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ABSTRACT
The Concepts in Secondary Mathematics and Science (CSMS) and Strategies and Errors in Secondary Mathematics (SESM) research projects based at Chelsa College, England, have shown the marked reluctance of secondary school students to use fractions when solving mathematical problems, even though they have been taught the topic for a number of years. One of the topics CSMS investigated was ratio and proportion; students appeared to have used the incorrect addition strategy identified by other researchers. SESM probed more deeply into such errors, and the meanings children give to a/b. For the first stage of the project, 23 children were interviewed individually about their reactions to tasks involving fractions. Then teaching modules were developed and evaluated with 59 children aged 13 and 14. Most children avoided using fractions on the pretest: they refused to acknowledge the existence of fractions. After instruction, about one-third of the children still avoided fractions in responding to test questions. Results are given for a variety of items, with the conclusion that the difficulties appeared to be fundamental. (MNS)

AVOIDANCE OF FRACTIONS

Kathleen Hart
Daphne Kerslake

The Concepts in Secondary Mathematics and Science (CSMS) and Strategies and Errors in Secondary Mathematics (SESM) research projects based at Chelsea College have shown the marked reluctance of secondary school children to use fractions when solving mathematical problems even though they have been taught the topic for a number of years. SESM has investigated the incorrect addition strategy in Ratio and Proportion and the meanings children give to $^a/_b$. D. Kerslake reports on interviews with children and test results which reveal their view of fractions to be limited.

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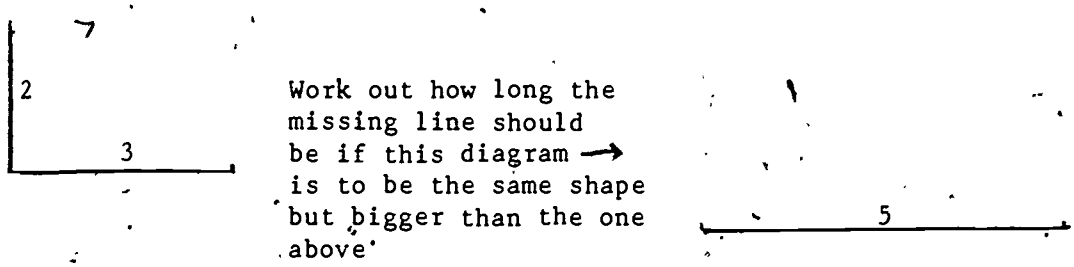
The research carried out at Chelsea College over the last eight years, Concepts in Secondary Mathematics and Science (Hart, 1980, 1981) and Strategies and Errors in Secondary Mathematics (Hart, 1983; Booth, 1983) has been concerned with secondary school children's understanding of various mathematical topics which appear in the British school curriculum. In the first project hierarchies of understanding in eleven topics were formed on the basis of data from pencil and paper tests (n = 10,000) and interviews (n = 300). The problems in each topic which were difficult were either more complex (in that they required several steps for completion) or they entailed the manipulation of fractions or decimals, or both. Although the recognition of a fraction and the naming of a shaded region appeared to cause little difficulty to the sample (aged 12-16), the realisation of the need to use non-integers and then the successful application of these was accomplished by less than half of those tested. Fractions and the four operations upon them are taught in British schools to children from about the age of nine. There is no lack of teaching of the topic but there is a marked degree of avoidance of the use of the taught algorithms.

One of the topics CSMS investigated was Ratio and Proportion and although all the problems could be regarded as requiring a knowledge of rational numbers, it was possible to distinguish between those that could be solved by a judicious use of whole numbers and halves and those that needed to utilise a fractional scale factor. A third of the sample which attempted the Ratio and Proportion problems (n = 2257) gave answers to four difficult questions, which suggested they had used the incorrect addition strategy identified by Piaget and Inhelder (1968) and Karplus et al (1975). This strategy is essentially that of forming an enlargement of a figure by adding a difference rather than multiplying by a scale factor as shown in Figure 1.



