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ABSTRACT

This document was initiated with the collection of 150 questions in telephone interviews of a national sampling of junior high school and high school teachers. A Mathematics Consultant Panel reviewed the questions, and selected topics were developed, for which there were research bases for answers. The material covers: (1) Effective Mathematics Teaching; (2) Individual Differences Among Mathematics Learners; (3) Communicating Mathematics; (4) Breaking Vicious Cycles: Remediation in Secondary School Mathematics; (5) Problem Solving: The Life Force of Mathematics Instruction; (6) Estimation: A Prerequisite for Success in Secondary School Mathematics; (7) The Calculator: An Essential Teaching Aid; (8) Understanding Fractions: A Prerequisite for Success in Secondary School Mathematics; (9) The Learning and Teaching of Algebra; (10) The Learning and Teaching of Geometry; and (11) The Path to Formal Proof. The document includes a 13-page bibliography. The document is set up so that each chapter may be read independently. Each begins with a question from a teacher, and the answer is constructed so that research information and classroom implications are clear. It is felt the material is of obvious use for individual study, and will prove useful for both pre- and in-service courses. It is hoped that teachers will take seriously the numerous invitations to replicate or validate the research cited. (MP)

Research Within Reach: Secondary School Mathematics

A Research-Guided Response to the Concerns of Educators

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Secondary School Mathematics

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FOREWORD

Educators and the public generally are increasingly vocal about a phenomenon which, they claim, is becoming a national scandal. secondary school students are less knowledgeable about and less interested in higher mathematics Some people have suggested we need to experience a national shock, on the order of the 1957 shock of the Soviet launch of Sputnik, to remind us of the value to our nation of mathematics and science education.

At the same time, attrition of the mathematics teaching corps is causing concern. The concern is magnified by the realization that fewer university students plan to become mathematics teachers. In order to reverse these trends, several plans have been tried, ranging from paying mathematics teachers higher salaries to retraining other teachers to equip them to teach secondary school mathematics

This gloomy picture becomes bleaker still when we realize that, according to the National Assessment of Educational Progress, while mathematics is the favorite subject of 9-year-olds surveyed, it is the least-preferred subject among 17-year-olds. What can be done to alter these conditions?

Obviously, no single plan will change the direction of mathematics teaching and learning. However, several initiatives sponsored by the federal government are being undertaken to mitigate the current situtation. The National Institute of Education has supported and continues to support research projects which have studied effective strategies for teaching and learning in mathematics Although much remains to be done in the area of research, the crisis in secondary school mathematics makes it essential that we focus carefully on dissemination of existing research information.

What Is RDIS?

The Research and Development Interpretation Service (RDIS), a project funded by the National Institute of Education, attempts to bring to teachers the research which they need. By focusing our efforts on answering questions which teachers pose, we hope to help teachers meet their most pressing needs. Of course, research information requires interpretation before it can be most effectively used This interpretation is required for several reasons:

- Researchers are not always clear about the implications of their research for the classroom.
- For a particular topic it is often difficult to get a comprehensive view of all the research that bears on that topic.

To date, RDIS, through its *Research Within Reach* series, has brought research to teachers in several basic skills areas elementary school reading, elementary school mathematics, oral and written communication, and, now, in secondary school mathematics.

How Does RDIS Work?

The fine details of development have varied for the four RDIS interpretive works, but the overall process has remained true to the Project's goal to make it possible for teachers and researchers to listen to and understand each other

The process used to develop *Research Within Reach: Secondary School Mathematics* illustrates how RDIS works. First, around 150 questions were collected in telephone interviews of a national sampling of junior high and high school teachers. Second, the RDIS staff met with the Mathematics Consultant Panel to review the questions. The panel members--Dr. Mary Grace Kantowski, Dr. Robert Reys, and Dr. Marilyn Suydam--identified the topics which were represented by the questions and for which there exists a research base for answers Next, literature searches were conducted and first drafts completed. Those drafts were reviewed by the Consultant Panel and by other mathematics educators Their reviews led to second drafts, and the process continued in this manner until the final drafts were completed and approved.

How Can This Book Be Used?

Research Within Reach, Secondary School Mathematics can be used in a number of ways. Each chapter begins with a question from a teacher. The answer is constructed so that research information and the classroom implications of that research are clear. Very often, the chapter includes recommendations for classroom practice. Each chapter concludes with a list of references on which the answer is based and to which readers may go for a more thorough understanding of the particular source. (Those references thought to be of special value for teachers are marked with an asterisk *.)

Each chapter is written in such a way that it may be read independently from the others. While this creates some repetition, it is our feeling that this provides greater flexibility both for the reader who wants to read about a particular topic, as well as for the reader who wants to survey the entire field. In any event, each reader is invited to read the chapters in whatever order seems best to him or her

In addition to its obvious use for study by individuals, *Research Within Reach Secondary School Mathematics* will prove useful for pre-service and in-service courses. Each chapter can be read in a relatively short time and then can be used as a basis for discussion in the 60 or 90 minutes periods which are often devoted to in-service activities. In-service leaders might use the chapters as the basis for the development of checklists of important research recommendations against which teachers can analyze their teaching and students learning.

Similarly, teacher candidates will find that the chapters may prepare them to consider actual classroom practice in a way which they have not before All of the recommended practices and much of the research reported are grounded in experience in real classrooms, an environment with which many teacher candidates remain relatively unfamiliar throughout their college career

Finally, it is hoped that readers of *Research Within Reach. Secondary School Mathematics* will take seriously the numerous invitations to replicate or validate the research cited here. One topic that seems especially conducive to teacher experimentation is the use of microcomputers. Although the available microcomputers research is cited in chapters where it is relevant, the body of research is too small to warrant a separate chapter. That situation must and will change; the topic is too important. The dearth of research will be corrected, but in the meantime, teachers' use of microcomputers will very much depend on the ingenuity and good sense of individual teachers. Teachers willing to undertake classroom research utilizing micros can perform a service both to their students and to the community of mathematics educators at large.

Classroom teachers work in environments rich with research potential While it is possible to view this call to research as one more burden, it is also possible to view it as a way to gain a new understanding of how our students learn and how we can be more effective teachers. It is our hope that *Research Within Reach Secondary School Mathematics* will help to ease the burden while it guides teachers toward that new understanding.

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It is our pleasure to acknowledge the contributions of many people First of all, as RDIS Director when this book was conceived, Linda Reed gave it the impetus and support it required to become a viable project. The consultant panel provided guidance in selecting teacher questions, helped to locate relevant research, and gave useful suggestions for revising each draft. The members of the panel are Mary Grace Kantowski, University of Florida; Robert E. Reys, University of Missouri-Columbia, Marilyn Suydam, Ohio State University

Colleagues at the regional labs assisted in several ways. We gratefully acknowledge the contributions of Sandra Orletsky. Merrill Meehan. and Joe E Shively, Appalachia Educational Laboratory; Carol Thomas. Alfreda Brown, Stephanie Siegel. and Nellie Harrison, CEMREL; Nancy Baker Jones. Southwest Eductional Development Laboratory, Janet Caldwell, Research for Better Schools; and Roger Scott, Southwest Regional Educational Laboratory We are grateful to RDIS staff members, Sandra Ruder, Karen Temmer and Sheila Walker for their support and help.

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Finally, we thank the teachers whose questions gave this book its initial momentum and direction. We give the book back to them and to their colleagues, with the hope that it helps to bring them more success and satisfaction in the classroom

EFFECTIVE MATHEMATICS TEACHING

What does research say about the effect of factors beyond subject content--for example, classroom configuration, teacher-student communication and teacher behavior--on students' learning of mathematics in secondary school? How do these factors affect student attitudes toward learning mathematics?

In recent years, mathematics teaching has become a profession in crisis. While the number of secondary school mathematics teachers dwindles to a dangerous level, many who are left in the ranks find themselves questioning their own commitment to teaching. Among the more distressing questions they ask themselves is one that pulls at the very roots of the profession: "Can the teacher really make a difference in the mathematics classroom?"

The answer, pieced together from a series of recent research studies, is clearly yes. No matter how teachers are identified as effective, whether by student achievement, by supervisors' recommendations, or by the testimony of students and classroom observers, it is evident that effective teachers of mathematics plan and behave differently from less effective teachers. More importantly, they do so in identifiable ways that can be learned by other mathematics teachers.

Too often, teacher training concentrates only on teacher behavior and ignores the influence on that behavior of teacher attitudes, opinions, expectations, and planning. Yet research clearly shows the strength of that influence. Consequently, this report is divided, somewhat loosely, into two parts: *ways in which effective and ineffective teachers behave* in the classroom, and *aspects of their thinking and planning* that influence their behavior.

Effective Behavior

In their study of 7th and 8th grade teachers, Evertson, Emmer, and Brophy were able to associate a certain style of class organization, used consistently, with effective mathematics teaching. (7) The researchers used two criteria for effectiveness--student achievement and student ratings--and found that, on the average, the more effective teachers devoted about half of each period to combined lecture, demonstration, and discussion. On the other hand, the less effective teachers used only about a fourth of each period for lecture, demonstration, and discussion, and over a half of each period for individual seatwork.

In another study of junior high mathematics classes, Evertson *et al.* found a similar pattern and were able to elaborate on the teachers' use of time: "the more successful mathematics teachers spent more time in class discussion or lecture, they asked more public questions (creating response opportunities), and response opportunities formed a greater proportion of their contacts with students. They did not, however, have fewer private contacts. Rather, they were simply more active." (6, p. 54)

Questioning. The larger amount of time effective teachers spend with their entire class means they have a greater opportunity to ask questions, and question-asking appears to be an important factor in the effective teaching of secondary level mathematics. In particular, Evertson, Emmer, and Brophy noted that effective teachers asked more so-called process questions (calling for explanations) and also more product questions (calling for short answers) than did less effective teachers, and that they asked more new questions after correct answers had been given. A study of classroom questions asked by geometry teachers showed that students' success in application tests in geometry is related to the frequency of their teachers' use of application questions in class. (8, 22)

A helpful discussion of the uses of classroom questioning is in Johnson and Rising. (12) Another good discussion of the roles and various uses of questions in mathematics teaching appears in the chapter entitled "Questioning" in *Didactics and Mathematics*, a volume devoted to the teaching of middle school mathematics. (15) In particular, it pinpoints more than eleven separate uses for teacher questions, including:

1. To motivate students to consider a new topic. ("What are some four-sided geometric figures you can think of?")
2. To challenge. ("What evidence do you have for thinking that?")
3. To provoke student interaction. ("Bill, do you agree with what Martha said?")
4. To get students to evaluate. ("How do you think your method would work on this next problem?")
5. To focus on process. ("What method did you use on that problem?")
6. To guide. ("Do you remember a problem similar to this one?")
7. To diagnose. ("How did you get that answer?")
8. To review. ("What are some of the things you've learned so far about triangles?")
9. To encourage exploration. ("Can you find a pattern in those numbers?")
10. To invite student questions. ("What questions does this information leave unanswered?")
11. To enhance transfer. ("How could you use that result in this new situation?")

While the quantity and variety of classroom teacher questions seem essential to teacher effectiveness, so does the control of wait-time between questions and answers--both the pause following a teacher's question and the pause following a student's response. In fact, Good's research in junior high classrooms has revealed that many teachers wait less time for their low-achieving students to respond to questions than they do for higher-achieving students, and that, in general, they provide these students fewer chances to participate in public discussions (9) In the end, Good maintains, most of these neglected students become totally passive learners.

Encouragement. Effective teachers try to keep most of their students--low achievers as well as high achievers--from slipping into passive learning. The effective junior high mathematics teachers observed by Evertson, Emmer, and Brophy were more encouraging and more receptive to student input than were their less effective colleagues. (7) In another study, Evertson *et al.* also noted that the higher the number of student-initiated questions and comments, the higher was student achievement in mathematics. (6)

As a result of his concern that so many students fall into a totally passive state of learning, Good recommends that "teachers who want to monitor and creatively examine their own behavior in order to reduce inappropriate behavior would do well to develop strategies for encouraging students to seek information as needed." (9, p. 419) If students are consistently denied such encouragement, the results can be disastrous, for, as we note in the *Research Within Reach* chapters "Understanding Fractions. A Prerequisite for Success in Secondary School Mathematics" and "Communicating Mathematics", research shows that by the time they reach high school, many students have acquired deep, enduring, yet nearly invisible, misconceptions about mathematics that seriously impair their learning and enjoyment of the subject in high school.

The basic strategy for teachers to develop, of course, is the consistent encouragement and reinforcement of questions and requests for help. Beyond that, it is a matter of setting a tone and a dynamic in classroom discussions which allow students to be curious and which lead them to ask questions.

Modeling. As the picture of teacher effectiveness unfolds from research, it is clear that it is not only through such direct means as asking questions, generating questions, and offering encouragement that teachers make their teaching more effective. Indirect means are also important, such as the teacher's own modeling of good learning behavior. Evertson, Emmer, and Brophy noted that more effective teachers "engaged in more problem-solving behavior" in their study. (7, p. 173) Thus, the teacher shows the way to problem solving through his or her own example. The problem-solving researcher Frank Lester has concluded: "Problem-solving instruction is most effective when students sense two things: (1) that the teacher regards problem solving as an important activity and (2) that the teacher actively engages in solving problems as a part of mathematics instruction." (14, p. 43)

A graphic illustration of the effect of teacher modeling emerged from the research of Gregory and Osborne. They discovered a clear correlation between the frequency of 7th-grade mathematics teachers' use of conditional reasoning in their speech (for example, "if-then" and "whether-then" statements), and their students' understanding of logical statements. (10)

Signs of teacher effectiveness show up in the selection and presentation of mathematical content as well as in the modeling of good learning behavior For example, Smith found in his study that effective mathematics teachers use a greater number of relevant examples than do their less effective colleagues. (21) In a research study using computer-assisted instruction, Shumway found that the use of counter-examples of mathematical statements, in conjunction with (positive) examples of those statements, resulted in higher achievement than did the use of examples alone. (20, 22) Thus, a discussion about isosceles triangles is best complemented by some identification of, and discussion about, non-isosceles triangles, discussions about rational numbers should be balanced with examples of irrational numbers, and so on.

As one aid in generating mathematical examples and counterexamples, teachers ought to take advantage of the handheld calculator. The Calculator Information Center prints a variety of such examples (11), and the *Research Within Reach* chapter "The Calculator. An Essential Teaching Aid" directs readers to other sources of calculator-based examples.

Clarity and Continuity. Campbell and Schoen conducted a study in the 7th and 8th grades in which they searched for correlations between the behavior of prealgebra teachers and student attitudes toward mathematics and toward the teachers. (3) Clarity, defined as the careful use of vocabulary and explaining the why with the how in solving problems, and showing the continuity of the mathematics curriculum were the two teacher qualities that correlated most positively with student attitudes. As the researchers noted, "Students who perceived their mathematics teacher as trying to remove the 'mysteries' of mathematics had more positive attitudes toward mathematics and the teacher." (3, p. 374)

McConnell focused on students' perceptions of teacher clarity in his study of 9th-grade algebra, and he found that they matched up fairly closely with the students' mathematical comprehension. (16) In fact, comprehension test scores were positively correlated with the ratings of teacher clarity given by the researchers as they observed classes, but those test scores were even more strongly correlated with student ratings of teacher clarity.

In several other studies of secondary mathematics instruction, clarity also emerges as an important component of teacher effectiveness. When Smith compared effective with ineffective teachers in his study, he noted a tendency among effective teachers to use fewer vague terms in their mathematical instruction (21) Bush *et al.* set up their study to try to capture the notion of clarity in terms of specific teacher behavior. (2) The behaviors they related to clarity were:

1. Taking time when explaining.
2. Stressing difficult points.
3. Explaining new words.
4. Demonstrating how to do something.
5. Working difficult problems on the board.
6. Givings students an example and letting them try to do it.

In a related study, Cruickshank *et al.* used junior high school students' perceptions to distinguish clear from unclear teachers. (5) Among the behaviors the students associated with clear, but not with unclear, teachers were:

1. Giving students individual help.
2. Explaining something and then allowing students to think about it.
3. Repeating questions and explanations if students do not understand.
4. Stressing difficult points.
5. Asking students before they begin a task if they know what they are supposed to do and how they are supposed to do it.
6. Taking time when explaining.

Planning for Effectiveness

No matter what gauge of effectiveness is used--student achievement, student ratings, or classroom observations--effective mathematics teachers behave in identifiable ways that set them apart. Their classes are structured for *consistency*—in particular, every class has some individual seatwork, but has more whole-class work than seatwork. They come to class prepared for *clarity* and *continuity*, aware and in control of their *questioning*, *modeling*, and *encouraging* students in the class. Such effective behavior requires preparation, and two specific areas that research has identified as important are *use of language* and *expectations*.

Use of Language. Judging from the testimony of classroom observers and from student testimony as well, teachers who wish to improve the clarity in their mathematics teaching would do well to measure the vagueness in the mathematical terms they plan to use in class. They should always plan to explain new words and terms and to spend enough time in class discussing the more difficult of them.

Some mathematical words and terms can have several meanings and they may confuse many students if the teacher's intended meaning is not made abundantly clear and held constant. In discussing changes in the meaning of terms during mathematical discussions, Kenme (13) provides the transcript of an algebra class in which the teacher poses a problem:

> *There is a certain number of students in the classroom. If there were twice as many and then another ten were added to it, then there would be 42. How many students are there?*

Several students arrive quickly and intuitively at the solution 16. The teacher, still hoping to use this problem to illustrate how to translate from word problems to equations, asked: "What kind of equation could you write in this case?" Since they knew the answer, several students answered, quite legitimately: "$x = 16$." The teacher, of course, wanted the equation $2x + 10 = 42$ as an answer. To the teacher, "equation" had a definite functional meaning--a tool for figuring out the solution. To the students, the term "equation" included the mere statement of the answer. Because of these different meanings, the class discussion turned into a verbal wrestling match, with the teacher trying to twist the desired equation from the students, while they remained unpersuaded and confused.

Examples of such misinterpretations abound in mathematics teaching For example, the term "large" can connote how far away from zero a particular number is. Hence, a sentence like, "4 is larger than -7," confuses many young people if they refer in their minds to distance from zero when they see "larger," and not to the two numbers' relative positioning on the number line. Similarly, if teachers are careless enough to portray rectangles almost exclusively as non-squares, their students can very easily be trapped into picturing only non-squares in situations where a more general conceptualization is appropriate. In such cases, they fail to recognize that all formulas and relationships connected to squares are special cases of formulas and relationships connected to rectangles.

Expectations. One critical aspect of teacher planning which research shows does not get the attention it demands is the area of teacher expectations about student performance. No one can say for certain that a teacher's expectations about a particular student will have a direct bearing on that student's academic achievement. What seems certain, however, is that teachers tend to behave differently toward high- and low-achieving students, that students can and do discern expectations from a teacher's behavior, and that they adjust their own attitudes, expectations, and behavior accordingly. Good cites his own research, as well as the supportive research of others, indicating that junior high students see their teachers behaving differently toward low-achieving students. In particular, high achievers are perceived by students to have "more choice of tasks and more time to complete work if they request it." (9, p. 421)

Other research has also focussed on ways in which many teachers behave differently toward high achievers than toward low achievers. Good summarizes that research, and though not all of the studies were conducted in the secondary school mathematics classroom, the full summary provides a valuable checklist for teachers and so we repeat it here: (9, p. 416)

1. Seating slow students farther from the teacher or in a group (making it harder to monitor low-achieving students or treat them as individuals).
2. Paying less attention to low-achievers in academic situations (smiling less often and maintaining less eye contact).
3. Calling on low-achievers less often to answer classroom questions or make public demonstrations.
4. Providing less time for low-achievers to answer questions.
5. Not staying with low-achievers in failure situations (providing fewer clues, not asking follow-up questions).
6. Criticizing low-achievers more frequently than high-achievers for incorrect public responses.
7. Praising low-achievers less frequently than high-achievers after successful public responses.
8. Praising low-achievers more frequently than high-achievers for marginal or inadequate public responses.

9. Providing low-achieving students with less accurate and less detailed feedback than high-achievers.
10. Failing to provide low-achievers with feedback about their responses more frequently than high-achievers.
11. Demanding less work and effort from low-achievers than from high-achievers.
12. Interrupting the performance of low-achievers more frequently than that of high-achievers.

The issue of whether teacher expectations affect girls' mathematical learning deserves special note. Parsons and her colleagues studied the effects of teacher expectations on interactions in 15 eighth-and ninth-grade mathematics class-rooms. (17) In the 5 classrooms in which teachers had the most different expectations for boys and girls, the researchers observed significantly more praise of boys' work than that of girls and fewer private student-teacher interactions. In contrast, when the researchers observed the 5 classrooms with the least sex-related differences in teacher expectations, they found that girls interacted more and received more praise, and that there was more one-to-one teacher-student interaction. (17, 19)

A set of firm and appropriate expectations, kept visible to students and appealed to regularly, is an essential component of effective teaching. A major study of British secondary schools revealed that high academic expectations make up one of several factors that set effective schools apart from ineffective schools, where effectiveness is measured by student achievement, attendance, behavior, and delinquency records. (18) The staffs in the study's effective schools communicated to their students that they expected most of their students to do well on exams, they assigned homework regularly, and they checked homework regularly. Less successful schools did none of these things as forcefully or regularly. Evertson, Emmer, and Brophy noticed a similar pattern in junior high mathematics classrooms: "The more effective teachers also manifested behaviors indicative of higher expectations for their students. They assigned homework more frequently, stated their concern for academic achievement more often, and gave more academic encouragement." (7, p. 176)

Conclusion

There can be no doubt that the effect of mathematics teachers on students is profound. As Bauersfeld describes it, "Teaching and learning mathematics is realized through *human interaction*. It is a kind of mutual influencing, an interdependence of the actions of both teacher and student on many levels. It is not a unilateral sender-receiver relation--the student's reconstruction of meaning is a construction via social negotiation about what is meant and about which performance of meaning gets the teacher's (or the peer's) sanction." (1, p. 25)

Only the teacher can determine whether the effect of those classroom interactions will be beneficial or harmful. The margin of benefit increases, however, with the care and planning teachers put into their clarity and the control they exert over their expectations and classroom efforts to welcome and to generate student input.

The importance of clarity and of involving students as much as possible is a message that emanates, not from one, but from several major research studies. Two recent studies, however, imply that we are far from heeding that message. In the first--a survey of research on patterns of instruction in American mathematics classrooms--the most noticeable pattern, in an overwhelming number of mathematics classrooms, involved a daily routine in which answers are given to the previous day's assignment, the more difficult problems are worked at the board, new material is covered briefly, assignments are given for the next day, and the rest of the period is spent on individual work or the homework assignment. (23)

Just as worrisome as patterns of instruction are patterns of student attitudes. Data from the second study, the National Assessment of Educational Progress, resulted in the following conclusion. "For the 9-year-olds, mathematics was the best liked of five academic subjects, mathematics was the second best-liked subject of the 13-year-olds and the least-liked subject of the 17-year-olds." (4, p. 134)

If we are to hold onto the interests of students as they begin to drift away in the early years of secondary school mathematics, then teachers must heed the message of research into teacher effectiveness and begin to adjust their planning, expectations, and behavior to create a classroom environment in which clarity is a constant goal and in which student input is at the center of the learning experience.

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INDIVIDUAL DIFFERENCES AMONG MATHEMATICS LEARNERS

What are some of the major differences in learning styles and levels of development among secondary school students? How should such differences be dealt with in the mathematics classroom?

Despite the fact that much of the secondary school curriculum is designed on the assumption that students all think in the same way, secondary level mathematics teachers know differently. Every year, they meet many students who seem unable to think logically, or who become confused whenever symbols are used to represent mathematical concepts. Furthermore, many students approach mathematical decisions without a sense of what is reasonable.

This report addresses individual differences in mathematics learning at the secondary level. It describes the major factors which research has linked to individual differences, lists the curriculum areas where those factors are likely to have a significant impact, and offers some suggestions to teachers for identifying and responding to those individual differences.

Individual Differences

There are several research perspectives on individual differences in mathematics learning. One group of researchers concentrates on the different stages of *cognitive development* through which children grow. Another group studies the various *cognitive styles* or ways of processing information among learners. A third group isolates the *curricular and environmental influences* to which learners respond differently, regardless of learning styles or developmental levels. Taken together, the three perspectives provide a more integrated picture of the individual mathematics learner than each provides separately.

Cognitive Development

Generally, developmental researchers draw their perspectives from the work of Jean Piaget and his followers. As Piaget describes it, adolescence is the period when children grow out of the stage of concrete operations where their thinking has been totally dependent on perceptions and concrete experiences. Readers who want a more comprehensive look at the concrete stage should refer to the report "The Bridge from Concrete to Abstract" in *Research Within Reach: Elementary School Mathematics*. This report focuses on the formal operation stage, the stage that follows the concrete operations stage, in which an individual can internalize thought, think about thinking, keep two or more variables in mind at one time, and see a concept as part of a larger system.

According to the original Piagetian hypothesis, a majority of children enter the stage of formal thinking between the ages of 12 and 14. Research has shown, however, that for many children the process is much slower. (22) There are individuals who begin to think formally on some tasks well before they can think formally on others, and some individuals *never* enter the stage of formal thinking for some tasks.

Piaget claims that the stage of concrete thinking ends and the formal thinking stage begins for a child when he or she can conserve the concept of volume. Conservation of volume is tested in several ways, but the goal in each case is to determine if a child understands the concept of volume well enough to ignore irrelevant attributes in volume problems. The irrelevance of the weight of an object to the volume of water it displaces when immersed provides an example. A student is shown two identical glass containers partially filled with equal amounts of water. Two metal cylinders of equal volume but different weight are then handed to the student. After the equal heights and thicknesses of the metal cylinders have been pointed out, the experimenter lowers the lighter cylinder into one of the two glass containers. Once the student notes the rise in water level, he or she is asked to predict the rise in water level when the heavier cylinder is lowered into the other glass container. A child who is at least in the early stage of formal thinking will recognize that the weight of the cylinder in this example has nothing to do with how much volume is displaced.

In his review of the research on cognitive development, Carpenter has described formal thought in the following way:

> "The most fundamental property of formal thought is the ability to consider the possible rather than being restricted to concrete reality. At this stage adolescents can identify all possible relations that can exist within a given situation and systematically generate and test hypotheses about these relations. They are capable of evaluating the logical structure of propositions independent of any concrete referents, and they are able to reflect on their own thought processes." (3, p. 176)

According to the Piaget model, two of the major facets of formal thought are:

1. Propositional logic. Individuals who have reached this level of thinking can understand "if...then" and "either...or" reasoning and can keep several variables in mind at one time. Flavell describes a study in which an experimenter shows poker chips of different colors to children, then hides one in his hand and says, "Either the chip in my hand is green or it is not green." The children are instructed to indicate whether they think the statement is true, false, or undecidable. Pre-formal thinkers tend to concentrate on their perceptions--in this case, a chip hidden from view--and indicate that they cannot tell whether the statement is true or false. Individuals who have attained the formal stage of thinking are more inclined to focus on the words, not their perceptions, and therefore to indicate that

the above statement is true. (11)

2. Proportional logic. Individuals who have reached this level of thinking can successfully compare ratios, as in the problem: "Which mixture would give the sweeter drink, one that is 6 parts orange syrup to 9 parts water or one that is 4 parts orange syrup to 8 parts water?" Preformal thinkers are generally unable to hold the two ratios together in their minds in a way that allows them to weigh one ratio against the other and then to adjust one ratio to make it proportional to the other. (6, 11)

Cognitive Style

Another perspective on individual differences among mathematics learners has developed from the work of researchers who study cognitive style, or how individuals differ in processing information. Many such cognitive styles have been identified; this report concentrates only on those styles that have the greatest implications for the teaching of secondary level mathematics.

As described by researchers, each cognitive style represents a continuum of style in information processing, and everyone has a place somewhere on that continuum. One such continuum is given by the two opposing cognitive styles *impulsivity* and *reflection*. Persons at the impulsive end of the continuum tend to pursue the first answer that comes to mind when they are asked a question or the first approach to a solution when they face a problem. Reflective individuals, on the other hand, are hesitant to respond or react quickly, and they are likely to reflect longer on the different possibilities for answers and problem solutions. (9)

Field dependence and *field independence* are the opposing ends of another cognitive style continuum. As Fennema and Behr describe it,

> "At the field dependence end of the continuum, activities and perceptions are global, that is, subjects focus on the total environment. At the field independence end of the continuum, activities and perceptions are analytical, that is, subjects perceive the environment in its component parts. At the one extreme of the performance range, perception and mental activities are dominated by the prevailing field; at the other extreme they are relatively independent of the surrounding field." (9, p. 331)

Other Factors

Other factors that contribute to individual differences among mathematics learners have arisen from the work of clinical researchers. Several clinical studies, in which students were interviewed as they worked through mathematical exercises and problems, have made the following hypothesis seem very plausible: as children proceed through school mathematics--in particular, as they approach secondary level mathematics--some develop a style of learning that relies on *memorizing rules and procedures*, while others develop a style that leads them to rely consistently on *intuition and common sense*. To illustrate these styles, Peck and Jencks reported on their interviews of two 7th graders, both fairly successful in mathematics, as measured by standardized tests and their placement in school. Both students were asked to work through exercises in the multiplication and comparison of decimals and in the addition of fractions. Each applied learned rules for these exercises, but one relied so heavily on memory that she was unable to sense when her memory had produced the wrong rule for her. For example, she worked through the following example:

$$\begin{array}{r} 1.1 \\ \times\ 2.23 \\ \hline .33 \\ 2.20 \\ 22.00 \\ \hline 24.53 \end{array}$$

Her description of the rule for locating the decimal point in the answer was, "You look at this bottom line and keep going straight down underneath for the answer." Her firm focus on the rule--which is appropriate for addition and not multiplication of decimals--blinded her to the unreasonableness of producing a number larger than 24 with the multiplication of two numbers less than 2 and 3, respectively. The second student, on the other hand, was consistently inclined to apply intuition to supporting and gauging the appropriateness of remembered rules. The two students were interviewed together and, as intelligent as she was, the first student was unable to make the leap from mathematics-as-memorized-rules to the common sense approach of the second student. (20)

Students often misapply rules and are unable to gauge whether a particular rule is appropriate. Though there probably is no single cause for such behavior, clinical research has clearly documented how deeply ingrained the reliance on memory becomes for some mathematics students and how narrow is their freedom to use their intuition when they are doing mathematics.

