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ABSTRACT

Development of geometric congruence and motion was studied through tasks that tapped transformational imagery, correspondence matching, measurement operations, and transformation combinations. Results showed even the youngest children studied could generate strategies for verifying congruence. The dominant strategy in younger children was edge matching. Findings are seen to support the view that young children are guided by rules that reflect knowledge of component parts of geometric figures. The dominant congruence-verifying strategy of 7- and 8-year-olds was superposition of one figure on another, indicating operation of a new rule set. Such results indicated development from knowledge of component parts to unified wholes and contradict theories which assume development of mathematical knowledge proceeds from wholes to parts. Data are noted to show the ability of young children to generate highly inventive measurement strategies when offered the opportunity for conventional and non-conventional means of measurement. Further inconsistent performance is thought to reinforce evidence that young children use less efficient sets of strategies than older children. The study demonstrates that an adequate account of the development of mathematical cognition requires both functional and structural analysis of performance and interrelation between structural and procedural knowledge. (MP)

NSF/SED-82020

Strategies and Structures in Understanding Geometry

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Part I: Project Administration

In our original proposal we planned to undertake three interrelated studies (1) Naturally occurring strategies in geometric problem solving, (2) On the relation between cognitive and instructional strategies, (3) Algorithmic and heuristic transfer. The latter studies (2 and 3) depended on the completion of the first study. Only the first study and two supplementary studies, were undertaken and completed.

There were multiple reasons for not undertaking the latter two studies. For one, the first study turned out to be more formidable, complex and detailed than originally anticipated. In addition, we were understaffed in relation to the magnitude of the undertaking. When originally proposed, the budget and staffing were very conservative. Nonetheless, at the suggestion of the NSF/NIE review panel, the budget was negotiated, with the reduction of one half-time Research Assistant (one third of the research staff request). Even though the original plan was not completed, only the voluntary contribution of a considerable amount of time and effort by the research assistants (and needless to say, the P.I.) made the studies that were completed possible. There were additional reasons for the time needed to complete the main study. The processing of videotapes for the analysis of strategies took considerably more time than anticipated, even though we were well aware of the problems earlier investigators have had in videotape analysis. In addition, without our own equipment, we experienced considerable delays when equipment in the central Graduate Center audiovisual pool was stolen and could not be replaced quickly. Furthermore, staff resignations, because of better job offers, with requisite training time for new assistants, and other standard personnel difficulties, often delayed the work. Despite all the difficulties we are very pleased with the outcome

of the research. We feel the results of our studies make a valuable contribution to knowledge of the problems we addressed and make our own committment of time and effort, as well as that of the granting agencies' contribution, very worthwhile.

Professional Personnel on Project

Harry Beilin, Principal Investigator and Project Director

Alice Klein, Research Associate (part-time for entire period of project)

Barbara Whitehurst, Research Assistant (part-time)

Sol Magzamen, Research Assistant (part-time)

Shelly Pazer, Research Assistant (part-time)

Fred Verdonik, Research Assistant (part-time)

Irwin Schonfeld, Research Assistant (part-time)

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Dr. Walter Prenowitz, New York City, NY (Professor Emeritus, Brooklyn College)

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Dr. Richard Lesh, Northwestern University

Dr. Diane Briars, Carnegie-Mellon University

Reports of research findings at Professional Meetings

Beilin, H. "Geometry structures and processing strategies in young children". Presented to the Fourth International Conference for the Psychology of Mathematics Education, Berkeley, California, August 16-17, 1980.

Beilin, H. & Klein, A. "Strategies in Geometric Problem Solving." Presented at the Biennial Meeting of the Society for Research in Child Development, Boston, April, 1981.

Beilin, H. & Klein, A. "Using what you know: Consistency and adaptability in problem solving strategies." Presented at the Annual Meeting of the Psychonomic Society, Philadelphia, November, 1981.

Klein, A., Beilin, H., & Whitehurst, B. "Strategies for solving geometry problems: A structural and functional analysis." Paper presented at the meeting of the Southeastern Conference on Human Development, Baltimore, April, 1982.

Beilin, H. & Klein, A. "Children's understanding of geometric congruence and motion." Presented in the symposium "Van Hielé levels and geometry learning", at the American Educational Research Association meeting, New York, March, 1982.

Papers for publication are in preparation.

Part II:

Strategies and Structures in Children's Transformational Geometry

Geometry instruction in elementary schools, following the lead of secondary school geometry curriculum, has been increasingly oriented to a "transformational approach."[3] This trend is occurring despite some evidence that there may be "inherent difficulties...in teaching 'motion geometry' to children (Lesh, 1976, pp. 185-186)." Despite these difficulties, which the present research documents as well, children have much greater competence for dealing with transformations than prior research indicates.

A number of theoretical considerations prompted the present research. In an earlier collaboration (Gholson & Beilin, 1979), we attempted to formulate a developmental theory of learning that sought to integrate structural and functional theory through strategy analysis, an information processing model and Piagetian structural theory. The viability of the theory was assessed in a series of experiments conducted by Gholson and his students on hypothesis testing in bivalued four-dimensional discrimination problems (Gholson, 1980; Tumblin & Gholson, 1981). Our desire in the present study was to extend the general form of a structural-functional analysis to more complex reasoning and problem solving behavior.

Another theoretical consideration bears on the structural features of both cognitive and mathematical systems and is related to the choice of transformation geometry as a domain of study. Until recently, one of the principal characteristics of Piaget's theory was its emphasis on the role of transformations and invariances in cognitive development. A salient characteristic of Piaget's theory that distinguished it from cognitive

theories of the past was its emphasis on the construction of invariances within a world in transformation. The first such invariances appear as the first abstract cognitive structures achieved by the child in sensorimotor development (e.g., the concept of a permanent object), followed by the invariances that characterize the child's conception of space as a system of interrelated structures (e.g., horizontal, vertical, oblique dimensions). In an important shift in emphasis, Piaget's theory now acknowledges the development of structures in which no transformations occur but depend on comparisons made between object states. These are said to result in the construction of "correspondences" and "morphisms" that are describable in the (mathematical) category theory of MacLane and Eilenberg (Piaget, 1979). Piaget's invariance and transformation notions, however, as they relate to space and geometry more generally parallel those of the Klein Erlanger Program, in which Felix Klein in 1872 compared and ordered geometries on the basis of invariant properties preserved under various transformation groups. A modern version of Klein's program proposes a theoretical hierarchy (of inclusion relationships) from topological to Euclidean (including projective and affine geometries and similarity transformations) as well as the non-Euclidean geometries. Piaget's early description of the development of children's conceptions of space and geometry paralleled the Klein mathematical ordering in its characterization of a developmental progression from topological notions to Euclidean and projective frameworks. A more recent characterization of geometric development in Geneva emphasizes development as proceeding from knowledge of intrafigural relations to interfigural relations and finally to transfigural relations. This is said to more adequately tie historical development to ontogenetic development than was the case with the earlier

description (Inhelder, 1978).

Criticism of Piaget's views on geometry (including transformational geometry) are extensively detailed in a critique by Fuson (1978). Piaget is said to create some confusion on the purported development from topological structures to the concurrently achieved Euclidean and projective structures by the very presentation of his ideas, which seem to contradict one another on the ordered relations among geometric concepts. Other comments by Piaget and Inhelder suggest, says Fuson, that after topological concepts, spatial concepts develop from projective to affine to similarities to Euclidean concepts.

One of the more telling criticisms directed at Piaget's application of the Klein program is that Piaget's use of mathematical terms differs from mathematical usage. An example is the term projective. Piaget uses the term in two senses, first as a general orientation of viewpoints, and then in the more restricted Kleinian sense. The tasks Piaget characterizes as projective are more concerned with projection than with projective concepts, with the exception of his straight line construction tasks. (They involve the projection of a 3-dimensional object onto a plane and depend on the point of projection or the point of view of the observer.) In addition, Piaget's "lazy tongs" task, which is ostensibly an affine task, confounds affine and Euclidean concepts.

In a critique of Piaget's proposals concerning topological geometry, Martin (1976b) claims that Piaget's mathematical concepts are not sufficiently precise to be able to label as topological the figures upon which he bases his developmental claims. All straight line figures are labelled as Euclidean and all curved figures are labeled as topological, but Martin identifies a group of

figures that are not classifiable unambiguously as either. Piaget himself reports that the ability to deal with some topological notions appears early, whereas others appear later, so that a precise ordered relation of topological to Euclidean cannot be asserted. Nonetheless, one can say that in general topological knowledge precedes Euclidean, even though some topological concepts appear late and some Euclidean early. This, we believe more clearly represents Piaget's position than one that holds that all topological concepts must appear before all Euclidean because logic demands it.

The way Piaget generally makes use of mathematical models is a complex issue that we will not elaborate on here, but it is evident, at least in respect to geometry and space, that Piaget has not used mathematical theory in a very precise manner. The critical question is what the consequences are of not doing so. Certainly, strong claims or inferences cannot be drawn from or in respect to the model in this circumstance, and the force or power of the model is thereby to some extent diminished. This is undoubtedly the case with Piaget's adoption of various mathematical theories, and the imprecise manner in which he uses them. Despite this, the metaphoric use of mathematical models does provide more than a rough approximation to developmental data. This is so despite the difficulties encountered when transformation tasks, which vary greatly, nevertheless, result in equivalent performance. These task difficulties relate to the complexity of the transformed figure, the size of the transformation and the number and combination of transformations. The effect of some of these variables are evident in recent research on transformational geometry.

Research on transformational geometry. Despite a number of difficulties in various applications of the Erlanger program, Martin (1976b), Weinzweig

(1978) and others, propose the adoption of the Erlanger program as a model of the child's developing conception of space, while eliminating the ambiguities and incorrect interpretations inherent in Piaget's use of the model. Martin (1976a) stresses that Klein's technique of classification is its most important feature, and the classification hierarchies themselves are secondary. Consequently, it is premature to make claims that any particular hierarchy models the sequence or structure of the child's construction of space. Alternative hierarchies can provide different models of how the child's concepts develop. Weinzweig (1978) points out further that many studies that purport to deal with transformations (motions, in the mathematical sense) do not; they deal with a different concept, that of movement or displacement. The geometric assertion for example, that any motion can be represented as a product of at most three reflections holds for a mathematical structure group but is not an assertion about figures in space. Weinzweig proposes that geometry, particularly as it applies to space, should be considered as a set of equivalence classes, starting with a space as a set of structures, with groups of transformations made consistent with this structure. In this type of formulation, he holds, hypothesizing intermediate physical states is appropriate in a way it is not in a group theory view of geometry. Geometry is then defined as "the structure we impose on space as a framework about which to organize our experiences so that we may account for them, explain them and predict what will happen if...(p. 173)." Studies of transformations in actual space are closer then to a Weinzweig geometry than to a group theory geometry on which the Klein model is built.

Kidder (1976), in common with other studies to be cited, applies the Klein model to the development of space. He starts with the assumption that knowledge

of the projective and Euclidean spatial framework of middle-school age children should have developed sufficiently to permit successful performance of Euclidean transformations. Specifically, the study investigated the child's ability to identify and reconstruct spatial transformations, which included a test of individual motions, compositions of motions, and inverse motions. Slide transformations were the easiest to perform with no difference between flips and rotations. Kidder found that from 59% to 70% of "motion" errors were failures in the ability to construct a congruent image of the original figure after the transformation. These errors were interpreted as due to the inability to conserve length, a rather distressing finding for subjects 9 to 13 years of age. The results were interpreted as disconfirming their own and Piaget's assertions concerning the achievement of Euclidean and projective concepts. With recognition of some of the study's limitations, the author however was loath to refute Piaget's theory on the basis of the study alone.

In a follow-up study (Kidder, 1978), subjects in the second (8 years), third (9 years) and fourth grades (9.9 years) were given a classical conservation of length test and a transformation task similar to the one used in the earlier study. All subjects now understood the concepts of slides, flips, and turns, in contrast to 18 of 90 older (13- year-old) subjects in the prior study who did not. Failures in the transformation task were attributed to ignoring size while subjects concentrated on where to position the transformed (i.e., displaced) objects. This study can be taken as evidence of the effect of task variables and instructions on performance. For in addition to the data cited, there were others to show that pretest

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(motion) instructions of how to place the objects resulted in the algorithm-like use of a superpositioning procedure which all the nonconservers who succeeded in the transformation task utilized faithfully.

The ability to recognize properties which remain invariant under rotation (turns), translation (slides), and reflection (flip) transformations was investigated by Thomas (1978). The properties investigated were invariance of length and invariance of incidence and orientation in the plane, employing alphabet letters as stimuli. For invariance of length, tested in various transformation contexts, there was a significant effect for Piagetian level, but not grade level, i.e., nonconserving subjects were inclined to judge that a transformation had changed the length of a figure (letter). The second set of tasks, locating points on a triangle under transformation, showed first graders placing a point on the correct side but in the wrong location. Third graders seemed more aware of the sides than the vertices, whereas sixth graders used both sides and vertices for location correctly. The results were taken to support the Piagetian topology to Euclidean progression. The alphabet tasks showed that difficulty level of a transformation task was affected by attributes of the configuration (e.g., symmetry) and that difficulty varied from isometry to isometry. Half turns on letters with rotational symmetry were very difficult at all grade levels. Overall, performance on the alphabet tasks varied with age, with third graders significantly poorer than the older subjects. Like the Thomas study, one by Shultz (1978) concerned the effects of complexity of configuration and complexity of transformation displacement on the difficulty of transformation tasks, in addition to the effect of the child's operational level. The dimensions of complexity studied by Shultz were direction of (horizontal or diagonal) displacement and size of configuration,

together with meaningfulness of the configuration. Subjects were 6 to 10 years old. The direction of translation affected subject' performance in that they centered on the extent of translation and ignored its orientation, which was to be preserved, with differences for horizontal and diagonal displacements. Short displacements were easier than long or overlapping translations (slides).

Although Piaget and Inhelder (1971) in their studies of mental imagery indicate that flips are not understood till eight years and turns about nine, others, Shah (1968) and Williford (1972), suggested that younger children (6- year-olds) are able to learn to deal with slides. Perham (1978), in a training experiment on slides, flips and turns (of three angular rotations), discovered that before instruction children conformed to the Piagetian norms. First graders understood both vertical and horizontal slides (in anticipation and representation), but could not deal with flips and turns of any degree. After instruction, however, the experimental first grade subjects dealt effectively with both turns and flips, but only in respect to anticipating the end points of a transformation (in a multiple choice test), and not in their drawings of various transformations. Diagonals were not comprehended after instruction, even when they were slides. Perham concluded that although instruction could help, there were developmental limitations on that achievement, as Piaget suggests.

Using a mathematical criterion for the relation between Euclidean and topological invariants, the topological would be more primitive and this fact motivates Piaget's psychological model for the acquisition of invariant structures, as already indicated. Moyer (Moyer & Johnson, 1978) attempted to test this order in a study with preschool to third grade children in which two

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plastic circular discs were placed before the child. The experimenter put a dot on the left disc that, like the right disc, had a black border around it, with some of the discs shaded half red, others were all one color. In some trials the subjects were shown a superposition motion for establishing congruence, in others not. In each condition, the left disc differed from the right by a turn, slide or flip transformation. After the experimenter placed his dot on one of the discs of a pair, the subject placed another at the corresponding location on the transformed disc. The result showed that topological enclosure was important at all ages while distance (Euclidean) became more important with age. In general, children responded to the topological features of the transformation prior to the Euclidean and projective, and Moyer concluded that the cognitive abilities acquired were in accord with the mathematical structures. According to the mathematical primitiveness of isometrics, the flip should be acquired first. The order of acquisition in this study, however, as with others, was slide, flip, then turn. In addition, children did not think of isometries as rigid motions in the way they are conceptualized mathematically as a relation between one static configuration and another static configuration, as Weinzweig (1978) and others emphasize. A second study (Moyer & Johnson, 1978) investigated the same general issues with an older group of students using one-dimensional configurations of wooden balls in different colors connected by wooden dowels embedded in one, two and three dimensional arrays. The three-dimensional cube was the most difficult to deal with although in terms of a logical analysis it was expected to be easier than the two-dimensional task, indicating again that psychological processes do not map exactly onto mathematical models.

Martin's (1978) study of the ratios of distances in one direction, as an affine invariant, is interesting on two counts. First, whereas most of the transformation studies cited concern Euclidean transformation, this study treats with affine geometry and at that with a property different from the parallel invariant studied by Piaget (in a study that Martin asserts confounded it with length). In the task, Martin used two figures, one square, the other a parallelagram that was also larger in size, made of rigid rods. (The increase in size had the effect of making the task a test of ratio of distances as opposed to a test of conservation of distance.) One rod figure was the model, the other the copy. For the model, beads were placed on two sides of the figure. In the transformed copies the beads were placed both at proportional distances and at incorrect distances. The subjects were assessed for their success in choosing copies that preserved the ratio of distances, marked off by the beads on the rods that defined the figure. A second, significant feature of the study was that Martin detailed the strategies subjects employ in carrying out the task and related them to patterns of success. For children from grades 3,4 and 5, none of the third graders answered more than 2 of 9 items correctly; of the fourth graders, none correctly answered more than 3 items and only 7 of the 20 fourth graders correctly answered all of the 9 items. At the same time, 8 of 10 third graders could copy (draw) the size of the model correctly. Available aids were made use of most fifth graders consistently and effectively, whereas the younger children could not use them. The most incorrect strategy evident, and it typified the younger subjects, was that subjects' conserved distance from one end of the rod to the bead. Martin concluded that the scores and strategies of the fifth graders were dramatically different from those of third and fourth graders. During a warmup period, all

subjects (young and old) used the aids ("helper stick"), but only the fifth graders used them in the actual experiment. Martin suggests that the fifth graders saw the aids as instruments with which to apply mental operations available to them, mental operations not available to the third and fourth graders.

In a study interpreted as an imagery for spatial movement experiment (Gruenreich & Herrman, 1981), six different types of movement (i.e., transformations) possible in a two-dimensional plane (horizontal, vertical, diagonal, rotation, size change and occlusion) were investigated in 4- and 5- year-old subjects. Significant differences in age were found according to transformation type, with size and occlusion most difficult. Imagery for movement appeared to develop more slowly than expected, and was in accord with Piaget's results, particularly in respect to the inability to visualize intermediate states in the transformation, although the tasks were easier than Piaget's.

The final study to be cited (Elman, 1973) investigated "sensivity" to related transformed stimuli that varied in orientation in 2-space, 3-space, and embedded context; i.e., orientation, rotation and embedding. Subjects included 9th graders who had no formal study of geometry and 10th and 11th graders who did. Able adolescents, regardless of instruction in geometry performed well in 2-space; they each ordered rotated figures in one cluster and embedded figures in another. There was also considerable consistency in preferred orderings of types of transformation. Younger and less experienced subjects were relatively insensitive, i.e., could not differentiate, projective transformations. The older and more experienced differentiated this transformation from rotations and embedded alternatives.

In sum, developmental studies in transformation geometry show consistent age differences in the ability to deal with transformations, with differences generally in accord with Piagetian formulations, and generally in accord with the Kleinian model as well, except that isometries do not follow the mathematical assumptions as to what is basic and derived. Lesh (1978) in an extensive examination of research and educational issues in transformational geometry makes a number of cogent observations. He first remarks on the close correspondence between general mathematical structures and cognitive structures, which he says is not surprising in that the method used to isolate each type of structure is nearly the same. Secondly, he makes the point that the isometries will not give consistent results relative to one another, since tasks will be different within a class of isometries, so that easy tasks of an otherwise difficult isometry may be easier than the difficult tasks of an easier isometry. The task elements that may be important are: the complexity of the transformed figure (so that properties perceived under a simple transformation may not be perceived under a difficult transformation), the size of the transformation (a large transformation may be more difficult than a small one), the number of compositions (a single transformation may be easier than the construction of two transformations, and the order in which individual transformations is composed may affect the difficulty of the composition). In summary, he claims that (1) a mathematically more general (or powerful) relation may not necessarily be more psychologically basic, (2) children often make mathematical judgments using qualitatively different systems of relations from those used by adults, (3) if operationally isomorphic tasks vary too much in difficulty it may be meaningless to equate tasks on the basis of operational structure, or to put it another way, task variables

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interact with operational structure to create performance differences. The literature also suggests, although few studies attempt to show this (Martin, 1978 is one of the exceptions), performance strategies at different ages appear to be related to the development of operational structures. One of the principal aims of the study now to be detailed was to show this relationship and to demonstrate it in children over a wide age range.

The focus of the present study. Research in mathematical cognition was dominated for a long period by attempts to characterize the structural basis of problem solving and reasoning (particularly in Piaget's theory). This state of affairs is rapidly changing as inteiest in a variety of performance models increases. These performance models are not represented by a single theory but by a common mathematical orientation that can be characterized as the New Functionalism (Beilin, 1981, in press). This functionalist orientation is "new" in its willingness to posit the presence of cognitive entities, but, as in previous functionalist appraoches, theorizing is kept close to observed data. Structuralism and functionalism, as metatheories, are not testable or falsifiable. It is not a question as to which point of view is correct. More to the point is whether specific theories within each domain make assertions challenging the truth values of the other.

Our own view, which underlies the geometric problem solving research to be reported is based on two assumptions. First, we hold that both functional and structural analyses of behavior are required for an adequate account of mathematical cognition, as well as all cognition. Second is our conviction that the functional characteristics of a cognitive system, such as performance strategies, are related to abstract structures that define the interrelations among functional systems.

The research model.

The research was based on a model in which two dimensions, one mathematical and the other psychological, were related. The mathematical dimension was based on the Kleinian classification of geometries, which, as indicated, is defined by the invariant properties of figures under a group of transformations, ranging from the topological to Euclidean. We chose for practical reasons to confine our research to planar or Euclidean geometry, because it lends itself best to experimentation with a variety of tasks over a broad developmental range, and it is the domain for which a fair amount of research is already available (Moyer, 1978).

The psychological dimension of the model dealt with cognitive processes. Since the properties of congruence (invariance) and motion (transformation) are central to an understanding of geometry (Prenowitz & Swain, 1966), the focus of our research was on the cognitive processes relevant to determining congruence between figures under planar transformation. An analysis of how the congruence of geometric figures (such as triangles) can be established in a spatial context, suggested the following cognitive processes as candidates for study, although they do not exhaust the set of possibilities. First, if graphic representation of geometric objects must be transformed to determine congruence, or physical realizations of geometric objects are transformed out of sight, some form of imagery process must be involved. Second, determining whether two figures are congruent in respect to geometric properties that have physical extension clearly requires measurement operations. Third, if Piaget's recent formulations concerning correspondences and morphisms are correct, a "precocious" understanding of congruence may be achieved through establishing correspondences between elements of figures, i.e., lines

and angles. Correspondence processes used to make comparisons between figures are presumed to be developmentally prior to measurement operations. Fourth, the ability to deal adequately with transformations suggests that some transformations may be more difficult to process than others. We were thus interested in the relative processing difficulty of individual transformations and the ability to integrate information from transformations. In summary, we studied four cognitive processes:

Transformational imagery. Processes basic to the ability to imagine (i.e., image) the terminal state of a figure resulting from a series of physical transformations when only the initial state (i.e., the figure) and the transformations are indicated.

Measurement operations. Processes basic to qualitative and quantitative measurement to determine the congruence or non-congruence of figures that differ by a transformation.

Correspondence by comparison. Processes by which corresponding elements of two figures (e.g., lines, angles) are established when the figures differ by a transformation.

Combination of motions. Processes involved in the ability to integrate information from different transformations.

These cognitive processes were investigated at two levels of analysis. First, we examined the competencies of children in respect to their knowledge of congruence, that is, whether children at different ages were successful in performing in tasks that exemplified these processes and thus could be presumed to have the cognitive resources to deal with them. Second, we were interested in the ways children at different ages go about solving the tasks, that is, in the behavioral strategies evident as they responded to the demands of the respective tasks.

METHOD

Research Design

The study employed a factorial design with repeated measures (tasks: 7 levels) on two between-subject factors: Age (8 levels) and Sex (2 levels). (See Appendi.. for details).

The order of task presentation was counterbalanced within age groups with the constraint that the imagery task was administered first so as not to confound the effects of other tasks on imagery, and the combination of motions task was always last. There were two presentation orders, and half of each sex within each age group received order 1, half order 2. If a subject required more than one testing session to complete the set of tasks, a subject was given the first and second tasks within an order during the first session, and the remaining third and fourth tasks in that order during the second session.

Presentation order in the imagery tasks for the three types of transformations, (flip, slide, turn) was rotated among subjects utilizing all six possible orders of the three transformations. In the measurement task, three dimensions were counterbalanced within each age group: (1) mobility of figures half the subjects responded to non-mobile figures (i.e., figures were "fixed" to the boards) and half to mobile figures (i.e., figures could be lifted off the boards and manipulated), (2) congruence of triangles - half the subjects judged congruent triangle and half judged non-congruent triangles, and (3) presentation order for the type of transformation (flip, turn and slide). The third set of tasks consisted of three correspondence tasks, lines, angles and points on a circle. The circle task was administered to all subjects, whereas half the subjects within each age group received the lines task and half received the

angles task. Presentation order for the two correspondence tasks and the type of transformation (flip, turn, and slide) was counterbalanced within each age group.

Materials

Imagery task. The Pretest employed a cookie cutter in the shape of an equilateral triangle and play-doh. In Phase 1 of the imagery task there were three sets of triangles (red, blue, and green) of 1/8 in. plexiglass. Each set of triangles contained a model triangle and four "selection" triangles. All model triangles were scalene right triangle. The selection triangles within each set consisted of three similar in shape to the model, one smaller than the model, one congruent with it and one larger than the model and one of non-similar shape. (The dimensions of the three sets of triangles are indicated in Table 1 of the Appendix.)

Phase 2 used the same three sets of triangles as phase 1, with the exception that two congruent model triangles were contained in each set. In addition, a 5 1/8 in. x 6 1/2 in. grey envelop without a flap was provided. Phase 3 materials were the same as phase 2.