Addition answer is 4cm for upright line

Figure 1 Enlargement item

The SESM research has probed more deeply into errors of this sort, which could be identified in the CSMS data. Children using the incorrect addition strategy on difficult questions (the 'adders') were found to be using addition (correctly) on the easier items, thus replacing multiplication by repeated addition or a process of halving. For example, given that in a recipe four onions were needed for eight people, they would reason that for four people one would halve and for six people one would need to halve again and then put the answers together. They regarded the topic of ratio and proportion as involving a series of additions, although they had certainly been taught methods of solution which relied heavily on multiplication by a fraction. One reason for this preference for addition was the stated desire to avoid fractions as illustrated here by Vernon:-

(Interviewer: I; Vernon: V)

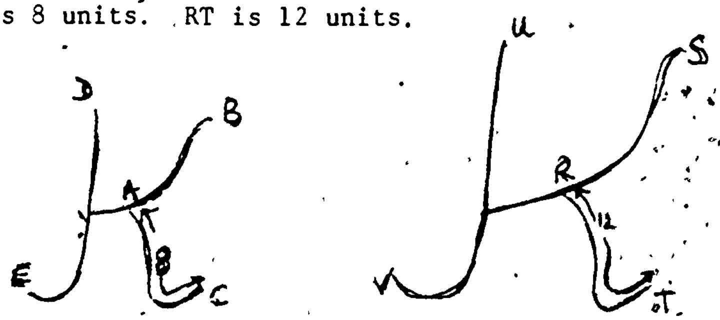
V: That's 18. That's 3. I don't know how to explain it. That's 2/3 and that's just a full one so that's 1/3 short so you divide 18 by ... take away 1/3 of 18 which is 6.
I: Could you have used that method on some of the other questions?
V: ..... No.
I: Why not? What's the difference between the questions?
V: Because you go into too much fractions.

Part of the SESM research was the formulation of a teaching module designed to eradicate the use of the incorrect addition strategy in enlargement problems by providing the specific mathematics missing from the adders' attempts to solve the problems. The first module, tried with small groups of adders, contained a section on multiplication of fractions in which the rule was introduced anew. The children were presented with enlargement problems which required them firstly to deal with integers multiplied by scale factors of 1½ and 2½ (which they could do) and then fractional dimensions were introduced. This led the group to ask for a method for multiplying fractions and this they were given. However, although immediate feedback suggested that they were able to compute the multiplications adequately, on the delayed post-test a number of children failed to solve the problems because their overall knowledge of fractions was weak. Thus, although the immediate difficulty of multiplying fractions may have been addressed, other misconceptions about the nature of fractions were still present. The subsequent versions of the module therefore contained a calculator replacement for the section involving multiplication of non-integers. This assumes that a distinction can be made between the understanding of what is needed to solve ratio and proportion problems and the necessary expertise for dealing with fractions.

The SESM sample chosen for the investigation of the addition strategy was taken from classes of average pupils. To provide a contrast and more information, 20 children (aged 12-16) who were considered very able by their teachers, were interviewed in order to ascertain the methods they used to solve 'hard' ratio problems, such as that shown in Figure 2.

These 2 letters are the same shape, one is larger than the other.

AC is 8 units. RT is 12 units.



The curve AB is 9 units. How long is the curve RS?

The curve UV is 18 units. How long is the curve DE?

Figure 2 Enlargement of letters

Noticeably these children, although considerably more flexible in their understanding and use of numbers, did not write down and compute fraction multiplications ($\frac{a}{b} \times \frac{c}{d}$). As illustrated by these replies to the problem shown in Figure 2, we can see their ingenuity.

Sarah: AC, which is the smaller one is 8. This curve here, and RT is 12 and AC is .... RT is half of AC plus AC. So half of AB is 4½ plus 9 is 13½.

Madeline: 8 to 12. 8 is two thirds of 12, so AB will be 2/3 of RS. So you find half of this which is 4½ and add it onto 9 which makes 13½.

Rose: The difference between that is .... that RT = AC + ½AC so if AB is 9, then RS is 9 plus half AB, which is 4½. Added together to make 13½.