Factors which grow out of the mathematics curriculum itself and the child's previous experience within that curriculum can also heighten differences among learners. For example, it seems clear from several studies that children vary in their success in bridging the system and language of mathematics with their real world system and everyday language. This is not a question of being far enough along in cognitive development to be able to handle mathematical symbols. Some children handle symbols well enough, but their impression of mathematics is that it is a system of rules divorced from the real world. Thus, they work through problems and exercises, using rules that may not, in certain situations, make any sense when measured against their real-world experience.

Erlwanger's study of this phenomenon was at the elementary school level, but it presents a graphic illustration of this disjointed view of mathematics, and is relevant to the teaching of secondary level mathematics. (8) It is tempting to assume that if a student has some basic misconceptions about mathematics, those misconceptions will show up quickly either through test results or teacher observation. Erlwanger, however, found sixth-grade children who were fairly successful in their individualized mathematics programs, but whose perceptions of mathematics as a system were very skewed. For example, one child named Benny had correctly observed that mathematical answers can be written in various forms, as in $1 = 4/4 = 1/2 + 1/2$, but had stretched his observation into a strange, basic belief about mathematics. Benny believed that in mathematics an answer can take apparently contradictory forms, because answers, in his words, work like "magic, because really they're just different answers which we think they're different but really they're the same." (8, p. 173) This belief made him adamant in defending such statements as $2/1 + 1/2 = 1$ and $5/10$ as a decimal is 1.5 "The one stands for 10; the decimal; then there's 5—shows how many ones." (8, p. 202) Amazingly, Benny had woven such misguided rules into a coherent system that permitted him to *succeed* in his individualized mathematics program.

Booth's work points to another variation of this lack of success in bridging mathematics and the real world. He reports a significant frequency of secondary-level students who may compute well enough using formal mathematical procedures such as long division, but who fall back on what Booth calls "child-methods," such as counting, when they work on mathematical word problems. (2) At no point along the way have they come to see how formalized computational procedures can be used as strategies for solving mathematical problems. While it is clear that all students use mathematical methods of some sort to solve real problems, it is equally clear that for many of these students a chasm stands between those methods and the mathematics they use for computational exercises.

Critical Topics

According to the research outlined above, every secondary level mathematics student (1) is somewhere on the continuum between concrete thinking and full formal thinking, (2) has a position on each of several cognitive style continuums, and (3) differs from many other students in the kind of bridge he or she has built—with language, intuition, and the formation of personal rules--between mathematics and the real world.

Whether a student understands a particular mathematics topic or not may depend on how properly the presentation of the topic fits the needs of that student. With the profile of individual differences developed above, it is possible to clarify some of those needs and to pinpoint topics in the secondary mathematics curriculum where the matching of presentations to individual differences is likely to be critical to success in learning.

Throughout the range of secondary level mathematics topics, teachers must be aware of one overriding limitation imposed on pre-formal thinkers and should tailor their choice and presentation of mathematics examples accordingly. The pre-formal child, in the words of Flavell, "usually begins with reality and moves reluctantly, if at all, to possibility." (11, p. 103) Hence teachers should not disregard students who do not respond very smoothly to "what if...?" questions or to challenges to make hypotheses. Those students may not have settled into the level of formal thinking where such tasks are a comfortable matter of course

Here are some specific mathematical topics where individual differences have been detected. They are presented with suggestions for adjusting instruction

Proportion and Fractions. Preformal thinkers have considerable difficulty with proportion problems, which compare two or more ratios. But what about the components of those problems, the ratios themselves? A ratio is, of course, one of the representations of the concept of fraction, and most students have been exposed to fractions from early elementary school. Thus most secondary teachers have high expectations about the facility in using fractions that students bring to secondary mathematics.

It turns out, however, that a firm understanding of fractions depends on the development of formal thinking. McBride and Chiappetta studied ninth-graders' understanding of equivalent fractions (for example, four-sixths equals how many ninths?), and they were led to conclude that facility in using the concept increases as proportional reasoning increases. (19) Many ninth-graders have not yet settled into full formal thinking, in particular, into proportional reasoning, so teachers at that level must watch for weaknesses in their understanding of equivalent fractions.

Students who are in transition from concrete to formal thinking can benefit from instruction in crucial topics like fractions. Like preformal thinkers, many early formal thinkers show their inability to do proportional thinking by adding inappropriately when they are asked to adjust one ratio to make it proportional to another. For example, consider the problem. When Bill made lemonade he used 4 spoonfuls of sugar and 10 spoonfuls of lemon juice. Mary made lemonade with 6 spoonfuls of sugar. How many spoonfuls of lemon juice must she use so that her lemonade will taste the same as Bill's?

It has been an established fact that a common strategy of preformal thinkers is to subtract 4 from 6 to get 2, then add to 10 to yield the answer 12. The correct answer, of course, is 15 and it is produced by proportional thinking—for example, $4/10 = 6/15$.

Recently, Karplus and his colleagues have confirmed that early formal thinkers may also use such an additive strategy. (13) In this study, nearly 25 percent of the sixth and eighth graders they tested and interviewed alternated between additive and proportional strategies in solving a string of such lemonade problems The researchers' conclusion, avoidance of fractions, rather than cognitive development, seemed to be the major obstacle in the way of these students' mastery of quantitative proportion problems. In other words, there are many students who could move fully into proportional thinking if the appropriate instruction were made available to them.

Kurtz and Karplus designed some instruction that can affect proportional reasoning. (16) In a laboratory setting with prealgebra students they used tables of data as the vehicle for illustrating the concepts, such as constant ratios, that are the basis of proportional reasoning. The lessons extended over fourteen class periods. When the students were tested before the experiment, one-sixth of the students in the experimental group exhibited proportional reasoning; after the instruction, roughly two-thirds of the students exhibited proportional reasoning.

Proof. A formal mathematical proof is a complex cognitive task. The individual constructing the proof must weigh several deductive paths at the same time. Once the path for the proof has been chosen, both the goal and the established evidence must come together to inform the steps of the proof. These are taxing, if not futile, demands on pre-formal students and there are many such students in the ninth and tenth grades, where formal proof is a frequent objective in the curriculum. (For further information about proof, see the chapter "The Path to Formal Proof".)

Algebra. Both cognitive style and cognitive development have a direct bearing on the learning of algebra. Cognitive style seems to come into play in the strategies students choose to solve algebraic equations. Petitto found that ninth-graders fell into two strategy groups: those who leaned toward an intuitive approach that tried to capture the numerical relationships among the numbers in an equation without transforming the equation itself, and those who relied on memorized or routinized step-by-step procedures (algorithms) for transforming the equation and producing an answer. (21) Some students moved easily between the intuitive and algorithmic approaches, and they tended to be the most successful.

For example, students in the intuitive group solved $3/5 = x/30$ by noticing that 30 is 6 times 5, so x must be 6 times 3, or 18. Students in the algorithmic group, on the other hand, multiplied both sides of the equation by 5 to produce a new equation, multiplied both sides of that by 6, etc. From observing these two styles in action, as she gradually increased the difficulty of the equations given to the students, Petitto learned that neither strategy, used alone, was foolproof. Students with the intuitive style faltered as the number relationships grew in complexity, as in $14/23 = 56/(x+2)$, while students in the other group occasionally failed to adapt their algorithmic procedures when the structure of the equations changed. As a result, they not only produced a wrong answer, but were unable to check to see if the answer fit in the equation.

Since the most successful subjects in the study were those students who moved easily between the intuitive and algorithmic approaches, Petitto suggests that algebra teachers give equal stress to step-by-step solution procedures and to the consistent intuitive assessment of equations to see if the numbers in them relate to each other in ways that suggest solutions.

Formal proof has already been mentioned as an aspect of algebra affected by cognitive development. Another is the concept of equation. In particular, Wagner studied how well 12-, 14-, and 17-year olds conserve the concept of equation. (26, 27). Once again, conservation was the measure of cognitive development and, just as in the case of conservation of volume, the task was to assess the understanding of a concept "by determining whether or not a person realizes that the critical attribute, the essence of the concept, is invariant under transformations of certain irrelevant attributes." In the case of equation, Wagner presented each student with two equations, (in the second equation W is seen through a small window from which it can be removed):

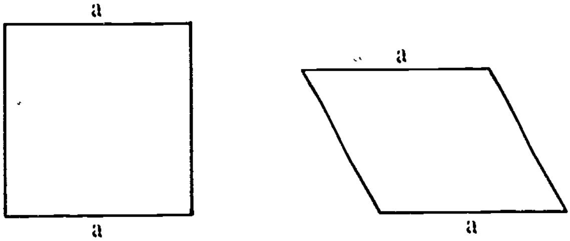
$$7 \times W + 22 = 109$$

$$7 \times \underline{W} + 22 = 109$$

Once the student acknowledged that they both have the same solution, the researcher replaced W by N in the second equation and asked "Which solution will be larger?" Students whose response indicated that they thought the solutions would be different were deemed nonconservers. Wagner reported that one-third of the interviewed students who had successfully completed algebra failed to conserve on this task. Her experiment makes it very clear that a sizeable number of children complete algebra but do not conserve the concept of equation.

Geometry and Measurement. Any geometric problem that requires a student either to construct a formal proof or to hold more than one variable in mind at a time will severely tax the child who is in the preformal reasoning stage To illustrate, Kidder conducted a study that required 13-year olds to consider the invariants when a triangle in a plane is rotated through a fixed angle. (14) The students' answers were a mixture of correct and incorrect conclusions, and Kidder ascribed this to their having to focus on one variable (e.g., length of sides) and then another (e.g., the positions of the triangle vertices with respect to the origin of the rotation). In particular, they generally showed an inability to conserve length in the task--that is, to understand that in the rotation of a triangle the side lengths do not change.

If conservation of length and conservation of volume are weak points for individuals who have not reached the level of full formal thought, it would seem likely that conservation of area would be a weak point, too. Indeed it is, as Szetela learned. He gave seventh- and eighth-graders the task of deciding whether or not deforming the perimeter of a shape affects its area (24) Of course, it usually does (for example, changing a square into a rhombus with the same side-length does not change the perimeter, but it changes the area), but Szetela found that the pre-formal thinkers in this group of subjects were inclined to believe that areas are invariant under such transformations of perimeter.



Problem Solving. A study of the processes used by eighth-graders to solve algebra problems showed that pre-formal thinkers used fewer processes than formal thinkers. (7) In particular, the study examined performance on problems such as:

> *Jeff bought 5 oranges and 10 apples for $1.65. An apple and an orange together cost 20 cents. How much does one apple cost? How much does one orange cost?*

Both groups contained students who used diagrams in their approaches to the problems as well as students who tried to recall similar problems, but students who were formal thinkers used the following processes not used by the others: deductive reasoning, use of successive approximations, estimation, checking of conditions, checking of manipulations, checking by retracing steps.

Cognitive style also affects the types of processes and strategies used to solve problems. In fact, Adi and Pulos related field independence—field dependence to flexibility in situations where two problem-solving strategies are in conflict: "One strategy had been used in the past, or is relatively simple, and the other strategy must be constructed, or is relatively complex. In both cases, the simple strategy is considered first, but only the field-independent subject 'drops' this to consider and construct the alternative strategy." (1, p. 150)

Firm connections between impulsivity-reflection and problem solving have not been made, but some plausible hypotheses have been offered For example, since reflective individuals are more inclined to take time to reflect, perhaps they are also more inclined to use strategies that flow from reflection, such as understanding the problem by identifying the unknown, or redefining the problem by constructing a simpler, but similar, problem.

Implications for Instruction

Two messages emerge from the research outlined so far in this chapter there exist significant individual differences in learning styles among secondary level students and there are numerous areas of the secondary mathematics curriculum where those differences are likely to affect learning. In the face of these two messages, it is natural to wonder what role mathematics teachers can play in assuring that learning takes place despite individual differences.

As a start, teachers can sharpen their diagnostic skills and be more alert for the kinds of differences described in this report. Careful observation, coupled with careful listening, will increase each teacher's sensitivity to individual differences.

Recognizing that Piaget's individual interviews are too time-consuming for teachers who want to gauge the emergence of early formal thought among their students, Renner *et al.* developed a paper-pencil test that comes very close to Piaget's Displacement Volume task interview in measuring early formal thought (23) For an additional example of how researchers probe for levels of propositional and proportional reasoning, readers should see Phillips's article (22) Even if teachers choose not to adapt the questions in the article for formal classroom diagnosis, they will at least become more aware of the kinds of questions that illuminate the formal thinking stage.

Another rich source of diagnostic wisdom is the several case studies mentioned in this report, whose interview transcripts provide a lively and relevant account of children's different approaches to learning mathematics. The carefully chosen, non-directive questions of the researcher-interviewers can serve teachers as models for their own classroom diagnosis of extreme dependencies on memory, skewed impressions of mathematical rules, and so on. With their observational and diagnostic skills sharpened, these teachers can then focus on appropriate instruction.

Some educators have been skeptical about the effectiveness of instruction in significantly increasing the pace of cognitive development. In recent years, however, several researchers have argued against this skepticism. Klausmeier has tested his own Cognitive Learning and Development (CLD) Theory and disagrees with those who are content to wait for children to develop without stimulating that development. (15) According to CLD Theory, the transitional period between concrete thought and formal thought is much longer than many previously imagined, and instruction can hasten the transition for many individuals.

Yeotis and Hosticka discuss the teaching of students who are in developmental transition, and they point out that what sets apart the thinking of concrete operational individuals from the thinking of formal operational individuals is not the processing of information (how knowledge is organized in the mind and memory for later use), but the ways in which knowledge is acquired and represented. "For the concrete operational learners ideas must be abstracted from their experiences with the physical world and their actions performed on objects, whereas the formal operational learner is able to work in a hypothetical deductive manner in which reasoning processes can be applied to any chosen set of premises." (29, p. 558)

Consequently, they suggest a three-phase approach to problem-solving instruction in the middle grades that accommodates the many students who are in transition from concrete to formal operations. Phase I stresses *cue attendance*, or having the students attend to all the relevant details potentially useful in solving a problem. During Phase II students practice *verbalizing* their problem-solving processes and strategies as they work on solving problems. Finally, Phase III takes the verbalizing one step further, and students are trained and required to *diagram* their problem-solving steps. Flow-charting is one recommended procedure for this, suitable in the way it represents problem solving, for both concrete and formal operational thinkers.

Phillips also addressed the role of the teacher in students' development. First of all, she listed the factors involved in development as they have unfolded from the work of Piaget and others: maturation, social interaction, equilibration, and experience. (22) *Maturation*, of course, is that aspect of development that derives from an individual's own interior clock and genetic scheduling. Individuals also develop through *social interaction*, especially with peers, and teachers ought to nurture this aspect of development through classroom discussion.

Equilibration is the two-way process an individual uses in cognitive development, first to absorb a new learning experience and adapt it to the conceptual framework the individual uses to intepret the world and second, to restructure that conceptual framework in light of the new learning experience Thus, as we develop, we are constantly adapting new learnings to our world-view, and changing our world-view in response to new learnings. Phillips argues that, with careful challenging, a teacher can help students in this process by providing "an environment both familiar and novel, comfortable and uncomfortable " (22, p 8) Thus, for example, as long as teachers stay mindful of students' varying capacities for propositional reasoning, occasional experiences with "what if..." questions and "if...then" statements are liable to help students in their development. In fact, one study of seventh-graders' logical reasoning skills found a high correlation between the frequency of teachers' use of conditional reasoning (e.g., "If--then" sentences) and the conditional reasoning skills of the students. (12)

About *experience* Phillips writes. "Too often the high school subject area specialist assumes that someone else has provided the concrete experiences and action-learning necessary" (for strengthening and moving cognitive development) (22, p.8) As an example she points to proportional reasoning: "Measurement activities using real tools and objects, making comparisons, using symbols for measurement terms, using objects to demonstrate fractional representations, are all ways to introduce understanding of proportion." (22, p. 9) Similarity of triangles is a concept related to proportional reasoning, and it is a concept many high school students find difficult to apply to mathematical problems. (4) The kinds of activities suggested by Phillips, focussed on similar triangles, can guide students to a full understanding of the concept.

The Szetela study cited earlier also pointed to the role concrete experiences should play in cognitive development at the secondary level. After determining the kinds of misunderstandings about area and volume that are common among secondary students, Szetela wrote. "The use of formulas to obtain areas and volumes should be delayed until students have had sufficient experiences to acquire better understanding of the seemingly simple, yet complex, concepts of area and capacity." (24, p. 11) Insofar as it is possible, such experiences should involve students in manipulating area and volume changes, to compare and determine what are the relevant and irrelevant variables in the change processes. To cite some earlier examples, students can experiment with the effects of weight change on volume displacement or the effects of shape changes on perimeter and area

According to a study by Threadgill-Sowder *et al.*, manipulatives can also be valuable for some junior high students in understanding logical connectives such as "and," "or," "not." In particular, students in the study who had scored low on standardized achievement tests benefited from instruction in the use of logical connectives that employed color-coded cards and attribute blocks. (25) In light of a recent national survey (10, 28) which reported that nearly 40% of all mathematics teachers in grades 7-12 *never* use manipulatives in class, it is clear that many secondary-level students might never have a full chance to learn mathematics

Conclusion

Because of current brain research and research into information processing (for example see 5 and 18), the future holds some exciting prospects for adapting mathematics instruction to accommodate individual differences In the meantime, insofar as they are able, teachers should try to ensure that each student's learning is consistent with that student's individual development and learning style As has been pointed out, students will differ from one another in the ways they perceive the world; in particular, in the ways they perceive the connection between mathematics and their world. Therefore, to the extent that teachers can help students put geometric concepts and facts, the manipulation of equations, and so on, into their own words and world-view, they will be helping them to make a bridge between their everyday world and the world of mathematics.

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COMMUNICATING MATHEMATICS

How precise do we teachers need to be in our use of mathematical language? On the one hand, texts seem almost too precise—for example, making a fine distinction between "reciprocal" and "multiplicative inverse." On the other hand, many students seem to be easily confused by terms like "least common multiple." They see "least" and look for the smallest number.

Language serves a dual purpose in education. It is, of course, the primary means by which thoughts are communicated by one person to another It is also the means by which thinking itself is done—when we think, we speak to ourselves and process our thoughts with silent words and sentences.

Mathematics is no one's native language, and so no one thinks or communicates totally in mathematics. Yet, more than any other discipline, mathematics requires careful translation, much as any foreign language does If the translation breaks down, misconceptions grow and mathematical thinking suffers

Research into the interplay between language and mathematics is in its infancy, although it is conceivable that future studies will uncover significant relationships between the learning and use of language and the learning of mathematics. (1)

This report focuses on the language of mathematics and on the effective communication of that language, dividing the relevant results into two parts: communication through reading and writing, as happens with textbooks and tests; and communication through speaking and listening, when students interact with teachers or peers. No matter what the mode of communication, however, the results reported below bear out the central role of teachers in communicating mathematics: they must not only monitor what is communicated, but also *how* it is communicated.

Communicating through Reading and Writing

Difficulties that arise in the translation of mathematics are not due solely to confusion about vocabulary terms such as "quadrilateral" or "least common multiple." Kane (15) points out that mathematical English differs from ordinary English in several ways, among which are:

1. Letter, word, and syntactical redundancies differ. For example, single letters such as x and y appear frequently in mathematical English, as do words such as "infinite" or "greater," and sentences built around the conditional phrase "if and only if."
2. Names of mathematical objects usually have a single denotation, unlike

nouns in ordinary English. For example, "point" denotes only one thing in mathematics, but in everyday language it can have a variety of meanings, from the tip of a cone to a seaside promontory.

In their assessment of British students' understanding of mathematics, Hart and her colleagues discovered several vocabulary misconceptions that arise typically when students cross from ordinary English to mathematical English (13) The following excerpt of an interview with a 14-year-old girl illustrates the misconceptions: (13, p. 213)

Interviewer: 10 sweets are shared between two boys so that one has 4 more than the other. How many does each get?

Faith: That's wrong, if you *share* they each have 5, one can't have 4 more.

Kemme (16) points out two more language difficulties that arise for students of secondary school mathematics:

3. Many mathematical expressions are hypothetical references, and most adolescents find hypothetical reasoning very difficult until they are between 14 and 16 years old. For example, both teachers and texts begin the solution of many algebra problems with "Let the unknown number be x." Taking the perspective of an adolescent, Kemme says, "Why should you name things that are yet unknown to you? That's a very unusual use of language for 7th grade pupils. Moreover, it's a type of hypothetical reasoning with unknown objects." (16, p. 46)
4. Many mathematical expressions refer to concepts that are new. In some cases, a concept can remain unfamiliar to students even when both teacher and textbook have taken its familiarity for granted. For example, Hart reports about the British assessment: "It was apparent when interviewing fourteen year olds that the words 'perimeter' and 'area' were not part of their normal vocabulary and had to be redefined." (13, p. 213) In the United States, the recent National Assessment of Educational Progress revealed that more than half of all thirteen year olds confused the concepts of area and perimeter. (6)

Compounding the communication problem are the occasional textbook definitions built around terms and concepts which are themselves not well understood For example, consider the following definition:

Polyhedron — a three-dimensional figure all of whose faces are polygonal regions.

In the definition, "three-dimensional", "faces", "polygonal", and "regions" are all terms which could be misunderstood by many students.

Krulik (19) points out some examples of yet another difference between mathematical and ordinary language:

5. Reading mathematical language often does not rely on left-to-right eye movement. As an example, consider what your eyes do in a careful reading of $\frac{5}{2}x - \frac{1}{2} = \frac{2}{3}x$.
   More than likely, they move from the 5 to the 2 beneath it to the x, and so on in a combined downward-then-to-the-right movement.

In their article about the solving of mathematics textbook problems, Barnett, Sowder, and Vos (2) point out several other differences between ordinary language and mathematical language:

6. Mathematical word problems are more compact and conceptually dense than ordinary prose. Often, several important ideas are squeezed into a single sentence, thus requiring a more aggressive and thorough kind of reading than ordinarily required outside of mathematics. For example: "To raise money for new playground equipment, Mrs. Maple's fifth-grade class sold 180 boxes of candy at $1.50 a box and 40 T-shirts at $2.00 each. If each box of candy and each T-shirt costs the class $1.20 and the students wish to award $3.00 in prize money, how much profit did the class make on the sale?"
7. Ordinary prose usually possesses a continuity of subject and ideas from sentence to sentence and paragraph to paragraph. In textbooks, word problems usually appear in groups of similar problems and students develop tendencies to process each problem in the same way, tendencies which are hard to break when other groups of problems are encountered.

As a result of their mostly silent encounters with the quirks of mathematical language, many students develop their own errant rules for mathematical language. For example, Kent describes a misconception he has seen among students, one that is especially insidious because it is as subtle as it is misleading. Challenged to simplify an algebraic expression like $\frac{2yh + h^2}{h}$ some students will offer the answer $2y + h^2$, instead of the correct $2y + h$. It might have been easy for Kent to ascribe this to a mere oversight, if he had not probed further with one particular student. The interview revealed that the student, whose grades in mathematics were not bad, treated operators (such as +) the same as variables (such as y and h). As a result, a numerator like $2yh + h^2$ becomes nothing more than a string of symbols, and if you eliminate an h from the denominator, you need only eliminate one h from the numerator. Thus, a seemingly innocent but very common mistake revealed a gross misunderstanding of mathematical language. (17)

From their observations and interviews of ninth-grade general mathematics students, Confrey and Lanier reported similar misunderstandings:

"Decimals were strings of numbers to be treated slightly differently, rather than wholes and parts. Fractions were only one number over another, without any image of pies, or ratios, or segments, or partitions of sets. Mathematics seemed to be a set of symbols to operate on within rules and if those rules failed to fit perfectly or errors were made, then the student was left with no recourse, except to perhaps try to manipulate the numbers more, and hope some answer would come out even." (9, p. 555)

Another language-related misconception that is resistant to change is "to multiply means to make bigger." Bell and his colleagues found that even after instruction aimed at correcting the misconception, when students between the ages of 12 and 16 ran into trouble when they were asked to compute the cost, at $1.20 per gallon, of filling a 0.22 gallon can, two-thirds avoided multiplication "because you've got a lesser amount. It's under $1.20, so obviously it's 1.20/0.22 or something like that." (4, p. 405)

Because of the differences between ordinary language and mathematical language cited above, researchers have generally shied away from using ordinary reading tests to measure the difficulty of reading mathematical language. Teachers and others who evaluate textbooks should be just as cautious with statistics that presumably reflect the reading level of a particular textbook. If the reading gauge applied is general in nature, it may be inappropriate for that particular mathematics textbook.

On the other hand, there is textbook reading research that can be helpful. Earp and Tanner conducted a study with a 6th grade textbook, but their results are likely to be relevant to reading mathematics in higher grades. (10) They first counted all the words in the text that could be classified as "mathematical words"—that is, words used in a technical way, such as "average," "commutative," "quadrilateral." There were almost 200 such words in the text, and the researchers' interviews indicated that there was only a 50 percent accuracy in the students' comprehension of the mathematical words. When the students were shown the words in context, however, their comprehension increased. The first context was in the form of sentences from the text ("Some customary units for measuring *volume* are the cubic inch, the cubic foot, and the cubic yard."), and the students' overall comprehension accuracy increased by 8 percent. When sentences provided stronger contexts ("*Volume* is a way of telling about the amount of space in something such as a box or container."), the accuracy increased by another 15 percent. There seem to be two implications for teachers, faced with explaining the meaning of common mathematical terms in their text, students may be inaccurate on many of them, second, their accuracy can improve considerably if they are allowed to discuss the definitions of mathematical terms and to consider them in context.

Cohen and Stover conducted a three-part study of obstacles to students' comprehension of word problems. The research focused primarily on sixth-graders, but the results have implications beyond the sixth grade. (8)

In the first part of the study the researchers asked a group of gifted sixth- and eighth-graders to review typical textbook word problems and to rewrite them (in the words directed to the students) to "make them easier for other students who have trouble with math." From the rewritten problems Cohen and Stover isolated three format variables from among those that dominated the attention of the student reviewers: the absence of a diagram; the presence of extraneous information; and the presentation of numbers in an order other than the order in which the numbers are computed.

In the second part of the study, the researchers tested two groups of average sixth-graders to assess the influence of the three variables. One group worked on problems like type A below and the other group worked on problems like type B When they compared the two groups the researchers concluded that the three variables did indeed affect the difficulty of word problems for average students

a. *the absence of a diagram*
   A. (no diagram) In Amy's class, 8 students have brown eyes. This is 25% of all the students in the class. How many students are in the class?
   B. (diagram) Same problem with 

b. *the presence of extraneous information*
   A. (extraneous information) Mr. Hopkins' total commission for the month of September was $216, of which he gave $108 to his son. $81 of the commission came from the sale of two color televisions and one short wave radio. What percent of the total commission was the commission from the sale of this electronic equipment?
   B. (no extraneous information) $81 is what percent of $216?

c. *the presentation of numbers in the word problem in an order other than that required for the appropriate computational solution*
   A. (non-matching order) The Kant family has driven 270 miles since they started their trip. The whole trip is 583 miles long. How many miles do they have left to go?
   B. (matching order) The Kant family is driving on a 583-mile trip. They have driven 270 miles since they started. How many miles do they have left to go?

In the third and final part of the study Cohen and Stover showed that students can be trained to adjust word problems to decrease the difficulty presented by the three variables. In the words of the researchers: "Instruction consisted simply of alerting students to the fact that they should check to see if a word problem could be diagrammed, or if extraneous information could be extracted, or if numbers needed to be reordered in order to fit the algorithm required to solve the problem This was then followed by drills in which each treatment group practiced the modification . . . That training lasted only three class periods; the differences between experimentals and controls were, nevertheless, substantial." (8, pp. 194-95) This suggests one clear implication for secondary school: students generally can benefit from discussions and training that help them to develop a skill in sorting out incoming data and for converting the data into meaningful information

Diagramming as a factor in solving problems appeared in the study by Bell and his colleagues. In the first part of the study, the researchers interviewed students between the ages of 12 and 16 and reported. "All of the pupils who were interviewed were completely unfamiliar with the notion of using an abstract diagram to enable them to decide which particular arithmetic operation is appropriate." (4, p. 407) When the students were provided with diagrams, however, the researchers reported that the diagrams enabled the students to estimate solutions and that they frequently led to a possible strategy for solving a particular problem, often one that was not derived from a standard algorithm For example, diagrams often led students to choose repeated addition in preference to multiplication

As several studies have noted, it is not only the weaker students who suffer because of the special and often unfamilar demands of reading and writing mathematics. In his summary of three surveys in mathematics education (11), Fey quoted a teacher interviewed in one of the surveys. "There is abundant evidence to show that we are encouraging superficial learning in some of our (best students) Sure, they do well on the tests. Our materials on hand encourage this. The algebra book, for instance, is pure abstraction. The really good memorizer can go right through and not really have it at all." (11, p. 498) The validity of this teacher's suspicions has been established in a recent study by Clement *et al.* (7) They asked college students to do the following problem:

> Write an equation for the following statement. "There are six times as many students as professors at this university." Use S for the number of students and P for the number of professors.

On a written test with 150 calculus students, 37 percent missed the problem Among 47 nonscience majors taking college algebra, the error rate was 57 percent. The majority of students who had responded incorrectly had written $6S = P$, instead of $6P = S$, and the researchers used interviews to determine the source of this reversal. They found two sources. Some students followed "word-order matching," a literal, direct mapping of the words of English into the symbols of algebra. For example, since "professors" follows "students" which follows the number 6 in the problem, the equation becomes $6S = P$. Another group of students appeared to know that there were more students than professors, but still wrote $6S = P$. In the words of the researchers who interviewed them. "Apparently the expression '6S' is used to indicate the larger group and 'P' to indicate the smaller group. The letter S is not understood as a variable that represents the number of students but rather is treated like a label or unit attached to the *number* 6." (7, p. 288) Like the teacher quoted in the Fey summary, Clement and his colleagues find some fault in secondary mathematics textbooks. They even point out that some popular secondary textbooks explicitly instruct students to translate word problems into equations by the often misguided word-order matching Instead, these researchers say, secondary students need more training in translating reliably "between algebra and other symbol systems, such as English, data tables, and pictures." (7, p. 289)

Kieran's research has highlighted another common misconception in students' experience of algebra, one which seems to arise from a mistranslation between real world experiences and the use of mathematical symbols. Kieran calls the conceptual scheme responsible for the misconception the "redistribution scheme", and it is based on the notion that "taking something off one number and adding it on to another does not change anything" (18, p. 7), a notion that causes the following error: $37 + x = 168$ becomes $47 + x = 158$.