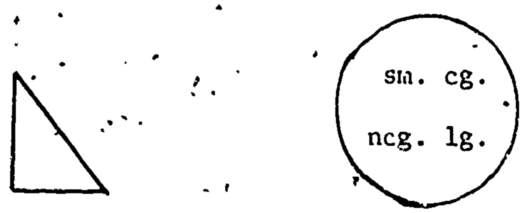
Measurement task. All stimulus triangles were presented on a 3ft x 2ft piece of heavy mounting board. There were three pairs of congruent triangles (red, blue, and green) and three pairs of non-congruent triangles (red, blue, green), all of 1/8 in. plexiglass. All pairs of congruent triangles were scalene right triangles. (Dimensions are in Table 1, appendix). Three sets of solution aids (red, blue, green) were available for use with the three stimulus triangles of the same color. Each set of solution aids contained the following materials: one selection triangle was identical in dimensions to the congruent triangles of the same color, four pieces of string

Figure 1a

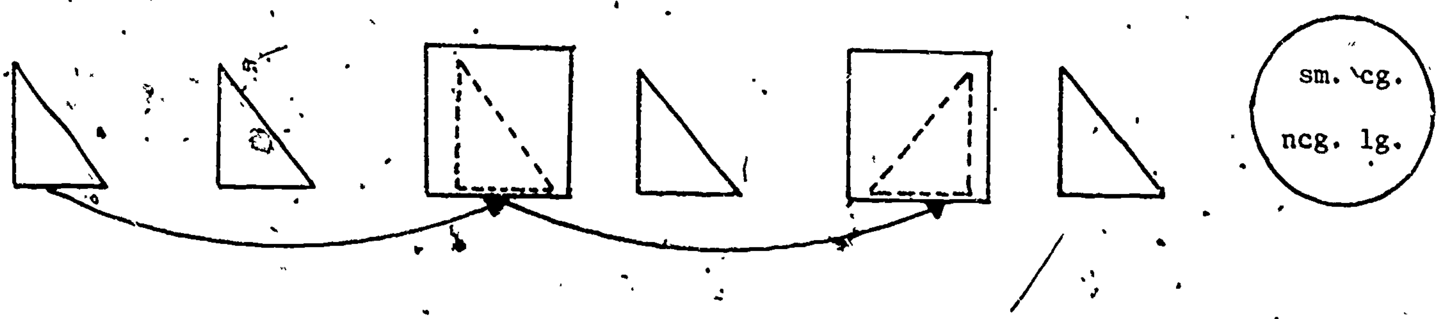
IMAGERY TASK:

PLACEMENT OF TRIANGLES IN FIRST, SECOND, AND THIRD PHASES OF THE FLIP TRIAL

First Phase



Second Phase



Third Phase

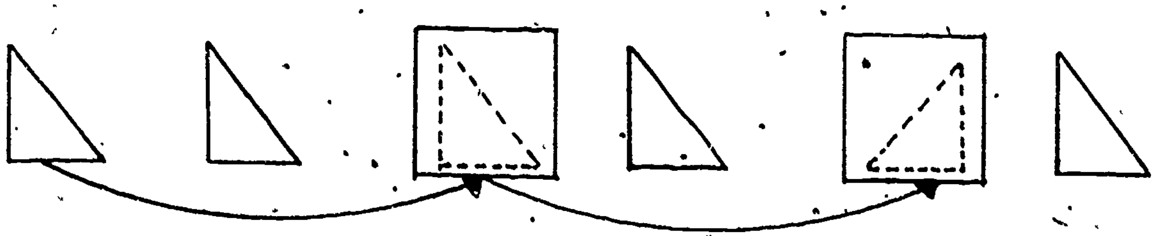
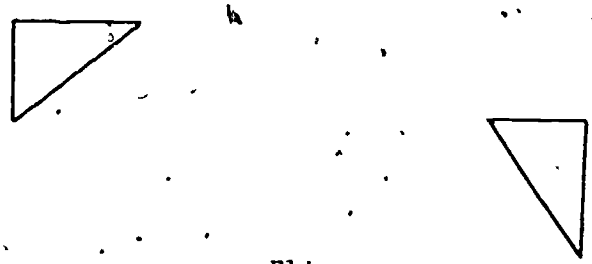


Figure 1b

MEASUREMENT TASK:

PLACEMENT OF CONGRUENT TRIANGLES FOR ROTATION, FLIP, AND SLIDE TRIALS

Rotation



Flip

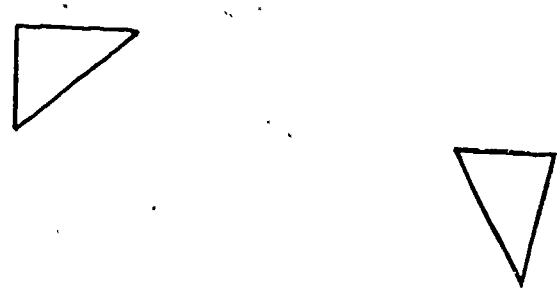
Slide

Figure 1c

MEASUREMENT TASK:

PLACEMENT OF NON-CONGRUENT TRIANGLES FOR ROTATION, FLIP, AND SLIDE TRIALS

Rotation



Flip

Slide

Figure 1d

CORRESPONDENCE TASK - LINE SEGMENTS:

PLACEMENT OF TRIANGLES FOR ROTATION, FLIP, AND SLIDE TRIALS

Rotation

Flip

Slide

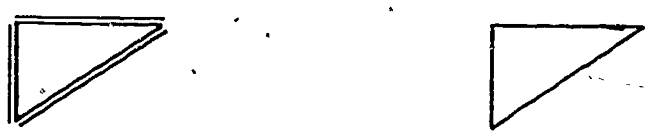


Figure 1e

CORRESPONDENCE TASK - ANGLE SEGMENTS:

PLACEMENT OF TRIANGLES FOR ROTATION, FLIP, AND SLIDE TRIALS

Rotation

Flip

Slide

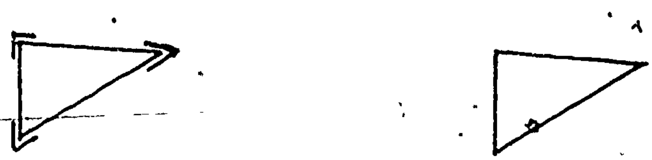


Figure 1f

CORRESPONDENCE TASK - POINTS ON A CIRCLE:

PLACEMENT OF DOTS ON CIRCLES FOR ROTATION, FLIP, AND SLIDE TRIALS

Rotation

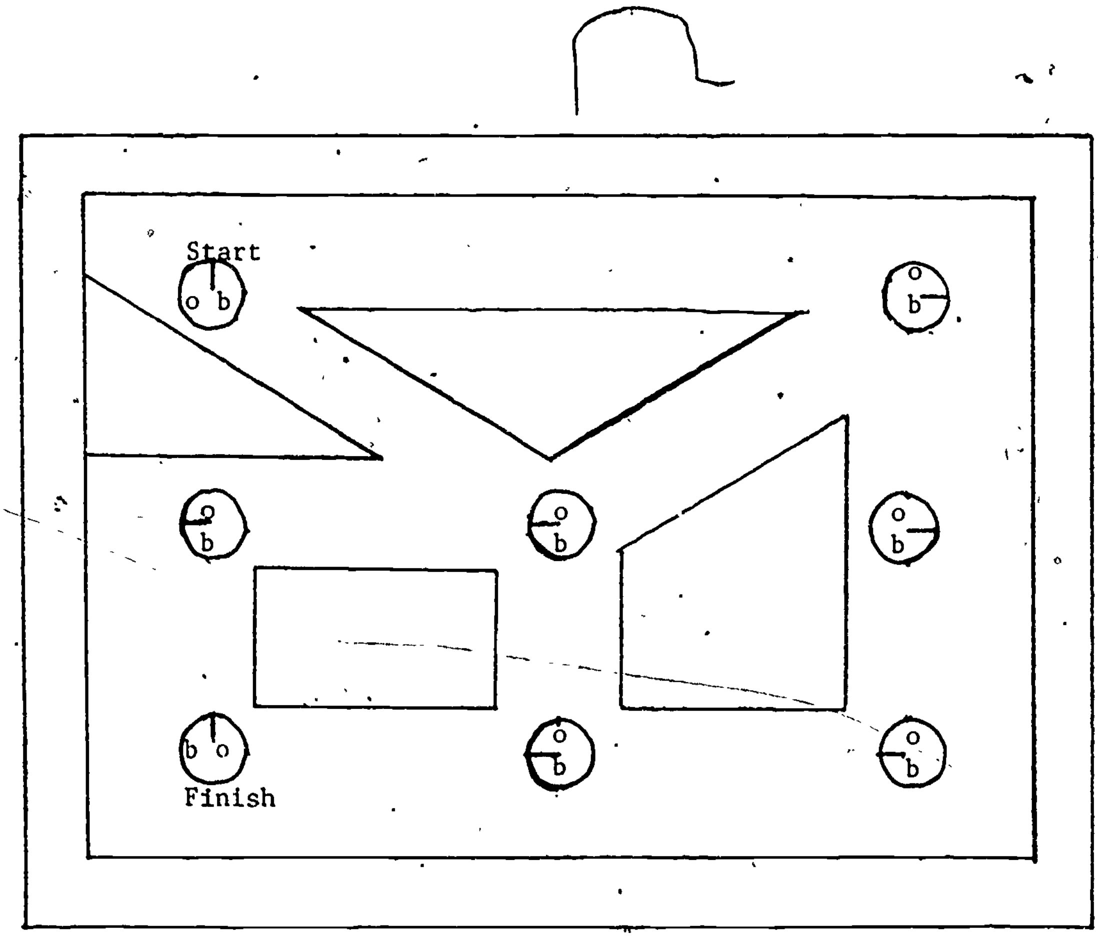
Flip

Slide

Figure 13

COMBINATION OF MOTIONS TASK:

PLACEMENT OF SIGNS ON THE TASK BOARD



with three pieces that corresponded in length to the sides of the congruent triangles of the same color, four plastic sticks that corresponded in length to the four pieces of string of the same color, one pair of scissors, a ruler, a roll of white twine, a protractor, a pencil, six sheets of unlined paper, and a tray to hold the solution aids.

Correspondence task: Line segments. Three pieces of mounting board were used in this task. On each board a pair of scalene triangles, which corresponded despite a transformation (flip, turn, or slide) was drawn. For the flip and turn transformations, the triangles were drawn approximately 5 in. apart. For the slide transformation they were drawn 10 in. apart. There were 3 sets of 3 plastic line segments (each 1/8 in. wide), a different color for each transformation, that corresponded in length to the sides of the experimenter's triangle on each board. (Dimensions in Table 1, Appendix.)

Correspondence task: Angles. Three different pieces of mounting board were used in the task that paralleled the line segments task materials: A set of three plastic angles with a different color for each transformation.

Correspondence Task: Points on a Circle. There were three mounting boards each with a drawing of two circles (diameters = 4 in.). Affixed to the circumference of one circle (the test circle) were 3 white "dots" (circular clear labels), and one red and one blue non-mobile dot. There were other color combinations for different transformations. All dots were ½ in. in diameter. On the other circle, which differed by a transformation, there were five colored dots ordered around the circumference, two of which corresponded to the dots of the test circle. The test circle was positioned so that the non-white dots were located at the 12 and 7 o'clock positions.

There were boxes of 10 movable dots of various colors (see Appendix 2) available to the subject that could be affixed to the white dot areas of the test circle.

In the Combinations of Motions task, the principal material was a large maze-like mounting board (20 in. x 30 in.) on which were drawn a series of paths from the "start sign" in the upper left hand corner to the "goal sign," with a series of "signs" along the way at the intersection of paths (see Figure 1). Surrounding the paths were "green grass" areas. The "signs" were circular 3 in. discs with a pattern made by a single black radius line that divided the two halves of the disc; each half was in a different color (the black line did not extend the entire length of the diameter). There were three identical game discs, called "movers" (3 in. in diameter). The "movers" were similar to the signs, except that they had a corresponding pattern on the other side of the disc. The "signs" on route intersections and "start" and "goal" points differed by rotations and flips from one another. A subject could take alternate routes in the game; some routes entailed traversing more signs than others. There was a "path barrier", a stick, used to block a route.

Procedures. (A synopsis of the procedure is given here, detailed procedures for each task appear in the Appendix 2, Experimenter's manual).

The pretest which preceded the imagery task was designed to ensure that the child understood the concept of congruence in at least one sense, relating to same size and shape. A pretest was necessary since all tasks assumed that the child understood the basic congruence question asked about the various figures that differed by a transformation. In the pretest the child was given some "play-doh" and a cookie cutter in the form of a equilateral triangle. The experimenter first demonstrated how the cookie cutter worked. Then the child was allowed to make his own triangular "cookies" with the cookie cutter. The experimenter then pointed out that all the resulting triangles were the same,

and asked why this was so. If the child had difficulty with this question, it was explained that the triangles were the same "because they were cut from the same cookie cutter."

The Imagery Task had three phases, but only the last two phases were assumed to entail imagery processes (see Figure 1a). The first phase assessed how the child would initially establish the equality of two triangles, since the second and third phases started with equivalent, i.e., congruent, triangles. The experimenter set out a model triangle and pointing to the selection tray, asked the child to "find the triangle in the tray that was cut by the same cookie cutter." The child's strategies in selecting a triangle that was equivalent to the model were recorded (for the first 16 cases by video tape, and for the remainder of the subjects by a classification scheme derived from the video tapes). After the child's choice was made he was questioned about the basis of his choice. (See Appendix 3 for classification code.)

The second phase began with the presentation of two identical model triangles placed on the table in the same orientation, with the assertion that they were cut from the same cookie cutter. The experimenter then slipped an opaque envelope over one of the triangles and performed either a slide, turn (of 90%) or flip transformation on the envelope containing the triangle. The child was instructed to select a triangle from the tray cut from the same cookie cutter as the triangle hidden in the envelope. The child's strategies in selecting the congruent triangle and his answers to questions about the basis of his choice were recorded as in the first phase. The initial procedures of the second phase were repeated. However, after the experimenter performed the transformation, the subject was requested to do the same with the visible model triangle, "so that it will look the way the triangle hidden in the envelope looks now."

The measurement task utilized two triangles (congruent or non congruent) that were placed on a board by the experimenter while the subject closed his eyes (see Figure 1b for triangle placement). The child was told that sometimes the triangle would be from the same cookie cutter and sometimes not, and that the child was to find out whether or nor they were from the same cookie cutter. The experimenter provided a tray of solution aids for "use to help find out whether these two triangles are cut from the same cookie cutter or not." In the fixed non-mobile condition the child was cautioned that the triangles could not be moved off the board. In the mobile condition the child was told he could move them to make his determination.

After making an initial judgment about the congruence or non congruence of the triangles the child was asked another way of finding out whether the triangle were from the same cookie cutter. A third request, which emphasized the use of "something (else) on the table" followed the second judgment. As in all of the tasks, strategies were recorded or coded and questions asked about the child's choice of strategies (See Appendix 2 - Experimenter's manual).

The Correspondence Task: line segments began by placing a board on which were drawn a pair of congruent triangles in front of the child. The pair of triangles on each board differed by one of three transformations. (For the placement of the triangles see Figure 1c). The experimenter indicated that the two triangles were "cut from the same cookie cutter," and then proceeded to "decorate" his triangle. After doing so, the experimenter after pointing to one side asked the child to point to the side of his own triangle "that is the same as this side." The child then moved the plastic line segment from the experimenter's triangle onto his triangle, for each succeeding side.

A follow-up series of questions ended each trial (see Appendix 2 - Experimenter's manual).

The Correspondence: Angles task was conducted in the same manner with only the wording of the instructions changed, as appropriate, i.e., "can you point to the corner of your triangle that is the same as this one." (Figure 1e .)

The Correspondence: Points on a Circle task presented the child with a board on which was drawn a pair of 4 in. circles. Five 1/2 in. "dots" were affixed in clock positions around the periphery of both circles. The pair of circles on each board (total of 3 boards) differed by one of three transformations. (See Figure 1f).

The task began with the experimenter asserting that the circles were "cut from the same cookie cutter." It was pointed out that the two colored dots on the child's circle (the test circle) corresponded to two dots with the same colors on the experimenter's circle. The experimenter then "decorated" her circle by placing three additional colored dots in their appropriate locations around the periphery of the circle, and suggested that they decorate the child's circle. The experimenter pointed to one of the three remaining dots on her circle and said, "can you point to the spot on your circle that is the same as this one?" (which was one of the 3 blank white spots on the child's circle). The child then relocated the dot from the experimenter's circle to the selected spot on his circle for each of the three dots.

The Combination of Motions task began with three "movers in non-corresponding orientations, and the child was asked to make them "look the same." If the child had difficulty, one hint was given that turning them over might help. If the child succeeded he was told to choose one for the game. The purpose of this pretest procedure was to familiarize the child with the process,

of matching one mover to another which was basic to sign matching in the combination of motions game.

The game board was then placed on the table, and the child instructed to begin at the "start sign" by making a match between the mover and the sign. He was given help if they did not match. He was then told to move his mover along one of the roads, taking the shortest route to the goal and not proceeding beyond a sign without first matching his mover to it. After one trial a follow-up question was asked (see Appendix 2 - Manual) and a path barrier was placed on the road the child took in the game. Then the child was asked to go through the game a second trial using another route to the goal. Follow-up questions were again asked. (Figure 1g).

Subjects Eighty children between 4 years and 11 years of age participated in the study. There were 10 at each of 8 age levels equally divided by sex. The 8 age levels were chosen to encompass a wide developmental range from the period when one-way mapping operations are said to characterize the child's performance (4 years) to the period when formal operations are said to develop (11 years) (Piaget et al.,1977).

The mean ages and range of the 8 age levels were: 4 years 2 months (range 3, 7 to 4,5); 5 years 0 months (4,6 to 5,4); 6 years 0 months (5,7 to 6,5); 7 years 1 month (6,7 to 7,5); 8 years 7 month (7,6 to 8,4); 9 years 1 month (8,9 to 9,3); 10 years 0 months (9,7 to 10,4); 11 years 0 months (10,8 to 11,4). These age levels are referred in the report as 4- to 11-year-olds, in yearly intervals.

Children were selected and tested in two phases. The first was a videotaped phase in which 2 children at each of the 8 age levels were tested in the university laboratory. The videotapes from this testing phase were

analyzed for the strategies manifested by the children in solving the tasks. These analyses formed the basis for the strategy coding schemes used in the second testing phase. The second phase was conducted at the school from which the remaining 64 children were drawn. Children in both phases were tested individually by the experimenter while a trained coder recorded. In the second phase, there were 6 additional subjects tested for a study of the reliability of the coding schemes.

The first 16 videotaped subjects were obtained by referral contacts. These subjects were from middle-class professional families and were above average in intelligence. The second group (64) were obtained by a random selection from the preschool, kindergarten and elementary school classes in a special school for gifted children, staffed by school system of a large city, but operated by a publically supported university. The intellectual status of the second group was well above average, and although this group was predominantly from middle-class homes, their ethnic status reflected an affirmative action policy in recruitment. The racial composition of the subjects was mixed. The total sample of the first and second groups consisted of 77.5% whites; 22.5% non-whites (blacks, hispanics and orientals).

Results

There were three types of data to be reported: (1) Strategies children employed in tasks designed to test their knowledge of geometric congruence and transformation, (2) the consistency of performance across tasks, with respect to accurate performance and strategy deployment, and (3) the relation between cross-task consistency and the strategies evident at different ages.

Some of the findings are striking. Children's strategies for establishing geometric congruence are evident in children as young as two and a half years

of age. Children appear to utilize rules that indicate knowledge of the constituent parts of geometric figures at very young ages. Strategies supercede one another in development becoming more powerful, sophisticated, geometrical, and accurate. A high measure of consistency appears within sets of tasks suggesting the availability of increasingly general cognitive structures with development.

These and other findings will be detailed and discussed in respect to analyses conducted on each of the tasks.

Strategy analyses

Coding Reliability

The tasks, as indicated, were first administered to 16 subjects, 2 at each of 8 levels at a university laboratory provided with a one-way vision room. Children's performance was videotaped and from the videotaped data a coding scheme was developed for the classification of strategy responses. The collection of data from the remaining 64 subjects took place in the field (i.e., schools) with responses classified according to the coding scheme. The reliability of the coding scheme was high. A study of the reliability of the coding of strategy responses was made with two judges. A sample of 7.5% of subject responses was made in the age range from 5 to 10 years. In the imagery task, 192 responses were rated with 98% agreement; in the measurement task there was 97% agreement; correspondence: lines/angles, 94%; points on a circle, 98%, and combination of motions, 98%. Because of the high reliability, data from the videotapes and the field data were pooled. The first data of interest are the strategies manifest in the tasks.

Imagery task strategies

Four main tasks were administered in the study, but as some of the tasks contained subtasks, there were in fact seven tasks from which strategy data were obtained. Three of these were in the imagery task. The first phase

of this task involved the presentation of a model triangle and a "selection" tray containing four triangles. The child was asked to select a triangle from the tray "cut from the same cookie cutter" as the model triangle. There were three such trials, each representing a different transformation of the triangle (slide, turn and flip).

Subjects approached the task with a behavior sequence of two parts, a selection phase, followed by a verification phase. The selection phase entailed either visual inspection of the materials or haphazard selection. As the labels indicate, visual inspection involved a deliberate process of looking over the model triangle and the tray triangles, after which a selection was made of one or more of them. This was followed by the verification phase in which the child checked the validity of his choice by one of a variety of methods. If selection was haphazard, it involved no deliberate visual examination of the materials on the table, with a "grab" for one of the tray triangles.

The selection phase was followed by verification in which two types of strategy were evident. The edge matching strategy (EM) involved the placement of a tray triangle so that one of its edges was placed parallel to or abutted an edge of the model triangle. A judgment of congruence was made on the basis of a single edge match or multiple edge matches. If the latter, they involved either two or three sides. Whether edge matches were made with corresponding or non-corresponding sides of triangles was noted, inasmuch as this action was seen as a significant indicator of subjects' geometric knowledge.

The second strategy was superposing, in which either the selection or model triangle was placed on the others so as to cover it. This could be accomplished by an anticipation movement with one triangle held a distance above the other, or their congruence judged by actual contact of one on the other. Sometimes prior to making a congruence judgment. Some subjects used a combination of

superposing and edge matching. The strategies observed were as follows (not all logical possibilities are indicated, only those observed):

I. Selection --> (followed by) No verification

A. Visual inspection: one triangle

B. Visual inspection: more than one triangle

C. Haphazard

II. Selection --> Edge matching Verification (Single edge matching only)

A. Visual inspection --> single edge matching: corresponding sides

B. Visual inspection --> single edge matching: non-corresponding sides or both corresponding and none-corresponding

C. Haphazard --> single edge matching: corresponding sides

III. Selection --> Superposing Verification

A. Visual inspection --> superposing: anticipatory (collapsed over single and more than one triangle categories)

B. Visual inspection --> superposing: contact (collapsed as above)

C. Visual inspection --> superposing: both anticipatory and contact (collapsed as above)

IV. Selection --> Combination of edge matching and superposing verification

A. Visual inspection --> single edge matching and multiple edge matching

B. Visual inspection --> single edge matching and superposing

Table 1

The data in Table 1 are based on the first pattern evident in each transformation trial, with the data summed over transformation trials (thus 3 responses per subject and 10 subjects per age level).

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The data reveal the following:

1. There are clear age trends in the use of strategies.

2. The predominant strategy in the younger age groups was edge-matching, in particular single edge matching; in the older subjects it was superposing.

3. A shift from edge matching to superposing appeared between the sixth and seventh years (57% EM at 6, 17% at 7; 20% superposing at 6; 83% at 7 years).

Table 1

Imagery Task - Phase 1: Frequencies of Strategy Patterns and Correct Congruence Judgments by Age[a]

	I[b]					II					III					IV			
Age	A	B	C	Total	% Total	A	B	C	Total	% Total	A	B	C	Total	% Total	A	B	Total	% Total
4	3	1	0	4	13	15 (93)	6 (33)	2 (100)	23	77	2	1	0	3	10	0	0	0	0
5	2 (50)	0	0	2	7	22 (95)	1 (100)	0	23	77	4 (100)	0	0	4	13	1 (100)	0	1	3
6	6 (50)	1	0	7	23	13 (100)	3 (67)	1 (100)	17	57	6 (100)	0	0	6	20	0	0	0	0
7	0	0	0	0	0	4 (100)	1 (100)	0	5	17	16 (100)	4 (100)	5 (100)	25	83	0	0	0	0
8	2 (50)	3 (67)	0	5	17	10 (100)	1 (100)	0	11	36	10 (100)	1 (100)	1 (100)	12	40	1 (100)	1 (100)	2	7
9	2 (100)	3 (100)	0	5	7	7 (100)	0	0	7	23	18 (100)	0	0	18	60	0	0	0	0
10	6 (83)	1 (100)	0	7	23	4 (100)	3 (67)	0	7	23	16 (100)	0	0	16	53	0	0	0	0
11	2 (100)	1 (100)	0	3	10	0	0	0	0	0	25 (96)	2 (50)	0	27	90	0	0	0	0
Total	23	10	0	33		75	15	3			97	8	6			2	1	3	
Total % Correct	61	60	0			97	60	100			98	75	100				100		

[a] Percentages of correct congruence judgments appear in parentheses, below frequencies.

[b] Category numbers and letters are defined in the text. Note: 10 subjects per age group X 3 responses = 30 responses per age group.

4. Combinations of strategies were rare; subjects persisted, for the most part, in one type of strategy.

5. Selections made without verification were relatively few and had an erratic course over the ages tested.

6. There was a drop in superposing strategy use at age 8 and a subsequent rise to predominant use of superposing at 11 years, with the reverse in the edge matching strategies.

7. Edge matching when it occurred was predominantly with corresponding sides of triangles; mainly the hypotenuses of the matched triangles.

8. Haphazard selection was rare in the data, and only occurred at the youngest ages studied.

Correct congruence judgments were also recorded and are indicated in Table 1. The following results are evident:

1. Children made correct congruence judgments even at the youngest age studied (4 years), providing that they used single edge matching (which 77% did). Eighteen of 23 judgments were correct when based on edge matching at this age; none was correct if made on another basis.

2. All superposing judgments from age 5 to age 11 were correct; none of the 4 year-olds' were (3 such judgments).

3. Visual inspection without verification was an inaccurate strategy up to age 9, at which point its employment led to correct congruence judgments.

4. At the point at which single edge matching declined and superposing began to predominate (from 6 years to 7) subjects are equally successful in the use of each strategy (i.e., 100% correct).

In sum, determining congruence for even the youngest subjects was not a difficult task, as long as a transformation was not involved, which was the case in phase one. The results indicate too that most children understood what the experimenter meant in the congruence questions. The change in

strategies with age (from single EM to superposing) is striking. In this phase they occur even though children are highly successful with edge matching at young ages.

In Phase two a different procedure was used from Phase one. A model triangle was placed on the table with a matching (congruent) triangle located at a distance to the right of it. The selection tray with four triangles was also on the table. A manila envelope was slipped over the matching triangle and a transformation performed on it (a flip, turn or slide). The subject was asked, following the transformation, to select a triangle that matched the triangle in the envelope. This procedure, which forms the basis of the next phase as well, was designed in part to determine whether a transformation performed on a hidden triangle would affect judgment as to whether size and shape properties of the hidden triangle were conserved. Thomas' (1978) study suggested that with a transformation size would not be conserved. The strategies employed in selection and verification of the tray triangle chosen as congruent with the hidden triangle were observed and recorded as was success in choosing the congruent triangle.

The strategies were the same as those in phase one, <u>visual inspection</u> and <u>haphazard</u> selection strategies, and <u>edge matching</u> and <u>superposing</u> verification strategies. In addition, the change in procedure prompted reliance on <u>memory of the hidden figure</u> with attempts to deal with the hidden figure by superposing the tray triangle on the envelope. Both edge matching and superposing on the common figure (the model triangle), in order to make inferences about the hidden figure, were taken to be an indication of the use of transitive inferences in that a judgment of congruence between the tray triangle and the hidden triangle by matching it with the model triangle (which had been shown to be congruent with the hidden triangle before it was covered and transformed)

Table 2

Imagery Task - Phase II: Frequency of Strategy Patterns and Correct Congruence Judgments by Age[a]

	I[b]					II				III		IV						V					VI							
Age	A	B	C	Total	%Total	A	B	Total	%Total	A	Z	A	B	C	D	Total	%Total	A	B	C	Total	%Total	A	B	C	D	E	F	Total	%Total
4	6 (7)	2	2	10	33	6 (33)	2	8	27	1	3	8 (63)	0	0	2 (100)	10	33	1	0	0	1	3	0	0	0	0	0	0	0	0
5	8 (50)	2 (100)	0	10	33	0	0	0	0	0	0	17 (82)	0	0	0	17	57	0	0	0	0	0	1	0	0	2 (100)	0	0	3	10
6	7 (71)	0	4	10	33	4 (75)	0	4	13	0	0	8 (88)	1 (100)	0	0	9	30	6 (100)	0	0	6	20	0	0	0	0	0	0	0	0
7	1 (100)	1 (100)	0	2	7	0	2	2	7	0	0	2 (100)	0	0	0	2	7	22 (100)	1 (100)	1 (100)	24	80	0	0	0	0	0	0	0	0
8	4 (50)	3 (100)	0	7	23	1 (100)	0	1	3	0	0	9 (100)	1 (100)	0	0	10	33	8 (100)	3 (33)	0	11	37	0	0	0	0	1 (100)	0	1	3
9	6 (100)	3 (100)	0	9	30	0	0	0	0	0	0	4 (100)	0	0	0	4	13	14 (100)	0	0	14	47	0	1 (100)	1 (100)	0	1 (100)	0	3	10
10	12 (83)	1 (100)	0	13	43	1 (100)	0	1	3	2 (100)	7	3 (100)	0	0	0	3	10	8 (100)	0	1	9	30	0	0	0	0	1 (100)	1 (100)	2	7
11	6 (50)	2 (100)	0	8	27	1	0	1	3	3 (100)	10	0	0	0	0	0	0	16 (94)	0	1 (100)	17	57	0	0	0	0	1 (100)	0	1	3
Total	50	14	6			13	4			6		51	2	0	2			75	4	3			1	1	1	2	4	1		
% Total Correct	64	86	0			57	0			83		86	100	0	100			99	50	67			0	100	100	100	100	100		

[a]Percentages of correct congruence judgments appear in parentheses, below frequencies.