The difference between the replies of those who add (incorrectly) and these mathematically able girls appears partly to be the interchangeability of the meanings given to $^a/_b$. So although they do not use a taught algorithm they are able to deal with a fraction as a number as well as its representation as a name for a part.

Another study carried out with the SESM framework has been on 'Fractions'. Some of the findings are described below by the researcher.

Booth, L.R. Strategies and Errors in Secondary Mathematics: Algebra. (Available 1983)

Hart, K. Secondary School Children's Understanding of Mathematics. (Ed. D. Johnson) London: Chelsea College, Centre for Science and Mathematics Education Research Monographs, 1980

Hart, K. Children's Understanding of Mathematics: 11-16. London: Murray, 1981

Hart, K. Strategies and Errors in Secondary Mathematics: Ratio. (Available 1983)

Karplus, R., Karplus, E., Formisano, M., & Paulsen, A.C. Proportional Reasoning and Control of Variable in Seven Countries. Advancing education through science programs, Report ID-65, 1975.

Piaget, J., & Inhelder, B. The Child's Conception of Space. Routledge & Kegan Paul, London 1967.

Avoidance of Fractions

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The CSMS results of the Fractions tests (Hart, 1981) suggested that there are many children of 12 to 14 years of age of average ability in Mathematics who are reluctant to acknowledge the existence of Fractions and avoid them where possible.

Over 1,000 children in this age-group were tested, one item in the test is shown in Table 1.

Table 1: CSMS Item and Results

A piece of ribbon 17cm long has to be cut into 4 equal pieces. Tick the answer you think is most accurate for the length of each piece:

a) 4cm remainder 1 piece
b) 4cm remainder 1cm
c) $4\frac{1}{4}$cm
d) 4/17cm

Age	$4\frac{1}{4}$cm	4cm rem 1cm	4cm rem 1 piece	4/17cm
12 yrs	43.0	37.4	8.5	9.3 per cent
13 yrs	54.4	29.8	6.8	6.5
14 yrs	61.4	26.6	4.2	5.8
15 yrs	61.4	27.4	3.7	6.0

It can be seen that nearly one-half of the 11 year olds and a third of the 14 year olds chose answers that used remainders rather than fractions.

Similarly, when asked to evaluate $3 \div 5$, 18.3 per cent of the 11 year olds and 17.5 per cent of the 12 year olds gave '1 remainder 2'. This is a particularly interesting response, since those children appear to have avoided fractions by first reversing the order of the division, and then using a remainder instead of a fraction. This avoidance of fractions, or failure to accept that fractions are numbers, is part of a more general phenomenon that has been described by, for example, Hasemann (1981), who, after using the CSMS tests in Osnabrück, said: " .... our investigation shows clearly that the main problem of the poorer scholars is precisely the lack of an idea of fractions." Kieran (1976) observed that rational numbers present difficulties for children because they are having to deal with 'structures which do not have an obvious basis in natural thought.' Behr, Post and Wachsmuth, in their paper presented at the 1982 AERA meeting said " ... with fractions children explore a new realm of the numbers which in various ways remind them of the whole numbers already familiar to them. Thus they may evolve strategies used in connection with whole numbers, being triggered by the whole number symbols, and apply them to fractional symbols."

One of the aspects of fractions studied as part of the SESM project was children's reluctance to acknowledge that fractions are numbers, and their strategy of avoiding them altogether where possible. For the first stage of the project, 23 children were interviewed individually about their reactions to tasks involving fractions. In one task, the children were shown a number of embodiments of the fraction 'three-quarters'. Some of the examples were of the geometric 'part of a whole' type, but one was $3 \div 4$. Only one child saw any connection between 3/4 and $3 \div 4$. Thirteen of them proceeded to share 4 by 3, although some read it the right way round. For example:

KP '3 shared by 4 .... that's 3's into 4 .... 1 remainder 1'
CD '3 divided by 4 .... 1 remainder 1. 3 goes into 4 once, and there's 1 left over'
GB 'I don't know ... 3 goes into 4, 1 left over.'

