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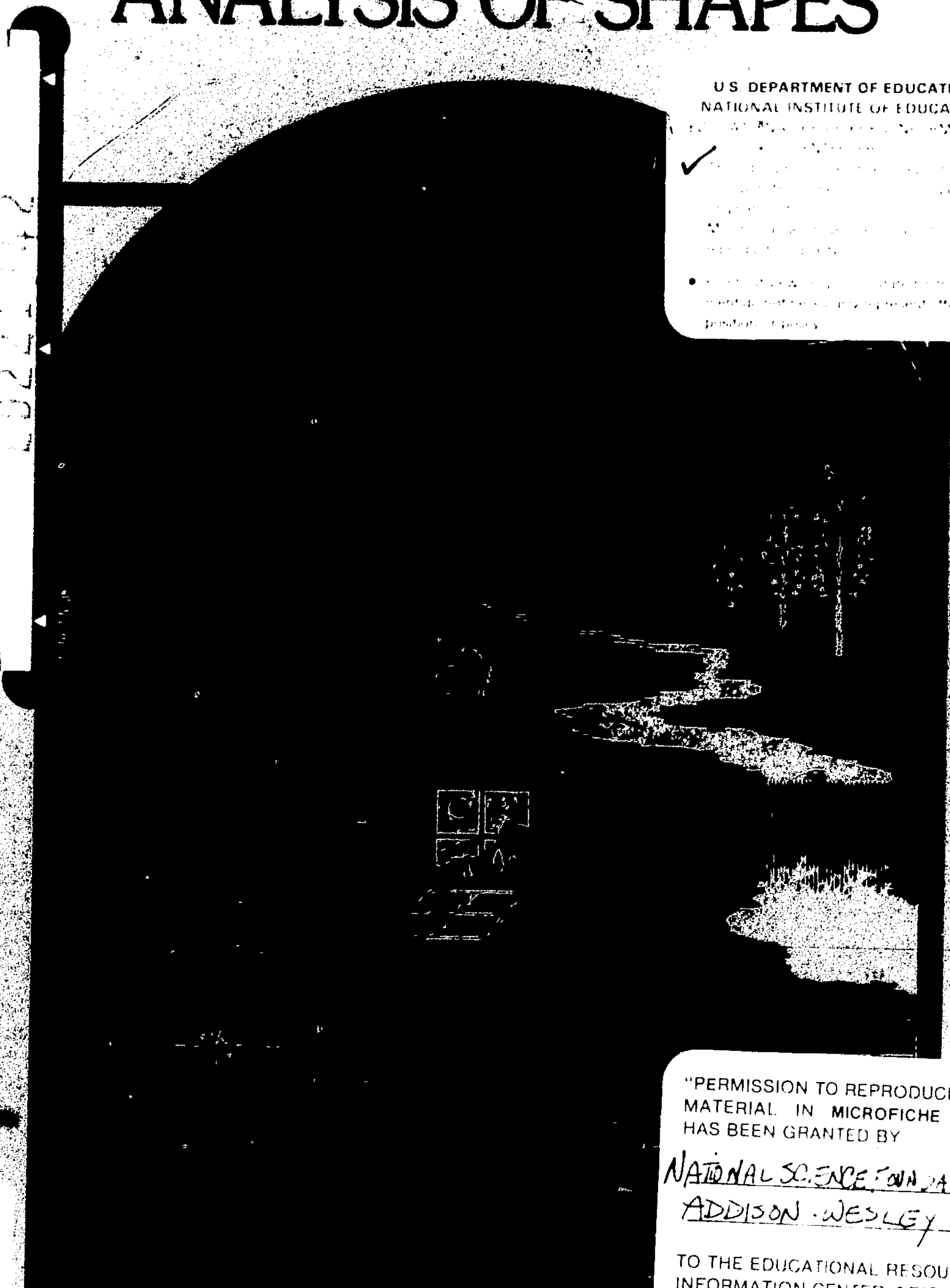
ABSTRACT

This unit is 1 of 12 developed for the university classroom portion of the Mathematics-Methods Program (MMP), created by the Indiana University Mathematics Education Development Center (MEDC) as an innovative program for the mathematics training of prospective elementary school teachers (PSTs). Each unit is written in an activity format that involves the PST in doing mathematics with an eye toward application of that mathematics in the elementary school. This document is one of four units that are devoted to geometry in the elementary school. In addition to an introduction to the geometry units of the MMP and to the shapes unit itself, the student text presents an overview and sections on straight lines, triangles, circles, and verification. The instructor's manual parallels the pupil text and provides information on the content of the unit, timetable suggestions, major questions, rules for materials preparation, comments and suggested procedure, and answers for problems in the student document. (MP)

ANALYSIS OF SHAPES



Mathematics-Methods Program
units written under the direction of

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MATHEMATICS EDUCATION DEVELOPMENT CENTER

INDIANA UNIVERSITY

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PREFACE

The Mathematics-Methods Program (MMP) has been developed by the Indiana University Mathematics Education Development Center (MEDC) during the years 1971-75. The development of the MMP was funded by the UPSTEP program of the National Science Foundation, with the goal of producing an innovative program for the mathematics training of prospective elementary school teachers (PSTs).

The primary features of the MMP are:

- It combines the mathematics training and the methods training of PSTs.
- It promotes a hands-on, laboratory approach to teaching in which PSTs learn mathematics and methods by doing rather than by listening, taking notes or memorizing.
- It involves the PST in using techniques and materials that are appropriate for use with children.
- It focuses on the real-world mathematical concerns of children and the real-world mathematical and pedagogical concerns of PSTs.

The MMP, as developed at the MEDC, involves a university classroom component and a related public school teaching component. The university classroom component combines the mathematics content courses and methods courses normally taken by PSTs, while the public school teaching component provides the PST with a chance to gain experience with children and insight into their mathematical thinking.

A model has been developed for the implementation of the public school teaching component of the MMP. Materials have been developed for the university classroom portion of the MMP. These include 12 instructional units with the following titles:

<u>Numeration</u>
<u>Addition and Subtraction</u>
<u>Multiplication and Division</u>
<u>Rational Numbers with Integers and Reals</u>
<u>Awareness Geometry</u>
<u>Transformational Geometry</u>
<u>Analysis of Shapes</u>
<u>Measurement</u>
<u>Number Theory</u>
<u>Probability and Statistics</u>
<u>Graphs: the Picturing of Information</u>
<u>Experiences in Problem Solving</u>

These units are written in an activity format that involves the PST in doing mathematics with an eye toward the application of that mathematics in the elementary school. The units are almost entirely independent of one another, and any selection of them can be done, in any order. It is worth noting that the first four units listed pertain to the basic number work in the elementary school; the second four to the geometry of the elementary school; and the final four to mathematical topics for the elementary teacher.

For purposes of formative evaluation and dissemination, the MMP has been field-tested at over 40 colleges and universities. The field implementation formats have varied widely. They include the following:

- Use in mathematics department as the mathematics content program, or as a portion of that program;
- Use in the education school as the methods program, or as a portion of that program,
- Combined mathematics content and methods program taught in

either the mathematics department, or the education school, or jointly;

- Any of the above, with or without the public school teaching experience.

Common to most of the field implementations was a small-group format for the university classroom experience and an emphasis on the use of concrete materials. The various centers that have implemented all or part of the MMP have made a number of suggestions for change, many of which are reflected in the final form of the program. It is fair to say that there has been a general feeling of satisfaction with, and enthusiasm for, MMP from those who have been involved in field-testing.

A list of the field-test centers of the MMP is as follows:

ALVIN JUNIOR COLLEGE
Alvin, Texas

BLUE MOUNTAIN COMMUNITY COLLEGE
Pendleton, Oregon

BOISE STATE UNIVERSITY
Boise, Idaho

BRIDGEWATER COLLEGE
Bridgewater, Virginia

CALIFORNIA STATE UNIVERSITY, CHICO

CALIFORNIA STATE UNIVERSITY, NORTHRIDGE

CLARKE COLLEGE
Dubuque, Iowa

UNIVERSITY OF COLORADO
Boulder, Colorado

UNIVERSITY OF COLORADO AT DENVER

CONCORDIA TEACHERS COLLEGE
River Forest, Illinois

GRAMBLING STATE UNIVERSITY
Grambling, Louisiana

ILLINOIS STATE UNIVERSITY
Normal, Illinois

INDIANA STATE UNIVERSITY EVANSVILLE

INDIANA STATE UNIVERSITY
Terre Haute, Indiana

INDIANA UNIVERSITY
Bloomington, Indiana

INDIANA UNIVERSITY NORTHWEST
Gary, Indiana

MACALESTER COLLEGE
St. Paul, Minnesota

UNIVERSITY OF MAINE AT FARMINGTON

UNIVERSITY OF MAINE AT PORTLAND-GORHAM

THE UNIVERSITY OF MANITOBA
Winnipeg, Manitoba, CANADA

MICHIGAN STATE UNIVERSITY
East Lansing, Michigan

UNIVERSITY OF NORTHERN IOWA
Cedar Falls, Iowa

NORTHERN MICHIGAN UNIVERSITY
Marquette, Michigan

NORTHWEST MISSOURI STATE UNIVERSITY
Maryville, Missouri

NORTHWESTERN UNIVERSITY
Evanston, Illinois

OAKLAND CITY COLLEGE
Oakland City, Indiana

UNIVERSITY OF OREGON
Eugene, Oregon

RHODE ISLAND COLLEGE
Providence, Rhode Island

SAINT XAVIER COLLEGE
Chicago, Illinois

SAN DIEGO STATE UNIVERSITY
San Diego, California

SAN FRANCISCO STATE UNIVERSITY
San Francisco, California

SHELBY STATE COMMUNITY COLLEGE
Memphis, Tennessee

UNIVERSITY OF SOUTHERN MISSISSIPPI
Hattiesburg, Mississippi

SYRACUSE UNIVERSITY
Syracuse, New York

TEXAS SOUTHERN UNIVERSITY
Houston, Texas

WALTERS STATE COMMUNITY COLLEGE
Morristown, Tennessee

WARTBURG COLLEGE
Waverly, Iowa

WESTERN MICHIGAN UNIVERSITY
Kalamazoo, Michigan

WHITTIER COLLEGE
Whittier, California

UNIVERSITY OF WISCONSIN--RIVER FALLS

UNIVERSITY OF WISCONSIN/STEVENS POINT

THE UNIVERSITY OF WYOMING
Laramie, Wyoming

CONTENTS

INTRODUCTION TO THE GEOMETRY UNITS OF THE MATHEMATICS-METHODS PROGRAM

Geometry to most people just means proving theorems about angles opposite equal sides, squares of hypotenuses, and such. This is natural since most people have their only exposure to geometry in high school where the traditional course has been built around such proofs. Geometry has been gradually working its way into the elementary school. Today's new textbooks contain a considerable amount of geometry.* Much of this material is being ignored or badly taught since many teachers see little relevance of this geometry to their own lives, to other aspects of the elementary school curriculum, or to the lives of their pupils. Moreover, some of the topics that are currently contained in textbooks were not taught when the teacher went to school and, therefore, are not fully understood by the teacher.

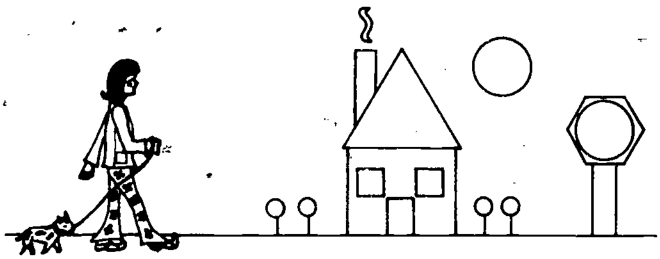
The geometry units of the Mathematics-Methods Program attempt to present geometry from a point of view that will bring out the potential for geometry with children. Geometry is presented as the <u>study of space experiences</u>. This point of view is not only consistent with the historical development of geometry, but it also keeps the focus on the relationship between geometry and the objects and shapes in our environment.

*Paul R. Trafton and John F. LeBlanc, "Informal Geometry in Grades K-6," in <u>The 36th Yearbook of the National Council of Teachers of Mathematics, 1973: Geometry in the Mathematics Curriculum</u>.

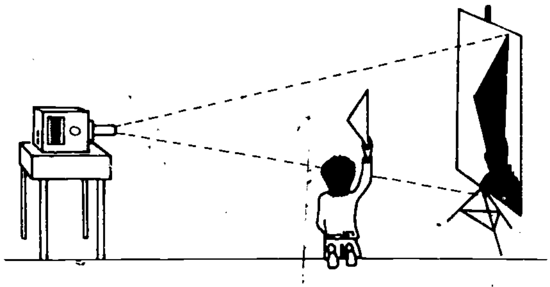
The study of space experiences addresses itself mainly to shapes. Shapes are abstractions from the environment. They can be informally investigated and analyzed. One can also study the changes (or transformations) that shapes undergo.

To effect this study of space experiences, four units have been developed.

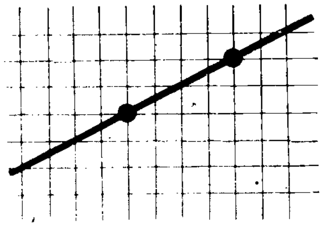
- The Awareness Geometry unit is designed to orient the prospective teacher to the informal study of geometry. In this unit one looks carefully at the environment, experiments with shapes that are observed there, and informally analyzes certain shapes. At the end of the unit, one is given experience with planning for geometry lessons with children.



- The Transformational Geometry unit studies changes that shapes can undergo. The unit is organized into the study of rigid transformations, projective transformations, and topological transformations. The presentation is informal and the focus is on concrete real-world examples of the concepts.

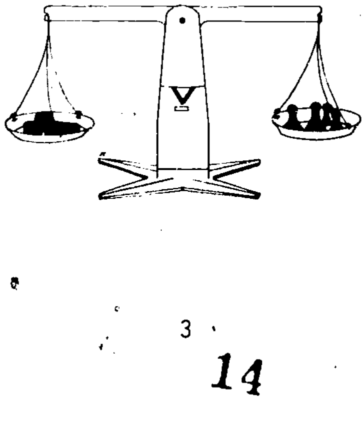


• The Analysis of Shapes unit studies straight lines, triangles and circles. The real-world occurrences and importance of each shape are investigated; each shape is informally analyzed to determine some of its important properties; and then the fruits of these analyses are applied to real-world problems. Many of the traditional topics of Euclidean geometry, including coordinate geometry, are considered here from a nontraditional point of view. There is also a section which deals with problems of verification and places into perspective the informal methods of elementary school geometry and the formal approach to high school geometry.



$$\frac{\text{up}}{\text{out}} = \frac{2}{4} = \frac{1}{2}$$

• The Measurement unit provides experiences with identifying attributes, choosing unit quantities of attributes, and determining numbers through comparisons. The emphasis is on informal, concrete, conceptual activities. There is a separate section which is devoted to child readiness and the planning of measurement activities for children. Metric units are used throughout. While measurement could have been included in the Analysis of Shapes unit, it has been placed in a separate unit because of its importance in the elementary school curriculum and in order to provide flexibility in the use of the units.

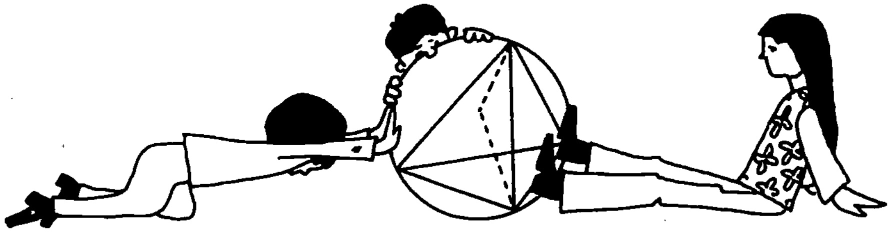


The four geometry units of the Mathematics-Methods Program are independent of one another. Any number of them can be used in any order. They can be used in a separate geometry course; they can be interspersed among other units of the Mathematics-Methods Program; or they can be used in conjunction with other materials.

These geometry units, like the other units of the Mathematics-Methods Program, involve one as an adult learner in activities which have implications for teaching children. One works with concepts that children might learn, with materials that children might use, and on activities that might be modified for use with children. The objective is to provide growth in understanding and enjoyment of geometry along with increased ability and desire to teach geometry to children.

INTRODUCTION TO THE ANALYSIS OF SHAPES UNIT

Geometry is the study of space experiences. In the "Introduction to the Geometry Units of the Mathematics-Methods Program," different aspects of the study of space experiences were outlined, namely awareness geometry, transformational geometry, measurement, and the analysis of shapes. In the Analysis of Shapes unit you will encounter many topics that are studied in high school geometry courses. The procedures and point of view are, however, quite different from those in high school geometry. The procedures are informal and experimental, contrasting with the formal theorem-proof format of most high school geometry courses. The point of view is one that attempts to provide motivation and direction to the analysis of shapes by relating it to the real-world occurrences of shapes and to the real-world applications of the analysis of the shapes.



After the Overview, the unit has four sections. Each of the first three sections contains the analysis of a single shape (straight line, triangle, and circle), and the fourth section explores the topic of verification--trying to put the informal procedures of the unit into context with the formal procedures of high school geometry.

OVERVIEW

FOCUS:

The geometry in this unit appears in many elementary school mathematics texts, and varying amounts of this material are taught in elementary school classrooms. This activity presents an overview of the role of the content of this unit in the elementary school curriculum.

MATERIALS:

(Optional). The Mathematics-Methods Program slide-tape overview entitled "Analysis of Shapes."

DIRECTIONS:

Do one of (1) or (3) below and then discuss the questions in (2).

1. View the slide-tape overview entitled "Analysis of Shapes."

2. Discuss the following questions:

   a) What do you recall from high school geometry concerning straight lines, triangles, and circles? Are you surprised at the level of elementary school geometry discussed in the overview?

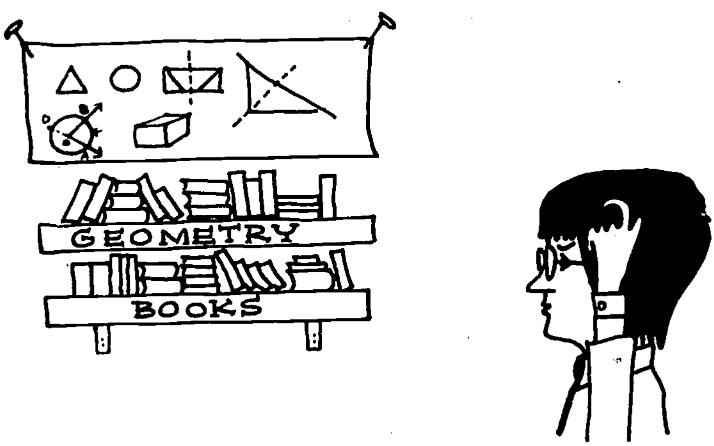
   b) Is geometry useful? If so, for what? How would you compare the importance of geometry with that of arithmetic? Would the parents of an elementary school child be likely to agree with you? Do you see any possible development that might change the relative importance of arithmetic and geometry?

   c) In your school experiences with geometry, has there been any emphasis placed on the occurrences of shapes or on the application of geometry learnings to real-world problems?

3. Read the short essay which follows. The point of this brief essay is to provide you with some idea of how the content of this unit relates to your goal of becoming an elementary school teacher.

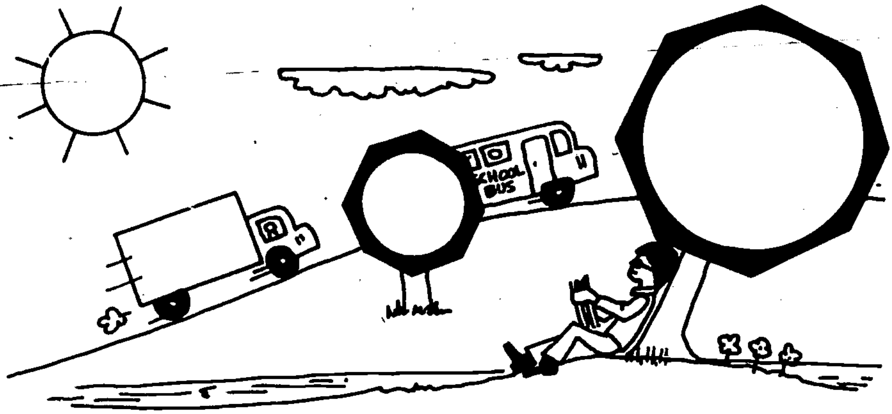
AN OVERVIEW OF
THE ANALYSIS OF SHAPES IN THE ELEMENTARY SCHOOL

Geometry is unlike arithmetic. Most people agree that both subjects are important. With arithmetic they not only agree that addition, subtraction, multiplication, and division should be taught; they even agree, to some extent, on when and how they should be taught. On the other hand, with geometry there are many different points of view as to how much, when, and why it should be taught.



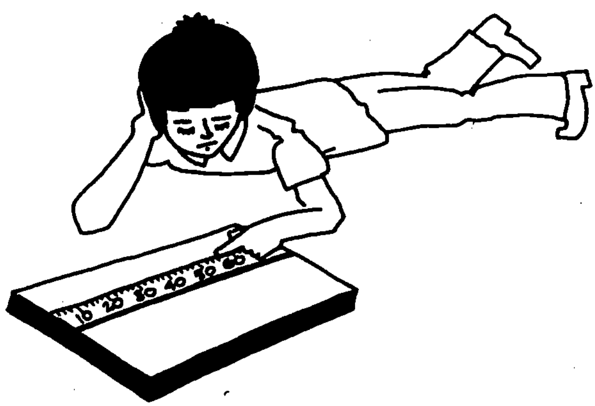
Geometry is on the upswing in the elementary school. Despite the lack of consensus over goals and procedures, more and more geometry is making its way into the elementary curriculum. It is not clear what the future will bring, but you will need to be prepared to teach it, whatever it is.

Current trends include increased emphasis on the relevance of geometry to the environment. Instead of just being presented with a shape to study, children abstract shapes from the shapes of objects around them.



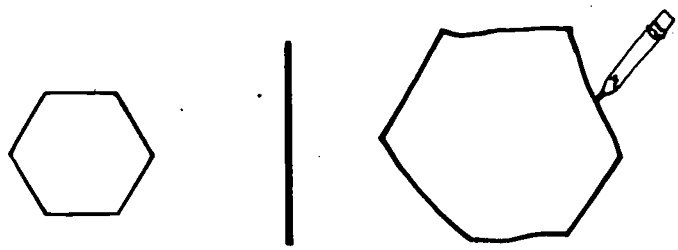
Another recent development is the appearance of transformational geometry in the curriculum. Some current texts include a few explorations into rigid transformations (changes of location, not size or shape). Other geometry material can be studied from a transformational point of view and may be, in the future.

The U.S. changeover to the metric system of measurement units has brought considerable recent emphasis to the topic of measurement in the elementary school. While this emphasis may abate, the utility of measurement as well as the potential for concrete child-centered measurement activities will assure it of a place in the curriculum.

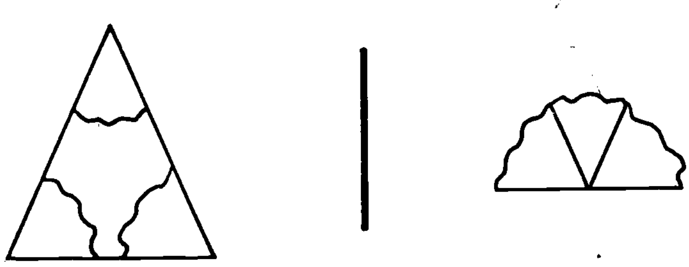


It may surprise you that, in spite of the new topics and new emphases, a large portion of the elementary geometry curriculum consists of the analysis of shapes. In this analysis the same topics are treated informally that are treated formally in high school geometry courses. Before we see what some of these topics are, it is worth noting that because texts vary so widely as to geometry content, and because treatments of these texts by individual teachers vary so widely, we can only indicate here a few of the geometry experiences that children <u>might</u> have during their schooling.

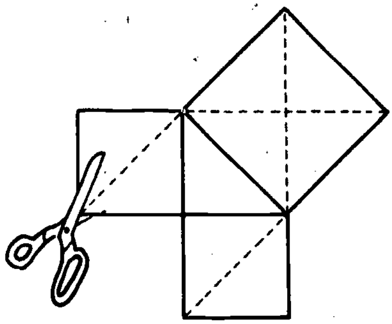
After some early awareness activities, in which children learn to recognize and name such shapes as straight lines, triangles, rectangles, and circles, children may get some very informal experience with the concept of similarity. For example, they could be asked to choose or to draw figures which have the same shape as one that they are given.



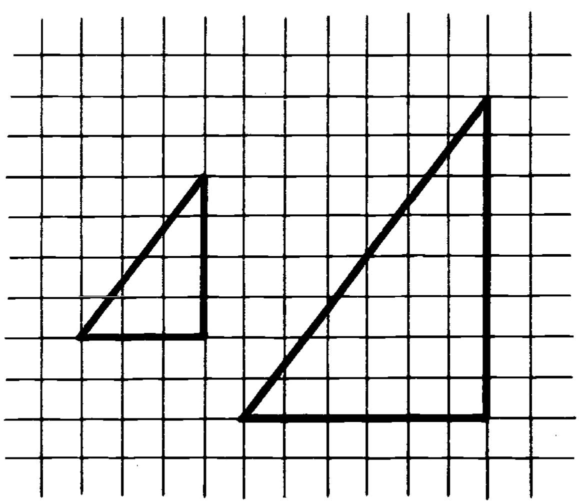
You may remember from high school that the sum of the measures of the angles of any triangle is 180°. Young children certainly are not up to a proof of this fact, but they can experiment with the angles of paper triangles to see that they can be pieced together to form a straight angle.



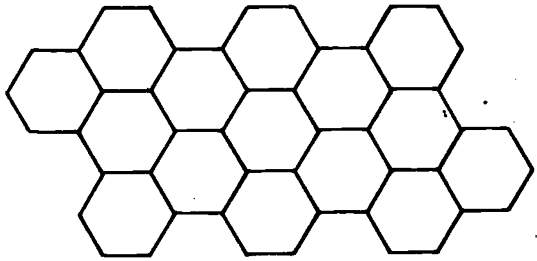
Do you remember the theorem about the square of the hypotenuse equaling the sum of the squares of the other two sides (the Pythagorean Theorem)? Children in the middle grades may be introduced to the Pythagorean Theorem (again with cutouts).



They may also study the angles that result from intersecting lines (vertical angles) and the location of points using ordered pairs of numbers (coordinate geometry). In the upper grades, some of the earlier topics may be revisited and refined. For example, scale drawing provides a nice vehicle for studying the relationship between equivalent ratios and similar shapes.



The numerical aspects of the Pythagorean Theorem can also be developed, and elementary school children can even get into graphing simple functions. Some texts are also beginning to study the tessellation (or tiling) property that some shapes have.

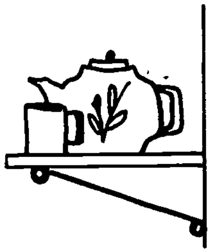


From even this brief sample you can see that there is quite a bit of geometry in the elementary curriculum. One serious concern is that this geometry be presented in a connected and coherent manner, so that the child develops competency and a sense of direction. The lack of this coherence in the sample above is probably representative of the experiences that most children have with geometry. This contrasts, by the way, with the arithmetic sequence, which tends to be very carefully developed and very meticulously followed by the teacher.

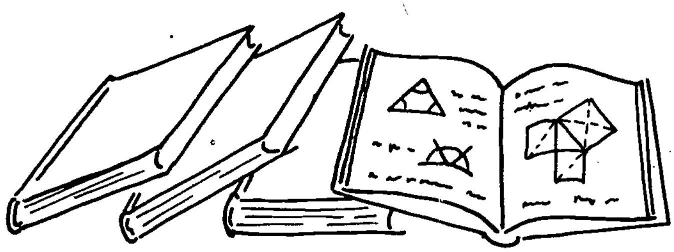
Since geometry is the study of real-world shapes, and since the analysis of these shapes has real-world applications, it seems appropriate to take advantage of this real-world relevance to give coherence and direction to the analysis of shapes. With each shape one can

- Observe occurrences and uses of the shape in the real world;
- Analyze the shape;
- Apply the analysis to real-world problems.

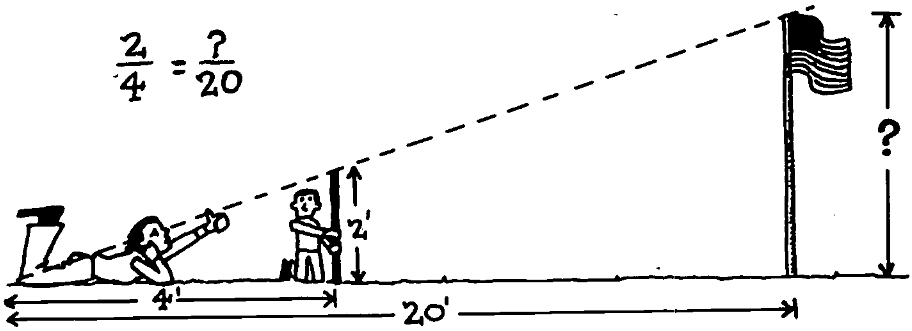
For example, with triangles one can observe their occurrences as sources of structural stability.



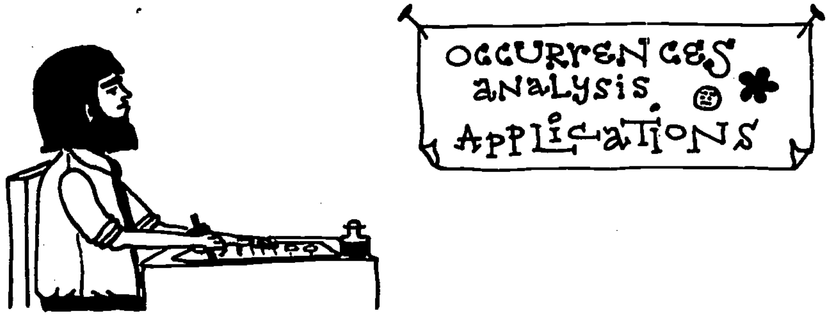
One can then analyze triangles with respect to sides, angles, etc.