[b]Category numbers and letters are defined in the text.

Note - 10 subjects per age level x 3 responses = 30 responses per age group.

would imply that the subject assumed that A=B (tray triangle = model triangle), B=C (model triangle is congruent with hidden triangle), therefore, A=C (tray triangle is congruent with hidden triangle).

Table 2
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Inasmuch as there were three transformations there were three opportunities to observe strategy use. The data in Table 2 are based then on three responses per trial (30 per age group). The strategy patterns observed were as follows:

I. Selection --> No verification

A. Visual inspection: one triangle

B. Visual inspection: more than one triangle (picked up and inspected)

C. Haphazard

II. Selection -> Memory verification (indicated by subject's verbalization)

A. Visual inspection: (one triangle) -> Memory match

B. Visual inspection: (more than one triangle) -> Memory match

III. Selection --> Hidden figure superposing

A. Visual inspection (1 triangle and more than one triangle) --> Hidden figure superposing

IV. Selection --> Common figure (single) edge matching

A. Visual inspection -> Common-figure single edge matching: corresponding sides

B. Visual inspection -> Common-figure single edge matching: non-corresponding sides and non-corresponding +corresponding sides

C. Haphazard -> Common-figure single edge matching: corresponding sides

D. Haphazard -> Common-figure single edge matching -- non-corresponding sides & corresponding + non-corresponding sides

V. Selection -> Common Figure Superposing

A. Visual inspection -> Common-figure superposing: Anticipation (1 triangle; +>1)

B. Visual inspection -> Common-figure superposing: Contact (1 triangle +>1)

C. Visual inspection -- Common-figure superposing: Anticipation & contact (1 T +> 1)

VI. Selection → Combination of verification strategies

A. Visual inspection → Memory match + hidden-figure superposing

B. Visual inspection → Memory match + common-figure superposing

C. Visual inspection → Hidden-figure superposing + common-figure superposing.

D. Visual inspection → Common-figure single edge matching + common-figure multiple edge matching.

E. Visual inspection → Common-figure single edge matching + common-figure superposing.

F. Visual inspection → Memory match + hidden-figure superposing + common-figure superposing.

The results were as follows:

In respect to strategy behavior -

1. Visual inspection without verification constituted fully a third of the subject's strategies at 4 years of age. That proportion was substantially the same through age 11, with a drop only at age 7.

2. Visual inspection occurred mostly with a single triangle in contrast to more than one (50 vs. 14), that is, the subject picked up and inspected only one triangle and made no attempt to match it with the model triangle.

3. Haphazard selections occurred up to age 6 and were few in number.

4. Visual inspections involving memory matches, that is, the child said he remembered what the hidden triangle looked like and selected the congruent triangle on that basis, occurred primarily in the younger subjects (27% at 4 years; 13% at 6).

5. Single edge matching was the dominant strategy up to age six reaching a peak at 5 years (57% of the responses). It dropped substantially at 7 years (7%) although it rose briefly at age 8.

6. Single edge matching, at all ages, including the youngest 4-year-olds, was with corresponding sides of the triangles, most often the hypotenuse.

7. As in Phase one, there was a transition at age seven in which the use of single edge matching was superseded by superposing (80% of responses) on the exposed model triangle.

8. Most of the superposing occurred without contact of the tray triangle with the model, instead it entailed holding one triangle at a distance above the other and thus judging their congruence.

9. A few subjects attempted to superpose the tray triangle on the hidden triangle, ostensibly, this was to the perceived or imagined outline of the figure.

10. As in Phase one, subjects stayed with a single strategy, either edge matching or superposing, and infrequently were they combined.

In respect to accuracy in judging congruence:

1. Visual inspection alone did not lead to correct judgments at ages 4 and 5. At older ages it was successful, although when used by even older subjects (10 and 11 years) it was less successfully used.

2. Single edge matching was less accurate at 4 years (63%), but by 5 years and 6 years was a fairly successful strategy (82% and 88% correct).

3. Superposing, when it appeared at 6 years, was wholly accurate although its very limited use at four years was not.

In general, the pattern of performance in phase 2 strongly paralleled that of Phase 1 both in strategies and in patterns of success in judging congruence, although there was somewhat less success in judging congruence in this condition. The fact that a substantial number of children at 5 years of age and even at 4 years (in the edge matching condition) made successful congruence judgments where a relocating transformation occurred attests to their ability to conserve those properties of the object on which the congruence judgments are based (i.e., size and shape) and are not in accord with the Thomas (1978) findings in respect to the effects of transformation on size judgments.

The appearance of edge matching and some superposing to the common triangle at ages four and five (57% of the responses) shows that at least a portion of the subjects were capable of making transitive inferences at these ages.

Phase 3 of the imagery task was designed to be the heart of the task. Its purpose was to assess the child's ability to invoke transformational imagery in the process of arriving at congruence judgments in the face of geometric transformation. In this phase, the transformations took place with the geometric figures hidden so that the child was required to imagine how the figure appeared in the process of transformation as it went through its trajectory to a termination point.

The task was a repeat of the procedures of Phase two, with modifications. Once two triangles were established as congruent, one of them was inserted into an envelope by the experimenter. The envelope was then transformed by a flip, slide or turn. In Phase three, the subject was then asked to carry out the same action with the uncovered triangle. The transformation however, was to be undertaken without an envelope to permit observation of the triangle's direction of movement, orientation, etc., and to deny spatial cues to the subject who might merely mimic and match the orientation of the envelope. Two aspects of the action were observed and noted. First, how the child placed the triangles. Second, how the child carried out the transformation.

Essentially, the child carried out a transformation or he did not. If the transformation was carried out, it was noted whether it was a relevant transformation or an irrelevant one. If it was relevant, it could be complete or not complete and if the latter it could be partial or overextended. If the transformation was relevant, it could be carried out in the wrong direction or it could be correctly transformed. The success of a child's performance could be interpreted in different ways. Three criteria were developed,

Table 3

Imagery Task - Phase 3: Transformation Strategies and Successful Performance by Three Criteria as a Function of Age (In percentage of totals)[a]

Age	Transformation Strategies I[b]	II	III	IV	V A	V B	Success Criterion (% Correct) Strong	Moderate	Weak
4	30	27	7	37	35	65	3	20	37
5	13	6	0	80	54	46	7	33	80
6	0	11	7	81	86	14	27	30	73
7	10	3	3	83	78	22	34	52	83
8	7	3	0	90	93	7	40	43	90
9	3	0	0	97	90	10	40	50	97
10	10	3	0	86	92	8	52	52	86
11	7	0	0	93	86	14	50	63	93
Total	9	7	2	82	78	22	32	43	80

[a]Based on 30 responses (10 subjects x 3 responses per subject) per age group.

[b]Coding scheme described in text.

strong, weak and moderate. The categories applied to the data were as follows, and appear in Table 3.

Table 3 Triangle Transformation Strategies

I. Transformation not performed; ambiguous (random)

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II. Irrelevant Transformation

III. Irrelevant & Relevant Transformation

A. Relevant Transformation: Partial

B. Relevant Transformation: Overextended

C. Relevant Transformation: Full

IV. Relevant Transformation

A. Partial

B. Overextended

C. Full

V. Direction of transformation

A. Same

B. Opposite

Accuracy of Transformation Responses

Strong Criterion: relevant [full] transformation + same direction.

Weak Criterion: any relevant transformation - regardless of direction.

Moderate Criterion: a full transformation irrespective of direction.

The results are as follows:

1. With the choice of placing the triangle in the original (i.e., proper) position or an altered one, children overwhelmingly placed it correctly (96%) even at 4 years of age.

2. With no instructions to the contrary, a substantial proportion of responses (48%) indicated that children took the instructions, "do what I [experimenter] did" literally, and mimicked the insertion of the triangle into an imaginary envelope. There were no age differences in this respect.

3. A sizeable proportion of subject's responses (37%), even at 4 years of age, were relevant with respect to the transformation (i.e., they were partial, full, or overextended).

4. A marked transition occurred between 4 and 5 years of age in the proportion of relevant transformations (from 37% of responses to 80%).

5. A comparable transition occurred with respect to the direction of the transformation, although at a later age (6 years). At 4 years of age, 65% of the responses were in the opposite direction from the model transformation, at 5 years it was approximately 50%, but at 6 years of age 86% were in the correct direction.

6. Successful transformational imagery performance may be judged in a number of ways: with a weak criterion, carrying out a relevant transformation alone (e.g., a slide, flip or rotation); with a strong criterion, based on a complete, but not overextended relevant transformation, plus carrying out the transformation in the same direction as the model transformation; and, a moderate criterion, set between the extremes, based on a complete transformation carried out irrespective of direction of transformation. The moderate criterion was used in later analyses of the consistency of performance.

Judged by the weak criterion, the proportion of success was essentially the same as category IV in Table 3 (relevant transformations), that is, a transition to a high level of successful performance (from 37% to 80%) occurred at 5 years of age. With the strong criterion, based on a complete transformation and correct direction of transformation, correct subjects responses did not exceed 52% even at 10 and 11 years of age.

With the moderate criterion (complete transformation irrespective of direction), performance showed two changes in magnitude, at 5 years (from 20% at 4 years to 50% of responses correct at 5 years), and at 7 years (from 30% at 6 years to 52% at 7 years). Performance, however, never reached a high level, even at 11 years of age.

Thus, by both the weak and moderate criteria a significant change occurred between 4 and 5 years in the ability to image the appropriate transformation, although achieving accurate performance, judged by a complete transformation in the appropriate direction was difficult for subjects and is to be accounted for by the nature of the task and the nature of the transformation involved.

An analysis of successful performance with each transformation revealed that in carrying out a complete relevant transformation the slide transformation was easiest, with 40% of responses correct at 4 years to 80% at 11 years. The flip was next, with 0% success at 4 years to 90% of responses correct at 11 years. Rotation was most difficult in respect to full transformation, with 20% of responses successful at 4 years, 50% at 7 years and a decline to 20% at 11 years. Concommittantly, however, relevant overextensions in rotation increased from 30% at 4 years to 80% at 11 years, whereas, because of the nature of the task, there were very few such overextensions (2 responses of 30) in the flip condition. Again, probably because of the nature of the tasks, there was a large number of partial flips, particularly in the middle years (at 8 years, 80% of the responses were partial flips), whereas in the rotations there were hardly any partial transformations at any age (1 response). In essence, it would appear that the flip and rotation were about equally difficult, with different transformation strategies emerging as a function of the nature of the task, although at four years of age the contrast between the two transformations showed a slight superiority of the rotation (complete and overextensions = 50% of responses correct) over the flips (comparable percentage: complete plus partial was 20%).

Discussion - Imagery Task

The data from this task reveal some important features of geometrical cognition. The most significant, aside from the very existence of geometric strategies in the very young child, is the shift from edge matching to superposing that occurred betweeen the sixth and seventh year groups. Edge matching although it precedes superposing and can be considered less advanced for a number of responses, would be a rather sophisticated method for verifying congruence if it were carried out on all sides of a triangle. It would provide a psychological counterpart for Euclid's SSS (side, side, side) proposition for establishing congruence. That is, if the sides of two triangles are congruent then the triangles are congruent. The method to instantiate this proposition would ideally be progressive edge matching, in which each side of the triangle is matched, one after the other. Subjects did not follow this procedure. For the most part they used one side only, usually the hypotenuse, the most salient corresponding side. The procedure they followed (in Phase one, for example) was to first scan the model triangle then the array of available triangles in the selection tray, apparently making some kind of mental or image match, finally selecting one from the tray that they thought would match the model. (Reports of younger children in Phase two that their judgments were based on memory matches lends support to this possibility.) Thus, the image of the triangle

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could provide both relational form data and size data. Whether they selected on the basis of a single edge match or more than one is no more evident than whether they matched at all; nevertheless, after selecting the best possible choice they proceeded to physically match one edge of the tray triangle with one from the model. The hypotenuse of the triangle was the most likely candidate for matching, possibly because it was the longest edge. Once the child verified that the length of one side of each triangle matched, the child jumped to the conclusion that the triangles were congruent. Younger subjects were more likely to be incorrect in their judgments of congruence with edge matching either because of incorrect relational form judgments, that is, they selected triangles whose formal features were not identical, lacking an appropriate relational matching rule, or their size choices were incorrect and instead of replacing the triangle in the selection tray to test another, terminated their search prematurely with an incorrect judgment. Older children tended to be more correct in their judgements, although they too jumped to concluding judgements with a single edge match; even though to be certain of congruence a check of all three sides would have been necessary. Their judgements were more likely to be correct because of correct visual matches on relational features and because of size. This is suggested by the fact that older children were more likely to be correct with visual inspection alone. That is, they were making correct mental (or image) matches on the basis of visual scanning in respect to both formal triangle and size relations, with sufficient assurance that actual matches, even of one side, were not necessary.

At the age when single edge matching became quite efficient (6 years), superposing began to be used, and by the next age level (7 years) was the predominant strategy. When the superposping strategy was used it invariably resulted in correct congruence judgements. The reasons may be sought in the

fact that the procedure involves, in one act, a test of the formal correspondence of the sides of the figures, the lengths of the sides and the surface areas of the figures, the three principal components that constitute the basis of congruence. The edge matching strategy is able to achieve these in only piecemeal fashion. These properties of the procedure may in fact account for why it is acquired later than the edge matching strategy, for to employ superposition requires the integration of these properties, which may not be possible till the age of 6 or 7 years. That period corresponds to the age when Piaget proposes that concrete operations are acquired.

The superposing strategy has another feature of significance relative to the edge matching strategy. Edge matching, particularly if it is multiple edge matching, as indicated, maps onto the Euclidean SSS proposition for asserting the congruence of triangles. There are other propositions, such as SAS (side, angle, side), but there is no evidence from the imagery task that young children were taking angles into account in establishing congruence, as is evident in the predominance of single edge matching. Euclid's SSS congruence proposition, however, applies only to triangles, it does not apply to other figures, such as quadrilaterals, for even though all the sides of two quadrilaterals might be equal in length they would not be congruent if their angles were not congruent, a condition that does not obtain with triangles whose angles of necessity are congruent if their corresponding sides are congruent. The superposing procedure for establishing congruence, on the other hand, applies to all figures and maps onto the proposition that two geometric figures are congruent if by a motion the figures can be shown to be isometric; although it can be covered more simply by Euclid's notion that two figures are congruent if they can be made to coincide (Euclid, 1956, p. 227). Whatever the theoretical relation between the Euclidean propositions, from an empirical point of view the edge matching strategy appears

to be simpler, and at the same time, a more limited method for establishing congruence, whereas superposition is more general (in applying to all figures) and more certain as a method of congruence verification.

The strategy data are significant in yet another respect. A persisting question in the literature on perception is as to the course of development in the perception of figures or objects. Two views prevail. One is that at the earliest age at which figures are perceived, the infant takes in the entire figure and development results in progressive differentiation of an object's features or components. The other position holds that the earliest engagement with the figure is of its parts and that development involves progressive integration of the parts to a final integration or composition of the figure. The former theories are identified as differentiation theories, the latter constructivist theories. The evidence from infant perception research designed to test each position has been hampered by technical difficulties (Hainline, 1982), so that an adequate characterization of perceptual development has not been forthcoming. Nevertheless, from other evidence, the Gestaltists and others (e.g., Gibson, 1969) have held to differentiation theories and others (like Hochberg, 1972) have argued for the constructivist thesis. The controversy has a bearing on the development of mathematical cognition, as is evidenced by P. van Hiele's theory of mathematical instruction (Wirszup, 1976; van Hiele, in press), which is based on gestalist differentiation assumptions as to the processing of geometric information. Van Hiele holds that in learning of properties of geometric figures, the child progresses from considering the figure as a whole to knowledge of its parts and its properties. In essence, it is a view of learning in which learning order follows genetic (developmental) order, although the learning he refers to occurs at school learning age. If developmental order of acquisition is to be used as a model for learning then

empirical findings as to developmental order have clear relevance for educational practice and instructional technologies.

The imagery task data speak to the question of developmental order of part-whole relationships, at least in respect to congruence verification strategies. What we find is evidence consistent with the constructivist view. Early single edge matching shows that subjects are attending to parts of the figure and not to the figure as a whole in judging the congruence of two figures. Taking the figure as a whole into account appears later with the appearance and almost exclusive use of the superposition strategy, which we suggest is based on the integration of parts and features of the figure. As will be reported later, two supplementary studies show even earlier appearance of the edge matching strategy. We want it to be clear, however, that we are not suggesting that these data directly concern perceptual examination of the figure as a part or whole, inasmuch as the strategies we observed appear after visual inspection of the figures, and short of eye movement data of the visual inspection process, no assertions as to the nature of what occurs in that scanning is possible.

We do refer, however, to the cognitive processes in making judgments of congruence and in this context the developmental order is as reported. To the extent that mathematical knowledge is more likely to be acquired by cognitive processes, we suggest that a more appropriate model for instruction may be these naturally occuring strategies than a developmental order defined by perceptual differentiation theory.

The data also reflect on a variety of children's competencies at young ages. The data of Phase two show that a substantial number of children at 4 and 5 years of age are capable of correct congruence judgments in the face of relocating transformations, which attest to the ability to conserve size and shape properties. The appearance of these abilities in younger children is

consistent with Piaget's more recent theory of conservation based on commutativity and correspondences.

It is also evident that at 4 and 5 years a portion of the subjects are capable of making transitive inferences in the use of edge matching and in superposing the common triangle in Phase two.

Whereas the data of Phases one and two suggest that imagery enters into the processes by which congruence judgments are made, Phase three, which more directly tested transformational imagery, showed that the capacity for performing accurately is more difficult when transformations are manipulated directly and representation is necessary through reconstructive action. It would appear that the nature of the task played an important role in making the task a demanding one for subjects. Using a criterion based on carrying out the relevant transformation, a substantial number of subjects were competent at 5 years of age (80% of responses correct). With a criterion based on modeling the transformation performed by the experimenter and approximating it in both extent and direction, success was only moderate, even at 11 years of age. This was clearly a demanding requirement particularly in respect to rotations and flips, whereas success was considerable with the slide transformation.

Thus, the data from Phase 3 are consistent with the general notion of a marked transition, around 5 years in competence in transformational imagery. The data also show that the nature of the transformation has a marked effect on the difficulty of transformational imagery so that a simple generalization about transformations is not possible without taking into account the nature of the transformation, as others have noted (Lesh, 1978). In some of Piaget's studies, the nature of the transformation that enters into the task is not always taken into account in explaining the difficulty of the results. This may be so, for example, in Piaget's falling pencil (around a fulcrum) demonstration,

and also in water level experiments, in which some orientations (in a rotation of the bottle) are more difficult than others.

Measurement Task Strategies

In the measurement task, two triangle differing by a transformation (slide, flip and turn) were placed before the child. The triangles were either congruent or noncongruent and were presented under moveable or fixed conditions. A set of materials that could be used as measuring instruments was laid out in a tray on the table. The child was asked to find whether or not the triangles were congruent. The child's strategies and measurement operations were observed in their efforts to accurately assess congruence.

The strategies evident in the task, consistent with what appeared in the imagery task, were visual inspection, single and multiple edge matching, and superposing, in addition, there was the use of objects as templates and specific measurement techniques used in conventional and nonconventional ways single and multiple edge matching occurred with corresponding or non corresponding sides of the figures or in combinations of these. With mobile figures, certain strategies were possible that were not possible with fixed figures. With the two test triangles on the table, measurements or comparisons between them occurred through direct comparisons between the figures or by some indirect method. There were three trials per subject.

Classification of measurement strategies (see Table 4).

Table 4

I. No response.

II. Estimation strategies

A. Visual inspection

B. Corner matching (only corners of triangles are approximated)

III. Single edge matching

A. Single edge matching: corresponding sides

Table 4

Measurement Task: Strategies by Age

(In frequencies)

Age	I[a]	II		III		IV		V		VI		VII		VIII		IX		X		XI	
		A	B	A	B	A	B	A	B	A	B	A	B	A	B	A	B	A	B	A	B
4	13	13	1	14	2	1		3	4			3		29	15			1			
5	6	12		6	1	1		4	4		1	10	6	16	30	8		3		1	
6	8	4		5	4	1	3	4	7	1		3	14	11	45			1			
7		1	1		1				4				15	5	57	6		10		4	
8		5				1			2		1	1	13	1	59	6		11			
9		2		1					5				12	1	55	5		19			
10		3		3	3				2				6	1	56	6		15			
11		8		4								1	6	1	52	10		13			
Total	27	48	2	33	11	4	3	11	28	1	2	18	72	65	369	41		74			
%	3.3	5.8	.02	4.0	1.3	.04	.03	1.3	3.4	.01	.02	2.2	8.8	7.9	45.6	5.0		9.0		5.06	
% Total Correct	0	81	50	88	64	50	33	36	82	0	100	50	83	26	81	88		91		80	

e: 3 trials per subject: multiple strategies per trial were possible.

[a] Strategy coding described in text.

III. B. Single edge matching: non-corresponding; corresponding and non-corresponding sides

IV. Multiple edge matching

A. Multiple edge matching: corresponding sides (fixed and progressive combined)

B. Multiple edge matching: non-corresponding sides; corresponding and non-corresponding (fixed and progressive combined)

V. Tray Triangle - single edge matching

A. No direct comparisons (corresponding and non-corresponding sides combined)

B. Direct comparison (corresponding and non-corresponding sides combined)

VI. Tray Triangle - multiple edge matching

A. No direct comparison (corresponding and non-corresponding combined)

B. Direct comparison (corresponding and non-corresponding combined)

VII. Template Matching (pencil tracing, string tracing, and figure cutout)

A. No direct comparison

B. Direct comparison

VIII. Measurements (conventional and nonconventional)

A. No direct comparison

B. Direct comparison

IX. Superposing

X. Tray triangle + superposing

XI. Combinational of single edge matching and measurement strategies

Analysis of subject's responses suggested the following:

1. The most frequent means of establishing congruence in this task was through the materials made available for the purpose of measuring one or both test triangles. There were two types of such material, conventional measuring instruments (ruler, protractor), and materials that could be used for non-

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Table 5

Measurement task: Mobile vs. Non-mobile condition; Mean correct[a] (percentage correct in parentheses)

AGE	MOBILE	NON-MOBILE
4	2.3(78)	.8 (27)
5	2.5(82)	1.7 (58)
6	2.6(82)	1.9 (64)
7	2.5(82)	2.5 (84)
8	3.0(100)	2.8 (93)
9	2.87(96)	2.9 (98)
10	2.87(96)	3.0 (100)
11	3.0 (100)	2.87 (96)

[a] 3 responses per subject

conventional measurement (that is, by methods not taught in school or home and not acquired from an adult, such as, string, sticks and body parts. Across age, 45% of the response patterns were measurement strategies. Among the younger children the tendency was to use nondirect comparisons, i.e., involving attempts at measurement with only one of the test triangles. These dropped out so that by 7 years direct comparisons between the two test triangles were made with measuring instruments. This was seen, for example, when a child aligned a piece of string with the edge of one triangle and that length of string was transported to the corresponding side of the other triangle. Non-direct comparisons tended to be unsuccessful (26% correct) and direct measurement, successful (81% correct overall).

2. In the mobile condition, the condition in which the test triangles could be moved by the subject, the strategies employed were the same as those in the imagery task, that is edge matching and superposition. Edge matching of corresponding sides was used by the younger subjects (4 to 6 year-olds) and less so with non-corresponding sides. The former led to more correct congruence judgments (88% vs. 64%). There was very little multiple edge matching.

Table 5

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3. The nonmobile condition was more difficult for subjects to judge correctly, at least from 4 to 6 years of age. This was probably because they could not use edge matching - with their usual success. Even 4-year-olds by contrast were highly successful (78%) in judging congruence when they could move the test triangles. Subjects had measurement strategies at an early age but these tended to involve nondirect comparisons which usually resulted in incorrect congruence judgments (Table 5).

4. A triangle was among the available measurement objects. It was used by a portion of the subjects, particularly in the non-mobile condition, for single edge matching, multiple edge matching and for superposing. It was used as a

common measure in comparisons between the two triangles almost across the entire age range. It was successfully used in direct comparisons, and not very successfully by the younger subjects, predominantly, in non-direct comparisons. Again, multiple edge matches were infrequent. Tray triangle superposing followed the general pattern seen in the imagery task. It appeared infrequently before 7 years, but after 7 was second only to the "measurement" strategies. It also yielded the most consistently correct congruence judgments (91%). Tray triangle superposing in the non-mobile condition usually involved placing the tray triangle on both test triangles, unlike mobile condition superposing in which one test triangle was placed on the other. Tray triangle superposing occurred much more frequently in the non-mobile condition than in the mobile, prompted no doubt by the nature of the respective task demands. Tray triangle edge matching was more likely to be used by younger subjects and tray triangle superposing by the older, in a pattern similar to that found in the imagery task.

5. One group of strategies shows the considerable inventiveness possible on the part of these subjects in determining the congruence of triangles. These strategies involved the creation of templates out of available materials to use in the comparison of test figures. One such template was created by pencil tracing the outline of one triangle on paper and then fitting the second triangle within the figure traced on the paper. Another template was created by tracing the outline of a triangle with string, sometimes with a stick as the anchor at one end in a usually awkward attempt to keep the outline as taut as possible, removing the triangle and replacing it with the second triangle. A third method was cutting a paper pattern with one triangle and then either placing the pattern on the second triangle or if the triangle were

movable inserting the triangle in the cutout left by the pattern's removal. These strategies were used in two ways, either as direct comparisons where the template was used as a common measure between the two test triangle or with no direct comparisons, that is, the judgment of congruence was made with one triangle alone. The latter application of the strategy was evident primarily in younger subjects (50%). Direct comparisons, typical of the older subjects (from 6 years on), were more likely to lead to correct judgments (83%).