These replies were characterised by pauses between the initial statement as presented in the question and the restatement in the reverse order. Others reversed the order immediately:

VC '3's into 4. 3's go into 4 one, and 1 remainder.'
GP '3's into 4. 1 remainder 1'

Three of the children who read the division the correct way round said that it was impossible to do:

TH '3 shared by 4. You can't do that. 4 is bigger than 3.'
SB '3 shared by 4. You couldn't do it. Well, it wouldn't ...'

The children who had seen no link between 3/4 and $3 \div 4$ were then given three cakes to share between 4 children. All but four of them then acknowledged the connection.

The SESM project set out, not just to identify problems and errors, but also to produce some appropriate teaching activities. These teaching modules were evaluated by means of a pre-test, immediate post-test and a delayed post-test, taken some six weeks later. The teaching module on fractions was addressed to three specific difficulties that children had been seen to experience during the interviews. The first was their reluctance to think of the fraction a/b as $a \div b$. The second problem concerned equivalence of fractions. Although most of the children interviewed were able to recognise or construct simple pairs of equivalent fractions, none was able to explain the process used or to use the idea in the solution of simple problems. The third difficulty that the teaching module was concerned with was the notion of fractions as numbers. For most of the children interviewed, fractions were seen just as 'part of a whole' and not as an extension to the number system.

The results for the pre-test on Fractions are discussed first. Fifty-nine children, aged 13 and 14 completed the three tests and the teaching module. The schools which they attended grouped their pupils by ability in mathematics, and these children were in middle-ability groups. One item on the test was:

$$12 \div 4 = ? \; ; \quad 4 \div 12 = ?$$

These questions were given as a pair, as the interviews had suggested that many children reversed the order of the division when a small number was to be divided by a larger. By having both, it was thought that the children would be forced into considering a fractional answer. However, many of the children gave the same answer, 3, for both, and so avoided the use of fractions. Others gave the answer '3' for one and '0' for the other. The choice of '0' suggests another stategy for avoiding fractions, and possibly stems from the use of the phrase, often heard in primary schools in England, 'twelves into four won't go' or 'twelves into four you can't'.

It seems likely that the phrases 'won't go' or 'you can't' get translated into 'zero'. The numbers of children giving each response at the pre-test is shown in Table 2:

Table 2: Response to 12 ÷ 4 and 4 ÷ 12 at pre-test. (n=59)

Both answers correct	8
Same answer (3) for both	25
One answer '0', the other '3'	15

Thus, forty of the fifty-nine children avoided using fractions. There were similar results for the pair of questions 3 ÷ 4 and 4 ÷ 3, the difference being this time, of course, that both give rise to a fraction. The responses '1 remainder 1' and '0' occurred again, and are shown in Table 3.

Table 3: Responses to 3 ÷ 4 and 4 ÷ 3

3 ÷ 4		4 ÷ 3	
Correct	12	Correct	17
1 rem. 1	8	1 rem. 1	7
Zero	6	Zero	4
1 1/3	10	3/4	1
No response	9	No response	12
One	4	One	2

Ten children appear to have reversed 3 ÷ 4 to give an answer 1 1/3, but of these, five gave the answer 1 1/3 to 4 ÷ 3 as well. It seems that a division which gives an answer of more than one is more acceptable than one with an answer of less than one. Only five of the children gave the same answer to both this time, there being no advantage in reversing the order. These five responses included two 'twelves' and two 'ones'. There was, though, an increased number of children who made no response to the question. If this number is added to those who chose '1 remainder 1', '0' or '1', then, of the 59 children, the number who avoided fractions was 27 for 3 ÷ 4 and 25 for 4 ÷ 3.

Another item on the pre-test was as shown in Table 4.

These four items were preceded by two using integers only, the results for which are not discussed here. For the items that demand fractions, the number of children who indicated that there were no possible solutions was again quite large.