Then one can apply this analysis to such real-world problems as finding distances, stabilizing structures, and constructing square corners.



This same sequence (occurrences, analysis, applications) can be followed for straight lines, circles--in fact, for any shape one wishes to study. The teacher who keeps these steps in mind in developing geometry lessons for children will be more likely to provide geometry instruction that has coherence and direction.



Section I

STRAIGHT LINES

Straight lines are among the most basic shapes to be analyzed. In this section, lines and the relationships between lines will be analyzed. In Activity 1 occurrences of straight lines in the real world will be presented. Then in Activities 2, 3, 4, and 5 you will engage in the analysis of straight lines. The analysis includes the study of the angles between intersecting lines, straightedge-and-compass constructions, and coordinate geometry. Finally in Activity 6 you will apply your analysis of straight lines to some real-world problems.

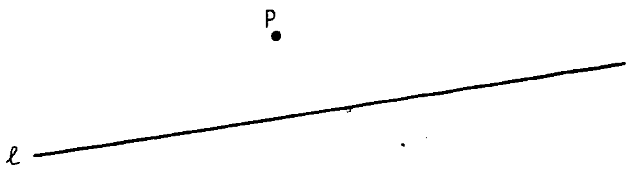
To help you keep track of what you are doing, each activity in this section and the next two will be introduced by a small matrix like the following one

	/	△	○
Occur.			
Anly.			
Appl.			

"Occur.," "Anly.," "Appl." are abbreviations for "occurrences," "analysis," and "applications," to indicate what the focus of the activity is. The symbols across the top designate whether the activity is concerned with straight lines, triangles, or circles. The black square in the matrix on the preceding page indicates that the activity is concerned with applications of circles.

MAJOR QUESTIONS

As a result of Section I you should be able to provide adequate answers to the following questions.

1. 

   Use a straightedge and compass to construct a straight line through point P parallel to the line ℓ.

2. Graph the pairs (x, y) that are solutions to the equation $2x - y = 4$.

3. Explain what straight lines and the great circles on a sphere have in common.

ACTIVITY 1
STRAIGHTNESS

FOCUS:

Your life is full of straight lines. In this activity, you will be asked to identify how and why straight lines occur.

MATERIALS:

Globe.

DISCUSSION:

The cable between a tow truck and a car is straight because it is pulled taut by forces acting on it. The pathway worn in the grass across a school yard is straight because it provides the shortest possible route for students to get to where they want to go. The edges of many buildings are straight because they are the intersections of the plane sides of the buildings. Straight lines may be used by an artist, e.g., for separation, direction, and balance. Windflow, congruence, momentum, and many other phenomena inspire the use of straightness. As you look around and think, you will be able to determine why many lines are straight. Others may be straight for no reason, by chance, or just by tradition.

DIRECTIONS:

1. Identify as many occurrences of straight lines in your life as possible. For each occurrence:

   a) Describe the occurrence;

   b) Analyze why you think the line is straight.

2. Try to describe a straight line to a classmate without relating to physical objects.

3. Briefly list those attributes of a straight line which distinguish it from a curved line. In what situations is each most useful? For example, which is easier to measure and why?

4. *OPTIONAL: Read* ***The Dot and the Line: A Romance in Lower Mathematics****, by Norton Juster, published by Random House in 1963.*

5. Discuss a strategy that you would use to help young children learn what "straight" means.

6. What attributes of a straight line is Johnny Cash evoking when he sings, "Because you're mine, I'll walk the line"?

7. If you were walking on the equator, you would think that you were walking a straight line. Experiment with a globe to determine which curves on the globe share which properties of straight lines. For example, on a sphere what are the analogues to parallel lines?

ACTIVITY 2
STRAIGHT LINES AND THEIR INTERSECTIONS

	/	△	O
Occur.			
Anly.	■		
Appl.			

FOCUS:

Having studied the <u>occurrences</u> of straight lines, you will now have an opportunity to <u>analyze</u> straight lines. In particular, you will be concerned with how straight lines interact with each other to form geometric shapes.

DIRECTIONS:

1. Two (infinitely extended) straight lines can have four possible relationships with each other in space.

    a) Describe each of the four relationships.

    b) With pencil and paper illustrate the relationships described in (a).

    c) How many different planes could contain a single line? Given two lines, how many different planes could contain both of them? (You may want to consider different cases for the two lines.)

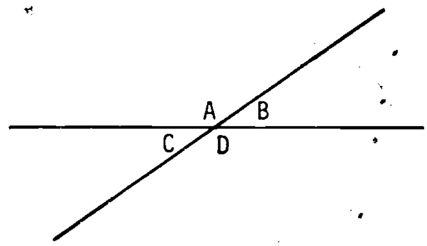
    d) Which of the relationships described in (a) can hold in a (two-dimensional) plane?

    e) How would you describe a point? Could you use two straight lines in your description? How many different straight lines can pass through one point? How many through two points? What is the situation with three points?

2. When two straight lines intersect in a point, several angles are formed. Are all of the angles always the same? Are all of the angles sometimes the same? Are some of the angles always the same? Are some of the angles sometimes the same? Experiment with some straight lines in order to arrive at answers to these questions. Illustrate your answers with examples.

3. Solve each of the following problems.

a) Two straight lines intersect in such a way that the measure of angle A is 145°. Find the measures of angles B, C, and D. Use reasoning rather than a protractor to arrive at your answer.



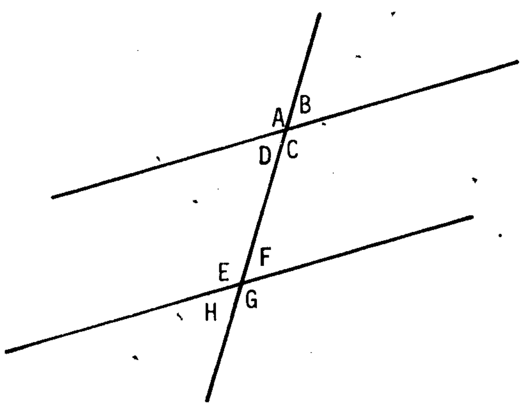
Measure of angle A = 145°

Measure of angle B =

Measure of angle C =

Measure of angle D =

b) Two parallel lines are intersected by a transversal in such a way that the measure of angle F is 58°. Find the measures of the other lettered angles.



Measure of angle A =

Measure of angle B =

Measure of angle C =

Measure of angle D =

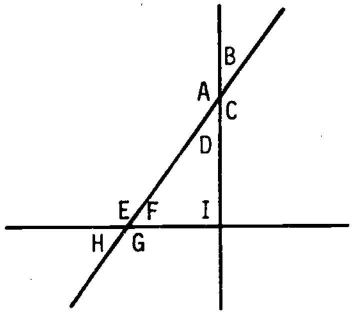
Measure of angle E =

Measure of angle F = 58°

Measure of angle G =

Measure of angle H =

c) Two perpendicular lines are intersected by a transversal in such a way that the measure of angle D is 36°. Find the measures of the other lettered angles.



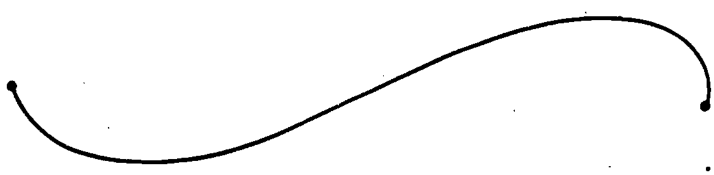
Measure of angle A =	Measure of angle F =
Measure of angle B =	Measure of angle G =
Measure of angle C =	Measure of angle H =
Measure of angle D = 36°	Measure of angle I =
Measure of angle E =	

4. So far in this activity we have been analyzing relationships between infinitely extended straight lines. Most of the straight lines that occur in your life are finite and are often referred to as line segments. Describe and illustrate the possible relationships between two line segments in space. Check around to make sure that you haven't overlooked any.

5. Length is an attribute of line segments.

   a) How would you describe length to a first-grader?

   b) How might you use the concept of length of line segments to determine an estimate for the length of the curve below?



   c) Describe as many different ways as you can think of to determine which of two line segments is longer.

ACTIVITY 3
STRAIGHTEDGE-AND-COMPASS CONSTRUCTION

	/	△	O
Occur.			
Anly.			
Appl.			

FOCUS:

You will be introduced to straightedge-and-compass constructions of various shapes which consist of straight lines. These constructions will not only play a role in the analysis of straight lines, but they will also be useful in the analysis of triangles and circles.

MATERIALS:

Straightedge, compass, and current elementary mathematics text series.

DISCUSSION:

As a practical matter, one sometimes wants to reproduce accurately a certain geometric shape. A straightedge and compass can be convenient for doing this. Straightedge-and-compass constructions are also involved in many of the classical problems of Euclidean geometry and are studied in high school geometry courses. Moreover, straightedge-and-compass constructions can be fun for children, since they involve them in actually doing something and since they can produce an attractive and accurate result if some care is exercised. In this activity you will be introduced to straightedge-and-compass construction and will have an opportunity to practice it on shapes which consist of straight lines.

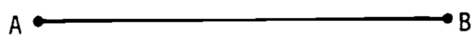
Straightedge-and-compass construction is a game* which has historical, theoretical, and practical aspects. The following are the rules of the game:

a) You can use the compass to make an arc of a circle.

*We use the term "game" since the rules are somewhat arbitrary and restrictive. For example, you are not allowed to use a ruler, i.e., a straightedge with length marks.

b) You can use the straightedge to draw an "infinite" line or to join points. (No measuring or marking on the straightedge is allowed.)

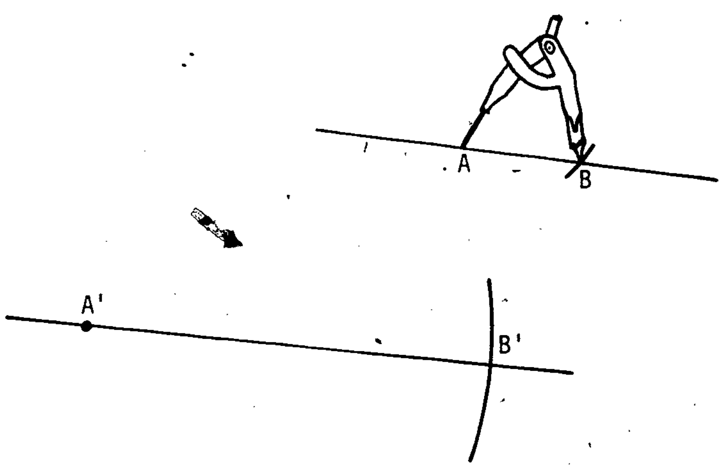
As a comment, we note that points are made by intersecting arcs, by intersecting arcs with lines, or by intersecting lines. For example, suppose that you wanted to construct a line segment <u>congruent to</u> (same size and shape as) AB.



There are several ways to proceed. Here is one.

Step 1: Draw a line with your straightedge.

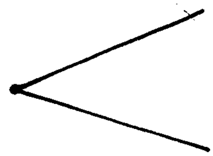
Step 2: Pick any point on this line and label it "A'." Starting with point A', make an arc with radius equal to the length of the line segment AB. Call a point where the arc intersects the line B'. Then you have constructed A'B' congruent to AB.



DIRECTIONS:

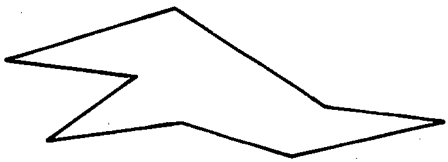
Use a straightedge and compass to do each of the following.

1. Accurately reproduce this angle. Check your results by laying your constructed angle on top of this printed angle.



2. Construct the bisector of the angle that you constructed in (1). (The bisector of an angle is a line that separates the angle into two congruent angles.)

3. *OPTIONAL: Accurately reproduce this shape. (Check your construction.)*



4. Construct two perpendicular lines.

5. Construct two parallel lines.

6. Choose a current elementary mathematics text series and determine how and at what level straightedge-and-compass constructions are introduced.

ACTIVITY 4

POINTS AND NUMBER PAIRS

	/	Δ	O
Occur.			
Anly.	■		
Appl.			

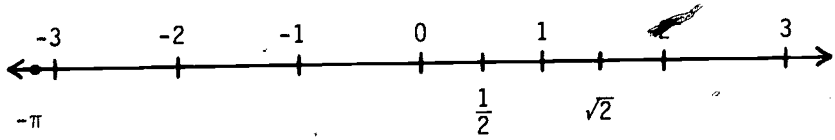
FOCUS:

In order to bring numbers and, eventually, algebra into your analysis of straight lines, you will need to have skill with locating geometric points by means of number pairs. This activity will serve as an introduction to or a review of that skill.

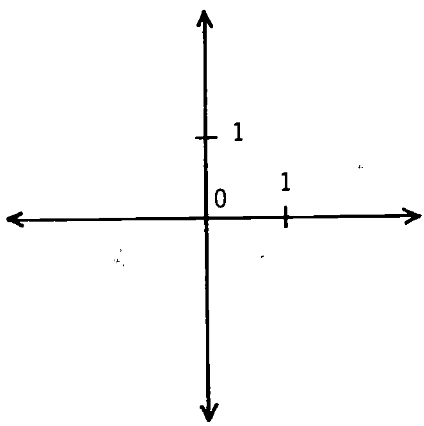
MATERIALS:

Graph paper.

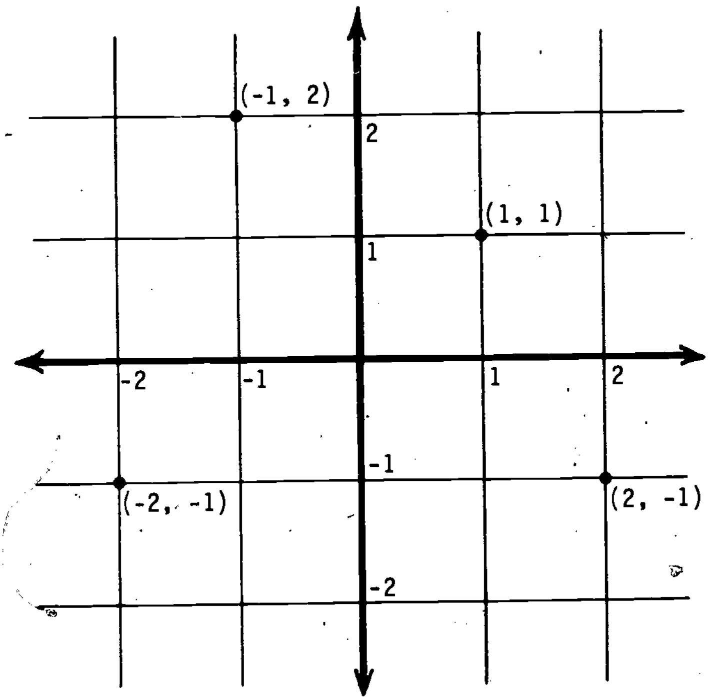
DISCUSSION:



You should be familiar with the number line as pictured above. Every point on that line can be designated with a real number, and every real number designates a point on the line. Once the locations of 0 and 1 have been chosen, the location of every other real number is determined.



Two such number lines that are perpendicular to each other and that intersect at their zero points can be used to locate points in a plane (not just on one line). We will illustrate the procedure for doing this with examples.

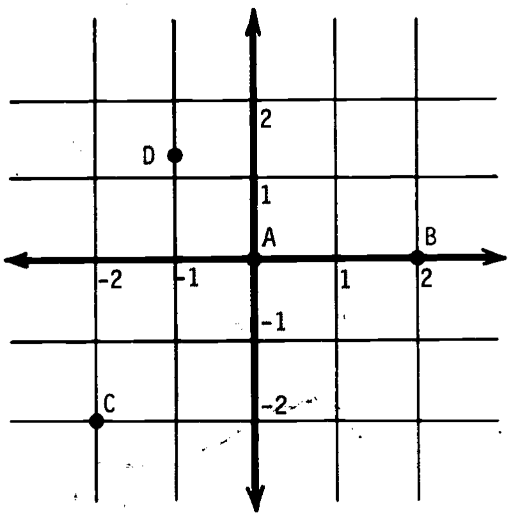


The point (1, 1) is located by traveling to 1 on the horizontal number line (x-axis) and then traveling up (parallel to the y-axis) one unit. To get to the point (-1, 2) one travels to the left on the x-axis to -1 and then up parallel to the y-axis two units. Notice that the point (2, -1) is in a different location, since the first number (x-coordinate) locates the point relative to the x-axis, and the second number (y-coordinate) locates the point relative to the y-axis.

DIRECTIONS:

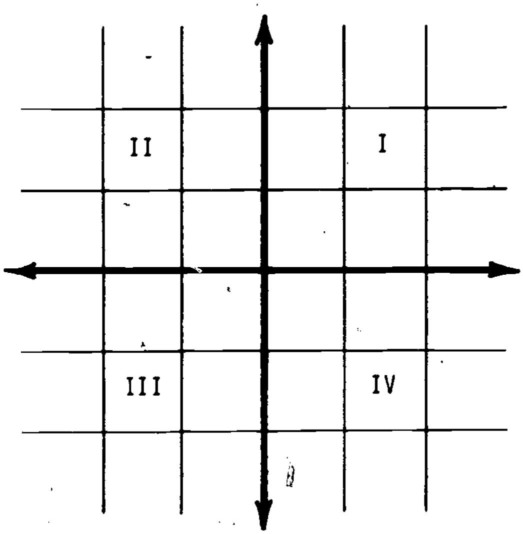
1. Draw coordinate axes (x-axis and y-axis) on some graph paper and locate the points which correspond to the following pairs of numbers: (2, 2), $(-\frac{3}{2}, 1)$, $(-\frac{1}{2}, -\sqrt{2})$, (3, -1).

2.



Estimate what you think the x- and y-coordinates of A, B, C, and D are.

3.



Coordinate axes divide the plane into four quadrants, which have been labeled I, II, III, and IV above.

a) In which quadrant are both coordinates always negative?

b) In which are both always positive?

c) In which quadrant is the first coordinate always positive and the second always negative?

4. The intersections of two walls of your room with the floor can be used as coordinate axes. One corner can be used as (0,0). Your pace can be used as the unit of length.

   a) Locate several of your classmates in the room by means of ordered pairs.

   b) Have someone close his or her eyes. When he opens them again, give him the ordered pair that locates a classmate, and see if he can identify the classmate.

5. The houses in most towns, the seats in a theater, or the rooms in a school building are located according to a coordinate system of sorts. In each of those cases, try to see the analogy with the coordinate system discussed here.

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ACTIVITY 5

EQUATIONS AND LINES

	/	△	○
Occur.			
Anly.	■		
Appl.			

FOCUS:

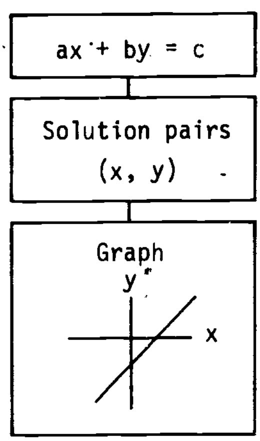
In this activity the relationship between a family of very simple equations (linear equations) and a family of very simple geometric shapes (straight lines) will be established. This relationship will be developed further in Activity 14, once the concept of similar triangles has been established.

MATERIALS:

Graph paper, straightedge and current elementary mathematics text series.

DISCUSSION:

At first glance, algebra and geometry may seem to be unrelated. It is to the benefit of each that they are <u>not</u> unrelated. In this activity you will see that if one has an equation in x and y--for example, $x + y = 2$, $2x - y = 0$, or $-x + y = 0$, the solutions to such an equation are number pairs (x, y). All of the number pairs that are solutions to one of these equations form a nice pattern (or graph), namely, a straight line.



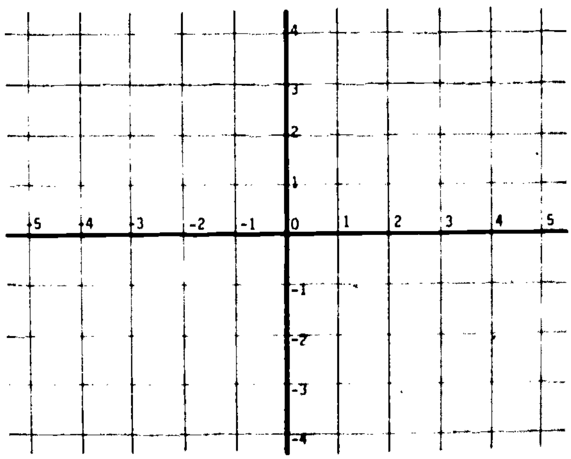
Once such a relationship between geometry and algebra has been established, one can use the various rules, theorems, and manipulations from algebra to analyze geometric shapes, and one can use geometric shapes to illustrate certain algebraic relationships.

DIRECTIONS:

1. Let's start with the equation $2x + y = 1$. Fill in the table below and the corresponding list of ordered pairs of solutions to the equation. (There are infinitely many possible pairs. You are free to choose any solution pairs that you like.)

x	y	
1	-1	(1, -1)
		(,)
		(,)
		(,)
		(,)

Locate on the set of coordinate axes below the points that correspond to the ordered pairs. Do the points fall on a straight line? If not, check your work, because they should.



2. Repeat the procedure in (1) with:

   a) $x - y = 0$;

   b) $2x + 3y =$

   Draw the graphs on graph paper.

3. a) If you have not already done so, draw the straight-line graph for each of the three tables that you prepared in (1) and (2).

   b) There are infinitely many points on each of the straight lines that you have drawn, even though you have labeled only a finite number of them. Does each of these points correspond to a pair of numbers which solve the equation? To get a feeling for the answer to this question, find solution pairs for which $x = \frac{1}{2}$, $x = 2$, $x = \frac{17}{11}$, for each of the equations in (1) and (2). Could you find a solution pair no matter which x you choose? (Try a little algebra.)

4. Show that each of the three equations above is of the form $ax + by = c$. That is, name the a, b, and c for each equation.

....

What we are hinting at here is the following statement (which is true): <u>The solution pairs of any equation of the form $ax + by = c$ make a straight line when they are graphed</u>.

....

5. Find an equation of a straight line which goes through the points (0, 1) and (-1, 0). Hint: Look for one of the form $ax + by = c$.

6. Two points determine a straight line. Use this fact and what you know about the graph of $ax + by = c$ to quickly graph each of the following.

   a) $4x + 3y = 9$

   b) $x - y = 1$

   c) $x = 3$ (You want <u>all</u> (x, y) where $x = 3$.)

   d) $y = 3$ (You want <u>all</u> (x, y) where $y = 3$.)

7. Choose a current elementary text series to see if, how and at what level linear equations are introduced.

ACTIVITY 6

PLAYING IT STRAIGHT

	/	△	○
Occur.			
Anly.			
Appl.			

FOCUS:

Now that you have identified instances of straightness and have had an opportunity to analyze a few properties of straight lines, you can consider a few simple ways in which your knowledge about straight lines affects your decisions and actions.

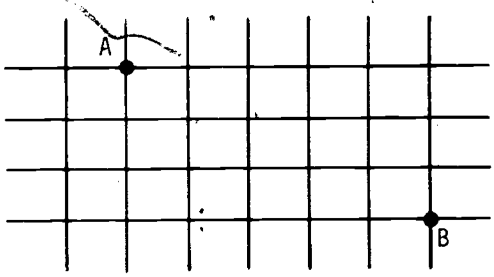
DIRECTIONS:

1. If you want to nail a board to your wall, what is the smallest number of nails that you can use and still be sure that it is secure? Which of the facts studied in Activity 1 helped you make your decision?

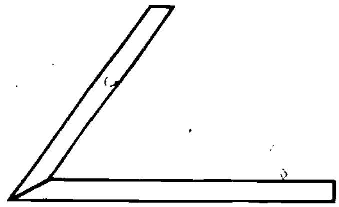
2. The following trivial application of knowledge about straight lines is frequently ignored by landscape architects. How would you determine the location and shape of walks to be placed in a heavily traveled area in order to avoid having people walk on the grass?

3. Is it possible for the intersection of two straight roads to form two 45-degree angles and two 110-degree angles?

4. Find the shortest route from A to B following the city streets in the diagram below. Is there only one? What would the "city" equivalent of a straight line be?



5. Suppose that you are a carpenter and that you are following blueprints in the construction of a home. The blueprints show two walls meeting in the angle pictured below. Devise a strategy for determining whether or not your walls actually do meet in the pictured angle. See if you can come up with several different ways of being sure.



6. If your sixth-grade class had a football game coming up and wanted to mark off a football field, how would you advise the kids to make sure that the opposite sides of their field are parallel?

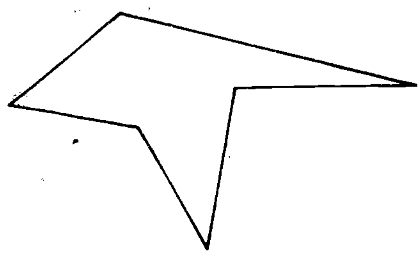
Section II

TRIANGLES

Triangles are simple shapes, and they are important in the real world as well as in the study of geometry. In Activity 7, occurrences of triangles will be studied. In subsequent activities (8, 9, 11, 12, 13) the angles and sides of triangles and their relationships are analyzed. In Activity 10, the Pythagorean Theorem is applied to the concept of length. In Activity 14, the equivalent ratios property of similar triangles is applied, to aid in finding the equation of a straight line, given two points on the line. Then in Activity 15 other applications of triangle learnings are investigated.

MAJOR QUESTIONS

1. Find the sum of the angles of the polygon below without measuring the angles.



2. Below are some sets of three lengths each. Indicate which could be the sides of triangles. Among those, indicate which could be the sides of <u>right</u> triangles, and identify any pairs of <u>similar</u> triangles.

1 cm, 3 cm, 1 cm	6 cm, 10 cm, 8 cm
1 cm, 2 cm, 3 cm	5 cm, 3 cm, 4 cm
4 cm, 4 cm, 5 cm	2 cm, 2 cm, 2.5 cm

3. Discuss the stability of triangles giving attention to:

   a) Its application in the real world;

   b) The mathematical reasons for it. (I.e., using theorems, give an argument which shows that triangles are stable.)

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ACTIVITY 7

THE IMPORTANCE OF TRIANGLES

	/	△	O
Occur.		■	
Anly.			
Appl.			

FOCUS:

In Activities 1, 2, and 3, straight lines were studied. First some of the occurrences and properties of straight lines were noted. Then straight lines and line segments were analyzed in order to determine certain relationships. Finally, some attention was given to the use that you might make of what you had learned about straight lines. In the next activities, you will go through a similar pattern with triangles.

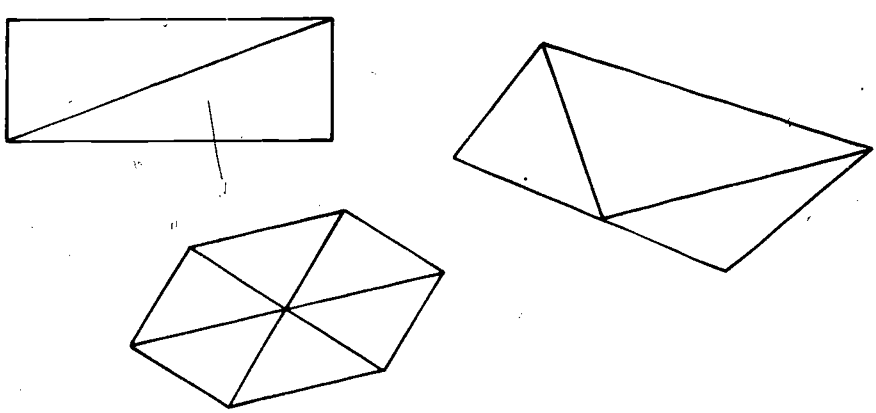
This activity focuses on the occurrences and properties of triangles. In other words, it addresses itself to the question of why triangles are worth studying.

MATERIALS:

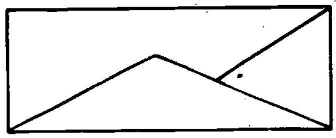
Paper, ruler, scissors, construction paper strips (at least twelve 1-inch x 8-inch strips per student).

DIRECTIONS:

1. Each of the following geometric shapes has been triangulated.

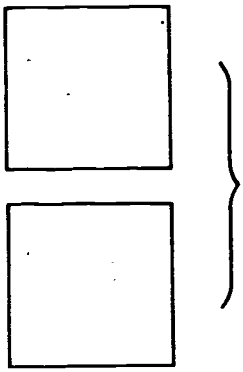


This shape has not been triangulated.

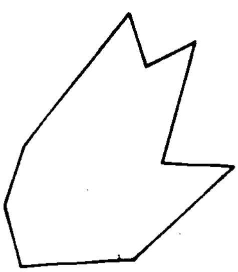


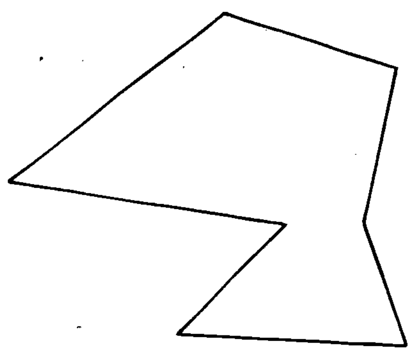
Do you see what is meant by "triangulate"?

a) Triangulate each of the following geometric shapes:



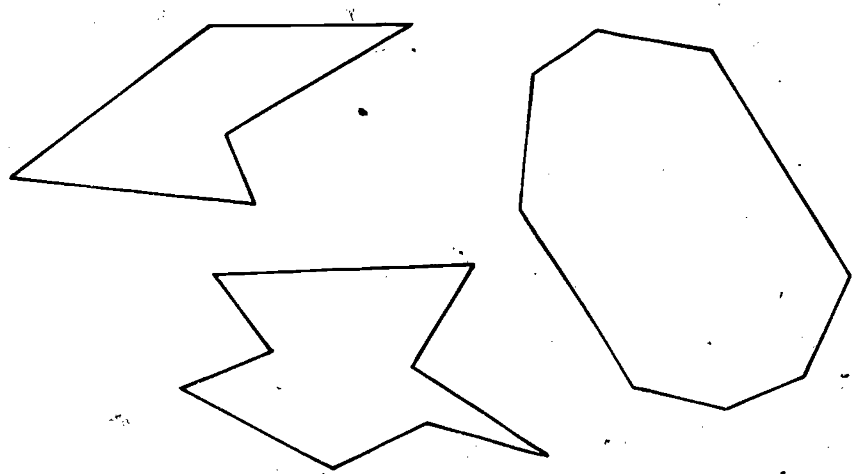
Triangulate in two different ways.