Discussion

The data from the measurement task support the strategy data findings of the imagery task. In fact, phases one and two of the imagery task are closer to aspects of qualitative measurement than to imagery, in that the notion of a common measure and transitive inference utilized in performing in those tasks are fundamental to the measurement concept (Piaget, Inhelder & Szeminska, 1960) and were intimately involved in edge-matching and superposition procedures in establishing congruence. The measurement task in having both mobile and non-mobile test triangles created special problems but also opportunities for subjects, particularly the non-mobile condition, in which objects other than the test triangles themselves had to be used to establish congruence. The availability of the measurement objects almost demanded their use, even in the mobile condition. It was often maddening to the experimenters to see a child working diligently at a complex measurement procedure, such as the creation of a template to test for congruence, when the experimenter knew how simple it would have been to lift one of the test triangles and superpose it onto the other to achieve the same objective. This speaks to a number of contested issues in cognitive development. One concerns the basic competencies of the

youngest age group studied (4-year-olds). Recently investigators have attempted to make much of the skills of very young children, either in asserting they share the same processes and structures as adults (e.g., Bryant & Trabasso, 1971), and differ only in secondary skills, or that they have hitherto undetected competencies (Gelman, 1972). Our data both support and contradict these views. First, they do show the availability of interesting and unexpected strategies, such as those of edge matching and template constructions, which support the early competence position. A fully logical use of edge matching, however, would result in perfect congruence judgments, but only if the procedure were multiple edge matching, in that to assert congruence on the basis of a SSS proposition, measurement would be necessary of all the sides. Instead, children mostly used only one side for verification in arriving at a judgment. The fact that they concentrated principally on the hypotenuse as a significant component of the triangle suggests non-trivial knowledge of triangle characteristics at 4 years of age, but that knowledge did not lead to very accurate performance and it was not till later that a more fully adequate strategy appeared, namely superposition that was almost fool-proof in verifying congruence. Clearly the integrative competencies explicit in this procedure were not available to the younger child to use spontaneously and in this case one has a set of competencies not available to the younger child.

Again, very young children do have other measurement skills at their disposal at 4 years associated with conventional uses of measuring instruments and non-conventional uses as well, but the 4-year-old was unable to perform well; i.e., above 50%, in making such judgments, whereas at 5 years a substantial

increase in correct responses occurred (75%). The inventive strategies associated with template construction appeared in small numbers at 4 years and more so at 5 years, but the strategy was not in place at 5, as evidenced by its use with only one of the two test triangles. It was not till 6 years that the strategy was effectively used in a direct comparison between the two test triangles. Thus, in respect to measurement operations we again see a general transition to a new level of competence occurring at about 6 and 7 years of age in the use of more elegant and general strategies. Again, whereas very young children exhibit competencies not previously reported, they nevertheless lack the power and sophistication to put these competencies to effective use.

Correspondence Task: Lines and angles

In this task there were two triangles, the subject's and the experimenter's, which differed by a transformation (slide, flip and turn). The experimenter placed 3 line segments of plastic on the sides of his triangle or 3 angles at the vertices, also of plastic, and asked the subject to transfer each line or angle to his triangle. There were three trials and three transformations per trial per subject, i.e., 9 responses per subject. Explanations of the basis of their responses were sought. Multiple explanations were possible per trial so that the number of explanations exceeded 90 in some cases.

Subject responses to the task were of two kinds. First, was the actual placement of the line segments and angles on the triangle, second were the verbal explanations given as to why a placement was made as it was. The classification of responses was as follows (Table 6):

Table 6 Placement Strategies

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Table 6

Correspondence Task: Lines/Angles; Strategies and Explanations by Age

(In percentages)[a]

Age	Placement[b] Immediate	Trial/ Error	Number	None	Process of Elimination	Explanations Comparisons Between 2T	Within 1T	Property	Trans- formation	Errors
4	74	26	84	21	1	55	23			9
5	92	8	91	4	10	74	7	2	3	
6	96	4	89	12	6	57	12	11	1	2
7	98	2	90	1	9	66	10	4	10	-
8	94	6	93	-	8	58	26	6	2	-
9	97	3	90	-	2	68	28	2		-
10	98	2	91	-		68	4	7	21	-
11	100	0	91 719	-		51	13	15	21	-
% Total	94	6	719	5	4	62	15 77	6	7	1

[a]Summed over three transformations - 3 trials per transformation; 90 responses per age group (720 grand total).

[b]Numbers of explanations vary.

I. Immediate Placement

II. Trial and error placement

Explanations

III. No explanation or irrelevant explanation. ("I don't know," "Because I'm not smart.")

IV. Process of elimination. ("It's the last place left.")

V. Comparisons:

A. Between two triangles. ("It's the shortest side on yours and its the shortest side on mine.")

B. Within one triangle. ("It's too big for that angle, but it fits this angle.")

VI. Property identification. ("It's the square corner.")

VII. Transformation. ("It's the same place as on yours, but it's turned around.")

1. Performance in this phase of the correspondence task was highly accurate. There were 9 errors (in 90 responses) at the 4-year-level and 2 at the 6-year-level.

2. Only at the 4-year-level was there any appreciable trial and error placement of corresponding lines and angles (26% of responses), a proportion that was reduced at the 5-year-level to 8%. Although a substantial number required trial and error searching, most of these subjects nevertheless made no errors in their placements (again, there were only 9 errors in the total group).

3. The dominant type of explanation was of a comparison of parts either across the two triangles or of parts within a single triangle, with the across-triangle comparison predominating. There appeared to be no age trend in these explanations. Geometric property and transformation explanations were much fewer, although

Table 7

Correspondence Task: Points on Circle; Strategies and Explanations by Age

(In percentages)[a]

AGE	PLACEMENT Immediate	PLACEMENT Trial/Error	LOCATION Number	LOCATION None	LOCATION Position	LOCATION Proximity	LOCATION Transformation	EXPLANATIONS Number[b]	EXPLANATIONS None	EXPLANATIONS Single Comparison	EXPLANATIONS Comparison 2 dots	EXPLANATIONS Comparison >2 dots	EXPLANATIONS Transformation	EXPLANATIONS Misc.	EXPLANATIONS Error
4	92	8	96	35	26	6	32	87	16	79	3	-	-	1	68
5	87	13	102	20	39	6	35	90	8	71	9	6	-	7	62
6	92	8	98	15	36	3	46	91	9	48	30	9	2	2	52
7	93	7	96	6	51	2	41	93	-	51	32	15	1	1	59
8	98	2	92	4	40	2	53	91	-	42	35	10	2	10	47
9	94	6	95	4	34	-	62	98	2	35	22	38	2	1	36
10	97	3	93	-	33	1	66	90	-	49	17	22	11	-	33
11	97	3	93	3	22	-	75	99	1	30	20	25	20	3	22
			765					739							
% Total	94	6	765	11	35	3	51	739	4	50	27 (37)	16	5	3	

[a]Summed over three transformations - 3 trials per transformation: Total 90 responses per age group. Except 4 years.

[b]Variations in number of explanations due to multiple responses per subject.

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these explanations increased appreciably in older subjects (10- and 11-year olds) particularly the transformation explanation.

Discussion

This task apparently taps a set of processes that are well instated in the young child's repertoire. The nature of the task was designed to expose the comparison processes that lead to establishing correspondence between geometric figures. Even the 4-year-old subjects in the study appeared to use comparison processes with great success. The only limitation in their performance, besides a somewhat higher error rate was greater trial and error search and test. This suggests that older subjects were able to process the same data at a greater speed, eliminating potential errors covertly and terminating their search quickly and efficiently.

The correspondence task: points on a circle, which we now describe, shows however that comparison processes and correspondence capacities can be greatly inhibited in a task devoid of the geometric properties of triangles.

Correspondence Task: Points on a circle

The interest here was to see how children use comparison methods in a context where the angles and straight sides of triangles were missing and where locating a point relative to others would have to be based on order and transformation relations. In this task 5 colored dots were placed in clock locations on the periphery of the experimenter's circle. Two colored reference dots were located on the child's circle. His task was to locate the three remaining dots in positions to match the series on the transformed experimenter's circle. There were three trials for each of the three transformations, i.e., 9 trials per subject. Three types of data were recorded (see Table 7). First, the placement strategy used, either trial and error or immediate placement. Second, was the

Table 7

location strategy or type of dot placement. Three response types created errors; one type of response led to successful placement. Third, were subjects explanations. A miscellaneous category indicated no meaningful explanation.

Strategies and Explanations

Placement Strategies

I. Immediate Placement

II. Trial and error placement

Location strategies

Error producing strategies:

III. No correspondence. Dot was placed so that it did not correspond to the experimenter's (for flip and rotation trials).

IV. Position correspondence. In flip and rotation trials, correspondence to the relative position of experimenter's dot, ignoring transformation of the circle.

V. Proximity correspondence. In slide and rotation trials, dot was placed in space closest to experimenter's circle.

Correct response strategy:

VI. Transformation correspondence. Correct relative position taking into account transformation of circle - in slide, flip and turn trials.

Explanations

VII. None or irrelevant

VIII. Single comparison ("It's in the same spot on yours and on mine.")

IX. Ordered comparison

A. Two dots ("It's next to the red dots on yours and it's next to the red dot on mine.")

B. More than two dots. ("It's between the red and the green dots on yours and it's between the red and the green dots on mine.")

X. Transformation. ("It's the same spot on yours but it's facing the opposite direction.")

The results were as follows:

1. Immediate placement was consistently high from the earliest age; this however was unrelated to error rate inasmuch as there were considerable errors in both categories.

2. Nevertheless, even in the youngest age group there was a fair amount of success in placement (32%), appearing predominantly in the slide condition. Success overall kept increasing with age, but never exceeded 75% of responses.

3. When errors were made they were first made with near equally random placement (the "no correspondence" category) and "position responses." Random responses reduced with age. Position responses increased and then reduced somewhat, although even at 11 years a fair number of subjects were employing what could be considered a stereotypic response (Gholson, 1980).

4. Most subjects (83% of responses) gave explanations for their placements, a large proportion of which were single comparisons (79%). That proportion decreased with age (to 30% at 11 years), whereas multiple comparisons increased. Two dot comparisons increased with age, and then decreased. At a point where they began to decrease, "2 dot plus" comparisons reached their peak (38%) at which point explanations begun to increase (age 10). At age 11 years, explanations were about evenly distributed among all categories except the random and miscellaneous group.

7J

Discussion

The change to a circle from a triangle entailed two changes in properties, the loss of angles and the shift from relations among straight sides of different length to order relations among points on the periphery of the circle. These stimulus changes required changes in strategies that from the other data were known to be in the children's repertoires, but which were not used very efficiently when applied to the circle task.

On the whole, the data show that very young children (4-year-olds) are able to employ comparisons, but have difficulty making comparisons in all transformation conditions. Difficulties were soon resolved for the slide (by 6 years), but difficulties with the flip and turn transformations continued so that by 11 years between 30% and 40% of responses were not correct. When incorrect, subjects made responses that maintained position relative to the experimenter's circle but ignored the transformation that had occurred. Developmental influences were evident in this task in all transformation conditions but particularly for the flip and rotation. For the most part, at least where the points on a circle task is concerned, it is not till the 11 year level that one could say that a substantial proportion of subjects had mastered congruence in the face of these transformations.

Combination of motions task - strategies

This task employed a simple maze-type board on which a movable disc with a pattern on each side was moved by subjects from the "start sign" to the "goal sign" along different paths matching the disc pattern to signs along the route in order to be permitted to proceed. Each sign was a rotated or flipped transformation relative to the start sign or preceding sign (Figure 1g).

The task itself was preceded by a disc matching "pretest" in which the child was asked to make these discs "look the same." (The discs differed by a flip, or rotation). The discs and signs along the routes had a radius line that differed in orientation [horizontal or vertical] and the two halves of the disc differed in color.)

There were two trials. The second trial differed from the first in that a barrier was set up on one of the paths of the maze - the path taken on the first trial - so as to force the subject to take another route with a different pattern of route sign transformations.

In the initial disc match, which preceded performance in the task, subjects could make three types of error: flip, rotation or order rearrangement errors. The percentage errors were 42% for flip, 49.5% for rotation and 8.5% for slide (averaged over the number of responses necessary to make a correct initial match, which for 4-year-olds was a maximum of 5 responses and 2 for 11-year-olds). On the final match of the initial disc matching series it was noted as to whether the child required assistance to achieve the match. Only 41% required no assistance and of the 59% who did, 55% required assistance on the flip, 1% on the rotation and 3% on both.

At 4 years, "start sign" matching errors (after the experience of the initial (pretest) matching, were only moderate and they gradually dropped off to 5% or none, at 8 to 10 years. Inexplicably they rose to 20% at 11 years.

Of 386 responses in the task itself across all ages, 159 (41%) were errors. Errors were of two kinds, relevant errors or irrelevant. Relevant errors were those of omission, e.g., a flip was required but not carried out. Irrelevant erros were those of commission, e.g., a flip was required but a rotation was carried out. Route sign errors of the relevant type were mostly

flips (25%), with 13% of rotations and 1% slides not carried out. The predominant irrelevant transformation was to a rotation (35%) with 13% flips and 3% slides. Omission errors on rotations dropped off after 4 years (from 32% to 7% at age 5) and flip errors after 5 years (from 41% to 23% at 6 years).

Discussion

This task, which tapped the child's ability to carry out a series of transformations, principally flips or rotations, was difficult, principally for the youngest subjects. As was evident in the correspondence task (points) as well, the flip and rotation transformations were about equally difficult; with the rotation somewhat more so (particularly for the 4-year-olds; 39% flip, 53% rotation errors). For 11-year-olds the proportions were equal, 50% each, although the number of errors decreased appreciably in older subjects.

Strategy Consistency Analyses

As the data on individual tasks show, a wide variety of strategies are capable of being generated by children in attempting to make geometrical congruence judgments in the face of spatial transformations. Some of the same strategies appeared in different tasks. This enabled us to make some inferences as to the consistency of performance, but clearly a specific vehicle was needed to establish a common base from which to judge subject consistency across tasks, one that would be amenable to traditional types of statistical analysis that conventional strategy data as a rule tend to resist.

The approach selected for the analysis was suggested by the fact that developmental data on strategy deployment pointed to different levels of rule use. An analysis yielded a set of levels of rule efficiency and economy

that could be applied to most of the strategies observed in the study.[1]

Thus, the intent of the rank ordering of strategies was to provide a common basis for equating strategies. The assumption was that strategies of equal level are characterizable by their algorithmic nature, defined by the level of complexity of the rules and processing components of the strategies. In that way a measurement instrument was developed for determining the extent to which individual subjects were consistent in their use of strategies. This is particularly necessary in a multiple task study where different processes are tapped and different strategies are brought to bear on problem solution. Where a common knowledge structure is investigated, such as congruence and geometric transformation, this offers a useful device for assesing cross task consistency, if not unity.

Strategy Levels

As the earlier analyses indicated, the strategies children use to establish congruence appeared to fall into three levels. An examination of the strategies suggested that they differ according to whether the strategies were based on the use of a rule-governed procedure, and within the rule-governed procedures by their economy of effort and generality in use. The strategies observed were ranked according to the following characteristics.

Level I - No Algorithm

The child appears to lack a rule to guide problem solving behavior. They are the least economical or efficient strategies in that they do not organize problem-solving behavior into a set of systematic procedures to be followed to solution. Evidence for partial rules may exist but these partial rules are not followed to solution.

[1] Due to the difficulties in identifying a hierarchy of strategies in the combination of motions tasks, the data from that task were omitted from strategy level analysis.

Level II - Power Algorithms

The child's problem solving behavior appears to be guided by a rule or rules that if correctly executed lead to congruence problem solution. These rules have limited economy or efficiency in that they entail more procedures (steps) than more efficient strategies, and yield less information about the geometric properties of figures than maximum power strategies.

Level III - Power 2 Algorithms

The child's problem solving behavior appears to be guided by a rule or rules that if correctly executed lead to congruence problem solution. These strategies appear to require a minimum number of steps in their execution and yield a maximum of information of the geometric properties of figures.

Strategies seen in each task were classified by algorithm rank as follows:

Imagery Task - Phase 1

Level	Strategy Group	Strategies
1	Selection without verification	Visual inspection - Haphazard selection
2	Edge matching	Single edge matching Single plus multiple edge matching
3	Superposing	Superposing Single edge matching plus superposing

Imagery Task - Phase 2

1	Selection without verification	Visual inspection Haphazard selection
1	Memory match verification	Visual inspection combined with memory match
1	Hidden figure superposing	Hidden figure superposing Memory match plus hidden figure superposing
2	Edge matching	Single edge matching Single plus multiple edge matching

3	Superposing (common figure)	Superposing Superposing plus any other strategy

Imagery Task - Phase 3*

1	No (scorable) response	Ambiguous transformation Transformation not performed
1	Irrelevant transformation	Error based response
2	Irrelevant and relevant transformation	Relevant response - partial, overextended or complete transformation
2	Relevant transformation	Error based relevant response - partial or overextended transformation
3	Relevant transformation	Correct response - complete transformation

Measurement Task

1	Estimation	Visual inspection Corner matching
1	Measurement	Conventional/nonconventional (all patterns without direct comparison)
1	Edge matching	Single plus multiple edge matching with tray triangle - without direct comparison
1	Template matching	Pencil trace/string trace/cutout - without direct comparison
2	Measurement	Conventional/noncoventional - with direct comparisons Single edge matching plus measurement.
2	Edge matching	Single plus multiple edge matching Single plus multiple edge matching with tray triangle with direct comparison
3	Template matching	Pencil trace/string trace/cutout
3	Superposing	Superposing Superposing with tray triangle

*Based on "moderate" criterion for correct response: Relevant, complete transformation regardless of direction (see strategy section for discussion).

Correspondence Task: Lines/Angles - Explanations

1	No (scorable) response	No explanation Irrelevant explanation
1	Process of elimination	Process of elimination
2	Comparisons	Comparisons between 2 triangles Comparisons within one triangle
2	Geometric Property	Geometric property explanation
3	Transformations	Transformation explanation

Points on circle - Explanations

1	No (scorable) response	No explanation Irrelevant explanation
1	Process of elimination	Process of elimination
1	Alternative choice	Alternative choice
2	Comparisons	Single comparison Ordered comparisons - 2 dots; more than 2 dots
3	Transformation	Transformation explanation

To test for consistency of strategy rank, children's strategic responses across tasks were classified by the child's predominant strategy level (I, II or III), which was defined as 67% of the child's strategies across tasks falling in the same level. Inconsistent patterns were defined as those with less than 67% of strategies in the same level. There were two types of inconsistent patterns: 2-level patterns, and 3-level patterns. In the 2-level pattern, a majority of the child's strategic response were in one strategy level but less than 67%, and 67% of the remaining responses were in a second strategy level. In the 3-level pattern, a majority of responses but less than 67% were in one level, and no other level contained at least 67% of the remaining responses.

Consistency analysis results across tasks.

Table 8

Consistent and Inconsistent Strategy Patterns for Algorithmic Levels Across Tasks by Age

(In numbers of subjects)

Age	Consistent Strategy Patterns			Inconsistent Strategy Pattern		
	I[a]	II	III	I/II	II/III	I/II/III
4	1	4		5		
5		6		2		2
6	1	8		1		
7		5			5	
8		8			1	1
9		8			2	
10		5			4	1
11		2	1	1	5	1
Total (%) by rank	2(4)	46(94)	1(2)	9(29)	17(55)	5(16)
Total (%)		49(61)			31(39)	

$\chi^2 = 4.05$; df = 1, $p < .05$.

[a]For description of ranks see text.

As Table 8 shows, there were significantly more consistent strategy patterns than inconsistent patterns across all age groups. The predominant consistent

Table 8 — 76

strategy pattern was level II (94%), and the predominant inconsistent pattern ranged over levels II and III (55%). The inconsistent patterns show an age trend in the shift from levels I (the lowest rank) in the younger ages to level III in the older ages. Thus there is a shift from the least economical and efficient response level to the most efficient with age. Children younger than 6 years were distributed between using level II strategies and a combination of level I and level II. All children between 6 and 9 years who were consistent relied on level II strategies. Children older than 9 years were distributed between consistent use of level II strategies and a combination of level II and level III. Consistent strategy use was significant, however, at only one age level - level 6 ($p < .05$) and approached significance at 8 years ($p < .10$) and 9 years ($p < .10$). This suggests that there was stability at certain ages and that between those ages there was change or transition. The causes of inconsistency we considered were these: (1) children are consistent within sets of tasks, but not across all tasks, (2) they are consistent only within individual tasks, or (3) they were inconsistent both within and across tasks. To test for these possibilities, consistency data within tasks was analyzed.

Consistency analysis results within tasks

Consistent and inconsistent patterns were determined within individual tasks with the same consistency criteria used in the cross-task analysis.

Table 9 — 78

Table 9 shows the numbers (and percentages) of subjects with consistent patterns on each task. Strategy consistency ranged from 80% to 100% on all tasks except the measurement task (overall Chi Square, $p < .001$). This result

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Table 9

Consistent and Inconsistent Strategy Patterns by Task

(In numbers of subjects; percentages in parentheses)

TASK	CONSISTENT STRATEGY PATTERNS			INCONSISTENT STRATEGY PATTERNS				
	Levels			Levels				
	I	II	III	I/II	I/III	II/III	I/II/III	p
Imagery Phase 1	9(11)	31(39)	39(49)	0(0)	0(0)		1(1)	
Total		79(99)				1(1)		<.001
Phase 2	30(38)	20(25)	30(38)					
Total		80(100)						<.001
Phase 3	9(11)	32(40)	30(38)		1(1)		8(10)	
Total		71(89)				9(11)		<.001
Measurement	10(13)	40(50)	7(9)	3(4)	0	14(18)	6(8)	
Total		57(71)				23(29)		<.001
Correspondence: Lines/Angles	2(3)	68(85)	2(3)	4(6)	0	3(4)	1(1)	
Total		72(90)				8(10)		<.001
Points	2(3)	72(90)	3(4)	3(4)	-	-		
Total		77(96)				3(4)		<.001

eliminates possibility (3) that children are inconsistent both across- and within- tasks. To eliminate one of the two remaining possibilities, a task profile of inconsistent strategy subjects was prepared which revealed that the form of inconsistency changed over age, as already suggested in the cross-task analysis. Below six years, children were consistent in strategy use with 2 sets of tasks. They used level II strategies on phase 1 of the imagery and correspondence tasks, but level I (the lowest level) strategies in phases 2 and 3 of the imagery task and the measurement task. Above six years, children were consistent within three sets of tasks and their strategy levels differed from the younger subjects. They manifested level III (the most efficient) strategies on all phases of the imagery task, level II strategies in correspondence tasks and a combination of strategy levels II and III on the measurement task. Thus, children were consistent in their strategy use within sets of tasks, but not across all tasks. Second, the basis of children's inconsistent strategy performance was not the same at all ages. We believe that strategy inconsistency utilizing both levels I and II in younger children reveals limitations in their conceptual knowledge of congruence, whereas in older children strategy inconsistency in levels II and III attests to the adaptation of procedural knowledge to specific task demands.

Discussion

The results of the strategy level consistency analyses suggest several conclusions about the relationship between conceptual knowledge and procedural knowledge. First, children are consistent in strategy deployment within sets of tasks, although on different sets of tasks at different ages. Second, the development of problem solving strategies proceeds at two complementary levels: on one level, knowledge of congruence improves with age, as indicated in the

Table 10

Successful Congruence Responses for Tasks as a Function of Age and Sex

(In mean percentages correct)

Age	Tasks: 1 Imagery I (30)[a]	2 Imagery II (30)	3 Imagery III (30)	3 Imagery III M	3 Imagery III F	4 Measurement (90)	5 Correspondence Lines/angles (90)	6 Correspondence Points (90)	7 Combinations of Motion (varies up to 5)
4	60	33	20	20	20	52	91	32	58
5	93	74	33	46	20	70	100	37	79
6	83	74	30	40	20	72	98	49	88
7	100	93	53	67	40	83	100	41	90
8	90	90	43	60	27	97	100	53	95
9	100	100	50	46	53	97	100	64	93
10	93	90	52	60	43	98	100	67	94
11	90	87	64	74	53	99	100	78	93
Total	89	80	43	52	35	84	99	53	86

[a] Numbers in parentheses are responses per age group (10 subjects x 3 trials). Responses in Combinations of Motions task ...es per subject (up to a total of 5).

...e sex differences were significant only in Imagery III task, in ANOVA the sexes are collapsed for all other tasks.

use of increasingly efficient strategies, although some young children did generate general strategies in solving congruence problems that were sometimes successful.

At the same time, children did not simply become more systematic in strategy use with age, as some studies have suggested (e.g., Richards & Siegler, 1981). Rather they become better able to adapt their knowledge of congruence to particular task demands. These conclusions support the view that conceptual knowledge and procedural knowledge are interdependent rather than independent, as is often assumed.

Success in congruence performance by age, sex and task

An assessment of the effect of age and sex upon correct congruence judgements was undertaken with a multivariate analysis of variance (MANOVA). The MANOVA was performed on the correct congruence judgment data for each task, expressed in percentage of correct responses. The multivariate F test was computed on the ROY largest root criterion (Morrison, 1967) (see Table 10).

Table 10

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The main effects were Age (8), Sex (2) and Task (7). Both Age and Task were significant ($p < .05$), Sex was not. Sheffé post hoc individual comparison tests showed that for Age, 4-year-olds were significantly different from the other age groups, which did not differ from each other. Task comparisons showed three levels of task performance. The most difficult level included the Imagery III and Correspondence/Points on a Circle tasks. The second level included the Imagery I, Imagery II, Measurement, and Combination of Motions tasks, which did not differ from each other. The easiest task was the Correspondence Task: Lines/Angles). These levels indicate three different patterns of performance over age that is reflected in the significant Age x Tasks interaction ($p < .05$). A significant Sex by Task interaction ($p < .05$) was due, as ANOVAS on individual tasks indicate, to a significant sex difference over age in the Imagery III Task,

and not in any of the others. Boys show superior performance to girls at each age level, but one. This task, however, is the only one that showed no significant age effect in an ANOVA (Age x Sex).

Discussion

Significant age differences in performance in respect to successful congruence judgments and actions were evident in all but one task, and in the one exception there was an age progression as expected, but it never reached the levels of success indicated in the other tasks. The only significant sex differences occurred in the Imagery III task, as well. This is the one (labeled) imagery task that depends on imagery processes per se, and is complicated by some transitive inference demands as well. The latter may account for the limited performance in this task over most of the age range, although male performance at age 11 (74%) was strong, if not high.

The fact that significant age changes occur between 4 and 5 years shows considerable congruence knowledge at the 5 year-old level, but the fact that the knowledge is concentrated in one transformation, the slide, as the transformation data show, qualifies that generalization.

Overall, one would say that sex differences in those data are minimal, and are confined to one task that involves transformational imagery. The present analysis details three levels of performance among tasks and indicates a discontinuity or transition points between the 4th and 5th years, and the 7th and 8th years. The first level is defined by simple correspondence abilities. Middle level performance relates generally to measurement abilities, two sub levels represented by single edge matching on the one hand, and superposing on the other. The third level, is defined by the ability to deal competently with the more complex transformations, the flip and rotations and reflects the ability to

Table 11

Relation of Strategy Level to Correct Congruence Judgments by Task

(Frequencies and percentage in parentheses for subjects who pass/fail)[a]

Task	Strategy Level 1	Strategy Level 2	Strategy Level 3	No Predominant Level[b]	Contingency Coefficient	p
Imagery						
Phase I	6(8)/ 3(4)	29(36)/ 2(3)	37(46)/ 2(3)	1(1)/ 0(0)	.25	< .05
Phase II	23(29)/ 7(9)	18(23)/ 2(2)	29(36)/ 1(1)	0/0	.28	< .05
Measurement	5(6)/ 5(6)	36(45)/ 4(5)	6(8)/ 1(1)	18(23)/ 5(6)	.12	< .02
Correspondence: lines/angles	1(1) 1(1)	61(76)/ 7(9)	2(2)/ 0	7(9) 1(1)	.22	< .05
points	1(1)/ 1(1)	30(38)/ 42(53)	3(4)/ 0	2(3)/ 1(1)	.22	< .05

[a]Imagery Phase 3 omitted from analysis in that strategy and success are by definition correlated.

[b]This category omitted in computing contingency coefficients.

utilize transformational imagery effectively. Transition to this level occurred at about 8 years of age.

Relation of strategy level to accurate performance within tasks

The purpose of this analysis was to determine whether strategy level III, based on the efficiency of algorithmic strategies possessed by subjects, related to their success in a task.