Table 4: The 'Missing Number' Item and Results

Put the missing numbers in the boxes. If there is no number, write 'no' in the box

c) 2 x □ = 7, d) 4 x □ = 10, e) 2 x □ = 1, f) 8 x □ = 5

Item	Number with correct answer	Number saying 'none'
c)	28	28
d)	16	32
e)	24	27
f)	0	42

The results in the second column suggest that there are many children who say, for example, that there is no number that 2 can be multiplied by in order to give the answer 1, and so they avoid fractions again.

The phenomenon was also observed in a question on equivalence shown in Table 5.

Table 5: Equivalence Items and Results (n=59)

Put the missing numbers in the boxes. If there is no number, number, write 'no' in the box

a) 3/4 = □/12 b) 5/3 = 15/□ c) 9/12 = 12/□ d) 14/12 = □/24

	Response		
Item	Correct	'None'	No response
a)	40	0	3
b)	35	5	6
c)	4	17	6
d)	6	13	8

Where the 'missing number' was an integer, there were few children who thought there was no possible solution. But, when the multiplying factor is a fraction, there were many more who said that no such number existed.

The last question to be discussed concerns the plotting of fractions on a number-line. The question is shown in Table 6.

Table 6: The Number-Line Item and Results (n=59)

Mark on the line, and label, three numbers between 1 and 2. If you think there are none, write 'none'

1 ________________________ 2

Correct	'None'	No response
36	10	9

There was an appreciable number of children again who refused to acknowledge the existence of fractions. The next part of the question was 'How many numbers do you think there are between 1 and 2?' This time, eight children thought there were none, while 18 made no response at all. The most popular answer, given by 12 children, was '3': this, no doubt, refers to the three numbers they had been asked to plot. Four children said there was one number, presumably the half-way point.

The evidence from these and from some other questions suggests that there is indeed a number of pupils of secondary school age in middle-ability groups who have little confidence in the existence of fractions and who will avoid them wherever possible.

The fifty-nine children then took part in the teaching experiment. A group of six children were removed from each of the three classes to work with the researcher, while the rest of the class was taught in the same way by a student teacher.

These three student teachers were on teaching practice in the schools and normally taught the classes concerned. The teaching material consisted of a set of worksheets of a practical nature at which the children were encouraged to work in pairs or small groups. In this way, they were able to discuss their results both with each other and with the teacher. The worksheets were not intended as an introduction to fractions as the children had met them before, and were concerned with the three aspects of fractions to which reference has already been made, namely: 1) that the fraction a/b can be thought of as a÷b, 2) the meaning of equivalence, and 3) that fractions are numbers. It was felt important that a number of models of each aspect should be used, and that the worksheets should encourage the children to notice patterns and make some generalisations. Use was made of calculators, and games were introduced in order to give the children a variety of approach.

Some of the teaching activities are described very briefly now.

1. Sharing activities using discrete objects such as cakes, so that fractional amounts result.
2. Sharing activities using continuous quantities such as milk into jugs so that fractional amounts result.
3. Use of a calculator and some known decimal equivalents to show that $1/2 = 1 \div 2$, $3/4 = 3 \div 4$ etc.
4. Use of a number-line to plot fractions, order fractions and observe equivalences.
5. Use of a calculator to illustrate equivalent fractions.
6. Games involving multiplication and division that give rise to numbers less than one, using a calculator.
7. Multiplying by numbers greater and less than one.

The effectiveness of the teaching module was then judged by comparison of the children's performance at two post-tests with that at the pre-test. The first post-test was taken immediately after completion of the teaching module and consisted of items very similar to those of the pre-test. The delayed post-test was taken some six weeks later. The pre-test was used again this time, because of the problem of finding items of comparable difficulty. Those results that concern children's avoidance of fractions are now discussed. The first concerned the division items and the number of children giving the various responses is shown in Table 7.