Kieran suspects that the scheme can be traced back to a real-life redistribution scheme practiced by small children: "We can envision the following scenario: Three children dipping into a bag of candy and pulling out 5, 3, and 4 candies respectively. One child (perhaps the one who pulled out three) suggests that the child with 5 candies give away one of his to the child with only three. Then the candies become more evenly distributed. In one sense, nothing has changed; the total number of candies has remained the same." (18, p. 16) In other words, taking one number off and adding it to another has not changed anything.

Communicating through Speaking & Listening

Because their roots are in reading research, textbook research, or paper-pencil testing, must of the studies discussed above have focused primarily on the written word or symbol. These results indicate that translation skills should be a significant part of a student's training in secondary level mathematics. Therefore, listening and speaking should be as much part of that training as reading and writing. As Bauersfeld describes it: "Teaching and learning mathematics is realized through *human interaction*. It is a kind of mutual influencing, an interdependence of the actions of both teacher and student on many levels. It is not a unilateral sender-receiver relation . . . The student's reconstruction of meaning is a construction via social negotiation about what is meant and about which performance of meaning gets the teacher's (or the peer's) sanction." (3, p. 25)

One aspect of communicating mathematics where this social interaction is importan is logical reasoning. In their study of seventh graders' logical reasoning skills, Gregory and Osborne found a high correlation between the frequency of teachers' use of conditional reasoning (e.g., "If . . . then" sentences) and the conditional reasoning skills of the students. (12) Generally, the interplay of logic and language—as in the use of "some," "all," "neither," "nor"—is a vulnerable area for adolescents and results in confusion as well as frequent misuse. They need modelling from teachers as well as ample opportunities to use logic and language. Bye reports a study in which high school students were shown a variety of shapes—circles, squares, and triangles—in two sizes and three colors, each labelled with a letter. The students were given the following task: "Write the letters of all the shapes that are neither small and red nor big and green." Eighty percent of tenth-graders and sixty-five percent of students in grades eleven and twelve were unable to complete the task correctly. (5)

Because their modelling is so important in communicating mathematics, teachers should cultivate consistency and an awareness of their own patterns of language. A term like $x^2 + b$ can be read as "x squared plus b", "the square of x plus b", "b added to x squared", "b added to the square of x." Although a teacher's inconsistent usage of these phrases without explanation could confuse students, a discussion of the equivalence of such phrases can help the students' mathematical communication skills.

Conclusion

The interplay of language with mathematics is a subject requiring much more research. A comprehensive bibliography of the research done so far appears in "Language and Mathematical Education" by Austin and Howson. (1)

Future research may highlight critical aspects of communicating mathematics, but it will not lessen teachers' responsibilities. Because students can develop deep yet surprisingly hidden misconceptions about mathematics, guiding students to articulate their experience of mathematics and listening carefully to them will always be a major responsibility of mathematics teachers. Because textbooks fall very short of guiding students to translate between symbolic mathematics and other systems such as English, data tables, and pictures, teachers must bear the major responsibility for helping students to develop these translation skills Finally, because students model much of their behavior in communicating mathematics on what they experience in classroom interactions, mathematics teachers must be ever alert to their own patterns in mathematical translation and communication.

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BREAKING VICIOUS CYCLES: REMEDIATION IN SECONDARY SCHOOL MATHEMATICS

I am confused about teaching mathematics to high school students with fifth- and sixth-grade skills. Sometimes I think we expect too much of them in the mathematics classroom; at other times, I think we expect too little of them. Is there any research information about teaching these needy students?

The dictionary's definition of "remedial" is clear and simple: "intended to correct or improve one's skill in a specified area." As it applies to the learning of mathematics, however, the definition is incomplete. It implies that *mathematical skills* may be all that need correction or improvement, and for a large number of secondary school students, that is an oversimplified prescription. They need to correct and improve their skills, but their needs for correction run deeper, to the levels of *understanding concepts* and *approaches to the learning of mathematics*.

Unfortunately, some secondary school remedial programs neglect the deeper levels and target only the skill level, and even then they usually consider only those skills that can be measured easily through standardized testing. Yet research has been clear in its implications about students' needs in mathematics: conceptual misunderstandings and skewed approaches to learning mathematics are so common among teenagers that instruction which ignores them can only be partially successful in the long run.

Because most of the other chapters in this book address various student mistakes and misconceptions, the theme of remediation runs through the entire book. The purpose of this chapter is to bring together the results and recommendations from those other chapters in a way that may highlight the threads that bind them together and the recommendations for instruction that follow from them. Consequently, the chapter is organized in the following way. First, topic by topic, we review some of the major areas of need for students as the other chapters have described them. Second, some themes that bind these needs together will be identified. As Confrey has pointed out, a student's performance in mathematics has two aspects: a private one where comprehension resides, and a public one where performance is judged. (7) Those teachers who are interested in remediation must probe both aspects; therefore, the third section lists major recommendations for instruction that can touch both the public and private aspects of learning mathematics.

Areas of Need

Solving problems. By the time they reach high school, most students compute well enough with whole numbers so that the need for remediation in this area is not as pronounced as it is for other topics. However, when it comes to *using* computational algorithms to slve problems—even one-step word problems—many secondary students need considerable help. The help they need is often in the strategies used, not in finding the correct answer. Booth found that many British students, at all secondary levels, avoid using the four operations for whole-number problems whenever they can. Instead, they rely on "child-methods" such as counting, when computational algorithms for the four operations would serve them better. Such child-methods often lead to correct answers when whole-number problems are involved and counting is possible, but students' lack of understanding of the uses of the four operations hurts them when the situation changes, for example, when fractions are brought into play. Without a sense of the meaning of addition and without any apparent recourse to a strategy like counting, many students become lost with exercises like $\frac{2}{3} + \frac{1}{2} = ?$. They often draw a wrong analogy with whole-number addition, which they have learned but whose meaning they have never fully grasped, and they add numerators and denominators: $\frac{2}{3} + \frac{1}{2} = \frac{3}{5}$. (2) In order to learn the full range of remedial needs common to the topic of problem solving, teachers should read the chapter "Problem Solving. The Life Force of Mathematics Instruction, Part One." In brief, several researchers have described how they perceive weak problem solvers differing from strong problem solvers and we repeat their lists.

In Whimbey's summary, weak problem solvers stand out in the following ways:

1. They fail to observe and use all the relevant facts of a problem.
2. They fail to approach problems in a systematic, step-by-step manner. They make illogical leaps, jumping to conclusions without checking them.
3. They fail to spell out fully any relationships within a particular problem
4. They are sloppy and inaccurate in collecting information and in carrying out mental activities. (23, 31)

Confrey, Lanier, and their colleagues have conducted a study of ninth-grade general mathematics courses. (8, 21) One facet of the study concerned the abilities that the Russian researcher Krutetskii has associated with good mathematics students:

1. *Information gathering*. the ability to discern the mathematical structure in a given problem.
2. *Generalization*. the ability to place a particular case under a known general concept or to see something general from particular cases—that is, to form a concept.
3. *Reversibility*. the ability to change from one train of thought to its reverse, to reverse mathematical processes, such as inverse operations (e.g., multiplication/division, addition/subtraction), and direct and converse theorems.

4. *Flexibility*, the ability to accept a variety of methods and to develop ease and efficiency with them.
5. *Curtailment:* the ability to shorten mathematical processes—for example, noting the cancellation possibilities in ⅔ x ⅘ x ¼ and concluding quickly that the product is ⅖. (8, 18)

The researchers studied general mathematics students in the light of the five abilities and they found the following patterns common to the students:

1. *Information gathering.* When general mathematics students were given problems with essential information missing, problems with superfluous information, or problems with no questions attached at all, they would frequently begin to calculate wildly, in any way possible using the available numbers. Often they didn't even notice that a particular problem had no question.
2. *Generalization.* Irrelevent variables—for example, the position of a triangle on the problem paper—often distracted general mathematics students.
3. *Reversibility.* Given 17 x 13 = 221 and asked to find 221 ÷ 13 = ?, many general mathematics students attempted the entire calculation rather than simply reversing and immediately noting that the answer must be 17.
4. *Flexibility.* Shown several methods for solving problems, many of the students could not keep the different methods straight and, in fact, concentrating on a second method often hindered their reconstruction of the first method.
5. *Curtailment.* Many of the general mathematics students observed and interviewed in the study were unable or unwilling to shorten the logical progression of steps required for solving a problem and, when they did shorten a logical progression successfully, they were often unable to reconstruct the steps they had used. (8)

Fractions. One of the critical areas of need for many secondary students is fractions, which is the topic of a separate chapter: "Understanding Fractions: A Prerequisite for Success in Secondary School Mathematics." According to the National Assessment of Educational Progress (NAEP), only about 40 percent of seventeen-year-olds have mastered basic computation with fractions, with particular trouble occurring in the exercises involving unlike denominators and mixed fractions. (3) Even when computation with fractions is done correctly, however, it is often done with little understanding, as evidenced by the poor student performance on NAEP exercises involving the estimation of the sum of two fractions.

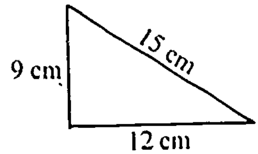
The variety of errors made by students in computing with fractions makes it a particularly knotty topic for remedial instruction. Lankford documented 22 *different* errors students make in figuring out ¾ − ½ = ? (22). Furthermore, once the type of error has been identified, it is also important to trace the source. For example, Vinner *et al.* presented to students between the ages of 13 and 15 several straightforward fraction addition problems such as $\frac{1}{2} + \frac{2}{3} = ?$. (29) Even among those who made the mistake of adding numerators and adding denominators, the researchers were able to isolate several sources, including a wrong analogy with fraction multiplication ("If $\frac{1}{2} \times \frac{2}{3} = \frac{1 \times 2}{2 \times 3} = \frac{2}{6}$, then probably $\frac{1}{2} + \frac{2}{3} = \frac{1 + 2}{2 + 3} = \frac{3}{5}$ and a wrong interpretation of symbols (no real meaning is attached to the fraction line between numerator and denominator, so since addition is called for, students figure they might as well add the things that can be paired together, namely, numerators with numerators, denominators with denominators).

Decimals. The NAEP assessment revealed that about than 60 percent of seventeen-year-olds cannot identify .625 as the decimal equivalent of $\frac{5}{8}$. Thus, fraction-decimal equivalency is a topic where many secondary students need remedial help, as is the topic of decimal place value. Bell and his colleagues conducted a study among less able fourteen-year-old British students to identify, then to remediate, common mistakes and misconceptions about decimals. (1) Their interviews revealed place-value misconceptions such as ".45 hours is 45 minutes" and "0.8 . . . that's about an eighth."

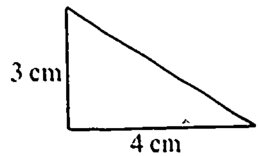
Another rather deep misconception identified by Bell et al. was the conviction that, no matter the numbers being multiplied, "multiplication gives an answer bigger than either of the numbers being multiplied." This misconception tripped up many students on problems like "If gasoline is \$1.20 a gallon, what is the cost of filling a 0.22 gallon can?" Even students who had recognized multiplication as the route to solution when the capacity of the can was 8.6 gallons, were thrown off by getting a price smaller than \$1.20 when they multiplied \$1.20x0.22. They were inclined to pull back from multiplication and to look for a way to produce a price bigger than \$1.20. The research team designed some remedial instruction aimed at these decimal difficulties, we will discuss their strategies later in this chapter.

Percents. According to the NAEP, the overall performance of secondary students on percent exercises was extremely low. (3) In particular, only about half of seventeen-year-olds responded correctly to basic concept exercises like, "Express 9/100 as a percent," and only about a third of seventeen-year-olds were successful on exercises involving any sort of operation with, or application of, percents - for example, "What is 4 percent of 75?"

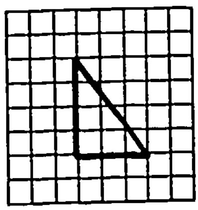
Measurement. According to the researchers who reported the results of the National Assessment of Educational Progress in mathematics, "Performance on perimeter, area, and volume exercises was among the poorest of any content area on the assessment." (3, p. 98) The frequency and severity of measurement mistakes and misunderstandings is clear from the following NAEP result. less than 20 percent of seventeen-year-olds were successful in finding the area of the right triangle:



Booth reported that a disappointingly small percentage of British teenagers were successful on a similar exercise, but he sensed the influence of the child-method of counting on many students' approaches to area problems and, in fact, to all measurement problems. (2) Whereas less than 50 percent of fourteen-year-olds were able to find the area of



during interviews, nearly twice as many found the area of a similar triangle when it was presented on a grid, because, Booth claims, counting of units was possible



Geometry. For a more thorough treatment of the facets of geometry where students struggle most frequently, readers should refer to the chapters "The Path to Formal Proof" and "The Learning and Teaching of Geometry." Perhaps the area where need shows up most clearly is proof, and though proof is not a topic to which the word "remediation" is usually attached, it is a topic where many students' skills are in dire need of correction and improvement. The study by Senk and Usiskin showed that even among students who have had a year of high school geometry, only about half can do more than simple geometric proofs. (25, 28) Worse yet, their study showed that more than a third of students who enter high school geometry courses do so without the appropriate prerequisite knowledge and skills, such as knowledge of the various properties of geometric figures ("The sum of the angles of any triangle equals 180°") and the interrelationships among geometric figures ("Any square is a rectangle, but not all rectangles are squares.")

Algebra. Again, readers who want a more detailed description of student difficulties with algebra should read the chapter "The Learning and Teaching of Algebra." Briefly, the difficulties that seem the deepest and most resistant to change are students' understanding of what an equation is and what a variable is. Kieran's research has led her to conclude that many students enter algebra without an appropriate bridge with arithmetic, and since arithmetic equations are action statements in which numbers are combined to produce an answer, the same conception takes root in algebra. (13, 17) What these students miss is the aspect of *equivalence* in the use of equations, and so fall into numerous traps as they struggle through beginning algebra. For example, a common mistake arising from overemphasizing the action aspect of equations is to approach an equation like $3x + 2 = 14 + 12$ by marching through from left to right and solving $3x + 2 = 14$ as $x = 4$, and ignoring the influence of $+ 12$ on the statement of equivalence.

The work of Wagner, the work of Hart, and the work of Clement, Lochhead and their colleagues have illustrated the variety of difficulties that can arise for students if their understanding of the meaning of variable is at all skewed (6, 12, 24, 30) For example, there are many students for whom variables in equations are labels for objects, rather than number representatives. Under the influence of that misconception, a statement like, "There are 10 times as many words (W) as sentences (S)," is often translated into the equation $10W = S$, rather than the correct $10S = W$.

Behind the Mistakes

By the time students reach high school, they have behind them enough years of mathematics to have strong and hardened impressions about the subject what it is about, how it works, why it is taught, and how it should be learned. For many students, those impressions are far from conducive to good mathematics learning. As Confrey found out in her student interviews, it is not uncommon for students to maintain that rules such as those for lining up decimal points in addition and for "counting in" decimal points in multiplication could just as easily be reversed. (7)

Thus, the mistakes that require remediation in secondary school mathematics are often the outgrowths of impoverished impressions of mathematics. Furthermore, as Lochhead points out, older students who are slow in learning mathematics are probably stuck with poor mathematical learning skills. (23) Therefore, in order for an attempt at remediation to have any hope for success, it must deal with *false impressions* and *poor learning skills* as well as with the *mistakes* they engender. The task is a difficult one, graphically described by Lochhead. "Poor learning and thinking habits can perpetuate misconceptions about mathematics These misconceptions can in turn act to discourage careful thinking by making it appear unrewarding. The task faced by the teacher of remedial mathematics is to break this vicious cycle." (23, p.3) Whimbey's list, cited earlier in this chapter, encapsulates some of those poor learning skills. inefficient observation and use of relevant facts, sloppy and inaccurate collecting of information, a failure to proceed through solutions in a step-by-step fashion, settling instead on illogical leaps to conclusion without checking.

Lankford noted a special case of the last phenomenon when he interviewed seventh-graders to determine the major behavioral differences between students who are good at computation and students who are poor at it. He found that the poor computers he observed often switched to something else that would produce an answer whenever they ran into difficulties using a computational procedure. Frequently, their chosen procedure was remote from the proper procedure, but getting an answer seemed the dominant goal. (22)

The belief that the *primary aim in mathematics is to get answers* is widespread among mathematics students, even among those who are relatively successful mathematics learners. It is especially common among the less successful, however, and is an imposing obstacle to effective remediation. In their study of ninth-grade general mathematics students, Confrey and Lanier reported: "We found evidence of this focus on the answer in students' pursuit of the problems in information gathering, in their lack of flexibility and subsequent preference for a single method, in their quick, but local generalizations and in their erroneous curtailment." (8, p. 554)

A related theme identified by Confrey and Lanier was *symbolic manipulation:* "Mathematics seemed to be a set of symbols to be operated on within rules and if those rules failed to fit perfectly or errors were made, then the student was left with no recourse, except to perhaps try to manipulate the numbers more, and hope some answer would come out even." (8, p. 555) As reported in the chapter "Individual Differences Among Mathematics Learners," research has shown that many students *at all* secondary levels are still at the developmental level where learning must be facilitated by concrete and pictorial representations of concepts. What Confrey and Lanier have observed is the nightmare that can arise when those representations are not a consistent part of classroom instruction for the students who require them.

Remedial Teaching Strategies

Hart reported on the extensive British study, similar in purpose and scope to NAEP, called Concepts in Secondary Mathematics and Science (CSMS). The subjects were students between the ages of 11 and 16, and one of the more striking conclusions was the widespread need for concrete and pictorial, as well as symbolic, representations of concepts. Hart wrote: "It is *impossible* to present abstract mathematics to all types of children and expect them to get something out of it. It is much more likely that half the class will ignore what is being said because the base on which the abstraction can be built does not exist." (12, p. 210)

Several studies have shown the effectiveness of using concrete and pictorial representations to teach adolescents fractions and concepts related to fractions. Karplus, Kurtz, and their colleagues have conducted a series of studies of proportional reasoning. Two conclusions from their work are:

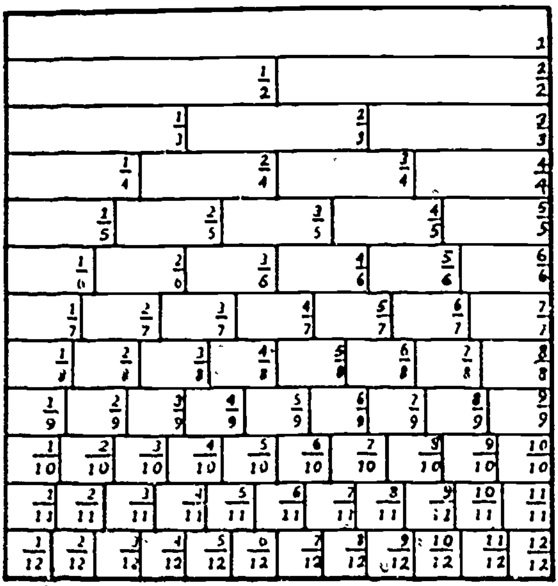
1. Adolescents' progress with proportional reasoning skills does not hinge on cognitive development alone. Often their ignorance of or inability to use

fractions stands in their way also. (16)

2. Using concrete aids to measure proportional amounts, many adolescents can improve their proportional reasoning skills significantly, and in a way that leaves them more motivated to learn such skills than are students who approach proportional learning without concrete aids. (20)

Dees worked with remedial students in grades 10, 11, and 12. She divided the students into two groups and administered two tests to both groups on fraction concepts and skills. equivalence and comparison of fractions, area and number line models of fractions, and addition and subtraction of fractions. One of the tests was written, the other involved only concrete or manipulable tasks. One group took the concrete test first, then the written test, the other group took the tests in reverse order. Students who took the concrete test first were more successful on the written test than the other group, implying that learning probably occurred during the administration of the concrete test. (10)

In a remedial program developed from their research, Hershkowitz *et al.* used the following chart, with success, when teaching students how to expand, compare and add fractions. They suggest that its use encourages students to see fractions as quantities, and not as separate whole numbers paired together. (14)



As described in the chapter "The Learning and Teaching of Algebra," Kieran and Herscovics determined that many difficulties encountered by students in algebra result from their having made some false generalizations from, and weak bridges to, arithmetic. Since the focus of the use of the equal sign in arithmetic is to, describe action resulting from numerical computations, it is not always a straightforward adjustment for students to deal with equations as equivalence relations. Kieran and Herscovics report success using a carefully planned instructional sequence that bridges arithmetic and algebra. At one critical juncture, when the students are able to construct arithmetic identities like $5 \times 5 = 30 - 5$, they are asked to consider identities with a number covered up by a finger:

 $+ 6 - 2 = 10 + 8 - 5$

Later, more pictorial and symbolic representations of equations are used, but the researchers are convinced that it is essential to lead in to the later representations with this more concrete representation. (13, 17)

Also reported in the algebra chapter is the finding by Rosnick and Clement that, by the time many students reach college, they have swung too far the other way in their understanding of the meaning of equations. By then they have lost much of the sense of action in equations, seeing them instead as static comparisons of labels. Thus $6S = P$ is read as "There are six times as many S's as P's," rather than the mathematically correct "P is the number equal to 6 times S" or "There are six times as many P's as S's." On the basis of their research Rosnick and Clement make the following two recommendations:

1. Teachers should emphasize that variables stand for number. They must be consistent in this emphasis, even watching that they set up equations with statements like "Let A stand for the number of apples," and not "Let A stand for apples."
2. Teachers should watch for an inclination on the part of students to view equations in a static way and to emphasize to them that equations represent active operations on variables that create an equality. (24)

Threadgill-Sowder and Juilf conducted a study in which they compared concrete versus symbolic materials in teaching logical connectives like "or", "and", and "not" to 7th-graders. The concrete materials included attribute blocks of various shapes. The instruction took three class sessions and, when it was complete, a test on logical connectives was taken by the students. Among the lower achievers in the group, those who learned through the use of the concrete materials did better on the test than the group of low achievers who learned through the symbolic approach. (26) Suggestions for activities using attribute blocks and pictorial methods to teach logical connectives to junior high students can be found in the book *Activities for TOPS, A Program in the Teaching of Problem Solving* (4)

Many secondary students require remediation for their arithmetic skills. Kuhm has compiled a collection of suggestions for remediating the arithmetic skills of ninth-grade general mathematics students. (19)

According to the study by Confrey and Lanier, remedial success in the general mathematics classroom may require some careful self-examination on the part of general mathematics teachers. In particular, when the researchers observed teachers who taught both general mathematics and algebra, and compared their behavior in each type of class, they found that they tend to act differently from one class to the other:

1. They give the general mathematics students much less direct instruction (In light of the evidence, described in the chapter "Effective Mathematics Teaching", that successful junior high mathematics teachers spend more time in class discussion and lecture, this phenomenon in general mathematics classrooms needs to be changed.)
2. Though general mathematics students spend most of their class time doing homework and individual problems, they get less assistance with their seatwork than do their counterparts in algebra. They also get less encouragement and less opportunity for discussion. (21)

Lanier has pointed out that general mathematics classes lack the "ripple effect" seen in algebra classes, where a core group of students often grasp content quickly, respond to the teacher at critical moments, and, in doing so, help to communicate the content to other students. (21)

According to Lochhead, successful remedial programs must be quite different from the typical general mathematics program described by Lanier In such programs, "students must be shaken out of the memorize-regurgitate cycle," they must "place major emphasis on getting students to think actively," and their students "must learn to discover their own approaches to solving simple problems before they can appreciate the more elegant designs of others " (23, p. 14)

The kind of student-centered, process-enriched approach to remedial teaching that Lochhead recommends was at the center of an experiment in instruction carried out in the Calgary Junior High School Mathematics Project Although its purpose was not remediation, its design could be adapted and, since the focus was the difficult topic of fractions, remedial teachers should take note of the results.

The objective of the eleven and one-half week program was to facilitate and enrich seventh-graders' learning of fractions through mathematical investigations. In particular, the students experimented with concrete materials, recorded what happened in the experiments, formulated questions, wrote up accounts of the experimental results, and applied the results to practical situations Not only did the experimental group's achievement improve significantly when they were compared with a group of students learning from a regular textbook, but they also displayed significantly greater enjoyment of fractions than the students in the regular group. Furthermore, the researchers noted a significant improvement in the experimental students' ability to give explanations, probably due to their regular recording of experimental results. (11)

All of the research cited so far leads to the conclusion that educators must design remedial programs in which students are taught to think, to experiment, and to discuss. These are not simple goals, and teachers should take advantage of every proven instructional aid they can. Concrete materials have proven their worth in remedial instruction and so, in recent years, have handheld calculators

In their study of student difficulties with decimals, Bell *et al.* designed some calculator-based instruction to remediate two of the most glaring difficulties: a lack of understanding of decimal place value and a deep-rooted conviction that "multiplication always makes bigger." (1) They were able to produce significant improvement in the students' understanding of place value by involving them in exercises like the game called Getting Closer, played in pairs, in which one student chooses a low number, the other a high number, and each puts his or her number on a calculator screen. The students then take turns, with the first adding any *non-whole* number each time to the number on the screen while the second student, with the higher number, subtracts non-whole numbers. Thus, the numbers on the two calculator screens approach each other. The first player to pass the number currently on the other player's screen is the loser. The players learn quickly that a knowledge of place value is an important advantage when the two numbers are close to each other.

There was also improvement, though not as significant, in students' understanding of the effects of multiplication. One of the teaching strategies was to involve the students in a game called Target. Here are the rules.

1. Player 1 enters any number onto the calculator.
2. Player 2 has to *multiply* this by another number so that the answer will be as near to the target number, 100, as possible.
3. Player 1 then *multiplies* this new answer, trying to get still *nearer to 100.*
4. The players take turns until one player "hits" the target by getting any number between 100 and 101 on the calculator display.

Creswell and Vaughn designed calculator materials to teach decimals and percents, over an eight-week period, to ninth-grade general mathematics students. On a posttest measuring the level of achievement over the eight weeks, the calculator group scored significantly higher than a group working during the same period with a standard textbook. (9)

Ninth-grade general mathematics was also the focus of Toole's calculator study For six months, students were taught as usual, with the exception of one day a week when they used calculators. In the six months between pretests and posttests, the students gained eight months more than a similar group who used no calculators. The breakdown into subtest gains was as follows: 7-month gain in computation, 5-month gain in concepts, 1-year gain in applications. (27)

Microcomputers also promise help in remediation Howe and his colleagues conducted a study of the effects of integrating the teaching of programming in the LOGO language into a remedial mathematics program for middle-school students. The researchers supplemented the students' normal quota of mathematics work with microcomputer work one hour per week. Programming in LOGO was taught during the first year; mathematical applications of LOGO were taught the second year. At the end of the experiment, the LOGO students were marginally better than their non-computer peers in algebra topics like solving for x and forming equations. Considering the growing evidence of the importance of student awareness, thinking, and discussion in remedial classrooms, perhaps the most significant result of the study was the observed greater propensity of the experimental students to discuss mathematical issues and to explain their own mathematical difficulties. (15)

Conclusion

Research that touches on remediation in secondary school mathematics leads to one overriding conclusion. in order to correct and improve students' mathematical learning, it is not enough to concentrate an isolated mistakes or on isolated skills.

Short-term efforts produce, at best, short-term effects, and we mathematics educators must aim for effects that last longer. The teaching that will produce those effects must include careful observation and diagnosis of the sources of mathematical difficulties and efforts to change remedial students' thinking skills and their ways of approaching mathematics, as well as efforts to remediate their skills in finding correct mathematical answers.

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PROBLEM SOLVING: THE LIFE FORCE OF MATHEMATICS INSTRUCTION

PART ONE

Many of my students make quick attempts to solve problems. Whenever the attempts fail, they just as quickly give up on solving the problems. What are some of the underlying reasons for this tendency? More generally, what sets apart persistent and successful problem solvers from unsuccessful problem solvers?

Problem solving is the direction toward which all mathematics instruction should point, with teachers always alert for opportunities to widen, deepen, and enrich their students' problem solving ability. Problem solving is, understandably, one of the major concerns of secondary school mathematics teachers, with close ties to all of their other curricular concerns. Thus, it is also a theme that runs through all of these *Research Within Reach* chapters, touching some topics directly, as in the case of geometric proof, and touching others indirectly, as in the discussions about estimation and remediation.

As a topic for research, problem solving has aroused intense interest among varied groups. The results, opinions, and speculations of researchers writing about problem solving fill volumes. Often the questions they have raised and the tasks they have undertaken seem far removed from the classroom For example, what role does memory play in problem solving, and how should information be organized in memory for use in problem solving? What can be learned from the artificial intelligence of sophisticated computers about the efficient organization of information for solving mathematical problems? What are the relationships between mathematical problem solving and problem solving in other areas— for example, in science?

Other researchers have studied the numerous aspects of the classroom teaching of problem solving. Ironically, the widespread interest in problem solving raises a problem for teachers. how to extract from the mass of articles, reports, and books on problem solving what is most relevant to their classroom mathematics instruction. The teacher whose question heads this report has focused on a very basic concern in problem solving instruction. students not thinking through, or even about, the mathematical problems they encounter That this condition is widespread was confirmed by the most recent National Assessment of Educational Progress (NAEP) in mathematics, which revealed that many students attempt to apply a single operation to all the numbers in any problem they confront, even numbers that are extraneous to the solution. Thus, nearly a quarter of the thirteen-year-olds tested solved the following word problem by multiplying $2 \times 5 \times 52 = 520$. "One rabbit eats 2 pounds of food each week. There are 52 weeks in a year. How much food will 5 rabbits eat in one week?" Another exercise asked the students to decide on missing information. "Maria left at noon to take a trip on her bicycle. She rode 5 miles each hour. Later that afternoon, Amanda decided to go after her. Amanda rode 10 miles each hour. What else would you need to know in order to find our how far the two girls rode before Amanda caught Maria?" More than half of the thirteen-year-olds and almost a third of the seventeen-year-olds could not identify what additional information would be needed to solve the problem. (4)

In light of the NAEP results and the allied concerns of the teachers who were interviewed for *Research Within Reach. Secondary School Mathematics*, this chapter and the one that follows have one primary focus What can secondary school teachers do to expand and enrich their students' thinking about mathematical problems? As a first step, we discuss what is known about students' thinking as they face mathematical problems.

In this discussion a "problem" refers to a situation "in which an individual or group is called upon to perform a task for which there is no readily accessible algorithm which determines completely the method of solution " (13, p 287) hus, problem solving refers to new terrain for an individual, where no immediate path to solution appears. According to this definition, a textbook word problem may or may not be a problem for a particular student, depending on whether that student has a routine procedure that can lead directly to a solution.