For this purpose, success in a task was defined by a criterion of 2 or 3 transformation trials correct. Strategy level was defined by the criterion that 2/3 (67%) of the child's strategic responses were classified in the same level. Subjects who could not be coded for a predominant strategy were omitted from this analysis.

Contingency coefficients were computed for each task (except for the Imagery Task III, where strategy and success are by definition correlated, and the Combination of Motions task, which did not lend itself to this analysis).

Contingency coefficients for 4 of the 5 analyses ranged from .22 to .28, the fifth was .12 (measurement). Only the Imagery phase II and measurement coefficients were statistically significant.

The pattern of results indicates that for only one task, Imagery II was there a substantial possibility for success (29% of the subjects) with strategies of the lowest level (I). In all other instances subjects had to have strategies of the second (II) level, with predominant pattern of success evident with level II for 3 of the 5 tasks, and level II as a significant base for the other two. Level I strategy success was highly related in Imagery I and II to success, as was the case with the superposing strategy in those tasks, which ensured almost complete

Table 11

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success in its application. As was shown before, however, other strategies including edge matching and even visual inspection alone could lead to success. Since it is not known what is entailed in visual inspection alone -- that is, what mental processes are involved, it could be that more sophisticated mental processing is in fact involved (Table 11). Correspondence tasks and the measurement task can be succeeded in very clearly with level II strategies (7.6% - 38%), although their deployment, as in the points task, is also associated with significant amounts of failure (53%).

Discussion

There is then a fair relationship between strategy level, or the algorithmic power of a strategy and success in congruence tasks, but the relationship is mediated by the nature of the task and its cognitive demands. In this vein correspondence tasks (lines/angles) require less powerful strategies for success than the imagery tasks, and are thus solved by younger children. The imagery I and II tasks entail greater cognitive demands, particularly with some transformations (rotations and flips) and require more sophisticated strategies for fully successful processing.

Effect of transformation type on performance

Congruence judgments were assessed and analyzed in each task as a function of one of three transformations, slide, flip or turn. Strategies already indicated, were affected by the nature of the transformation, as was the extent of correct performance. To assess its effect more systematically, the effect of transformation type and age was studied in all of the tasks, and its relation to sex was studied in one of the tasks. A further analysis of the effect of the non-mobile condition was also undertaken. For these analyses a series of MANOVAS

Table 12

Summary of Effect of Transformation Type, Age, Sex and Mobility Condition on Performance in Various Tasks

(MANOVAS based on number of correct responses)

Task		Analysis	p	Significant differences
Imagery	Phase I	Age x transformation type		
		Age	$< .05$	4 years $<$ 5-11 years
		Transformation	N.S.	
		Age x Transformation	N.S.	
	Phase II	Age x transformation type		
		Age	$< .05$	4 years $<$ 5-11 years
		Transformation I x II	N.S.	
		Age x Transformation type	N.S.	
	Phase III	Age x Sex x transformation type		
		Age	$< .05$	4 to 6 years $<$ 7 to 10 years $<$ 11 years
		Sex	$< .05$	female $<$ male
		Transformation	$< .05$	S $>$ R
		Age x Sex	N.S.	order of difficulty: S $<$ F $<$ R
		Age x Transformation	$< .05$	S $>$ R+F at 4,6,8; S=R=F at 5,7; S+F $>$ R at 9,10
		Sex x Transformation	N.S.	
Measurement		Age x Mobility Condition x Transformation		
		Age	$< .05$	4 years $<$ 5 to 6 = 7 to 11 years
		Condition	$< .05$	Non-mobile $<$ mobile
		Transformation	N.S.	
		Age x Condition	N.S.	
		Age x Transformation	N.S.	
		Condition x Transformation	N.S.	
Correspondence	Lines/Angles	Age x Transformation		
		Age	$< .05$	(overall effect significant; no individual comparison significant)
		Transformations	N.S.	
	Points	Age x Transformation		
		Age	$< .05$	(overall effect significant, not individual comparisons)
		Transformation	$< .05$	S $>$ R

Table 13

Imagery Task I: Correct Congruence Responses as a Function of Transformation Type and Age (mean % correct)

	Transformation[a]		
Age	Slide	Rotation	Flip
4	60	60	60
5	100	90	90
6	80	90	80
7	100	100	100
8	90	90	90
9	100	100	100
10	100	80	80
11	90	90	90
Total	90	88	86

[a] Each transformation is based on one response per subject, ten responses per age group.

Table 14

Imagery II Task: Correct Congruence Responses as a Function of Transformation Type and Age (mean % correct)

Age	Transformation[a] Slide	Rotation	Flip
4	20	60	20
5	80	70	70
6	80	70	70
7	90	100	90
8	90	90	90
9	100	100	100
10	80	100	90
11	100	80	70
Total	80	84	75

[a]Each transformation is based on one response per subject, ten responses per age group..

Table 15

Imagery Task III: Correct Congruence Performance as a Function of Transformation Type, Sex and Age (mean % correct)

	Transformation[a]/Sex[b]								
	Slide			Rotation			Flip		
Age	M	F	M/F	M	F	M/F	M	F	M/F
4	40	40	40	20	20	20	0	0	0
5	60	20	40	40	20	30	40	20	30
6	60	60	60	40	0	20	20	0	10
7	40	40	40	80	20	50	60	40	50
8	80	60	70	60	20	40	40	0	20
9	60	80	70	20	20	20	60	60	60
10	60	60	60	40	20	30	80	40	60
11	100	60	80	20	20	20	100	80	90
Total	63	53	58	40	18	29	50	30	40

[a]Each transformation (collapsed over sex) is based on one response per subject, ten responses per age group.

[b]Each sex condition within a transformation is based on one response per subject, five responses sex condition.

Table 16

Measurement Task: Correct Congruence Performance as a Function of Transformation Type, Mobile/Non-Mobile Condition and Age

(mean % correct)

Age	Transformation[a]/Condition[b]								
	Slide			Rotation			Flip		
	Mobile	Non-Mobile	Mobile/ N-Mobile[b]	Mobile	Non-Mobile	Mobile/ N-Mobile	Mobile	Non-Mobile	Mobile/ N-Mobile
4	87	33	60	73	33	53	73	13	43
5	67	53	60	93	67	80	87	53	70
6	73	60	67	87	60	73	87	73	80
7	73	73	73	80	87	83	93	93	93
8	100	87	93	100	93	97	100	100	100
9	100	93	97	87	100	93	100	100	100
10	93	100	97	100	100	100	93	100	97
11	100	93	97	100	93	97	100	100	100
Total	87	74	81	90	79	85	92	79	85

[a]Each transformation (collapsed over conditions) is based on three responses per subject, 30 responses per age group.

[b]Each condition within a transformation is based on three responses per subject, 15 responses per condition.

(multivariate analyses of variance) were performed, based on the number of responses correct in a task.

Imagery tasks. In the imagery tasks, phase I and II, in an Age x Transformation
Type Multivariate analysis of variance, only Age was significant ($p < .05$). The
Tables 12, 13, 14 principal individual comparison difference was between 4-year-old performance
and the other age groups (see Tables 12, 13 and 14).

86, 87 88 Phase III data were subjected to an Age x Sex x Transformation type
analysis, inasmuch as the data appeared to suggest sex differences. All three
Table 15 main effects were significant ($p < .05$). Age levels divided significantly into
three groups: the 4- to 6- year-olds, 7- to 10- year olds, and 11- year-olds.
89 Sex differences were significant with males exceeding females (the only task
in which this was so). The significant transformation type effect showed the
order of difficulty was slide $<$ flip $<$ rotation (from least to most difficult).
The Age by Transformation interaction was significant ($p < .05$), the other
interactions not. The significant interaction was accounted for mainly by
the fact that at age seven the flip was easier than the slide, whereas at all
other ages the reverse was true (and at 10 years they were equal) (Tables 12
and 15).

Measurement task. For the measurement task analysis, Age, Mobility condition
(mobile vs. non-mobile stimuli) and Transformation type were the main effects.
Age and Mobility condition were significant, Transformation not, nor were any
interactions significant. In individual age comparisons the 4- year-olds
Table 16 were significantly different from the other age levels and the non-mobile
condition was significantly more difficult than the mobile (see Tables 12
90 and 16).

Table 17

Correspondence-Lines/Angles Task: Correct Congruence Performance as a Function of Transformation Type and Age (mean % correct)

	Transformation[a]		
Age	Slide	Rotation	Flip
4	90	87	83
5	100	100	100
6	100	93	100
7	97	100	100
8	100	100	100
9	100	100	100
10	100	100	100
11	100	100	100
Total	98	98	98

[a]Each transformation is based on three responses per subject, 30 responses per age group.

Table 18

Correspondences-Points Task: Correct Congruence Performance as a Function of Transformation Type and Age (mean % correct)

	Transformation[a]		
Age	Slide	Rotation	Flip
4	53	13	30
5	80	0	33
6	90	20	37
7	100	7	17
8	90	33	37
9	100	60	33
10	100	50	50
11	100	60	73
Total	89	30	39

[a] Each transformation is based on three responses per subject, 30 responses per age group.

Tables 17, 18
92, 93

Correspondence tasks. In the Correspondence task (lines/angles), the Age effect was significant ($p < .05$) although no individual age group comparison was significant. Transformation type was not significant, and there were no significant interactions (see Tables 12 and 17). A significant Age effect appeared in the Correspondence (points) task ($p < .05$), with no significant individual comparison. Transformation type was significant as well ($p < .05$) with the slide easier than the flip and turn, with the latter transformations not different from one another (Tables 12 and 18).

Effect of race in performance

The sample of subjects contained a substantial number of non-white (black, hispanic and oriental) as well as white subjects. To determine whether there were racial differences (white vs. non-white) in correct performance on the tasks studied, an analysis of covariance was performed with percent correct responses summed over tasks as the dependent measure with Age as the covariate and Race as a main effect. None of the effects assessed (equality of slopes, means, or linearity of performance) showed any significant difference as result of race.

Discussion

It is evident that transformation type affected accurate performance inconsistently. In some tasks it made no difference at all; in other tasks it did affect results. The task in which the effect was most salient was in the Imagery (phase 3) and the Correspondence (points) tasks. In these, the rotations and flips were most difficult to image, with the rotation providing particular difficulty in the imagery task; but this was so mostly for girls. In the Points on a circle task both transformations were difficult,

with only fair successful performance at 11 years of age (60% rotation; 73% flip). Thus, the effect of transformations was not uniform. None was consistently more difficult than another, except that on the whole the slide was easier than the other two transformations. Flips and turns were of near equal difficulty, except that the particular type of task affected which was more difficult. In sum, there is a complex interaction between transformation type, the type of psychological process involved in a particular task, and the age of the child.

Supplemental Studies: I. Varied Geometric Figures

The data of the main study led to two conclusions. First, a well-defined set of strategies for determining the congruence of two geometric figures (right triangles) exists in children as young as four years of age. Even if the application of these strategies is not as successful as later-appearing strategies, the evidence suggests that young children are performing according to rules for congruence testing. Second, young children responded to geometric figures not as unanalyzed entities but to components such as edges with at least limited knowledge of such properties as length. What was not evident from this study was whether the manifest stategies were a function of the figures employed as stimuli (right triangles), or whether other figures would provoke the generation of different strategies. It was quite possible, for example, that quadrilaterals would occasion the use of other than single edge matching, in that the SSS (side, side, side) rule for congruence offers a necessarily correct solution only with triangles. Consequently, the use of

an edge matching strategy with quadrilaterals would not be adaptive, if used by either younger or older children. We assumed instead that quadrilaterals would prompt the use of multiple or progressive edge matching, in which all four sides of the figure would be matched developmentally prior to the appearance of the superposing strategy. To test these assumptions, the Imagery task, Phase I of the main study was administered by Barbara Whitehurst to a new group of subjects, with a new set of materials.

Subjects

Forty subjects, 10 each from 5-, 7-, 8- and 10 year-old age groups were selected at random from a large urban parochial school. (Mean ages: 5- year-olds; 5,1; 7- year-olds: 7,0: 8- year-olds: 8,0; 10- year-olds; 9,11.) These age levels were chosen because they approximated ages at which significant strategy changes were likely to become evident, considering the findings from the main study. The school has a racially mixed population; our sample contained, overall, 45% whites and 55% non-whites (principally black and hispanic). In the 5- year-old group the white/non-white proportion was 70/30; 7- year-olds: 50/50; 8- and 10- year-olds: 30/70.

Materials

Four different geometric figures, constructed from the same type of plastic, were used as models: right triangle, equilateral triangle, square and quadrilateral. Two of the figures are three-sided, and two four-sided. They divide further into figures with symmetrical sides (the equilateral triangle and the square) and non-symmetrical sides (the right triangle and the quadrilateral).

For each of the model figures, there was a selection tray set of figures that differed within each set in size as well as shape. Three of the figures

Table 19

Strategies by Age (and Correct Responses) for 3-Sided and 4-Sided Geometric Figures -- Summed Over Four Geometric Shapes
(Frequencies)[a]

Age	Visual Inspection	Edge Matching: Single	Edge Matching: Multiple	Superposing	Multiple Strategies
5	6(2)	21(12)	2(1)	7(2)	4(1)
7	14(4)	17(16)	1(1)	3(3)	5(4)
8		23(12)	1(1)	11(11)	5(4)
10		2(1)	2(2)	27(27)	9(8)
Total	20(6)	63(41)	6(5)	48(43)	23(17)
		69(46)			
% Total (% Correct)	13(30)	43(67)		30(93)	14(74)

[a] Number correct in parentheses

differed in size, one differed in shape. There was a different color for each set. (The dimensions of the materials appear in Appendix 4)

Procedure

The procedure followed was comparable to that for Phase I of the imagery task in the main study, except that a subject was presented with each of the different geometric figures for testing congruence and there was only one trial per geometric figure. That is, each subject was required to pick from the selection tray a figure "cut from the same cookie cutter" as the model, and there were four such choices, one for each geometric figure.

Results

The data were analyzed in respect to both strategies and the accuracy of performance.

The findings in respect to strategies were as follows:

Table 19

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1. The pattern of strategies was the same, over the four geometric figures, as it was for the right triangle alone. There was visual inspection (VI), edge matching (EM) (including single edge matching, multiple edge matching and combinations of the two), superposing (SUP) and combinations of superposing and edge matching (MULT) (Table 19).

2. Strikingly, the predominant strategy for the 5-, 7- and 8- year-old subjects was single edge matching (68%, 48% and 65% of their responses, respectively). At 10 years the proportion diminished dramatically (to 10%).

Multiple edge matching, contrary to the thesis that prompted the supplementary study, at no age entered as a significant factor in establishing congruence for any of the shapes, including the two 4-sided figures.

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Table 20

Strategies (and Correct Responses) as a Function of Geometric Figure and Age

(In percentages)[a]

Age	Right Triangle VI	Right Triangle EM[b]	Right Triangle SUP	Right Triangle Mult	Right Triangle (overall correct)	Equilateral Triangle VI	Equilateral Triangle EM[b]	Equilateral Triangle SUP	Equilateral Triangle Mult	Equilateral Triangle (overall correct)	Square VI	Square EM[b]	Square SUP	Square Mult	Square (overall correct)	Quadrilateral VI	Quadrilateral EM[b]	Quadrilateral SUP	Quadrilateral Mul	Quadrilateral (overall correct)
5	10 (100)	70 (57)	10 (0)		(40)	10 (0)	70 (57)	20 (50)		(50)		80 (38)	20 (50)		(40)	40 (25)	50 (60)	10 (0)		(40)
7	30 (0)	50 (100)	10 (100)	10 (100)	(70)	40 (50)	50 (100)		10 (100)	(80)	30 (33)	50 (100)	10 (100)	10 (100)	(80)	40 (25)	40 (75)	10 (100)	10 (0)	(50)
8		70 (57)	30 (100)		(70)		70 (86)	30 (100)		(90)		60 (33)	30 (100)	10 (0)	(50)		60 (50)	20 (100)	20 (100)	(70)
10		10 (100)	60 (100)	30 (100)	(100)		10 (100)	70 (100)	20 (100)	(100)		10 (0)	70 (100)	20 (50)	(80)		10 (100)	70 (100)	20 (100)	(100)
Total % Correct	10 (25)	50 (70)	30 (83)	10 (100)	(70)	13 (40)	50 (80)	30 (92)	7 (100)	(80)	8 (33)	50 (50)	32 (92)	10 (50)	(63)	20 (25)	40 (63)	28 (91)	12 (80)	(65)

[a]Based on four responses per subject -- ten subjects per age group = 40 responses per geometric shape. Data for sex groups have been collapsed. Percentages of correct responses appear in parentheses.

[b]Includes (single-edge matching/multiple-edge matching) multiple strategy subjects

3. Superposing increased with age until it became the predominant strategy for the 10-year-olds (91%). (97% of the responses were correct.)

4. Visual inspection alone was evident only in the 5- and 7-year-olds (15% and 35% responses, respectively).

5. The strategy pattern for each type of figure was essentially the same at each age level as the overall (across age) patterns, suggesting that subjects' strategies were either highly consistent irrespective of the type of geometric figure or that differences in geometric figures had little effect (Table 20).

The accuracy of performance findings:

A multivariate analysis of variance (MANOVA) was performed on the accuracy data represented in correct congruence judgments for the four geometric shapes by age and sex distributions [Age (4) x Geometric figures (4) x Sex (2)] (See Table 20).

Table 20
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1. Only the Age main effect was significant ($p < .05$). Post hoc comparisons among the age groups showed only the difference between the 5- and 10- year-old groups to be significantly different (mean % correct from 4 to 10 years was 43; 70; 70; 95). There was a significant Age x Sex interaction ($p < .05$) due primarily to the drop in female performance at age 8, a difference possibly due to sampling. No other interaction was significant.

Thus, overall, there was a significant increase in accurate congruence judgments with age with a large gain evident between the 5th and 7th year and again between the 8th and 10th year.

2. Sex differences did not appear to play a sizeable role in successful performance.

3. There was a difference between accuracy of congruence judgments in

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these data and those of the main study, with less successful right triangle performance evident for visual inspection and edge matching strategies in this sample (VI: 61% vs. 25%; EM 94% vs. 70%). This result may be due to the nature of the selection triangles in the supplementary study, which were less discriminable than in the main study -- except for the non-similar triangle. In fact, 82% of error responses resulted from choice of the non-similar triangle. A majority of the subjects who chose the dissimilar triangle used a single edge matching strategy; another 22% used visual inspection. These data suggest that subjects were focusing on matching the lengths of one side of the triangle and incorrectly estimating or ignoring the other sides. It implies that edge matching subjects were using a rule to the effect that if the length of one side of the figure (usually the most salient) appeared equal to the model's, then the entire figure was to be identified as congruent.

4. The same error choice pattern that appeared for right triangles was evident in each of the other geometric shapes. When subjects were in error, it was principally (86%, equilateral triangle; 73%, square; 71%, quadrilateral) in choosing the non-similar figure, and again it was based on the use of a single edge matching strategy or visual inspection.

Thus, in general the supplementary study shows that the patterns of development for strategies and accurate congruence judgments were not a function of the use of right triangles. It also shows the striking absence of what would be an effective and adaptive strategy, multiple edge matching, for four-sided geometric figures.

Supplemental Study: II. Early appearance of congruence strategies.

As the main study indicates, the seemingly least sophisticated responses, denoted as Level I, and evident in haphazard guessing and visual inspection without verification, were those in which children did not appear to be using a rule that

when properly executed could lead to accurate congruence judgments. These non-algorithmic strategies were most evident in the youngest age group tested, 4-year olds. At the same time, a substantive number of 4-year-olds used Level II edge matching strategies in Imagery Task Phase I and the measurement task. It was not known from the main study as to the earliest age at which congruence strategies would appear in the child's behavior. It was important to determine this in order to provide a more critical test of the thesis that the developmental progression of edge matching to superposition, was initiated by the child's early knowledge of the congruent parts of fugures, in particular, their edges. A supplemental study was undertaken then by Steven Guberman to determine the nature of strategic behavior in children as young as 2 years 6 months. Two tasks from the main study were used for this purpose, (Imagery Phase I and the Measurement task) since these appeared to be within the range of competencies of very young children.

Subjects

There were ten children, equally divided by sex, ranging in age from 2 years 6 months to 3 years 5 months, with a mean age of 3 years 2 months. All were from middle-class families. Nine were tested in a preschool and one at the subject's home.

Materials

The materials were the same as those used in the main study for the Imagery Phase I, and Measurement tasks.

Procedure

The procedures of the main study were modified in minor respects to accomodate to the younger age of the subjects, principally in the instructions and in reductions in the number of trials.

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Results

Both subjects' ability to carry out the tasks correctly and the strategies employed in approaching the tasks were assessed, as in the main study.

Contrary to our expectations, all 10 subjects used single edge matching, but only half of them arrived at a correct congruence judgment. Nevertheless, six (60%) of the subjects attempted to edge match with the corresponding sides of the right triangles (two subjects ended by making no judgment, however), and for the six subjects, 34 of the 35 matches that were made were of the hypotenuse of one triangle with the hypotenuse of the other. Of the subjects matching corresponding sides, 67% arrived at a correct congruence judgment, the remainder made no choice, claiming that all the triangles in the selection tray were the same. Four of the ten subjects matched both corresponding and non-corresponding sides, but only one made a correct congruence choice; the others asserted that all the selection triangles were the same.

The measurement task provided similar strategy behavior. In this task each subject had three opportunities to measure the two (mobile) model triangles. In these 30 responses, 9 (30%) were edge matching strategies. Of the remaining responses, 3 (10%) also employed edge matching, but of the tray triangle, and 10 (33%) used measurement strategies that involved no direct comparison, another 8 (27%) gave no response. Thus 30% of the responses could be considered Level II strategies -- (those involving edge matching between the model triangles), the others were Level I strategies. Considering the first measurement request alone, 90% of the subjects used single edge matching; of these subjects, 60% matched corresponding sides leading to 83% correct judgments. Only one subject who matched corresponding

sides arrived at an incorrect judgment. All three subjects who matched non-corresponding sides made incorrect congruence judgments. While various solution aids were used by subjects (33% of all responses), no subject made a direct comparison of both test triangles with any of the available aids.

There was a high degree of consistency across tasks. Of those subjects who used single edge matching of corresponding sides exclusively in the imagery task (40%), all used the same procedure in the measurement task. The four who aligned non-corresponding sides did the same in the measurement task. Furthermore, all subjects who made correct congruence judgments in the imagery task did so in the measurement task. The five subjects who made no judgments on the imagery did the same in the measurements task.

Discussion

While the subjects of this age group (2,6 to 3,5) did have a strategy for the determination of congruence between scalene right triangles, a striking finding in itself, the strategy was used effectively by only half the subjects. The unsuccessful subjects appeared to have a procedural rule without sufficient understanding of how to use it. The very high incidence of single edge matching in these young subjects supports our earlier conclusion that geometric knowledge in young children is based on the component parts of figures. Thus, 77% of all matches (across both tasks and all trials) were comparisons of corresponding sides, indicating that children, even this young, attend to a figure's component parts, and a full 59% of the subjects could use this strategy consistently to arrive at correct congruence judgments, for both congruent and non-congruent figures.

Of particular interest are those subjects who used single edge matching in the imagery task and asserted that all the selection triangles were the same.

This occurred despite the care that was taken to ensure that subjects understood that only one selection triangle could be the same, and that the subject understood "same" and "different" in the sense intended. Of the 5 subjects who did so, three (60%) matched both corresponding and non-corresponding sides. In matching, it was noticed that several subjects first aligned the top vertices of non-congruent triangles, then seeing that the bottom vertices were not aligned, moved the triangle to align the bottom vertex. Several went back and forth aligning first the top then bottom then top again and so on. This finding is in accord with other observations that young children do not conserve length when one of the ends is displaced, because "they do not take account of both ends simultaneously" (Piaget, Inhelder & Szeminska, 1960).

Paradoxically, for this group the measurement data did not accord with the imagery data. With 4 of the 5 subjects, three (incorrectly) judged congruence on the basis of comparisons of non-corresponding sides of congruent triangles, and one (correctly) judged non-congruence using edge matching of corresponding sides. A fifth was consistent in ignoring length on both tasks. Subjects apparently responded differently in the measurement task because of the differences in task demands. The imagery task required successive judgments in scanning the model triangle and the four selection triangles; the measurement task, on the other hand, required only one judgment of the two model triangles. Using only a partial algorithm in the latter task, (aligning sides and noting length but not of corresponding parts) the subjects could arrive at a judgment, even if not a correct one.

These considerations indicate that a consistent strategy for the determination of congruence is first emerging in the age group studied. While 50% of the subjects had mastered a strategy another 50% were still putting together strategy

components into an integrated whole.

General Discussion

Structural - Procedural knowledge relations. The data of these studies address a number of issues germane to what constitutes an adequate account of cognitive development as well as to the nature of mathematical cognition.

Theories that claim to describe and explain cognitive development have in general taken two forms in recent years. The most influential until only a short while ago were structuralist in nature. In these theories, universal dimensions of cognitive structure are described, with structures typically characterized as undergoing change in development and said to explain developmental differences in reasoning, problem solving and the like, manifest in settings as divergent as naturalistic environments and laboratory experiments. Observable behaviors, that is, the surface level organization of behavior is linked to covert cognitive organization of highly abstract form which is believed to generate the surface behaviors. The most important developmental theory of this kind is Piaget's, and the most influential of the non-developmental theories is Chomsky's account of linguistic structure.

Considerable dissatisfaction has developed with this class of structuralist theories to the point where alternative explanations of cognitive functioning and development have been sought. The model that has captured major attention is information processing, but this model takes many forms and it is more to the point to characterize the alternative explanations to Piagetian and other structuralist theories as functionalist in nature (Beilin, 1981, in press). Functionalist accounts tend to characterize cognitive performance in terms of procedural knowledge, rules, strategies, schemata, and scripts that are said

to define the fundamental relations in observed behaviors, i.e., the same behavioral domain accounted for by structuralist theories. A number of developmentalists have made important contributions to functionalist studies, including Gelman, Klahr, Siegler, Nelson and Paris. The overemphasis on cognitive universals in development has been recognized by Genevan investigators for some time, and a shift in theoretical and research priorities toward functional analysis has been evident in the products of Genevan research (Inhelder, Sinclair & Bovet, 1974; Karmiloff-Smith & Inhelder, 1975; Mounod, 1981). Attempts at establishing some type of integration of structuralist and information processing models has been evident in the work of Pascual-Leone (1976) and Case (1978), although these investigators have tended more to reduce Piagetian theory to an information processing model than to provide a balance between the two. The justification for maintaining equal balance between Piagetian theory and information processing models is that most functionalist accounts of cognitive development do not provide adequate theoretical explanations of how developmental change occurs. To date the Piagetian view still dominates the field in this respect, despite its many difficulties and despite valiant efforts by some information processing theorists to provide alternative models (e.g., Klahr & Wallace, 1976). It appears to us then that an adequate account of cognitive development and learning requires two types of analysis, structural analysis and functional analysis. The latter details the nature of procedural knowledge necessary for a response to task demands, the former to detail the consistency in performance across tasks and cognitive contexts. But additionally, and equally important is the need to define the relationship between procedural and structural or logico-mathematical knowledge. This was attempted earlier in respect to the strategies involved in bivalued discrimination problem solving tasks (Gholson & Beilin, 1978).

In an extensive series of studies, Gholson (1980) has shown that strategies and stereotyped responses employed in experimental problem solving contexts map onto structural levels defined by the classic Piagetian model. We attempted in the present study to show in parallel fashion that structural and procedural knowledge are related in tasks of a richer and more varied nature.

