purchase to product marketing and incorporates a wide range of techniques — costing, supply and demand curves, queueing theory, optimisation, linear programming, and so on.

Calculators, Computers, and Applications

As yet calculators and computers have made little impact on the teaching of applications in schools. The availability of calculators to all pupils makes the application of mathematics to realistic problems possible in that the tedious computation is eliminated, and there is evidence that increasingly they are being taken advantage of for this purpose, but the practice is not general.

The use of computers in schools currently centres around learning programming at the junior levels. At senior levels, where the construction of algorithms is important, modelling is clearly involved. There is a small amount of usage of computers in other subject areas, but usually in this case the computer is merely a vehicle for simulation packages.

Ideal and Realization

As long ago as 1970, an international conference of educators claimed that ". . . present trends suggest that mathematics will increasingly be seen, for the majority of students, as comprising a useful set of techniques to be applied in practical contexts, rather than as an abstract study in its own right" [6].

Statements in the aims of mathematics syllabuses and prescriptions about applications, rhetoric at gatherings of mathematics teachers, and the acceptance of some materials and courses (for lower ability students) based on practical use of mathematics indicate that teachers in New Zealand perceive the desirability of giving more emphasis to applications in their teaching. Yet it would not be true to say that the majority of New Zealand school-leavers have developed facility in applying mathematics, or even been exposed to teaching methods which would be effective in enabling pupils to achieve this goal.

There are several factors that may have contributed to this state of affairs:

(1) A sharp distinction between "applied mathematics" and "mathematics" made in course structures in schools and universities may have been perpetuated in day-to-day teaching in the classroom.

(2) An apparent assumption that the appropriate role of applications is to make mathematics more intelligible to less-able pupils.

(3) Failure to appreciate that the procedures and thought processes necessary in applying mathematics must be taught explicitly. For example, it seems often to have been assumed that a two-dimensional diagram representing a solid is connected in a meaningful way for the pupil with the real object it represents. Similarly, there is a quantum leap for many pupils between a statement such as "The cost of printing cards is made up of a fixed cost and a cost per card" and $C = F + nc$.

(4) An assumption that the pupils' view of the world coincides with that of the teacher and other adults. A 14-year-old may not perceive any more relevance in house mortgages than in quadratic equations.

(5) Many traditional "applications" exercises are what might be termed intellectual applications. They refer to situations remote from the pupils' experience and which cannot be intuitively grasped so that although the required cues and responses are learned, transfer to other (mathematically) similar problems cannot be made.

(6) Success for pupils in learning to apply mathematical techniques pre-supposes mastery of those techniques. Where there is an unsuccessful attempt to apply half-learned skills, it is likely that serious interference with both learning to apply mathematics and learning the required skills occurs.

(7) Little attention has been paid in school mathematics curricula to the notion of mathematical modelling. Activities directed towards giving pupils the ability to translate verbal descriptions or physical situations into the diagrams, equations, expressions to which mathematical techniques can be applied and,

conversely, activities in which pupils describe circumstances and objects for which given equations, expressions, and diagrams are analogues can and should be made an integral part of teaching practice.

Teaching of applications has a role in motivating students, in ensuring the transfer of mathematical ideas and skills, for illustrating the role of mathematics in our society, for teaching problem solving, and, not the least important, for giving insights into the nature and power of mathematics itself [7]. Prerequisites for effective learning in this area include pupil confidence in constructing a mathematical model, mastery of the skills and techniques needed for a given application, sufficient familiarity with the area of application to be able to interpret results of the application. Ormell [8], in discussing social aims of "mathematics for the majority", suggested that the useful residue left when all technical work is forgotten should include:

(a) An appreciation of the modelling role of mathematics and what it achieves,

(b) A feeling for the kinds of problems and proposals which can benefit from mathematical analysis,

(c) Innovative mindedness, and

(d) A capacity for logical thinking about real problems.

Although considerable effort has been made in New Zealand to emphasise the role of applications in mathematics, and although teachers are becoming increasingly aware of the issues involved, there is still some way to go before there can be confidence that aims such as those listed above will have a chance of being achieved.

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NORWAY

Gunnar Gjone

In the period 1955 to 1976, Norway went through a profound change in its educational system. Compulsory education was increased in length from 7 to 9 years, and secondary education (grades 10 to 12) was reshaped. Several different types of schools — the gymnas and various vocational schools — were merged into one "type". The earlier different types of schools emerged as "areas of study" in the new "Upper Secondary Schools"[1].

These changes also had consequences for mathematics education. As had most of the other countries in Western Europe, Norway experienced "new math" in the same period as the school reform took place. Several projects were initiated, and by 1971, it seemed that the "new math" would influence strongly the curriculum for all grades, 1 through 12. However, before the curriculum reform of the new school system was completed, reactions to the "new math" movement were underway, and by the completion of the curricula (around 1975) the curriculum in mathematics was considered by the "new math" proponents to be very "traditional". But "new math" has survived to a varying extent for the various grades:

> <u>Primary education</u> (grades 1-6) contains some topics usually associated with "new math", especially the methodical approach to introducing concepts.
>
> <u>Secondary - I</u> (grades 7-9) includes basically traditional mathematics.
>
> <u>"Upper Secondary"</u> (grades 10-12) retains some "new math" in the "General area of study". This is mainly some content in the "University preparation courses".

The "new math" in Scandinavia strongly emphasised the principle of applying mathematics, and the opponents attacked it for not being able to implement this principle in the actual texts.

The Present Situation

In order to understand some of the problems facing Norwegian mathematics educators, it is necessary to know some of the important elements in the Norwegian school system.

Norway has a very centralized school system. Textbooks are commercially produced, but they are all subject to government approval; i.e., they must all cover the material set down in the curriculum plans. The curriculum plans are, moreover, common to the whole country. Another factor of interest should be mentioned: it has been shown by various surveys that the teaching, especially in mathematics, is very much textbook-dominated. Therefore, the curriculum plans have some effect on everyday teaching through the textbooks.

After the intense debate and development in the preceding period, mathematics education saw a quiet time from 1975 to about 1980-81. People and the authorities felt that mathematics had gotten its fair share of government funding and attention; therefore, other school subjects were put into focus. This state of affairs resulted in very few experimental projects. Such projects have to be government-approved, because of the final exams, taking place at the end of grades 9, 10, 11, and 12. It should also be noted that these national exams are by many considered to be a conservative element in the school, restricting extracurricular activity to a large extent.

However, around 1981, things started to change. The reasons for this change are difficult to assert, but some factors were clearly visible. An increasing number of teachers complained that the curriculum (and textbooks) were too theoretical, with too few real applications. One asked questions about how the present mathematics education enabled pupils to use mathematics in daily life. Another factor is that Norwegian mathematics teachers increasingly have become aware of the developments in other countries. But before we consider these new tendencies, let us see how application of mathematics is considered in our present curriculum.

The Curriculum Plans

Mathematics education is governed by two curriculum plans, one for the Basic School (grades 1-9)[2] and one for the "Upper Secondary".

In the formulation of goals for school mathematics in the Basic School we find:

> exercises to apply mathematics in daily life problems, and problems from other school subjects,

and in the commentary this aspect is further strengthened:

> the main aim of mathematics education in the compulsory school is to enable the pupils to solve problems which arise in daily life, in society or in various vocations (which are not too well specified).