Students' Thinking Processes

Researchers have been able to learn much about the thinking used by problem solvers. In brief, students' success at problem solving seems to be affected by their cognitive development and by their previous experience in, and impressions of, mathematics. Successful problem solvers have much in common, but they can differ from one another in their styles and approaches to problems Finally, students can improve their problem-solving performance by attending to appropriate guidelines, in particular, Polya's four phases of good problem solving: (10, 14, 16)

1. understanding the problem
2. devising a plan
3. carrying out the plan
4. looking back at the solution

These conclusions are the results of many studies over the last couple of decades, studies that used a variety of research methods. Observation of individual problem solvers at work has been used extensively, and techniques are now sophisticated enough to allow for observation of whole groups. Interviews of problem solvers during and after problem solving sessions have formed the core of many studies. Other studies have compared different instructional approaches to problem solving, then compared the problem solving performance of students after the instruction.

Attempts to relate problem solving ability to other cognitive abilities—such as spatial ability—have yielded few firm conclusions about the nature of problem solving, except to suggest that problem-solving ability is not a single trait. That is, different mixtures of abilities are needed for different classes of problems. (1) Problem-solving ability and computational ability have been found to be related for younger children, but they are only remotely related for students of college age. (13)

Comparing the behaviors, thought processes' and strategies of successful problem solvers with those of less successful problem solvers has yielded some promising results. In recent years, this approach has grown more useful because observation techniques have been developed that can capture some of the subtleties of successful problem solving. Thirty years ago, before the techniques were developed and long before NAEP underscored the faulty thinking of unsuccessful problem solvers, Bloom and Broder conducted a study of problem solving among college students and were able to pinpoint some of the differences between good and poor problem solvers. (2) They noted some of the same phenomena that are evident from NAEP. For example, unsuccessful problem solvers spent little time considering questions but chose answers on the basis of a few clues, such as a feeling, an impression, or a guess. In contrast, good problem solvers pulled key ideas out of problems and brought relevant information to bear on the problems, poor problem solvers did not, even though they often knew the needed information. In short, good problem solvers were much more active than poor problem solvers. In a recent article discussing the Bloom-Broder study, Whimbey has suggested that there are "two major characteristics that distinguish successful from unsuccessful students, the step-by-step approach; and carefulness—the concern and quick retracking when ideas become confusing, the rechecking, reviewing, and rereading to be sure that errors haven't crept in, that nothing is overlooked." (21, p. 561)

This research led to studies of the thought processes that set successful problem solvers apart. The model for much of this work is that proposed by the Russian researcher Krutetskii, whose observations of gifted mathematics students led to his conclusion that a major difference between good and poor problem solvers lies in their perception of the important elements of problems. In particular, Krutetskii noted the following about problem perception:

1. Good problem solvers can distinguish relevant from irrelevant information in problems.
2. Good problem solvers can see quickly and accurately the mathematical structure of a problem. In fact, talented problem solvers have what Krutetskii termed a mathematical frame of mind, that is, the tendency to impose a mathematical structure on their perceptions of the world.
3. Good problem solvers can generalize across a wide range of problems. Thus, they might recognize the comparison of similar triangles as a common thread that runs through a variety of problems, and so would be inclined to look

for that thread in many geometric problems.

4. Good problem solvers can remember a problem's mathematical structure for a long time. Thus, if a good problem solver has solved, or seen solved, a problem in which two or more similar triangles are compared, and if the same problem is posed again several weeks later, he or she will be inclined to recognize quickly that similar triangles are involved (11, 13)

Though good problem solvers have characteristics in common that set them apart from less successful problem solvers, research shows that there is plenty of room for individual styles in problem-solving In fact, Krutetskii found some "very capable" students in mathematical problem solving who could work only in a symbolic mode, while other equally capable students could solve problems only through the use of diagrams and pictures. The students went to considerable lengths to use their preferred styles, even on problems where Krutetskii did not think them appropriate. (1,11) Several North American researchers have produced similar findings. For example, Silver asked eighth-graders to separate a collection of problems into categories of problems which they judged to be mathematically related. The study confirmed a relationship between students' perceptions of mathematical structure in problems and their problem-solving competence Specifically, unsuccessful problem solvers were more inclined to sort problems according to question form, contextual details, or the presence of a common concept than according to mathematical structure. For example, the two problems below are not closely related in mathematical structure. The first involves a direct application of least common multiples where time is the unknown quantity and the second involves an equation in one variable where the number of students is the unknown. Both problems do involve time, however, and Silver found that unsuccessful problem solvers were more inclined than successful problem solvers to group two such problems together as mathematically related. (18, 19)

A. Nickolai and Natashe are trained circus bears who perform their act while riding bicycles around a circus ring. Natashe can complete the circle in 4 minutes, but it takes Nickolai [illegible] minutes to make the entire trip They start at the same point, and their act is over when they again reach the starting point at the same time. How long does their act last?

B. There are 8 boys and 16 girls at an eleventh grade committee meeting Every few minutes, one boy and one girl leave together. How many boy-girl pairs must leave so that there are exactly three times as many girls as boys left at the meeting?

There are other characteristics that set successful problem solvers apart from unsuccessful problem solvers. For example, the range of strategies used in solving problems appears to be important. Webb worked with forty high school students on an individual basis, asking them to think aloud as they solved a series of problems from algebra, geometry, and analytic geometry. The interview data, matched against the students' problem solving performance, led Webb to conclude that better problem solvers use a wider range of strategies and techniques than do poorer problem solvers. (20)

Goal-Oriented Planning

Reflecting on her own research and on the research of others, Kantowski reported that goal-oriented planning is closely related to successful problem solving in areas where it has been closely studied, namely, in geometry and in number theory. Goal-oriented planning refers to several thought processes: identifying the goal of the problem, identifying intermediate goals, if the ultimate goal cannot be reached directly, setting down a plan of attack—possibly through trial-and-error, making a table, or searching for a pattern. (8)

Goal-oriented planning is akin to what is called "thinking through problems." Research confirms that it is an important part of problem solving. But how can teachers nurture goal-oriented planning among their students as well as other important parts of problem solving? Before proposing strategies, it is important to take note of some of the obstacles to problem solving which many secondary school students face.

The first obstacle to consider is cognitive development. As we discuss in the chapter "Individual Differences Among Mathematics Learners," many teenagers are slow to develop cognitively into the stage called the formal operational stage, wherein conditional thinking ("if-then" thinking) comes more easily to them and they are not forced to "center" on single thoughts or variables: that is, they can hold two or more variables in mind at the same time. Several Soviet studies have confirmed that such centering does exist even among older teenagers, and Lesh pointed to the bearing this might have on traditional classroom problem solving: "For example, persons reading a new mathematics text for the first time will center on some points and neglect others, and they will reinterpret and perhaps distort many ideas to fit their previous conceptualizations of the subject." Thus, an inclination to center might be one obstacle to students' thinking through mathematical problems. (12, p. 159)

Another possible obstacle, alluded to by Lesh, is a student's previous mathematical experience and the conceptualization of mathematics that grows from that experience. For many students this conceptualization leaves little room for thinking through problems, based as it is on memorization, regurgitation, and the conviction that the sole purpose for doing any mathematical problem is to get the right answer and, furthermore, that for each problem there is only one right way to reach the answer. Almost 50 percent of the thirteen- and seventeen-year-olds in the recent NAEP assessment agreed with the statement, "Learning mathematics is mostly memorizing;" almost 90 percent agreed with the statement, "There is always a rule to follow in solving mathematics problems." (4).

This narrow and distorted conceptualization of mathematics is widespread among secondary school students. It is a phenomenon we deal with at length in the chapters "Communicating Mathematics" and "Individual Differences Among Mathematics Learners." Lochhead also touches on it in his discussion of the

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Bloom-Broder study cited earlier. He emphasizes especially the conclusion that good problem solvers, quite simply, *do* more than poor problem solvers- more planning, more checking, more reviewing, and so on:

> The inactivity of poor problem solvers could be attributed to laziness but there is an alternative explanation. Poor problem solvers are less active because they do not believe there is anything for them to do. Their view of both problem solving and learning places them in the passive role of absorbing information and repeating it back. They think you either know the answer to a question or you do not. (15, p. 2)

Kantowski saw signs of the same phenomenon in her research. In particular, she had designed a teaching experiment that stressed several problem-solving strategies with students. One was "looking back," whereby students were encouraged, once they thought they had reached a solution to a problem, to review what they had done, both with an eye toward checking and also toward simplifying the solution, changing to a different solution, or posing a new question The students used the other strategies that had been stressed, but there was scant use of looking back. Kantowski offered a possible explanation by pointing out that most students come to expect one solution and one solution process for each problem and so see little sense in looking further. (9) As the NAEP results cited above indicate, this narrow and mechanical appreciation of mathematics is widespread among secondary level students. (4)

If working with problems becomes mechanical for students, with little understanding of underlying concepts, some fundamental misconceptions can arise and persist for a long time. In their study, Clement, *et al.* asked 150 calculus-level college students to write an equation for the following statement: "There are six times as many students as professors at this university." An appropriate answer, of course, is 6P = S, but thirty-seven percent of the students missed the problem and two-thirds of the errors took the form of a reversal of variables 6S = P Interviews of the students revealed what the researchers called a "self-generated, stable, and persistent misconception concerning the meaning of variables and equations." As a result, when the format of a problem fails to fit the mechanical processes these students have come to depend on, their skill in dealing with mathematical problems begins to crumble. (6)

The alternative to having students rely on mechanical approaches is to develop in them a variety of problem-solving processes from which they can draw, depending on what is most appropriate for particular problems. Unfortunately, the quality of problems usually encountered in the classroom tends to make this alternative less feasible than it should be. Days and his colleagues compared the problem-solving processes used by eighth-graders who are formal operational thinkers with those used by eighth-graders who are still concrete-operational thinkers. In particular, they compared the processes used on problems with simple and complex structures: (7)

Example 1. *Simple structure.* A cow and pig together cost 56 dollars. The cow cost 30 dollars more than the pig. How much does each cost?

Example 2: *Complex structure.* Jeff bought 5 oranges and 10 apples for $1.65. An apple and an orange together cost 20 cents. How much does one apple cost? How much does one orange cost?

The researchers discovered that on the complex structure problems the formal operational students used the following processes not used by the others: deductive reasoning, use of successive approximations, estimation, checking of conditions, checking of manipulations, and checking by retracing steps. On the simple structure problems, however, both groups tended not to differ in their use of processes—for example, they drew diagrams and tried to recall similar problems. The researchers made the following comment about the comparison: (7, p. 144)

> The fact that the concrete and formal subjects for the most part did not differ in process use on the simple structure problems suggests that the simple structure problems may not have evoked the use of "high level" processes. If this was the case, then many textbook problems probably fail to elicit the use of "high level" processes, also. The latter statement is based on the fact that the simple structure problems were typical of many of the problems found in seventh-and eighth-grade mathematics textbooks.

Like the popular textbooks, commercial problem-solving tests are also a long way from emphasizing appropriate problem-solving processes. In particular, Zalewski reviewed commercial tests to gauge their value in studying problem solving. (22) He found them not to be very valuable in such studies for three reasons:

a. Commercial tests overemphasize story problems.
b. Scoring focuses on correct responses only, not on the processes used by the problem solvers.
c. The tests are tied to time limits which are too short.

Several problem-solving projects have produced activities, appropriate at the middle school and junior high school levels, which are designed to encourage the use of what Days and colleagues call "high level" processes. Interested teachers at those levels can enlist the aid of the projects. (5, 17) Teachers at all levels can benefit from the suggestions and problems in the National Council Teachers of Mathematics (NCTM) Problem Solving Yearbook. (10)

Conclusion

Some differences between successful and unsuccessful problem solvers, then, are clear. Successful problem solvers are more active, use more problem-solving processes and strategies, and have a different impression of and appreciation of the experience of learning mathematics. "Part II of Problem Solving: The Life Force of Mathematics Instruction" looks more closely at the role of teachers in making these qualities available to all students.

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PROBLEM SOLVING: THE LIFE FORCE OF MATHEMATICS INSTRUCTION

PART TWO

How can teachers increase their students' ability and willingness to stick with solving problems, to think problems through, and to appreciate that alternative methods of solution do exist for most problems?

This is the second of two chapters on problem solving. The first chapter sketched a profile of the kinds of thinking involved in both good and poor problem solving. The following table lists briefly the highlights of that sketch.

Successful Problem Solvers . . .	*Unsuccessful Problem Solvers . . .*
1. *do* more re-reading, rechecking, reviewing.	1. proceed on the basis of few clues.
2. are able to pull key ideas from a problem, to distinguish between relevant and irrelevant information, and to bring relevant information to bear on a problem.	2. often know what is relevant, but even when they do, they do not bring the information to bear on solving problems.
3. exhibit goal-oriented planning—that is, they identify a solution and a plan of attach.	3. do not, as often or as well.
4. use a wide variety of problem solving processes, including estimation, recalling similar problems.	4. tend to apply one operation in the solution of a word problem to all the numbers in the problem. They perceive mathematics as primarily based on memorization.
5. perceive the mathematical structures of problems	5. tend to focus on question form or context (e.g., time problems or distance problems).
6. can remember the mathematical structure of problems.	6. cannot, as well
7. can generalize across problems, seeing mathematical threads.	7. cannot, as well.

This chapter describes what teachers can do to affect the [illegible]th and quality of their students' thinking about mathematical problems. In approaching this instructional challenge—perhaps their greatest—teachers need to be aware of two things. First of all, research is clear in concluding that students of all ages and all achievement levels can be induced to assume many of the behaviors and thought processes associated with effective problem solving, often, their problem-solving achievement scores will improve at the same time.

Second, in order to improve student problem solving, teachers must integrate problem solving with three instructional roles. they must *model* some aspects of problem solving, they must *teach* directly some aspects of problem solving; and, finally, they must *facilitate* some aspects of problem solving

Modeling Problem Solving

Teachers must *model* problem solving for their students, who should see their teachers posing problems, actively using strategies to push them through to solution, and then posing new problems that spring from the ones just solved As models, teachers need to be alert to the values they communicate to students In this regard, Lester has written, "Problem-solving instruction is most effective when students sense two things, (1) that the teacher regards problem solving as an important activity and (2) that the teacher actively engages in solving problems as a part of mathematics instruction." (17, p.43) The recent work of Lochhead and Whimbey leads to one additional value to be communicated (3) that the teacher values each student as a problem solver—that is, wants t know and *accept* the thought processes each student applies to mathematical problems, regardless of how much refinement those processes seem to need. (18,19,20)

Thus, before students can learn to be good problem solvers in the mathematics classroom, they need something more than direct instruction They need to see teachers modelling appropriate behaviors and they need to sense in their teachers appropriate attitudes about problem solving. Schoen and his colleagues conducted a study to evaluate the effectiveness of problem-solving materials they had developed for grades 5 through 8. They found some corroborating evidence about teacher attitudes, namely, that a teacher's attitude toward problem solving was related positively to the problem-solving ability of that teacher's class (26,27)

Teaching Directly

Certain aspects of problem solving are appropriate for direct teaching For example, Vos conducted a study in which he taught sixth-, seventh-, and eighth-graders three techniques to be used to organize their approach to problems: drawing a diagram, approximating and verifying, and constructing a chart He found that not only did the students use the techniques once they had been taught them, but there was also a relationship between the careful use of the three organizing techniques and success in problem solving. (32)

Such specific techniques are called *tool-skills* by researchers, and they are problem-solving prerequisites, the groundwork upon which effective problem-solving strategies can be built. Included with the three just mentioned should be the tool-skills of writing an equation, using a formula, and making numerical estimates. The results found by Vos and mentioned in related research recommend the teaching and regular reinforcement of these skills as mechanisms for students to use to attack problems in the most organized fashion possible

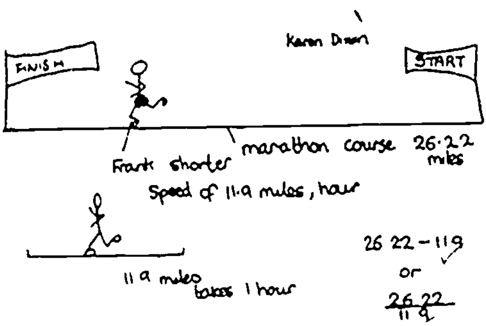
Bell and his colleagues discovered several interesting things about the strategy of drawing diagrams. They conducted a study of secondary level students' difficulties with word problems involving decimal numbers (2) In the first stage of the study, the researchers interviewed students between the ages of 12 and 16, asking them to work on a set of problems and watching for misconceptions that arose and what strategies, if any, the students applied. The last stage of the study involved the use of calculator-enriched teaching materials designed to remedy the identified misconceptions.

During the interviews the researchers encouraged the use of diagrams as aids in solving the problems. The students' skills in making appropriate diagrams were extremely limited. In fact, all of the students were "completely unfamiliar with the notion of using an abstract diagram to enable them to decide which particular arithmetic operation is appropriate." (2, p.407) Diagrams drawn by the interviewers, however, were found to be useful for three reasons:

1. They removed the words from the problem and were then able to be used as an independent, uncluttered statement of the problem.
2. They enabled the students to estimate solutions.
3. They frequently led to a possible strategy for solving the problem, but this was rarely one of the "standard" algorithms. (For example, diagrams often led pupils to choose repeated addition in preference to multiplication) (2, p. 408)

Part of the study's last phase involved training in drawing appropriate diagrams Though it proved to be a difficult strategy to use, diagramming served the students well in clarifying problems—for example, in inducing discussions concerning whether or not the operation to be performed was dependent on the numbers involved in a particular problem. The example below illustrates how one student used a diagram and a calculator to solve the problem:

A marathon is 26.22 miles long. Frank Shorter runs 11.9 miles per hour in a marathon How long does it take him to complete the marathon?



Heuristics are more direct problem-solving strategies. A common definition is the following: "A heuristic is a general suggestion or strategy, independent of subject matter, that helps problem solvers approach, understand, and/or efficiently marshal their resources in solving problems." (28, p. 315)) Some of the more commonly discussed heuristics have already been mentioned or alluded to in this report: goal-oriented planning, trial-and-error, searching memory for similar problems, searching for patterns, working backward using a known objective to construct a solution, looking back and posing a new yet related problem. In the past ten or fifteen years there have been numerous studies aimed at determining the effectiveness of teaching the use of such heuristics. Generally, the results have been positive. For example, Lucas conducted an intricate study of the effects of heuristics teaching on the problem-solving skills of college calculus students. His results, probably applicable to secondary school students as well, indicate that students who were taught heuristics regularly and in a variety of contexts, and who were reinforced in their use of heuristics, approached problems in a more organized fashion than students who were not given such training. (21) Going a bit farther, Schoenfeld has learned from his research that in order to benefit from heuristics training, students need to be taught not only *how* to use heuristics, but *when*. (28) Thus, for example, searching for a pattern is appropriate for some types of problems and not for others. As they develop pattern-searching as an approach to problem solving in their students, teachers should also discuss with them the proper contexts in which the strategy should be used.

As a heuristic, trial-and-error is popular, especially among novice problem solvers, and can be a building block for the other heuristics. Webb's research has established that trial-and-error is valuable as a process supplementing the use of equations, but that it loses its value as a problem-solving aid if it is allowed to replace the use of equations. (34) For example, word problems that involve only whole numbers, like the following problem, can often be solved through trial and error.

> Sam has a roll of five-dollar bills that still leaves him 10 dollars short of paying a 90-dollar grocery bill. How many five-dollar bills does he have?

No matter how adept students become at solving such problems through trial and error, they will probably falter quickly in the face of similar problems that are not restricted to whole numbers, unless they are skilled in setting up and using equations.

> Sam is walking 5 miles an hour in a 90-mile hike. How long has he been walking when he stops to camp 22 miles from the finish?

Furthermore, teachers need to be generally cautious about trial and error. It is useful, but it can overstay its welcome. Kantowski has stated that "without some instruction (in heuristics) students generally revert to trial-and-error in solving problems or do not attempt to solve them at all." (10, p. 13)

One other aspect of mathematical problem solving should be taught directly

Students need to learn that reading mathematical word problems is different from reading ordinary prose. Often multiple readings are required, with attention being paid to vocabulary and relationships among variables. (1)

In one recent study, Cohen and Stover asked gifted sixth-graders to identify what they thought were some characteristics of word problems that are most difficult for average mathematics students. The researchers selected three of the most frequently mentioned:

a. the absence of a diagram;

b. the presence of extraneous information;

c. the presentation of numbers in the word problem in an order other than that required for the appropriate computational solution.

In the second part of the study the researchers were able to conclude, by giving a word problem test to a group of average sixth-graders, that these three variables did indeed affect the difficulty of word problems for average students. In the third and final part of the study, Cohen and Stover showed that students can be trained to adjust word problems to decrease the difficulty represented by the three variables. In the words of the researchers:

> Instruction consisted simply of alerting students to the fact that they should check to see if a word problem could be diagrammed, or if extraneous information could be extracted, or if numbers needed to be reordered . . . This was then followed by drills in which each treatment group practiced the modification . . . That training lasted only three class periods; the differences between experimentals and controls were, nevertheless, substantial (7, pp 194-95)"

Although this study concerned sixth-graders, secondary school teachers can adapt the techniques to provide the same sort of experience in analyzing the reading of mathematics. Another suggested technique is to have students compose, and then solve, their own word problems.

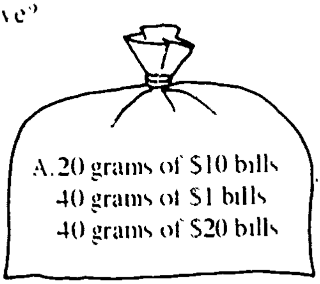
Facilitating Problem Solving

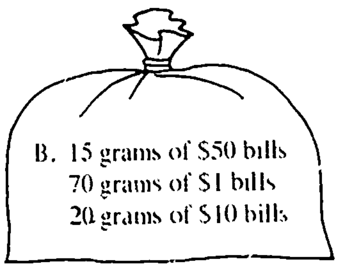
Some aspects of problem solving should not be taught directly. Rather, they must grow in students from their encounters with problems and from their classroom interactions with their teachers and peers. Goal-oriented planning is an aspect of problem solving that needs to be nurtured in this way. It must develop in problem solvers from a growing awareness in them of the nature of mathematical problems and of their own thought processes as they approach problems. The research of Lochhead and Whimbey and of others has confirmed that a student's choice of method in approaching a mathematical problem is not always conscious—indeed, often it is quite unconscious—and so awareness is critical to that student's success. (18,19,20,35)

Perhaps the greatest boost a teacher can offer students toward developing such awareness is a classroom environment where they are regularly encouraged to *verbalize* their problem-solving experiences. Putting words to one's thinking often brings that thinking to the conscious level, only then can a problem solver evaluate and refine the techniques used to solve mathematical problems In one research study, students who were asked to verbalize what strategies they had used on a practice set of problems were more successful on a subsequent set of related problems than students who had not been asked to put their strategies into words (22)

The researchers of the Iowa Problem Solving Project designed another technique to heighten students' awareness of appropriate strategies for solving word problems. (26, 27) In essense, the technique adapts Polya's four stages of problem solving (25) to a training program designed around calculators and a set of problem cards, each of which contains a problem and a set of questions The students, from grades 5 through 8, work in pairs on the cards and are expected to attend to each of the four stages in turn. It is the hope of the researchers that the technique will provide "a language whereby students can communicate what they are doing and where they are having difficulty as well as a general framework for attacking a problem " (27, p.7) Here is an example from one of the cards:

> A one-dollar bill, a ten-dollar bill, a 20-dollar bill, and a 50-dollar bill each weigh about 1 gram. Of course you would rather have 10 grams of $10 bills than 10 grams of one dollar bills Which of these two bags would you rather have?





1 Get to know the problem.	What does one 20-dollar bill weigh?
	Will your answer be a number of grams, a number of bills, or one of the bags?
2. Choose what to do	How will you find the amount for bagA?
3 Do it	Find the value of the money in each bag.
4. Look back over what was done.	Did you find that both bags contained more than $1,000?
	Write a problem similar to this one.

In a comparison between a group working with this training program and a group of students in a traditional program, the researchers found that the program

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produced an attitude toward solving word problems that was significantly more favorable than the attitude in the traditional group. (27)

Bloom and Broder had unsuccessful problem solvers work in small groups with tutors, taking turns solving problems out loud and reading the solutions of more successful problem solvers. The researchers reported that the students generally became more aware of the gaps and inadequacies in their own thinking, they read problems with more care, and they reasoned more actively and more accurately (3)

Inspired by the research of Lochhead and Whimbey, Whimbey has developed an approach to learning problem solving in which students work together in pairs, one student solving the problem out loud, the other student checking on the accuracy of the work and insisting that the first student keep verbalizing. Further research needs to be done to determine the program's effectiveness, which Whimbey maintains does *not* teach a method of problem solving, but rather develops certain attitudes, including:

1. a faith in persistent systematic analysis of problems;
2. a concern for accuracy;
3. the patience to employ a step-by-step procedure,
4. an avoidance of wild guessing,
5. a determination to become actively involved with a problem

(18, 35)

In real problem-solving situations outside the classroom, good problem solvers do not always work in isolation. Quite often, they are very good question-askers who thrive on talking through solutions to problems. Learning to ask appropriate questions about mathematical problems is not a simple task for many people. Recognizing this, Lesh has recommended small group activities in the classroom, especially for less successful students. In his words, "Many individual problem solving strategies are quite difficult for average or below-average ability youngsters. But, when these internal processes are externalized in the context of small group activities, they are often easier to describe in a form that is understandable to lower ability problem solvers." (15, p.157) Small group activities monitored by the teacher can free students from narrow, perhaps even distorted, ways of looking at a problem and help them to see the problem in a new light

Kantowski suggests another technique to facilitate deeper thinking by students about problems, posing problems with missing information, followed by questioning the students to categorize the missing information (12) In general, as the NAEP reviewers were firm in recommending, a steady diet of teacher heuristic questions can do wonders to facilitate problem solving—for example, "Can the problem be solved with the given information?"; "Have you seen a similar problem before?" (5) Teachers can benefit from the materials of three programs that have been developed to integrate the use of heuristics into problem-solving activities (6, 14, 27)

Sources of Mathematical Problems

In order to honor the suggestions given so far in this report, teachers must look beyond the boundaries of textbook word problems. Several years ago, there seemed to be few sources of such *real* or *non-routine* problems for teachers, but that situation has changed. Sources such as references 8, 9, 13, 23, 24 suggest problems, while references 16 and 13 provide listings of further sources of problems. Furthermore, there are now more guidelines to help teachers develop their own problems. For example, research has shown that students can transfer problem-solving skills, such as the use of heuristics, from one problem to an other if the two problems are at least moderately related mathematically (29). Hence, teachers should support their instruction with sequences of related problems, provided they and their students do not get caught in the trap of working only in clumps of related problems, a habit that discourages flexibility and encourages a rote approach to problem solving. Using one problem to pose questions that result in a new problem is one way to construct a sensible sequence of related problems. Interested teachers might find the work of Walter and Brown on problem posing to be a source of inspiration and an aid to developing this skill. (33)

Kantowski provides a simple example of transforming a routine textbook problem into a problem that would be non-routine for many secondary school students. The transformation from Problem 1 to Problem 2 can serve as a model for writing more advanced non-routine problems (12):

Problem 1. Maria bought a hamburger for \$.90 and a coke for \$.30. If the local sales tax is 5%, how much change should she receive if she gives the clerk \$2.00?

Problem 2. Maria has exactly \$2.00 and would like to spend it all on her lunch. The menu includes hamburgers at \$.90, hot dogs at \$.80, onion rings at \$.60, french fries at \$.50, and colas at \$.30, \$.40, or \$.50. The sales tax is 5%. What could Maria have for lunch?

Conclusion

Among mathematics teachers, nothing evokes an appreciation of the reward and the challenge of teaching as much as problem solving. When students who have done little more than memorize and imitate in their previous years of school mathematics, begin to *think* about mathematical problems, the satisfaction for teachers is enormous. While the research outlined in this chapter says clearly that such rewards are within the reach of secondary school teachers, it also leaves no doubt about the scope of the accompanying challenge. Teachers who want to improve their students' thinking about mathematical problems must employ an approach to problem solving instruction that is highly structured yet open-ended. It must be structured to provide regular teaching of problem-solving tool-skills and heuristics, the consistent modelling by the teachers of the behaviors and attitudes associated with good problem solving, and a ready access to a variety of non-routive problems. At the same time, the teachers and their classroom environment must be open to students' becoming more aware of their own thinking and open to their experimenting with that thinking in the context of mathematical problem solving.

In such a classroom environment, problem solving is valued as a *process* as well as a means to arrive at answers, and teachers should include problem-solving process with problem-solving achievement in their student evaluation. To help in this endeavor, researchers are beginning to experiment with paper-pencil instruments for describing and evaluating the processes used by students in solving problems. In a report of his recent study, Schoenfeld includes several such instruments and interested teachers can perhaps draw some guidance from them (29). The area of developing ways to evaluate problem-solving processes is an exciting one and, in light of the research of Brandau and Dossey, which shows that different problems elicit the use of different thought processes and different heuristics (5), the area is also a challenging one.

Further research should increase the excitement about problem solving. For example, researchers will build on the work of Silver (30,31), Schoenfeld (28,29) and others to clarify how previously-solved problems affect a problem solver's approach to related and unrelated problems. Novel instructional techniques, such as Whimbey's pair-problem solving, will be looked at more closely.

Finally, as we move well into the nineteen-eighties and both calculators and microcomputers become readily available in schools, technology will probably play a greater role in problem-solving research and instruction. The research studies of Bell *et al.* and Schoen *et al.*, cited in this chapter, speak well for the role of calculators. As for microcomputers, some educators envision student-computer-teacher dialogues in which the students can experiment with new problem-solving strategies, while the computer stands ready to provide hints and to remind the students of the strategy options which are available, and the teacher helps the students to integrate the newly-practiced strategies into their broader experience of mathematics. (11)

As with all of the unfinished business of problem-solving researchers, we will have to wait to see how realistic this vision is. In the meantime, it is a refreshing vision to hold onto, as are all the visions this chapter may evoke of students thinking more deeply about mathematical problems because of their experience with mathematics in the classroom.

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ESTIMATION: A PREREQUISITE FOR SUCCESS IN SECONDARY SCHOOL MATHEMATICS

I know estimation is important, but except for rounding, I don't know what to do. Is there specific instruction in estimation that should be done?