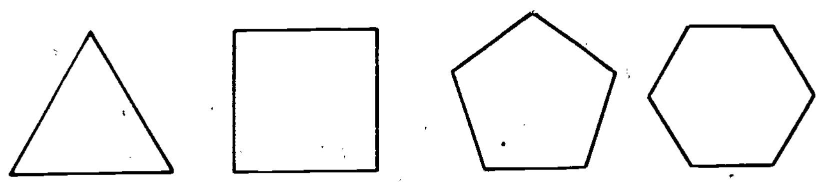


b) Can you give some examples of shapes that cannot be triangulated?

c) You may recall that the sum of the angles of a triangle is 180°. Use this fact and your newly acquired skill of triangulation to find the sum of the angles of the following shapes.

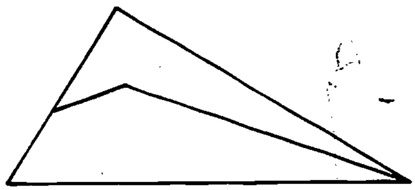


d) *OPTIONAL: A <u>regular polygon</u> is a polygon all of whose sides are equal and all of whose angles are equal. Below are pictured regular polygons with three, four, five, and six sides.*



Determine the sum of the angles of each of these polygons. Then try to generate a formula for the sum of the angles of a regular polygon with n *sides, for each positive integer* $n \geq 3$. *(Check to see if your formula works on the four regular polygons above.)*

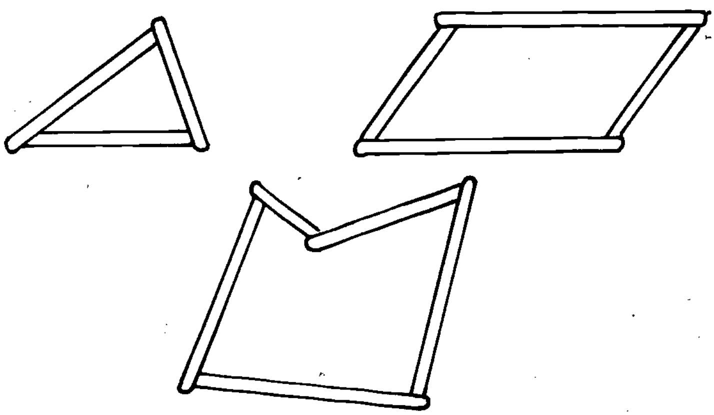
e) Is the quadrilateral as basic as the triangle? That is, could one "quadrilaterate" the same shapes that one can triangulate? For example, the following triangle is quadrilaterated.



What about other polygons? Can you determine what the sum of the angles of any quadrilateral is?

2. Triangles have the unique property among polygons of being <u>rigid</u>. As you will see, this fact, together with the fact that all polygons can be triangulated, enables one to stabilize many different structures.

   a) To see what it means to say that <u>triangles are rigid</u>, make each of the following shapes out of construction paper strips and brads. Try to change the shape of the figure without tearing or bending the sides.



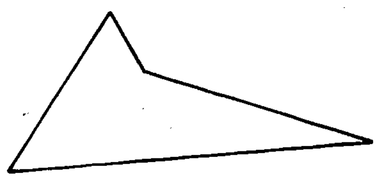
b) See if you can stabilize the unstable shapes by triangulation (or even partial triangulation).

c) Think of as many real-world instances as you can where triangles have been used to stabilize shapes.

3. Triangles have another virtue that gives them real-world significance. Namely, triangles tessellate. Roughly speaking, this means that triangles are good shapes to make floor tiles out of.

a) Put five or six sheets of paper together and cut out five or six congruent triangles. Place the five or six triangles together into a pattern with no holes or overlaps.

b) You can do the same thing with a square. Do you think that you could do it with any old quadrilateral such as this one?



c) What about with circles?*

4. Look around you and find as many occurrences of triangles as you can. Try to determine which (if any) of the above properties (i.e., triangulation, rigidity, and tessellation) gave rise to each occurrence that you find.

---

*Tessellation is a fascinating subject in its own right, and there are many interesting shapes that tessellate. The point of this exercise is that triangles are useful because any triangle will tessellate. For further study of tessellation, you are referred to the Transformational Geometry unit and the Measurement unit of the Mathematics-Methods Program.

ACTIVITY 8

ANALYSIS OF TRIANGLES--SIDES

	/	△	○
Occur.			
Anly.			
Appl.			

FOCUS:

Having recalled that triangles do occur in your life, and having seen that triangulation, tessellation, and structural stability give triangles special reasons for occurring, you will now begin your analysis of triangles. In this actviity you will see just which combinations of lengths qualify as the three sides of some triangle.

MATERIALS:

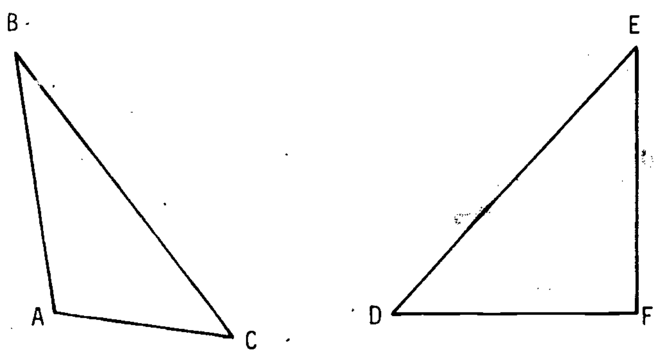
Straightedge, compass.

DISCUSSION:

In this activity you will begin your analysis of triangular shapes by using a straightedge and compass to construct certain triangles. Then you will be asked to discover or recall (with help if needed) a relationship that exists between the lengths of the sides of any triangle.

DIRECTIONS:

1. Use straightedge and compass to construct a triangle congruent (same size and shape) to each of the following two triangles.



2. The principal ingredient of a triangle is its <u>sides</u>. Some combinations of lengths can be the sides of a triangle, some cannot.

   a) Try to construct (with straightedge and compass) a triangle with each of the following two combinations of lengths as sides.

2 cm	1.5 cm
2.5 cm	2 cm
3.75 cm	4 cm

   b) You should have succeeded with one combination and failed with the other. Try to analyze what it is that distinguishes combinations that work from those that don't.

3. There is a rule (or theorem) which predicts which combinations of lengths can be used to construct a triangle.

   a) Write down the rule. (This may simply involve your remembering the rule from high school geometry. It may involve your analyzing your constructions above. It may involve talking with your classmates or instructor.)

   b) Test your rule on each of the combinations of lengths that you used in (1) and (2) to see if it accurately predicts success and failure.

   c) Modify your rule if it proves faulty.

4. Use your rule to predict which of the following combinations of lengths would not work as the sides of a triangle.

   6 cm, 2 cm, 3 cm
   3 cm, 4 cm, 5 cm
   3 cm, 4 cm, 7 cm
   8 cm, 4 cm, 6 cm

In this activity

a) You had some experiences with constructing triangles from certain combinations of lengths.

b) You stated a rule or generalization about that experience.

c) You tested that rule to see if it was consistent with your experience.

d) You applied the rule to new situations.

This sequence of steps is often found as part of an effective strategy for developing geometry concepts with children. Reflect on it now. It will be discussed further later.

ACTIVITY 9

ANALYSIS OF RIGHT TRIANGLES

FOCUS:

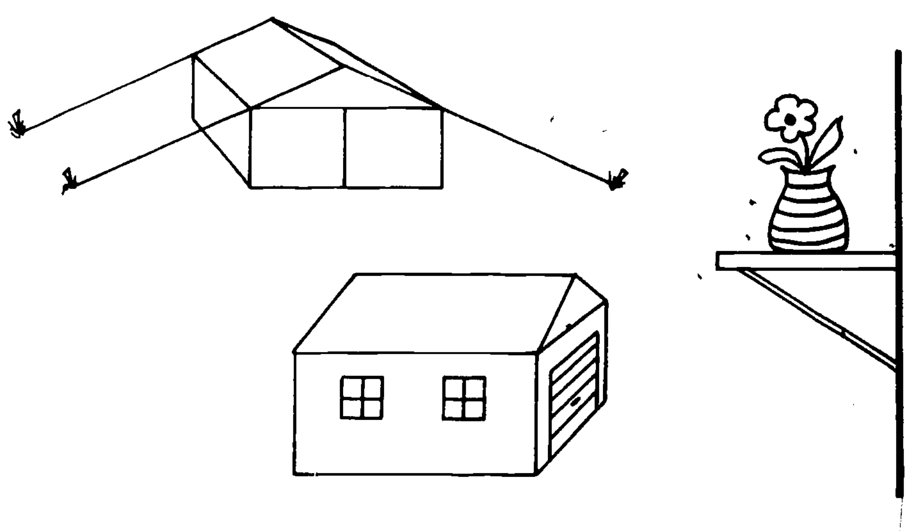
You will continue your analysis of triangles by analyzing the sides of right triangles and by working with the relationship between the sides of a right triangle that is described by the Pythagorean Theorem.

MATERIALS:

Graph paper, metric ruler, straightedge and compass.

DISCUSSION:

Right triangles are special triangles. One encounters them in shoring up a sagging door and, for that matter, in many situations where a heavy weight needs to be supported against falling straight down and against toppling sideways. For example, look at the tent illustrated below. Do you see the structural roles played by right triangles?

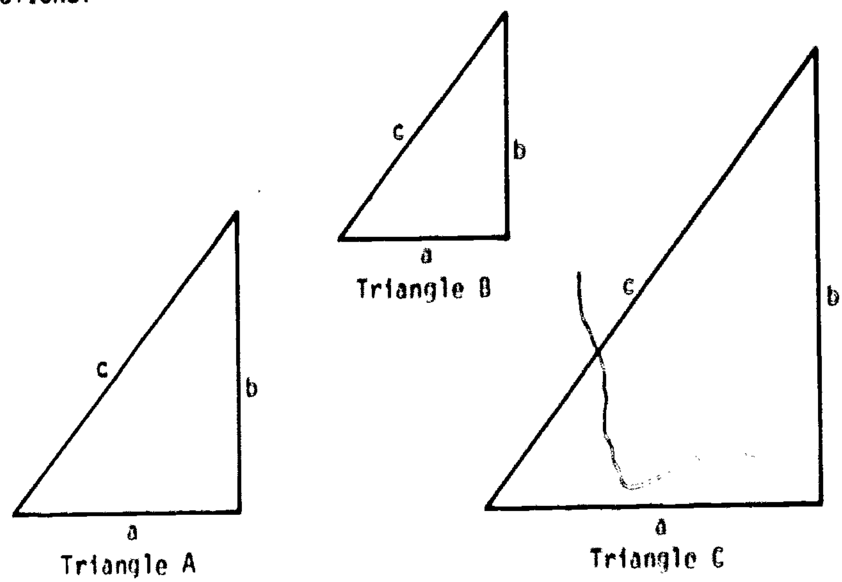


Right triangles have a very special mathematical property which is described by the Pythagorean Theorem.

In this activity you will do some experiments that you might have an upper elementary child do in order to discover or confirm the Pythagorean relationship. Hopefully, in addition to gaining experience with some activities that you might do with children, you will gain further insights into the Pythagorean relationship and into informal approaches to verifying it.

DIRECTIONS:

1.



Each of the above triangles is a right triangle. Measure the lengths in centimeters of the sides and fill in the table below.

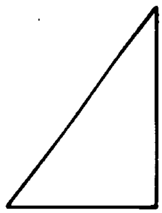
	a	a^2	b	b^2	a^2+b^2	c	c^2
Triangle A							
Triangle B							
Triangle C							

Do your measurements (allowing for inevitable error from estimation) support the Pythagorean Theorem, which says that $a^2 + b^2 = c^2$?

2. a) Write down a careful statement of the Pythagorean Theorem. (Get help if you need it.)

   b) Check your theorem or rule against the examples in (1).

   c) Apply your rule to the following triangle to see if you think it is a right triangle.



3.

Use a straightedge and compass to construct a triangle with sides congruent to the above line segments.

   a) Did it turn out to be a right triangle?

   b) From the Pythagorean Theorem, would you have expected it to be a right triangle? Notice that here we are using $a^2 + b^2 = c^2$ a different way. In (1) and (2) we were observing that if a, b, and c are the lengths of the appropriate sides of a right triangle, then $a^2 + b^2 = c^2$. Here we are saying that since $a^2 + b^2 = c^2$, we would expect a triangle with sides a, b, and c to be a right triangle. Both of these uses are legitimate for the Pythagorean Theorem.

4. The Pythagorean Theorem is valid, and you have gone through some steps that were designed to increase your confidence in its

validity. Read and think for a moment about the following statement.

The lengths of the sides of every right triangle are multiples of 3, 4, and 5.

Do you believe the statement?

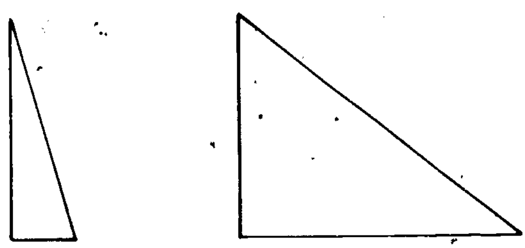
a) Check back through the examples that you did in (1), (2), and (3) above. See if, indeed, each of the right triangles has sides that are (allowing for imprecision of measurement) multiples of 3, 4, and 5 (e.g., 6, 8, and 10; 1.5, 2, and 2.5).

b) Since you based your belief in the Pythagorean Theorem on those triangles, shouldn't you believe the above statement on the basis of those triangles?

c) Take a piece of graph paper and draw several right triangles without paying particular attention to the length of the sides (they are easy to draw on graph paper). Does each of your triangles have sides which are multiples of 3, 4, and 5?

....

Obviously, the statement is false. There are many right triangles with sides that are not multiples of 3, 4, and 5. Consider the following:



But every example that you were given in (1), (2), and (3) had sides that were (allowing for measurement error) multiples of 3, 4, and 5. So you had, on the basis of these examples, just as much reason to believe the statement in (4) as you did the Pythagorean Theorem. Why did we try to mislead you? . . . There are two points to be made, (i) a pedagogical point, and (ii) a point concerning the verification of generalizations.

i) When choosing examples to illustrate a concept, a teacher must be careful to avoid developing misconceptions in students.

ii) There is a need for some form of verification or proof that goes beyond generalizing from examples. One's examples may not be representative of the general situation. None of this negates the fact that the Pythagorean Theorem is true and that the statement in (4) is false. It just illustrates a potential pitfall in generalizing from examples and a precaution that one should take in choosing examples.

....

5. Indicate how you would change (1), (2), and (3) above in order to avoid possible misconceptions.

6. a) Indicate which of the following number triples are the sides of a right triangle.

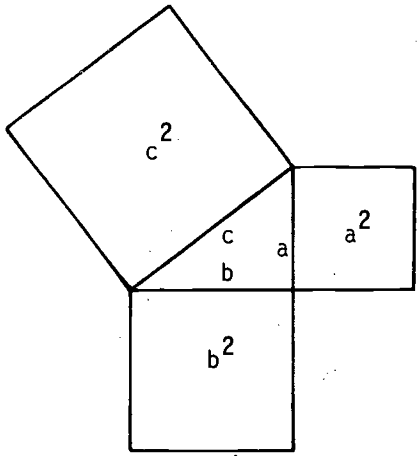
   1.8, 2.4, 3;　　　　5, 1, 2;

   1, 2, 1;　　　　5, 12, 13;

   $a, b, (a^2 + b^2)$ where $a > 0$ and $b > 0$.

   b) Generate two "Pythagorean triples" of your own; i.e., find two more sets of positive integers, a, b, and c, which can be the lengths of sides of a right triangle. Be sure that at least one of the triples is not a multiple of 3, 4, and 5.

7.



The picture above suggests an interpretation of the Pythagorean Theorem in terms of the areas of squares on the three sides of a right triangle. Draw such a right triangle on a sheet of paper, and "verify" that $a^2 + b^2 = c^2$ by cutting up the smaller squares and fitting them on top of the larger square.

TEACHER TEASER

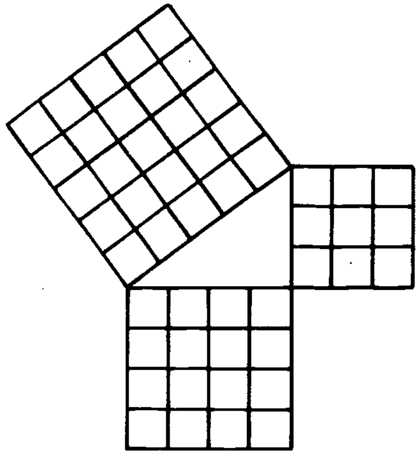
Generating Pythagorean Triples

1. (3, 4, 5) is a Pythagorean triple since $3^2 + 4^2 = 5^2$. Find another Pythagorean triple.

2. There is a rule that says that if a and b are numbers so that $b + (b + 1) = a^2$, then the numbers a, b, and $(b + 1)$ are a Pythagorean triple. For example, if you let $a = 3$ and $b = 4$, $b + (b + 1) = 4 + 5 = 9 = a^2$. Use this rule to find two more Pythagorean triples.

3. Show <u>why</u> the rule works.

8.



a) What does the above picture have to do with the Pythagorean Theorem?

b) Discuss the advantages and limitations of the approach illustrated by this picture for convincing a child of the truth of the Pythagorean Theorem? (E.g., which right triangles can it be applied to?)

9. *OPTIONAL: In exercises 1, 2, and 3 you were given examples designed to convince you of the truth of the Pythagorean Theorem. In (7) and (8) you did an experiment that also tended to make the theorem seem reasonable. In high school geometry you probably saw a deductive proof of the Pythagorean Theorem. Discuss with your classmates what you might want to do with fifth-graders to convince them of the truth of the Pythagorean Theorem. (This matter will be dealt with in more depth in the discussion of verification in the fourth section.)*

ACTIVITY 10

AN APPLICATION OF THE PYTHAGOREAN THEOREM TO LENGTH AND COORDINATES

	/	△	○
Occur.			
Anly.			
Appl.		■	

FOCUS:

We will digress from our analysis of triangles for a moment to apply the results of the last activity. Here you will learn to use the Pythagorean Theorem to find the distance between two points whose coordinates you know.

DIRECTIONS:

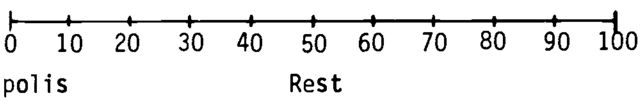
1. Suppose that you are driving in a straight line away from Indianapolis in your car.

   a) If you start 5 miles away, how far have you traveled when you are 12 miles away?

   b) If you rest for a while 50 miles from Indianapolis, and then drive until you are 95 miles away, how far did you drive after your rest?

   c) If you start m miles away and drive until you are x miles away, write an expression telling how far you have driven.

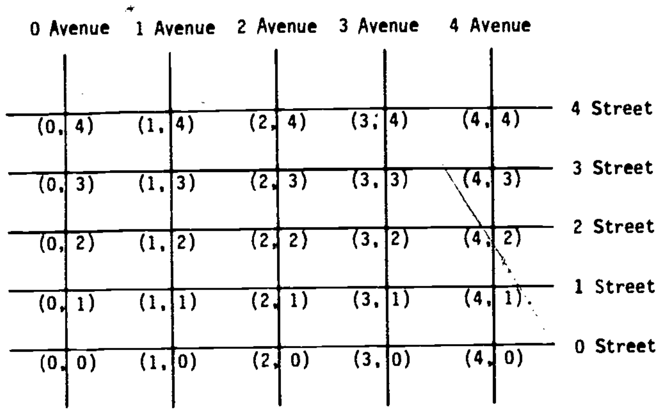
   d) Illustrate each of (a), (b), and (c) on the number line below.



   e) What is the distance between the number 1.23 and the number 4.32 on the number line?

   f) What is the distance between the number a and the number x on the number line? (Be a little careful here-- a may be to the left of x or to the right of x).

2. Suppose that the grid on page 53, represents the map of a carefully laid out city, so that to get from one place to another one must travel along the streets.

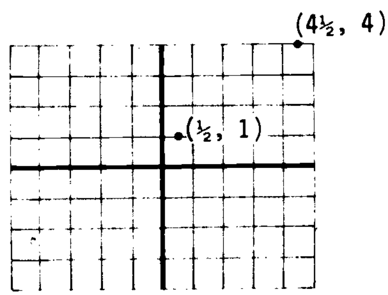


a) How many blocks is it from the corner of 0 Avenue and 0 Street (i.e., (0, 0)) to the corner of 1 Avenue and 3 Street (i.e., (1, 3))? Does it matter which direct path you take?

b) How many blocks is it from (2, 4) to (1, 2)?

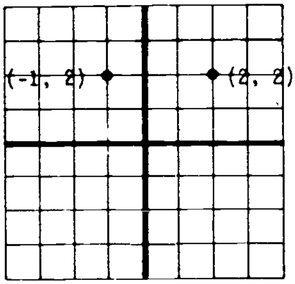
c) Can you come up with a general formula for the (shortest) distance from (a, b) to (x, y)?

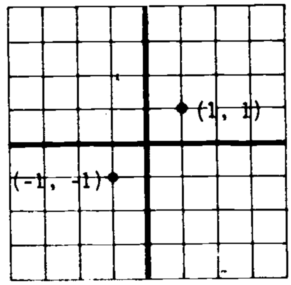
3. Let's get off the city streets and travel <u>as the crow flies</u>.

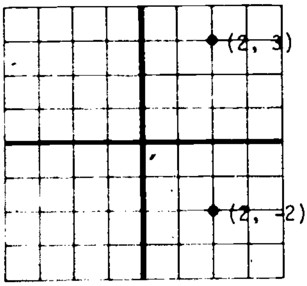


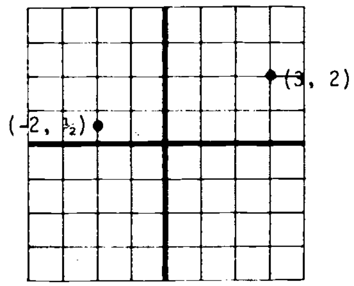
Find the exact distance between the two points above. (You may have to think about this some. Make a triangle and use the Pythagorean Theorem. Measurement with a ruler is not sufficiently accurate here.)

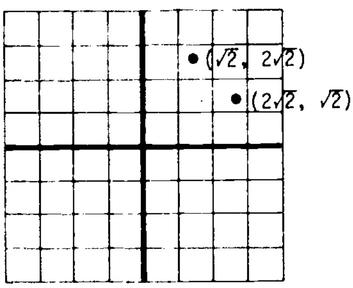
4. Find the distance between each of the pairs of points represented on the next page.

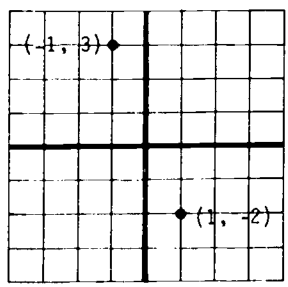












5. a) Write down an expression for the distance between the points (a, b) and (x, y).

   b) Apply your expression to two of the pairs of points in exercise 4 to see if you get the same results.

6. In this activity you developed a procedure for finding distances. Then in 5(a) you arrived at a formula which can now be used "mechanically." Are there any advantages in arriving at the formula in this way, rather than being given the formula and being asked to apply it? Discuss.

ACTIVITY 11

ANALYSIS OF TRIANGLES--ANGLES

	/	△	○
Occur.			
Anly.		■	
Appl.			

FOCUS:

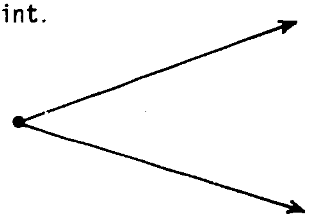
Each triangle, of course, has three sides and three angles. You have had some opportunities to analyze the lengths of the sides of triangles. In this activity you will determine which combinations of angles can be the angles of a triangle.

MATERIALS:

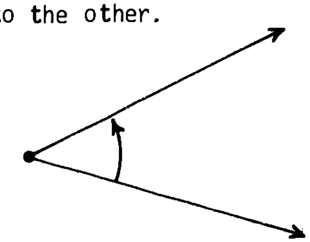
Paper, scissors, ruler and protractor.

DISCUSSION:

An angle consists of the union of two rays (half-infinite lines) which have a common endpoint.

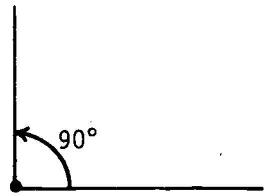


The measure of an angle is a number which describes the amount of turn to get from one ray to the other.

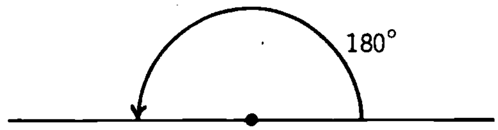


One unit of angle measure is the degree. The degree is $\frac{1}{360}$th of a turn through a complete circle. The measures of two common angles follow:

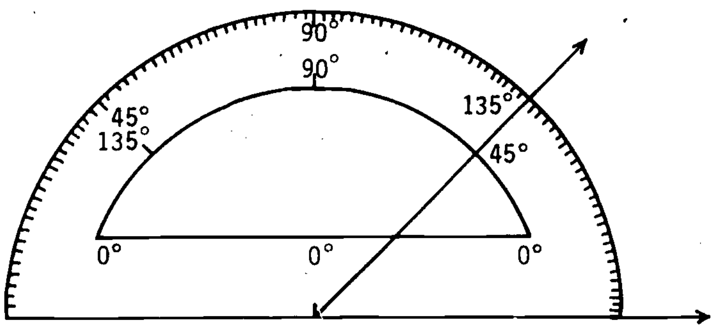
Right Angle



Straight Angle



The most common device for measuring angles is the protractor.



Here the protractor is measuring a 45° angle.

In this activity you will study those combinations of angles which can be the angles of a triangle. You probably already know, or at one time knew, that any three angles whose measures add up to 180° can be the angles of a triangle; so you will have some experiences here to remind you of this fact and to apply it. You will see that one might plan some similar experiences for children.

DIRECTIONS:

1. a) Draw and cut out several triangles of different sizes and shapes.

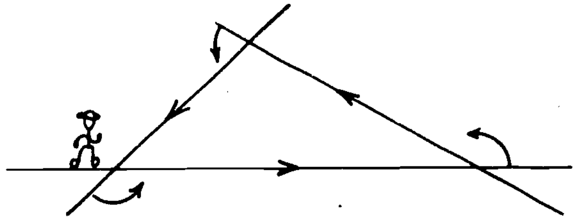
b) Measure the angles with a protractor and fill in the following table.

	Measure of Angle A	Measure of Angle B	Measure of Angle C	Sum of the measures of the angles
1				
2				
3				
4				
5				

c) Write down a careful statement of the rule that describes which sets of three angles "qualify" as the angles for some triangle.

2. Tear off the vertices of one of your triangles and piece them together to get a physical model of your rule in 1(c). (Remember that the measure of a straight angle is 180°.) Repeat this process with several of your triangles. Do your results confirm your rule?

3. OPTIONAL: *If you start at one corner of a triangle and start walking around the triangle, you will turn a complete circle (360°) by the time you return to your original position. Use this fact to "derive" a rule for the sum of the angles of a triangle. This is a little tricky, but if you think it through carefully, you may be able to see it.*



4. a) Which of the following sets of measures could be the angles of a triangle?

1°, 62°, 117°	30°, 30°, 30°
45°, 45°, 45°	29°15', 37°40', 113°5' *
60°, 60°, 60°	

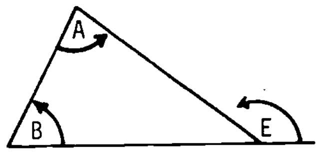
b) Use a protractor and ruler to draw triangles with two of the above sets of angles.

c) Complete each of the following sets of measures so that the resulting set could be the measures of the angles of a triangle.

40°, 40°, ____	18°18', _____, _____
90°, 90°, ____	

5. The following is a rule or theorem which can be verified directly by using the rule that the sum of the measures of the angles of any triangle equals the measure of a straight angle:

> For any triangle the measure of an exterior angle equals the sum of the measures of the opposite angles of the triangle.



The rule says that the measure of E equals the sum of the measures of A and B. <u>Your task</u> is to verify that this is true for every triangle, using what you know about the angles of a triangle.

---

*15', for example, designates 15 minutes where 1 minute is $\frac{1}{60}$th of a degree.

6. Plan and outline a development for fifth-graders of the rule that the sum of the measures of the angles of a triangle equals the measure of a straight angle. Give some thought to what a child should know before your lesson starts. Also consider how you would make sure that a child had learned the rule.

7. Here you have measured, cut, and walked your way to a rule for the sum of the angles of any triangle. In high school geometry you probably proved a theorem which stated the same rule. There are several ways that people become convinced of the truth of statements:

   - People accept statements on the <u>authority</u> of others.
   - People <u>generalize</u> from specific instances or experiences.
   - People are convinced by <u>plausibility</u> arguments.
   - People construct deductive <u>proofs</u>.