Procedural and structural knowledge can be related as follows: (1) procedural knowledge can be accounted for completely by structural knowledge, (2) structural knowledge is accounted for or reduced to procedural knowledge, (3) structural knowledge is independent of procedural knowledge, and (4) structural knowledge and procedural knowledge are interdependent components of problem solving behavior. The evidence from the present study is consistent with the last possibility. Supporting data come first from the consistency in children's strategy deployment evident within sets of tasks. In this respect we found that children's geometric strategies were classifiable into three levels of algorithmic efficiency, or degrees of rule use. With a criterion of consistent strategy use across tasks (67% of the child's strategies at the same algorithmic level) it was evident that children were significantly more consistent than inconsistent in performance over all ages. With the exception of one task (measurement), there was also considerable consistency (80% to 100%) within tasks. The fact that children exhibited strategy consistency within certain sets of tasks but not others suggests that where inconsistent strategy patterns were evident, they did not reflect random problem solving behavior but instead adjustment of strategies to task demands. Performance consistency was also age-related. Children at 6, 8, and 9 years of age, the ages at which significant strategy consistency was demonstrated, employed Level II strategies across tasks. Children in the incon-

sistent strategy age groups, however, showed two different patterns. Those below 6 years used Level II strategies and a combination of Level II and Level I. Those above 6 years used Level II, and a combination of Level II and Level III strategies. Thus, younger and older children showed what could be called task tuning. The younger children if they could apply the more sophisticated strategies (Level II) did so, but if faced with more demanding tasks reverted to simpler strategies. Older children performed consistently and effectively in accord with task demands. The developmental evidence suggests that inconsistent strategy deployment in younger children reflects underlying structural knowledge, whereas in older children it can be attributed to improvements in procedural knowledge. Thus, we propose two principles to define the interdependence of procedural and structural knowledge: (1) task tuning, which holds that children utilize the most advanced strategy available in their procedural repertoire to maximize the match of structural knowledge to problem demands, and (2) cognitive economy, which holds that children adapt their procedural repertoire to specific task demands, and use a less advanced strategy to minimize effort in solving a problem. These processes are complementary, and to an extent, apply concurrently.

Thus, in addition to the data of consistent strategy deployment within sets of tasks, the evidence that strategy development reflects emerging knowledge of geometric congruence, as well as the adaptation of that knowledge to particular tasks, argues for a view of cognitive development that engages procedural and structural knowledge in an interactive relation.

Knowledge of geometric congruence. As already indicated, there are two aspects to knowledge of geometric congruence and transformations, structural and procedural. When applied to actual performance there is no clear cut

distinction possible between them, as in the classification of strategies. Strategies are ordinarily treated as procedural, but when a strategy shift occurs, as in the present studies, from single edge matching to superposing at a particular age, it is fair to assume that the shift occurs either because of changes in structural knowledge, that is, logico-mathematical knowledge, or the way the strategies themselves are composed, at least in part, of that knowledge. Either way, as "dependent upon" or "composed in part of," there is a significant interdependent relation between logico-mathematical knowledge and procedural knowledge. Considered in terms of a mathematical, i.e., geometric, model, the aforementioned strategies represent different mathematical possibilities. For example, if one were to adopt the Kleinian geometric model as a model for psychological processing, as Piaget at one time has done, there is a certain explanatory advantage, but it also engenders some difficulties, inasmuch as the geometric model itself is far from perfect (Fuson, 1978), and is undergoing change with developments in mathematics itself. As applied by Piaget, these difficulties are due to incorrect and ambiguous uses of mathematical terms and concepts, and the model maps on poorly to empirical findings. The Kleinian model, however, is useful in the present context in the following way.

According to a Euclidean conception of congruence, the edge matching strategy is one possible strategy for determining the congruence of two triangles, because it maps onto the SSS (side, side, side) proposition, which states broadly that if the sides of two triangles are congruent, then the triangles themselves are congruent. It is surprising, as evident from our data, that children as young as 2 years 6 months have sufficient knowledge of congruence to match edges for establishing or verifying the congruence of two triangles that differ by a

transformation. That children this young possess even the rudiments of such rules contradicts the assumptions of many mathematics educators, and educational and developmental psychologists, as to the mathematical, particularly geometric, competencies of young children. Not only do the data show that young children have a procedure for establishing congruence, but additionally, some mathematical rule knowledge to which the edge matching strategy relates. If one adopts a mathematical model for cognitive development, however, it also raises a number of questions about psychological competencies and their development. For an ideal application of the Euclidean SSS proposition, a multiple edge matching procedure is necessary. That is, all three sides of the triangle have to be matched. Why even older children very infrequently use this procedure is a puzzle. Why multiple edge matching is hardly ever used even with quadrilaterals is an even greater puzzle in that the SSS rule does not apply to quadrilaterals (i.e., if the sides of two quadrilaterals are equal and their angles unequal the quadrilaterals will not be congruent).

Another puzzle is why checking angles does not appear among the strategies, since a SAS (side, angle, side) rule would also establish congruence, at least in triangles. One possible answer is that children, in fact, do take information about angles into account, principally during visual inspection, but there is no way of establishing this, short of eye movement studies. Some evidence that angles are taken into account comes from one of our supplemental studies that shows some matching at the vertices of the triangles. It is not clear, however, whether angles are being checked or whether vertices are being used as terminal points of triangle edges. The latter appears more likely in the light of other of the child's actions.

The SSS rule is sophisticated and at the same time limited in respect to congruence. It is sophisticated in that is provides a geometric proposition and procedure for the congruence of triangles, but as already shown, it is limited in its application to certain classes of geometric figures. The <u>superposing</u> strategy, on the other hand, maps onto a proposition that does apply to all figures, namely, if two figures can be made to coincide the figures are congruent. This proposition, which is consistent with Euclidean geometry (Euclid, 1956, p. 227), also accords with the Kleinian model. Again, as a psychological model it provokes a number of questions. Superposing, which provides a procedure for establishing that two triangles are isometric in all its critical properties, i.e., length of line segments, angles, and areas, etc., when it appears in the child's repertoire does so relatively quickly. As expected, considering the sophistication and generality of the procedure and its underlying mathematical logic, it almost always leads to a correct congruence judgment, a condition that does not prevail for edge matching, even when applied to triangles. What accounts for the sudden appearance of this strategy? Is it linked to new logico-mathematical knowledge, or does the strategy arise from the construction of skills already available with the child's repertoire? A clear cut answer is not available. Before offering a hypothetical account of the progression from single edge matching to superposing, a set of findings should be highlighted, which also bears on a controversy as to how geometrical ideas develop.

Two views have been offered as to how objects, including geometric figures, are perceived. In one interpretation, perception first involves engagement of the whole figure, followed by progressive differentiation of its component parts.

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An alternative view is that the total figure is constructed progressively from visual or haptic engagement with the figure's components. Gestalt and differentiation theories favor the first interpretation, constructivist theories the second. Data on perceptual processes in infants provide, at present, no way of deciding the issue between these alternatives (Hainline, 1982). Our data, however, are consistent with one of these interpretations. Single edge matching, in even the youngest subjects studied (2 years 7 months to 3 years 5 months) shows that a substantial portion of the single edge matches were made with the corresponding sides of figures, usually the hypotenuse. This offers evidence that young children are attending to the component parts of the figures in an informed way. At the same time, the fact that they jump to a judgment of congruence on the basis of the match of only one side, from what one can tell, indicates at least in the early years, that the entire figure has not been constructed from an integration of component parts. A multiple edge match would indicate just such a construction, but it is rarely observed. Superposing, however, by implication presupposes such an integration of component parts. Thus, we propose the following development to occur:

The earliest perceptual engagements of the infant with the world presupposes the differentiation of edges, enabling the infant to perceive lines and to differentiate figure from ground. These inborn capacities are indicated by the work of Hubel and Weisel (Barlow, 1982) and succeeding investigators, and from research, on the congenitally blind. These inborn visual capacities are nevertheless almost immediately affected by the child's visual experience.

Within the first few months of life the child is able to respond to and even

recognize total figures such as faces so that the organization of component parts into total entities is already possible. These developments are principally perceptual in nature, but by the age of 18 months children can recognize and name objects in pictorial form, even though they have not been exposed to pictures before (Hochberg & Brooks, 1962). In addition, by 2 years 6 months, the age of our youngest subjects, one could assume a wealth of prior experience in manipulating geometrically fashioned objects such as blocks, through stacking, sorting, fitting and so on. When asked to make and verify a congruence judgment, these subjects used only one edge of the figures for matching, usually the sides that corresponded in length to a side of the other figure, usually the hypotenuse. The children appear to be following a rule that says, "If the most critical (salient) feature matches, the entire figure is congruent with the other." Even though children at this age have had extensive experience with total figures and responded to them as such, and to edges as well (a Japanese study reports 2-year-olds as conscious, in drawing, of the contour of the face (edges) as well as eyes and mouth inside the contour they draw [Yokochi, K., 1980]), they nevertheless choose only one side to verify. It is tempting to say the reason this is so is that it followed visual inspection of the figures, in which the total figure was surveyed and its most salient feature selected for test, but we have only crude observational data to attest to this. The single edge matching rule then appears to override all other experience for the young child, even that which some children must have in fitting triangular figures into cutouts of various shapes that are common in "puzzle boxes" and many educational toys.

The first rule: Two triangles are equal if one side of each is equal, is followed then by a second state. The child in this state, in which visual

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inspection appears, combines measurement and estimation operations. The child first surveys the figure, picks the most likely candidate for checking and measures the length of one edge, visually estimating the lengths of the others.

In this instance the child understands that all three edges must be compared, but only measures one and estimates the others. The rule followed appears to be, "If one set of corresponding edges matches, visually inspect the other edges and estimate whether they do as well. If they appear equal, then the triangles are congruent."

Some children, who know three edges need to be compared to determine congruence, develop a strategy for determining this. Instead of measuring one edge and (visually) estimating the others, measure all three. This is the multiple edge matching strategy. The rule here is "To determine the congruence of figures, edge match the corresponding sides; if they match, the figures are congruent." Although one would expect more of this to appear as children get older, what occurs instead is that estimation alone increases, and as a consequence one observes increasing amounts of visual inspection alone appearing with no actual measurement occurring.

With increasing visual inspection, that is, visual estimation of lengths and other features of the triangles, information concerning constituent parts and edges, angles and surfaces (the same Japanese study [Yokochi, 1980] shows that 3-year-olds become conscious of surfaces) is integrated into a totality that manifests itself in the superposing strategy of laying one triangle onto the other. It presupposes other knowledge including the fact that lengths must not overextend on one side (i.e., a length must coincide at both vertices in the triangle). The latter knowledge, in turn, is related to conservation of length, which appears generally at about the age children in this study manifest the superposing strategy and use

it with consistent success. The rule followed now is: "To determine the congruence of figures, lay one figure on the other, spatially transforming the figure so that corresponding edges, angles and surfaces match; if they do, and no edges are overextended, the figures are congruent." In sum, the development of strategic behavior we propose entails an integration of both logical (structural) knowledge and procedural knowledge bearing on congruence and transformation.

Structure and process development. The tasks presented to subjects were selected on the assumption that they tapped processes closely related to the ability to make congruence judgments of geometric forms that differed by one or more transformations. These processes, transformational imagery, correspondence matching, measurement, and the integration of transformations, were considered to be equivalent in developmental level, except for correspondence, which according to recent Piagetian data is an earlier achieved process. Information processing theorists argue that most cognitive processes are in place early in development, and that age differences are due primarily to differences in skill attainment, skill complexity, encoding skills and the like. This is not borne out by the data of this study. Instead, we find for most of the tasks investigated, a developmental progression in the qualitative nature and power of the strategies generated, as well as change in the general level of rule structure that defines performance across tasks. Consistent with our expectations, accurate correspondence matching was evident in the youngest age group (4-year-olds) in the main study. This type and level of performance is in accord with Piagetian one-way mapping structures of preoperational thought. The remaining processes appear to divide into two levels of perfomance. The measurement, transformational imagery and transformation-dealing competencies, which are generally considered in the Piagetian model to relate to concrete operational

structures, appear developmentally at two levels, and are achieved one following the other. First, measurement knowledge and procedures, including transitive inference appear, then transformational imagery operations and processes that entail processing the more demanding transformations (flips and turns).

Although these developmental processes and operations appear in a developmental sequence, the data of the study show, as do a number of recent functionalist studies, and which Piaget himself noted although only in passing, the nature of the task affects performance in addition to the apparent age of acquisition of an operation. However, functionalists tend to neglect the internal structure of an operation, independent of the task context in which it appears, which also affects performance. This is evident in performance differences that appear with the various geometric transformations. Geometric transformations (slides, flip and turn) are more than task variations. They entail conceptual elements related to the conception of space (surfaces, distances, etc.), and mathematical (i.e., geometric) relations. As such they engage the subject's logico-mathematical and physical knowledge systems, both in respect to strategies and performance consistency.

In sum, we find that psychological processes brought to bear in geometric reasoning and problem solving tasks concerning congruence and motion (transformations), are achieved in a developmental order and appear to be a function of both the child's progressively acquired structural (logico-mathematical) knowledge and procedural (strategic and task sensitive) knowledge.

Implications for education. Educators have looked to psychology for many reasons. They seek models for curriculum development. They seek understanding

of the child's thought, action and feelings, then look for insights into the child as a learner, and search for sophisticated means of assessing children's capacities and achievement. With the significant shift in psychology's own interests to cognitive processes and knowledge structu es, cognitive developmental theories have been adopted by educators as models for curriculum development or simply as justification for particular types of curriculum models of their own. It is no secret that for many years educators, particularly mathematics and science educators, looked to Piaget's theory and research as the source of insight into the child's cognitive development and as a model for curriculum development. Despite Piaget's insistent plea that he had little to recommend in the way of educational practice, educators and others have nonetheless tried to apply the theory to mathematics and other fields of education. A fairly large number of educators have also Piagetian-type research. Despite Piaget's reluctance to apply his theory, there were many self-appointed interpreters who attempted the job for him. Most proposals that resulted have taken a rather general form: to foster the child's own activity, to take the child's cognitive level into account in teaching, to order the curriculum in accord with the order suggested by the child's own development, and to place less emphasis on language as a means of learning in the early years. Some recommendations have been more specific, however. For example, mathematics educators took seriously Piaget's reports of a developmental progression from topological to both euclidean and projective concepts, although a sizeable number interpreted the progression incorrectly as a linear progression. A large number of studies conducted by psychologists and mathematics educators, in the main, supported Piaget's claim, as our own reviews indicate (Beilin, in press). More

recently, the reports divide between confirmatory and critical assessments, with commentators increasingly critical of Piaget's use of Klein's geometric theory and of the developmental ordering of geometric concepts. The Genevans, apparently have not been oblivious of these criticisms, and more recent characterizations of the developmental progression (Inhelder, 1978; Montangero, 1976) are in terms of a transition from intrafigural to interfigural and finally to transfigural geometric relations.

In the mathematics education research literature that concerns geometry (although it is true for other domains, as well), one sees in the past five years or so, if not a mass defection from Piagetian theory, at least a sizeable migration. The migration for the most part is to information processing. The appeal of information processing for the mathematics educator (Resnick & Ford, 1981) is in providing a language, principally a computer language, for the representation of symbolic processing, a method of analysis of both procedural and declarative knowledge, and a vehicle for testing a variety of theories. What is also seen as a contribution is a concern for careful task analysis (Hiebert, 1981), attention to prior knowledge of the task environment, and to problem solving strategies.

Mathematics educators have reacted to these developments with a response ranging from caution, "It is too early to tell what implications the research in this area might have for mathematics curriculum (Hiebert, 1981, p. 48)", to considerable skepticism (Bell, 1979), because of the tendency to reduce problem solving and reasoning to algorithmic procedures, and to the practice of not dealing with "real problems with real data." This critic (Bell, 1979, p.11) concludes, "I don't think we can look to present or foreseeable information processing models as guides to instruction for applied problem solving."

With apparent rejection of the two dominant models (Piagetian and information processing) of contemporary developmental and educational psychology, mathematics educators appear to be opting instead for instructional models based on a technology that bears only a general relation to psychological theory. They are moving toward models of the kind that engineering provides relative to physics or classical mechanics, and medicine relative to molecular biology and immunology. In other words, the goal is to develop curriculum models on the basis of principles that derive primarily from educational practice, the means of instruction, the nature of subject matter, and the nature of developing cognition, but the last is only one component in the system and not the principal focus of curriculum construction. The educational context rather than the child himself, is the principal causal agency that defines the source of mathematical knowledge. In our view this is a healthy and desirable trend. An example of this trend is the wide ranging interest and test of an instructional model proposed by the Dutch mathematics educators, P.M. and D. van Hiele (van Hiele, in press; van Hiele, 1959). The van Hiele model has been described as influenced by a combination of Gestalt and Piagetian ideas. While influenced by these theoretical traditions, at its core is another focus. P.M. van Hiele writes as follows:

"A psychological theory will never supply enough data and could never be used on a basis of teaching unless it is rooted in the study of practical teaching: didactics can never be considered as the application of a psychology. No one can solve the problem unless he combines in his person the psychologist and didactician and unless he applies his psychological knowledge in the didactical practice (1959, p. 2)." Also, "...it has now become clear that psychology is in itself,

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insufficient to supply the data of which didactics has need. Whoever wishes to develop didactics will have to start from didactic experience themselves to direct his attention to school learning situations. A reasonable knowledge of psychology is without doubt necessary for this study; but psychological results obtained independently of school learning situations will only be fruitful if integrated with psychological results which have been obtained within didactics (p. 17)."

Van Hiele's emphasis is on stages of learning in each instructional context and less on the structuring of thought itself, although the instructional model itself parallels to a degree Piaget's developmental (cognitive) levels. The van Hiele levels, however, constitute a hierarchy of knowledge and skills which instruction should facilitate ascending. The levels differ from Piagetian structural levels, which are modeled on logical or mathematical models and not on van Hiele's sense of didactics.

Although we would endorse van Hiele's general aim of developing instructional models based largely on the nature of school related functions (i.e., the logical structure of the subject matter, teacher characteristics, instructional technology and the like) and take account of the developing character of the child's cognitive resources, we take exception to van Hiele's use of the latter. For one, the model is overly simplistic in its characterization of the course of development. Second, in its application to adolescent learning it applies assumptions derived from a differentiation theory of perceptual development. Learning (of geometry), in essence, is treated as a microgenetic series of changes, proceeding from treating figures as wholes to the recognition and treatment of geometric parts differentiated from that whole.

Although an instructional model may succeed on these principles and assumptions, they may in fact be inferior to an instructional model based on what we have discovered as the spontaneously generated strategies for dealing with geometric objects. The developmental order we observed shows cognitive development in respect to some geometric concepts to occur in the reverse order from that assumed by the van Hieles. That is, it develops from the components of a figure to dealing with the figure as an integrated entity. We are not suggesting that the developmental sequence we have found, because it is closer to empirical observation, will of necessity lead to superior instructional outcomes to the one proposed by the van Hieles. The reason for our caution derives from reviews of past research on the training of Piagetian logical operations (Beilin, 1971, 1978). In these reviews we have shown that almost any type of training procedure results in significant acquisition of conservation knowledge (despite Piaget's early skepticism of that possibility). Furthermore, one method, the most "unnatural" of all in Piaget's terms, verbal rule instruction, was superior to all others, including the method that Piaget holds comes closest to the manner in which such knowledge is acquired naturally by children (cognitive conflict and equilibration). It suggests strongly that instructional methods need not model themselves either on cognitive developmental theory or empirical demonstrations of cognitive development. The relation of instructional technology to psychological theory should parallel that of bridge building to physics and medicine to biology. These technologies cannot ignore the facts and theories offered by these sciences, but the sciences do not build bridges, cure poliomyelitis, or teach children.

On a more specific note, our study offers some suggestions for instruction in geometry. Geometry is typically taught in high schools, although increasingly

it is found in junior high instructional units and in the later years of elementary school. Teachers tend to the view that instruction in geometry should be introduced in elementary school but not in formal logical terms, that is, in terms of formal mathematical proofs more typically characteristic of high school instruction. We would concur in this view. At the same time, if it is the case that young children 2 years 6 months of age have rudimentary notions of congruence and motion and by 7 years are able to generate strategies (like superposing) that enable them to deal with congruence in more sophisticated ways, it suggests that children are capable of dealing with geometric ideas to a greater extent than has been assumed up to now. There has been so little research on children's knowledge of geometry (only a fraction of what has been devoted to number and arithmetic in the early years) that the range of their knowledge can only be hinted at.

At the same time, it is evident that many children as old as eleven have considerable difficulty with certain geometric ideas, particularly in connection with transformations. Attention to having children acquire knowledge of the fundamental logical and spatial bases of rotational (turn) and reflectional (flip) transformations should lead to a firmer foundation for logical and algebraic treatments of geometric theorems and propositions. This poses a challenge to present instructional technology.

Our last point concerns the development of naturally occurring strategies. Research on mathematical cognition demonstrates to an increasing extent (e.g., Heibert, 1979; Resnick & Ford, 1981) the inventiveness and significance of the strategies children develop in problem solving tasks.

Rather than develop instructional technologies based solely on the logical

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structure of the subject matter or mathematical models of mind, greater attention might be given to the strategies generated by children themselves in the solution to problems. There is a reverse side to this as well that may aid instructional goals. At an age when children are generating and using single edge matching strategies, for example, in verifying congruence relations, it could very well be that instruction in the utility of a multiple edge matching procedure, which children at any age, almost never generate by themselves, could progress the child's knowledge of congruence in a way the child is not able to achieve alone.

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Appendix I

Table 1

Dimensions of Stimulus Materials Used in the Imagery, Measurement, and Correspondence Tasks

(In inches)

Color	Imagery		Measurement			Correspondence	
	Model Triangles	Selection Triangles	Congruent Triangles	Non-Congruent Triangles	Selection Triangle	Line Segments	Angles
Red	3 3/8 X 4 1/2 X 5 5/8	3 X 4 X 5 3 3/4 X 5 X 6 1/4 4 1/2 X 3 3/4 X 5	3 3/8 X 4 1/2 X 5 5/8	3 3/8 X 4 1/2 X 5 5/8 (L) 3 3/4 X 4 1/2 X 5 (R)	3 3/8 X 4 1/2 X 5 5/8	3 3/4 X 5 3/16 X 6 1/16	3 1/2 X 4 11/16 X 5 3/8
Blue	3 X 4 X 5	2 5/8 X 3 1/2 X 4 3/8 3 3/8 X 4 1/2 X 5 5/8 4 X 3 1/2 X 3 3/4	3 X 4 X 5	3 X 4 X 5 (L) 3 1/2 X 4 X 4 5/8 (R)	3 X 4 X 5	3 1/4 X 4 5/6 X 4 11/16	3 3/4 X 5 3/16 X 6 1/16
Green	3 3/4 X 5 X 6 1/4	3 3/8 X 4 1/2 X 5 5/8 4 1/8 X 5 1/2 X 6 7/8 5 X 4 1/2 X 4 1/4	3 3/4 X 5 X 6 1/4	3 3/4 X 5 X 6 1/4 (L) 4 1/8 X 5 X 6 (R)	3 3/4 X 5 X 6 1/4	3 1/2 X 4 11/16 X 5 3/8	3 1/4 X 4 5/16 X 4 11/16

APPENDIX I

Experimental Design

I. General design of study: Factorial design with repeated measures (on Task) with Age and Sex between factors (MANOVA): Age (8) x Sex (2) x Task (7).

A. Age (8 levels) = 4 years through 11 years of age.

B. Sex (2 levels) - Boys, girls.

C. Tasks (7 levels) - (1) Imagery: Phase 1;(2) Phase 2;(3) Phase 3; (4) Measurement; (5) Correspondence: Lines/Angles;(6) Points;(7) Combination of Motions.

II. Counterbalancing Procedures: Order of tasks.

For counterbalancing purposes tasks were grouped into four sets: Imagery, Measurement, Correspondence Matching Parts, Combination of Motions.

A. Two (2) task orders were generated within the constraint that the Imagery tasks were always administered first and the Combination of Motions task was always last.

Order I: Pretest, Imagery, Measurement, Correspondence, Combinations of Motions.

Order II: Pretest, Imagery, Correspondence, Measurement, Combination of Motions.

B. Task order was counterblanced by sex within each age group; i.e., one-half of boys received Order I, one-half received Order II, etc.

C. When subjects required more than one testing session to complete tests (most subjects below 7 years), first and second tasks within an order were given in first session and third and fourth tasks were given in the second session.

III. Counterbalancing within tasks:

A. Imagery Tasks. Type of transformation varied (slide, flip, turn). Order of presentation for type of transformation was varied across subject, as follows:

1. All six possible order combinations of the three transformations (slide, flip, turn) were used.

2. Each subject was assigned to one of the six presentation orders by rotating the six orders among subjects from the youngest to oldest age group.

B. Measurement Task. Three dimensions were varied within the task:

1. Mobility of Figures: mobile vs. non-mobile

2. Congruence of Triangles: congruent vs. non-congruent

3. Type of Transformation: slide, flip, turn.

Each of the variables was counterbalanced across subjects, as follows:

1. Mobility of Figures.

a. Within each age group, half of the subjects were assigned to the mobile figures condition, half to the non-mobile condition.

b. Assignment of subjects to mobile and non-mobile conditions was counterbalanced for task order and sex, i.e., for task order I (order II) half of the subjects were in the mobile condition and half were in the non-mobile condition. Similarly, for the boys (girls), half were in the mobile condition and half were in the non-mobile condition.

2. Congruence of Triangles

a. Within each age group, half the subjects received congruent triangles, half non-congruent triangles.

b. Presentation of congruent and non-congruent triangles was counterbalanced across subjects for task order, mobility of figures and sex,

i.e., for the task order I (order II), half of the subjects received congruent triangles and half non-congruent triangles. Similarly for boys (girls) half received congruent and half received non-congruent triangles.

3. Type of Transformation

a. Presentation order of the transformations was counterbalanced by generating 5 random orders of the three transformations.

b. Within each age group, one of the 5 orders was assigned to each subject in the mobile figures condition (Total = 5Ss), and the same 5 orders were assigned to subjects in the non-mobile condition.

C. Correspondence Tasks. Two dimensions were varied in this task:

1. Type of Parts Task: line segments, angles and points.

2. Type of Transformation: slide, flip, turn.

Each variable was counterbalanced across subjects, as follows:

1. Type of Parts Task

a. Within each age group, all subjects received the Points task, but only half of the subjects received the line segments task and half the angles task. Performance was later combined in the data analysis because subjects performed equivalently on these two tasks.

b. Assignment of subjects to the Line segments and Angles task was counterbalanced for task order and sex.

c. Presentation order of the two sets of correspondence tasks was counterbalanced by assignment the Points task first to half the subjects and last to the other half of the subjects in an age group.

2. Type of Transformation

All six possible orders among the three transformations were rotated among subjects. Presentation orders of the transformations within each correspondence task was the same as in the imagery task.

APPENDIX - 2

EXPERIMENTER'S MANUAL

Pretest

Materials Checklist: 1. One cookie cutter
2. Eight play-doh patties

Directions To Interviewer:

The play-doh patties and the cookie cutter are on the table in front of the interviewer and the child. The interviewer and the child are seated next to each other, with the interviewer on the child's left, facing in the same direction.

Interviewer says:

I: "Hello ______. Today we are going to play a game with some shapes." "Here is some play-doh." (Point to play-doh).

I: "Do you know what this is?" (Point to cookie cutter).

I: "(Yes, that's right) It's a cookie cutter in the shape of a triangle. I'll show you how it works." (Demonstrate how cookie cutter works by cutting out play-doh triangles from two patties).

I: "You try." (Give child cookie cutter to cut some triangles. If child uses cookie cutter correctly, interviewer says: "Good, that's very good.")

(If child fails to stamp out some triangles with cookie cutter, the interviewer is to help the child cut them out.)

I: "Look at all the triangles. They're all the same. Why is that?"