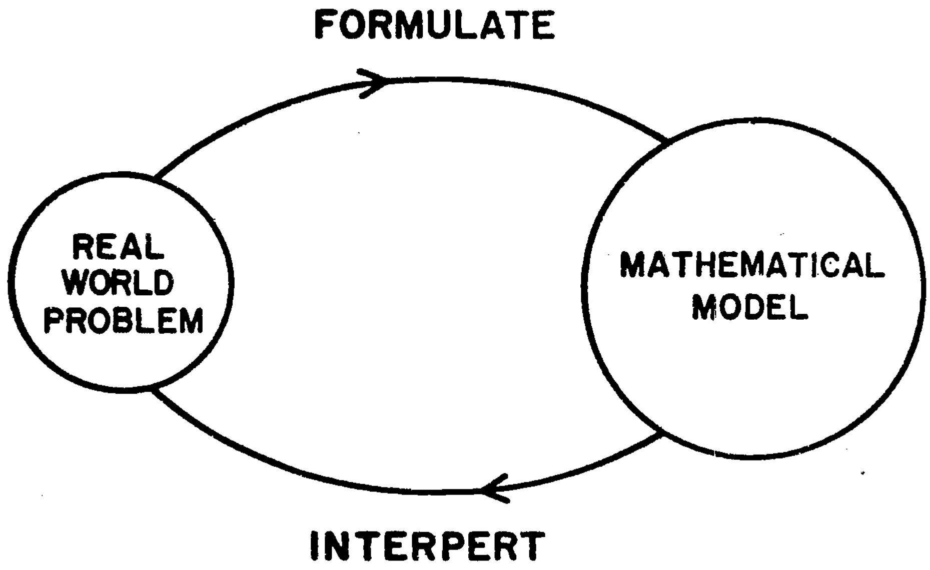
Also, the commentary seems to suggest the "model" in Figure 1 for applications of mathematics in schools (cf. [1].

How are these general formulations expressed in the list of topics (content)? One major topic for the Basic School mathematics is "Application of mathematics" — the others being numbers and arithmetic, algebra, equations, functions, and geometry. This organisation has been criticized as isolating applications from the rest of the mathematical topics, so that it does not become integrated into the curriculum.

A consequence of this division in the curriculum plan is that most textbooks do tend to be edited the same way as the curriculum plan, expecially in the upper grades. This on the one hand makes the books more suitable for "projects"; on the other hand, it is easier to skip this material if time runs short.

One more thing should be noted. In the curriculum plan, little emphasis is placed on model building/formulation:

More advanced mathematics education



Introductory mathematics education

Figure 1. Model for applications in school mathematics.

> When the pupils encounter a problem in the mathematics class, the problem formulation should include relevant information about the subject matter necessary for solving the problem.

This we read in the curriculum plan for the Basic School. Hence, one is not encouraged to do much exploring and model-building oneself. We also find that model-building has not been much tested in the exams. However, this is one area in which one expects to find large differences in the actual teaching practices.

As a conclusion we might say that the curriculum plan for the Basic School stresses applications, but it is questionable to what extent it is being realised in mathematics education.

Concerning the curriculum plan for the "Upper Secondary School", we find a quite different situation. The school is very different from the Basic School. It has several different "study areas". We will not consider here the various vocational study areas, but concentrate on "the area of general studies"[3]. In this area we also find several "branches" (social studies, natural science, and languages). The first year (grade 10), however, is common, with a common mathematics course for all pupils. Only the natural science branch and the social studies branch have advanced mathematics courses, but a certain freedom of choice is also possible.

The natural science branch has advanced courses preparing the pupils for further studies. Applications would be mainly in other school subjects, notably physics. The social studies branch would contain a fairly large proportion of mathematics in use in society and economics. These advanced courses may be regarded as specialized; the most interesting case would be grade 10.

It should here be remarked that in the "goal-formulation" for the curriculum, applications as such are hardly mentioned explicitly. We find only formulations like the following:

> necessary knowledge and skills . . . concerning what is needed in a modern society.

The phrase "mathematical method" also enters the goal-formulation, but it is not made explicit what this means.

Mathematics in grade 10 — "the first common year" — has been a very problematic issue. To construct a course that on the one hand provides a basis for further study and on the other hand a meaningfully terminated course has been very difficult. The textbooks being used contain many interesting examples of applications of mathematics to society and natural science. To a lesser extent than the Basic School, the "Upper Secondary" focuses on general applications; i.e., model formulation, interpreting, etc.

The picture outlined above is by far the most common concerning mathematics education in Norway, but as noted earlier we have seen some changes in recent years.

New Textbooks and Projects

New texts, both experimental and regular texts, have been written with an emphasis on applications of mathematics. The rest of this article will examine two of these texts in more detail.

First (for both texts mentioned), they have been very well-received by teachers. Moreover, the authors are well aware of international tendencies and developments in other countries.

The first text we will consider is a regular[4] text by the authors Garmannslund and Winther [2] called "The Math Book". It has so far only been issued for grades 4, 1, and 5 (in that order). In the teachers' commentary to the fourth-grade book, the authors explicitly state their philosophy [3]:

- To start in the reality that surrounds the pupils every day

- To find situations and themes that are such that the relevant mathematical subject matter is "natural" to the situation

- To show the pupils that we need the use of mathematics and use it every day — it is not merely what they do in class and on exams, but something they can use here and now

- To concentrate on material which we feel is a necessary basis to function in modern society.

How has this programme found its form in the text? If we consider the book for fourth grade, it consists of 10 themes or projects:

(1) At home

(2) The clock and the calendar

(3) Traffic

(4) The shopping centre

(5) Preparations for Christmas

(6) Our school

(7) Sports and recreations

(8) Dogkeeping

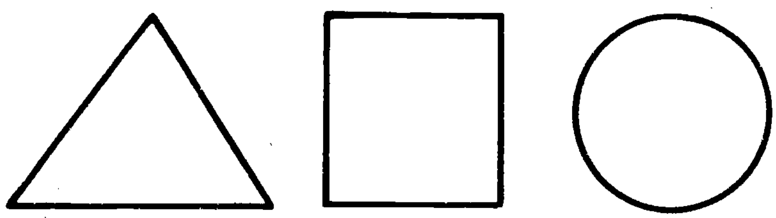
(9) The grocer's storage room

(10) The class tour (end of the school year)

As can be seen from the authors' programme, modelling is not the dominant trait. However, it is a question to what extent it can be introduced at this stage. It is probably more correct to say that they show how mathematics is used and enter into situations supposed to be familiar to the pupils.

Let us consider one theme in more detail: Traffic, which has as one of its subtitles "Geometry in traffic". The first section is on identifying various forms and shapes in the "traffic picture" — circles, triangles, etc. This is then related to the meaning of the road signs.

The next section is a brief introduction to combinatorics by looking at ways to put up road signs (of different shapes):



Then, in the last section, pictures of bending roads are used to introduce the concept of curves, lines, and line segments. This is in one sense not strictly applied mathematics, but at this stage the material serves the purpose of introducing mathematical concepts, i.e., formulating a mathematical model. We might say it represents the "first" half of the picture we presented above:

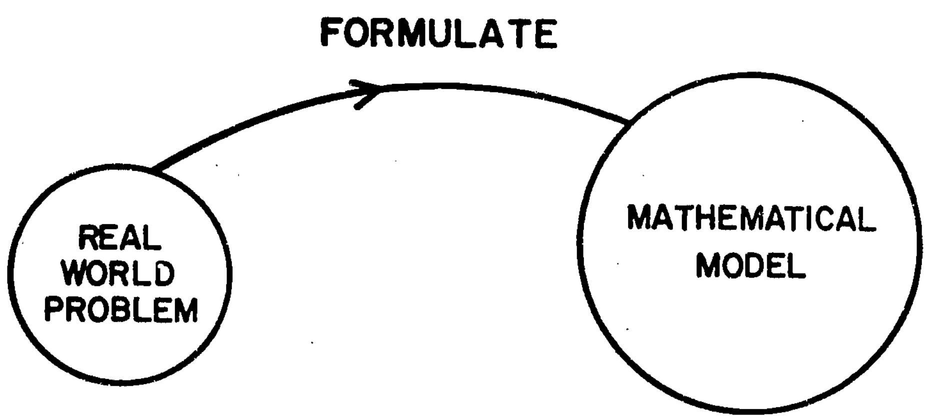


Figure 2.