   Discuss:

   a) Was what you did in (1), (2), and (3) a proof? If not, what was it?

   b) Was what you did in (5) a proof?

   c) Are you <u>sure</u> that the sum of the measures of the angles of any triangle equals the measure of a straight angle? If so, what convinced you? If not, what would it take to convince you?

   d) Why do you suppose that a child, a high school student, and a professional mathematician might be respectively harder to convince of the truth of a mathematical rule? (More attention will be given to these questions later.)

8. OPTIONAL: *Think about it. Are you surprised that every triangle has the same angle sum? If you were to draw triangles on a sphere, would you find the same things? Try it.*

ACTIVITY 12
ANALYSIS OF TRIANGLES--SAS, ASA, ETC.

FOCUS:

You have analyzed the sides of triangles and the angles of triangles. Now you will put them together to analyze some relationships among the sides and the angles of triangles.

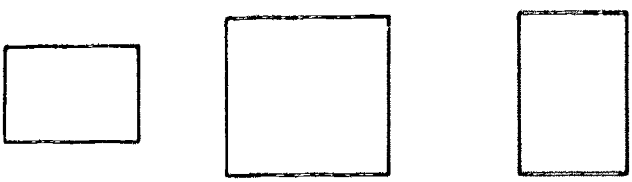
MATERIALS:

Metric ruler, protractor, straightedge, compass, scissors, graph paper, and tracing paper.

DISCUSSION:

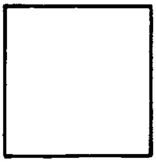
We are studying triangles here. However, we open with a brief discussion of quadrilaterals, in order to provide a basis for comparison. One can have many different quadrilaterals with the same sides. For example, these two quadrilaterals

have sides of the same length. Is the same thing true for triangles? One can also have many different quadrilaterals with the same angles. For example, these three rectangles



all have the same angles. Note, however, that they don't even have the same shape, let alone the same size! Is the situation the same for triangles?

There is only one quadrilateral that has four 2-cm sides and four right angles, namely, the <u>square</u> below.



In this case, we say that the four sides given and the four angles given <u>determine</u> a quadrilateral. We saw above that giving four sides alone does not determine a quadrilateral, since many quadrilaterals can be made with the same four sides.

In this activity you will analyze which combinations of sides and angles determine a triangle and which do not. In particular, you will study the ASA and SAS combinations that are familiar from high school geometry courses.

DIRECTIONS:

Do each of the following in order to discover what conditions determine a triangle.

1. In (a), (b), (c), (d), (e), and (f) below, you will be given various combinations of sides and angles.

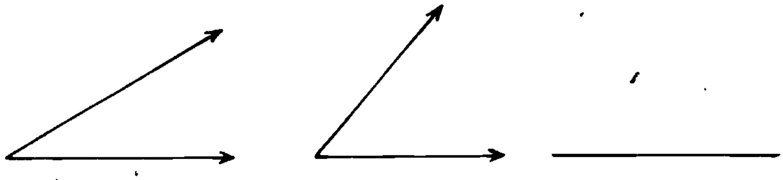
   - In each case you should try to draw at least one triangle that has the given sides and angles. (You may want to use tracing paper, scissors, protractor, compass, or graph paper.)
   - In each case you should decide which of the following is true.

     i) No triangle is possible with those sides and angles.

     ii) <u>Exactly one</u> triangle is possible; that is, the particular combination <u>determines</u> a triangle.

     iii) Two or more noncongruent triangles are possible with the particular combination of sides and angles.

   a) Attempt to draw a triangle or triangles with the following combination of two angles and a side, with the side between the two angles. Then indicate which of (i), (ii), and (iii) above is true about this particular combination.

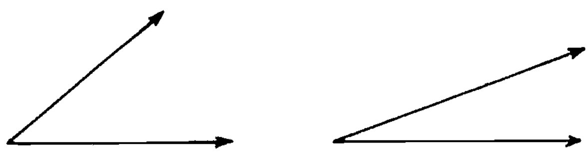


b) Answer the same question with the same side and angles as in (a) but with the side not between the two angles.

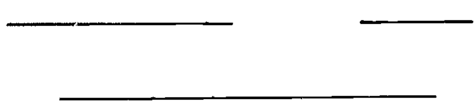
c) Do as in (a) and (b) for the following combination of two sides and one angle, with the angle between the two sides.

d) Answer the same question with the same angle and sides as in (c) but with the angle not between the two sides.

e) Do as above with the two angles below.



f) Do as above with the three sides below.



2. To summarize your findings indicate which of the following combinations of ingredients seem to determine a unique triangle. (You should expect to have to do additional experimentation for some of your answers.)

a) Two sides and the angle between them (i.e., the included angle), referred to as SAS.

b) Two sides and an angle which is not between them, SSA.

c) Two angles and the side between them, ASA.

d) Two angles and a side that is not between them, AAS.

e) Three angles the sum of whose measures is 180°, AAA.

f) Three sides, the sum of any two of which is greater than the third, SSS.

3. Answer the following questions:

   a) If one side is 3 cm long, another is 4 cm long, and the angle between them is 90°, what must the length of the third side of the triangle be? Draw the triangle and estimate what the measures of the other two angles must be.

   b) If a triangle has angles of 30° and 40° and if the side between them is 5 cm long, what must the measure of the other angle and the lengths of the other sides be? (Estimate the lengths of the sides after drawing the triangle.)

   c) Make up another question in the spirit of (a) and (b), and give it to a classmate to do.

4. Discuss with your classmates any implications that can be drawn from the above concerning the rigidity of triangles. (Is it possible to change the shape of a triangle without changing its sides?)

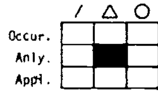
5. *OPTIONAL: Answer the following questions, which relate back to the "Discussion" at the beginning of this activity.*

   a) As was pointed out in the "Discussion," two quadrilaterals can have the same angles and yet have a different shape. Is that possible for triangles?

   b) What are some minimal conditions that determine a quadrilateral?

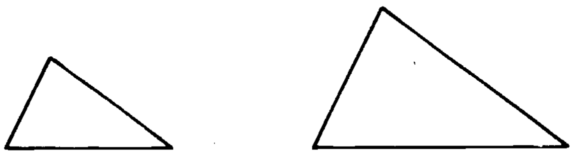
ACTIVITY 13

ANALYSIS OF SIMILARITY

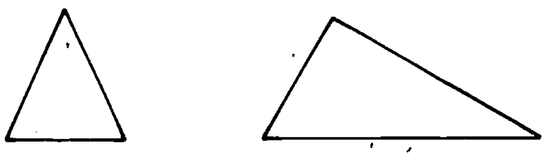


FOCUS:

These two triangles are similar (same shape).



These two are not.



In this activity we will continue the analysis of triangles by investigating the ingredients of similarities.

MATERIALS:

Graph paper, protractor, and ruler.

DIRECTIONS:

1. a) Draw a triangle on graph paper. (Be sure that its sides are small enough so that you can carry out the next step.)

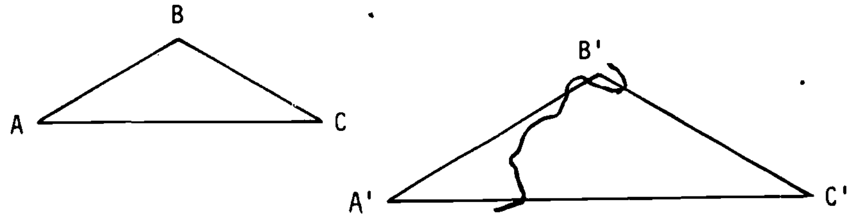
   b) Draw another triangle each of whose sides is exactly <u>twice</u> as long as those of the first triangle.

   c) Compare the appearance of the two triangles. Are they similar? That is, do they have the same shape?

   d) Measure the corresponding angles of the two triangles.

   e) Repeat the above steps making the lengths of the sides of the second triangle one-third of those of the first.

2. There is another way of expressing the fact that the lengths of the sides are twice as long or one-third as long.



If the sides of the larger triangle are three-halves as long as those of the smaller one, that means that $\frac{|A'B'|}{|AB|} = \frac{3}{2}$, $\frac{|B'C'|}{|BC|} = \frac{3}{2}$, and $\frac{|A'C'|}{|AC|} = \frac{3}{2}$.* Here we see that the <u>ratios of the lengths of corresponding sides of the two triangles are the same</u>; i.e., $\frac{|A'B'|}{|AB|} = \frac{|B'C'|}{|BC|} = \frac{|A'C'|}{|AC|}$.

Go back and test the ratios of the lengths of corresponding sides of the pairs of triangles in (1).

3. a) Draw two noncongruent triangles which have angles measuring 30°, 60°, and 90°. Measure the sides of each, and compare the ratios of the lengths of corresponding sides.

   b) Do as in (a) with angles of your own choosing.

....

Your experimentation in (1) and (2) was designed to suggest two rules:

- When we say that two triangles <u>have the same shape</u> (are similar), we are saying that their <u>corresponding angles are the same</u>.
- If the ratios of corresponding sides of two triangles are the same, the triangles are similar.

Your experimentation in (3) was to suggest that:

- If two triangles are similar, the ratios of corresponding sides are equal.

---

*|AB| is the length of the line segment AB which joins the points A and B.

In high school geometry, the first of these three rules is usually taken as a <u>definition</u> of similarity of triangles, and the other two are proved as theorems.

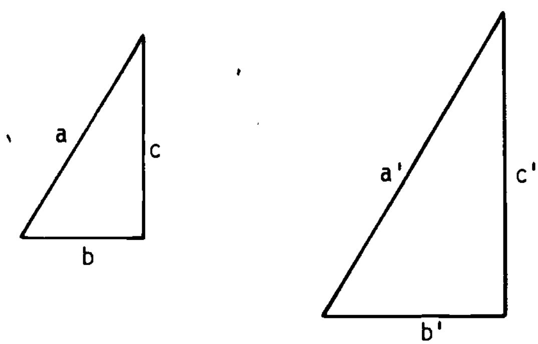
4. Which of the following sets of dimensions are the sides of similar triangles?

   a) 2, 3, 4, and 3, 4, 5
   b) 3, 4, 5, and 6, 8, 10
   c) 1, $1\frac{1}{2}$, 2, and 4, 6, 8
   d) 1, 1, 1, and .5, .5, .5
   e) 2, 3, 2, and 3, 4, 3.
   f) $\sqrt{2}$, 3, 4, and 2, $3\sqrt{2}$, $4\sqrt{2}$.
   g)* 1, 2, 4, and $\frac{1}{2}$, 1, 2.

5. a) If a triangle has sides of lengths 3, 4, and 5 centimeters, what would be the lengths of the other two sides of a similar triangle whose shortest side has a length of 1 inch?

   b) If a triangle with sides of lengths 1, 1, $\sqrt{2}$ inches has angles whose measures are 45°, 45° and 90°, what would be the measures of the angles of a triangle whose sides are 4, 4, $\sqrt{32}$ inches?

6.



Use the fact that the ratios of corresponding sides of similar

*Be careful.

triangles are equal to show that $\frac{a}{c} = \frac{a'}{c'}$ for any two similar triangles.*

7. Similarity was invented in order to talk about the everyday concept of "same shape." For triangles it turned out that two different characterizations of similarity were possible, namely:

   - Two triangles are similar if the corresponding angles have the same measure.
   - Two triangles are similar if the lengths of corresponding sides have the same ratios.

   What about quadrilaterals? Does either of these characterizations work? That is, do either same angles or equal ratios guarantee you similarity? (You may want to look back at Activity 12. You also may want to sketch a few simple quadrilaterals and experiment.)

8. *OPTIONAL: Which of the following statements are true for <u>all</u> polygons?*

   - *If two polygons have the same shape, corresponding angles have the same measure and the lengths of corresponding sides have the same ratios.*
   - *If the corresponding angles of two polygons have equal measure, the polygons have the same shape.*
   - *If the lengths of corresponding sides of two polygons have equal ratios, the polygons have the same shape.*
   - *If the corresponding angles of two polygons have equal measure, and the lengths of corresponding sides have equal ratios, then the polygons have the same shape.*

*Note: This fact is the basis of the subject of trigonometry.

ACTIVITY 14

STRAIGHT LINES REVISITED

FOCUS:

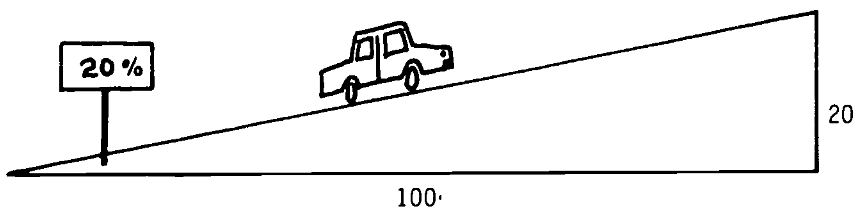
In Activity 6 you had experience with graphing the solution pairs for equations of the form ax + by = c. You found that each of the graphs was a straight line. In this activity the relationship between similar triangles and equal ratios will be applied to pursue the reverse process. That is, given a straight line (at least, two points on it), you will find an equation of the form ax + by = c.

MATERIALS:

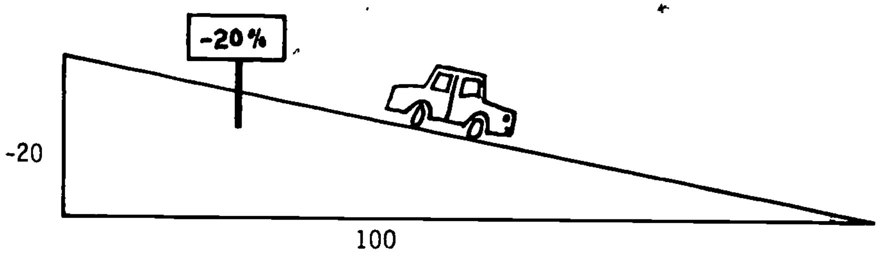
Graph paper, straightedge.

DISCUSSION:

If you are traveling up a mountain, you might see a sign saying, "Caution 20% slope."



This sign means that for every 100 units traveled on the horizontal, 20 units are traveled vertically. So that ratio of "up" to "out" is 20 to 100, i.e., $\frac{20}{100} = 0.20$. So, if you traveled 5 kilometers horizontally, you would go vertically 1 kilometer. If you traveled 10 kilometers horizontally, you would go vertically 2. In general, if you went x kilometers horizontally, you would go vertically y where $y = 0.20x$; 0.20 is called the slope of the road.



If the sign were at the top of the hill and you were about to travel down, the sign might warn you of a <u>negative</u> 20% slope. Now your ratio of "up" and "out" is $\frac{-20}{100} = -0.20$, since you travel -20 units up when you travel 100 units out. So we say that this road has a slope of -0.20. In your work below you will see what all this has to do with $ax + by = c$.

DIRECTIONS:

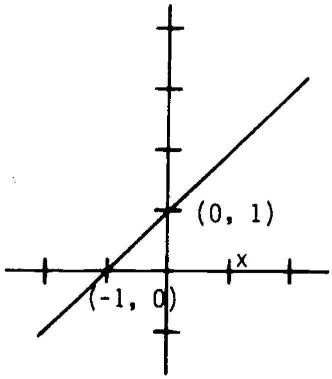
1. Suppose that you are at the top of a hill and suppose that for each 50 meters of your travel, you descend 5 meters.

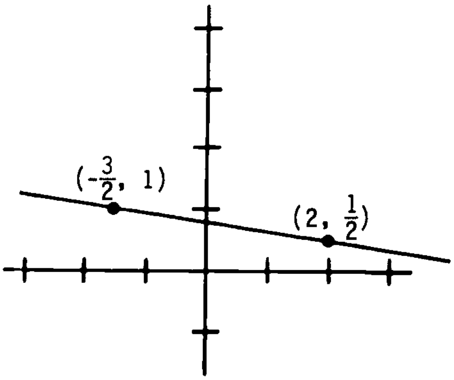
   a) Sketch a picture like those above.

   b) Find the slope of the hill.

   c) How far out would you have to go in order to descend 75 kilometers?

2. Find the <u>slopes</u> of the two lines below. (Treat them like hills on which you will drive from left to right.)

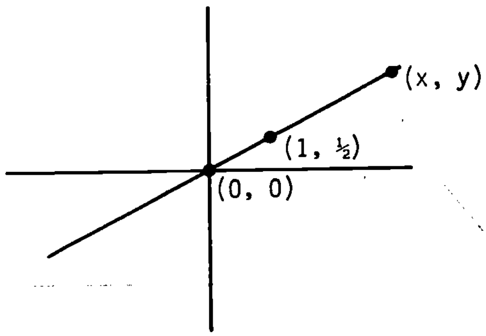




3. a) Find the slope of the line below.

   b) What does $\frac{y}{x}$ equal for any (x, y) on the line?

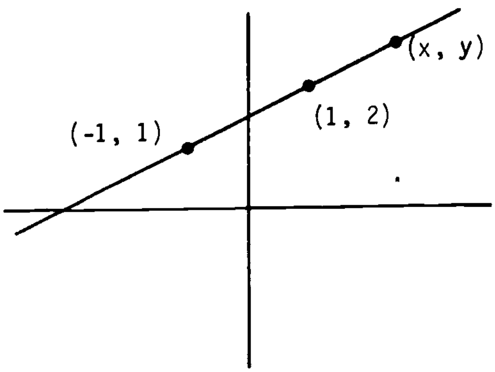
   c) When x = 100, what does y equal?



4. a) Find the slope of the line below.

   b) Put any point (x, y) on the line (e.g., the one drawn). What does $\frac{y-2}{x-1}$ equal? (You may want to look back at the development of distance in Activity 10.)

   c) Explain your answer to (b) in terms of "up" and "out."



5. Consider the straight line through the points (-1, -1) and (3, 1).

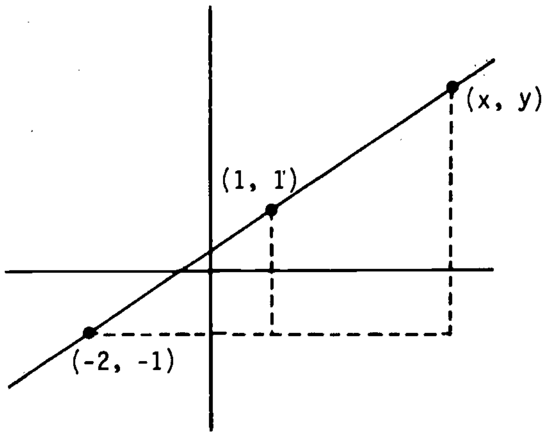
   a) Find the slope of the line.

b) What does $\frac{y-(-1)}{x-(-1)}$ equal for any point (x, y) on the line?

c) Again, explain your answer in terms of "up" and "out."

6. a) In terms of the <u>similar</u> triangles pictured below and ratios of lengths of corresponding sides, explain <u>why</u> $\frac{y-(-1)}{x-(-2)} = \frac{1-(-1)}{1-(-2)}$.

b) Discuss how your answer to 6(a) provides a justification for your answers to 4(b) and 5(b).



7. a) In (3), (4), (5), and (6) you produced equations involving x and y. Take each of these equations and put it in the form of ax + by = c. (You may want to get some help on the algebra.)

b) Describe a general procedure for doing the following. Given two points, find an equation of the form ax + by = c whose graph is the straight line through two points.

c) Summarize what you have learned in this activity and in Activity 5 about the relationship between equations of the form ax + by = c and straight lines.

8. a) Find an equation of the straight line through the points (1, 2) and (3, 2). For what value of y is (5, y) on the line?

b) Find an equation of the straight line through the points (1, 2) and (3, -2). Draw the line.

c) Find an equation of the straight line through (1, 2) and (1, 3). Draw the line.

ACTIVITY 15

APPLICATIONS OF TRIANGLE LEARNINGS

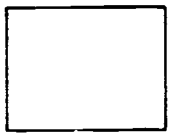
FOCUS:

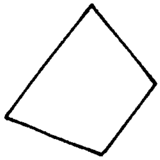
In this section you have been studying triangles. You have seen that they do occur in your life. You have analyzed various aspects of triangles. You have applied your triangle learnings to the analysis of straight lines. In this activity you will investigate further applications of your analysis of triangles.

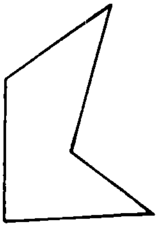
DIRECTIONS:

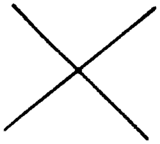
Solve each of the following problems. With each solution indicate what previous learnings you have applied.

1. How would you "stabilize" each of the following shapes with the smallest possible number of straight lines?

a) 

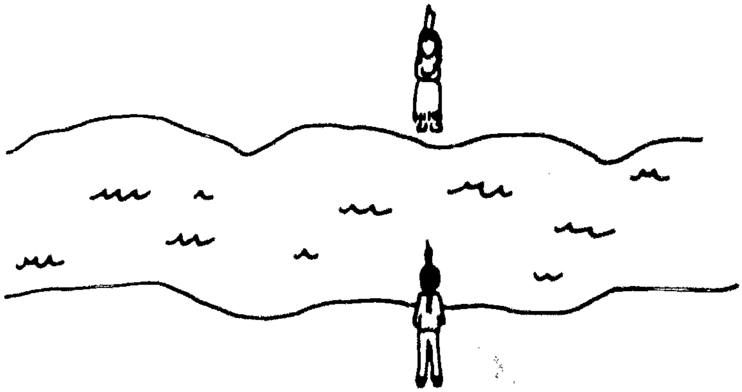
b) 

c) 

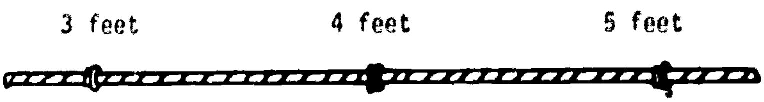
d) 

2. Little White Dove saw her lover across the raging river. Lest she drown in her attempt to swim the torrent, she wanted to determine how wide the river was. She was armed solely with a compass (with which she could measure distances on her side of

the river), and a knowledge about theorems involving equal ratios and similar triangles. Can you devise a strategy whereby she could measure the width of the river?

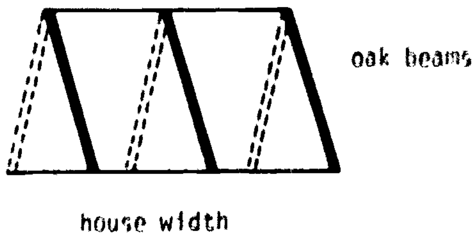


3. A carpenter might keep a rope handy which has knots as follows:

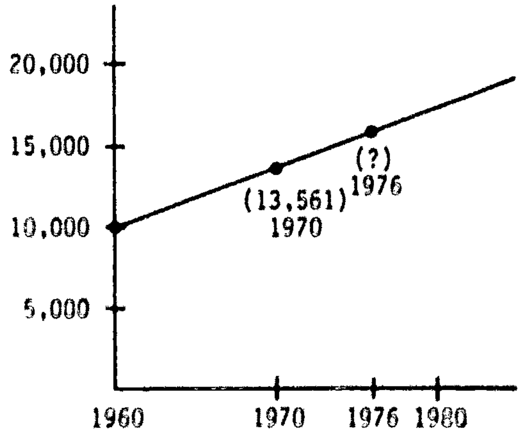


How could such a rope be useful in making "square" corners (i.e., ones with a right angle)?

4. A man who lives in snow country wants to build a house with a pitched (i.e., triangular, like an inverted V) roof so that the snow won't build up on it. He also wants the roof to be supported with some very lovely oak beams which are available locally in lengths up to 20 feet. Discuss what he must consider in deciding how wide his house can be.

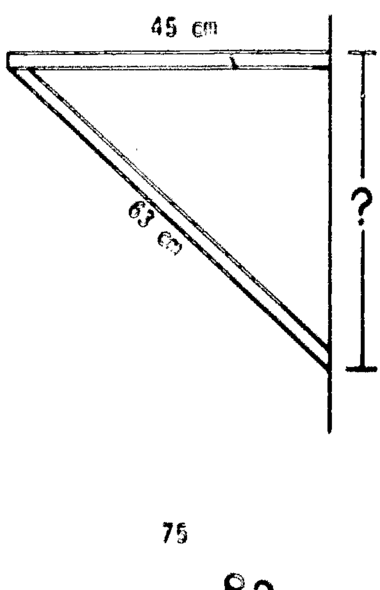


5. In 1960 the enrollment of a certain college was 10,000. In 1970 it was 13,561. What will the enrollment be in 1976 if the growth is linear?



You can compute this exactly using what you have learned about slopes of lines. Predictions like this based on past data are called extrapolations.

6. Suppose that you are building a wall shelf which is 45 cm deep, and suppose that your supports are 63 cm long. How far down the wall from the shelf will the supports be attached?



7. Suppose that you are stabilizing a TV antenna by placing three guy wires on it. You have been advised that the wires should be attached to the antenna 3 meters up the pole and they should be attached to your (horizontal) roof 2 meters from the base of the pole. Allowing 10 cm for each tie in the wire, exactly how much wire should you purchase?

Section III

CIRCLES

Circles are around you everywhere! In fact, man seems to have an eternal dedication to reinventing the wheel! Levity aside, circles are important shapes and have some very interesting properties to analyze.

Activity 16 engages you in observing the occurrences, both physical and conceptual, of circles in your life. Activities 17 through 20 analyze circles and their relationships with points, lines, Cartesian coordinates, etc. In Activity 21 you will have a chance to make a somewhat unusual application of what you have learned about circles.

Activity 22 is separate, in that it asks you to summarize your experiences with Sections I, II, and III by analyzing, discussing, and developing experiences for children.

MAJOR QUESTIONS

1. Describe three different uses of circles and indicate the properties of circles that make them appropriate for each use.

2. Of what circle is $x^2 + y^2 = 4$ the equation? (Draw it.) Explain what it means when we say that $x^2 + y^2 = 4$ is the equation of a circle.

3. Use a straightedge and compass to construct a circle and then to construct a tangent to the circle.

ACTIVITY 16

CIRCLES: THEIR ROLE IN THE WORLD

FOCUS:

With straight lines and with triangles, you have gone through a three-step analysis:

1. You have observed the occurrence and importance of the shape in the world.
2. You have analyzed various properties of the shape.
3. You have considered ways in which the knowledge you have gained about the shape in steps 1 and 2 could be applied.

In this activity and the next few, you will be following the same steps for circles. In particular, this activity focuses on the many ways in which circles enter your life.

DIRECTIONS:

1. Brainstorm: List several occurrences of circular shapes. You might want to keep separate lists of those that occur in nature and those that are manmade.

2. There are many reasons why a circular shape might be used in a particular situation. For example:

   - aesthetics
   - tradition
   - ease of construction
   - laws of physics
   - utility of a specific attribute
   - mystical or religious forces

   [illegible] the following [illegible] occurrences of circles, [illegible] why circles appear [illegible] insights that you have.

3. [illegible] an object that all want [illegible]

[illegible]

b) Puddles of water tend to be round.

c) Most dinner plates are round.

d) Car wheels are circular.

e) Handmade bowls are usually round.

f) Basketballs have circular cross sections.

g) So do pumpkins.

h) A Master Charge card has intersecting circles on it.

3. Each shape has certain structural attributes. We have included below a list of attributes. Discuss the way in which each attribute applies to circles (or does not apply). Compare and contrast with other shapes.

- has corners or vertices
- has a "balance" point
- has a radial symmetry
- has constant width
- tessellates

4. List as many different ways as you can think of to draw or make a circle.

5. On the lighter side,

a) You have heard of "big wheels" and of "social circles." Can you think of other clichés or sayings that have reference to circular or round shapes? What properties of the shape have relevance in these sayings?

b) You might have fun trying to conjure up other phrases or sayings which are [illegible]. For example, have you heard of "the eternal triangle" or "a straight arrow"?

ACTIVITY 17

CIRCLES AND POINTS

FOCUS:

As a part of the analysis of the properties of circles, you will investigate the relationship between circles and points.

MATERIALS:

Ruler, compass, and straightedge.

DIRECTIONS:

1. Mark a point on a piece of paper.

   a) How many different circles can have that point as center? Draw a couple of them.

   b) How many different circles can pass through (contain) the point? Draw a couple of them.

2. Now mark two points on a piece of paper.

   a) How many circles can be drawn through both points? Draw a few of them. Have you less freedom than you did with one point? Can you describe any property that is common to all of the circles through the two points?

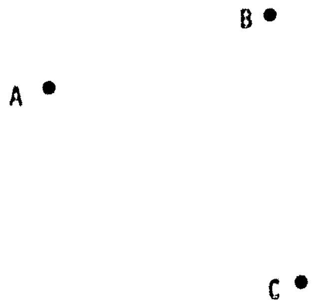
   b) Describe how to find the center of any circle which passes through both points.

3. a) How about three points? Put three points on a piece of paper. How many circles can you put through all three? Draw them.

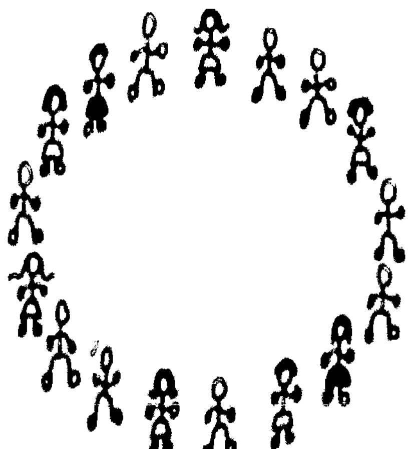
   b) How can one find the center of any circle which passes through three points?

   c) How about four points?

4. It is possible to use a straightedge and compass to construct a circle through the points A, B, and C below. Try to devise a technique for doing this.