I: (If the child responds correctly to the question, say: "That's right. They're all the same because they were cut from the same cookie cutter."

If the child responds incorrectly, say: "They're the same because they were cut from the same cookie cutter.

If the child fails to respond, say: "They're the same because they were cut from the same cookie cutter."

I: "Now we are going to play a game with some triangles that have been cut from cookie cutters."

End of Pretest.

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EXPERIMENTER'S MANUAL

Imagery Tasks Slide Rotation Flip (Circle One)

PHASE 1

Materials Checklist 1. Standard RED BLUE GREEN triangle (circle one used in task)

2. Tray holding RED BLUE GREEN selection triangles (circle one)

Directions To Interviewer:

The standard RED BLUE GREEN triangle A, is in front of child such that its longer and shorter legs form the letter "L". The tray holding the RED BLUE GREEN selection triangles is placed on the table about 3 inches above triangle A. The interviewer and the child are seated next to each other, with the interviewer on the child's left facing in the same direction (See Figure).

Interviewer says:

I: "Look at this triangle." (Point to standard triangle A.)

I: "And look at the triangles in the tray." (Point to selection triangles in tray.)

I: "One of these triangles in the tray was cut from the same cookie cutter as this triangle (point to A). Can you find the triangle in the tray (point to tray) that was cut from the same cookie cutter as this triangle (point to triangle A)?"

(If the child asks to see the cookie cutter, interviewer replies: "We don't have the cookie cutter here, but the same cookie cutter made this triangle (point to triangle A) and one triangle in the tray."

Response Measures:

1. Record child's selection strategy.

2. Record triangle selected by child (accurate/inaccurate and, if inaccurate, type of error). Code: 1= smallest triangle
2= congruent triangle
3= longest triangle
4= non-similar triangle

EXPERIMENTER'S MANUAL

Standard Post - Phase 1 Questions:

I: 1. "Why do you think these two triangles (point to triangle A and the triangle A and the triangle selected by the child) are cut from the same cookie cutter?"

I: 2. "Why didn't you pick one of the other triangles (point to the triangles remaining in the selection tray)?"
Prompt: "What is it about the other triangles (point to the triangles remaining in the selection tray) that makes them not the same as this triangle (point to triangle A)?"

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Imagery Tasks Slide Rotation Flip (Circle One)

Phase 2

Materials Checklist.

1. Yellow manilla rectangular envelope without a flap.
2. Tray to hold selection set of triangles.
3. Three different sets of triangles - RED set, BLUE set, and GREEN set (circle one used in task).

Directions to Interviewer:

Place two identical RED BLUE GREEN (circle one) triangles, A (which serves as the standard) and B, in front of the child with standard triangle A on the right. Each triangle is aligned so that its longer and shorter legs form an "L". The shorter legs of each triangle are collinear and 8 inches from the edge of the table. The nearest vertices of triangles A and B are 10 inches apart with triangle A on the right (See Figure).

Interviewer says:

I. "Here we have two triangles. These triangles were cut from the same cookie cutter so they are the same." (Point to triangle A and triangle B.) (If the child asks to see the cookie cutter, point to triangle A and triangle B and reply: "We don't have the cookie cutter here, but the same cookie cutter made both these triangles.")

I: "Watch what I am going to do." (Hold triangle B steady, and slip, from top to bottom, the envelope without disturbing its position or orientation. The envelope is positioned such that triangle B is in the center of the envelope, and the left and right sides of the envelope are parallel to the vertical leg of triangle B.)

I: "Now watch very carefully what I do to the envelope." Perform a planar transformation on the envelope as specified below:

	Imagery Task	Task 1: Slide the envelope 5 in. to the right.
(Circle one used in task)	Imagery Task	Task 2: Rotate the envelope 90^0 clockwise with the lower left corner of the envelope as the center of rotation.
	Imagery Task	Task 3: Flip the envelope about the right vertical side of the envelope.

I: "Look at these triangles in the tray." (Present the tray holding four RED BLUE GREEN (circle one) selection triangles.)

I: "One of these triangles in the tray was cut from the same cookie cutter as the triangle hidden in the envelope. Can you find the triangle in the tray (point to tray) that was cut from the same cookie cutter as the triangle in the envelope (point to envelope containing triangle B)?"

(If the child asks to see the cookie cutter, reply: "We don't have the cookie cutter here, but the same cookie cutter made the triangle in the envelope and one triangle in the tray.")

(If the child attempts to pick up the envelope, say: "Try to find a triangle in the tray without touching the envelope.")

Response Measures:
1. Record child's selection strategy.
2. Record triangle selected by child (accurate/inaccurate and, if inaccurate, type of error.)

Code: 1 = smallest triangle
2 = congruent triangle
3 = largest triangle
4 = non-similar triangle

Standard Post-Phase 2 Questions

I: 1. "Why do you think this triangle (point to triangle selected by the child) is cut from the same cookie cutter as the triangle hidden in the envelope (point to envelope)?"

I: 2. "Why didn't you pick one of the other triangles (point to the triangles remaining in the selection tray)?"
Prompt: "What is it about the other triangles (point to the triangles remaining in the selection tray) that makes them not the same as this triangle (point to triangle A)?"

Conclude Phase 2 by returning the child's selection to the tray and removing triangle B from the envelope. Interviewer should now be ready to begin Phase 3.

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EXPERIMENTER'S MANUAL

Imagery Tasks   Slide   Rotation   Flip   (Circle One)

Phase 3

Material Checklist:
1. Yellow manilla rectangular envelope without a flap.
2. Tray to hold selection sets of triangles.
3. Three different sets of triangles - RED set, BLUE set, and GREEN set (circle one used in task).

Directions to Interviewer:

Place two identical RED BLUE GREEN (circle one) triangles, A (which serves as the standard) and B, in front of the child with standard triangle A on the right. Each triangle is aligned so that its longer and shorter legs form an "L." The shorter legs of each triangle are collinear and 8 inches from the edge of the table. The nearest vertices of triangles A and B are 10 inches apart with triangle A on the right (See Figure).

Interviewer Says:

I: "Here are two triangles again. These triangles were cut from the same cookie cutter so that they are the same." (Point to triangle A and triangle B).

(If the child asks to see the cookie cutter, point to triangle A and triangle B and reply:
"We don't have the cookie cutter here, but the same cookie cutter made both these triangles.")

I: "Watch what I am going to do." (Hold triangle B steady, and slip from top to bottom the envelope around triangle B so that triangle B is completely hidden inside the envelope without disturbing its position or orientation. The envelope is positioned so that triangle B is in the center of the envelope, and the left and right sides of the envelope are parallel to the vertical leg of triangle B.)

I: "Now watch very carefully what I do to the envelope." Perform a planar transformation on the envelope as specified below:

Imagery Task 1: Slide the envelope 5 in. to the right.

Imagery Task 2: Rotate the envelope 90° clockwise with the lower left corner of the envelope as the center of rotation.

(Circle one used in task)

EXPERIMENTER'S MANUAL

Imagery Task 3: Flip the envelope about the right vertical side of the envelope.

I: "Look at your triangle." (Point to triangle A)

I: "Can you do with your triangle what I did with mine so that it will look the way the triangle hidden looks now."

(If the child attempts to pick up the envelope, say: "Try to put your triangle without touching the envelope.")

Response Measures:

1. Record child's transformation strategy.

2. Record accurate/inaccurate transformation of triangle A by child and, if inaccurate, type of error committed.

Standard Post-Phase 3 Questions:

I: 1. "What did you do with this triangle (point to triangle A) to make it look like the triangle hidden in the envelope (point to envelope)?"

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EXPERIMENTER'S MANUAL

Measurement: MOBILE FIGURES NON-MOBILE FIGURES (circle one)

Task: ROTATION FLIP SLIDE (circle one)

Trial: CONGRUENT NON-CONGRUENT (circle one)

Materials Checklist. 1. One 3 ft. x 2 ft. piece of Bainbridge board.

2. Tray to hold the sets of solution aids.

3. Pair of RED. BLUE GREEN stimulus triangles (circle one)

4. Set of RED BLUE GREEN solution aids (circle one) containing:

(a) one RED BLUE GREEN selection triangle (circle one)

(b) one set of four pieces of RED BLUE GREEN string (circle one)

(c) one set of four RED BLUE GREEN plastic sticks (circle one)

(d) one pair of scissors

(e) one ruler

(f) one roll of white twine

(g) one protractor

(h) one pencil

(i) six sheets of unlined paper

Directions to Interviewer

The interviewer and the child are seated next to each other with the interviewer on the child's left. The Bainbridge board is placed on the table directly in front of the child such that its 3-foot side is 3 in. from the bottom edge of the table. The tray to hold the set of solution aids for a particular trial is placed to the right of the board. Triangle L in each pair of stimulus triangles is placed on the left half of the board according to the following specifications: the upper left vertex of triangle L is a right angle, the longer leg of triangle L is 15 in. from the bottom edge of the board, and the shorter leg of triangle L is 9 in. from the left edge of the board. The other stimulus triangle in each pair, triangle R, is placed on the right half of the board according to the following specifications for each trial (See Figures and):

Rotation-Congruent Trial

Place red triangle R on the right half of the board about 13 1/2 in. to the right of triangle L in the same orientation as triangle L. Rotate triangle R 90° clockwise with the lower left vertex as the center of rotation.

Rotation-Non-Congruent Trial

Place red triangle R on the right half of the board about 13 1/2 in. to the right of triangle L in approximately the same orientation as triangle R in the rotation-congruent trial. Make the 4 1/2 in side of triangle R in the rotation-non-congruent trial coincide with the placement of the 4 1/2 in. side of triangle R in the rotation-congruent trial.

Flip-Congruent Trial

Place blue triangle 2 on triangle L in the same orientation as triangle L. Flip triangle R about an imaginary vertical line that is 7 inches from the upper right vertex of triangle L.

Flip-Non-Congruent Trial

Place blue triangle R on the right half of the board about 14 in. to the right of triangle L in approximately the same orientation as triangle R in the flip-congruent trial. Make the 4 in. side of triangle R in the flip-non-congruent trial coincide with the placement of the 4 in. side of triangle R in the flip-congruent trial.

Slide-Congruent Trial

Place green triangle R on triangle L in the same orientation as triangle L. Slide triangle R 13 in. to the right of triangle L.

Slide-Non-Congruent Trial

Place green triangle R on the right half of the board about 13 in. to the right of triangle L in approximately the same orientation as triangle R in the slide-congruent trial. Make the 5 in. side of triangle R in the slide-non-congruent trial coincide with the placement of the 5 in. side of triangle R in the slide-congruent trial.

(If the Equivalence of Figures tasks are administered as the second task of Session I, the interviewer says just before introducing the first Equivalence of Figures trial: "Remember the game we played with the triangles? Now we are going to play another game with triangles.")

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Interviewer says:

I: "Please close your eyes while I get the triangles ready." The pair of stimulus triangles is placed on the board according to the specifications for each trial.

I: "You can open your eyes now. Look at these triangles." (Point to triangle L and triangle R).

I: "The way we play this game is that you have to find out whether these two triangles (point to triangle L and triangle R) are cut from the same cookie cutter or not."

(Omit the above statement after the first trial on the Equivalence of Figures tasks.)

I: "Sometimes the two triangles are cut from the same cookie cutter and sometimes the two triangles are not cut from the same cookie cutter - you a have to try to find out."

I: "Here are some things you can use to help you find out whether these two triangles are cut from the same cookie cutter or not." (Place set of RED BLUE GREEN (circle one) solution aids in tray.)

I: In mobile figures condition say: "You may also move the triangles to help you find out whether these two triangles are cut from the same cookie cutter or not".

In non-mobile figures condition say: "You may not move the triangle to help you find out whether these two triangles are cut from the same cookie cutter or not."

I: 1. "Now, can you show me how you would find out if these two triangles are cut from the same cookie cutter or not?"
(Stop child from additional strategies after he/she has made a judgment about the congruence of the two triangles in question)

I: 2. "Can you show me how you would find out in another way?"

I. 3. "Can you show me how you would find out using something (else) on the table?"

Response Measures:
1. Record child's judgment (accurate/inaccurate).

2. Record child's solution strategy.

Post-Trial Questions:

1. "What is the best way to find out if the triangles are (not) the same?"

Prompt: "How did you find out if they were (not) the same?"

EXPERIMENTER'S MANUAL

Correspondence - Line Segments SLIDE ROTATION FLIP (circle one)

Materials Checklist.. 1. Drawings of pair of congruent triangles that correspond by a SLIDE ROTATION FLIP (circle one)

2. Set of plastic line segments
GREEN - SLIDE
RED - ROTATION
BLUE - FLIP

Directions to Interviewer:

The interviewer and the child are seated next to each other with the interviewer on the child's left. Each pair of triangles is drawn on a piece of Bristol board with the transformed triangle, triangle E, in front of the interviewer, and the untransformed triangle, triangle C, in front of the child. Triangle C is positioned 8 inches from the bottom edge of the Bristol board, such that its shortest side is vertical, and its longest side is diagonal in orientation anchored by endpoints at the lower right and upper left. In each pair of triangles, triangle E corresponds to triangle C by the following specifications for each trial (see Figure):

Rotation Trial

Triangle E is drawn approximately 5 in. to the left of triangle C and corresponds to triangle C by a left 5 in. slide and a 90° clockwise rotation with the leftmost vertex as the center of rotation;

Flip Trial

Triangle E is drawn approximately 5 in. to the left of triangle C and corresponds to triangle C by a flip about an imaginary vertical line that is equidistant from the vertical sides of triangles C and E;

Slide Trial

Triangle E corresponds to triangle C by a slide 10 in. to the left of triangle C;

Interviewer says:

(If the Correspondence Line Segment Tasks are administered as the second task of Session I, and prior to the administration of the Angles task, the Interviewer says just before introduction the First of the three Line Segments Tasks: "Remember the game we played with the triangles? Now we are going to play another game with triangles.")

I: "Look at these triangles." (Point to the triangles.)

I: "We're going to play a game with these two triangles. Both of them were cut from the same cookie cutter."

EXPERIMENTER'S MANUAL

I: "This triangle (point to triangle C) is your triangle, and this (point to triangle E) is my triangle."

I: "Watch what I'm going to do. I'm going to decorate my triangle like this." (Place BLUE, GREEN RED plastic line segments on the appropriate sides of triangle E).

I: "We want to decorate your triangle so that it is exactly the same as my triangle is now."

I: (Point to smallest side of triangle E.) "Can you point to the side of your triangle that is the same as this side?"

I: (After the child has identified a side of triangle C, remove the smallest plastic line segment from triangle E and place it on the side of triangle C identified by the child.)

Response Measures:

1. Record child's selection strategy. (accurate/inaccurate)

2. Record first and if necessary second side selected by child.

Follow-up Questions:

I: 1. "How did you know to pick that side on your triangle?"
Prompt: "How did you figure out that the two sides (pointing to the appropriate side on triangle E and the corresponding side on triangle C identified by the child) are the same?"

The one plastic line segment now on triangle C is left in place while the Interviewer procedes.

I: (Points to middle-sized side of triangle E) "Can you point to the side of your triangle that is the same as this side?"

Response Measures:

1. Record child's selection strategy (accurate/inaccurate)

2. Record first and if necessary second side selected by child.

Follow-up Questions:

I: 1. "How did you know to pick that side on your triangles?"
Prompt: "How did you figure out that the two sides (pointing to the appropriate side on triangle E and the corresponding side on triangle C identified by the child) are the same?"

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The two plastic line segments on triangle C are left in place while the Interviewer procedes.

I: (Points to larger side of triangle E) "Can you point to the side of your triangle that is the same as this side?"

Response Measures:

1. Record child's selection strategy. (accurate/inaccurate)

2. Record first and if necessary second side selected by child.

Follow-up Questions:

I: 1. "How did you know to pick that side of your triangle?"
Prompt: "How did you figure out that the two sides (pointing to the appropriate side on triangle E and the corresponding side on triangle C identified by the child) are the same?"

Post-Task Questions:

I: 1. "Are the triangles the same?"

I: 2. "Why?"

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EXPERIMENTER'S MANUAL

Correspondence - Angles Tasks SLIDE ROTATION FLIP (circle one)

Materials Checklist. 1. Drawings of pair of congruent triangles that correspond by a SLIDE ROTATION FLIP (circle one)

2. Set of plastic angles
BLUE - SLIDE
GREEN - ROTATION (circle one)
RED - FLIP

Directions to Interviewer

The interviewer and the child are seated next to each other with the interviewer on the child's left. Each pair of triangles is drawn on a piece of Bristol board with the transformed triangle, triangle E, in front of the interviewer, and the untransformed triangle, triangle C, in front of the child. Triangle C is positioned 8 inches from the bottom edge of the Bristol board, such that its shortest side is vertical, and its longest side is diagonal in orientation anchored by end points at the lower right and upper left. In each pair of triangles, triangle E corresponds to triangle C by the following specifications for each trial (see Figure):

Flip Trial

Triangle E is drawn approximately 5 in. to the left of triangle C and corresponds to triangle C by a flip about an imaginary vertical line that is equidistant from the vertical sides of triangles C and E.

Slide Trial

Triangle E corresponds to triangle C by a slide 10 in. to the left of triangle C.

Rotation Trial

Triangle E is drawn approximately 5 in. to the left of triangle C and corresponds to triangle C by a left 5 in. slide and a 90° clockwise rotation with the leftmost vertex as the center of rotation.

Interviewer Says:

(If the Correspondence Angles Tasks are administered as the second task of Session I, and prior to the administration of the Line Segments Task, the Interviewer says just before introducing the First of the three Angles Tasks: "Remember the game we played with the triangles? Now we are going to play another game with the triangles.")

I: "Look at these triangles." (Point to the triangles.)

I: "We're going to play a game with these two triangles. Both of them were cut from the same cookie cutter."

EXPERIMENTER'S MANUAL

I: "This triangle (point to triangle C) is your triangle, and this triangle (point to triangle E) is my triangle."

I: "Watch what I'm going to do. I'm going to decorate my triangle like this." (Place BLUE GREEN RED plastic angles on the appropriate vertices triangle E).

I: "We want to decorate your triangle so that it is exactly the same as my triangle is now."

I: (Point to smallest angle of triangle E.) "Can you point to the corner of your triangle that is the same as this corner?"

I: (After the child has identified a vertex of triangle C, remove the smallest plastic angle from triangle E and place it on the vertex of triangle C identified by the child.)

Response Measures:

1. Record child's selection strategy. (accurate/inaccurate)

2. Record first and, if necessary, second vertex selected by child.

Follow-up Questions:

I: 1. "How did you know to pick that corner on your triangle?"
Prompt: "How did you figure out that the two corners (pointing to the appropriate vertex on triangle E and the corresponding vertex on triangle C identified by the child) are the same?"

The one plastic angle now on triangle C is left in place while the Interviewer procedes.

I: (Points to middle-sized angle or triangle E) "Can you point to the corner of your triangle that is the same as this corner?"

Response Measures:

1. Record child's selection strategy. (accurate/inaccurate)

2. Record first and, if necessary, second vertex selected by child.

Follow-up Questions:

I: 1. "How did you know to pick that corner on your triangle?"
Prompt: "How did you figure out that the two corners (pointing to the appropriate vertex on triangle E and the corresponding vertex on triangle C identified by the child) are the same?"

EXPERIMENTER'S MANUAL

The two plastic angles on triangle C are left in place while the Interviewer procedes.

I: (Points to largest angle of triangle E) "Can you point to the corner of your triangle that is the same as this corner?"

Response Measures:

1. Record child's selection strategy. (accurate/inaccurate)

2. Record first and, if necessary, second vertex selected by child.

Follow-up Questions:

I: 1. "How did you know to pick that corner on your triangle?"
Prompt: "How did you figure out that the two corners (pointing to the appropriate vertex on triangle E and the corresponding vertex on triangle C identified by the child) are the same?"

Post-Task Questions:

I: 1. "Are the triangles the same?"

I: 2. "Why?"

EXPERIMENTER'S MANUAL

Correspondence - Points on a Circle

SLIDE ROTATION FLIP (circle one)

Materials Checklist. 1. Task board containing drawing of a pair of circles that correspond by a SLIDE ROTATION FLIP (circle one). Affixed to circle C are the following non-movable dots:

3 white
1 red light-blue black (circle one)
1 blue brown green (circle one)

2. Affixed to circle E are the following nonmovable dots:

1 red	light-blue	black
1 green	orange	red
1 yellow	green	blue
1 dark blue	brown	green
1 orange	red	yellow

(circle one column)

3. Set of eight boxes each containing ten movable dots of the following colors:

dark blue
light blue
green
red
yellow
orange
black
tan

Directions to Interviewer

The interviewer and the child are seated next to each other with the interviewer on the child's left. Each pair of circles is drawn on a piece of bristol board with the transformed circle, circle E, in front of the interviewer, and the untransformed circle, circle C, in front of the child. The boxes of movable dots are located at the interviewer's left. The centers of the circles C and E are 10 inches from the bottom edge of the Bristol board. Circle C is positioned such that the two non-white dots affixed to it are located at the 12 and 7 o'clock positions. In each pair of congruent circles, circle E corresponds to circle C by the following specifications for each trial (see Figure):

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Slide Trial
Circle E corresponds to circle C by a slide 10 in. to the left of circle C.

Rotation Trial
Circle E is drawn approximately 5 in. to the left of circle C and corresponds to circle C by a left 5 in. slide and a 90° clockwise rotation with the red dot as the center of rotation.

Flip Trial
Circle E is drawn approximately 5 in. to the left of circle C and corresponds to circle C by a flip about an imaginary vertical line that is equidistant from the circle's centers.

Interviewer says:

I: "Look at these circles" (point to the circles).

I: "We're going to play a game with them. Both of them were cut from the same cookie cutter."

I: "This circle (point to circle E) is my circle, and this circle (point to circle C) is your circle."

I: "Look where the RED AND BLUE BLUE AND BROWN BLACK AND GREEN dots are on your circle, and look where the dots are on my circle."

I: "Watch what I'm going to do. I'm going to decorate my circle like this." (Place GREEN-YELLOW-ORANGE ORANGE-GREEN-RED RED-BLUE-YELLOW dots over the appropriate nonmovable colored dots on circle E).

I: "We want to decorate your circle so that it is exactly the same as mine is now."

I: (Point to GREEN ORANGE RED movable dot) "Can you point to the spot on your circle that is the same as this one?"

I: (After the child has indicated the location of a nonmovable dot on circle C, remove the GREEN ORANGE RED dot from circle E and place it on the spot on circle C identified by the child).

Response Measures:

1. Record child's selection strategy (accurate/inaccurate).

2. Record first, and if necessary, second spot selected by the child.

Follow-up Questions:

I: 1. "How did you know to pick that spot on your circle?"
Prompt: "How did you figure out that these two spots (pointing to the appropriate dot on circle E and the corresponding dot on circle C identified by the child) are the same?"

EXPERIMENTER'S MANUAL

The one movable dot now on circle C is left in place while the interviewer procedes.

I: (Points to the YELLOW GREEN BLUE movable dot on circle E) "Can you point to the spot on your circle that is the same as this spot?"

Response Measures:

1. Record child's selection strategy. (accurate/inaccurate)

2. Record first, and if necessary, second dots selected by child.

Follow-up Questions:

I: "How did you know to pick that spot on your circle."
Prompt: "How did you figure out that these two spots (pointing to the appropriate dot on circle E and the dot on circle C identified by the child) are the same?"

Post-Task Questions:

I: 1. Are the circles the same?

2. Why?

Response Measures:

1. Record childs "rearrangement" strategy.

EXPERIMENTER'S MANUAL

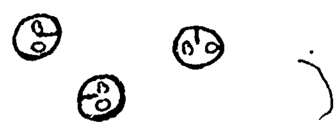
Combinations of Motions Task

Materials Checklist.

1. One task boardwith "signs" (see Figure) indicated on them.
2. Three identical game pieces, called "discs".
3. One "path barrier".

Phase 1. Directions to Interviewer

The three game discs are placed in front of the child (the game board should not be on the table). The interviewer is seated at the child's left. The pieces are placed as follows:



Interviewer says:

I: "These are called 'movers.'" (Interviewer points to discs).

I: "We're going to play a game with one of these movers (points to disc). First, let's see if all the movers are the same. Try to put them so they all look the same."

(If child is having difficulty, ask the child to look at the other side and/or to turn one in order to facilitate the matching process; "Do you think it would help to turn any of them over?" No other kind of assistance should be given.)

I: After the subject has matched the three discs, say: "See if they are all the same on the other side."

I: "Now look at the three movers. They are all the same."

I: "Now choose one of them."

I: "This one will be your mover for playing the game."

Response Measures:

1. Record child's strategies.
2. Record any difficulties in matching.

EXPERIMENTER'S MANUAL

Phase 2. Directions to Interviewer.

Place task board on table (see Figure). The child is seated facing the center of the task board, directly in front of sign #2, with sign #1 on the subject's left. The interviewer is seated to the left of the child.

I: "Now we are going to play a game with your mover and this game board. First, let's look at the board. These are the roads (point to all the roads) and these green places are grass (point to grass). These are signs along the roads (point to all the signs). This is the "start" sign (point to start sign) and this is the "finish" sign (point to finish sign)."

Interviewer says:

I: "To start, put your mover on this sign." (point to start)

I: "Be sure your mover is just like the sign."

Now give child mover in the following orientation: 

(If necessary, provide prompt: ask child if the colors on the mover are the same as the colors on the start sign).

If incorrect after one prompt, interviewer places disc on start sign correctly: "See this is how it should be."

After match of disc to starting sign is made, give the rules as follows:

I: "Now, here is how you play the game."

I: "You move your mover on any of these roads." (point to the two roads between signs #6&F and #4&F).

I: "You cannot move on the green grass (point to green areas)."

I: "You move your mover from the start (point to start) where it is now, to the finish (point to finish)."

I: "You should try to find the shortest road to the finish. This means you should pass the smallest (littlest) number of signs you can."

I: "You must make your mover look like every sign it goes by just like you did here (point to the disc on start). Remember your mover must look like every sign you pass, like when you pass here or here (point to signs #4 and #6)."

I: "Now start to move your mover to the finish on the shortest path."

EXPERIMENTER'S MANUAL

During the game the child will move a disc along the "roads" drawn on the board. A subject may reverse directions to return to a position, but such moves must be made in the same way and following the same rules as all other moves. The "disc" must follow the roads at all time.)

Response Measures:

1. Record the order of positions passed by the gamepiece as the child moves it from start to finish.

2. Record tactics utilized by child as he(she) moves the game piece between positions.

3. Record responses to questions.

Post-Phase 2 Questions:

After trial is completed, Interviewer removes the disc from the board and asks:

I: 1. "What did you do with your mover to make it look like every sign you passed (along the road you took)?"

Interviewer should provide prompts if child does not explain how he/she matched the mover to all of the signs along the road taken.

Phase 3. Directions to Interviewer.

Place the "path barrier" across a road at one of the three positions (A, B, or C) in Figure , depending upon road taken by subject in phase 2:

1) on the path between "start" and sign #2 (point "A" in figure 1) if child moved disc on this path.

2) on the path between signs #1 and #2 (point "B" in figure 1) if child moved disc on this path.

3) on the path between signs #1 and #3 (point "C" in figure 1) if child moved piece on this path.

4) if child moved disc on more than one of these three paths, place barrier in accordance with the last of the paths the child moved disc on.

Interviewer says:

I: "Now, let's play the game again. Put your mover here (point to start), on start. But this time you can't move on this road because there is grass here."

EXPERIMENTER'S MANUAL

Give child the mover in the following orientation: (B'C)

I: "Remember, make your disc look like every sign you pass, and pass the smallest (littlest) number of signs you can."