The other example is an experimental text[5] for grade 10, by Ommundsen and Solvang [5], titled Mathematics and Society.

The concept of function is considered as a unifying concept, and the text considers the use of functions in various situations. For example, equations arise when we know the function value, but not the (independent) variable. Unlike the first text, the chapter headings of this book contain mathematical themes:

(1) Mathematics and society — an introduction

(2) Functions of the second degree

(3) Hyperbolas

(4) Rational expressions

(5) Equations

(6) The mathematics of growth

As an example we will consider how the topic "Functions of the second degree" is introduced.

We are introduced to the topic with an example — how long a person can survive in the sea depending on water temperature (a relevant problem in Norway at the end of the bathing season). Initially, we are presented with a newspaper advertisement, among other things showing a table giving values for how long a time it will take for a person to get exhausted (die) with decreasing temperature. This table is then plotted into a coordinate system, giving something which "does not look like a linear function". Then the idea of a second-degree function is elaborated, and uses are explained through examples.

The picture we get from this text is roughly the full picture of modelling:

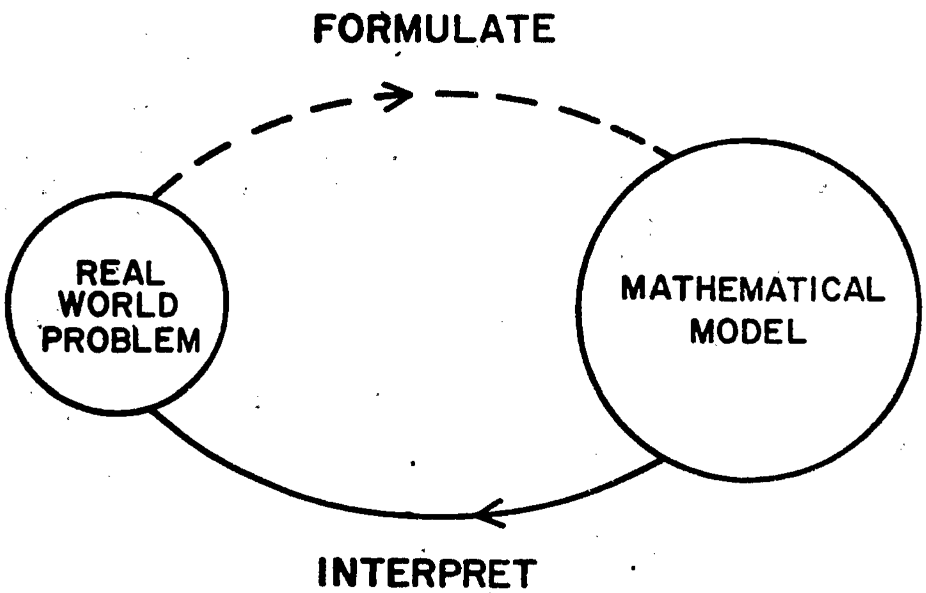


Figure 3.

but with interpreting the model as the most dominant factor. However, one aspect which is covered to a very little extent is improvement of the model.

We also evaluate this text as foremost in showing how mathematics is used in society. Mathematics is found in many situations and we can use it beneficially. That model building, as such, is not the primary target of the text can be explained by the fact that this is an introductory text. On the one hand it will be followed by advanced (specialized) mathematics courses; on the other hand it should also be self-sufficient.

Concluding Remarks

These two books are examples of the current trend in Norwegian textbook production, and given that mathematics teaching is textbook-dominated, they might have some effect on the teaching of mathematics.

For the Basic School, several textbooks have been recently revised, and applications of mathematics have increased in these books. For the "Upper Secondary", several new texts, building upon the same philosophy as that of Ommundsen and Solvang, have been announced. Therefore, we might well talk about a trend toward applications in Norway.

A general feature of these texts seems to be that they are related to the use of mathematics in society. The mathematical models are used to explain everyday phenomena — the electricity, or what it costs to operate a car. Thus far, there have not been many applications to natural science.

That applications of mathematics will be an important part of school mathematics in the coming years in Norway is a likely assumption. In a discussion paper on the mathematics for the Basic School, issued by the Basic Schools Council [4], applications are considered as one of the major factors to determine the content of school mathematics.

Reference Notes

1) It should be noted that "Upper Secondary School" will be established as the English language notation for this type of school in Norway for pupils of age 16-19.

2) This is called The Model Plan for the Basic School.

3) The mathematics curricula for the various vocational areas of study contain a fairly large portion of applications of mathematics within the different vocations.

4) That means it is approved by the government to be in correspondence with the curriculum plan.

5) An experimental text does not have to follow the curriculum plan; therefore these pupils will have separate exams. It is approved as an experiment on the grounds that it also gives the pupils the necessary qualifications for further studies.

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[2] Garmannslund, K. and Winther, S. Matteboka ("The Math Book"), 4.klasses ("4th grade"). Fabritius/Gyldendal, Oslo, 1981.

[3] Garmannslund, K. and Winther, S. Matteboks, Lærerveiledning ("Teachers' Commentary"). Fabritius/Glydendal, Oslo, 1981.

[4] Grunnskoleradet. Matematikk for grunnskolen ("Mathematics for the Basic School"). Oslo, 1982.

[5] Ommundsen, J. B. and Solvang, R. Matematikk og samfunn ("Mathematics and Society"). Prøvetekst ("Experimental text"). Cappelen, Oslo, 1982.

TANZANIA

G. Mmari

Introduction

Since school systems differ, it is important from the very outset to describe the school system in Tanzania in order to place the rest of this short paper in its proper perspective.

In effect since 1967, the year the Arusha Declaration[1] was announced, official policy requires that children go to school at the age of seven. These children remain in school at the Primary level for the next seven years, until they are 14-plus. A small proportion of these children then proceed to the next level — the Secondary level — for another four years. The proportion has been declining since, whereas the number of children going into secondary schools has been increasing, this increase has not been large enough to offset the larger increase of those completing the primary school level. The third stage is the Advanced Secondary Level, which takes two years to complete. Diagrammatically this would correspond to the following:

Years	7 years	4 years	2 years	1+
Level	Primary	Secondary (O-level)	Secondary (A-level)	Post-Secondary Education or Training
Age Range	7 - 14	14 - 18	18 - 20	20+

The description above highlights the following points:

(1) That each school level is terminal, with a few proceeding to the next stage.

(2) That boys and girls attending schools in Tanzania are on the average two years older than their peers in other countries, e.g., the U.K.

(3) Although this is not clear from the description above, these school groups are not homogeneous. Not all pupils begin school at exactly seven years of age. Some are as old as ten, while a few are six years old, e.g., in some urban areas.

Historical Background

Mathematics has historically been developed out of real-life situations. Before any contact with the outside world, people found it necessary to count their possessions, e.g., cattle; measure time; describe distances, weight, and capacity; etc. There are studies to show how different societies did this in Africa[2] and also in Tanzania. One can therefore state that mathematics has an applied origin and application in Tanzania as elsewhere.