5. Suppose that you were teaching a class and that you wanted the children to stand in a particular circle. If you placed Sally and Joe where you wanted them to stand and then asked the rest of the class to form a circle which contained Sally and Joe, could you be sure that the class could form the circle that you wanted? Explain.



ACTIVITY 18

CIRCLES AND LINES

FOCUS:

We have had a glimpse of the interaction between circles and points. Now we will continue our analysis of circles by investigating the interaction between circles and lines.

MATERIALS:

Straightedge, compass, and protractor.

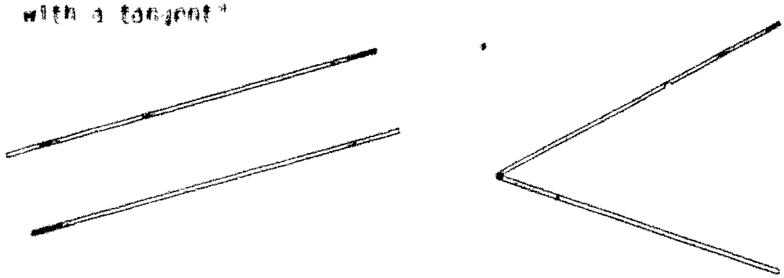
DIRECTIONS:

1. This is a good exercise to sharpen your skills with straightedge-and-compass constructions.

   a) Construct a circle and construct a radius of the circle using your straightedge and compass.

   b) Use your straightedge and compass to construct a tangent to the circle.

2. Have you ever been accused of "going off on a tangent" or "flying off the handle"? If a handle were spinning you around very fast and you were to fly off, what would your motion have to do with a tangent?



[illegible] of the two pairs of straight lines above,

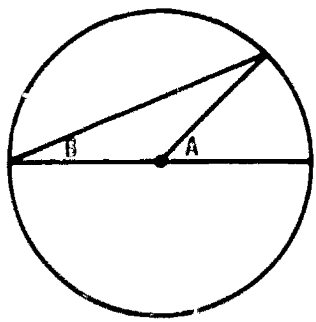
[illegible] three circles that are tangent to both lines

b) Describe where the centers of all such circles would be.

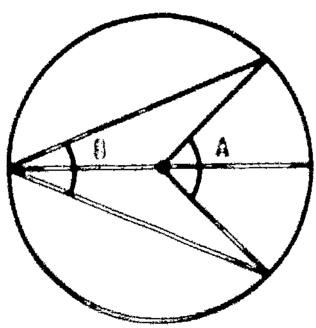
4. A yo-yo on its string almost has the configuration of a circle with a tangent. Can you think of others?

5. One way that lines and circles interact is through angles inscribed in circles. There is a nice rule concerning such inscribed angles that you can develop here.

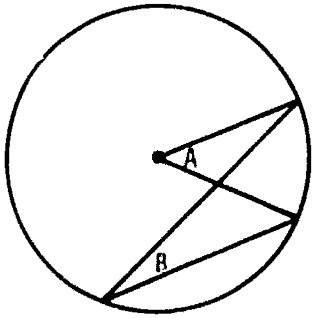
   a) Use what you know about angles and triangles to convince a classmate that the measure of angle B is one-half of the measure of angle A.



   b) Do the same thing as in (a), using the fact that a diameter of the circle bisects both angle A and angle B.



   c) The rule holds in a much more general situation like, for example, the one pictured on the following page. It is a little tricky to prove. If you can, look up a proof in a high school geometry book. Otherwise, write a careful statement of the rule, and then draw several different examples and use your protractor to convince yourself that the rule holds.



6. You may have been flirting with proof once or twice in exercise 5. Discuss the kind of verification that you did use in 5(a), (b), and (c). 5(c) surprises many people. Are you convinced? Did experimentation help to convince you? Was it authority, experience, plausibility, or deduction which convinced you?

ACTIVITY 19

EQUATION OF A CIRCLE

FOCUS:

You found out in Activities 5 and 14 that for each equation of the form $ax + by = c$, there is a straight line (its graph) and for each straight line there is an equation of the form $ax + by = c$. In this activity you will learn a similar relationship for circles.

MATERIALS:

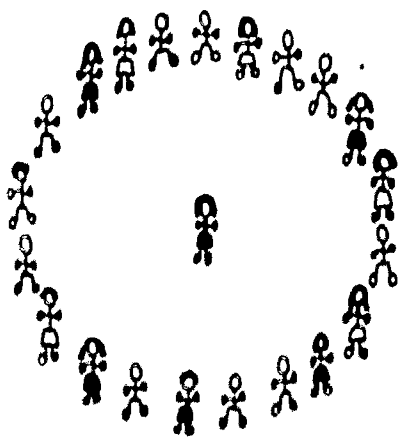
Compass and graph paper.

DISCUSSION:

You may want to refer back to Activity 4 where you discovered how to find the distance between pairs of points in terms of the coordinates of those points.

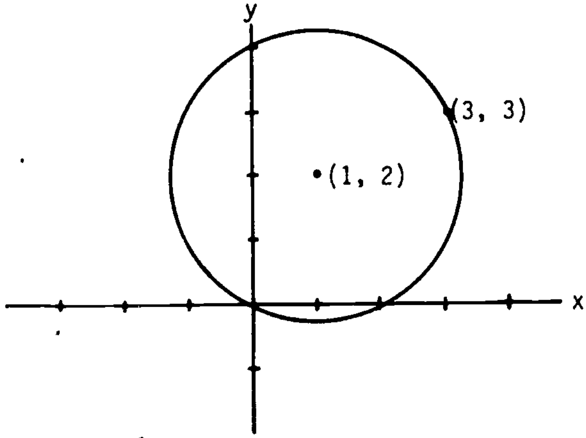
DIRECTIONS:

1.



Carol stands at the center of a circle of people. Which of the people is farthest from Carol? Discuss.

2. a) Suppose that (1, 2) is at the center of a circle and (3, 3) is on the circle. What is the radius of the circle?

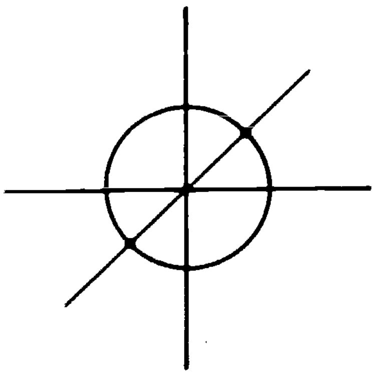


b) In terms of the distance between (1. 2) and (x, y) write down an equation that must be satisfied by any point (x, y) on the circle.

c) Find the coordinates of two points on the circle other than (3, 3).

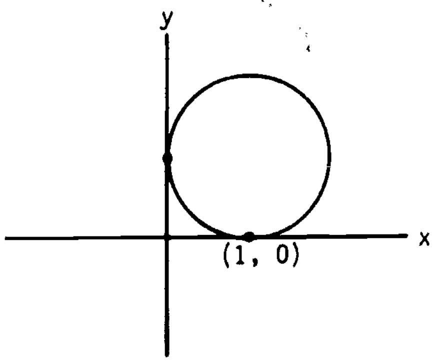
3. If a circle has radius r and center (a, b), find an equation which must be satisfied by any point (x, y) on the circle.

4. The graph of $y = x$ is the straight line shown passing diagonally through the center of the circle below, and the graph of



$x^2 + y^2 = 1$ is a circle with radius 1 and center (0, 0). Find the coordinates of the points where the straight line and the circle intersect.

5. Find the equation of the circle below which is tangent to the x-axis at (1, 0) and which is tangent to the y-axis. (Check a couple of points to see if you are right.)



6. Find the graph of the circle whose equation is

$$(x - 1)^2 + (y - 3)^2 = 9.$$

7. One way to build up geometric concepts is to start out with points as the most simple objects, and to think of all other geometric figures (lines, planes, circles, etc.) as consisting of points.

   a) Do you think it would make sense to a young child (first- or second-grader) to tell him or her that a circle is made up of points? Why or why not?

   b) Does it make sense to you? Relate your answer to this activity.

ACTIVITY 20

AREA VS. PERIMETER

FOCUS:

The perimeter of a shape is the distance around it, i.e., the length of its boundary. The area of a shape is a measure of its surface. The ratio of the perimeter to the area is an indication of the efficiency with which the boundary bounds the area. In this activity we will analyze the very special position that circles occupy with respect to perimeter/area ratio.

MATERIALS:

50 cm of string, graph paper.

DIRECTIONS:

1. Suppose that you had a large plot of land and had 50 meters of fencing material and 20 fence posts. You desire to fence in a plot of land for a garden. Your problem is to determine what shape you should make your plot in order for it to contain the most land.

   a) Make a guess. Give some thought to the way that other people have solved the problem.

   b) Model the problem with a 50-cm piece of string and a piece of graph paper. Try making shapes including ones like those on pages 90 and 91. (You can lay the string on the graph paper in the desired shape, marking certain dots as fence posts.)

   Fill in the table on the following page as you go. You may want to estimate the area of each shape by determining how many square centimeters in one square on the graph paper and then counting the number of squares inside the shape.

Shape	Perimeter	Area	Perimeter/Area
A			
B			
C			
D			
E			
F			
G			
H			

c) Can you characterize the shapes that seemed to enclose the greatest area for the given perimeter? That is, which shapes give the smallest perimeter/area ratio? Write out a statement of the rule that you have observed.

2. For a perimeter of 100, what exactly is the perimeter/area ratio of a

   a) circle?

   b) square?

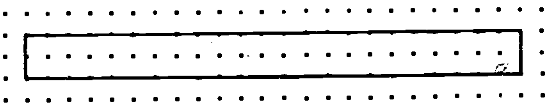
   c) a rectangle whose length is twice its width?

3. Spheres play the same role with respect to the surface area/volume ratio as circles do with respect to the perimeter/area ratio.

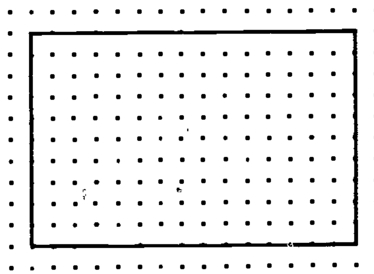
   a) In the light of this fact, conjecture why a cat curls up into a "ball" on a cold day.

   b) List several real-world examples where the special role of circles and spheres with respect to perimeter/area and surface area/volume is employed; e.g., consider the circular cross section of tin cans.

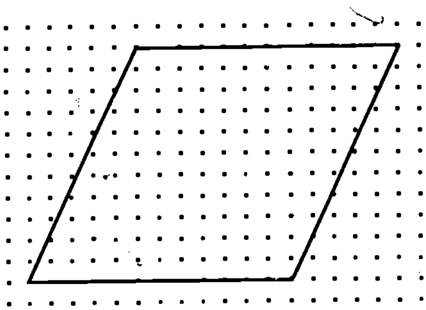
Shape A



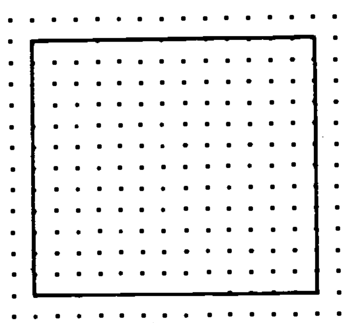
Shape B



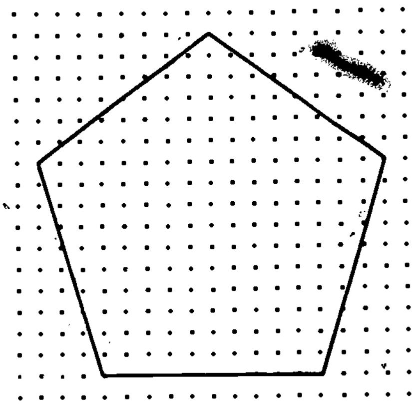
Shape C



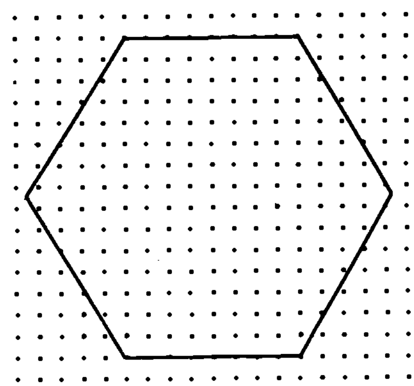
Shape D



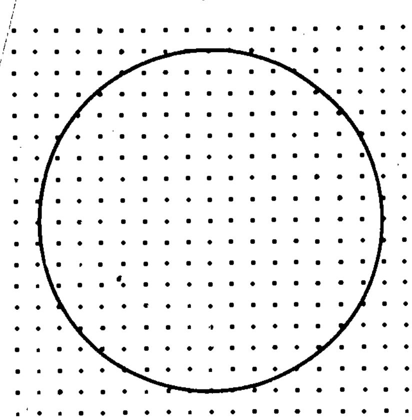
Shape E



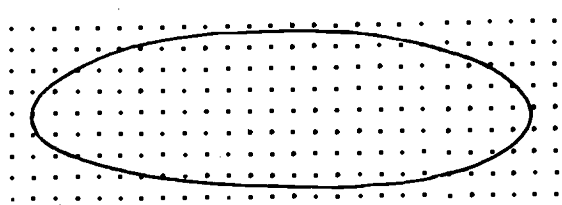
Shape F



Shape G



Shape H



ACTIVITY 21

CIRCLES AROUND YOU

	/	△	○
Occur.			
Anly.			
Appl.			■

FOCUS:

Circles have many fascinating properties. You have briefly analyzed a few of them. In this activity you will have a chance to apply some of what you have learned about circles to a real problem.

DIRECTIONS:

Adventuresome architects have been experimenting with the shapes of buildings. Some have even gone so far as to build circular homes. Here you should:

1. Draw a floor plan for a circular home. Make its complexity suit the time that you have.
2. Consider the implications of circular home design for
   a) Building material costs and heating and cooling costs;
   b) Construction problems (put yourself in the carpenter's shoes);
   c) Aesthetics--what would and would not be possible in decorating?
   d) Human usage--what would be the implications of roundness for the lifestyle of the occupants?
   e) Access and peripheral structures--what about driveways, garages and the like?

ACTIVITY 22

HOW TO TEACH GEOMETRY

FOCUS:

In this activity you will compare the approach of this unit to teaching geometry with the approach of elementary mathematics texts. Then you will plan for teaching a geometry topic to children.

MATERIALS:

Current elementary mathematics text series.

DISCUSSION:

For straight lines, for triangles, and for circles each of the following steps was taken:

- The <u>occurrences</u> and importance of the shape in the real world were noted.
- The shape was <u>analyzed</u> in order to establish certain properties.
- The properties of the shape were <u>applied</u> to some real-world situations.

Various strategies were used to convince you of the validity of certain properties. They included:

- authority,
- experience,
- plausibility argument.

Deductive proof was all but omitted. Keep all of this in mind as you do the following.

DIRECTIONS:

The tasks on the following page can be done in a class discussion, in small groups, or as an individual homework assignment.

1. Select (possibly with the aid of your instructor) an elementary text series. Choose a geometric shape such as a triangle or a circle, and trace the study of that shape through the series. Compare the series approach to study of that shape with that of this unit. In particular, did the text develop:

   - The occurrence and importance of the shape in the child's real world?
   - The analysis of the shape?
   - The applications of the properties of the shape?

2. Outline a series of lessons on quadrilaterals that you might do with children. Fashion your lessons after the approach used for straight lines, triangles, and circles in this unit. Indicate how you would convince the children of the truth of various statements.

3. *OPTIONAL: Only plane shapes have been dealt with in this unit.* Yet the approaches and the point of view of this unit are equally applicable to solid shapes. Choose a solid shape and outline a series of lessons on that shape that you might do with children. Fashion your lessons after the approach in this unit.*

4. *OPTIONAL: If you have done (2) or (3) above, try the first two of your lessons on an appropriate small group of elementary school children.*

---

*More attention is given to solid shapes in the Awareness Geometry unit and the Measurement unit of the Mathematics-Methods Program.

Section IV

VERIFICATION

Most teaching and, for that matter, most communication involves convincing someone that something is true. Many decisions that one faces in life involve deciding what is true. The techniques that one uses to convince others and the evidence that an individual will accept for truth vary widely from person to person and from situation to situation. For example, contrast the methods of convincing used in Sections I, II, and III of this unit with the proofs of high school geometry. In this section you will study these questions of verification with an eye toward choosing appropriate forms of verification for teaching children.

In Activity 23 you will be asked to analyze the different ways in which one comes to believe the truth of something. In Activity 24 you will analyze just what it takes to convince you of something. The special role of proof in verification is the subject of Activity 25. And, finally, in Activity 26 you will analyze the implications of all of this for teaching children.

MAJOR QUESTIONS

1. Outline an example of verification of each of the following kinds:
   - authority,
   - experience,
   - plausibility argument,
   - deduction.

2. Give a complete discussion of the verification situation in Activity 9. Be sure to make it clear that you understand the point that was being made and the message of the activity for pedagogy.

ACTIVITY 23

HOW DO YOU KNOW IT'S TRUE?

FOCUS:

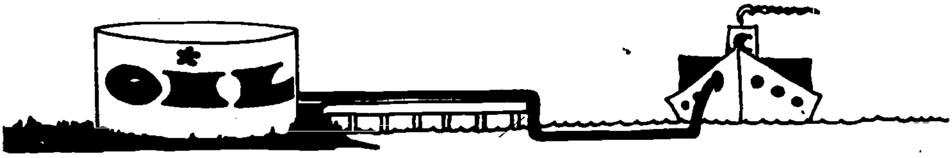
Did your teacher tell you? Did you hear it on the radio? Did you actually see it? Or did you deduce it from other things that you knew to be true? In this activity you will have some experiences with different kinds of verification. You are doing this as a step toward analyzing appropriate verification for children and toward understanding the role of proof in high school geometry.

MATERIALS:

Current elementary mathematics text series.

DISCUSSION:

As an illustration, consider the following ways of knowing that increased foreign oil prices have contributed greatly to inflation in the U.S.



Way #1: Secretary of State Kissinger says it's true, and I believe him.

Analysis of Way #1: In this case you are accepting the truth of this statement on someone else's authority. For certain kinds of knowledge, authority may be the only reasonable source (e.g., the date when Columbus discovered America). It is obvious, however, that your authority may be wrong--either inadvertently or intentionally.

Way #2: I run a plastics company, and we purchase foreign oil. Because of increased oil prices, we have had to raise the prices of all of our products. My prices have risen because of the price of oil, so oil must be behind U.S. inflation.

Analysis of Way #2: In this case you are generalizing from your own experience. Whether or not experience is an appropriate source of truth depends both on the kind of truth being sought and on how representative the experience is. For example, if your company were a small one and no other company was having your experience, your generalization would not be appropriate.

Way #3: The government says that it's true. It's happening to me. I know that the U.S. imports a lot of oil, and I know that increased commodity prices usually lead to increased prices in general. It just makes sense to me.

Analysis of Way #3: In this case you are saying that the statement is consistent with what you know. It seems reasonable or plausible. Most of us use this mode for deciding truth in many different situations. It is still possible, however, that you are overlooking something. It might be that increased wages due to union demands had such a widespread effect on the world economics that reducing foreign oil prices would do little to abate inflation.

Way #4: High foreign oil prices lead to large trade deficits. Large trade deficits lead to weakened currency. Weakened currency leads to inflation. So high foreign oil prices lead to inflation.

Analysis of Way #4: This is a deductive argument in which you derive new truths from accepted truths. It is not always possible to get a matter sufficiently clarified and simplified to be able to subject it to deductive proof. But deduction is generally regarded as the highest form of verification. One must beware, however. The truth of a deduced conclusion depends on the truth of the statements from which it was deduced. If, for example, a large trade deficit would lead to a declaration of war instead of weakening the currency, the economy might strengthen considerably. In such a case our deductive argument would be valid, but it would be based on a false assumption, so the argument would not lead us to truth.

This example was intended to bring to your attention the fact that different people in different circumstances may come to accept the truth of different statements on the basis of:

- authority
- experience
- plausibility
- deduction.

It was also intended to illustrate the fact that doubt can be raised concerning truth that has been accepted on any basis.

DIRECTIONS:

1. a) List some things in your life that you accept on authority.

   b) List some things in your life that you accept as true on the basis of experience.

   c) List some things in your life that you accept as true because they seem plausible.

   d) List some things in your life that you have come to accept on the basis of a deductive argument.

2. Choose a current event, and analyze the ways in which individuals might have come to "know" about it. (Watergate certainly abounds with examples where most forms of verification proved unreliable.)

3. In the first three sections of this unit, efforts were made to convince you of the truth (i.e., to verify) various geometric facts. Different techniques were used to verify these facts. Look back and:

   a) Find an instance where <u>authority</u> was evoked to verify.

   b) Find an instance where <u>experience</u> was provided as a step in verification.

   c) Find an instance where a <u>plausibility</u> argument was intended to be the basis of verification.

   d) Was <u>deduction</u> used? Look carefully.

4. Go to an elementary mathematics textbook and find instances where

   a) authority

   b) experience

   c) plausibility

   d) deduction*

   were used as techniques of verification. Your examples need not come from geometry.

*You may not succeed in finding deduction. It depends on the text series and on the grade level.

ACTIVITY 24

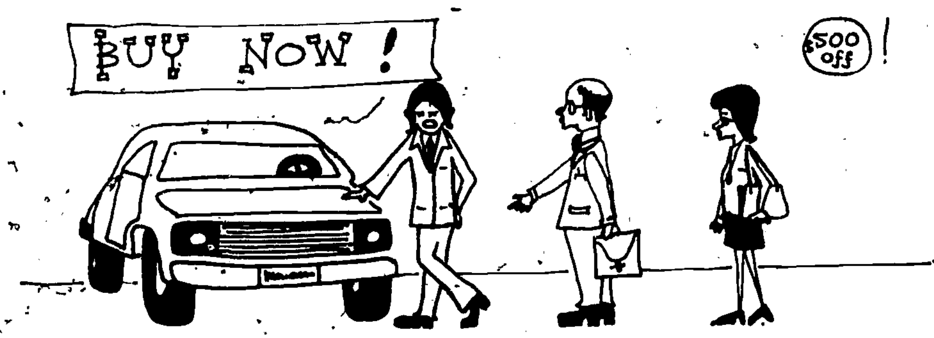
WHAT WILL IT TAKE TO CONVINCE YOU?

FOCUS:

You have seen that there are different ways in which one can verify that something is true. Now you will briefly confront the issue of who will accept what kind of verification under what circumstances. This issue has direct relevance to planning instruction for children.

DISCUSSION:

Sophistication seems to be a major variable in determining what it takes to convince someone of something. Two usual ingredients of sophistication are knowledge and experience. It seems that the more knowledge and experience one has in an area, the more that person is going to be aware of alternatives and subtleties which may have something to do with the truth of something. Consider the following for examples.



New car salesman: This model gets the best mileage available--30 miles per gallon.

First buyer: How do you know that?

Salesman: This fact was the result of an independent study.

First buyer: Great. I'll take it.

Second buyer: I have a couple of additional questions.

- At what speed, over what terrain, and at what temperatures were these results found?
- What other cars were tested?
- Who financed and who supervised the test?
- Will you guarantee that the particular car that I buy will get that kind of mileage?

The second buyer may be no smarter and no older than the first, but he or she certainly has more information concerning the various things that could affect a mileage rating. The last question asked suggests that the second buyer may have learned from experience that an individual car may fail to live up to even the valid claims for a line of cars.

DIRECTIONS:

1. Write out a scenario like the one above for some other circumstance.
2. Recount some "fact" which you once accepted easily in your own life and for which you required increasing verification as your knowledge and experience related to it grew.
3. Let $f(x) = x^4 - 6x^3 + 11x^2 - 6x$. Is $f(x)$ equal to 0 for all x?

   a) Try 0, 1, 2, 3.

   b) On the basis of what you have done so far, do you think that some people might be convinced that the answer is "Yes"?

   c) If someone knew that a fourth-degree polynomial has at most four roots, what would his or her reaction to (a) be?
4. Go back to Activity 9. You will recall that we tried to trick you there.

   a) How did we try to convince you there? What was wrong?

b) What is the message from that activity for choosing examples to convince children?

c) Think up a false generalization that you could make believable by presenting carefully chosen examples. It needn't be in geometry, or even in mathematics for that matter.

ACTIVITY 25

PROOF

FOCUS:

It may have occurred to you that the word "proof" has not come up in either of the two previous activities. "Proof" is a name that is given to formal deductive arguments based on stated assumptions or axioms. Such proofs are a major part of most high school geometry courses but are not common in most individuals' everyday lives. This activity is designed to give you a brief insight into the nature of proof and the role of proof in mathematics.

DISCUSSION:

Way back in Activity 9 we may have led you to make a false generalization concerning right triangles based on your experience. We have seen that both authority and plausibility can also lead one to false conclusions. The following example will remind you that deduction can also result in false conclusions.

Think about the following two statements: "If you are interested in this course, you will do well in it," and "Those who do well in this course major in mathematics." From these two statements we conclude that, "If you are interested in this course, you will be a mathematics major." Do you agree with the conclusion? If it is false, is it because the logic is wrong? No. The logic is correct. From statements of the form, "If A, then B," and "If B, then C," it is legitimate to conclude, "If A, then C." What is wrong then? Let's analyze the truth of the statements themselves. It is pretty clearly false that every student who is interested in this course will do well. A student could easily have a poor background, or could have had some periods of illness, or encountered other circumstances that would preclude his doing well. The second statement is false also. Very few students who take this course become mathematics majors. Certainly there are many students who do well in this course but who do not become mathematics majors. So while our deduction was valid, our conclusion was false since it had been derived from false statements.

If deduction can lead to false conclusions just as other forms of verification can, why is proof such a big deal? The answer is that while deduction can lead to false conclusions, deduction will not deduce false conclusions from true assumptions. The truth of deduced conclusions depends on the truth of the statements from which the conclusions are deduced. In high school geometry, for example, the course starts with certain assumptions called axioms. Then deductive arguments are used to derive conclusions, called theorems, from the axioms. In this way one shows that the truth of the theorem depends on the truth of the axioms.

Such formal deduction is a common activity of mathematicians. In fact, Bertrand Russell once said, "A mathematician never knows what he is talking about nor whether what he is saying is true." Russell's point was just the one that we have been trying to make. Namely, deductive proofs do not establish the truth of a statement (theorem) in the real world. They only establish that the truth of the statement depends on the truth of certain assumptions (axioms). You may still quite reasonably ask, "Why spend all of the time on proofs that we do in high school geometry?" There is some disagreement concerning the answer to that question. Here is one reasonable answer.

> The organization of a body of knowledge by deductively deriving it from acknowledged assumptions is an important human mental activity. Just as writing poetry and performing scientific experiments are important, so is carefully building a system of statements by deduction. Most people should have this experience.

You ask, "If the main objective is experience with a certain kind of thought process, why should it happen in geometry?" That is a good question. It could happen in algebra. It could happen in a course on philosophy, since deductive reasoning has been an important tool of philosophers. It happens that some 2000 years ago, Euclid organized geometry by deriving the theorems which were known at that time from a few very reasonable axioms. Eudlid's work has been refined, and it provides one of the easiest examples of an interesting axiom system to work with.

In order to gain brief experience with an axiom system different from Euclid's, do the following.

DIRECTIONS:

Consider the following assumptions about hedges and bushes.

Axiom 1: Every hedge is made up of bushes and has at least two bushes in it.

Axiom 2: There are altogether at least three bushes.

Axiom 3: Each pair of bushes is part of one hedge and only one hedge.

Axiom 4: If one has a hedge and one has a bush that is not in the hedge, there is one and only one hedge which contains the bush and does not touch the original hedge.

1. Using only the axioms, prove the following theorem.

   Theorem 1. If there are two hedges, then there are three hedges.

2. Once you have proved Theorem 1, prove

   Theorem 2. If there are three hedges, then there are four hedges.

3. Do Theorems 1 and 2 have anything to do with true facts about bushes and hedges? Explain.

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ACTIVITY 26

VERIFICATION FOR KIDS

FOCUS:

What do the last three activities have to say about teaching kids? It would not be realistic to expect a clearcut answer concerning the teaching of mathematics to children. You should, however, be gaining some insights into appropriate kinds of verification to use in teaching children.

DIRECTIONS:

Address yourself to the following questions in a class discussion:

1. Authority — Plausibility
   Experience — Deduction

   are all approaches that can be used to convince someone of the truth of something. Moreover, the kind of convincing that a person requires in a situation depends on the knowledge and sophistication that the person brings to the situation. Discuss the kind of verification that you would use in each of the following situations:

   a) To convince a third-grader that the diagonals of a parallelogram bisect each other.

   b) To convince a bright and skeptical eighth-grader of the validity of the Pythagorean Theorem.

   c) To convince a second-grader that all four sides of every square have equal length.

   d) To convince a third-grader that the sum of the lengths of two sides of each triangle is greater than the length of the third side.

2. What do you think about the way high school geometry is taught? Would proofs be better presented in a course on logic? Did you get anything out of high school geometry? What?

REQUIRED MATERIALS

ACTIVITY	AUDIO-VISUAL AND OTHER RESOURCES	MANIPULATIVE AIDS
Overview	Slide-tape: "Analysis of Shapes," cassette recorder and projector. (Optional)	
1	Juster, Norton. The Dot and the Line: A Romance in Lower Mathematics. New York: Random House, 1963. (Optional)	A globe.
3	Current elementary school mathematics textbook series.	Straightedge, compass.
4		Graph paper.
5	Current elementary school mathematics textbook series.	Graph paper, straightedge.
7		Paper, ruler, scissors, construction paper strips, brads.
8		Straightedge, compass.
9		Graph paper, metric ruler, straightedge, compass.
11		Paper, scissors, ruler, protractor.