Response Measures:

1. Record the order of positions passed by the game piece as the child moves it from start to finish.

2. Record tactics utilized by child as he (she) moves the game piece between positions.

3. Record responses to questions.

Post Phase 3 Questions:

I: 1. "What did you do with your mover to make it look like every sign you passed [along the road you took]?"

Interviewer should provide prompts if child does not explain how she/he matched the mover to all of the signs along the road taken.

If child has not taken the road between start and sign #2 is phase 2 or phase 3, ask the following question:

I: 2. "If you take this road between here (point to start) and here (point to sign #2), what do you have to do with your mover to make it look like this sign (point to sign #2)?"

APPENDIX - 3

Coding Manual - Definitions

I. Imagery Task

Coding applies to slide, rotation and flip tasks (Indicate One)

A. Selection Strategies

There are two dimensions to the strategy coding form layout. The horizontal dimension classifies the strategies that are observed in the tapes or in performance. The vertical dimension indicates the phases in the experiment and within each phase the specific trials.

In classifying indicate for each trial in each phase all of the strategies observed by checking the appropriate boxes. Be complete, classify all observed behavior. Where necessary detail your comments in addition to the check-mark coding.

Codable responses are those in which an action is initiated by the child and carried out until terminated. Termination will be determined by the objectives of the trial (i.e., in phase 1 it is the selection of a triangle).

Responses may also be verbal. It has to be decided in each case as to whether the verbal response provides the kind of information that is codable as a strategy.

Code		Definition
NR	No Response	No action or verbal reply to the experimenter's instructions or probes, or irrelevant reply.
HP	Haphazard Selection	Selection of one or more triangles is made from the tray in a manner that suggests no apparent rule or method of selection.
CM	Corner Matching	The child lays out a selection triangle and model triangle with corners touching so as to make one the mirror image of the other. A single corner may be matched (in which case 1 is placed in box) or it may be 2 or 3 (in which case these numbers are recorded).
Visual Insp.	Visual Inspection	Inspection of triangle evidenced by head or eye movements which accompanied selection of 1 or more Δ's from tray.
DC	Direct Comparison	Materials in the selection tray are compared with model triangle on board - Indicate: comparison with
1T	one triangle	comparison with one triangle, or
> 1Δ-Suc	more than one suc	(more than one triangle in succession), or
> 1Δ-Sim	more than one sim	(more than one triangle simultaneously)

		If there is more than one triangle, search may be:
E	Exhaustive	Exhaustive.
N-E	Non-Exhaustive	Not Exhaustive.
R/E or R/N-E	Redundant	Redundant/Exhaustive or Redundant/Non-Exhaustive.
Tactile Insp.	Tactile Inspection	Selection is made by the child's finger tracing of triangles in the selection tray.
DC	Direct Comparison	Child traces with finger along edge of tray triangle and traces with finger around model triangle. There are two aspects to this: There is (1)
1T	one triangle	
>1T	more than one triangle	
		(2) and in each case the tracing is either:
Com.	Complete	Around entire triangle.
Inc.	Incomplete	Only partially around triangle.
No DC	No Direct Comparison	Finger tracing is along edge of tray triangle, but not on model triangle. Again, there is (1)
1T	one triangle	
>1T	more than one triangle traced	
		(2) Tracing is either:
Com.	Complete	Around the triangle
Inc.	Incomplete	Partial trace of triangle
		B. Verifying Strategies
MM	Match to Memory.	Based on verbal report in which subject says he remembers what model looks like.
HDFG	Hidden Figure Match	The tray triangle is matched to the hidden figure either by:
	Tactile Inspection -	In which the child feels or finger traces the hidden figure as well as the selection figure, and/or
	Superposes -	places the selection figure on the envelope with the figure hidden.
Com.Fg.	Common Figure Match -	The visible figure is matched to the selection tray triangle and (usually) by verbal report it is indicated that transitive relation holds among selection triangle, visible matching figure and hidden figure. Match is made between selection triangle and visible triangle by:
Edge M.	Edge Match	Edges of triangles are aligned.
B-P M	B.P. Match	Body Parts match. The child uses parts of his own body (hand, fingers) to establish commonality between selection figure and visible triangle.

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Super	Superpose	Selection triangle is superposed on visible triangle.
Edge Match	Edge Matching	Triangles are compared by placing selection triangle edge next to edge of model triangle. Matching could be single (1 edge) or multiple (more than one edge)
Single Cor	Single Corresponding Edges	Single edge matching can involve: (1) Edges are of corresponding (i.e. relative to) triangles. Specify if combined figures result in:
T	triangle	
D	deltoid	
R	rectangle	
P	parallelogram	
NR	non-regular figure	
N-Cor	Non-corresponding edges -	edges from non-corresponding sides of triangles are combined - always results in non-regular figure.
		(2) In addition to whether single edge matching involves corresponding or non-corresponding sides of triangles, indicate whether match is:
	Static	Triangles are simply placed next to one another, (Score Search: E,N-E,R) OR
	Transformational	One or both of the triangles is transformed in the process of matching - i.e., preserving matched edge. (Score Search: E,N-E,R)
Multiple	Multiple Edge Matching	Two or more edges are matched between each of the triangles. The matching is either with one triangle fixed or matching is progressive:
Fixed Cor	Fixed T all corresponding -	One triangle is fixed, other is rotated, (See E of F details) all corresponding edges are compared.
N-Cor	all non-corresponding -	all non-corresponding edges are compared.
Cor/N-cor	corresponding and non-corresponding -	corresponding and non-corresponding edges compared
Prog	Progressive	Both triangles are rotated, (See E of F details)
Cor	all corresponding	all corresponding edges are compared.
N-cor	all non-corresponding	all non-corresponding edges are compared.
Cor/N-cor	corresponding and non-corresponding	corresponding and non-corresponding edges compared.
		In addition, indicate whether judgment of congruence is made after:
1-step		after first edge-match
2-step		after second edge-match
3-step		after third edge-match
3 -step		beyond third edge-match

static	static (E,N-E,R)	edges are matched (Search: E,N-E,R)
trans'f	transformed (E,N-E,R)'	transformed with preserved match edge - (with flip or rotation) (Search: E, N-E,R)
B-P M	Body Parts Matching	A part of body (i.e., finger, hand) used as a measure of congruence between triangles. Comparison may be direct or not direct.
DC	Direct Comparison	The body part is transposed from the selection triangle to the model triangle - with:
1T	One Triangle	(selection)
>1T	More than one selection triangle	
$\overline{DC}$	No Direct Comparison	The body part is set against the selection triangle(s) only:
1T	One Triangle	
>1T	More than one selection triangle	
Super	Superposing	The selection triangle is placed on top of the model triangle - one or more triangles may be selected.
1T	One Triangle (selection)	Placed in following relations to model:
Antic.	Anticipatory superposing	Figure is rotated or flipped before setting down on model, or
Contact	Contact superposing	Placed directly on model and rotated or flipped on the model.
>1T	More than one selection triangle	
Antic.	Anticipatory superposing	Figure is rotated or flipped before setting down on model, or
Contact	Contact superposing	Placed directly on model and rotated or flipped on the model.

C. Transformation Strategies

T place	Initial Triangle Placement	With respect to transformation that occurs, indicate how subject places triangle prior to transformation. Initial placement is in:
orig.	Original	Original position
alter	Altered	Position is changed
envel. insert	Envelope Insert	Mimics envelope insertion
T transform	Transformation of Triangle	The indications of triangle transformation are in respect to the direction of the transformation and how the transformation is executed.

exec.	Execution	may be:
N perf.	Not perform transf.	not performed
rand.	random	no apparent rule
irrel.	irrelevant	rule governed, but not relevant to model transformation
Relev.	Relevant	Transformations may be:
partial	partial	Partial
full	full	complete (accurate) transformation
over	over	overextended transformation
Dir	Direction	
same	same	same
opp.	opp.	opposite

Amb	Ambiguous	Where actions or verbalizations are unclear as to their intent, scope, organization or meaning.
	comments	Add statements that will clarify any of the strategies - add to strategies, or point to difficulties in classification.

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II. Measurement Task

Indicate on top of form whether the condition is (1) Mobile or Non-Mobile, (2) Rotation - Flip - Slide, and (3) whether figures are Congruent or Non-Congruent.

Note: Strategies NR, CM, and VI are estimation strategies. All other strategies are measurement related or measurement strategies.

CODE	DESCRIPTION	Definition
		If definitions are the same as in the anticipation tasks, it will be indicated as SAB (same as before)
NR	No Response	SAB
CM	Corner Matching	SAB
VI	Visual Inspection	Visual inspection entails head/eye movements which accompanied, verbal report. In this task, visual inspection always entails a direct comparison between the two triangles, but do not score search. Inferred from child's verbal report ("same") and must be followed by a judgement of a n-cong/cong. to constitute a response to a request.
Edge Match	Edge Matching	SAB
Single	Single	SAB
Cor (T,D, R,N-R)	Corresponding Edges	SAB (specify: Triangle, Deltoid, Rectangle, Parallelogram, Non-Regular)
N-Cor	Non-Corresponding edges	SAB
Static	Static	SAB. Do not score search.
Transf.	Transformation	SAB. Do not score search.
Multiple	Multiple Edge Matching	SAB
Fixed	Fixed Triangle	One triangle is fixed. The other triangle is rotated such that its sides are edge-matched successively with the sides of the fixed triangle.
Cor	Corresponding edges	SAB
N-Cor	Non-norresponding edges	SAB

Cor/N-Cor	Both Corresponding and Non-Corresponding	SAB
Progressive	Progressive Edge Matching	Both triangles are rotated such that the sides of 1 triangle are edge-matched successively with the sides of the other triangle.
Cor.	Corresponding edges	SAB
N-Cor	Non-Corresponding edges	SAB
Cor/N-Cor	Corresponding and Non-Corresponding	SAB
1-step	one step	Decision of congruence is made after one, two, etc. steps.
2-step	two step	
3-step	three step	
>3-step	more than 3 steps	
static	static match	Static match in which triangles are not transformed by rotation or flip transformation.
transform.	transformational match	Transformation of one or both triangles preserving matched edge.
Super	Superposing	One triangle is placed on second triangle to establish congruence (for mobil condition).
Antic.	anticipation	Triangle is transformed prior to placement on model.
Contact	contact	Triangle is transformed after it is placed on the model.
Pencil Trace	Pencil Trace	
DC	Direct Comparison	Trace around one triangle is made on paper with pencil, and second triangle is placed within trace of first triangle.
$\overline{DC}$	No Direct Comparison	Outline of one or two triangles is traced, but not compared with each other by common measure.
1T	One Triangle	One Triangle is traced and judgment made.
Compl. / Inc.	complete / incomplete	Triangle trace is either complete (3 sides) or incomplete (one or two sides).
2T	Two Triangles	Two triangles are traced on same or different pieces of paper.

compl. / inc.	complete / incomplete	They are either complete or incomplete.
String Trace	<u>String Trace</u>	
DC	Direct Comparison	Outline of one triangle is traced with string (possibly using stick as anchor). First triangle is removed from string while outline is kept as rigid as possible. Second triangle is placed in string outline of first triangle.
$\overline{DC}$	No Direct Comparison	(Comparable to DC in pencil trace)
Fig. Cut	<u>Figure Cut Out</u>	One or two triangles are cut out of paper.
DC	Direct Comparison	Paper cutout(s) of triangle 1 and/or triangle 2 is used as a common measure.
1T	One Triangle Cutout	One triangle cutout is placed on second triangle.
2T	Two Triangle Cutout	Two triangle cutouts are directly placed on one another.
DC	No Direct Comparison	Child does <u>not</u> use the cutout(s) to make judgment about equivalence of triangles.
1T	One Triangle	One triangle is cutout and used for judgment.
2T	Two Triangles	Two triangles are cutout and used for judgment.
FI	<u>Figure Insertion</u>	The pattern left by making cutout of one triangle is used as basis for judgment of equivalence. Second triangle is inserted in pattern of first triangle.
Tray T.	<u>Tray Triangle</u>	Tray triangle is used as common measure.
DC	<u>Direct Comparison</u>	Tray triangle used for direct comparison between two triangles on table.
edge 2T	Edge match to each triangle on table.	By edge matching. Also score edge matching strategy.
super 2T	Superposing to each triangle on table.	By superposing. Also score superposing strategy.
DC	<u>No Direct Comparison</u>	Tray triangle is compared to only <u>one</u> of the triangles on the table.
Edge 1T	Edge matching to one triangle	By edge matching. Also score edge matching strategy.
Super 1T	Superposing to one triangle	By superposing. Also score superposing strategy.

Conventional	Conventional Instruments	Instruments with unit subdivisions used in conventional or non-conventional ways for measurement. "Conventional" ways refer to used learned in school or from an adult.
Ruler/Protractor	Ruler/Protractor	(specify R or P in response box)
align w/t	alignment with triangle	Ruler and/or protractor is aligned with triangle. No attempt at measurement.
1T	one triangle	One triangle is measured.
comp/inc.	complete/incomplete	The measurement is complete (all sides) or incomplete (2 or 1 side)
2T	2 triangles	
cor.	corresponding sides	Corresponding sides are measured (i.e.,S,M,L)
comp/inc.	complete/incomplete	Measurement is complete (all sides) or incomplete (2 or 1 side)
N-Conventional	Non-Conventional Instruments	Instruments or objects that are adapted or used in conventional or non-conventional ways for measurement. "Non-conventional" ways are other than traditionally taught in school or at home.
string/B.P.	String/Body Parts	(Specify Sg. or BP or Sk. in response box) Uncut string is used which may be aligned with the sides of one or both triangles on board. Child may or may not cut string to equal sides of triangle(s). The alignment may be complete (all sides) or incomplete (1 or 2 sides). Similarly, the child may use a body part (e.g., finger) as a measurement aid by aligning it with one or both triangles, etc.
align w/t	alignment with triangle	No attempt at measurement.
1T	One triangle	SAB
com/inc.	complete/incomplete	SAB
2T	Two triangles	SAB
com/inc.	complete/incomplete	SAB
Cut stg/stick	Cut string/sticks	(Specify Sg or Sk in response box) Child uses either cut string or available cut sticks.
align w/t	alignment with triangle	No attempt at measurement.
1T	One triangle	SAB
com/inc.	complete/incomplete	SAB
2T	Two triangles	SAB
com/inc.	complete/incomplete	SAB
AMB	Ambiguous	Responses are ambiguous cannot be classified properly

III. Correspondence Tasks - Matching Lines Task and Matching Angles Task

Indicate on top of coding form whether Lines or Angles are being tested by circling one.

Next, indicate whether test is with a Slide, Rotation, or Flip motion by circling one.

A. Behavior Strategies

These are strategies based on observable behaviors alone. They are not accompanied by any explanation

Selection Strategies

B-P M	Body Parts Match	Part of body is used as a measuring unit.
S	Simple	Single part of figure (e.g., 1 side (angle) at a time).
C	Complex	Combined parts of figure (e.g., 2 sides plus 1 inclusive angle is matched at time).
Transf.M.	Transformation Match	Represents or demonstrates transformational relation between two figures.
Gest.	Gestures transformation	Uses hands to represent transformation of triangle
Enact	Enacts transformation	Demonstrates transformation of triangle.
Self	Self	Enacts (shows) motion with own body (e.g., walks around board).
Board	Board	Shows motion by moving board.

Placement Strategies

NR	No Response	No action on the part of the subject.
HAP	Haphazard	Haphazard selection of matching part.
T/E	Trial/Error Placement	Apparent uncertainty about placement of part. Subject attempts several placements before settling on one selection.
I	Immediate-Placement	Subject makes placement immediately.

B. Explanatory Strategies

These are strategies based on explanations made in response to the interviewer's question, "How did you pick that side (corner)?". Explanatory strategies may or may not be accompanied by observable behaviors (e.g., pointing, gestures).

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NE	No explanation	Child does not respond to interviewer's question or responds by saying, "I'm just guessing" or "I don't know".
Irrel.	Irrelevant Explanation	Child's response is ambiguous or unrelated to the interviewer's question. E.g., "I thought it was going to be wrong; but it wasn't."
Elim.*	Process of Elimination	Child says, "Only one left".
Transf. V.	Transformation Verbalization	Explanation indicates awareness of the transformation relation between two figures or refers to mental transformation of one of the figures.
Comp.	Comparison	Explanation indicates a comparison of the sides (angle either within the child's triangle or between the child's and the interviewer's triangles.
W	Within	Explanation compares selected side (angle) with other sides (angles) on the child's triangle. E.g., "It's too big for that side". Smallest side points only to his triangle.
B	Between	Explanation compares selected side (angle) on the child's triangle with side (angle) on the interviewer's triangle. E.g., "Exact same...exact same," or "They look the same".
PI	Property Identification	Explanation does not entail a comparison between the child's and the interviewer's triangles. Instead, explanation refers to a particular geometric property of the selected side (angle) of the child's triangle. E.g., "Small side," "Square corner".

*Note: 1) If elimination was first response and second response was spontaneously given or elicited by experimenter, drop elimination and code only second response. 2) If elimination was second response, drop it and code only first strategy.

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III. Correspondence Tasks - Points on a Circle Task

Indicate on top of coding form that Points on a Circle is being tested.

Next, indicate whether test is with a Slide, Rotation, or Flip motion by circling one.

A. Behavior Strategies

These are strategies based on observable behaviors alone. They are not accompanied by any explanations.

Selection Strategies

Transf. M.	Transformation Match	Represents or demonstrates transformational relation between two figures.
Gest.	Gestures transformation	Uses hands to represent transformation of circle.
Enact Self	Enacts transformation Self	Demonstrates transformation of circle. Enacts (shows) motion with own body (e.g., walks around board).
Board	Board	Shows motion by moving board.

Placement Strategies

NR	No Response	No action on the part of the subject.
HAP	Haphazard	Haphazard selection of matching part.
Pos. Cor.	Position Correspondence	Error Category: Specific to rotation and flip. Places dot on child's circle such that it corresponds in relative position to dot on interviewer's circle. Note that this strategy yields an inaccurate response for the flip and rotation transformations. Responses to slide transformation are not coded in this category.
N-Cor	No Correspondence	Error Category: Places dot on child's circle such that it does not correspond in relative position (slide), sense (flip) or orientation (rotation) to the dot on interviewer's circle.
Prx.Cor	Proximity Correspondence	Error category: specific to slide and rotation. Places dot on child's circle which is closest available space.
Transf. Cor.	Transformation Correspondence	Accurate Response Category: Places dot on child's circle such that it corresponds in sense, orientation of position, to dot on the interviewer's circle by a flip, rotation, or slide transformation respectively. Note

		that this strategy yields an accurate response for the flip, rotation, and slide transformations.
T/E	Trial/Error Placement	Apparent uncertainty about placement of part. Subject attempts several placements before settling on one selection.
I	Immediate Placement	Subject makes placement immediately.

B. Explanatory Strategies

These are strategies based on explanations made in response to the interviewer's question, "How did you pick that spot?" They may or may not be accompanied by observable behaviors (e.g., pointing, gestures).

NE	No Explanation	Child does not respond to interviewer's question or responds by saying, "I'm just guessing" or "I don't know."
Irrel.	Irrelevant Explanation	Child's response is ambiguous or unrelated to the interviewer's question. E.g., "I thought it was going to be wrong, but it wasn't."
Elim.*	Process of Elimination	Child says, "Only one left."
Transf.V.	Transformation Verbalization	Explanation indicates awareness of the transformational relation between two figures or refers to mental transformation of one of the figures.
Comp.	Comparison	Explanation indicates a comparison of one or more dots either between the child's and the interviewer's circles or within the child's circle.
Single	Single Dot Comparison	Explanation compares selected dot on child's circle with one dot on interviewer's circle. E.g., "It is in the same spot."
Order	Ordered Comparison	Explanation is based on the order relation among two or more dots within the child's circle, or between the child's and interviewer's circles.
2	Two Dot Comparison	Order relation between two dots.
Adjacent	Adjacent dots	Explanation compares two dots that are adjacent to each other on the child's circle and/or the interviewer's circle.
Across	Across dots	Explanation compares two dots that are across from each other on the child's circle and/or the interviewer's circle.

>2	More than Two-Dot Comparison	Order relation among more than two dots.
Adjacent	Adjacent dots	Three or more dots adjacent to each other on the child's circle and/or the interviewer's circle are compares. Explanation refers to either the "between relation" or the "perimeter relation" among the dots. E.g., "It is between the red and green dots", or "It goes like this (child finger traces semi-perimeter of circle)".
3 6	3 dots 6 dots	Specify: 3 dots compared (i.e., 1 "between rel. or 1 "semi-perimeter relation"). 6 dots compared (i.e., double "semi-perimeter relation").
Across	Across dots	Three or more dots across from each other on the child's circle and/or the interviewer's circle are compared. E.g., "It's across from these two."
3 6	3 dots 6 dots	Specify: 3 dots compared (i.e., 1 triangle). 6 dots compared (i.e., 2 different triangles).
Alter.	Alternative Choice	Explanation indicates that child's preferred choice for dot placement was blocked and, therefore, the nearest available spot was selected. E.g., "I wanted to put it in this position (7 o'clock) but I couldn't so I put it in the nearest spot not taken (10 o'clock).

*Note:

1) If elimination was first response and second response was spontaneously given or elicited by experimenter, drop elimination and code only second response.

2) If elimination was second response, drop it and code only first strategy.

IV. Combination of Motions Task

NR	No Response	SAB
HAP	Haphazard	SAB

Phase I: Disc Matching

The criteria for a disc matching response are: 1) the child ceases to manipulate the discs, or 2) the child indicates that all the discs "look the same". For every disc matching response, it is necessary to code both the dimension(s) on which the discs are matched and all the transformations performed in making the response. On the final disc matching response, note in the comments section whether the interviewer assisted the child in making the matching response, or whether the child acheived it without assistance from the interviewer.

SP.OR.	Spatial Orientation	Spatial orientation of discs as placed by child.
H	Horizontal	Discs are placed horizontally.
V	Vertical	Discs are placed vertically.
Disc M.	Disc Match	Discs are matched or not matched on the dimensions of line orientation and color.
M	Match	Discs are matched on both line orientation and color.
Match	Not Matched	Discs are not matched. Specify the dimension(s) on which they do not match:
Line	Line Orientation	Discs not matched by line orientation.
Color	Color	Discs not matched by color.
Line/ Color	Line Orientation and Color	Discs not matched by line orientation nor by color.
Disc T.	Disc Transformation	One or more of the following transformations are performed on the discs to achieve a disc matching response:
Flip	Flip	Flip
Rotation	Rotation	Rotation
Rearrange	Rearrange	Rearrange of the order of the discs.

Phase II and Phase III: Sign Matching Without Barrier (Phase II) and With Barrier (Phase III)

The criterion for a sign matching response is that it must be the first response that the child makes in attempting to match the mover to a sign. This criterion excludes self-correction responses and responses which follow assistance from the interviewer. If the first sign matching response is an error, note in the comments section whether the child spontaneously corrected the error or whether the interviewer assisted the child in correcting the error.

S Sign	Start Sign	Mover is placed at the start sign. Mover is matched or not matched to the start sign on the dimensions of line orientation and color.
M	Match	Mover is matched to start sign on both line orientation and color.
$\overline{\text{Match}}$	Not Matched	Mover is not matched to start sign. Specify the dimension(s) on which the mover does not match the start sign:
Line	Line Orientation	Mover not match sign by line orientation.
Color	Color	Mover not match sign by color.
Line/Color	Line Orientation and Color	Mover not match sign by line orientation nor color.
Route Signs	Route Signs	Mover passes route signs along path taken by the child. One or more planar transformations are performed on the mover in order to match it with a route sign.
SR	Slide Response	A slide transformation is performed on the mover in order to match it with a route sign. No other planar transformations are performed.
Slide Only	Slide-Only Signs	Route signs which require only a slide transformation of the mover in order to acheive a match.
>Slide	More-Than-Slide Signs	Route signs which require a slide transformation as well as one or two additional transformations (flip/and/or rotation) of the mover in order to acheive a match. Code with error category.
Anticipatory R.	Anticipatory Response	Two or more transformations are performed on the mover in order to match it with a route sign. These transformations are initiated before reaching the route sign. All anticipatory responses entail a slide transformation combined with a flip and/or rotation transformation(s). An anticipatory response only applies to slide route signs (see definition above).
Execution	Execution	Specify the point at which the anticipatory response is made. This category only applies to the flip and/or rotation transformation(s). The slide transformation is assumed, but not coded in this category.
Early	Early	In the path, but nearer to the preceding sign.
Middle	Middle	In the path midway between the preceding sign and the next sign.
Late	Late	In the path, but nearer to the next sign.

Rel.T.	Relevant Transformations	Specify the number of relevant transformations that are performed on the mover in order to match it with a route sign. A relevant transformation is defined as a transformation of the mover which is required in order toacheive a match with a particular route sign. This category only applies to the flip and/or rotation transformation is assumed, but not coded in this category.
All	All	All of the transformations relevant to a route sign.
Part	Part	Part of the transformations relevant to a route sign.
Contact R.	<u>Contact Response</u>	Two or more transformations are performed on the mover in order to match it with a route sign. These transformations are initiated on contact with a route sign. All contact responses entail a slide transformation combined with a flip and/or rotation transformation(s). A contact response only applies to slide route signs (see definition above).
Rel.T.	<u>Relevant Transformations</u>	Specify the number of relevant transformations that are performed on the mover in order to match it with a route sign. A <u>relevant</u> transformation is defined as a transformation of the mover which is required in order to achieve a match with a particular route sign. This category only applies to the flip and/or rotation transformation(s). The slide transformation is assumed, but not coded in this category.
All	All	All of the transformations relevant to a route sign.
Part	Part	Part of the transformations relevant to a route sign.
Error	<u>Error Response</u>	An error is defined as a failure to acheive a <u>complete</u> match between the mover and a route sign. Errors can be either relevant or irrelevant.
Relevant	Relevant Errors	Failure to perform all of the relevant transformations on the mover which are required to acheive a match with a route sign. These are errors of omission. Specify the transformation(s) <u>not</u> performed.
R	Rotation	
F	Flip	
R/F	Rotation and Flip	

Irrel.	Irrelevant Errors	Performance of one or more irrelevant transformations on the mover. An <u>irrelevant</u> transformation is defined as a transformation of the mover which is <u>not</u> required in order to acheive a match with a particular route sign. These are <u>errors of commission</u>. Specify the irrelevant transformation(s) performed:
R	Rotation	
F	Flip	
R/F	Rotation and Flip	
AMB.	<u>Ambiguous Responses</u>	

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Appendix 4

Supplementary Study I: Dimensions of the Four Sets of Geometric Shapes

(In inches)

Type and color of each set of shapes	Model and identical shapes	Selection shape similar to model but smaller	Selection shape similar to model but larger	Selection shape with at least one side equal to one of model
Right Triangle (Red)	6 3/8 x 5 x 4	6 x 4 3/4 x 3 3/4	6 3/4 x 5 1/4 x 4 1/4	6 3/8 x 5 3/16 x 3 3/4
Equilateral Triangle (Blue)	5 1/2 x 5 1/2 x 5 1/2	5 1/4 x 5 1/4 x 5 1/4	5 3/4 x 5 3/4 x 5 3/4	5 11/16 x 5 1/2 x 5 5/16
Square (Green)	5 x 5 x 5 x 5	4 3/4 x 4 3/4 x 4 3/4 x 4 3/4	5 1/4 x 5 1/4 x 5 1/4 x 5 1/4	5 x 5 x 5 x 5 (Parallelogram)
Quadrilaterl (Orange)	5 x 4 x 3 1/8 x 3 7/8	4 1/4 x 3 3/4 x 3 x 3 5/8	5 1/4 x 4 1/4 x 3 1/4 x 4 1/8	5 x 3 3/4 x 2 7/8 x 4 1/8