The practical orientation of mathematics persisted when foreign contacts were made. Arab traders from the Middle East used units of measure to sell their wares and paid in cash or in kind for the articles they carried away from Tanzania and the rest of Eastern Africa. The Arab influence exists to this day in the form of the mathematical terms used. Even the word for mathematics has an Arabic origin.[3]

The Arabs interacted with the native population so much that their cultural values and practices found a lasting place in this part of the world. Formal schooling in the form of Koranic schools paved the way for further formal schooling. The German and British colonial governments, Christian missionaries, and others continued with the formal schooling throughout Tanzania. Their curricula were practical-oriented since their main interests were the training of skilled workers for government service, for trading transactions, for farming undertakings and such other practical activities.

Attempts to relate the practical examples to the real-life situations started quite early. Books written in the local language and for the Primary level were full of exercises on currency, travel, weights and measures, etc.[4] Those used at the Secondary level were imported and written in English. The practical examples in the exercises were foreign in character and unfamiliar to the ordinary boy and girl with a local upbringing. Games and sports, currency, cultural problems, etc., were set in strange environments and hence unsuitable.

At political independence in 1961, Tanzania had inherited a system that was based on Kiswahili during the first four to six years of Primary education and based on English during the last two years of Primary education (it was then an eight-year Primary school system). English was the only language of instruction at Secondary level. As pointed out above, many of the textbooks were inappropriate, since they were written outside Tanzania and for children in a foreign environment with a foreign culture.

Nationalistic feelings demanded "localization" of the curricula used in all subjects.[5] Use of local geographical names, reference to local economic and social settings, and greater use of the local language were emphasized. The climax was reached when the policy document "Education for Self-Reliance" was issued in March 1967. Changes were embarked upon in earnest in all subjects. It is no wonder, therefore, that one began to see syllabuses which were different from those of the past.

While these changes were taking place, the "Modern Mathematics" influence made its way into East Africa. In 1962 an international workshop was organized in Uganda at Entebbe to write the first of a series of books in modern mathematics at both the Primary and Secondary levels. Two strong programmes affected these developments in East Africa. One was the American SMSG programme and the other was the British SMP programme. Both were allowed to influence activities in East Africa. As a result, two parallel programmes were allowed to co-exist, the Entebbe Mathematics Project and the SMP Project. The latter changed its name to SMPEA and finally to SMEA.[6] Tanzania decided to merge the two in the early 1970s into simply Modern Mathematics (S and E).

While modern mathematics programmes were being introduced, traditional programmes continued to be taught. It did not take long, however, before the anti-modern mathematics feelings caught up with Tanzania. These were resolved by evolving a single mathematics curriculum referred to simply as Mathematics. This takes into account the best from both worlds, but with a strong influence of the 1967 policy paper on Education for Self-Reliance which stresses the practical aspects of knowledge.

Another important factor which has influenced the teaching of mathematics has been the national urge to diversify education into practical biases. Four biases have been officially launched and these are (1) Technical, (2) Agriculture, (3) Commercial, and (4) Domestic Science. Every secondary school is supposed to offer one of these four practical biases and each is required to spend up to one-third of its teaching programmes on the practical bias assigned to it. It goes without saying that mathematics lessons taught in such schools will be greatly influenced by the school atmosphere.

November 1974 is yet another important milestone in the history of education in Tanzania. On that date, the National Executive Committee of the political party in power decided to revisit the policy paper on Education for Self-Reliance. Among other things, the Party meeting directed that practical emphasis should be stepped up in the whole educational system. It also decided that theoretical examinations should be played down in favour of other methods of assessment[7]. These other types of assessment were to include project work, field work, practical work, etc. Final assessment was to be determined by performance in both theory and these practical exercises.

This historical sketch now needs qualifying through examples which will illustrate the principles outlined above.

Examples of Applications

(1) From Policy Statements

After the policy paper "Education for Self-Reliance", it was decided that ". . . all subjects taught in the primary school should be related to agriculture. . ."[8]. Teachers of mathematics were not left alone and from 1967 onwards, therefore, textbooks, teachers' guides, examination questions, and syllabuses were expected to show how mathematics can be used and is used to solve problems of an agricultural character.

Referring to the teaching of mathematics, the 1967 document states:

> . . . local flavour is brought in by using objects and examples that are familiar to pupils . . . the application of mathematics to Agriculture. Profit and loss, credit, and application of mathematics to science in general must be given due prominence . . . [9]

On examinations, the documents states ". . . include a greater emphasis on socialism, cooperative unions, agriculture and civics . . ."[10].

(2) From Syllabuses

The Primary School Mathematics Syllabus drafted in 1969, i.e., two years after the 1967 ESR policy paper, is based, according to the preamble, on:

(i) New approaches and ideas in the teaching of mathematics

(ii) The need to provide relevant education which reflects their environment in order to prepare them for life as stipulated in the overall aims of Primary education.

It will be noted here that, whereas new ideas are accepted, there is an emphasis on skills for life after school. This implies mathematics and its applications to real-life problems.

The 1974 Secondary School Mathematics syllabuses, on the other hand, were drafted seven years after the 1967 ESR document and at around the 1974 Party policy meeting. The objectives of teaching mathematics are listed as:

- to develop mathematical skills among pupils which will enable them to function in all practical affairs of life
- to provide pupils with a mathematical tool which they can apply in other subjects
- to develop pupils' abilities to discover mathematical concepts and ideas and also their ability to think logically
- to prepare pupils for higher studies [11]

The syllabuses, which cover both Ordinary and Advanced levels of secondary education, specify which topics must be taught. They also indicate in the "Notes" column what practical projects ought to be carried out to drive home the possible applications of the concepts covered in the various topics. This is not possible for each topic, since not all topics lend themselves easily to this kind of treatment.

It should be noted that even before the current syllabuses were drafted, the emphasis on applications had been given prominence. Statements like "The emphasis of the examination will be on the understanding of basic mathematical concepts and their applications, rather than on skill in performing lengthy manipulations . . ."[12] occur on old examination syllabuses used in Tanzania and East Africa. Examinations have a great influence on teaching and hence the statement above would imply that applications have had a role in school mathematics teaching.

(3) From Textbooks

The mathematics syllabus at Primary school level is usually taught using a series "Mathematics in Tanzania" Books 1-7, one for each year of the Primary school. Whereas Books 1-6 have mathematical topics many of which reflect real-life activities, Book 7 is almost exclusively real-life situations. It does not need much to see that Primary 7, which is the terminal year of the Primary school cycle, is used here as a preparatory year for life after school.

The same can be said of the Secondary school textbooks. Book Three of the series "Secondary Mathematics" is a typical example of how emphasis has been given to applications of mathematical concepts. Chapter 8 is on Plan and Elevation. Examples and exercises are from real-life buildings, structures, and facilities which are technically handled using concepts of plan and elevation.

(4) From Experimental Programmes

The Higher School Certificate Subsidiary Mathematics syllabus was traditionally used to teach mathematics to boys and girls (aged 18-20) at the Advanced level who were not doing mathematics as a principal subject. Those doing combinations involving geography, chemistry, physics, economics, etc., were required to do HSC Subsidiary Mathematics. This was detested by the pupils because they did not like taking the extra load and also because they did not see how this was related to their subjects.

A questionnaire was sent to teachers of subjects other than mathematics and requested them to indicate which topics they considered essential for proper handling of their own subjects. These questionnaires were also sent to teachers of engineering subjects in universities and colleges. Other scientists and users of mathematics were also requested to indicate their observations.

Returns from the questionnaires were summarised and formed the basis of an experimental syllabus for non-mathematicians[13]. This approach indicated that mathematical topics were useful as tools for grasping techniques and concepts in other fields. It was possible to convince pupils that these were not an imposition but necessary for their own success.