Estimation has held a rather strange place in the curriculum Although it has appeared on numerous lists of important skills, neither textbooks nor training programs have shown teachers how to teach it, what to stress, or even why estimation is so important. Furthermore, because it is difficult to capture all of the thought processes of a person who is estimating the answer to a mathematical question, there has been relatively little research done on the topic Lately, however, using research methods that make thought processes more accessible, several studies have appeared which scrutinize estimators' thinking Their results have implications for instruction.

The Importance of Estimation

Why should estimating with a sense of reasonableness be considered a prerequisite for success in secondary school mathematics? One answer is that estimation is a close cousin of problem solving, and problem solving is at the core of secondary school mathematics. In fact, Trafton suggests that estimation and mental arithmetic (mental arithmetic is exact computation done without pencil and paper) probably help students to develop problem-solving skills because they provide practice in making mathematical decisions (10) ("How far off would I be if I rounded those two numbers to the nearest tens?"; "Does this estimate take me above or keep me below the exact answer?") Paull's research study revealed a correlation between the ability to estimate answers to numerical computation and the ability to solve problems by trial and error. (5) Trial and error is a very basic yet important problem-solving strategy because teachers can use it to help students build more powerful and more efficient strategies. The presence in the classroom of estimation as well as trial-and-error procedures is a sign of a mathematically healthy environment and there should be frequent opportunities for both.Many students fail to connect their classroom mathematical experiences with mathematical experiences outside the classroom. Reys and his colleagues interviewed good estimators and found that most of them thought of estimation as a skill learned and practiced outside the classroom; mathematics classes, in their view, always demand exact answers. (6)

If, as many educators suspect, a reliance on paper-and-pencil solutions breeds thoughtless, automatic, and often mistaken computation on the part of students, it would seem likely that a regular stress on estimation and mental arithmetic could help students to break away from such thoughtlessness. While all the benefits of teaching estimation and mental arithmetic are not yet known, the available evidence does point to an influencing of students away from thoughtlessness. In his review of the relevant research, Zepp points to the improved mathematics achievement among 6th, 7th, and 8th graders that resulted from training programs in mental arithmetic. (11)

Buchanan points to four potential benefits that he sees in estimation instruction: first, it can produce a sense of reasonableness about computation; second, it can result in students having a greater appreciation for number size and the structure of the number system; third, since calculator users can never be sure when they will hit the wrong keys, or if a particular calculator is totally trustworthy, estimating can complement the use of calculators; finally, as we've noted above, it can facilitate the learning of problem-solving skills. (3)

Profile of Good Estimators

Just as profiles of good problem solvers are emerging from recent research studies, a profile of good estimators has begun to take shape from the study by Reys and his colleagues. The research team selected a group of recognized good estimators (adults, as well as students from grades 7 through 12), observed them as they worked through several sets of estimation exercises, and then interviewed them to determine their thinking processes, their strategies, their attitudes, and other characteristics that set them apart from less successful numerical estimators. (6) The study helps to clarify the nature of good estimating and has important implications for preparing students for secondary school mathematics.

First of all, the researchers isolated three key estimation processes from their observations and interviews:

1. *Translation.* By this process the estimator changes the mathematical structure or the equation into a more manageable form. Thus, an unwieldy addition problem might be more readily estimated with a different structure imposed, say multiplication:

$$\begin{array}{r} 87,419 \\ 92,765 \\ 90,045 \\ 81,974 \\ +98,102 \\ \hline \end{array} \qquad \text{is estimated as } 90,000 \times 5 = 450,000$$

2. *Reformulation.* Whereas the mathematical structure is changed through translation, reformulation changes the numerical data into a more mentally manageable form, and the structure is left alone. For example, an estimator might attack the five-number sum above by adding together the first digits of the five numbers (8 + 9 + 9 + 8 + 9 = 43), and concluding

that a reasonable estimate would be a bit more than 430,000, say 440,000 or 450,000

3. *Compensation* This is the process of making adjustments to compensate for the inaccuracies accrued through translation or reformulation The last step in the previous example adding 1 or 2 to 43 —is an example of compensation As another example, a good estimator might estimate the following sum with some compensating before the end of the problem

73,655	
86,421	"Round all of the numbers to 100,000 except the top one.
91,943	Drop this one to make up for rounding the others This leaves
96,509	the estimated answer somewhere between 500,000 and
93,421	600,000 "
106,409	

One strategy consistently used by the estimators in this study is the so-called "front-end strategy." In one of its variations it appears in the example of reformulation above, focus only on the first digits, operate on those digits, then do what compensation seems necessary to make the final estimate reasonable Many of the people in the study who used the front-end strategy both regularly and well could not recall having been taught the strategy in school.

Another strategy which the study identified and which teachers should note was the use of compatible numbers, or what some students referred to as "nice" numbers This is particularly appropriate in estimations involving long division, fractions, and decimals. For example, faced with estimating the answer to a problem like $28\overline{)5657}$, many students would change it to $30\overline{)6000}$ and so estimate the answer as 200. In another example, students in the study were asked.

"The Thompson's dinner bill totaled $28 75 Mr. Thompson wants to leave a tip of about 15% About how much should he leave for the tip?"

Among the students in grades 7-10 who converted the problem to a fraction approximation, those who changed 15% to $\frac{1}{7}$ were inclined to change $28.75 to the compatible $28 00, and therefore estimate the answer as $\frac{1}{7} \times \$28.00 = \$4.00$, while those who changed 15% to $\frac{1}{6}$ were inclined to change $28 75 to the compatible $30.00, and so give the still acceptable estimate of $\frac{1}{6} \times \$30.00 = \$5.00$. These students have learned that they do not always have to aim for one right answer and that a variety of strategies will allow them to stay within an acceptable range of answers.

Students learn rounding as a mechanical skill As is often the case with skills learned mechanically, mechanical rounding doesn't always serve students well in real estimating situations. A much richer and more flexible form of rounding is the use of compatible numbers noted by Reys and his colleagues Buchanan also argues against mechanical rounding, preferring rounding skills to be an extension of the concepts "between" and "closer", as in "346 is between 300 and 400, but it is closer to 300, so I can round it to 300." (3)

An important objective for teachers as they offer their students instruction and practice in estimation is that the students come to value estimates in their own right, as distinct from exact answers. When students are asked to estimate a product like 28 × 49, they often multiply, find the exact answer 1372, and then round the answer to 1400 Such an approach reduces estimation to a role inferior to exact computation, and teachers should take pains that students not develop this impression of estimation.

Reys and his colleagues were able to isolate some other characteristics of good estimators from their study. In general, good estimators are quick and accurate with paper-pencil computation and they use a variety of strategies to estimate In fact, they frequently consider several different strategies before deciding on a particular one. This calculated flexibility is essential to estimating well and, indeed, to doing all mathematics well, and it should be part of every student's instruction

Good estimators' judgment and sense of reasonableness not only apply to their estimated answers, but also to the process of estimation itself They are able to judge when an exact answer is needed and when estimated answers are sufficient or appropriate Finally, their clear sense that there are many situations in which it is all right not to strain for an exact answer leaves them less afraid to be wrong than their peers who are not as successful at estimating.

As noted in the *Research Within Reach* chapters on problem solving, problem solving research has made, and continues to make, a strong case for releasing students from the burden of thinking that mathematics is a rigid system which leaves little room for an individual's own ideas and strategies The same is true for estimation. Teachers must work to convince secondary-level students that there is room for them as individual thinkers in the mathematics classroom, that the individual stamps they put on their estimating will be prized at the highest level of classroom achievement. A good source of activities to help teachers in this effort is the book by Reys and Reys (7)

As in problem solving, however, students require regular practice Since estimating is foreign to many students, teachers should start by offering frequent opportunities for them to choose estimates from among several options, then discuss the most appropriate choice with the students, the factors that make it the most appropriate choice, and so on. (10) The front-end and compatible-number strategies noted by Reys *et al.* are examples of strategies that students need exposure to and instruction in.

Another such strategy has emerged from the study by Siegel and his colleagues. Through interviews and observations of individuals at all school levels, as well as adults, they attempted to define a flow-chart model of the process used to approach the kinds of estimation problems that begin in the physical world and end with a rough numerical solution. For example, "About how many names are there on this page of the phone book?" The researchers found that more than half of the estimators—of *all* ages—used perceptually-based strategies ("It *looks* like there are a lot of words, probably 300, on the page."), when a more reliable "decomposition" strategy was appropriate ("There are probably 100 names in a row, and there are 5 rows on the page, so I'd say there are about 500 names on the page.''). The researchers concluded that many individuals are not aware of such strategies for decomposing an estimate into a grouping of more manageable subestimates, and so could benefit from instruction and practice in them (9)

Calculators and microcomputers offer the promise of help for teachers in providing estimation instruction and practice In a recent British study, Bell and his colleagues were able to use calculators first to identify, then to remediate, some common mathematical difficulties of students between the ages of 12 and 16 (1) Among the most prominent of the identified difficulties was a lack of understanding of place value in decimal numbers, for example, the researchers' interviews produced comments like ''. . . 0.8 . . . that's about an eighth'' and ''1 07 lbs. is 1 lb. 7 ounces.'' A companion difficulty of this ignorance of place value was an inability to estimate with a sense of reasonableness.

During the teaching phase of their study the researchers were able to produce significant improvement in the students' understanding of place value by involving them in calculator exercises like the game called Getting Closer, played in pairs, in which one student chooses a low number, the other a high number, and each puts his or her number on a calculator screen. The students then take turns, with the first repeatedly adding any *non whole* number to the lower starting number while the second student subtracts similarly from the higher starting number Thus, the numbers on the two calculator screens approach each other. The first player to pass the other player's number is the loser. The players learn quickly that a knowledge of place value and skills in estimating are important advantages when the two numbers are close to each other.

Levin has pointed out that individuals differ according to their mental images of numbers, and that the microcomputer can help them to use those images in estimating—for example, in combining lengths of segments of the number line to estimate sums of numbers. He discusses several computer programs that have been developed to sharpen estimating skills using as an example, one in which students estimate numbers by shooting a ''harpoon'' at the number line or Cartesian plane. (4)

Conclusion

That estimating skills ought to be taught and practiced on a regular basis is an undeniable conclusion of all the research reviewed in this chapter Several studies have shown that successful estimation instruction need not consume much time in the classroom. For example, Schoen and his colleagues worked with students in grades 4 through 6 and showed that ''estimation in whole number computation can be taught in a short period of time.'' (8, p. 176)

In their estimation study, Bestgen and her colleagues worked with prospective elementary school teachers, giving one group weekly training and practice in estimating strategies, while another group received just weekly practice, and a third group received no training or practice in estimation as all (2) The first group the training and practice groups emerged from the ten-week study with a greater understanding of and respect for estimation than the other two. Yet this was a program that lasted a mere ten weeks. If secondary-level teachers will commit themselves to regular classroom practice and reinforcement of estimating skills, the effects on students as they advance through secondary-level mathematics will be astounding.

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THE CALCULATOR: AN ESSENTIAL TEACHING AID

How can hand-held calculators be used to enhance the learning of secondary level mathematics?

Because of the convenience and motivation it provides and because of the latitude it affords teachers in skill and concept development, the handheld calculator has moved into the front ranks as an aid in the teaching and learning of secondary school mathematics. The attitude of most mathematics teachers toward classroom calculators has changed rapidly from caution to enthusiasm. In fact, a strong case can be made from research evidence that the calculator should be an integral part of the teaching and learning of secondary school mathematics. That evidence is the subject of this chapter.

The issue of the calculator's potential harm to students' learning has never been as great among secondary school teachers as it has been among elementary school teachers. Even so, it is important to clear the air of any doubts, and to note that research has firmly established the acceptability of hand-held calculators at all school levels--elementary and secondary. Roberts looked at thirteen studies of the effects of calculator use in the secondary school mathematics classroom (21) Eight of those studies measured effects on concept attainment, nine measured effects on attitude, and eleven measured effects on computational skills. None of the studies favored non-use of calculators for any of the three categories. One study favored calculators for concept attainment, two for attitudes, and six for computational achievement. In the other studies, no significant differences showed up between calculator use and non-use.

Roberts remarked in his review that the attitude studies tended to be too short in time to gauge any significant attitude changes, and he asserted that "the learning settings in which these studies were conducted did not generally emphasize concept-formation skills." (21, p. 84) Thus, he pointed out, educators will not know the true power of calculators in the changing of attitudes about mathematics and in the learning and teaching of mathematical concepts until there are studies that take advantage of the unique capabilities of calculators and studies that measure calculator effects over longer periods of time.

Robert's observations raise two questions:

- What is the extent of calculator use and the commitment to calculator use in our secondary schools?

- What are the unique instructional capabilities offered by calculators?

The statistics available reflect a continually growing role for the calculator in the classroom. In 1980, Reys surveyed teachers in Missouri and found that just over 60 percent of senior high mathematics teachers had used calculators in the classroom. (17). Kasten's survey of teachers in four states showed that the percentage of secondary teachers and principals in those states who believe that calculators should be included as a topic on high school competency tests ranged from just over 40 percent to 66 percent. (13)

Another survey underscored the need to identify and exploit the unique pedagogical capabilities of calculators. Wyatt interviewed teachers who had never used calculators in the classroom and found that they seemed primarily aware of two uses, computation and checking. (32) Clearly, while the majority of secondary teachers favor a role for calculators in the classroom, there are still many teachers who are unaware of the wide benefits of calculator use or of how calculators might be integrated into their teaching.

Benefits of Use of Calculators

As they unfold from research and from teacher experimentation in the classroom, these benefits seem to fall into three categories and we will discuss each in turn:

- Calculators provide a powerful tool for evaluating the depth of students' understanding of mathematics and for diagnosing mathematical misconceptions and difficulties.
- Calculators permit teachers to adopt more freely and comfortably some of the classroom behaviors that research has associated with the effective teaching of mathematics.
- Calculators facilitate the teaching and learning of several concepts and skills which have traditionally been stumbling blocks in secondary school mathematics.

Diagnosis and Evaluation. In ways often incidental to their primary objectives, several research studies have shown that the calculator can be used as a lens by researchers and teachers to assess students' understanding and to pinpoint areas of weakness. The topic of division provides a good example. The recent National Assessment of Educational Progress (NAEP) posed an exercise similar to the following one to groups of 13- and 17-year-olds who were allowed to use calculators and to groups of 13- and 17-year-olds who worked the exercise by paper and pencil:

$$.04\overline{)8.4}$$

Among the 13-year-olds who used calculators on this exercise, nearly 30 percent reversed the divisor and dividend, while just over 10 percent of the 17-year-olds did the same. (5) Similarly, an extensive study of British adolescents' understanding of various mathematical concepts reported that under 10 percent of the 15-year-olds tested were "consistently able to press the buttons on their calculator in the correct order in solving simple division problems." (8, p 47)

No doubt, some of these errors can be ascribed to a lack of familiarity with symbols, perhaps with the working procedures of the calculators Mainly, however, the results reflect a basic lack of understanding among many teenagers of the concept of division.

Hart urges teachers to capitalize on incorrect calculator answers and to probe students' conceptual understanding and mathematical sense of reasonableness (8) For example, when a student works 5 ÷ 100 on the calculator, ask her what she expects the answer to be. Quite often, 20 is the expected answer, so when 05 shows up instead, ask for an opinion as to what might have happened. In other words, use the calculator as a catalyst for mathematical dialogues with students

The NAEP testers also asked students to order a set of fractions according to size:

$\frac{5}{8}, \frac{3}{10}, \frac{3}{5}, \frac{1}{4}, \frac{2}{3}, \frac{1}{2}$

The success rate was very low among both 13- and 17-year olds, whether they used calculators or not (2 percent success for both groups of 13-year-olds; just over 10 percent for both groups of 17-year-olds). (5) If the students generally understood how to convert fractions to decimals, the calculator groups should have scored higher on this exercise, since the calculator makes such non-routine computation much less risky. The scores were uniformly low, however, so we have strong evidence that many teenagers do not know how to convert correctly fractions to decimals.

In the area of problem solving, the NAEP researchers were able, with the help of calculators, to identify and call attention to a crisis that pervades the entire mathematics curriculum. even when computation is removed as an obstacle, most teenagers cannot think sensibly about mathematics word problems In particular, the researchers compared the performances of calculator users and non-users on several problems like the following:

A man has 1310 baseballs to pack into boxes which hold 24 baseballs each How many baseballs will be left over after the man has filled as many boxes as he can?

Because calculators record division remainders in decimal form, calculator users were obliged to translate the machine's answer for 1310 ÷ 24 = ? into a whole-numbered remainder. With or without calculators, few students solved this problem correctly, but calculator users fared especially badly: 29 percent of 13-year-old non-users were successful, as opposed to only 6 percent of the 13-year-old calculator users. Among 17-year-olds, only 19 percent of the users were able to obtain the correct answer (5).

Zepp considered a similar issue. the role of computational skills in proportional thinking. (33) In particular, his research sought to identify how much computational difficulties contribute to the difficulties many students have in answering questions like the following:

Bill made lemonade with 12 lemons and 9 teaspoons of sugar. Sandy starts with 20 lemons. How many teaspoons of sugar should she use so that her lemonade tastes the same as Bill's?

Zepp worked with groups of 9th graders and college freshman and divided them into a calculator group which used calculators on all practice activities and on a posttest, and a non-calculator group which used no calculators at all on the same practice activities and posttest. Because there was no significant difference on posttest achievement between the two groups, Zepp concluded that we should not be looking to computational difficulties as the major obstacle to proportional thinking.

Reys and his colleagues used a "broken" calculator to help them gauge the level of good estimators' confidence in their own estimates. (18, 19) The researchers secretly programmed a calculator to be wrong by varying degrees, asked their subjects to make some computational estimates and to check their estimates against the calculator's computations. During individual interviews, each subject was given a set of estimation exercises and, as they checked with the calculator, the error range of the calculator was allowed to increase progressively from answers about 10 percent greater than a reasonable estimated upper bound, to 25 percent, then to 50 percent. Even though almost 90 percent of their estimates were within an acceptable range, 36 percent of the subjects went all the way through the experiment without concluding that the calculator results were unreasonable Instead, they chose to indict their own estimates. The lesson of the experiment is clear and, as our culture becomes more tied to technological devices, it is all the more pressing, we need an increase at all levels of the curriculum in activities that develop estimating skills and in classroom dialogues that develop confidence in the use of those skills.

Here is an example, taken from (7), of a calculator exercise that encourages estimating. Similar exercises can be found in (4), (15) and (16):

Starting with 15, how many successive multiplications will it take to get an answer in the interval (10,000, 10,500)?

One way of getting there in 4 steps is:

1. 15 × 600 = 9,000
2. 9,000 × 1.1 = 9,900
3. 9,900 × 1.1 = 10,890
4. 10,890 × .95 = 10,345.5

While the Reys study used the calculator to show that even good estimators lack confidence in their own estimates skills, Blume used the calculator to view the solution process of students when they tackle problems with and without the aid of the machines (2). For example, there are at least two ways to set up the solution of the following problem:

The star basketball player scored a total of 297 points during the first nine games. Jill averaged 14 points fewer per game than the star. How many points did Jill score during the first nine games?

A $((297 \div 9) - 14) \times 9 = ?$

B $297 - (14 \times 9) = ?$

Method A first determines the per-game average of the star, subtracts Jill's average, then multiplies by 9 to obtain Jill's total. Method B shortcuts this process by multiplying the difference between the star's and Jill's per-game averages and subtracting that total difference from the star's total to give Jill's total.

Either method is valid and sound, but Blume found that a group of seventh-grade students tended to ignore the shortcut solutions more and favored the longer solutions more when they used calculators than when they used only paper and pencil. The implication for teachers is clear: allow students to use calculators to solve mathematical problems, but help them through discussion, to become aware of their own solution processes as well as the variety of solution processes available to them.

Blume and Mitchell worked with 7th graders and trained them in the use of calculators with Reverse Polish Notation (RPN) logic, a bracket-free machine logic used by many scientists and engineers. (3). Once the students had learned to operate the RPN calculators, they were tested on several computations. The majority showed they had mastered the new machine logic. In fact, 81 percent were correct on the following example. $(25.97 + 57.78) - 13.4 = N$ ~~When the~~ parentheses were missing, however, and it was up to them to decide on the hierarchy of operations, the students did not fare as well. Only 20 percent gave the correct answer 70.5 to the following exercise: $83.3 - 54.4 - 4.25 = N$. Most subtracted first, then divided, rather than the reverse, thus revealing how confused most students are about the notion of operational hierarchies—that is, which operation must be performed before others in a computation.

Especially as they head into algebra, students must be comfortable with operational hierarchies. Lappan (15) suggests using the calculator as a tool for helping students to acquire skill in manipulating parentheses in equations, through problems like:

1. *Insert parentheses to make these true:*

$16 \times 15 - 7 = 233$	$16 \times 15 + 7 = 247$
$16 \times 15 - 7 = 128$	$16 \times 15 + 7 = 352$

2. *Insert +, −, x, ÷, and parentheses, if needed, to make these true:*

29	15	13 = 57
29	15	13 = 5655
29	15	13 = 448
29	15	13 = 1
29	15	13 = 0.1487179

If you use such examples to help the students' transition to algebra go more smoothly, make sure you have them discuss the roles of the parentheses in the exercises that allow different answers to arise each time.

In a recent British study, Bell and his colleagues were able to use calculators first to identify, and then to remediate, some common mathematical difficulties of students between the ages of 12 and 16. (1) Among the most prominent of the identified difficulties was a lack of understanding of place value in decimal numbers, for example, the researchers' interviews produced comments like "0 8 . . . that's about an eighth" and "1.07 lbs. is 1 lb. 7 ounces."

During the teaching phase of their study the researchers were able to produce significant improvement in the students' understanding of place value by involving them in calculator exercises like the game called Getting Closer, played in pairs, in which one student chooses a low number, the other a high number, and each puts his or her number on a calculator screen. The students then take turns, with the first repeatedly adding any *non-whole* number to the lower starting number while the second student subtracts similarly from the higher starting number Thus, the numbers on the two calculator screens approach each other. The first player to pass the other player's number is the loser. The players learn quickly that a knowledge of place value is an important advantage when the two numbers are close to each other.

Effective Teaching Behaviors. Recent research has made it more possible than ever to describe effective mathematics teaching, by identifying those classroom behaviors that contribute to effectiveness. That description is dealt with in depth in the chapter, "Effective Mathematics Teaching." There is a specific connection between effectiveness research and calculator research, however. Calculators can facilitate the learning and use of effective teaching behaviors, and it is that connection we describe in this section.

Research shows that effective teachers spend more time than less effective teachers on whole-class lecture, discussion, and demonstration. When Reys and his colleagues interviewed teachers who had begun to use calculators in the classroom, the teachers reported that they were able to cover more topics with the aid of calculators and that they dealt more with concept development and less with computation during their mathematics classes. (17) As they become more adept with calculators in the classroom, teachers can apparently use them to create environments which invite more lecture, discussion, and demonstration. Another trademark of effective teaching which is related to allocation of time is the amount of time teachers keep their students engaged in learning tasks. In this vein, Szetela noted at the end of a study involving the use of calculators to teach ratio to seventh-graders that, in the study's posttest, "students using calculators appeared more motivated, were more industrious, and spent less time idling" (28, p. 70) It is likely that classroom teachers could create the same effects when they use handheld calculators.

Another characteristic of effective teaching that has emerged from research is question-asking--the number and quality of questions asked by the teacher and the number of opportunities made available to students for their own questions. The calculator is a natural inducer of curiosity and of inclinations to experiment and to ask questions. Shirey's study illustrated this inducement to experiment among calculator users in grades 10 through 12. (22) It was a brief study, comparing one group learning a unit on home mortgages via computer-augmented instruction with a group learning the unit with the aid of handheld calculators. Shirey noted that "more calculator students performed some experimentation beyond the minimum when compared to the computer group." Piaget has urged the development of environments around young children that are filled with objects to pique their curiosity. In the secondary classroom, calculators can apparently serve a similar function for older students.

By allowing students to manipulate numbers and to observe number patterns without the tedium that often accompanies paper-pencil computation, calculators make it possible for students to turn their questions into conjectures and their conjectures into mathematical argument and proof. If such a process becomes a regular part of classroom activity, it leads students to construct a view of number and mathematics in the same way that, according to Piaget, they develop their world-view: by interacting and experimenting with the objects around them. The primary role of the teacher in this process is to help them to formulate their questions and conjectures and, of course, to make sure they have frequent opportunities to use the calculators in this way.

Krist illustrates the use of calculators to engage students in a dialogue leading to conjecture with the following example. (14) First, the teacher notes that $6^2 - 6 = 5^2 + 5$ and asks the students to look for a pattern that might extend this equation into a conjecture. Do other numbers fit into the same sort of equation? How could you check it for other numbers? What short-cut expression might state the conjecture that the pattern exists for all whole numbers? (e.g., $N^2 - N = (N - 1)^2 + (N - 1)$). If the class is versed at all in algebra, the next question might be: How could you show that this is true for all whole numbers N?

The ability to communicate the continuity of mathematics from topic to topic is another characteristic of effective mathematics teachers. (See the chapter "Effective Mathematics Teaching.") Hiatt points out that one aspect of that continuity—the mathematical method of inquiry—is communicated clearly and consistently when calculators are used well (9). As Hiatt describes it, the method of inquiry has five steps, each of which can be seen in the Krist example above:

1. making observations;
2. organizing observations into patterns, conjectures;
3. specializing and generalizing through inductive or analogous reasoning;
4. inventing symbolism for the generalized conjecture;
5. proving the conjectures.

Concept and Skill Development. In his overview of calculator research and development in mathematics education. Weaver points out that the true power of the calculator is that it can transform (and in some classrooms *has* transformed) the process of learning mathematics. (30) Traditionally, how teachers develop mathematical ideas, applications, and problem-solving skills in their students has been interwoven—often tangled—with their development of student proficiency in paper-pencil computational algorithms. Weaver writes that, although the two development processes should work in parallel and not in conflict, parallel development has never been the rule in secondary mathematics classrooms. Now the calculator offers an opportunity to make the two processes truly parallel and to keep them that way. The message to teachers. teach paper-pencil computational algorithms, but also take advantage of the handheld calculator as a means for developing mathematical ideas, applications, and problem-solving skills

If the picture Weaver paints is accurate, then we might expect to see some clear evidence of the effects of instructional calculator use on concept and skill development. Unfortunately, calculator research is young and so has produced a relatively modest, though extremely promising, set of results. Once researchers have more time for exploration, however, it is possible that the power of the calculator will be felt throughout the mathematics curriculum. In fact Jewell analyzed a set of typical secondary school textbooks and concluded that approximately one-half of the content of algebra, geometry, and elementary functions texts and one-eighth of an algebra-trigonometry text could be appropriate for calculator applications that contribute to mathematical understanding. (11)

In their search for specific topics that are especially ripe for calculator use, researchers have looked for concepts whose learning is often impeded by the computational difficulties involved. Ratio is one of those concepts. Szetela designed a seventh-grade study with the hypothesis that the measurement situations that quite often form the basis of ratio instruction can be swept clean of distracting calculations if calculators are used. (28)

The study involved eleven days of ratio instruction for calculator and non-calculator groups. Included in the instruction were measurement of circles to determine the ratio of diameter to circumference, measurement of poles and their shadows, measurement of automobile width and length, coin tosses, and so on In the testing administered after the instruction was completed, the calculator group did better, though not significantly better, on two achievement tests and one attitude test. The calculator group did significantly better than the non-calculator group on a test on unfamiliar ratio problems, during which the calculator group was allowed to use calculators.

Moving to a later point in the curriculum, Szetela also studied ninth and tenth graders using calculators to learn trigonometric ratios. (27) At the heart of this study was some intensive instruction over a three-week period that centered on the development of abbreviated trigonometry tables through measurement activities with right triangles. One group of students worked through the table development with the aid of calculators, another group worked without calculators. There was no significant difference between the two groups on an achievement test administered after the instruction was over. Of special interest to teachers, however, may be the testimony of the teachers involved in the study that "teaching with calculators was much less onerous than teaching without calculators." (27, p. 118)

Wheatley conducted a problem solving study with calculators at the sixth-grade level which clearly invites similar investigations by secondary school educators. (31) In the study, two groups of students received the same training in the use of problem-solving strategies such as estimating, retracing steps, and checking the reasonableness of answers. One group used calculators in the training and the other did not. In a final problem-solving test, the calculator group used significantly more of these strategies than did the members of the noncalculator group.

One area of the curriculum where calculator researchers have been fairly active is ninth-grade general mathematics. Toole conducted one of the longer studies in this area, a 6-month study in which she compared a calculator-assisted program used one day a week with non-use of calculators in the same course. (29) In the six months between pretests and posttests, the calculator group gained eight months more on the total test than the non-calculator group. The breakdown into subtest gains was as follows: 7-month gain in computation, 5-month gain in concepts, 1-year gain in applications.

Creswell and Vaughn also conducted a calculator study among ninth-grade general mathematics students, based on eight weeks of instruction in decimals and percents. (6) Two groups of students were compared: calculator users and non-users. The non-users were taught from the standard textbook; the users received instruction based on materials designed by the researchers for the reinforcement of the concepts involved in decimals and percents. On the posttest that measured the level of achievement over the eight weeks, the calculator group scored significantly higher than the non-users.

Remarking on this difference, and noting the frequency of studies where no significant difference arose when the calculator was used merely to supplement the textbook for checking and calculation, Creswell and Vaughn ascribe the achievement difference between users and non-users in their study to the materials they developed to exploit the calculator. Both Roberts and Suydam have also taken note of the dearth of research studies and of curriculum materials that exploit the unique capabilities of the calculator (21, 26). Instead, we see curriculum materials that suggest only supplementary use. One major exception to this is a carefully developed eleventh- and twelfth-grade mathematics curriculum built around the programmable calculator by Rising and his colleagues. (20) Another exception is the ninth-grade course, based on concepts from statistics and on the use of the programmable calculator, which was developed by Hoffman and her colleagues (10). Without the proliferation of such materials, we may never see the real pedagogical potential of the calculator fulfilled.