Table 7: Results for Division Items Before and After Teaching

Question	Response	Pre-test	Post-test	Delayed Post-test
$12 \div 4$, $4 \div 12$	Both correct	8	24	10
	Same answer for both	25	21	27
	One answer '0'	15	2	9
$3 \div 4$	Correct	12	37	22
	1 rem. 1	8	1	4
	Zero	6	3	5
	No response	9	6	11
$4 \div 3$	Correct	17	29	17
	1 rem. 1	7	1	5
	Zero	4	2	3
	No response	12	4	7

The teaching module had success in obtaining more correct responses to the division items $12 \div 4$ and $4 \div 12$, but about a third of the children still said the two were the same, and so avoided fractions. The number giving '0' for one of the divisions was reduced. The success rate for the divisions $3 \div 4$ and $4 \div 3$ was also increased. For the division $3 \div 4$, fractions were avoided by the choice of the answers '1 remainder 1', '0', '1' or by failure to give any response. The number of children in this category went from 27 at the pre-test, down to 11 at the immediate post-test but up to 22 at the delayed post-test. For $4 \div 3$, the results were similar, with 25 avoiding fractions at the pre-test, 9 at the immediate post-test and up to 19 at the delayed post-test. This suggests that the children could be persuaded to think of fractions in the short-term, but that several regressed to their avoidance strategy after a few weeks.

The next item to be discussed is the 'missing number' item, in which fractional multiplying factors were required. The children were asked to indicate if they thought that no such number existed. These results are shown in Table 8.

Table 8: Results for 'Missing Number' Items Before and After Teaching

Question	Response	Pre-test	Post-test	Delayed Post-test
2 x □ = 7	Correct	28	35	36
	'None'	29	23	21
4 x □ = 10	Correct	16	22	25
	'None'	32	33	26
2 x □ = 1	Correct	24	43	40
	'None'	27	6	12
8 x □ = 5	Correct	0	1	0
	'None'	42	22	27

It can be seen that there was increased success at the first three items, particularly in the case of the third. The fourth item 8 x □ = 5, was seen to be much too difficult for these children. The number of children who rejected the existence of fractions by saying that there was no solution was reduced in each case.

The third question concerned equivalence and the children were again offered the option of saying that there is no number that could be placed in the box. These results are shown in Table 9.

Table 9: Results for Equivalence Items Before and After Teaching

Question	Response	Pre-test	Post-test	Delayed Post-test
3/4 = □/12	Correct	40	49	48
	'None'	0	0	0
	No response	3	3	5
5/3 = 15/□	Correct	35	41	45
	'None'	5	1	3
	No response	6	3	4
9/12 = 12/□	Correct	4	4	4
	'None'	17	25	29
	No response	6	7	5
14/16 = □/24	Correct	6	3	3
	'None'	13	25	14
	No response	8	7	5

The results for the equivalence items were less encouraging. The first two were relatively easy, and the success rate increased and there were very few responses of 'none'. The second two items, in which the multiplying factor is fractional unless the children simplified the first fraction, were seen to be very difficult, and the teaching module had no positive effect. The number of children who thought there was no answer actually increased.

The last question asked for three numbers to be marked between 1 and 2 on a number-line, but again offered the children the opportunity of saying that there were none. They were also asked how many numbers there are between 1 and 2. The results are shown in Table 10.

Table 10: Results for Number-Line Item Before and After Teaching

Question	Response	Pre-test	Post-test	Delayed Post-test
Mark in three numbers between 1 and 2	Correct	36	50	50
	'None'	10	3	7
	No response	9	5	3
How many numbers between 1 and 2?	'None'	8	5	6
	No response	18	10	9

The ability to mark in the three points was increased, but there remained a residual group of children who made no response, or who said that there were no numbers.

The reluctance of children to accept the existence of fractions and to avoid them where possible suggests that the difficulties often observed with fractions are fundamental. The algorithms for equivalence, addition and multiplication can have little meaning for a child who does not think that 1/2 is a number, that $12 \div 4$ is the same as $4 \div 12$, that there is no number that satisfies $9/12 = 12/?$ and that there are no numbers between 1 and 2 on a number-line. It seems that teaching strategies that offer practical instances of a wider variety of embodiments of a fraction than the ubiquitous 'part of a whole' might have some success in the development of the idea of a fraction.

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