It was also possible to explore areas of cooperation between mathematics teachers and those of other subjects, e.g., physics teachers handling the mechanics topics of the mathematics syllabus, in order to give the topic a physics orientation. The resulting syllabus has since been branded a "Basic Applied Mathematics" syllabus since the concepts covered were basic and since they were for application in other disciplines.

Topics in this syllabus include slide rule; sets — their properties and Venn diagrams; coordinates; algebraic solutions to equations; matrices; vectors; probability; elementary statistics; mappings, relations, and functions; differentiation and integration; numerical methods; kinematics; complex numbers; sequences and progressions; and logic. As pointed out above, these topics were mentioned by the non-mathematicians and received emphasis of varying degrees. This short paper cannot go into where each topic can be used in each field investigated.

Another experimental approach which has become the modus operandi is the curriculum development process itself. Subject Panels are at the core of the process. On each Panel sit members from a cross-section of mathematical interests both in a vertical sense and in a horizontal one. The drafts produced are then circulated to a wide range of potential users of the concepts taught. For example, the Primary School Geography curriculum developer was interested to see that bar graphs, pie-charts, proportions, and simple statistics are taught at around the same time that he is ready to apply these concepts and skills derived from their teaching in Geography. Not all syllabuses lend themselves to this type of treatment, but the principle has proved extremely useful.

Trends for the Future

From the description so far, it is obvious that practical applications of mathematics will continue to be emphasized in Tanzania. This will be both in regard to life situations outside school as well as within the school when teaching the other subjects.

The syllabuses will also see the inclusion of more and more practical topics. For example, book-keeping will be stressed so that school leavers can apply this skill either during self-employment or in their future vocations in public and private sectors.

Textbooks written for use in Tanzanian schools will have many examples related to the changing environment in this part of the world; e.g., statistics of exports and imports; construction industry; farm produce; etc.

The training of teachers for the future schools will stress the project approach which will enable the teacher trainees to integrate their college work and the real-life situations outside the college. This will also have to stress innovations, creativity, and imaginative character.

Finally, there will be need for greater cooperation btween mathematicians and non-mathematicians who apply mathematical techniques in their day-to-day activities. This will ensure closer development of programmes reflecting the actual environment.

Reference Notes

(1) The Arusha Declaration set the socialist and self-reliance path Tanzania was to follow. See entry [10] under References. It was promulgated by the Tanganyika African National Union, the party in power.

(2) Raum, O. F. Arithmetic in Africa, Evans Brothers Ltd., London, n.d. Other researchers include Zaslavsky, C. Black African Traditional Mathematics, Mathematics Teacher, April 1970, pp. 345-356.

(3) Mmari, G. R. V. Tanzania's Efforts and Experiences in Teaching Mathematics in a Foreign Language. Paper presented to a UNESCO Conference, Nairobi, 1974. Also published in Mathematical Association of Tanzania Bulletin, vol. 9, no. 3.

(4) See books by Raum, Carey, and Owen, whose series were used widely at the Primary school level. Those by Carey Francis and by Owen were used in East African schools. Other authors also stressed local examples of applications of mathematical concepts.

(5) See proceedings of the Conference convened to discuss how to implement the policy paper "Education for Self-Reliance", entry [9] under References. See also the official view in the national plans The Second Five Year Plan 1969 - 1974, Vol. I, United Republic of Tanzania, Dar es Salaam, Government Printer, 1969, chapter X.

(6) SMPEA is essentially the SMP as revised for East Africa by localizing geographical names, currency, etc. This was later replaced, after experimentation, with SMEA, i.e., School Mathematics for East Africa, which stressed the East African context of the topics, their teaching, and exercises.

(7) Specific reference to the need to downgrade written examinations and emphasize practical work is at paragraph no. 47 of the document referred to at [11] in the References.

(8) See Conference Report [9] of References, page 5.

(9) Ibid., p. 6.

(10) Ibid., p. 8.

(11) These are spelt out at the preamble to the syllabuses cited as [5] in the References.

(12) Practically each syllabus used prior to the 1974 published document has, as its introduction, this general statement. This was the case with the Cambridge Overseas School Certificate Examinations, the East African Examinations Council syllabuses, and the Tanzanian Provisional Examinations Syllabuses.

(13) For a detailed description of this experiment, see Mmari, G. R. V. Participants in Curriculum Development: A Case Study of H.S.C. Subsidiary Mathematics Syllabus. Studies in Curriculum Development, No. 2, 1972.

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[4] Ministry of National Education. Secondary Mathematics Book Three. Ministry of National Education, Dar es Salaam, 1972.

[5] Ministry of National Education. Syllabuses: Mathematical Subjects for Secondary Schools. Ministry of National Education, Dar es Salaam, January 1974.

[6] Mmari, G. R. V. New Mathematics. Studies in Curriculum Development, No. 1, pp. 117-128, Institute of Education, 1972.

[7] Mmari, G. R. V. The United Republic of Tanzania: Mathematics for Social Transformation. In Socialist Mathematics Education (F. Swetz, editor). Burgundy Press, Pennsylvania, USA, 1979. Chapter 8.

[8] Mmari, G. R. V. Secondary school Mathematics in the United Republic of Tanzania. Studies in Mathematics Education, Vol. I, UNESCO, 1980. Pp. 106-126.

[9] Nyerere, J. K. Education for Self-Reliance. In Freedom and Socialism. Oxford University Press, 1968. Pp. 267-290.

[10] TANU. The Arusha Declaration. In Freedom and Socialism. Oxford University Press, 1968. Pp. 231-250.

[11] Tanganyika African National Union "Directive on the Implementation of "Education for Self-Reliance". In Papers in Education and Development, No. 3, Department of Education, Dar es Salaam, 1976. Pp. 1-14.

UNITED STATES OF AMERICA

Max S. Bell

Introduction

An assessment of applications in school mathematics instruction should probably attend to at least five categories of availability and use of resources:

(1) Raw materials from which to construct applications of mathematics for school use should exist. These include expositions about the uses of mathematics written at a level accessible to teachers and to curriculum builders who are not themselves specialists in this or that application. They also include sources of interesting data and information about a variety of things.

(2) Such raw materials should be transformed into problem sets and units that honestly reflect uses of school mathamatics. Mere existence of such materials will not necessarily mean they will be incorporated into school instruction, but that is the eventual goal. They should be linked to specific school topics, use realistic data in realistic ways, and exist in large amounts to serve a wide range of student abilities and interests.

(3) Such problems and units should become a normal part of everyday instruction in mathematics at all levels. For example, widely used school textbooks should make good use of such materials. Official curriculum guides and widely used examinations should include attention to applications. Calculators and other tools to finesse mere computational drudgery in applications should be available at all levels to all students.

(4) Preservice and continuing education of teachers should help them link mathematics to its uses in their teaching.

(5) There should be active research and development concerned with the teaching of applications and with better understanding of how people learn to use what they know.

A brief summary of the teaching of applications in the United States can be put in that context: there are rich and quite accessible raw materials from which to construct applications. In recent years a variety of sourcebooks, articles, and units have been published that provide excellent applications for school use at all levels. Several well-established organizations work at the production of more such materials. At every school level, exemplary textbooks that attend well to applications have been published. Hence, a solid basis for substantial attention to the teaching of applications exists. But those exemplary materials are little used in schools and often go out of print within a few years of their publication. Teachers generally are quite poorly equipped to handle applications of mathematics. A number of promising research and development projects concerned with applications exist, but formerly generous federal funding for such projects has been essentially eliminated. Computation tools that in principle should make the teaching of applications much easier (calculators and, increasingly, microcomputers) are in very widespread use in the world outside of schools. Their use in mathematics education, however, is restricted pretty much to college and high school, and seldom in the service of substantial applications.