ACTIVITY	AUDIO-VISUAL AND OTHER RESOURCES	MANIPULATIVE AIDS
12		Metric ruler, protractor, straightedge, compass, scissors, graph paper, tracing paper.
13		Graph paper, protractor, ruler.
14		Graph paper, straightedge.
17		Compass, straightedge.
18		Straightedge, compass, protractor.
19		Compass, graph paper.
20		50 cm of string, graph paper.
22	An elementary school mathematics textbook series.	
23	An elementary school mathematics textbook series.	

Continued from inside front cover

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* * * * * * * * * *

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This unit integrates the content and methods components of the mathematical training of prospective elementary school teachers. It focuses on an area of mathematics content and on the methods of teaching that content to children. The format of the unit promotes a small-group, activity approach to learning. The titles of other units are *Numeration, Addition and Subtraction, Multiplication and Division, Rational Numbers with Integers and Reals, Awareness Geometry, Transformational Geometry, Measurement, Graphs: The Picturing of Information, Number Theory, Probability and Statistics,* and *Experiences in Problem Solving.*

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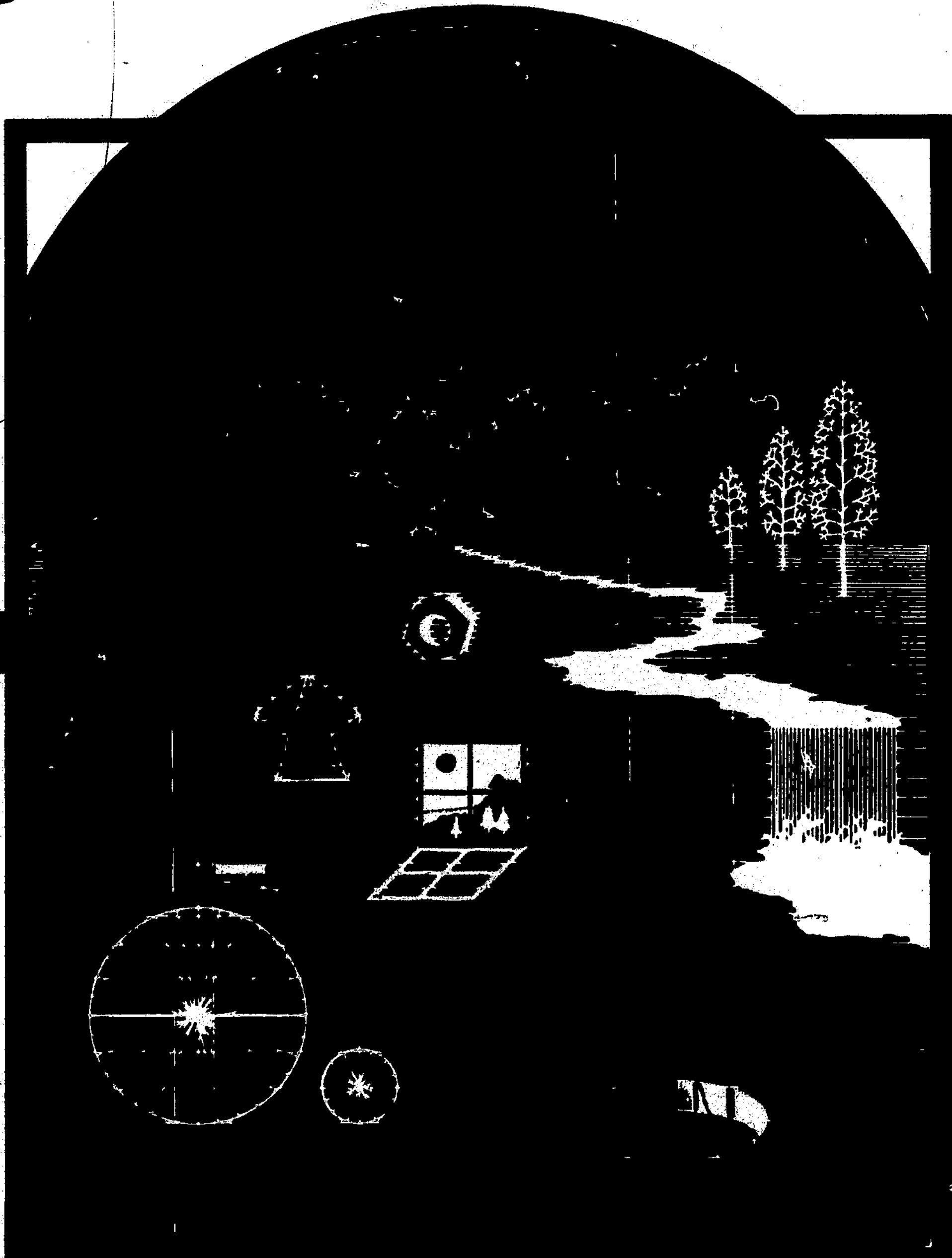
INSTRUCTOR'S MANUAL
to accompany

ANALYSIS OF SHAPES



Mathematics-Methods Program
units written under the direction of

John F. LeBlanc or Donald R. Kerr, Jr. or
Maynard Thompson

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Instructor's Manual to accompany

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Prepared by

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Under the direction of

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CONTENTS

ANALYSIS OF SHAPES
INSTRUCTOR'S MANUAL

This unit, as do other units of the Mathematics-Methods Program, involves one as an adult learner in activities which have implications for teaching children. One works with concepts that children might learn, with materials that children might use, and on activities that might be modified for use with children. The objective is to provide growth in understanding and enjoyment of geometry along with increased ability and desire to teach geometry to children.

The "Introduction to the Geometry Units of the Mathematics-Methods Program" which appears on pages 1-4 of the student's unit explains the spirit of the series of four geometry units and describes the allocation of topics to the units. It would be a good idea for the instructor to become acquainted with the contents of this introduction before deciding to use the unit.

THE CONTENT OF THE UNIT:

As is noted in the introduction, this unit studies straight lines, triangles, and circles, by investigating important real-world occurrences of them, analyzing them to determine their properties, and then applying the fruits of the analyses to real-world problems. The final section of the unit deals with problems of verification and places in perspective the informal methods of elementary school geometry and the formal approaches of high school geometry. The allocation of activities to these purposes may be summarized in the following table, whose columns correspond to the unit's four sections, whose rows define the approach to the topic, and whose entries are activity numbers.

	1: Lines	2: Triangles	3: Circles	4: Verification
Occurrences	1	7	16	23,24,25
Analysis	2,3,4,5,14	8,9,11,12,13	17,18,19,20	
Applications	6	10,14,15	21	
Teaching Children	Parts of most activities; all of 22			26

TIMETABLE SUGGESTIONS:

The time spent on this unit will depend upon a number of factors, including the mathematical background of the students, the time available for the unit, and the relative emphasis to be given to mathematics content and to teaching methods. The chart below suggests three alternatives for scheduling the work of the unit, each predicated on a different set of values and priorities. We have characterized the alternatives as:

A. Mathematics & Methods, Leisurely--for an integrated content and methods course in which there is time to deal with this unit in some detail. About 27-33 single periods would be needed.

B. Mathematics & Methods, Rushed--for an integrated content and methods course in which this unit has low priority or in which time is at a premium. About 15-18 single periods would be needed.

C. Mathematics Emphasis--for a course which is concerned mainly with mathematics content for prospective teachers. About 12-15 periods would be needed.

These are just three of many possible schedules; we hope they will be helpful in deciding how to use the unit. The numbers in the table below are estimates of the number of class periods needed for each activity. The symbol "HW" indicates that all or part of the activity could be done as homework. When "HW" precedes the number of periods, advance preparation by students is suggested; when "HW" follows the number of periods, homework to finish the activity is intended. "HW" alone indicates that the entire activity could be done outside of class.

Section	Activity	Alternative Timetables: A. Mathematics & Methods, Leisurely	B. Mathematics & Methods, Rushed	C. Mathematics Emphasis
1	1	.5	HW,.25	HW
	2	.5-1	HW,.75	HW,.25
	3	1	.5	HW,.75
	4	1	.5	.5
	5	HW,1-2,HW	HW,1,HW	HW,.5,HW
	6	1	HW,.5	HW
2	7	HW,1	HW,.5	HW,.25
	8	2	HW,1	HW,.75
	9	2	1-2	1
	10	2	HW,1	.5,HW
	11	1-2,HW	1,HW	HW,.5
	12	1-2,HW	HW,1	HW,.5
	13	1-2,HW	HW,1	HW,.5
	14	1-2,HW	1,HW	1,HW
	15	HW,1	HW,1	HW,1
3	16	HW,.5	HW	HW
	17	.5	HW,.5	HW,.25
	18	1-2	HW,.5-1	HW,.75
	19	HW,1	HW,.5	HW,.5
	20	1-2,HW	1,HW	HW,1,HW
	21	1	HW	HW
	22	1,HW	.5,HW	HW
4	23	HW,.5	HW	HW,.5
	24	HW,.5	HW,.5	HW,.5
	25	1,HW	omit	1,HW
	26	HW,1	omit	HW
Total Time Required		27-33	15-18	12-15

SECTION I
STRAIGHT LINES

INTRODUCTION:

This section is concerned with one of the most basic of geometric shapes, the straight line. Activity 1 deals with occurrences of straight lines in the real world. Activities 2, 3, 4, and 5 analyze straight lines by studying intersections of straight lines, angles, straightedge-and-compass construction, and coordinate geometry. Activity 6 concludes the section by applying some of the analyses to real-world problems.

MAJOR QUESTIONS (DISCUSSION):*

1. This question was chosen to test students' ability with straightedge-and-compass construction. Students may be expected to perform this construction neatly and accurately.
2. This question tests the skill developed in Activity 4. A quick way of checking the students' graphs is to see whether (0,-4) and (2,0) lie on the line.
3. An answer to this question is given in the answer to question 7, Activity 1.

---

* The major questions which appear at the beginning of each section attempt to capture the essence of the material which follows. As such, they may be seen as advance organizers. They may be put to a variety of specific uses; for example, they may be assigned as homework, or modified for use as examination items, or simply discussed in class. In any case, their purpose is to provoke thoughtful consideration of the major concepts of the section.

ACTIVITY 1
STRAIGHTNESS

MATERIALS PREPARATION:

Globe (more than one if possible); The Dot and the Line (book or movie)(optional)

COMMENTS AND SUGGESTED PROCEDURE:

Through the emphasis on occurrences, the students focus on real-world instances of something considered only in the abstract when covered in higher mathematics courses. If students are unaware of the meaning of "vector," they will lose a great part of the humor in the moral of Juster's story. Question 7 only looks easy and the answer must be carefully presented for the less visually oriented person. This is not a trick question but certainly a thought-provoking one.

ANSWERS:

1. Some samples are lines of type for easier reading, classes' lining up so that mischief makers are easy to spot, traffic lanes to share the road without argument.

2. We really couldn't come up with one unless we went to mathematics. The purpose of this question is to put the college student into the position of finding words to describe this concept which is so easy to visualize but so hard to define.

3. A straight line is (a) the shortest distance between two points in a plane; (b) it is symmetrical; (c) two points define it; (d) its length is unique. These properties are obviously useful when measuring. One example is measuring a board of unknown length with a short piece of string whose length is known. Suc-

cessive placement of the string along the board will result in knowledge of its length.

5. Ideas could include asking them to walk a distance with and without obstacles; having them walk a path with and without their eyes opened; having them pull a string that has one end tied; putting rubber bands on geoboards.

6. No deviations allowed!

7. In the shortest distance sense, only great circles are analogous to straight lines. --A line parallel to a great circle is not a great circle but latitude "lines" never intersect.

ACTIVITY 2
STRAIGHT LINES AND THEIR INTERSECTIONS

MATERIALS PREPARATION:

Visuals for illustrating skew lines.

COMMENTS AND PROCEDURE:

This can be a deceptive exercise. The questions may look trivial, but they ask the student to conceptualize spatial relationships. In question 2, we have assumed that only four angles (any two adjacent angles supplementary) are formed when two lines intersect. (See Diagram 1.) If one chooses to allow more angles, e.g., those shown in Diagram 2, some responses will be different.

Diagram 1

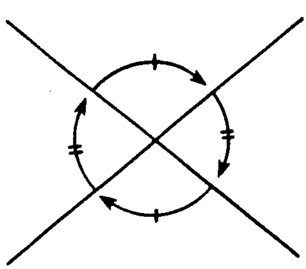
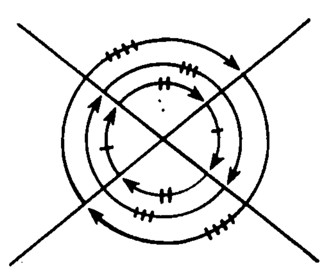


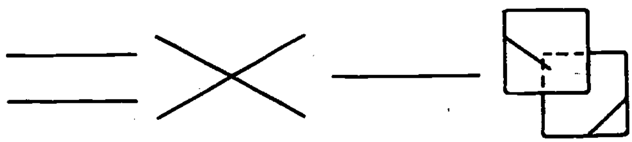
Diagram 2



ANSWERS:

1. a) Parallel, intersecting, coincident, skew.

   b)

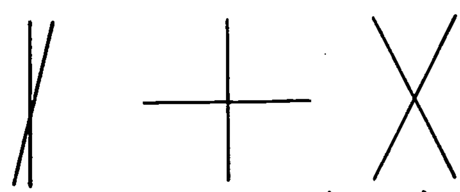


   c) An infinite number; if parallel, one; if intersecting, one; if skew, none; if coincident, an infinite number.

   d) Parallel, intersecting, coincident.

   e) The non-empty intersection of two distinct lines in a plane. Yes. An infinite number of lines pass through one point. Only one line passes through two points. Either one or no single line could pass through three points.

2. No, yes, yes, yes.

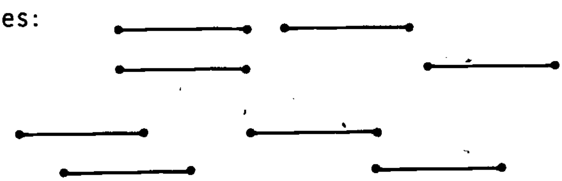


3. a) We know $\measuredangle A + \measuredangle B = \measuredangle A + \measuredangle C = 180°$. So $\measuredangle B = \measuredangle C = 35°$ and $\measuredangle D = 145°$.

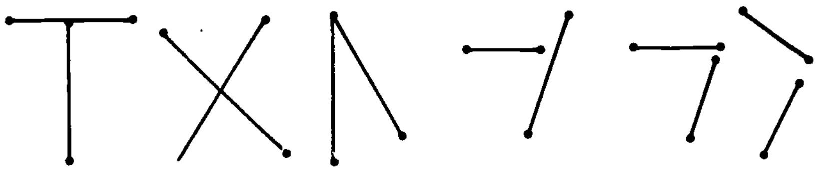
   b) $\measuredangle A = \measuredangle C = \measuredangle E = \measuredangle G = 122°$
   $\measuredangle B = \measuredangle D = \measuredangle F = \measuredangle H = 58°$

   It is useful here to remember that the transversal of parallel lines makes congruent angles of alternate interior angles; that vertical angles are equal; and that interior angles on the same side of the transversal are supplementary.

4. Consider two pencils as representing line segments. Now we have parallel cases:



intersecting cases:



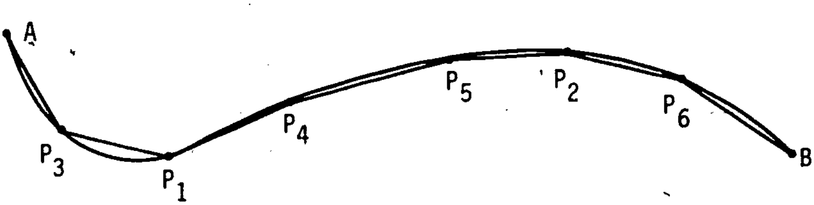
coincident cases:

Consider a room for skew lines. The "segment" made by the intersection of the ceiling and a wall, and the "segment" made by an adjacent wall intersecting a floor are skew.

5. a) When a child lies down, the distance from his head to his feet is a length. Length is how long something is, or how far away something is.

   b) A first-approximation length of the curve AB could be the total length of AP_1, P_1P_2, and P_2B.

A better approximation can be obtained by choosing more intermediate points.



The successive approximations with more and more intermediate points give total lengths which come closer and closer to a number which can be thought of as the "true" length.

c) There are many different ways that this can be done, and the point of the question is to encourage thought about the alternatives. Most methods involve either making a model of one segment and comparing it with the other, or using an instrument such as a ruler to measure both.

ACTIVITY 3
STRAIGHTEDGE-AND-COMPASS CONSTRUCTION

MATERIALS PREPARATION:

One straightedge and compass per student; copies of a current elementary mathematics text series.

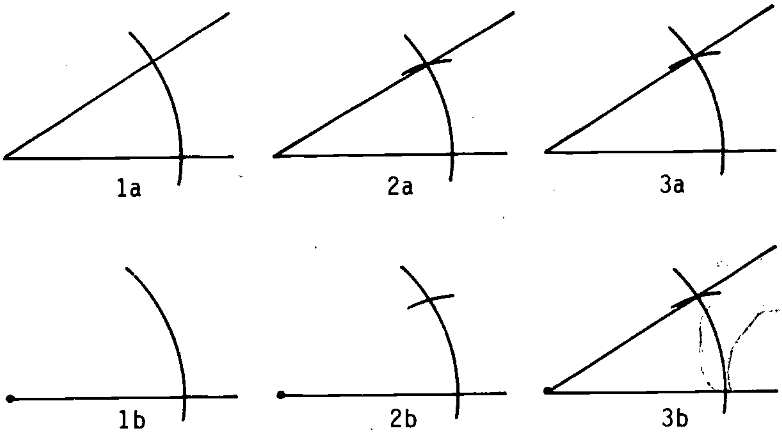
COMMENTS AND SUGGESTED PROCEDURE:

Students may wonder why they are asked to repeat constructions of high school geometry until they answer question 6. Because the elementary series are didactic in their constructions, the assignment to examine them comes at the end of the activity so that students would experiment. Question 6 is very important; students should realize that elementary school children are now being given geometrical experiences that were previously confined to high school.

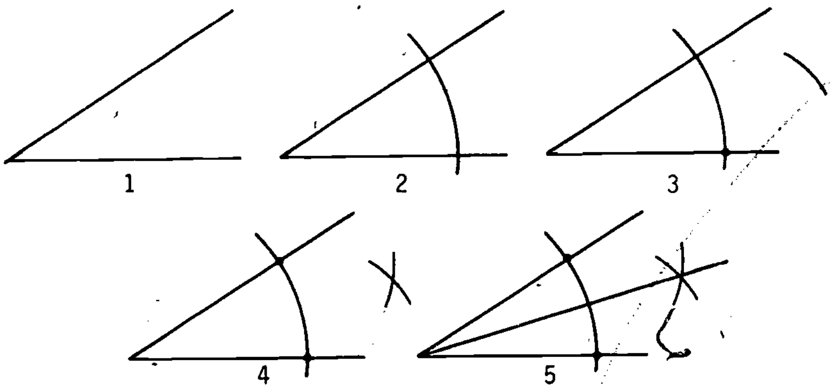
The answers given below show steps that can be used in carrying out the constructions. For students who are having difficulty with the constructions, a series of overlays for the overhead projector, one step to each layer, may be helpful.

ANSWERS:

1.

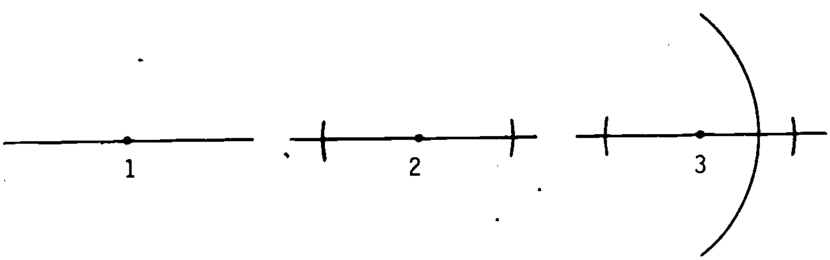


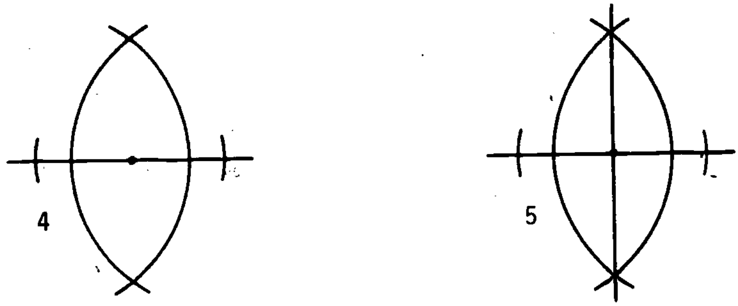
2.



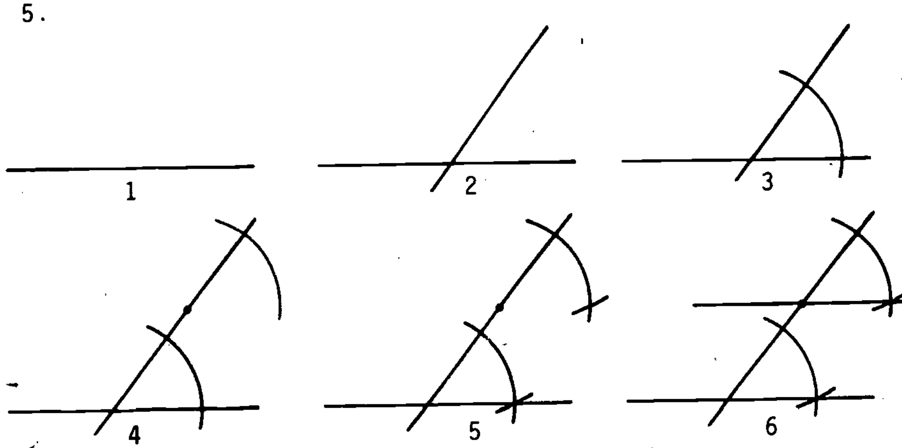
3. *OPTIONAL: This construction is difficult to show by a diagram, but it is done by reproducing each side and angle in turn.*

4.





5.



6. The answer to this question will depend upon the text series chosen. Some students may be surprised to find that the constructions they did in high school are now being done in the elementary school.

ACTIVITY 4
POINTS AND NUMBER PAIRS

MATERIALS PREPARATION:

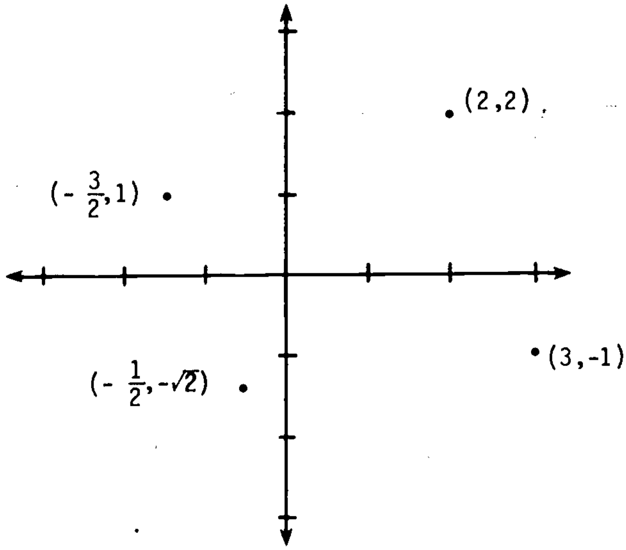
Graph paper, straightedge and compass.

COMMENTS AND SUGGESTED PROCEDURES:

Activity 4 and Activity 5 give a brief exposure to the Cartesian coordinates and the powerful contribution they make to geometry. In this activity, no reference is made to experiences children have in elementary school. Development of the coordinate system there is often done through the topic of graphs. Question 1 requires the plotting of $-\sqrt{2}$. Students may wish to construct the length $\sqrt{2}$ by taking the length of the hypotenuse of an isosceles right triangle or they may estimate it as 1.4. Question 4 is a game situation to add personal involvement to the coordinate system.

ANSWERS:

1.



2. A(0,0) B(2,0) C(-2,-2) $D(-1,1\frac{1}{3})$ or $D(-1,\frac{4}{3})$

3. a) III b) I c) IV

5. The analogy is that two references uniquely determine a location. There is only one theatre seat (17,W) and only one house (719, Cottage Street) or one city located at (37°N, 70°W).

ACTIVITY 5
EQUATIONS AND LINES

MATERIALS PREPARATION:

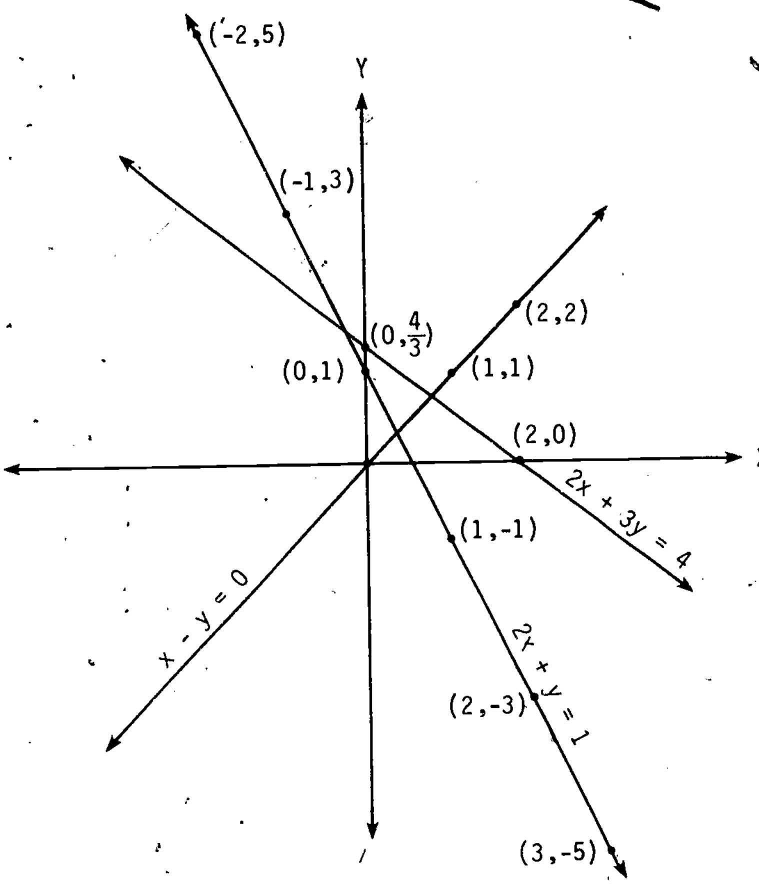
Graph paper, straightedge, elementary text series.

COMMENTS AND SUGGESTED PROCEDURE:

This activity might be called a snapshot of analytic geometry. Opportunities for different students to explain how they got their answers should be provided. In particular, question 5 should be thoroughly discussed. Question 7 will be an eye opener if the 1974 Ginn and 1974 Fields K-6 mathematics texts are among those surveyed.

ANSWERS:

1 & 2.



3. b) Yes

$2x + y = 1$	$x - y = 0$	$2x + 3y = 4$
$(\frac{1}{2},0)$	$(\frac{1}{2},-\frac{1}{2})$	$(\frac{1}{2},1)$
$(2,-3)$	$(2,-2)$	$(2,0)$
$(\frac{17}{11},-\frac{23}{11})$	$(\frac{17}{11},-\frac{17}{11})$	$(\frac{17}{11},\frac{10}{33})$

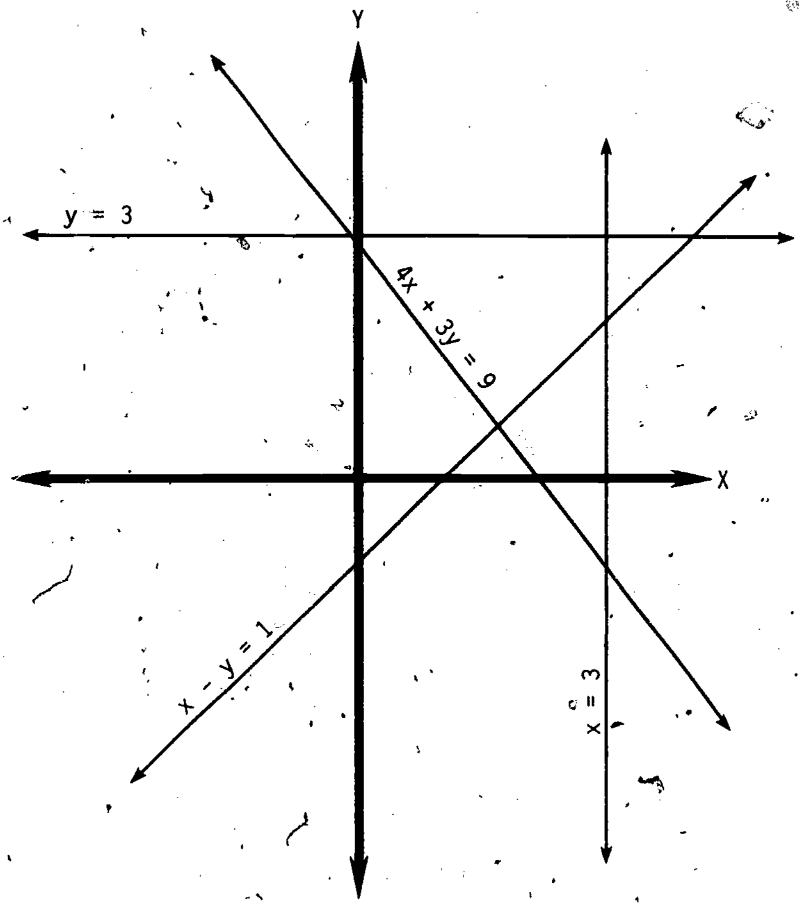
4. 2x + y = 1 a = 2 b = 1 c = 1

   x - y = 0 a = 1 b = -1 c = 0

   2x + 3y = 4 a = 2 b = 3 c = 4

5. The equation is x + y = +1.

6.