We also may not get a true reading of the usefulness of calculators if they are not welcomed into test-taking. A majority of secondary teachers are apparently ready to take this step. In their survey, Reys and his colleagues found that 67 percent of senior high school students and 52 percent of junior high school teachers would support the use of calculators on standardized tests measuring concepts or applications. (17)

Conclusion

The calculator has been cast into a peculiar situation. On the one hand, it has become a fixture among American teenagers the NAEP data shows that 80 percent of thirteen-year-olds and 85 percent of seventeen-year-olds either own their own calculators or have one available for use. (5) On the other hand, financial exigencies may keep the machine relatively invisible in the secondary school classroom. In particular, if calculators have to compete with microcomputers for funding, the curriculum materials needed to integrate the calculator into the curriculum (as opposed to its usual supplementary role) may never be developed. Furthermore, calculator training for teachers may be put aside for lack of funding, and the need there is critical. Reys *et al.* found that a large percentage of the teachers they surveyed said they wanted training in the use of calculators, but had never had any 71 percent of junior high school teachers and 63 percent of senior high school teachers wanted training, while only 13 percent and 17 percent, respectively, had already had some training. (17)

That the calculator should be an integral part of the curriculum has been established, but there are still many pieces missing from the calculator picture. How and at what specific points should it be integrated into the curriculum? Over the long term, how well can it facilitate students' use of problem-solving processes and strategies? How can teacher training be designed so that calculator use will make it easier for teachers to behave effectively in the classroom? These are questions that touch on the most important issues in mathematics instruction. They must not be ignored.

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UNDERSTANDING FRACTIONS:
A PREREQUISITE FOR SUCCESS IN
SECONDARY SCHOOL
MATHEMATICS

I teach mathematics in both the eighth and ninth grades, and I see a mysterious change in students' performance from the one grade to the next. Many students who seem to compute fairly well with fractions in the eighth grade appear to run into trouble when they face fractions in the context of algebraic equations in the ninth grade. Is there something in the change of context that throws the students off?

Secondary school mathematics must seem like a foreign language to students who are not fully prepared for it. Like the students described in the question above, they often founder even in areas where some of their skills seem secure. To many teachers this foundering is a signal for educators to re-examine the teaching of prerequisites for high school mathematics.

Among the prerequisites singled out for scrutiny by the teachers whose questions form the basis of *Research Within Reach: Secondary School Mathematics*, understanding fractions was one of the two most frequently mentioned. The other, estimating with a sense of reasonableness, is covered in a separate chapter.

A Difficult Concept

In describing the several important threshholds in the learning of mathematics, Steffe has underscored how very real and very critical is the dilemma faced by both the teacher who asked the question at the beginning of this report and by the students alluded to in the question: "There are (at least) three critical achievements in a child's mathematical life—the idea of ten as a unit, the idea of a fraction, and the idea of an unknown." (18, p. 20) Faced as they are with the double dilemma of stepping into the arena ofalgebraic unknowns—the third critical achievement—without the aid of the second critical achievement—an understanding of fractions—it is no wonder that many students' seeming mastery of fractions begins to fall apart.

How can it happen that so many eighth graders can seem to master fractions, only to stumble over them in the ninth grade? On one level, the answer is simple: there is much more to mastering fractions than mastering computation. On a deeper level, there are several subtle aspects of fractions which slip by many students as they prepare for secondary level mathematics. Some research in the past decade has helped to delineate those subtle aspects.

Payne's review of fraction research provides an overview of the process of learning fractions that reflects how long and winding the process is, and the variety of contexts in which fractions are encountered in the elementary school curriculum. (15) The early days of learning fractions are not so difficult. Payne cites evidence that most students from the age of eight on can master the initial fraction concepts and symbols in a two-week period, and that they tend to be quite enthusiastic about fractions during this initial learning period. Yet, later on, as they confront such concepts as equivalent fractions, even proven instructional strategies like paper-folding and using different-sized rods cannot offset the trouble most children have with those concepts. Some pictorial representations of fractions – in particular, on the number line—seem especially difficult for students in the intermediate grades. (15)

The recent National Assessment of Educational Progress (NAEP) illustrates some of the weaknesses in understanding that underlie many secondary students' experiences with fractions. In several exercises NAEP tested students' skills in estimating computations with fractions. The results indicated not only a general weakness in students' understanding of fractions, but also revealed that many students resort to memory—and often mis-remembered rules—to *compute* fractions instead of estimating. Thus, when asked to estimate the answer to 12/13 + 7/8 (the test did not allow enough time to figure this out with pencil and paper), fewer than 25 percent of the 13-year-olds and fewer than 40 percent of the 17-year-olds chose the correct estimate of 2. Many of those who were mistaken attempted to compute the answer without a check for reasonableness in their answers—in fact, 19 and 21 were common answers. (1)

In their summary of the status of secondary students' understanding of fractions and their skills in computing with fractions, the researchers who summarized the NAEP results made the following statement: "Overall, it appears that roughly two-thirds of the 13-year-olds and about three-fourths of the 17-year-olds have learned most of the very elementary fraction skills. However, only about half of this number can integrate these skills to solve some of the more involved calculations with unlike denominators and mixed numerals. In other words, only about 40 percent of the 17-year-olds appear to have mastered basic fraction computation." (1, p. 331)

One critical aspect of fractions that students often do not grasp is their flexible nature—they are quantities with real number value (½ is greater than ⅛); they also express relationships between quantities (¾ of 12 is 9). Kieren (8) has analyzed the various contexts in which we use rational numbers, the language and symbols that accompany each usage, and he has summarized the four contexts in this table (Note. *a rational number* is a number that can be expressed as the quotient of two integers. All rational numbers can be expressed as fractions, but not all fractions are rational numbers - for example, $\pi/3$):

Context	Language	Symbol
measure	three-fourths of a unit	¾
quotient	three divided by four	3 ÷ 4
ratio	three to four	3 : 4
operator	three for every four	3 for 4

As examples of the four aspects, consider the following:

measure: "The area of this region is 1 square meter plus ½ square meter plus ⅙ square meter, or $1\frac{4}{6} = 1\frac{2}{3}$ square meters."

quotient: "Sharing 3 candy bars among 5 people means each person gets ⅗ of a candy bar."

ratio: "A one-to-three mixture of flour and water has the same consistency as a two-to-six mixture, because ⅓ = $\frac{2}{6}$."

operator: "If a store shows a profit 2 out of every 3 days, then over a 30-day period, there will be 20 days of profit, because ⅔ of 30 is 20."

The researchers who summarized the recent NAEP results offered their assessment that most 13-year-olds see these four aspects of fractions as separate, unrelated topics, rather than as different contexts for the same concept. (1) The validity of that assessment is strengthened by the research of Noelting. (14,8) He found that different contexts of fractions draw qualitatively different responses from students. In particular, he asked students a series of questions using either the ratio context or the quotient context. Here are sample questions from the study:

Situation 1 (ratio number questions)
Which of the following mixtures has a stronger orange flavor, A or B?
A: One orange concentrate, three water
B: Two orange concentrate, six water
Which of the following mixtures has a stronger orange flavor, M or N?
M: Two orange, three water
N: Four orange, six water

Situation 2 (quotient number questions)
Some cookies are shared among two groups of boys. In which group will a boy get more cookies, A or B?
A: One cookie for three boys
B: Two cookies for six boys
In which group will a boy get more cookies, M or N?
M: Two cookies for three boys
N: Four cookies for six boys

Noelting found that students generally were able to answer the second ratio question if they were able to answer the first, but found the first of the quotient questions easier than the second, even though Situation 1 and Situation 2 are mathematically the same. Obviously, students think differently in the quotient context than in the ratio context.

Larson noted another facet of students' misunderstanding of fractions—the ability to distinguish between a fraction as an expression of part of a unit and a fraction as a number with a unique place on the number line. (11) She asked the seventh grade students in her study to locate the point on this line segment that can be named by the fraction 1/5.

0 1

In general, many of the students tended to use the rule, count the number of equivalent segments (in this case, 5) for the denominator and count the number of equivalent segments from zero until you reach the number which will combine as the numerator with the chosen denominator to yield the fraction ⅕ (in this case, 1). This rule served them well in the above problem, but approximately 20 percent of the students also used the algorithm to answer the same question about the following line segment, and they chose the indicated point.

0 . x . . 1 2

Thus, they chose the point representing $\frac{2}{10}$ or ⅕ of the whole line segent from 0 to 2, not the *number* ⅕.

In a similar vein, Ekenstam conducted a study among Swedish students and noted that more than half of the 15-year-olds tested were erratic in their selection of fractions less than 1 from a list of fractions of various sizes. (2)

Lankford set up a series of interviews of 7th graders, designed to determine the kinds of misconceptions young people have about fractions. (10) In one example, he carefully documented 22 *different* errors the students made in figuring out ¼ − ½ = ? Overall, the most common errors he noted in his study were:

1. Multiplying a mixed fraction times a whole number by multiplying the whole numbers and tagging the fraction on the end:
   3 ¼ × 5 = 15 ¼
2. Adding fractions by adding numerators and adding denominators.
   ½ + ¼ ≠ $\frac{2}{6}$.

Again, it is important to try to look at the roots of such errors. As Vinner and his colleagues point out, in order to know that ½ + ¼ = $\frac{2}{6}$, a student must know that $\frac{2}{6}$ = ⅓, that ⅓ < ½, and must understand what the addition of fractions means. (20) These are conceptual issues and cannot be settled through algorithmic training alone.

Kieren and Nelson conducted a study, based mostly on interviews of students in grades 4 through 10, the purpose of which was to delineate the development in young people of the notion of a fraction as an operator (for example, ⅗ of 20 is 12). (9) The students were asked to observe a "machine" into which a certain number of papers went in and a lesser number came out, and then to describe what rule ran the machine. Thus, they might see an input of 20, an output of 15; and input of 40, and output of 30, and so on, and conclude that the rule is ¼ ×. From their interviews the researchers hypothesized the following three levels of growth:

a. The students are ½-oriented. They can identify operations that are ½ ×, but are relatively fixed on ½, to the extent that they are inclined to identify other fractional operations, such as ⅓ ×, as ½ ×.

b. A transitional level, where the students can identify unit fractional operators—that is ½ ×, ⅓ ×, ¼ ×, ⅕ ×, and so on, and the composition of unit operators, for example, ½ × ⅓, ⅓ × ⅕.

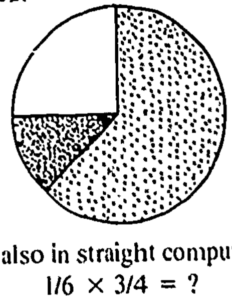
c. The students can identify all forms of fractional operators.

Role of Instruction

A very high percentage of the studies cited so far were based on student interviews, which should be a clear signal to teachers wanting to shore up their students' understanding of fractions. In order to learn which aspects of fractions are misunderstood by their students, teachers must encourage them to verbalize as much as possible and should take advantage of the ensuing classroom dialogues to develop a full understanding of fractions.

Lochhead (13) gives an example of such an instructional approach in the context of addition of fractions. Give no preliminary explanation of what adding fractions is all about. Give a simple question such as "½ + ¼ = ?," which will probably bring a correct answer. Ask the students to verbalize the rule by which the addition was carried out and which can be applied to further addition exercises in fractions. Now give another example that will test the student-offered rule and which might provide a counter-example to that rule if it was flawed. Finally, continue the discussion, having the students revise their rule if necessary.

Hasemann's study made it clear that instructional strategies can have a tremendous bearing on how well students understand fractions. (6,7) He worked with German adolescents who were relatively unsuccessful in mathematics. He presented fraction exercises in two forms: in diagram form and in computational form. Thus, a circle was shown with three-quarters of it covered with dots. The students were asked to shade in 1/6 of the dotted part and then to say the fraction of the circle that had been shaded.



The same exercise appeared also in straight computational form:

$$1/6 \times 3/4 = ?$$

Slightly more than 50 percent of the students in the study were successful in the computational exercise, while only 30 percent succeeded in the diagram version.

Hasemann contrasted his results with the results of a similar experiment with British student. in a program whose emphasis was on understanding fractional concepts through diagrams. The results were reversed. slightly more than 50 percent of the British students succeeded in the diagram exercises, while fewer than 25 percent were successful in the computational version. Pointing out that German schools emphasize the computational approach to fractions, in contrast to the diagram-oriented British program, Hasemann concluded that instructional emphasis and stress will affect many students' understanding of fractions.

Another possible factor with a bearing on instruction emerges from cognitive processing research, that is, the research that studies how learners process information, parcelling it out into the shelves of memory and gaining access to it when it is needed. As a result of his research, Greeno has offered the opinion that when learners compare two fractions by regions or diagrams they use spatial processing, but that they process the algorithm for comparing fractions--choosing a common denominator, then multiplying and dividing by the appropriate numbers--in a different way. (3, 15) In essence, says Greeno, the two ways of processing produce two different concepts of fractions.

Both Hasemann's work and Greeno's work imply that students need to see both approaches to fractions--visual and algorithmic--and that they need help in seeing how the two relate to each other. Instruction that puts a heavy stress on computational algorithms for fractions can lead students astray. Peck and Jencks interviewed sixth-graders as they worked on various fraction exercises, such as "Which is larger, 1/4 or 2/5?" "What is 2/3 + 1/4?" and "Can you draw a sketch of 1/3?" (16,17) The researchers noted "Almost all the children appeared to search their memories for rules and then to try to apply the rules. The rules were often misapplied, and the students could not tell that they had done so." (16, p.347)

In describing the results of the extensive British study entitled *Concepts in Secondary Mathematics and Science (CSMS)*, an assessment of the mathematical and scientific understanding of students between the ages of 12 and 16, Hart summed up the researchers' conclusions for the topic of ratio: "Finally, teaching an algorithm such as a/b = c/d is of little value unless the child understands the need for it and is capable of using it." (5, p.101)

The CSMS researchers found that, rather than using an algorithm that has no meaning for them, students often approach a ratio problem by "building up". For example, in approaching the problem "1/2-cup of cream is sufficient for 8 people. How much is sufficient for 12 people?", they reason that 12 is 4 more than 8, 4 is half of 8, so the answer is 1/2 + 1/2 (1/2) = 3/4 of a cup. The reasoning is valid, but the researchers pointed out: "The majority of children do not progress beyond doubling, halving and using doubles and halves to 'build up' to answers. This ability is no guide to how the child would tackle a ratio of say 5:3."

(5, p. 101)

The instructional implications of the research described so far are threefold:

1. Be alert to the multiple levels of meaning in the concept of fraction and to the different degrees of understanding associated with each.
2. Be aware of the limitations, some emerging from cognitive development, others from variations in spatial processing, that cause differences in students' understanding of fractions.
3. Do not rely totally on algorithms to teach fractions. Integrate the use of diagrams with the use of algorithms.

One promising instructional approach which is faithful to all three has been part of the Calgary Junior High School Mathematics Project. (4) The approach to teaching fractions was a "process" approach, through which seventh-grade students carried out a mathematical investigation. They experimented with concrete materials, recording what happened in the experiments, formulating questions, and writing up accounts of experimental results as well as applying the results to practical situations. The study lasted eleven and a half weeks. Not only did the experimental group's achievement improve significantly when they were compared with a group of students learning from a regular textbook, but they also displayed significantly greater enjoyment of fractions than did the students in the regular group. Furthermore, there was a significant improvement in their ability to give explanations, probably due to their experience in writing up accounts of experimental results.

Calculators are a valuable tool for teaching fractions. Szetela reports that seventh-graders who were taught the concept of ratio with the aid of calculators did better on an unfamiliar ratios test than a group who learned ratios without calculators. (19) Szetela also reported that the learning experience was "less tiring and frustrating for both teachers and students when calculators were used." (19, p. 70) As in all such uses of calculators, however, the machines cannot stand on their own. Teachers must be ready to deal with issues and questions which calculator use can generate. For example, many students do not understand why 1/9 becomes .1111 . . . on the calculator screen. Alert teachers will note the confusion and help students to see the meaning that binds the two representations.

Microcomputers also offer some exciting prospects for improving students' understanding of fractions. Although educators have known for decades that individuals vary widely in their mental pictures of number--in particular, of fractional numbers--there has been a dearth of instructional strategies that can support a wide variety of approaches to picturing and manipulating numbers. With its capacity for displaying different visual models of number concepts, such as in the program described by Levin which invites students to estimate numbers by shooting a "harpoon" at the number line or at the Cartesian plane, the computer promises to increase the supply of such strategies. (12)

Conclusion

The technological advances of the past decade that have resulted in the handheld calculator and the microcomputer have run parallel with the advances made by the set of fraction research projects described in this chapter. The analyses and strategies that have resulted, combined with the technological aids that have been developed, improve the chances of teachers for establishing an understanding of fractions as a real prerequisite to secondary school mathematics, one that can be fulfilled by most secondary school students.

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THE LEARNING AND TEACHING OF ALGEBRA

I have a two-part question concerning algebra. First, students in Algebra I seem to be at several different levels of understanding of variables and equations. What characterizes those different levels? Second, even when they are taking Algebra II, many students seem to have mastered only mechanical skills and they cannot adapt them to new situations. For example, they may recognize that factoring can be applied to $a^2 + 2ab + b^2$, but not see that it can also be applied to $a^4 + 2a^2b^2 + b^4$. What contributes to this inability to adapt algebraic techniques?

As many teachers know from experience, success in Algebra I does not guarantee success in Algebra II. The techniques learned in the first course often stall at the mechanical level and give little help in the second course's applications.

Furthermore, it appears from recent research that success in Algebra II does not guarantee an understanding of equations and variables deep enough to permit students entering college mathematics to translate freely between word statements and algebraic expressions. The algebraic skills and understanding of many of these students stall at a more advanced, yet still intermediate, level (4, 5, 6, 18)

The difficulties involved in developing a deep understanding of algebra result in part from algebra's having several different faces. On the one hand, it is a kind of generalized arithmetic, with central roles for addition, subtraction, multiplication, and division. On the other hand, it is a structured system for formulating and manipulating variables and formal mathematical statements. Because of misconceptions or slow cognitive development, young people may succeed in some applications but fail to connect algebra to its broader mathematical applications. The teacher's question that begins this report provides one such example, the results of the recent National Assessment of Educational Progress (NAEP) provide another.(3) Around 40 percent of 17-year-olds with one year of algebra were able to solve linear equations in one unknown. The comparable figure for 17-year-olds with two years of algebra was 60 percent. In both groups, however, the success rate for applying algebraic knowledge to word problems was consistently much lower than 40 percent. The researchers who interpreted the NAEP results noted: "It appears that although additional study in algebra may improve students' algebraic skills, it does little to help them learn to apply those skills to solve problems." (3, p. 60)

Recent research has provided a clearer picture of student misconceptions about algebra and of the nature of the various levels of algebraic understanding. This has been achieved through careful testing, followed by comprehensive student interviews and instruction that fits the student needs identified in the interviews. Rather than just focusing on *what* an algebra student learns, researchers can now focus on *how* algebra is learned, as well.

This report describes the major research findings and the recommendations drawn from them. In the first section we focus on the student and discuss the major misconceptions and errors that have been uncovered. In the second section we concentrate on the concepts in algebra and discuss the several levels of meaning of these concepts, as well as factors that may limit a student to one meaning level while blocking access to the other levels. The third section lists suggestions from researchers and other educators for eliminating misconceptions and errors and for broadening students' understanding of algebra.

Student Errors and Misconceptions

The concepts of variable and equation are central to algebraic understanding, and so misconceptions surrounding these two concepts are central to failure in algebra. Wagner conducted a study to delineate some of the early misconceptions that are commonly developed. (22) She interviewed 30 students from the ages of 10 to 18, and her focus was conservation of equation--that is, the ability of an individual to recognize the irrelevance of changing noncritical attributes in situations in which variables and equations appear. For example, each student was shown the equation 7 x W + 22 = 109. The interviewer then said, "I'm going to change this W to an N," and showed 7 x N + 22 = 109. The student was then asked which would be larger, W or N. Those who indicated correctly that the change made no difference were deemed conservers. The nonconservers looked upon the second equation as a whole new problem and, indeed, it was not uncommon for nonconservers to say that, if the two equations were solved, the first equation would yield the higher number because W comes later in the alphabet than N. Less than half of the students gave conserving responses to the task and, though there was little correlation between age and conservation, there was a significant correlation between conservation and completion of at least one semester of algebra.

Wagner noted one tendency among the older nonconservers that is related to an apparently common misconception about equations: they were convinced that they had to solve for W and N before they could answer the interviewer's question. The implication seems to be that many students view the equation sign as a signal to do something, rather than as a statement of relationship. This is a phenomenon that is familiar among elementary school students. Many can answer 6 + □ = 9 correctly, but are stymied by 9 = 6 + □, the equation sign not appearing in the latter case in the customary "action" position ("6 + 3 equals 9"). It now appears that the misconception lingers for many older students as well.

Another misconception about algebra has shown up in at least five unrelated studies from several countries, namely the persistent impression that variables are labels for objects, not number representatives. For example, in an assessment of the mathematical understanding of British children, there was the following task: "Blue pencils cost 5 pence, red pencils costs 6 pence. I buy 90 pence worth. If b is the number of blue pencils bought and r is the number of red pencils bought, what can you write down about b and r?" Nearly 20 percent of the 14-year-olds tested answered with the equation $b + r = 90$ which indicates they saw b and r as labels for the objects purchased. (8) Similarly, Ekenstam and Nilsson noted from their assessment of mathematical understanding among 16-year-old Swedish students. "It seems probable that almost every student would have given the correct answer to the problem 'Write in lowest terms 15/15,' but only about half of the students mastered a/a, a sign that it was not clear to the other half that the letter is used as a number representative." (7, p. 64)

In the United States, some recent research has shown that this misconception persists into adulthood. Clement and his colleagues asked a group of college engineering students to express the following sentence as an equation, using S for students and P for professors. "There are six times as many students as professors at this university." Only 63 percent of the students gave a correct answer like $S = 6P$, while a typical wrong answer was $6S = P$. During interviews, many of the students who responded with $6S = P$ maintained that the equation meant "For each 6 students there is 1 professor." To them, S apparently was a label for students, not a symbol for the number of students, and the equation sign signalled a correspondence, rather than a number equivalence. (4, 5, 6, 18)

NAEP uncovered a similar weakness in translating from word sentence into algebraic expression. Only 45 percent of American 17-year-olds were able to do the following translation problem correctly. "Carol earned D dollars during the week. She spent C dollars for clothes and F dollars for food. Write an expression using D, C, and F that shows the number of dollars she had left." (3)

Research into algebraic understanding has revealed some common pitfalls in manipulating and interpreting algebraic expressions:

- A tendency to mix numbers and letters. When asked to "add 4 onto 3N", nearly half of British 14-year-olds responded with either 7 or 7N. (8)
- A weakness in dealing with denominators in equations. While 70 percent of Swedish 16-year-olds were able to solve $3(3x - 2) = 2x$, less than 30 percent were able to solve $(3x - 2)/2 = x/3$. (7)
- A tendency to ignore operations in generalizations. Kieran calls "one of the most common errors made in algebra" the inappropriate generalization of $7a \div 7 = a$ to $7a - 7 = a$. (12)
- A tendency to ignore the hierarchy of operations in the solving of an equation. Thus $2 + 3 \times 5$ is read as a string from the left ("2 + 3 is 5 times 5 equals 25"), rather than an expression tied to a hierarchy ("$2 + 3 \times 5$ equals $2 + (3 \times 5)$ and that equals $2 + 15$ which equals 17"). (12)
- A weakness in interpreting inequalities. The NAEP researchers stated that most 13- and 17-year-olds "did not understand the special properties of inequalities and appeared to treat inequality relationships as equalities." For example, about 40 percent of the 17-year-olds failed to reverse the inequality by a negative number. (3, p. 57)

- A lack of familiarity with the notation associated with functions. In the NAEP assessment, 86 percent of the 17-year-olds correctly answered an exercise like "What is the value of $a + 7$ when $a = 5$?", but only half as many 17-year-olds were correct on the structurally similar exercise, "If $f(a) = a + 7$, what is $f(5)$?" (3, p. 67)

Some, if not most, of the misconceptions and errors that complicate the learning of algebra are rooted in students' first experiences in algebra and the conceptual frameworks they create to assimilate those experiences. Kieran's research has shown how students naturally build their frameworks on their arithmetic experiences and how, if this is not done carefully, it can lead to errors. (11, 12) She has identified several so-called "conceptual schemes" that underlie the initial learning of algebra, among them:

1. *Quasi-equality scheme*, which is based on the notion that the equal sign is an operator calling for action, rather than an indicator of equivalence. Two kinds of errors can result from too strong a reliance on this scheme. First, equations with an unknown on the left side are solved in terms of the first numeral on the right side. Thus, $4 + x - 2 + 5 = 11 + 3 - 5$ is solved by many students as if it were $4 + x - 2 + 5 = 11$ and they put in 4 for x: $4 + 4 - 2 + 5 = 11$. Second, students derive the notion that "it doesn't matter when you perform the operations, as long as they get totaled up sometime." Thus, Kieran reports that after writing $4 + 4 - 2 + 5 = 11$ for the above exercise, many students continue with $4 + 4 - 2 + 5 = 11 + 3 - 5 = 9$. In their minds, the task of finding a number for x and the task of combining the numbers on the right side of the equation are not as integrated as they should be.
2. *Redistribution scheme*, which is based on the notion that "taking something off one number and adding it to another does not change anything." Thus, $37 + b = 168$ could be transformed to $47 + b = 158$, where 10 is added to one side and removed from the other. Of course, the same scheme applied on one side of an equation is valid: $3x + 17 = 47$ is equivalent to $3x + 5 - 5 + 17 = 47$.

Levels of Meaning and Levels of Readiness

What makes the learning of algebra especially difficult, and so too the teaching of algebra, is the matching of levels of meaning with the levels of learner readiness. Each of the primary algebraic concepts—variable and equation—can have several different meanings, depending upon context, and a learner's ability to understand and make use of a particular meaning depends in good part on that learner's cognitive development. (For a more complete treatment of cognitive development, see the chapter "Individual Differences Among Mathematics Learners.") For the present it suffices to note that until they are in early adolescence (12 to 14), most children are concrete operational (in Piagetian terminology) and their thinking is largely tied to their perceptions. Once they enter the formal operational stage of cognitive development, they are able to do more hypothetical reasoning, keep two or more variables in mind at one time, think about their own thinking, and so on.

The lines of cognitive development are never clearly drawn. In any random group of teenagers, there are likely to be individuals who are in the early concrete operational stage, others who are late concrete operational, and still others who are early formal operational. When such a group is introduced to multilevelled concepts like algebraic equations and variables, it is not surprising that fundamental misconceptions arise. Furthermore, as Matz points out, when young people move from arithmetic to algebra in their schooling, they are quietly expected to take a giant leap in their mathematical problem-solving strategies, while they have learned to expect in arithmetic that merely applying algorithms like long division will see them through, in algebra they must compose and carry out plans for solution. (15)

Matz identifies several meanings for the concept of equation. First of all, there is the meaning that most elementary school children attach to it, namely, *a connection between a procedure and a result*—doing the operations on the left side of the equal sign produces the answer on the right side of the equal sign "Answer" is an essential component of this meaning of equation An example of what people have in mind when they apply this meaning is $6 + (7 \times 2) = 20$. Secondly, there is *tautological meaning*, with the equation used as an expression of equivalence between two algebraic expressions. Examples are $(x + 2)(x + 3) = x^2 + 5x + 6$ and $4x + 12 = 4(x + 3)$. Lastly, there are equations used to *express constraints on variables*, usually inviting solution. An example is the linear equation $3x + 3 = 2x + 7$.

From a transcript of a classroom lesson, Kemme illustrates how multiple interpretations of "equation" can drive a wedge into teacher-student communication. (10) In the transcripts the teacher posed a problem:

> *There is a certain number of students in the classroom. If there are twice as many and then another 10 were added to it, then there would be 42. How many students are there?*

Several students arrived quickly and intuitively at the solution 16. The teacher, still hoping to use this problem to illustrate how to translate from word problems to equations, asked, "What kind of equation could you write in this case?" Since they knew the solution, several students answered, quite legitimately: "$x = 16$." The teacher, of course, wanted the equation $2x + 10 = 42$ as an answer. To the teacher, "equation" had a definite functional meaning, a tool for figuring out the solution. To the students, the term "equation" included a tautological meaning: the mere statement of the answer. Because of these different meanings, the class discussion turned into a verbal wrestling match, with the teacher trying to twist his desired equation from the students, while they remained unpersuaded and confused.

Kemme's transcript illustrates a common trap for teachers of secondary school mathematics. It is all too tempting, as Herscovics and Kieran point out, to concentrate on training students to develop their skills in manipulating equations and to ignore an entirely different skill—constructing meaning for the concept of equation. Kieran's research has convinced her that facilitating the learning of this skill is not an easy task. The impression among most adolescents that equations are what they appeared to be in arithmetic—expressions of the process that begins with a computation and ends with an answer—is an impression that resists change. (12) The section of this chapter entitled "Teaching Algebra" discusses some proven methods for changing this impression.

At the same time they are constructing meaning for the concept of equation, students must also come to grips with the several levels of meaning for the concept of variable if they are to develop a deep understanding of and facility with algebra. Hart's report of England's extensive research program, Concepts in Secondary Mathematics and Science (CSMS), lists six different interpretations that algebra students must attach to the use of letters in equations (8) We list these interpretations and include examples of questions where each interpretation is appropriate.

1. The letter has a numerical value from the outset—"What can you say about M if M = 3N + 1 and N = 4?"
2. The letter is not used directly, and can be ignored to the extent that it need not be evaluated—"If A + B = 43, A + B + 2 = ?"
3. The letter is used as a shorthand for an object, or for an object in its own right. For example, "2A + 5A ≠ ?"
4. The letter is used as a specific but unknown number—"Add 4 onto 3N."
5. The letter is used as a generalized number, able to take on more than one value—"What can you say about C if C + D = 10 and C is less than D?"
6. The letter is used as a variable, that is, it represents a range of unspecified values, and a systematic relationship is seen to exist between two such sets of values—"Which is larger, 2N or N + 2?" To understand this question well enough to answer it, a student must be able to grasp how both 2N and N + 2 will *vary* as N varies.

The CSMS study revealed that the majority of British students aged 13, 14 or 15 were not able to cope consistently with exercises that called for Interpretations 4 through 6 above. To Hart, this implied they were still concrete-operational and that they would need to develop into the formal operational stage before they could move smoothly among these last three interpretations. Since a basic understanding of algebra depends at least on Interpretation 4 and later applications of algebra depend on Interpretations 5 and 6, it is clear that it is possible to overtax the readiness of many teenagers to solve algebraic problems. It is important to challenge algebra students, but it is equally important to align the challenges with their cognitive development.