In sum, in the United States most of the ingredients that would enable substantial attention to the teaching of applications appear to be present. Yet with few exceptions, applications are not a part of the school mathematics experience. The remainder of this review will expand on that summary and cite examples of activity or materials in each of the five categories listed above. Annotations of many of the materials referred to here as well as more information about sources and availability can be found in Bell [3].

Raw Materials from Which to Construct Applications

The starting place for any serious effort to link school mathematics to its uses is reliable information about genuine uses of mathematics and good sources of data to exploit in constructing problems. In the United States, at least, we are blessed with vast amounts of such material at all levels of application of mathematics and statistics. In part, this reflects the enormously expanded uses of mathematics in the past few decades. Most scholarly pursuits and many ordinary occupations apply mathematics at least at the level of coping with numerical and graphical information. Happily for us, experts in a wide variety of applications have served up accessible and often entertaining expositions about these applications. Vast amounts of numerical and graphical data are available on a wide range of subjects in newspapers and in general circulation or special interest magazines. Almanacs and other collections of facts (and trivia) exist and sell large numbers of copies. Information services accessible by relatively inexpensive personal computers are gathering subscribers. The net result is that it is quite easy to support construction of believable and varied applications for any school mathematics course.

With so many raw materials for constructing applications available, I will discuss here some useful bibliographies and collections of such materials, rather than attempting coverage of the materials themselves. A good place to begin is with the annotations of more than 150 expository articles and titles of more than 50 books in A Sourcebook of Applications of School Mathematics [6]. I regard this as a model of usefulness, since the annotations themselves give much useful information about a variety of applications. The same can be said of the pamphlet Books for Teachers of Statistics in Schools from the Joint Committee on the Curriculum in Statistics and Probability [20]. Another sort of model is provided by Statistics: A Guide to the Unknown [38], which includes several dozen brief, non-technical essays by expert statisticians about specific applications they have made of statistics. Collections of essays about applications of other mathematical disciplines at a similar level of exposition and easy accessibility to those not expert in that discipline would be very welcome, but seem not yet to have been attempted.

Each book or article in those bibliographies is likely to give additional sources of applications. The same is true of most of the books for teachers that will be discussed below. A steady stream of U.S. Government publications gives vast amounts of numerical and other information. There is certainly no shortage of raw materials from which to construct applications of school mathematics.

Collections of Applied Problems

With the assurance of ample raw materials from which to build applications, the next task would seem to be transformation of those into school-usable materials. There has been a lot of progress on that task in the U.S. during the past decade, and we will concentrate in this review on the published problem collections that have thus become increasingly available.

Several fine sources of applied problem material have resulted from projects funded by the National Science Foundation. Perhaps the richest of these is the Mathematics Resource Project [14]. This is a resource file in five parts, each of which includes a "didactics" section, a "classroom materials" section with large numbers of applied problems, and an excellent annotated bibliography. Permission to copy the materials comes with purchase of them, which makes available hundreds of pages of fine applications materials. Here are the titles of the five parts and the number of pages in each:

(1) Number Sense and Arithmetic Skills (832 pp.)

(2) Ratio, Proportion, and Scaling (516 pp.)

(3) Geometry and Visualization (830 pp.)

(4) Mathematics in Science and Society (464 pp.)

(5) Statistics and Information Organization (850 pp.)

Parts 4 and 5 are especially valuable with respect to applied problems. I regard this material as a nearly indispensable resource for what in the U.S. would be grades 5-9, and for teacher training.

Two book-length NSF-funded collections of carefully crafted "real problems with real data" are Mathematical Models and Uses in Our Everyday World by Bell [4] and A Sourcebook of Applications of School Mathematics by Bushaw and others [6]. The former is one of a limited number of School Mathematics Study Group (SMSG) publications still available, and consists mainly of applications of arithmetic built around various themes. The bulk of [6] consists of many brief problems (with solutions given) in six chapters titled "Advanced Arithmetic", "Algebra", "Geometry", "Combinatorics and Probability", and "Odds and Ends". A few longer projects round out the collection.

Other nice collections of short applied problems have been published through commercial or private channels. Present availability of them is sometimes uncertain, so one may need to rely on libraries or direct correspondence with the authors. These include Ahrendt [1], Texas Instruments [39, 40], Friebel and Gingerich [12], Judd [21], Saunders [33], Sloyer [37], others annotated in Bell [3], and no doubt many others.

Two projects with National Science Foundation funding have produced applications units at opposite ends of the curriculum. Unified Science and Mathematics for Elementary Schools (USMES) published about 30 units for grades K-8, each built around such problems as furniture design, making a school crossing safer, improving the school playground, and so on [10]. The Undergraduate Mathematics and its Applications Project (UMAP) has many applications "modules", mainly for calculus but also for topics both more and less advanced. These are used mainly in college work, but about 80 U.S. high schools have also served as field-test sites for some of the modules. At least a dozen exploit merely school algebra or arithmetic [41]. With the end of NSF support, the project has been incorporated as The Consortium for Mathematics and its Applications. That organization will continue development of modules, including some for pre-calculus courses, and publication of The UMAP Journal.

Curriculum materials for introducing "systems dynamics" computer simulation models have been developed (with U.S. Office of Education funding) in a project directed by Nancy Roberts at Lesley College [31]. These are said to consist of:

> . . . six student learning packages which integrate text material, completely worked through examples, and open-ended exercises. The packages take students from initial problem conceptualization to building a mathematical model of the problem under study, and simulating the model over time with the aid of a computer. [31].

The materials have been pilot-tested with good results with college students and capable high school students. They are to be published as a college text (by Addison-Wesley); the authors hope they will also be used in high school, and not just in mathematics courses. A special computer language (DYNAMO) has been developed to help in the simulations.

Another fine set of applications modules is Statistics by Example [27]. These were produced without government or publisher funding by a joint committee of the American Statistical Association and the National Council of Teachers of Mathematics. Royalties from the sale of these volumes and from Statistics: A Guide to the Unknown [38] have been used by ASA and NCTM for continued support of this committee — a possible model for continuing innovative work in education as government funding disappears.

Still another hopeful indication that applications of mathematics may play a significant role in school mathematics is the success of Scholastic Math Magazine [34]. This is a 16-page publication issued 14 times each school year for about grades 6-8. Most of the articles and games deal with interesting everyday uses of arithmetic. In two years it has achieved a paid circulation of more than 430,000; a companion magazine for younger children — Dynamath — was inaugurated in September 1982.

Another indication that school mathematics may become more closely linked to its uses is that many teachers are using the raw materials discussed in the first part of this review to devise materials for use in their own classes. We have no firm data, but it is no longer rare, for example, to observe a teacher building a newspaper item with mathematics content into a classroom exercise. We see quite a number of examples of locally produced resource and enrichment materials based on sports statistics or uses of mathematics in local businesses. These tend to be unknown and hence unavailable away from the place where they are produced, but they do indicate to me a demand for applications-oriented materials. Since the products published (and kept in print) by U.S. educational publishers are determined very largely by teacher demand, these local teacher initiatives and the success of Scholastic Math Magazine may encourage publication of more books for general school use such as are discussed in the next category.