ACTIVITY 6

PLAYING IT STRAIGHT

MATERIALS PREPARATION:

None

COMMENTS AND SUGGESTED PROCEDURE:

This activity seeks to clarify concepts of preceding activities by having the students apply their knowledge in new situations. Discussion of all answers is strongly encouraged. We provide one possible way for question 6.

ANSWERS:

1. Only two are needed to fix its position, because two points determine a unique line. Other nails may be added for greater adhesion.
2. Landscaping could be done after a period of time when the paths are formed by the traffic patterns. Paths are likely to be straight lines between important points such as entrances to buildings, gates, etc.
3. No, because $110° + 110° + 45° + 45° \neq 360°$.
4. There is no unique shortest path. Any path from A to B which goes only to the right and down will be a "shortest path."
5. There are several ways to tackle the problem. One set of strategies involves making templates of various kinds to compare the drawing with the building. Another approach would be to read off from the blueprint a "distance out" from the vertex and a corresponding "distance across" the corner at that distance.
6. One can use rope knotted at intervals of 3, 4, and 5 units to make a "rope stretchers triangle" containing a right angle. Another useful fact is that when the sides have the correct lengths and the diagonals are equal, the quadrilateral will be a rectangle.

SECTION II
TRIANGLES

INTRODUCTION:

This section contains nine activities related to triangles and their properties. Activity 7 deals with occurrences of triangles in the real world. Activities 8, 9, 11, 12, 13 analyze the angles and sides of triangles and the relations between triangles. Activity 10 applies the Pythagorean Theorem to the problem of finding lengths in coordinate geometry; Activity 14 uses properties of similar triangles to solve the problem of finding an equation for a line passing through two given points. Activity 15 concludes the section with other applications of the facts and principles concerning triangles that students have studied.

MAJOR QUESTIONS (DISCUSSION):

1. Protractors are not to be used to answer this question. Either a formula or a triangulation may be used to conclude that the sum is 540°.

2. {1 cm, 3 cm, 1 cm} and {1 cm, 2 cm, 3 cm} do not form triangles. {6 cm, 10 cm, 8 cm} and {5 cm, 3 cm, 4 cm} make similar triangles; as do {4 cm, 4 cm, 5 cm} and {2 cm, 2 cm, 2.5 cm}.

3. a) Many examples of triangles being used for stability may be found.

   b) The stability of the triangle is a consequence of the fact that three sides determine a unique triangle (SSS).

ACTIVITY 7
THE IMPORTANCE OF TRIANGLES

MATERIALS PREPARATION:

Rulers, one per student or pair of students; strips of construction paper or light cardboard approximately 2 cm x 20 cm, about ten per student; brads (brass paper fasteners), about 20 per student; scissors; scratch paper, about five to ten sheets per student.

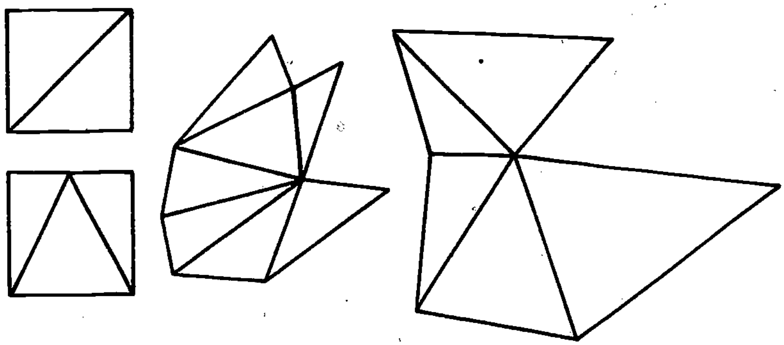
COMMENTS AND SUGGESTED PROCEDURE:

This activity has two main parts which cover three important facts about triangles. In the first part, triangulation is used to introduce a method for finding a relationship between the number of sides of a polygon and the sum of its interior angles and to introduce the idea of the stability of the triangle and of triangulated shapes. The second part is concerned with tessellation of triangles and other figures.

The amount of time an instructor spends on this work will depend upon his class' previous exposure to these ideas, through, for example, Activity 2 of the Awareness Geometry unit. For classes which have not done this kind of work before, the activity could be done in class with discussion of its three main ideas. For classes which have encountered this kind of work before, this activity could be done quickly in class or assigned as homework.

ANSWERS:

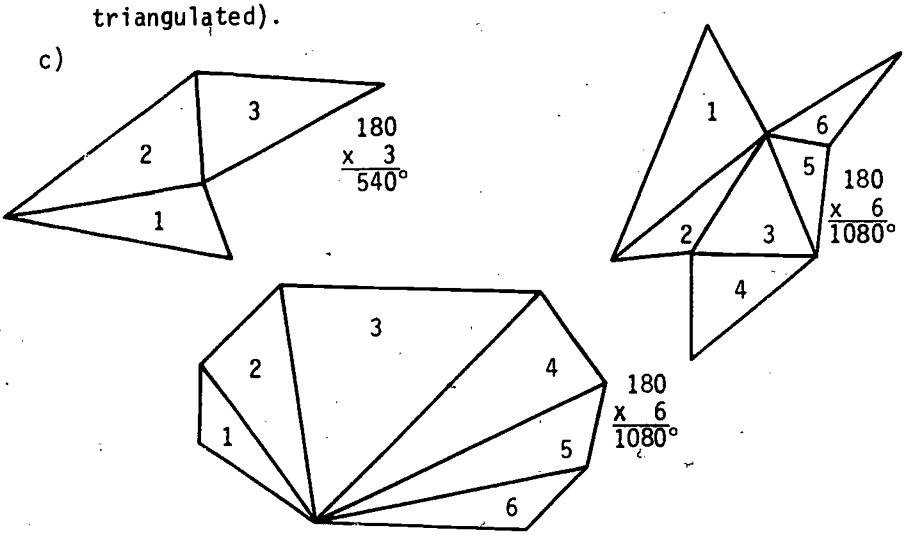
1. a)



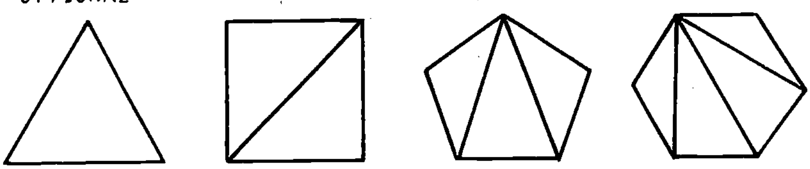
NB These are examples of correct solutions, not the only correct solutions.

b) Figures with curved sides (e.g., circles, ellipses cannot be triangulated).

c)



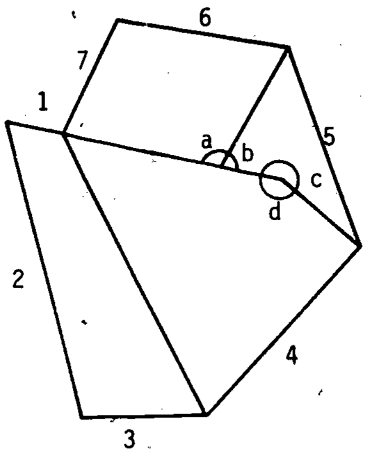
d) *OPTIONAL*



3 sides	*4 sides*	*5 sides*	*6 sides*
1 triangle	*2 triangles*	*3 triangles*	*4 triangles*
180°	*360°*	*540°*	*720°*

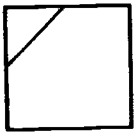
Formula: sum of measures of interior angles = (number of sides − 2) times 180°

e) Any polygon can be "quadrilaterated." The sum of the interior angles of any quadrilateral is 360°. These facts may be used to find the sum of the interior angles of any polygon, but there are complications, as the following example shows.

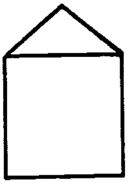


Take the septagon (7 sides) at left and "quadrilaterate" as shown. The 4 quadrilaterals have 4 x 360° = 1440° as the sum of their interior angles. However, angles a, b, c, and d whose sum is 180° + 360° = 540° are counted but are not interior angles of the polygon. So the sum of the interior angles of the polygon is 1440° - 540° = 900°. This is the same as the answer obtained by applying the formula (n - 2) x 180 where n = 7.

2. b) Complete triangulation will ensure complete stabilization, but partial triangulation may provide partial or total stabilization as is shown in the two examples below.



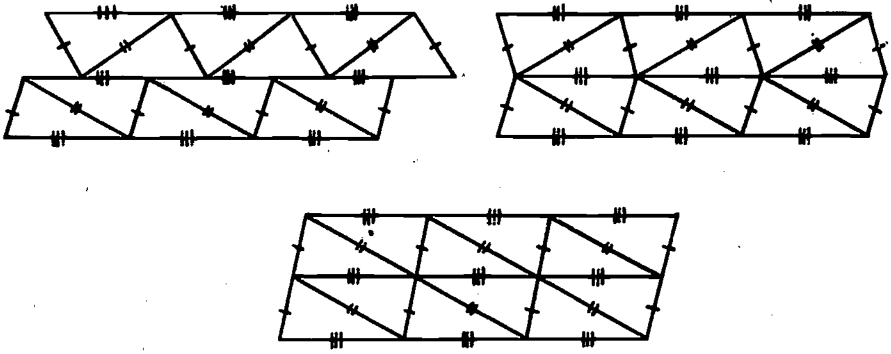
partial triangulation
complete stabilization



partial triangulation
partial stabilization

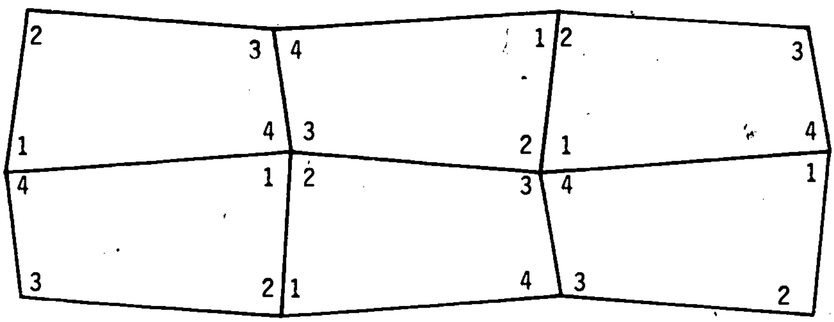
c) Roof structures, bridges, bicycle frames, and guy-wires are some obvious examples.

3. a) There are several different ways this can be done. For example,



b) Any quadrilateral will tessellate. Two examples are:

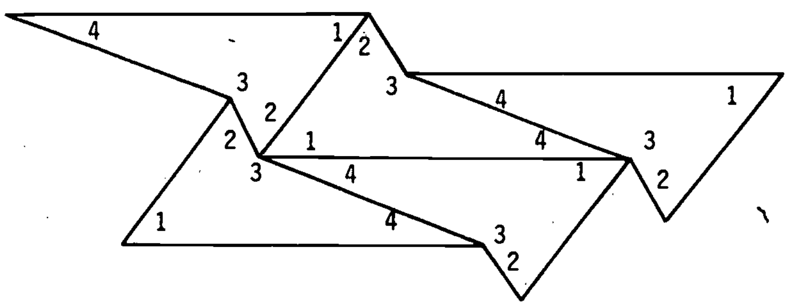
Convex Quadrilateral



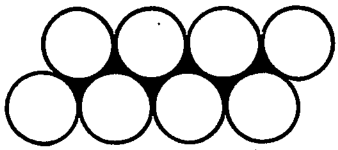
etc.

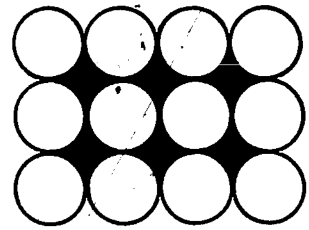
Concave Quadrilateral

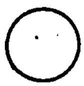
etc.

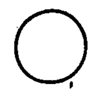


c) A circle by itself will not tessellate. However, here are two examples in which a circle together with another shape make a tessellation.





Tesselation of  and 

Tesselation of  and 

4. Most examples of the use of triangles involve triangulation for rigidity. See answer (2c). A three-dimensional example of a triangle tessellation may be seen in the surface of a geodesic dome. Also, chair caning may be done in such a way as to produce an approximation to a tessellation of triangles.

ACTIVITY 8
ANALYSIS OF TRIANGLES--SIDES

MATERIALS PREPARATION:

Straightedge and compass for each student.

COMMENTS AND SUGGESTED PROCEDURE:

This activity deals with the important relationship of the lengths of the three sides of a triangle. The four steps of this activity--experimentation, generalization, verification, and application--are easily done and make a good example of one type of concept learning common in elementary school mathematics. The Pedagogical Comment on page 44 deserves discussion.

ANSWERS:

1. Since this activity is about the sides of triangles, it is expected that students' constructions of triangles congruent to the given triangles will be done by reproducing the respective sides. Any combination of sides and angles which determines a triangle may be used.

2. a) 3 cm, 3.5 cm, 4.5 cm works; 2 cm, 3 cm, 6 cm does not.

   b) The combination which does not work has one side longer than the sum of the other two; this is not true of the combination that does work.

3. a) The sum of any two sides must be greater than the third side. Or, equivalently, no side may be longer than or equal to the sum of the other two.

4. 6 cm, 2 cm, 3 cm -- no
   3 cm, 4 cm, 5 cm -- yes
   3 cm, 4 cm, 7 cm -- no (a straight line segment)
   8 cm, 4 cm, 6 cm -- yes

ACTIVITY 9
ANALYSIS OF RIGHT TRIANGLES

MATERIALS PREPARATION:

Graph paper, one sheet per student; metric ruler, straightedge, compass, one per student.

COMMENTS AND SUGGESTED PROCEDURE:

This activity is concerned with the Pythagorean relationship. In addition to inferring the rule and using it to predict when a triangle will be a right triangle and to find the lengths of sides of right triangles, students also experience a very common pitfall in teaching--that of generalizing from a special case.

In order to make the point emphatically that examples must be chosen carefully so as not to be only special cases, the questions in the book should be followed closely. The discussion (see pp. 48-49) of this point deserves special attention.

The issue of accuracy and approximation may be raised in this activity especially in questions 2 and 7. When the problems don't work out exactly, most students tend to suspect their ability to measure precisely enough rather than question the truth of the theorem.

ANSWERS:

1.

	a	a^2	b	b^2	a^2+b^2	c	c^2
A	3	9	4	16	25	5	25
B	1.5	2.25	2.0	4.0	6.25	2.5	6.25
C	6	36	8	64	100	10	100

These measurements do support the Pythagorean Theorem.

2. a) The square of the length of the hypotenuse is equal to the sum of the squares of the lengths of the other two sides.

   b) The rule does apply to the examples in question 1.

   c) The triangle has sides 3.2 cm, 2.0 cm, 2.5 cm.

   $(2.0)^2 + (2.5)^2 \ \underline{?} \ (3.2)^2$

   $(4.00) + (6.25) \ \underline{?} \ (10.24)$

   $10.25 \cong 10.24$ ("close enough is good enough")

3. a) The triangle constructed from these sides is a right triangle.

   b) The given sides measure 3 cm, 4 cm, 5 cm, and $3^2 + 4^2 = 5^2$.

4. This statement is not true, but it is believable on the basis of the examples which precede it.

   a) Yes, all the examples have been chosen so that their sides are multiples of 3, 4, and 5.

b) The cases above support the hypothesis that having sides in the ratio of 3:4:5 is a sufficient condition for a right triangle, but more cases need to be studied to find out whether <u>all</u> right triangles have sides in that ratio.

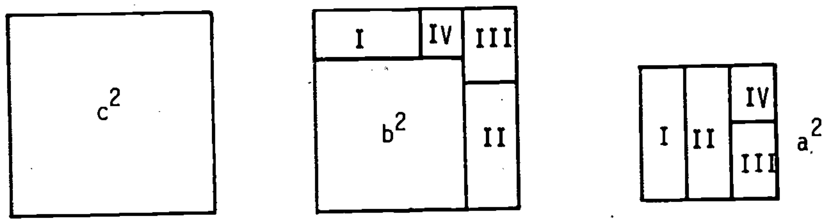
c) Using graph paper it is easy to show that a right triangle may have its two perpendicular sides in any ratio. The triangles are drawn with their two perpendicular sides along the lines of the grid.

5. The changes required in questions 1, 2, and 3 would involve including triangles, sets of sides, and sets of measurements in ratios other than 3:4:5. The strategy is good; only the examples need changing.

6. a) {1.8, 2.4, 3} makes a right triangle.
$\{a, b, (a^2 + b^2)\}$ where $a > 0$, $b > 0$ does <u>not</u>, <u>in</u> general, make a right triangle. However, the triple $(\frac{1}{2}, \frac{\sqrt{3}}{2}, 1)$ does satisfy this relationship and does form a right triangle.
{1, 2, 1} does not even make a triangle.
{5, 1, 2} does not even make a triangle.
{5, 12, 13} makes a right triangle.

b) One method of generating Pythagorean triples is based on the fact that $x^2 - y^2 = (x + y)(x - y)$. Rewriting this as $x^2 = y^2 + (x + y)(x - y)$ and considering only cases where x and y differ by one reduces the problem to $x^2 = y^2 + (x + y)$ which means that we must find two numbers that differ by one and whose sum is a perfect square. For example, consider 25, a perfect square. Two numbers which differ by one and sum to 25 are 12 and 13, so $(\sqrt{25}, 12, 13)$ or (5, 12, 13) is a Pythagorean triple.

7. There are several ways of cutting and fitting the smaller squares into the larger one, but all of them involve measurement errors in the neighborhood of ± 1 mm. One solution is



8. a) This diagram demonstrates a special case of the 3, 4, 5 right triangle, showing the triangle and the squares on the sides and hypotenuse.

   b) This approach has the advantage of being very clear, but it can only be used with right triangles all of whose sides are a whole number of units long. Whenever the lengths of the sides are not all whole numbers, there will be problems of estimation and accuracy (c.f., question 7.).

9. *OPTIONAL: This question is a preliminary and condensed version of Section 4. (q.v.) The discussion might include Authority, Experience Plausibility, and Deduction as methods of convincing.*

TEACHER TEASER, page 50

1. There are many Pythagorean triples; here are several which are relatively prime:
   (3,4,5) (5,12,13) (7,24,25) (8,15,17) (9,40,41)
   Other triples may be formed by multiplying each component of these triples by the same number.

2. The "first" five Pythagorean triples generated by this rule are
   (3,4,5) (5,12,13) (7,24,25) (9,40,41) (11,60,61)

3. To show that when $a^2 = b + (b + 1)$ that $(a, b, (b + 1))$ is a Pythagorean triple we need to show that $a^2 + b^2 = (b + 1)^2$. Substituting $a^2 = b + (b + 1)$ we have $b + (b + 1) + b^2 = (b + 1)^2$ which is the identity $(b + 1)^2 = b^2 + 2b + 1$.

ACTIVITY 10

AN APPLICATION OF

THE PYTHAGOREAN THEOREM TO LENGTH AND COORDINATES

MATERIALS PREPARATION:

None.

COMMENTS AND SUGGESTED PROCEDURE:

This activity extends the work done with the Pythagorean Theorem and applies it to the problem of finding the distance between two points in the Cartesian coordinate plane. The six questions are designed to evoke all the steps of the derivation of the distance formula. They are straightforward and clear or could be used as class discussion, small group work, and individual assignment.

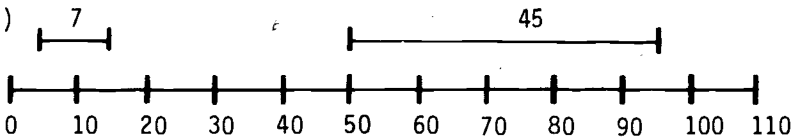
ANSWERS:

1. a) 7 miles

   b) 45 miles

   c) (x - m) miles

   d)



   e) 3.09

   f) $|x - a|$ or $|a - x|$ or $\sqrt{(x - a)^2}$ or $\sqrt{(a - x)^2}$ or words to this effect.

2. a) 4 blocks, no matter which direct route you take.

   b) 3 blocks

   c) $|a - x| + |b - y|$ or $|x - a| + |y - b|$ or equivalent

3. 5 units

4. (-1,2) to (2,2)   3

   (-1,-1) to (1,1) $\sqrt{8} = 2\sqrt{2} \doteq 2.8$

(2,3) to (2,-2) . 5

$(-2,\frac{1}{2})$ to (3,2) $\sqrt{27.25} \approx 5.2$

$(\sqrt{2},2\sqrt{2})$ to $(2\sqrt{2},\sqrt{2})$ 2

(-1,3) to (1,-2) $\sqrt{29} \approx 5.4$

5. $\sqrt{(x - a)^2 + (y - b)^2}$ or $\sqrt{(a - x)^2 + (b - y)^2}$. or equivalent

6. This kind of presentation not only shows how the rule is derived, it also forces the learner to search for appropriate reasons for each sub-step. Also, knowing why the rule is true may make it easier to recall the details of the formula, for example to remember that it is the square root of the sum of the squares.

ACTIVITY 11
ANALYSIS OF TRIANGLES--ANGLES

MATERIALS PREPARATION:

Scissors, ruler, protractor, plain paper for each student.

COMMENTS AND SUGGESTED PROCEDURE:

This activity deals with the fact that the sum of the measures of the interior angles of a triangle in a plane is 180°. The activity could be done quickly in class or assigned as homework if the intention were only to teach this well-known fact, but the activity also deals with methods of verification in general and particular methods suitable for teaching this topic in the elementary school.

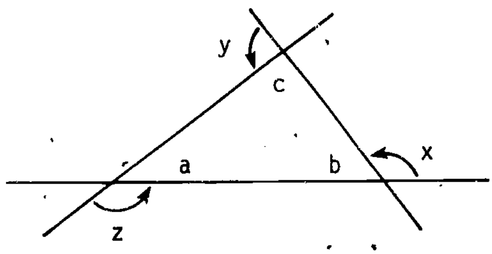
ANSWERS:

1. b) Numbers filled into the table will depend upon the triangles cut out. All entries in the "sum of the measures of the angles" column should be 180° but may differ from this by one or two degrees through measurement error.

c) When the sum of the measures of three angles is 180° (or a straight angle or a half-turn) then those angles can form a triangle.

2. This method of demonstrating the rule is common in elementary school curricula.

3. *OPTIONAL: Label the figure as follows:*



So that a, b, c are the three (interior) angles of the triangle and x, y, z are the three angles through which one must turn in traversing the triangle. Now $x = 180 - b$, $y = 180 - c$, $z = 180 - a$, *and* $x + y + z = 360°$, *so* $360° = (180 - a) + (180 - b) + (180 - c)$. *This reduces to* $360 = 540 - (a + b + c)$ *or* $a + b + c = 180°$.

4. a)

1°	62°	117°	yes
45°	45°	45°	no
60°	60°	60°	yes
30°	30°	120°	yes
29°15'	37°40'	113°5'	yes

c)

45°	45°	90°	
90°	30°	60°	
18°18'	61°42'	100°	(or any two whose sum is 161°42')

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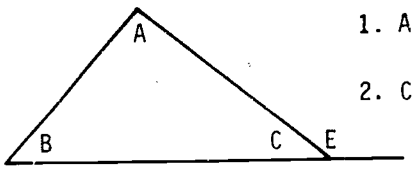
5. This can be proved deductively by



Statement	Reason
1. A + B + C = 180°	1. interior angles of a triangle
2. C + E = 180	2. exterior angle and interior angle are supplementary
3. ∴ A + B = E	3. subtract 2 from 1

6. A lesson on this topic for fifth graders would probably be centered on the activity of tearing of the angles and "summing" them as in question 2, p. 57. Among the ideas a child should have before starting this lesson are what an angle is, how angle sizes are compared, and why the lengths of the arms do not affect the size of the angle.

7. a) The activities done for questions 1, 2, and 3 were <u>not</u> proofs; 1 and 2 were generalizations from specific instances; 3 was a plausibility argument.

   b) The answer given in 5 above <u>does</u> constitute a proof.

   c) Most students probably believe this fact on the basis of authority or generalization from experience; plausibility arguments and proofs are probably just "icing on the cake."

   d) This question could be argued either way. A mathematician would probably demand a proof and be easily convinced by a correct one. A child could probably be convinced by generalization from experience or a statement by an authority.

8. OPTIONAL: *On a sphere the sum of the measures of the interior angles of a triangle will be between 180° and 540°.*

ACTIVITY 12

ANALYSIS OF TRIANGLES, SAS, ASA, ETC.

MATERIALS PREPARATION:

Each student should be supplied with a metric ruler, a protractor, a straightedge, a pair of scissors, some graph paper and some tracing paper. None of these items is required, but they all should be available at the students' choice.

COMMENTS AND SUGGESTED PROCEDURE:

In this activity students do some experiments to find out which combinations of sides and angles determine a triangle. The word "determine" is being used here in a restricted sense to mean that no other triangle not congruent to the given triangle can be constructed from the sides and angles given.

Although the facts to be learned are few and simple, there is much to be gained from actually going through the experiments carefully and discussing the findings.

ANSWERS:

1. a) (ii) determines a unique triangle
   b) (iii) two non-congruent triangles possible
   c) (ii) determines a unique triangle
   d) (iii) two non-congruent triangles are possible
   e) (iii) an infinite number of similar but non-congruent triangles is possible
   f) (i) no triangle is possible.
2. a) determines a unique triangle
   b) does not determine a unique triangle
   c) determines a unique triangle
   d) does not determine a unique triangle

e) describes an infinite set of similar triangles

f) determines a unique triangle.

3. a) 5 cm, 53°, 47°

   b) 110°, 3.5 cm, 2.7 cm

4. Through discussion the student should come to understand that the fact that the triangle is rigid is equivalent to the fact that SSS determines a triangle.

5. a) *OPTIONAL: No, two triangles which have the same angles will have the same shape.*

   b) *OPTIONAL: There are several conditions which determine a quadrilateral, including (a) 4 sides and one specified diagonal; (b) 3 sides and two included angles; and (c) 4 sides and one specified angle.*

ACTIVITY 13
ANALYSIS OF SIMILARITY

MATERIALS PREPARATION:

Graph paper, ruler, and protractor for each student

COMMENTS AND SUGGESTED PROCEDURE:

This activity begins with some examples of and a description of the idea of similarity. The concept is explored in greater depth and two rules are developed to describe conditions under which triangles are similar. The work follows the pattern of experimentation and generalization; discussion in groups or as a class is important in this process.

ANSWERS:

1. c) yes, the two triangles are similar

   d) the corresponding angles have equal measures

2. The ratios of the lengths of corresponding sides are all 2:1.
i.e., $\frac{|A'B'|}{|AB|} = \frac{|A'C'|}{|AC|} = \frac{|B'C'|}{|BC|} = 2$

3. a & b. The ratios of corresponding sides will be equal to a number that depends upon the figures drawn.

4. a) no

   b) yes

   c) yes

   d) yes

   e) no

   f) yes

   g) the shapes are similar, but neither of them is a triangle.

5. a) $1\frac{1}{3}$ inch $1\frac{2}{3}$ inch

   b) 45° 45° 90°

6. $\frac{a'}{a} = \frac{c'}{c} \Rightarrow a'c = ac' \Rightarrow \frac{a'}{c'} = \frac{a}{c}$

7. For two quadrilaterals to be similar, it must be the case both that the corresponding angles have the same measure and that the corresponding sides have the same ratio. Examples to support this statement can be found in Activity 12, pp. 60-63.

8. *OPTIONAL:*

   a) *true*

   b) *false*

   c) *false*

   d) *true*

ACTIVITY 14

STRAIGHT LINES REVISITED

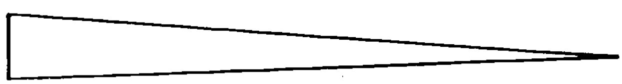
MATERIALS PREPARATION:

Graph paper, three or four sheets per student; straightedge

COMMENTS AND SUGGESTED PROCEDURE:

This activity which deals with the equations of straight line graphs is one of the more difficult ones in the unit. Unlike most of the other activities, it involves a fair amount of algebraic manipulation. Although some students may not feel comfortable with the algebra, the idea of slope is built on both "up and out" and on ratios in similar triangles, so that the computations are firmly rooted in the student's experience. The way in which this activity is done will depend a great deal upon the student's background and ability with high school algebra and geometry.

ANSWERS:

1. a) 5 

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b) $\frac{-5}{50} = -10\%$

c) 750 kilometers

2. a) $\frac{1}{1} = 1 = 100\%$

b) $\frac{-\frac{1}{2}}{3\frac{1}{2}} = \frac{-1}{7} \approx -14.3\%$

3. a) $\frac{1}{2}$

b) $\frac{1}{2}$

c) 50

4. a) $\frac{1}{2}$

b) $\frac{y - 2}{x - 1} = \frac{1}{2}$

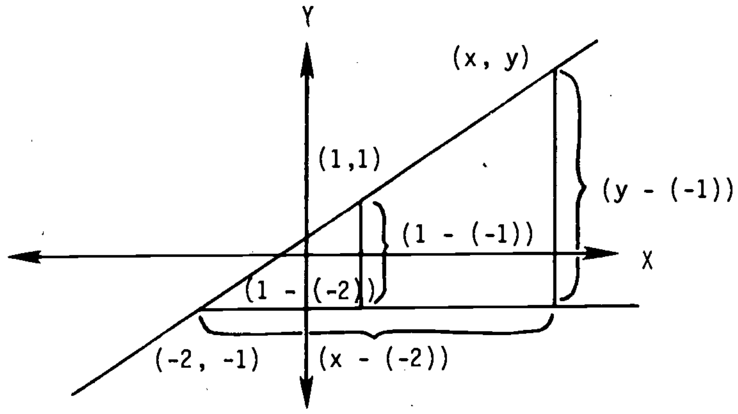
c) To get to (x,y) from (1,2) you must go up y - 2 and go out x - 1; however, since the line is straight it is always going up 1 and out 2, so $\frac{y-2}{x-1} = \frac{1}{2}$.