Some researchers have sought to define that alignment more clearly. From his work with engineering students described earlier, Clement reminds us that understanding an equation in two variables (S and P in his problem) appears to require an understanding of the concept of variable at a deeper level than that required for a one-variable equation. (4) Adi conducted a study to determine whether individuals were more successful with one approach to solving equations than with another, according to their level of cognitive development. (1) She considered two approaches to solving equations:

1. The reversal method: $5 - \frac{x-2}{3} = 2$

"What must I subtract from 5 to leave 2?", so ... $\frac{x-2}{3} = 3$

"What divided by 3 gives 3 as an answer?", so ...? $x - 2 = 9$

"What number, take away 2, leaves 9?", so... $x = 11$

2. *The compensation method* (if you act on one side of the equation, compensate by doing the same to the other side):

$$5 - \frac{x-2}{3} = 2$$

$$5 - \frac{x-2}{3} - 2 = 2 - 2$$

$$3 - \frac{x-2}{3} = 0$$

$$3 - \frac{x-2}{3} + \frac{x-2}{3} = \frac{x-2}{3}$$

$$3 = \frac{x-2}{3}$$

120

$$9 = 3\frac{(x - 2)}{3}$$

$$9 = x - 2$$

$$11 = x - 2 + 2$$

$$x = 11$$

Adi concluded from her study that, of the two methods, the reversal method was much easier to learn for individuals at the early concrete operational stage than for individuals at the early formal operational stage. For the latter group, the results did not favor either method over the other.

Errors and misconceptions in algebra are not random. It is a new terrain for students, quite different in its demands than the arithmetic they are used to, and the majority of students begin their algebra experience with developmental limitations. As a result, many overgeneralize the rules that have worked for them in arithmetic ("Which has the larger solution, $6N + 3 = 41$ or $6T + 3 = 41$? I won't know until I find the numbers that work.") or the rules that have worked for them before in algebra ("Solve $x^2 - 3x + 2 = 7$. When I had $x^2 - 3x + 2 = 0$, I set $x - 1 = 0$ and $x - 2 = 0$, so $x = 1$ or $x = 2$. Now I'll set $x - 1 = 7$ and $x - 2 = 7$, so $x = 8$ or $x = 9$"). Matz studied this phenomenon of over-generalization and concluded that adept problem solvers generally try to rewrite an unfamiliar problem so it can fit a relevant rule, while unsuccessful algebra problem solvers get hooked into altering the rule to fit the unfamiliar problem. Changing a rule to fit a problem isn't always wrong ("There probably is a rule that says that $ax + ay + az = a(x + y + z)$ since there is a rule that $ax + ay = a(x + y)$."), but Matz's work confirms that good problem solvers are not trapped into using it as a general strategy. (15)

Teaching Algebra

Many of the research studies that have investigated how young people learn algebra have also contained teaching components. Once the researchers have identified thought processes, successful strategies, errors, and misconceptions, they applied some experimental instruction in an attempt to eliminate the errors and misconceptions. For example, Herscovics and Kieran recognized how natural it is for teenagers to perceive algebra as generalized arithmetic and so they designed a sequence of instruction that can take advantage of this perception, while it minimizes some of the false generalizations many young people make. (9) In opting for this strategy, the researchers were on solid ground made evident by teacher effectiveness research, effective mathematics teachers identify and communicate the continuity of mathematics to their students. (2)

Herscovics and Kieran also chose to heed the research on cognitive development, and so avoided an early plunge into a totally symbolic approach to equations and unknowns. Instead, they began by working with their students (12- and 13-year-olds) on strictly arithmetic equations, focusing on the notion of equivalence and investigating the effects of various operations on equivalence. An essential component of their instruction was a sequence of nondirective questions aimed at giving students room to construct meaning for the concept of equation. Thus,

"Can you use the equal sign with an operation on both sides?"
produced

$5 \times 4 = 4 \times 5$
$2 + 6 = 6 + 2$

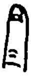
"Can you give me an example with a different operation on each side?"
produced

$5 + 5 = 5 \times 2$

"Can you give an example in which you have more than one operation on each side?"
produced

$4 \times 3 + 1 - 3 = 3 \times 2 + 4$
$3 + 5 + 4 = 12 - 4 + 4$

The researchers defined such identities as "arithmetic identities," leaving the term "equation" for the algebraic usage, and leaving themselves free to build the bridge from the familiar "arithmetic identity" to the less familiar "equation." Given the evidence that cognitive development is an everpresent influence on a young person's initial learning of algebra, the researchers recognized that bridging the two must parallel the bridge from concrete representations to abstract representations. Hence, the first step on the bridge was to cover one of the numbers in an arithmetic identity with a finger and to define "equation" as "an arithmetic identity with a hidden number."

"What's the hidden number in this equation?"
 $+ 5 + 4 = 12 - 4 + 4$

At the next level, the hidden number was represented pictorially, namely, with a box:

$6 + \square - 1 = 5 \times 2$

Finally, after working with equations represented in these concrete and pictorial ways, the students were ready to deal with the abstract representation:

$$2 \times a + 5 = 7 + 6$$

Interested teachers should read Kieran's papers (11, 12) and the article by Herscovics and Kieran. (9) Briefly, however, their research study led to the following two conclusions:

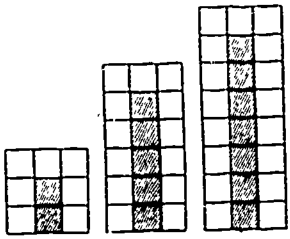
1. The throwback notion among young teenagers that equations are expressions of action in which numbers are acted upon and answers produced, though resistant to change, does tend to change after an instructional sequence like the one described above.
2. When Herscovics and Kieran tested their students after a summer layoff, they found that gathering unlike terms (e.g., saying that $7a + 5 = 40$ is equivalent to $12a = 40$) had become stronger, not weaker, as a conceptual scheme applied to simplify equations. As Kieran points out, however, research on how people process and retrieve information shows that old ideas die hard, even after instruction has seemed to put the wrong old ideas to rest. Renewed instruction—again based on the student's construction of meaning for the concept of equation—would seem to be necessary to allow many students to assimilate the appropriate schemes for handling equations.

The researchers made the following three recommendations:

1. A sequence like the one outlined above should precede the more traditional and typical initiations to algebra, like "think of a number" exercises and word problems.
2. To circumvent the sort of confusion about conserving variables that Wagner reported, a variety of letters should be used to represent hidden numbers in equations.
3. Teachers should not be too directive in teaching "Do the same operation to both sides of the equation." The students who worked with Herscovics and Kieran settled comfortably into this strategy as a way to *decide if two expressions were equivalent* ("Someone started to solve this equation: $6 + 39m - 4 + 2 = 43$. This was the way they started off: $6 + 35m + 2 = 43$. Is it alright to do this, or not?"), but the novice students, especially, did not find the strategy helpful in *solving equations*.

Cognitive development is an ongoing process of assimilating information into conceptual schemes and adjusting the conceptual schemes accordingly. It is not always a process that progresses smoothly, however, and so occasional lapses in students' algebraic skills and understanding should be expected by teachers.

When Hart and her colleagues realized from the CSMS study that so many British teenagers had little access to three of the six interpretations of letters used in equations, probably because of their not having developed yet beyond the concrete operational stage, they saw the need to recommend activities that are more pictorial than symbolic, and so allow concrete-operational students to construct meaning for the concept of variable. Here is a suggested exercise. (8, p. 118)



Ask students to find the number of white tiles needed for perhaps 10, 20, 40, and eventually 100 black tiles. Challenge them to come up with a rule that expresses the relationship between the numbers of black and white tiles.

With many pre-algebra students any discussion of patterns or rules need not be expressed symbolically or algebraically. What is more important is that they have the chance to become familiar with, and discuss, variable relationships such as the dependency in the above example of the number of black tiles on the number of white tiles.

The longer an algebraic misconception persists, the harder it is to remove it through instruction. Rosnick and Clement confirmed this principle when they worked with nine of the students who reversed the variables S and P in the student-professor equation. (18) They tried seven different ways to change the pattern of reversal, ranging from just telling the students that the reversal is incorrect to asking the students to draw graphs or to test the equations by plugging in numbers. At least seven of the nine students demonstrated to the researchers, in one way or another, that they maintained the reversal misconception even after the attempts at remediation.

What can teachers do to prevent misconceptions about algebra from becoming so deeply rooted? One strategy is to engage students in an *early*, pre-algebra process of constructing meaning for equations and variables, such as that proposed by Herscovics and Kieran. Teachers can also guard against the growth of misconceptions by carefully monitoring their own use of language in algebra classes. For example, Rosnick points out how easy it is for teachers to drift into careless remarks like "Let P = professors," rather than the more pedagogically sound "Let P = the number of professors." (17) As is clear from the work of the researchers cited in this report, both "equation" and "variable" (or "unknown") have multiple levels of meaning, and the meanings a teacher attaches to the concepts at any one time must match the meanings attached by his or her students.

Usiskin has developed an extensive program to incorporate meaning into the learning of algebra through the regular use of applications. (20) In particular, the goal of the program, Algebra Through Applications, is the construction of algebra out of real-world problems, rather than the application of an already-constructed algebra to real-world problems. Usiskin has eliminated some topics he believes need no attention in beginning algebra, such as traditional word problems ("John can shovel a walk in 3 hours and Mary can shovel it in 2 hours. How fast can they shovel the walk if they work together?"), trinomial factoring ($x^2 + 6x + 8 = (x + 4)(x + 2)$) and the manipulation of complex fractional expressions $\frac{x^3 - 1}{x^2 + 3x - 4}$.(21)

Instead, the program emphasized probability, statistics, operations, and problems and patterns arising from real situations, such as politics and various tasks in measurement.

The program has been independently evaluated, with groups using the program compared with groups taught traditionally in 17 schools. (19) For the most part, there was no significant difference in achievement between the two groups on achievement on several tests, although in 6 of the 17 schools the applications group did significantly better on a test designed to capture the materials in the program *and* did no worse than the traditional group on a standardized algebra test The researchers who conducted the evaluation concluded that, at the very least, the materials can be used with traditional first-year algebra textbooks as a source of relevant applications. They did, however, recommend that schools adopting the materials conduct a faculty seminar on their use.

Microcomputers promise to be a rich source of algebra learning activities. Since many algebraic investigations can become mired in lengthy computations, the computer can make such investigations more accessible.

Determining the solutions of polynomial equations provides one such example. Given a positive integer N and integers $A_0, A_1, —, A_N$, how could you find solutions for $A_N x^N + A_{N-1} x^{N-1} + \ldots + A_1 x + A_0 = 0$? For advanced algebra students the question is a rich one, loaded with potential mathematical learning, but prohibitive because of the computations required. Moursund has developed a program for the microcomputer that allows students to probe their way, using graphs and tables, to the discovery of solutions. (13, 16) Zabinski and Fine have shown how the computer can be used to develop a discovery approach to quadratic equations. (23) Landry has detailed how his students' use of microcomputers to approach a topic for which the computer is not particularly well-suited—solving linear equations like $3x + 2 = 4x - 7$—led with some incidental development to new and deeper insights into linear equations. (14)

The potential of the microcomputer as an algebra teaching and learning aid is undeniable. One study has confirmed the value. As a follow-up to their research into algebra misconceptions, such as the student-professor equation reversal, Clement, Lochhead, and Soloway attempted to find out if a basic introduction to computer programming, where variables clearly represent numbers and where an equation expresses the equivalence resulting from the interaction of variables and numbers, could change the misconceived tendency among many veteran algebra learners to perceive variables as labels for objects and equations as statements of correspondence between the labeled objects. (6) The researchers found that with just some introductory programming experience, most students who tended to reverse variables wrote equations *correctly* when they constructed them in a programming context ("At the last company party, for every 6 people who drank soda, there were 11 people who drank punch. Write a computer program in BASIC which will output the number of punch drinkers when supplied with the number of soda drinkers. Use S for the number of people who drank soda and P for the number of people who drank punch.") Concerning their results, the researchers hypothesized: "Computer programming apparently encourages an active, procedural view of equations that many students fail to use in the context of algebra." (6, p. 1)

The results of this study are exciting in themselves and they are even more exciting because of the compatability they hint at between computers and the teaching of algebra. Further research is imperative.

Conclusion

In this chapter we have focused on the concepts of variable and equation. Other topics are important to algebra teachers and students, of course, such as functions and graphing, but since the effective learning of all topics hinges on an understanding of variables and equations, we have chosen to concentrate on the two fundamental concepts in this report and to leave the others for treatment elsewhere.

Perhaps more during beginning algebra than at any other time in their school careers, secondary school students must see mathematics as a foreign language. There are multiple levels of meaning and various visual and symbolic representations. As with any foreign language, translation skills are essential to success, and algebra students must learn to translate between visual and symbolic representations and among the several levels of meaning for variables and equations. Helping students to learn these translation skills is one of the most difficult tasks faced by anyone in the teaching profession, and despite the promising textbooks and computer programs that have appeared and that will appear, the teacher remains at the center of that task.

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THE LEARNING AND TEACHING OF GEOMETRY

Does traditional deductive geometry have a future in the curriculum? Many students find it very difficult and I find it difficult to convince students they should take it.

Among adults' recollections of school mathematics, those connected with high school geometry are often the most vivid. The combinations of propositions, proofs, problems, and constructions that are encountered there seem to leave few individuals with lukewarm reactions. Either the experience was refreshing for them in its consistency and clarity, or it was painfully frustrating. Because the latter reaction is not uncommon, and because about 50 percent of high school students now choose not to enroll in geometry courses (14), some educators question the value of traditional Euclid-based geometry and wonder whether we shouldn't just let it disappear from the high school curriculum.

When it is viewed as the study of space and spatial relationships, and not just as the deductive system that Euclid built, geometry has an acknowledged solid footing in mathematics education. Usiskin (22) cites three reasons for this solid footing:

i) Geometry connects mathematics and the real world.

ii) Geometry enables ideas from other areas of mathematics to be pictured. For example, geometry lends visual aid to subjects like advanced algebra and calculus, and hence makes them more accessible to learners.

iii) Geometry is an example of a mathematical system--in fact, one of the earliest examples available to students.

The force of such standing in the mathematical family argues against eliminating geometry from the high school curriculum. Yet, undeniably, geometry lacks stature among high school students. In the recent National Assessment of Educational Progress (NAEP) in Mathematics, "doing proofs" received the lowest "I like" rating by seventeen-year-olds from a list of content topics. Worse yet, less than 50 percent saw the topic as important. (2)

Thus, the concern of teachers like the one whose questions opens this chapter is very realistic. It divides into two questions that may be addressed by researchers and curriculum developers. First, why do so many students have trouble learning deductive geometry? Second, what strategies and materials are available for making geometry understanding more accessible to students?

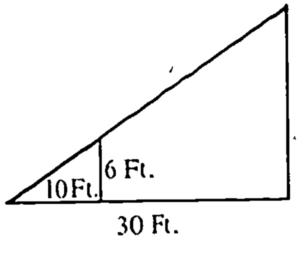
The remainder of this chapter treats each question in turn. In brief, an answer to the first question is taking shape from several research studies that have identified some apparent mismatches between traditional geometry instruction and the cognitive needs of most teenagers. Out of this and related research and development come answers to the second question. In particular, the strides made in delineating the several levels of geometric understanding, combined with the tremendous potential for using microcomputers to aid in geometry instruction, make it appear that geometry, somewhat changed in content and presentation will gain new life in the high school curriculum.

Difficulties in Learning Geometry

Young people can have a variety of difficulties in learning geometry, ranging from vocabulary to visualization, and from making deductions about the properties of geometric figures to applying those properties to real-world problems. During the past decade several studies have probed the nature of those difficulties, while others have sought their source.

In its geometry section the recent National Assessment of Educational Progress (NAEP) dealt primarily with geometry ideas students would probably encounter outside of a formal course in deductive geometry. The results showed that students could generally *recognize* geometry figures, but they were less successful in their knowledge about *properties* of those figures (for example, that the sum of the angle measures of a triangle is 180°). Furthermore, high school students who had taken geometry for a year generally scored much higher in knowledge about geometric properties than their peers with no formal training in geometry. Even in figure recognition, students tended to run into problems if certain vocabulary terms like "congruent" or "symmetric" were used. When a problem used the phrase "same size and shape" rather than "congruent," the success rate for the problem was considerably higher. (2)

Another NAEP conclusion dealt with problem solving. The majority of both 13-and 17-year-olds were unable to solve routine problems involving similarity of triangles or the Pythagorean Theorem. Among the students with a year's experience in a geometry course, slightly more than half solved the Pythagorean Theorem problem correctly, while two-thirds solved the following similarity problem correctly:

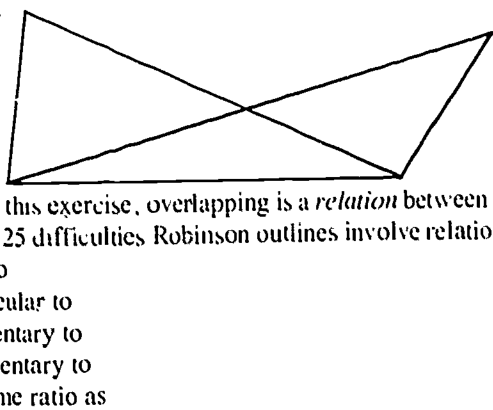


The picture shows how Jose used a short tree to find the height of the tall tree. What answer should Jose get?

Difficulties with Visualization and Vocabulary. The NAEP researchers were able to reach several other conclusions from students' incorrect answers. First, students can handle some geometric problems much better if they are able to deal directly with a visual representation of the problem than if they are required to work from an abstract representation of the problem. For example, when 13-year-olds were shown various triples of *line segments* and asked to select the triples that could serve as sides of a triangle, two-thirds of them were able to do the task successfully, yet when they were given the same task with only *number triples* to work with, almost 90 percent failed the task. More than 80 percent of seventeen-year-olds failed the same task. (2)

A second NAEP conclusion concerning students' interpretations of diagrams will come as no surprise to most geometry teachers. When hard evidence is lacking, students will often make conclusions based on appearances alone. For example, in a diagram in which insufficient information was given to allow students to deduce the size of a given angle, 30 percent of the seventeen-year-olds were duped by the angle *appearing* to be 90°. Among the subgroup of seventeen-year-olds with a year's geometry experience, however, two-thirds responded correctly that there was insufficient information.

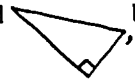
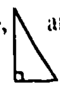
In a similar vein, Robinson has listed 25 common difficulties and misconceptions of students in geometry. (20) For example, she found that many students have trouble recognizing overlapping triangles, as in the task of pointing out why there are more than three triangles in the following picture:



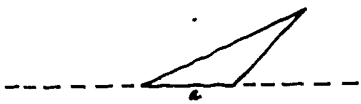
As it appears in this exercise, overlapping is a *relation* between triangles and, in fact, most of the 25 difficulties Robinson outlines involve relations, including:

is parallel to
is perpendicular to
is supplementary to
is complementary to
is in the same ratio as
is congruent to
bisects

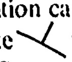
Vollrath studied one particular difficulty many students have, namely, recognizing *similar* geometric shapes. (26) Working with young people who ranged in age from 8 to 19, Vollrath set them to work on sorting tasks---exercises in which they were shown collection of shapes and asked to group together all shapes in a particular collection that fit a certain criterion. At times the task was expressed as "Put all similar figures together", at other times, the instruction was "Put all figures with the same shape together." Neither task proved easy at any age level, but an interesting phenomenon revealed itself, one that underscores the importance of teachers' awareness of their own use of language. Sorting according to "similar" led students to focus on attributes of particular figures within a grouping, while "same shape" led them to focus on broader patterns. For example, "similar" led to sorting out all rectangles in a group with the same width, while "same shape" led to sorting out *all* the rectangles, as if the students were identifying "same shape" with "shape-name".

Vinner and Hershkowitz tested over 500 students in grades 7, 8, and 9 to identify what kinds of images young people attach to certain geometric concepts (25) For example, when asked to circle in a group of drawings all of the *right triangles*, fewer than 70 percent included  and  barely 40 percent included . Apparently,  and  are images associated so tightly with the concept of right triangle that there is little room for variation.

Similarly, when they asked the students to draw *altitudes* to various triangles, fewer than 10 percent were able to do it correctly for side *a* in a triangle such as



The researchers concluded that, in the minds of most young people, the image of altitude to the base of an isosceles triangle replaces a more general image for the concept of altitude.

Fisher studied how students distort geometric concepts because of the influence of pictures. (5) In particular, she was concerned with the type of distortion in which students make incidental visual clues into essential features of a concept. Thus, a vertical-horizontal orientation can become so attached to the concept "perpendicular" that something like  will not seem to be an example of the concept. Fisher asked the question, "Do students form concepts that are biased in favor of upright figures?" Her study of 6th-graders, 9th-graders and college students convinced her that the answer is yes and that, regardless of instruction favoring upright figures or instruction favoring a variety of orientations, students at all levels can more easily recognize upright figures than tilted figures as examples of concepts.

Proof leads to student difficulties that are probably the most conspicious of all such difficulties to geometry teachers, in good part because proof is one of the most sophisticated challenges in all of school mathematics. The chapter "The Path to Formal Proof" focuses on the topic in detail. The present chapter views it in context, as one of the last in a line of difficulties, and concentrates on the kinds of concept and skill development young people must pass through in order to be ready for deductive proof.

One model of that development that has received considerable recent attention is the van Hiele model. According to the model, all children progress through several levels of geometric understanding, and the van Hieles have claimed that a combination of time, content, and teaching methods will carry each child from one level to the level following it. Several very good descriptions of the van Hiele model are available, and interested teachers can read about it in detail in References 3, 10, 20, 23, 24, 27. Hoffer's description (10) is very well written, and because he includes many examples with his description, we strongly urge teachers to read it. Briefly, as he describes them, the proposed levels are:

Level 1. *Recognition.* The student learns some vocabulary and recognizes a shape as a whole. For example, at this level a student will recognize a picture of a rectangle but usually won't be aware of many properties of rectangles, such as parallel opposite sides.

Level 2. *Analysis.* The student analyzes properties of figures. At this level a student may realize that the opposite sides of a rectangle are parallel and congruent, but will not yet notice how rectangles relate to squares or right triangles

Level 3. *Ordering.* The student can logically order geometric figures (for example, all squares are rectangles, but not all rectangles are squares), and understands interrelationships between figures and the importance of accurate definitions

Deductive thinking skills are not fully developed at this level. Although students at this level may be able to understand the relationship of the class of squares to the class of rectangles, and the relationship of the lat. r to the class of parallelograms, they may not be able to deduce why the diagonals of a rectangle are congruent.

Wirszup has described how deductive thinking begins to take shape at Level 3. As students collect the visual properties of various shapes, the growing collection asks for organization, and that is the start of deductive thinking (27)

Level 4. *Deduction.* The student understands the significance of deduction and the role of postulates, theorems, and proof.

At this level students will be able to use postulates to prove statements about rectangles and triangles, but this thinking may lack enough rigor for them to understand why the postulates are necessary.

Level 5. *Rigor.* The student understands the importance of precision in dealing with foundations, such as the collection of axioms and postulates at the foundation of Euclidean geometry.

This is a level of sophisticated thinking rarely reached by high school students. Later, in college mathematics, many will be able to reach an overview of Euclidean geometry that permits them to adjust to the different systems of non-Euclidean geometry, where rectangles, for example, do not exist. In the meantime, however, the vast majority of high school students never get beyond honing their deductive thinking skills at Level 4, if they reach that level at all.

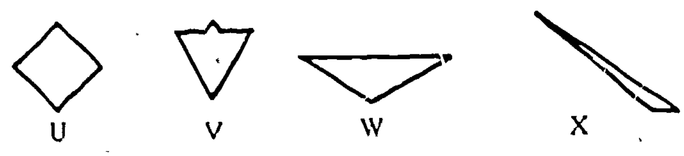
The proof skills that reach fruition at Level 4 begin to develop at earlier levels and evolve through their own set of sublevels. Van Dormolen (24) has described three such sublevels:

a. Proofs that are *very local* and *case-specific*. For example, if asked to describe the effect on points in the plane of a rotation through 90°, students at this sublevel will pick a point, say (2,3), carefully rotate it 90° to get (−3, 2), and stop there.
b. Proofs that are more general, focussing on *collections of similar objects or cases* and not on single cases. With the above example, students at this sublevel can look beyond the single case (2, 3). In fact, they might even deduce that (2, 3) rotates to (−3, 2) from the realization that (2, 0) rotates to (0, 2) and that (0, 3) rotates to (−3, 0).
c. Proofs that are *general*. At this sublevel, students can prove that any point (a, b) rotates to (−b, a).

It is important to point out that the van Hiele model of levels of geometric understanding is a hypothetical model. In fact, van Hiele himself has recently voiced doubts about the existence of Level 5. (23) However, several studies have shown that the model is a valuable tool for exploring geometric understanding.

Usiskin and Senk conducted a study of several thousand high school geometry students to determine what changes in van Hiele levels take place during the year of geometry, and to determine how well the van Hiele levels of students entering high school geometry can predict the level of their proof skills at the end of their year in geometry. (21, 23) Of course, in order to make these determinations, the researchers had to begin by determining how many of the students fit the van Hiele model and, of those who did, what their levels were. With sets of questions representing tasks at each level as the gauge, the researchers ruled out students who qualified for two levels but not the level in between. Here are two examples of the questions they asked, one for Level 1, the other for Level 3:

Level 1: Which of these are triangles?



(A) None of these are triangles.
(B) V only
(C) W only
(D) W and X only
(E) V and W only

Level 3: What do all rectangles have that some parallelograms do not have?

(A) opposite sides equal

(B) diagonals equal
(C) opposite sides parallel
(D) opposite angles equal
(E) none of (A)-(D)

According to their criterion, about three-quarters of the students fit the van Hiele model. Remarkably, over one-half of those students were at Level 1 or below!

The study determined that during the year of geometry, more than 50 percent of those students at the lowest level moved to Levels 2 or 3, but that about a third of them remained at Level 1. (23) Furthermore, the study found that after a full year of a geometry course with proof, only about half of the students could do more than simple proofs. (21) Finally, as a predictor of how well students will do with proof after a year-long geometry course, the van Hiele model proved to be somewhat successful. In particular, it appears that if a student enters geometry at Level 1 or below, there is little chance of success with proof. Level 2 implies a better than even chance of success, and Level 3 and above imply a good likelihood of success. (21, 23)

Since Mayberry's research indicates that a student may be on different van Hiele levels with respect to different topics within geometry (18, 23), it is clear that researchers have not finished the task of defining the development of geometric understanding. In general, however, the van Hiele model is proving to be a valuable tool for the task.

Instructional Strategies

Once teachers get a clearer view of students' difficulties with geometry and the sources of those difficulties, they can begin to adjust their teaching strategies accordingly. For example, Wirszup points out that one implication of work with the van Hiele model is that maturation in geometry is a process of apprenticeship, and not just of development. (27) Consequently, teachers need to explore ways to smooth their apprentices' learning at each level. Coxford suggests the following list for ctivities at the first 4 van Hiele levels (3):

Level 1. Individual figure-recognition, production, and naming.

Level 2. Determining properties of the figures.

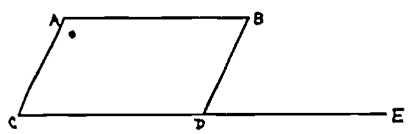
Level 3. Determining relationships between the figures and their properties. Thus, what are the properties of parallelograms, how do rectangles fit in with parallelograms, and so on.

Level 4. Use of Level 3 knowledge to study geometric facts from a deductive approach. Thus, what can be deduced about the interior angles of parallelograms once it has been established that opposite sides are parallel?

All of these activities should be carried out through class discussions, based on student observations and hypotheses. Reiterating what van Dormolen pointed out, a student settles into deductive proof only in stages, first concentrating on single instances of a phenomenon, later collecting similar instances into a patterned array, then finally arguing on the most general level. When students can verbalize their understanding, or lack of understanding, of phenomena, their piecing together of clues to discover patterns, and their logic in deducing properties, then their apprenticeships can proceed more smoothly.

Some recent research by Greeno and Magone promises to strengthen the teacher's hand in proof instruction. (6) At the heart of their study was a conviction that students must understand what proofs *are* before they can understand how to construct them. In particular, they must appreciate that the rules of formal deductive proof are more stringent than the rules of everyday argument. In the latter case, statements are expected to be reasonable and noncontradictory in the light of previous information. Each statement in a formal proof, however, must not only be consistent with previous statements that have been accepted, but must necessarily follow from them.

The researchers call this "the principle of deductive consequence" and they decided that a reasonable gauge of students' understanding of the principle would be a test of proof-checking, give students some completed geometric proofs with hidden errors and challenge them to find the errors. In some instances, the error might be the listing of a reason for which no geometry theorem or postulate exists, in others, the error might be the use of a theorem whose conditions had not been established, either in the statement of the original problem or in previous steps in the proof. In the following example, the reason listed for Statement 1 does not apply because it has not been established that AC is parallel to BD.



Given: $\overline{AB} \parallel \overline{CE}$ and $\overline{AB} \cong \overline{CD}$
Prove: $\angle ACD \cong \angle ABD$

Statement	Reason
1. $\angle ACD \cong \angle BDE$	corresponding $\angle$s
2. $\angle BDE \cong \angle ABD$	alternate interior $\angle$s
3. $\angle ACD \cong \angle ABD$	transitive property

The researchers designed a two-hour training program in checking proofs for 15 college students who had taken geometry in high school but who were not very good at checking proofs. The program taught the students to apply the following 5 steps whenever they were checking proofs:

1. Check if the reason given for a statement is a valid definition, theorem, or postulate.
2. Divide the reason into its "if" and "then" components.
3. Check the "if" part. Has it been shown previously in the given informa-

tion, in the diagram, or in the previous steps of the proof?

4. Check the "then" part. Does it match the relationship in the proof statement being checked?
5. If the staement is the last one in the proof, does it match the goal of the problem?

The students were tested in proof-checking along with 15 comparable students who did not take part in the two-day training session. The trained students were superior in their proof-checking. For example, 8 out of the 15 training program students were able to detect the error in the example given above, while only 1 out of the other 15 students was able to detect the error. Furthermore, observers reported that the program students appeared to read the statements of a proof more carefully than the others. Finally, in a test to see how well the 30 students could *construct* proofs, the 15 program students were superior to the others, prompting the researchers to remark: "It seems likely that the training in proof-checking gave subjects some skills that facilitated their performance in proof construction problems as well." (6, p. 36). The benefits of the two-hour training session seem remarkably rich, and geometry teachers should consider doing some similar training of their students in proof-checking.

In his article "Geometry Is More Than Proof," Hoffer (10) points out that learning geometry involves five kinds of skills :

- spatial
- verbal
- drawing
- logical
- applied

He describes the kinds of activities that typify each of these skills for each of the van Hiele levels, then lists activities that are appropriate for each of the skills at each level. For example, *a verbal skill* on *Level 3* is "Formulates sentences showing interrelationships between figures." A *verbal activity* for *Level 3* is "Write a careful and brief definition of the word 'rectangle'".