Applications in Everyday School Mathematics Instruction

In commenting elsewhere on the teaching of applications, I noted that we seem to have learned how to transform excellent raw materials into fine problem sets, but:

> Even given this new abundance of problem material, one of the clearest lessons we have learned . . . is that mere existence of [excellent] problems and units does not assure their use in schools. . . . Like the proverbial engineer who promised to learn the theory of relativity when it was printed in his engineering manual, many teachers say they will try to use applications in their teaching when they appear in textbooks and examinations. [3]

In the United States, mainstream textbooks remain completely inadequate in linking mathematics to its uses. On the other hand, there are little-used examples at each school level of books that do this well, and these will be discussed in this part of the review.

At the primary school level, there is a double bind that makes significant applications rare in school books. First, the only really important objective in U.S. arithmetic teaching is acquisition of fine calculation skills. Hence, the rich possibilities in non-calculation uses of numbers are ignored, as are highly applicable ideas about measure, probability, and geometry. Second, the U.S. primary school curriculum is very pessimistic about how fast children can learn calculation. Hence, applications involving any but trivial computations are ruled out until at least fourth grade, especially since use of calculators is pretty much forbidden in grades K-6. Generally speaking, the only primary school links to uses of mathematics are through stereotyped "word problems" of the "one-rule-under-your-nose" variety derided by Polya.

A few little-used "experimental" primary school textbooks depart from the situation just outlined. For example, Developing Mathematical Processes [32] includes much early work with applied measure and geometry. Another such innovative exception is CSMP Mathematics Program [30], which has nice materials on relations, graphs, and probability, and which has begun to exploit calculators. (Development of both these series was supported by the National Institute of Education.)

A relatively new entry into the "conventional" primary school textbook mainstream is the Real Math series [44]. These books include much richer content than is usual in U.S. books, including significant work from kindergarten on with measure, geometry, probability, estimation, and many non-trivial applications. These books also move calculation skills along significantly faster than the usual mainstream textbooks. Hence, they counter both the main trends mentioned above that inhibit applications in elementary schools. Their success in the marketplace is still uncertain.

During the 1960s, several experimental primary school science curricula were developed that included fine applications of mathematics, and not merely computation. (Examples include [10], [22], and [26].) However, any contribution of such books to the teaching of applications of mathematics has been made moot by the fact that science is rarely taught in grades K-6 in the United States.

The picture is scarcely less discouraging at present for what in the U.S. is called "middle school" (grades 7-8 or grades 6-8), and "general mathematics" courses in high school with similar content. During the 1970s, at least two major publishers marketed books with considerable stress on applications for those years; all those books are now out of print, and their replacements have little emphasis on applications [2, 18, 19]. The situation is less bleak than for primary schools, however, since science courses are often offered in those years. Also, the ban on calculator use begins to weaken by eighth grade and applications-oriented supplementary materials are common. Furthermore, it seems probable that textbooks with many applications will reappear for those years. Optimism is hazardous, but it is perhaps more justified here than elsewhere.

For mainstream U.S. "college preparatory" high school courses, the picture is somewhat mixed. It is clear that books used by the overwhelming majority of students have few links to applications beyond stereotyped one-rule-under-your-nose "word problems". Generally speaking, they use such simplified and phony data that students comes to feel that answers other than whole numbers or simple fractions are probably wrong! One exception is an NSF-funded experimental text, <u>Algebra Through Applications with Probability and Statistics</u> [43], which unfortunately has relatively few users. In the published mainstream for first-year algebra, a few books that do not stress applications do nevertheless include more than the usual number of interesting applied problems (for example, [7] and [15]). For the U.S. college preparatory plane geometry course, I know of no published text that substantially links geometry to its applications; Jacobs [16] has more applied problems than most. For our advanced algebra and elementary functions courses, <u>Algebra and Trigonometry: Functions and Applications</u> [11] is a fine, if lonely, example of standard content effectively linked to applications. Its publication by Addison-Wesley's Innovative Publishing Division (which has published a number of applications-oriented materials) indicates its status as "non-standard", even though its mathematical content is quite usual for U.S. second-year algebra courses.

Statistics courses are relatively rare in U.S. secondary schools, and when offered tend to be fourth-year options for the most capable students. These are also the courses best served by applications-oriented textbooks, which also serve the college market. <u>Statistics in the Real World</u> [24] is one among several textbooks heavily motivated by examples of actual uses of statistics.

Many of us believe that widespread offering of one or more non-technical, data-oriented, statistics courses as a secondary school option would serve many students well, including many college preparatory students who otherwise drop mathematics as soon as they have fulfilled minimum requirements for graduation or for college admission. The NCTM and ASA joint committee (JCCSP) referred to above has prepared a textbook for such a course. Its fate so far illustrates a sort of double bind that I would suppose is not confined to the United States: publishers say that in the absence of a widely offered course to provide a market, they can't risk publication, while schools say they need a textbook in order to offer the course. So far, what appear to be excellent materials thus remain unpublished.

With tens of millions of students, U.S. schools would seem to offer lucrative markets for quite a variety of books. But the courses actually offered in schools tend to be much the same everywhere. As in other countries, "standardized" achievement testing and college entrance examinations support this uniformity and, in particular, applications of mathematics play essentially no role in those examinations. Hence, the publisher caution just noted seems justified more often than not. For example, The Man Made World [8] is a really quite wonderful blend of pre-calculus mathematics, science, and technology that would support a most useful course. But with no prescribed slot in our standard high school mathematics curriculum or our standard science curriculum, the course is seldom offered.

On the other hand, Mathematics: A Human Endeavor [17] fit no standard school course when it was published, yet it has achieved widespread school use. This textbook is a beautifully motivated potpouri of "useful" mathematics including non-technical work from logic, sequences, functions, geometry, probability, statistics, and topology. Its success may hinge in part on two factors: first, even on first publication it had a home in college mathematics courses for the mathematically unwashed. Second, it was a plausible option in a definite high school department — mathematics — for a group of students that would probably not otherwise continue taking mathematics. (Since that would also seem to be true for the data-oriented statistics course just discussed, the difficulty in getting that option offered is puzzling.)

It will be interesting to see what happens with Sci-Math [13], still another nice outcome of NSF-supported development. The chemistry professor author tries to bridge the gap between mathematics as generally taught and mathematics as used in school science courses. That results in a text for a semester course (or perhaps brief units in other courses) that provides thorough applications-motivated treatments of measure, rates, ratios, direct and inverse proportions, graphing, and "units analysis". It fits nowhere in particular in our standard curriculum, so we should be pessimistic about its success, yet this material would be very useful to most middle school or high school students, including those in "college preparatory" courses. This book does at least have a publisher, so it will have a shot at the market, unlike some other promising materials.

In principle, calculators should finesse many of the barriers that formerly inhibited the inclusion of genuine applications in school textbooks.

In practice, calculators are effectively barred from elementary school use, are at best tolerated in middle schools for those who have proved they can compute without them, and are assumed to be available only in college preparatory courses. Innovative materials that emphasize applications usually assume that calculators will be used, but most newly published books have the same old word problems with the same cooked and phony data that offer little scope for use of calculators.

It is still too early to assess what effects the current school enthusiasm for computers and microcomputers will have on applications in school courses. So far few such effects are apparent.

In sum, it seems fair to say that the abundance of applications raw materials and of problems based on those have had little impact on textbooks used in schools and no impact on examinations that set the expectations of school work. So far, the textbooks that buck that trend have, in general, not succeeded in the school marketplace. Government funding that has kept innovation in applications-oriented materials flowing despite the market has essentially disappeared. There is much talk in the U.S. about making "problem solving" a major emphasis of school work [28], but that phrase means many things to many people and seldom suggests more attention to applications as such. Applications as a significant component of widely used textbooks and examinations in school mathematics will very likely remain the exception rather than the rule for some time to come.