Transformation Geometry

One vehicle that many educators propose for paving the way through the levels of geometric understanding is transformation geometry, that is, the study of reflections (sometimes call flips), rotations (spins), and translations (slides) in the plane. Robinson, whose work was cited earlier as revealing the prevalence among students of difficulties with relational terms, argues that by studying the effects of such transformations—in particular, what properties stay invariant under them—students can develop meaning for relational terms like "congruence", "perpendicularity" and so on.

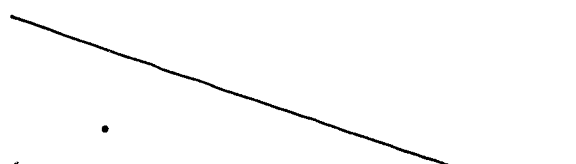
For example, congruence remains invariant under reflections, but orientation (clockwise vs counterclockwise) does not. By experimenting with combinations of transformations and discussing their effects, students can develop meaning for relational terms their teachers will introduce shortly thereafter.

Transformation geometry is not new to mathematics education. In fact, the nineteenth-century mathematician Felix Klein defined geometry as the study of what properties remain invariant under different sets of transformation. In the case of Euclid's geometry, that set comprises combinations of reflections, rotations, and translations in the plane. Transformation geometry gained some prominence in the so-called "modern mathematics" movement of the nineteen-sixties, and it has always been considered part of informal geometry, traditionally suggested for study in grades 7 and 8 as "the physical geometry of the space we live in, rather than as an abstract mathematical system." (19)

In most American school systems, if transformation geometry has been taught at all, it has been taught at the junior high level and separated by at least a year from formal deductive geometry. This is not the case in some other countries, nor is it the recommendation of most mathematics researchers in this country, who see a solid role for the topic in high school. Kartowski has pointed out that the Soviet mathematics curriculum gives equal weight to two aspects of geometry, the spatial-visual and the logico-deductive. (13) Wirszup has studied the Soviet experience in designing a geometry curriculum around the van Hiele model. They pay careful attention to the "apprentice" aspect of the theory, according to which a student's development through the various levels depends on instruction and curriculum content, as well as biological maturation. (27) Accordingly, the Soviets are quite specific in their geometry objectives, for example, aiming to begin Level 3 work in grade 4. Because of its natural combination of the spatial-visual with the logico-deductive, transformation geometry plays a crucial role in this curriculum.

In Great Britain the role of transformation geometry in the curriculum has not been as substantial as in the Soviet Union, but a recent major study has underscored the advisability of beefing up its role. Hart and her colleagues conducted the Concepts in Secondary Mathematics and Science (CSMS) study, a project similar in scope and in many of its goals to the NAEP study in the U.S. (7) The study consisted of interviews and tests of British students between the ages of 11 and 16. One of the mathematics sections contained a series of questions involving combinations of rotations and reflections. The tasks ranged from relatively simple questions like:

> What is the image of the given point when it is reflected through the given line?



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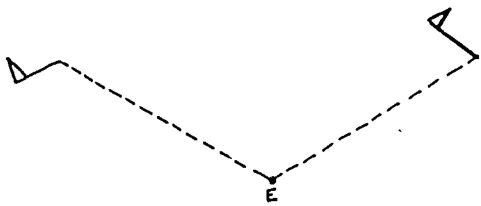
to a more difficult challenge such as:
"Explain why E is not the center of rotation in this picture."



The tasks that involved combined reflections and rotations were the most difficult of all for the students, so difficult, in fact, that Hart concludes that transformation geometry can be a viable and challenging topic in the curriculum, separate from but also complementing deductive geometry: "The approach being advocated is one that directs children toward discoveries from which the rules and properties of the transformations can be surmised and against which they can be tested." (7, p. 157) To this Hart adds her firm belief that "such activities are vital to the development of critical thinking." (7, p. 157)

Research confirms the complementary nature of the spatial-visual and the logico-deductive in the learning of geometry. In fact, the accepted description of concept development, especially relevant to geometry, pictures children first acting on objects, then internalizing the actions, and finally forming conceptual representations. (1) This process, for which there are no shortcuts, holds for older children as well as younger children. Consequently, students need extended opportunities to internalize geometrically-related actions *before* deductive geometry. Transformation geometry offers the framework for such opportunities As Hart describes it. "The transformations can be internalized in gradual steps, by focusing first on the actions themselves, then on their representation, and then on the representation of imagined actions. In addition, the resulting drawings can be checked at each step by a return to the actions or by reference to drawings of similar problems." (7, p. 157)

The task of integrating the spatial-visual with the logico-deductive in the geometry curriculum will not be simple, and it will require further research attention. As Fisher discovered in her study cited earlier, students form concepts that are biased in favor of upright, as opposed to tilted, figures. Although the bias doesn't necessarily stand in the way of geometry learning, it *is* resistant to instruction. Kidder's research into students' comprehension of reflections, rotations, and translations revealed that few eighth-graders were able to form a *mental image* of a figure in the plane and then to mentally perform one of the three transformations on it. (15). Mental representation of such geometric actions seems to demand more cognitive sophistication than is available to most eighth-graders, and this gives further impetus to the inclusion of transformation geometry in the high school curriculum, as a complement to deductive geometry.

If transformation geometry can be as valuable a complement to deductive geometry as research suggests, a question must naturally arise in the minds of educators. "Where do we make room for it in the high school curriculum?"

Usiskin has suggested a route toward making room by calling for the careful reduction of material ordinarily covered in deductive geometry. (22) In particular, he recommends eliminating:

a) Early vigorous proofs of obvious statements involving points, lines, and angles. He maintains that the facts should be covered, but informally.
b) Expectations that students will be competent with the same general proofs written in the same general way. Usiskin claims: "For judging a proof, clarity is a more important criterion than the amount of detail." (22, p. 421) As corroborating evidence he cites the Soviet research indicating that one of the hallmarks of a capable mathematics student is the ability and propensity to find shortcuts in mathematics arguments.
c) The least important theorems. Usiskin suggests the elimination of 6 sets of theorems, which he estimates as amounting to 2-3 weeks of class instruction.

Some may consider Usiskin's suggestions drastic, but their intention is to inject new life into the teaching of deductive geometry, an area of the curriculum that needs and deserves new life. The research summarized in this chapter has two main messages that have a bearing on this new life. the timely involvement of students in deductive geometry is a very important and irreplaceable mathematical experience, the timeliness of that involvement depends upon a carefully nurtured apprenticeship in the development of all the skills and understanding that must precede deductive geometry. As Wirszup has written. "The teacher's role in this apprenticeship includes choosing materials that can orient students toward becoming familiar with geometric structures, organizing conversation so that the structures can be uncovered, using customary terms once the structures have been uncovered, assigning tasks that can be carried out in different ways so that students can orient themselves freely, and finally guiding the students to integration by helping them to condense to a whole the domain their thought has explored" (27, p. 83)

Conclusion

In essence, Wirszup is underlining the essential role that the teacher plays in geometry instruction, a role that can be made a bit less weighty by some of the very good materials that are available for classroom use In the category of textbooks that aim toward complementing deductive geometry as outlined above are Hoffer's *Geometry, A Model of the Universe* (9) and Coxford and Usiskin's *Geometry. A Transformation Approach* (4). If teachers use other textbooks, there are several sources of ideas and activities that will enrich the teaching of deductive geometry. Hart's report on the CSMS study (7) contains a series of exercises in transformation geometry, as does the National Council of Teachers of Mathematics (NCTM) yearbook on the teaching of geometry (8). Krause's *Taxicab Geometry* (16) and the NCTM's *A Sourcebook of Applications of School Mathematics* (11) are both excellent sources of ideas for enriching geometry instruction.

One final source of support for teachers deserves mention, namely, the exciting potential of microcomputer software for improving the learning of geometry. In particular, the microcomputer could prove to be the best bridge yet between the spatial-visual aspects of geometry and the logico-deductive aspects Kantowski has explored this potential in the context of describing several programs she and her colleagues have developed. (12, 13)

One program shows students a polygon, with anywhere from three to seven sides. By manipulating the computer controls, the students can rotate, reflect, or translate the polygon on the screen to make it match the orientation of a second, identical polygon. In the course of their manipulation, the students must tackle such concepts as angle, parallel lines, and so on.

In the second program, the computer was programmed to list, at certain points during geometric proof, several categories from which a student could choose the type of information desired. For example, at each decision point, the student can choose to see a relevant diagram or a list of relevant theorems and definitions. The hints are provided by the computer, but the choice is the student's and he or she must learn what kinds of information will help the most at various points in a proof

There is no doubt that high school geometry has been suffering through a period of the doldrums. The research and development activities described in this report have opened up the very real possibility that geometry will once again become an exciting subject to learn and an exciting subject to teach.

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THE PATH TO FORMAL PROOF

Constructing proofs is a very difficult task for many of my students. They can't even get started on most proofs. How can I help them to analyze a question or problem well enough to discover a starting point for a proof?

On one level of learning, the role of proof is clear, it is the fundamental tool for extending the field of mathematics. Yet few secondary level students aspire to be mathematicians, so educators have had to search elsewhere to define a role for proof in secondary school mathematics. In varying degrees since the time of Euclid, proof has been touted as a means to discipline the mind to think in an orderly fashion, as a vehicle for improving logical thinking, and as a stimulus toward the kind of responsible, critical and reflective thinking that should be the mainstay of democratic life.

Today's secondary school teachers are asking pressing questions about the role of proof in the mathematics curriculum. Where, for example, should proof be taught in the curriculum? How tied should it be to the teaching of geometry? How can we motivate the many students we face who are reluctant to learn how to construct proofs? Do skills in proof-construction carry over to other mathematical thinking skills? Do they transfer to non-mathematical subject areas?

Research has shed some light on these issues, and this report reviews the results that have a direct bearing on the mathematics classroom and on the responsibility of the mathematics teacher. That responsibility is threefold. to induce students to appreciate the value of proof, to teach them what a proof is, how to follow one, and how to distinguish proof from non-proof, and, to help students develop skills in proof construction.

In general terms, proof is the process of reasoning from a set of premises through a series of connected inferences to a conclusion, in such a way that any doubt about the conclusion must be referred back to the premises, rather than to the logical necessity of the inferences. In mathematics there are five major methods of proof. *direct proof* (starting with proposition P, a chain of "If . . then" inferences arrives at proposition Q, so P implies Q); *proof by use of the contrapositive* (showing that the negation of proposition Q implies the negation of proposition P is equivalent to showing P implies Q), *reductio ad absurdum* (assuming that P does not imply Q will often produce a logical absurdity, thus assuring that P implies Q), *proof by enumeration* (in certain cases it is possible to prove a proposition by enumerating the instances it encompasses), finally, *proof* *by existence* (some propositions assert the existence of a mathematical phenomenon or situation under certain conditions, and proof entails the construction of that phenomenon or situation). (13) Besides these five methods of proof in mathematics, there are two methods of disproof: *disproof by contradiction* (to show that under certain assumptions a proposition is not true, it is often possible to show that the truth of the proposition would lead to a logical contradiction of one or more of the assumptions) and *disproof by counterexample* (to disprove a proposition it suffices to find one example that satisfies the conditions of the proposition but not its conclusion).

Each of these methods of proof and disproof appears in secondary level mathematics, yet mathematics teachers should take nothing for granted about their students' understanding of the use of these methods or even about their acceptance of the need to use them. Discussions about each of these methods appear in (1) and (4). The book by Baxandall *et al.* (1) also contains many examples of the use of these methods, especially in the contexts in which they appear in the upper secondary grades and in the first years of college mathematics.

Students and Proof

Most students come neither quickly nor naturally to the use of mathematical proof. The unrushed pace of cognitive development limits the ability of most children to reason hypothetically or deductively until they are between 13 and 15 years of age. Even then, apparently, many have little to appeal to in their experience when they face mathematical proof. Williams surveyed eleventh graders and found that fewer than 30 percent exhibited any understanding of the meaning of proof in mathematics, that approximately 60 percent were unwilling to argue, for the sake of argument, from any hypothesis they considered false, and that there was "no evidence that high school students understand that a mathematical statement and its contrapositive are equivalent." (19, p. 166) (For example, the following two statements are equivalent, with the second the contrapositive of the first: "If a four-sided figure is a rectangle, then its diagonals are congruent." "If the diagonals of a four-sided figure are not congruent, then it is not a rectangle.") Obviously, then, many secondary students are limited by their cognitive development, by their lack of prerequisite understanding, and by their lack of experience. In order to understand why so many students have difficulty with proof, it is essential to isolate developmental limitations and the ways in which they make themselves known, to determine the nature of prerequisite understanding, and to isolate those misconceptions about proof that are more social in nature and are due to inexperience.

Cognitive Development

For an extensive treatment of cognitive development we refer you to the chapter "Individual Differences in Secondary School Mathematics." The present chapter is limited to the implications of development for children's abilities to understand and construct proofs. To that end, it suffices to point out that during the period between ages 13 and 15--earlier for some children, later for others--children pass from the stage called by many developmental psychologists *concrete operational thinking*, when their thinking is almost completely tied to their perceptions, to the stage of *formal operational thinking*, when they can begin to think hypothetically and deductively, to hold more than one variable in mind at a time, and to think about their thinking. In one research study a group of secondary students was given the following exercise. "All successful scientists work hard, and Mr Smith is a scientist who works hard. Can we say from this that Mr. Smith is a successful scientist?" About 25 percent of the fourteen-year-olds who responded to this question answered "yes," and nearly 20 percent answered "no," but gave a poor reason for their "no" answer. (13) Commenting on this, Lovell remarked that although it is not impossible for concrete operational individuals to solve this kind of exercise, it does not lend itself to imaging very easily-- that is, the set of scientists who work hard but are not successful is never mentioned and an image of that set does not arise very readily from a reading of the problem. Thus, the exercise becomes very difficult for students whose thinking is still tied to sensory perception.

In several of its logic exercises the recent National Assessment of Educational Progress (NAEP) underscored the role of cognitive development in successful deductive thinking by revealing that the jump in performance between the 9-year-olds and the 13-year-olds tends to be much greater than the jump between the 13-year-olds and the 17-year-olds. (3) In particular, on an example similar in structure to the one used by Lovell in his experiment—"Every flyer is crazy. Chris is crazy"—a correct conclusion that there is not sufficient information to decide whether Chris is a flyer was reached by 25, 51, and 58 percent, respectively, of the 9-, 13-, and 17-year-olds. It is noteworthy that so many of the 17-year-olds—more than 40 percent-- were not able to deduce the appropriate answer.

Reys and Grouws looked at one particular topic where many students are challenged to understand a proof before they have developed to the stage of formal operations, namely, division by zero. (15) One recommended way for teachers to approach a question like "What is 6 divided by 0?" is to remind students of the close relationship between multiplication and division: $4 \times 3 = 12 \rightleftarrows 12 \div 4 = 3$. Thus, for a related pair of statements like $3 \times \square = 6 \rightleftarrows \square = 6 \div 3$, the same number, in this case 2, fits in both squares. Since $0 \times \square = 6 \rightleftarrows \square = 6 \div 0$ are related statements and since the left statement has no solution (0 times any number is 0), there is no solution for the right statement, either.

This is the kind of reasoning students must understand in order to understand why $6 \div 0 = ?$ has no solution. In their testing and interviewing of fourth, sixth, and eighth-graders, Reys and Grouws found that an understanding of such reasoning required a facility with translating back and forth from multiplication to division and an understanding of zero as a number. Furthermore, the logic and symbolism involved in the proof make cognitive development a factor the more cognitively mature students are, the greater are their chances of deducing that division by zero has no solution.

Bell points out that a child's path to formal proof begins with early attempts at making and establishing generalizations. Children can recognize patterns and relationships, even extend and describe them, but they cannot justify or deduce them. (2) Often the generalizations are unreasonable, too. In the research study described by Lovell, secondary level students were presented with a version of the famous, and unproven, Goldbach Conjecture: Every even number can be expressed as the sum of two prime numbers. The students were told to study the following list:

$2 = 1 + 1$	$10 = 5 + 5$
$4 = 3 + 1$	$12 = 7 + 5$
$6 = 3 + 3$	$14 = 7 + 7$
$8 = 5 + 3$	$16 = 11 + 5$

(and so on, to 16 instances concluding with $32 = 3 + 29$)

They were then asked, "Do these facts show that every even number can be put as the sum of two prime numbers?" Approximately [illegible]9 percent of the students in the 14- to 15-year old range answered, in effect, "Yes, enough evidence." To Lovell, these data implied that many of the students in this age range have not fully developed the deductive thinking skills that could prevent such hasty and unexamined generalizations. (13)

Oddly enough, although many secondary level students have a tendency to jump too quickly from patterns to generalizations, it is also common for adolescents to ignore the conclusions they could draw from counterexamples. Galbraith reports a study in which students, most of whom were between 13 and 15 years of age, were asked to evaluate a rule proposed by an imaginary student Brenda - for predicting which whole numbers have the property that the sum of their digits are divisible by 7 (for example, 43, 70, 383). (6) Brenda's rule was: "Every number with this property can be found by adding 9 to the previous number. You start with 7." When the students were alerted to a counterexample to Brenda's rule (for example, 59 has the property but 50 does not), approximately one-third of the students did not see the significance of the counterexample in refuting Brenda's rule. As one student said: "Could be a freak accident—one in a million chance."

In the years before children can fully take hold of formal proving skills, other developmental factors can affect their progress. For example, verbal and writing skills have a bearing on proof skills. Lester remarks: "An examination of research involving the logical-reasoning abilities of young children reveals that these abilities may be superior to their ability to put an argument in written form." (11, p. 15) And Hoffer warns, "Precise formulations may be thrust on students before they are ready—before they have an opportunity to describe concepts themselves and recognize the lack of precision in their statements." (8, p. 12)

Levels of Prerequisite Concepts and Skills

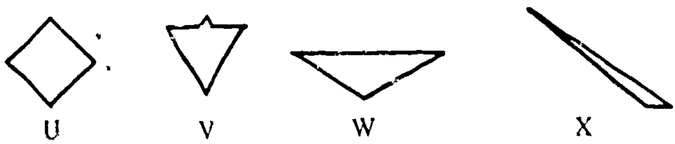
Most students have little experience with deductive proof before high school geometry. Even then, many lack the prerequisite skills and conceptual understanding that would permit them to understand and use formal deductive proofs Several research studies have attempted to delineate those prerequisites for deductive proof, and one result of those efforts is the van Hiele model of the levels of mental development in geometry, a model named after the Dutch researchers who first hypothesized it.

According to the model, all children progress through several levels of geometric understanding, and the van Hieles have claimed that a combination of time, content, and teaching methods will carry each child from one level to the level following it. As described by Hoffer (8), the proposed levels are:

> Level 1. *Recognition*. The student learns some vocabulary and recognizes a shape as a whole. For example, at this level a student will recognize a picture of a rectangle but usually won't be aware of many properties of rectangles, such as parallel opposite sides.
>
> Level 2. *Analysis*. The student analyzes properties of figures. At this level a student may realize that the opposite sides of a rectangle are parallel and congruent, but will not yet notice how rectangles relate to squares or right triangles.
>
> Level 3. *Ordering*. The student can logically order geometric figures (for example, all squares are rectangles, but not all rectangles are squares), and understands interrelationships between figures and the importance of accurate definitions.
>
> Deductive thinking skills are not fully developed at this level. Although students at this level may be able to understand the relationship of the class of squares to the class of rectangles, and the relationship of the latter to the class of parallelograms, they may not be able to deduce why the diagonals of a rectangle are congruent.
>
> Wirszup has described how deductive thinking begins to take shape at level 3. As students collect the visual properties of various shapes, the growing collection asks for organization, and that is the start of deductive thinking. (20)
>
> Level 4. *Deduction*. The student understands the significance of deduction and the role of postulates, theorems, and proof.
>
> At this level students will be able to use postulates to prove statements about rectangles and triangles, but this thinking may lack enough rigor for them to understand why the postulates are necessary.
>
> Level 5. *Rigor*. The student understands the importance of precision in dealing with foundations, such as the collection of axioms and postulates at the foundation of Euclidean geometry. This is a level of sophisticated thinking rarely reached by nigh school students.

Usiskin and Senk conducted a study of several thousand high school geometry students to determine what changes in van Hiele levels take place during the year of geometry, and to determine how well the van Hiele levels of students entering high school geometry can predict the level of their proof skills at the end of their year in geometry. (17, 18) In order to make these determinations, the researchers began by determining how many of the students fit the van Hiele model and, of those who did, what their levels were. About three-quarters of the students fit the model. Remarkably, over one-half of those students were at Level 1 or below. The following are two examples of the questions asked:

Level 1: Which of these are triangles?



(A) None of these are triangles.
(B) V only
(C) W only
(D) W and X only
(E) V and W only

Level 3. What do all rectangles have that some parallelograms do not have?

(A) opposite sides equal
(B) diagonals equal
(C) opposite sides parallel
(D) opposite angles equal
(E) none of (A)-(D)

The study determined that during the year of geometry, more than 50 percent of the students at the lowest level moved to Levels 2 or 3, but that about a third of them remained at Level 1. (18) Furthermore, the study found that after a full year of a geometry course with proof, only about half of the students could do more than simple proofs. (17) Finally, as a predictor of how well students would do with proof after a year-long geometry course, the van Hiele model proved to be somewhat successful. In particular, it appears that if a student enters geometry at Level 1 or below, there is little chance of success with proof. Level 2 implies a better than even chance of success, and Level 3 and above imply a good likelihood of success. (17, 18)

So far, the van Hiele model shows great promise as a tool for delineating the skills and conceptual understanding that must precede students' work with formal deductive proof. Further research must sharpen the delineation and also identify the classroom activities that are most appropriate for each level (Hoffer's article

(8) proposes some geometry activities for each level and interested teachers can refer to it.)

The research cited so far has made it clear that many adolescents contend with a variety of obstacles to the learning of the rules of formal proof:

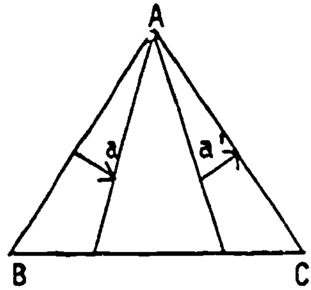
- an inability to think hypothetically or to express their reasoning in writing.
- a lack of clarity about the equivalence of a mathematical statement and its contrapositive.
- an unwillingness to accept the conclusive evidence presented by a counterexample.
- a tendency to generalize too quickly from recognized patterns.
- a lack of prerequisite skills and conceptual understanding.

There are other obstacles, too. In his research report, Galbraith noted a tendency of many adolescents to focus on only part of the statement of a proposition, and tendencies as well to change the conditions of a proposition to suit the direction of their own thinking or even to be subjective in their assessment of a proof ("Maybe Brenda didn't mean to say 'every'.") (6)

It is important for teachers to note that inexperience is also bound to affect a young person's work with proof. For much of their lives, adolescents have attempted to win arguments primarily on subjective grounds. Bell pointed to this influence of inexperience and the role of the teacher in dealing with it when he said: "It follows . . . that pupils will not use formal proof with appreciation of its purpose until they are aware of the public status of knowledge and the value of public verification. The most potent accelerator towards achievement of this is likely to be cooperative, research-type activity by the class. In this, investigation of a situation would lead to different conjectures by different pupils, and the resolution of conflicts by arguments and evidence." (2, p.25)

The Teacher and Proof

Perhaps the most extensive attempt to create the kind of classroom experience which Bell recommends was Fawcett's classic teaching experiment during the nineteen-thirties. (5) The experiment lasted two years and drew much of its life from Fawcett's conviction that, with appropriate guidance, secondary students can learn to think critically, reflectively, and deductively, and can learn to apply that thinking both to mathematics and to nonmathematical areas as well. The subject area was geometry, the teaching method mostly nondirective. The students were frequently and consistently challenged to develop, through argument and group agreement, their own system of geometric definitions, axioms and theorems—in fact, their own textbook. For example, the following question is typical of the teacher challenges to the class:



Assume that angle a = angle a'. What are the resulting implications?

Note that the question doesn't lead the students to any particular implication. When the students began to list implications, however, the teacher made them examine, debate, and justify each on the basis of previous work, and then to incorporate the implications they derived into their textbook.

Fawcett's account of the experiment is the 13th yearbook of the National Council of Teachers of Mathematics, *The Nature of Proof*. It is very readable and interested teachers should refer to it for a complete picture of the experiment. At the end of the two years the experimental students scored higher than students in traditional classes on a state geometry examination and both the experimental students and their parents claimed that the students' deductive thinking had improved in nonmathematical situations. Perhaps the most important outcome to Fawcett, however, was the proliferation in the experimental class of the behaviors he considered characteristic of students who understand proof and the value of proof and which can still serve as beacons to any teacher of mathemetics (5, p.11):

1. They will select the significant words and phrases in any statements that are important to them and ask that they be carefully defined.
2. They will require evidence in support of any conclusions they are pressed to accept.
3. They will analyze that evidence and distinguish fact from assumption.
4. They will recognize stated and unstated assumptions essential to the conclusions.
5. They will evaluate those assumptions, accepting some and rejecting others.
6. They will evaluate the arguments, accepting or rejecting the conclusions.
7. They will constantly re-examine the assumptions which are behind their beliefs and which guide their actions.

There are ways in which mathematics teachers can incorporate Fawcett's list into their own teaching objectives without investing in a two-year commitment. First, as much as possible, they should model the kind of reasoning they want from their students. In their study of seventh-graders' logical reasoning skills, Gregory and Osborne found a high correlation between the frequency of teachers' use of conditional reasoning (e.g., "If . . then" sentences) and the conditional reasoning skills of the students. (7)

Second, teachers should think aloud while attacking problems and constructing proofs. They should also create opportunities for students to think aloud In this way both teacher and student can examine the student's thinking--assumptions, use of evidence, the depth and comprehensiveness of criticism Four of the seven behaviors on Fawcett's list concern assumptions, and at least one research study has made it apparent that the assumptions many students bring to formal proof need airing and adjustment. In that study, reported by Lovell, attention was focused on the development of the concept of proof. When asked "What do we mean by an hypothesis?", more than 20 percent of the students between the ages of 16 and 18 gave answers such as, an hypothesis is a true statement; an untrue statement, a proved statement, a statement that cannot be proved. (13) With assumptions like those it is no wonder so many students have trouble with formal proof! Whether the topic for discussion is a particular mathematical proof or the process of proof itself, students need to be made aware of their own assumptions and those of others. That can only be done through regular classroom discussions among students, discussions that are guided by the teacher.

The theme of student involvement was at the heart of a study by Libeskind (12) The study centered on a short course in number theory for students from grades 9 through 11. Using study booklets and a sequence of mastery tests, the researchers guided the students through 25 hour-long sessions. The researchers' main focus was the effectiveness of the heuristic teaching of proof, whereby the teacher doesn't just appeal to axioms or previous results in the course of developing a proof, but shows why it is reasonable to start a particular proof in one way and not another, how one knows the way to proceed from one step to the next, and what alternative strategies there might be for developing a particular proof.

The researchers guaranteed students' involvement by asking them to suggest what the next step in a particular proof might be. The students wrote their suggestions in their notebooks and a step was adopted only if more than half of the students suggested it. Furthermore, to discourage memorization the students were required to write proofs in several forms: the traditional two-column form, a story (sentence-paragraph) form, and a diagram form. So-called "flow proofs" are examples of a diagram form and interested teachers can refer to McMurray. (14)

Nine students completed the course in Libeskind's study and all reached the mastery level. In particular, during the course the students developed the ability to reproduce proofs even though they had been discouraged from memorizing, the ability to recognize if a proof was valid, and the ability to apply the methods they had learned to prove statements they had not seen before. Libeskind concluded that the involvement of the students through the heuristic approach was central to their success.

Conclusion

It should be evident by now that becoming proficient at mathematical proof demands more than just a single skill. In fact, it appears to be the outgrowth of a mixed set of skills, habits, and attitudes, encompassing alertness to assumptions, listening to and evaluating arguments, recognizing patterns but also recognizing when a pattern has not been extended to a firm proof, and both the ability and the willingness to think hypothetically.

One other aspect of proof should not be ignored. A formal proof is usually a series of statements, but skills in proving are born in the asking of questions questions that allow one to analyze a concept or situation, to examine it from various vantage points, and to gather data about it. In one series of experiments, sixth and seventh grade students were trained for a year in a program of classroom learning called the Inquiry Method. (16) The method was used mainly in science classes, where students were shown events that tended to contradict preconceived notions, such as the larger of two blocks of wood floating in a liquid while the smaller piece sinks to the bottom. The students' task was to ask the teacher questions, answerable by "yes" or "no", until they felt they could explain why everything in a particular experiment had happened the way it did. Five years later, the researchers compared the Inquiry Method students with a comparable group of students who had not been exposed to the method. The inquiry-trained students were significantly more analytical than the other students, and were better in mathematics. During interviews, the inquiry-trained students made the connection between the year's training and their later experience with proofs in geometry. As the researchers reported, "Apparently techniques suggested in the strategy sessions, such as thinking of a 'start, middle, and end' to an experiment, getting 'all' the facts, or asking 'precise' questions, were the kinds of things to which the students referred in the questionnaire that were internalized and retained during the five years between the teaching regime and this investigation." (16, p. 142)

Some researchers see the microcomputer as a potential source of a similar kind of inquiry training. To do a geometric proof, for example, students must make a series of decisions about the kinds of information they need, visual information, known theorems and related results, etc. Researchers are investigating the effects of building into microcomputer programs the capacity to respond to a student's request for more information. In one project, for example, the computer was programmed to list, at certain points during a geometric proof, several categories from which the student could choose the type of information desired. (9) The hints came from the computer, but the direction of the hints came from the student.

In a similar vein, Krist conducted a study with students in grades 11 and 12, using a curriculum designed to be augmented by the programmable calculator. One of the major conclusions of the study was that calculators can contribute to building a bridge between the formal proof of basic theorems and students' understanding and acceptance of those basic theorems. (10)

From Fawcett's experiment in the 1930's to the use of calculators and microcomputers, the path to formal proof has not changed for secondary school students. What has changed is our picture of the path, which is clearer now than it ever has been in its delineation of the skills that underlie formal proof and in the portrayal of obstacles to learning formal proof. In particular, we have a clearer picture of which obstacles are developmental in nature and which are not, and what strategies are available to teachers who face these obstacles in the teaching of mathematics.

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