Teacher Preparation for Teaching Applications

In the late 1950s, the mathematician Saunders MacLane began an invited talk on "Modern Mathematics in the School Curriculum" with "My subject is vacuous...". In a similar vein, this section can be quite brief. Many of the materials already discussed would help teachers teach themselves about applications and could also support courses for preservice or continuing education of teachers. But textbooks for teacher training at all levels generally give no attention to genuine uses of mathematics. Partial but inadequate exceptions are Bell et al. [5] and O'Daffer and Clemens [29]. The teaching of "word problems" gets some attention in teacher training, but these word problems seldom reflect genuine applications. There are a few universities that regularly offer in-service courses on applications, but I would estimate that at most 150 U.S. teachers per year might take such courses — of 1.5 million or so teaching mathematics or arithmetic at the school level. Even if that is too pessimistic and the figure were something like 1500 (I know of no reliable data on this point), that would still be less than one teacher per thousand.

Professional journals such as <u>School Science and Mathematics</u>, the <u>Arithmetic Teacher</u>, and the <u>Mathematics Teacher</u> regularly publish articles offering specific applications and advice about teaching applications, and the <u>Mathematics Teacher</u> now has a monthly department featuring applications. The National Council of Teachers of Mathematics offers yearbooks and other publications concerned with applications [23, 35, 36]. That is, there are ways for teachers to help themselves learn more about applications, but few organized courses with that purpose.

It is often said that teachers tend to teach as they were taught and reform rhetoric often invokes changes in teacher training as one prerequisite for change. If so, it is unlikely that very much will be done with the teaching of applications in U.S. schools except as teachers may be persuaded to teach themselves. There will, of course, be some teachers and teachers of teachers who will teach applications wonderfully, and we should seek out and cherish such examples.

Research and Development on the Teaching of Applications

In addition to supporting invention of applications-oriented curriculum materials, the now-defunct Science Education Development and Research division of the National Science Foundation (SEDR/NSF) has supported a number of more speculative research or development projects aimed at improved teaching of applications. I will outline here three examples that take quite different approaches: inquiry into applied problem solving processes; conceptualization of a certain applications domain; and use of microcomputers to ease the teaching of applications. Descriptions of other such projects are in annual listings of NSF-supported projects.

(1) Applied Problem Solving in Middle School Mathematics (R. Lesh, Northwestern University)

This project seeks to investigate the processes used and skills needed as middle school students of average ability attempt to use elementary mathematics concepts to solve problems about realistic everyday situations. Much of the data comes from observations of a certain group of seventh graders. These students worked in small groups in 35-minute sessions twice a week on such problems as planning a vacation under certain constraints, buying wallpaper for a room, estimating the effects of inflation, etc. The youngsters could talk freely among themselves, consult various resources, use calculators, and in general act as people might in a non-school applied problem-solving situation. Project people observed each group and a complete transcript of each session was made from audio or video tapes. The data from that and a variety of other exercises is now being assessed. One tentative conclusion is that youngsters in such group problem-solving situations seem to "naturally" follow a mathematical modelling process such as has been described by professional users of mathematics. It was not uncommon for a group to run through six or more identifiable cycles of such a modelling process in a half-hour session. The mathematical models were sometimes a series of successive approximations and sometimes independent solutions. Another tentative conclusion is that this modelling and solution process is easily deflected by lack of what might be called "critical elements" for the problem at hand; for example, poor intuition about rates, or a shaky grip on decimals, or lack of skill with calculators. A series of reports and articles will be the main output from the project; one such is [25].

(2) Arithmetic and its Applications Project (Z. Usiskin and M. Bell, The University of Chicago)

This project aims to conceptualize the applications of arithmetic and provide an organizing taxonomy for them. The directors believe that such organization may enable arithmetic applications to compete successfully for a place in school instruction with the all-too-well-organized calculation curriculum.

One outcome of the project is the Handbook of Applications of Arithmetic [42], organized around three of the phases often identified with mathematical modelling: getting data; various transformations of data (what some statisticians call "fondling data"); and deciding what arithmetic operations to use. For each of these, it is argued that billions of individual uses can be sorted into a few categories. For example, there are billions of uses of numbers, but the Handbook suggests that there are just six "use classes" for those numbers: counts; measures; locations in reference frames; scalars (usually ratios); codes; and nominal, essentially meaningless, uses such as "lucky numbers". Similarly, users of arithmetic do many things to make numerical data more informative, but most of those things fall into five categories: renaming (e.g., 40% for 2/5); rounding and estimation; transformation of a data set (e.g., from raw scores to standard scores); and displaying data sets (e.g., tables and graphs). In the same general vein, there are many occasions on which one might push the addition button on a calculator, but most of those can be seen as falling into just three use-classes, with similar organization proposed for the other operations of arithmetic.

The Handbook is written at several levels to accommodate teachers, curriculum developers, and researchers. In addition, some student materials based on the handbook use-classes have been produced. The materials are completed, but not yet published; articles about the project are available from the directors.

(3) The Uses of Microelectronics Technology in Teaching Applications of Mathematics (J. Fey, The University of Maryland)

Professor Fey believes that people can understand and exploit certain applications of mathematics without necessarily having a good grip on the particular (and possibly tedious) calculations that go with them. Furthermore, he believes that doing applications greatly helps one understand the mathematics being applied. The idea is not dissimilar in principle to the notion that with calculators children might be able to handle, for example, division and its applications well before mastering the long division algorithm. But Fey believes that computers fail as tools analagous to calculators because of the complexities of programming them for specific applications. This project seeks to finesse that by providing and testing pre-programmed microcomputer diskettes as "tool kits" for various applications of mathematics. So far the idea has been tested with non-trivial linear programming, curve fitting, and statistical applications in a pre-calculus beginning college course for mathematically underprepared students. More information is available from the project director.

As noted above, these examples are representative of a number of "scholarly" projects exploring various aspects of using applications of mathematics in teaching. There are also in the U.S. many investigations of "problem solving", "information processing", and so on that are not much concerned with applications as such but that may provide interesting hypotheses about why some people can and others cannot use whatever mathematics they "know".

Summary

To close as we began, it seems fair to report that pretty much everything needed for excellent progress in using applications in mathematics instruction exists in the U.S., yet such emphases in teaching touch very few students. We have fine expository articles and books about applying mathematics and about particular applications. We have rich and readily available sources of interesting and realistic data about a great many things. Within the past decade, an abundance of fine problem sets and projects that honestly reflect applications have been published. Calculators are cheap and ubiquitous, so use of realistic data and fruitful trial-and-error methods are much easier than ever before. For many school levels and courses, exemplary textbooks exploiting applications exist. A variety of research and development projects inquire directly and indirectly into "the teaching of mathematics so as to be useful" (a fine phrase from Hans Freudenthal). Professional societies at all levels declare the importance of teaching applications of mathematics (and statistics) and provide for publication of materials consistent with those declarations.

For the most part, however, the fine raw materials, the fine problem sets, and the exemplary textbooks are not used in schools in the U.S., and applications are not included in the widespread annual rituals of achievement tests and college entrance examinations, except possibly for trivial "word problems". Teachers are poorly prepared to teach applications and there is little prospect that existing programs for preservice or continuing education of teachers will repair that for more than a few hundred of the million or more people teaching mathematics at some school level.

Remedies to bridge the gap between what essentially everyone agrees should be done and the facts of school instruction are fairly obvious: putting more and better applications in most school textbooks; routinely including applications in widely used examinations; perhaps some mass communication initiatives such as television courses for teachers (and parents?); and perhaps some special institutes for teachers of teachers. But optimism about such remedies being attempted would be foolish. The present neglect of teaching of applications in school mathematics in the United States seems very likely to persist.

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