5. a) $\frac{1}{2}$

b) $\frac{y-(-1)}{x-(-1)} = \frac{1}{2}$ for all (x,y) on the line

c) going from (-1,-1) to (3,1) you go up 2 and out 4, so the slope is $\frac{1}{2}$, going from (-1,-1) to (x,y) for (x,y) on the line you go up (y - (-1)) and out (x - (-1)) and the slope is still $\frac{1}{2}$.

6. a) This explanation is best given in terms of the following diagram:



The two triangles (-2,-1), (1,-1), (1,1) and (-2,-1), (x,-1), (x,y) are similar, and their corresponding sides have the same ratio.

b) In (4b) and (5b) other pairs of similar triangles are formed by the grid lines and the straight line graph.

7. a) (3) x - 2y = 0 (4) x - 2y = -3 (5) x - 2y = 1
(6) 2x - 3y = 1

b) Suppose you are given the points (a,b) and (c,d) and an arbitrary point (x,y) which also lies on the line. $\frac{d-b}{c-a}$ is the slope of the line between (a,b) and (c,d). $\frac{y-b}{x-a}$ is the slope of the line between (a,b) and (x,y). Since the line is straight the slopes are equal, so we put

$$\frac{d-b}{c-a} = \frac{y-b}{x-a} \text{ and solve}$$

$(x-a)(d-b) = (c-a)(y-b)$

$dx - ad - bx + ab = cy - cb - ay + ab$

$x(d-b) - ad = y(c-a) - cb$

$(d-b)x + (a-c)y = ad - cb$

c) 1. Any straight line graph is the graph of an equation which may be written in the form $ax + by = c$.

2. Any equation in the form $ax + by = c$ has a straight line graph.

8. a) The equation of the line is $y = 2$. $(5,y)$ is on the line only for $y = 2$.

b) The equation of the line is $2x + y = 4$.

c) The equation of the line is $x = 1$.

ACTIVITY 15

APPLICATIONS OF TRIANGLE LEARNINGS

MATERIALS PREPARATION:

None

COMMENTS AND SUGGESTED PROCEDURE:

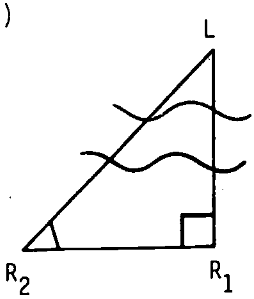
This activity contains a variety of problems which can be solved using the ideas about triangles which have been developed up to this point. If time permits, the problems could be discussed in class and additional ones of the same kind assigned. If time is limited, the problems could be assigned as homework to be checked over in class.

ANSWERS:

1. Most students will stabilize the figures by drawing diagonals to triangulate them. One line is required in each of (a), (b), and (d); two in (c). Note, however, that some of the shapes can be stabilized without being triangulated. For example, (a) may be stabilized by  without being triangulated at all and (b) may be stabilized by  even though it is only partially triangulated.

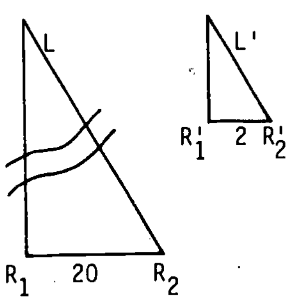
2. There are several possible solutions to this problem; two are shown below.

(1)



Running Bear standing at R_1 across the river from his lover at L walks down the bank of the river (toward R_2) in a direction perpendicular to the line R_1L. He stops at R_2 the point at which the angle between R_2R_1 and R_2L is 45°. Since the triangle he has formed is an isosceles right triangle, R_2R_1 is as long as R_1L, so he measures R_2R_1 to find out how far it is across the river.

(2)



Running Bear at R_1 is directly across the river from his lover at L. He walks 20 paces down the bank of the river to R_2 and uses his compass to measure the angles LR_1R_2 and LR_2R_1. Then he moves to another place and constructs another triangle $L'R_1'R_2'$ in which $R_1'R_2'$ is 2 paces long and angles $L'R_1'R_2'$ and $L'R_2'R_1'$ have the same measure, respectively, as LR_1R_2 and LR_2R_1. Finally, he measures the length of $L'R_1'$. This distance multiplied by ten is the distance from L to R.

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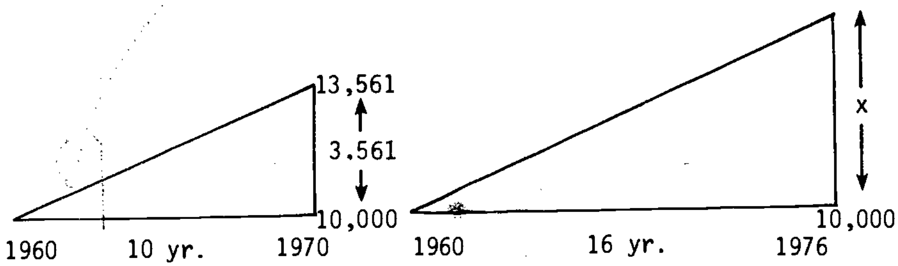
3. He could use the rope to make a 3, 4, 5 right triangle and could compare the angle between the 3-unit and 4-unit sides with the corners he wants to test for squareness. (This is the Egyptian rope stretchers' triangle.)

4. This is really a discussion question. Thinking about the problem in terms of the combinations of length of sides possible in a triangle tells us that the maximum width is 40', but in this case the roof would be flat. If the cross section of the roof were an equilateral triangle (roof angle 60°) then the width would be 20 feet. If the cross section of the roof were an isosceles right triangle (roof angle 45°), the width would be $20\sqrt{2}$ feet or 28 feet, approximately.

5. One solution to the problem was similar triangles.



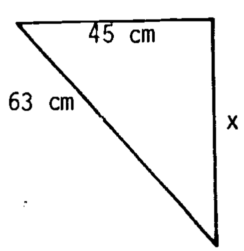
$$\frac{16 \text{ years}}{10 \text{ years}} = \frac{x}{3,561}$$

$$10x = 16 \times 3561$$

$$x = 5697.6 \approx 5700$$

So in 1976 there would be 10,000 + 5,700 = 15,700 students.

6.



$$(45)^2 + x^2 = (63)^2$$

$$x^2 = (63)^2 - (45)^2$$

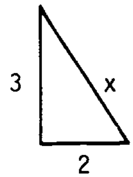
$$x^2 = 3969 - 2025 = 1944$$

$$x = \sqrt{1944} \approx 44 \text{ cm}$$

7. Each guy wire will be x meters long where

$$x^2 = 3^2 + 2^2 = 9 + 4$$

$$x = \sqrt{13} \approx 3.6 \text{ meters}$$



Allowing 20 cm for tying makes each wire 3.8 m so the total length of wire required for 3 guys is 11.4 meters.

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SECTION III
CIRCLES

INTRODUCTION:

This section contains six activities dealing with circles. Activity 16 considers physical and metaphorical circles in the real world. Activities 17 through 20 analyze circles and their relationships with points, lines, cartesian coordinates, and other figures. The work with circles concludes with Activity 21, an unusual application of concepts of circles designed to provoke some creative thought by students.

Activity 22 may be thought of as separate from the rest of the section. It serves as a summary of Sections I, II, and III, by putting the focus on geometry experiences for elementary school pupils.

MAJOR QUESTIONS (DISCUSSION):

1. Many different examples can be chosen as illustrations of the "equal distances from one point" property, the "rolling" property, and the "constant width" property.

2. The circle whose equation is $x^2 + y^2 = 9$ has radius 3 and center at the origin (0,0). When we say that $x^2 + y^2 = 9$ is the equation of the circle we mean that the circle is composed of points such that the sum of the square of the x-coordinate and the square of the y-coordinate is nine.

3. The construction of a circle and a tangent to it should be neat and accurate. The instructor should look for construction lines and arcs which show that the tangent has been constructed as a line perpendicular to the radius through the intersection of the radius and the circle.

ACTIVITY 16
CIRCLES: THEIR ROLE IN THE WORLD

MATERIALS PREPARATION:

None

COMMENTS AND SUGGESTED PROCEDURE:

This activity should be given as an outside assignment so the class can concentrate on sharing their insights. Occurrences of circles in our lives are so numerous we are often blind to them. The students are here given ample opportunity to look and see.

ANSWERS:

1. Here is a partial list.
   Manmade--watch faces, coins, balls, globes, knobs, wheels, light bulbs, rings, drinking glasses, movie reels, jar covers, buttons

   Natural--oranges, grapefruits, radish seeds, sunflowers, tree trunk, the planets, eyes, ripples on a pond, birds' nests, rain drops

2. a) Everyone equidistant, no one sees more than any other

   b) Chemical bonds of water molecules; surface tension minimal

   c) Traditional shape; can be picked up at any spot and always fit

   d) Smooth ride

   e) No corners to catch food; maximum volume for minimal clay

   f) They will bounce evenly no matter how they impact

g) Maximum volume for minimal skin

h) Aesthetics?

3. Lest anyone be caught napping by the obvious, "constant width" is a property of the principal part of the Wankel engine which is not circular. With respect to tessellation, a circle will not tessellate alone but may with other appropriate plane figures (c.f., Activity 7, question (3c).).

4. Circles may be approximated with varying degrees of satisfaction by many methods including tracing, using a template, pencil and string, eyedrop of liquid, pencil point, trial and error, and compass.

5. a) People who go around in circles always return to the same place. A round peg in a square hole is a misfit. A well-rounded person is "equally knowledgeable" in many areas. Circular reasoning goes nowhere.

   b) Some others we conjured up are a square deal, a "square" (a "straight" person), and parallel construction of sentences.

ACTIVITY 17
CIRCLES AND POINTS

MATERIALS PREPARATION:

Compass and straightedge per student, blackboard compass

COMMENTS AND SUGGESTED PROCEDURE:

This is another activity in which we suggest that the questions be investigated initially as an out-of-class activity. The discussions will be richer and more exciting.

ANSWERS:

1. a) An infinite number. The idea of concentric circles underlies this question.

   b) An infinite number. It is interesting to note that the circles passing through the given point may be of any size (see Figure 1) and may lie in any direction from the given point (see Figure 2).

Figure 1

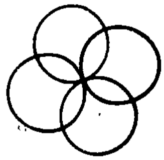


Figure 2

2. a) An infinite number. The constraint in this case is that the centers of the circles must lie on the perpendicular bisector of the line segment joining the two given points.

   b) The center of any circle passing through two given points will lie on the perpendicular bisector of line joining the two points. The center of any particular circle can be found by choosing a third point on the circle, drawing a chord (between the third point and one of the two given points) and constructing the perpendicular bisector of the chord. The perpendicular bisectors of the two chords intersect at the center of the circle.

3. a) If the three points are not collinear, one unique circle may be drawn containing them. If the points are collinear no circle contains them.

   b) The three perpendicular bisectors of the three lines joining the three points should intersect at the center.

   c) Through four points there may or may not be a circle. Since three points determine a circle, the constraint is that the fourth point must lie on the circle determined by the first three. Other (weaker) restrictions are that no three of the given points may be collinear and that no point may lie within the triangle formed by the other three.

4. This is an extension of questions (2b) and (3b) which may not be obvious to the students. The center is formed at the intersection of the perpendicular bisectors of the three line segments forming the three given points. The radius is the distance from the center to any one of the given points.

5. This is an application of question (2a). No unique circle is determined by 2 points; three are required (see Figure 3).

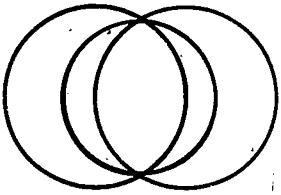


Figure 3

ACTIVITY 18
CIRCLES AND LINES

MATERIALS PREPARATION:

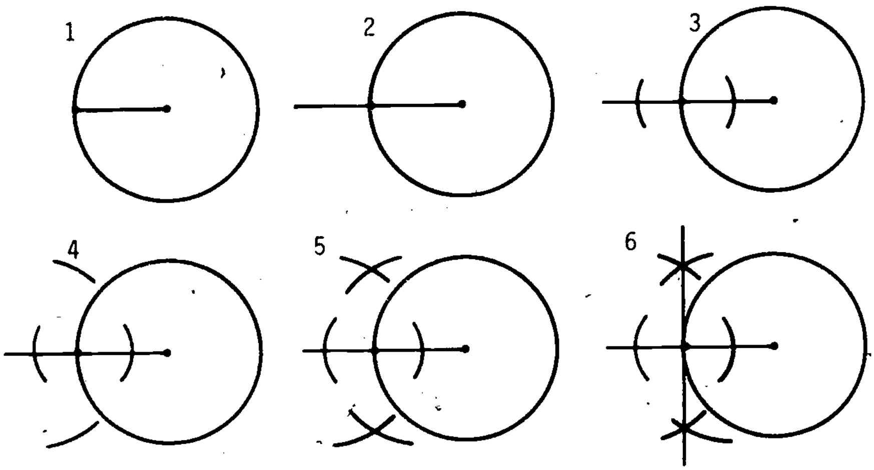
One straightedge, compass, and protractor per student; blackboard compass and protractor.

COMMENTS AND SUGGESTED PROCEDURE:

For this activity, organization of the class into small groups is recommended. With some thought, the first four questions should prove to be fairly easy for the students to solve. Question 5 is much less likely to be solved without reference to an appropriate text. Discussion is always appropriate in these analytical situations and should be encouraged here. An interesting and somewhat unusual approach to Question 5 may be found in Skemp, Richard R. <u>The Psychology of Learning Mathematics</u>, Penguin Books, 1971.

ANSWERS:

1. Sometimes students are unable to relate current problems to past ones. The situation presented here appears to be novel until one realizes that it is nothing more than the construction of a perpendicular to a line at a given point. The line here is the radius (which must be extended in the construction) and the given point is the point of tangency. The steps of the construction are shown below.



2. The motion would be in a straight line perpendicular to the radius at the point where it left the circle, hence, on a tangent. (See Figure 1.)

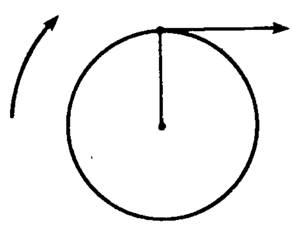
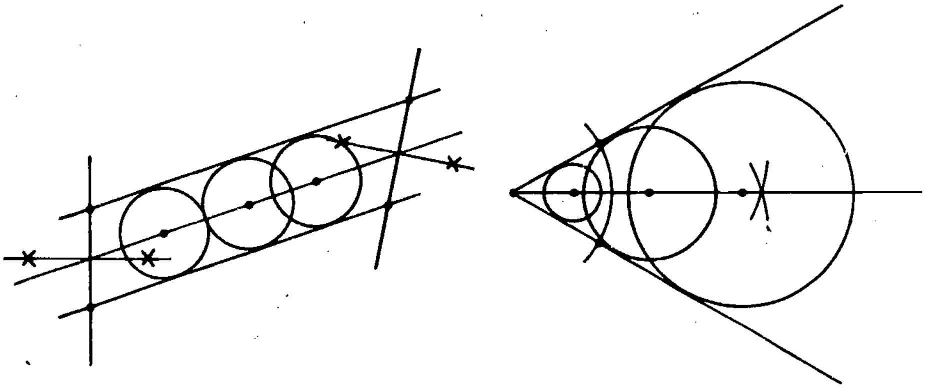


Figure 1

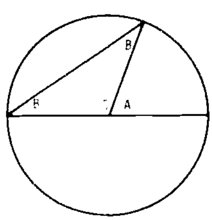
3. a) The solutions would probably look like this.



b) In the first case the construction entails finding a line parallel to the two given lines and midway between them. In the second case the bisector of the angle is needed. The centers of the required circles lie on these lines.

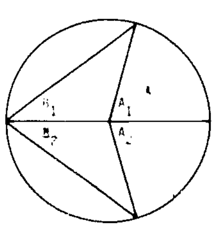
4. Examples might include a wheel on a road, a bowling ball rolling down an alley, or a plate being carried at arm's length.

5. a)



$$\begin{aligned} B + B + C &= 180^\circ \\ C + A &= 180^\circ \\ B + B + C &= C + A \\ B + B &= A \\ 2B &= A \end{aligned}$$

b)



$$\begin{aligned} A &= A_1 + A_2 & B &= B_1 + B_2 \\ 2B_1 &= A_1 & &\text{proved in part a} \\ 2B_2 &= A_2 & &\text{proved in part a} \\ 2B_1 + 2B_2 &= A_1 + A_2 \\ 2(B_1 + B_2) &= A_1 + A_2 \\ 2B &= A \end{aligned}$$

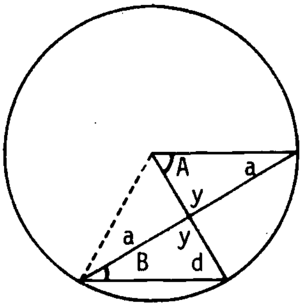
c) 

Figure 2

It is helpful to add to the given figure the (dashed) construction line and to name the angles formed as shown in Figure 2. Note that the two angles labelled a are equal because they are base angles of an isosceles triangle whose equal sides are radii of the circle. Similarly in another isosceles triangle a + B = d. Now A + a + y = 180 and y + d + B = 180 so A + a + y = y + d + B and A + a = d + B. Substituting a + B for d in this expression gives A + a = a + B + B, hence A = B + B or A = 2B.

6. The arguments given above are deductive. For many people the plausibility of the conclusion stems from the plausibility of the statements proved along the way to the final conclusion.

ACTIVITY 19
EQUATION OF A CIRCLE

MATERIALS PREPARATION:

Compass and graph paper for each student, blackboard compass and coordinate transparency (optional).

COMMENTS AND SUGGESTED PROCEDURE:

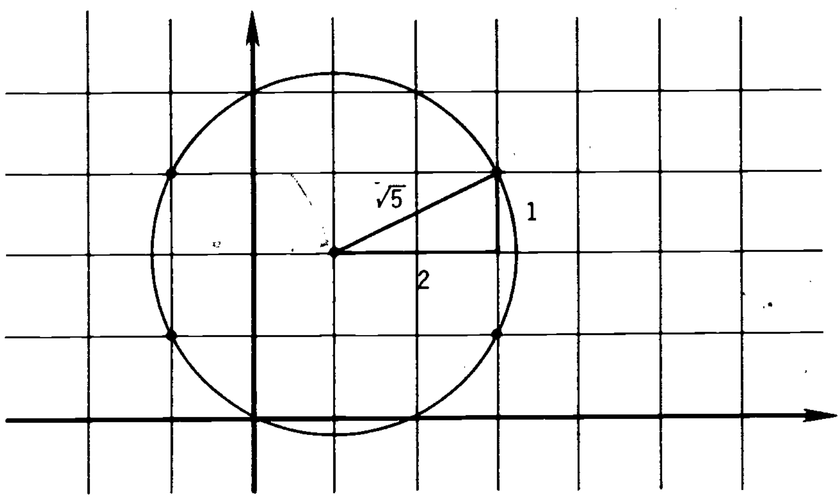
This activity contains a careful development of the equation for a circle from concepts which the students have been learning and using in preceding activities. For this reason it would be possible (perhaps as a change of pace) to conduct the lesson in class with prior out-of-class preparation.

Question 7 is a discussion question, the answer to which will depend upon students' background in mathematics, their experiences with children, and their personal opinions. The comments about this question given below do not comprise the only correct answer, but are, rather, considerations that might be raised (by the instructor, if need be) in the discussion.

ANSWERS:

1. All points on a circle are equidistant from the center. Hence, no person is nearer to (or farther from) Carol than any other.

2. a) As is shown, the vertical distance is (3 - 2) or 1 and the horizontal distance is (3 - 1) or 2. The direct-distance is the hypotenuse of the right triangle (see diagram). This is equal to $\sqrt{(3-1)^2 + (3-2)^2} = \sqrt{2^2 + 1^2} = \sqrt{5}$.



b) $(x - 1)^2 + (y - 2)^2 = 5$, or $(1 - x)^2 + (2 - y)^2 = 5$, or equivalent.

c) Some are (0,0) (0,4) (2,0) (3,1) (3,3)
(2,4) (-1,3) (-1,1)
$(1 + \sqrt{5}, 2)$ $(1 - \sqrt{5}, 2)$ and so on.

Each can be shown algebraically to satisfy the equation.

3. $(x - a)^2 + (y - b)^2 = r^2$, or $r = \sqrt{(x - a)^2 + (y - b)^2}$, or equivalent.

4. If $x = y$ intersects $x^2 + y^2 = 1$, then there exist coordinates (a,b) such that $a = b$ and $a^2 + b^2 = 1^2$. Hence

$$
\begin{aligned}
x^2 + y^2 &= 1^2 \\
x^2 + x^2 &= 1^2 \\
2x^2 &= 1 \\
x^2 &= \frac{1}{2} \\
x &= \sqrt{\frac{1}{2}} \\
x &= \pm \frac{\sqrt{2}}{2}
\end{aligned}
$$

Thus the coordinates are $\left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$ and $\left(\frac{-\sqrt{2}}{2}, \frac{-\sqrt{2}}{2}\right)$

5. It can be ascertained that the center is at (1,1) and that the radius is 1. The equation is $(x - 1)^2 + (y - 1)^2 = 1^2$.

6. The center is at (1,3) and the radius is 3. (1,0) (1,6) (4,3) and (-2,3) are on this circle.

7. a) Maybe and maybe not. Sometimes children focus on the smoothness of a line and deny that the line itself consists of points even though they can show you points on the line. On the other hand, older children who have heard about "360° in a circle" may think that there are exactly 360 points on a circle and can often be convinced that there could just as well be 720 points or even an infinite number.

   b) It is likely that many students will not have been introduced explicitly to the pointwise model of space. However, throughout the unit we have taken geometric figures to be sets of points and in this activity we have used algebraic expressions to describe general points and have assumed that the figures consist of all points which have the property that they satisfy the equations.

ACTIVITY 20
AREA VS. PERIMETER

MATERIALS PREPARATION:

One 50 cm length of string and two or three sheets of graph paper for each pair of students.

COMMENTS AND SUGGESTED PROCEDURE:

Organization of the class into groups of two students is recommended. In question 1 the number of fence posts used is not crucial, but has been chosen as a convenient number. Question 2 is algebraic, and may be difficult for students with weak backgrounds in this area.

ANSWERS:

1. a) This answer is the student's guess of the answer before attempting to work out a solution. Students can be asked to compare this answer with the answer to (1c).

   b) The entries in the table will depend upon the shapes students make and measure. The entries in the area column should all be less than 200, since the circle has the maximum area, approximately 199 cm^2. The entries in the Perimeter/Area column should all be greater than 0.25, the value in the case of the circle.

2. a) $P = 100$ $\quad A = \pi r^2$ $\quad \frac{P}{A} \approx \frac{100}{795} \approx 0.126$

   $P = 2\pi r$ $\quad A \approx 795$

   $r \approx 15.9$

   b) $P = 100$ $\quad A = s^2$ $\quad \frac{P}{A} = \frac{100}{625} \approx 0.160$

   $P = 4s$ $\quad A = 625$

   $s = 25$

   c) $P = 100$ $\quad A = \ell w$ $\quad \frac{P}{A} = \frac{100}{556} \approx 0.180$

   $P = 2\ell + 2w$ $\quad A = 2w^2$

   $P = 6w$ $\quad A \approx 556$

   $w \approx 16.6$

3. a) This is a move to conserve body heat. The least amount of surface is exposed to the air.

   b) We include rain drops, mercury when spilled, the human head, cups and glasses. An inflated tire tends to assume a round cross section.

ACTIVITY 21
CIRCLES AROUND YOU

MATERIALS PREPARATION:

None

COMMENTS AND SUGGESTED PROCEDURE:

Although our culture is generally "right-angled," there are peoples who build round shapes to live in. An igloo is more round than square, and some African tribes build round huts. This activity should be enjoyable, since it provides students with an opportunity to be creative, and to show their aesthetic sense and their practical sense.

Question 1 requires an original, creative design from the student, so no answer can be given. However, we have provided some discussion guidelines for question 2.

This activity could be given as a homework exercise, in class individually, or in small groups.

ANSWERS:

2. a) The costs for heating and cooling should be less. Ideally there would be <u>one</u> continuous outside wall which would expose less area to be affected by weather. Building costs could be greater, depending on any special equipment, material, or expertise needed. There could be a lot of waste since building materials are designed for use in rectangular configurations.

b) The basic shape in lumber is rectangular. One problem the carpenter would need to solve would be putting doors and windows (plane regions) into a curved surface. It would involve new shapes or building rectangular rooms inside the circle.

c) There would probably be difficulties in putting straight-sided furniture against curved walls.

d) Visitors may feel awkward in a round room. Would occupants need to be well-rounded? Would traffic patterns have to be circular?

e) A circular home puts some restrictions on "neat" division of peripheral areas. A rectangular garage could be semi-attached to some part of the house. Driveways should probably be semi-circular.

ACTIVITY 22
HOW TO TEACH GEOMETRY

MATERIALS PREPARATION:

Elementary mathematics text series

COMMENTS AND SUGGESTED PROCEDURE:

This could be an in-class, small group activity if one wished to reduce the out-of-class preparation time of his students. The decision should be based on whether or not the students will be trying out their lessons (optional question 4). Although sample answers are not given in this activity, included below are books and periodical articles which may be used as resources for both instructors and students.

The Arithmetic Teacher, October 1969 and October 1973.
Environmental Geometry, Nuffield, Wiley & Sons, New York City, 1969.

Geometry for Elementary Teachers, Young, John E. and Bush, Grace A. San Francisco Holden-Day, 1971.

The 36th Yearbook of the National Council of Teachers of Mathematics: Geometry in the Mathematics Classroom.

SECTION IV
VERIFICATION

INTRODUCTION:

In the traditional secondary school mathematics curriculum, geometry served as a vehicle for introducing students to deductive logical methods. This section considers alternate methods of verification (authority, experience, plausibility argument, deduction) which are appropriate in different teaching situations in mathematics.

MAJOR QUESTIONS (DISCUSSION):

1. The examples need not be extensive, but they should show clearly the distinctions between these four ways of knowing. The answers given here will be similar to those given to the questions in Activity 23.
2. The answer to this question is contained in the answer to question 4, Activity 24. The essential point is that generalization from a special case may lead to a false conclusion. Only one counterexample is needed to disprove a theorem.

ACTIVITY 23

HOW DO YOU KNOW IT'S TRUE?

MATERIALS PREPARATION:

Elementary school mathematics textbook series.

COMMENTS AND SUGGESTED PROCEDURE:

This activity gives the student some examples of four methods of verification (namely authority, experience, plausibility, and deduction), engages him in discussion of these "ways of knowing" that are used in everyone's daily experience, and refers him to other parts of this unit and to an elementary school mathematics textbook to find examples of verification techniques used in mathematics.

The first part of this activity (questions 1 and 2) is essentially discussion; the "exercise" (questions 3 and 4) is equally important.

ANSWERS:

1. There are many answers which might come up in the discussion of this question; some examples follow:

   a) historical facts, time of day, etc.

   b) the sun will come up tomorrow; a hot stove will give you a burn if you touch it, etc.

   c) a tax cut will stimulate the economy; weather forecast made on the basis of weather reports, etc.

   d) mathematical propositions; scientific predictions from theory, etc.

2. Whatever answer is given to this topical question, it is hoped that authority, experience, plausibility, and deduction would be considered as ways of arriving at an answer.

3. Many instances can be found in which the four different techniques of verification have been used. Consider, for example, Activity 11, Analysis of Triangles, Angles. The definitions of angle, measure of an angle, and degree, and the statement that any three angles whose measures add up to 180° can be the angles of a triangle were all stated as true on the basis of authority. The activities in questions 1 and 2 were arguments from experience. Question 3 is in effect a plausibility argument. The answer we gave to question 5 contained a deductive proof.

4. The instances found will depend upon the textbook chosen. Examples of authority abound in, for example, definitions. Experience is exemplified by lab-type activities using manipulative apparatus. Plausibility and deductive arguments may be hard to find.

ACTIVITY 24

WHAT WILL IT TAKE TO CONVINCE YOU?

MATERIALS PREPARATION:

None

COMMENTS AND SUGGESTED PROCEDURE:

This activity explores techniques of verification in greater detail by looking at some considerations which may affect the truth of a proposition. Again, as in Activity 23, a discussion provides the introduction, and an exercise in finding and analyzing fallacies culminates the activity.

ANSWERS:

3. a) $f(0) = 0$, $f(1) = 0$, $f(2) = 0$, $f(3) = 0$

   b) Some people might believe that $f(x) \doteq 0$ for all x, but students who have experienced Activity 9 will probably be wary.

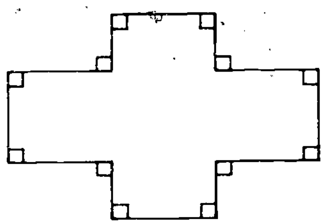
   c) Such a person would conclude that $f(x)$ is non-zero for all x not equal to 0 or 1 or 2, or 3.

4. a) The first part of Activity 9 was an argument from experience. All of the experiences were instances of a special case.

   b) The moral is that one must take care to choose a variety of examples not all of which are special cases.

   c) There are many possible answers to this question. Here is a sample.

   <u>Proposition</u>: Any closed plane figure all of whose sides are either parallel to or perpendicular to every other side is either a rectangle or a square. This can be disproved by the following counter-example.



ACTIVITY 25
PROOF

MATERIALS PREPARATION:

None

COMMENTS AND SUGGESTED PROCEDURE:

This activity which gives students some experiences with proof is set in the context of a simplified set of four axioms about "bushes" and "hedges." This setting serves several purposes. First, the arguments which are similar to the familiar ones about points and lines seem clearer because of the sparseness of the system. Secondly, because the mathematical objects are not conventional ones, the student may be reminded that any argument he makes is true and valid only insofar as his assumptions are true and valid. Third, this activity puts the student into the position of having to relate new abstract mathematical vocabulary with real-world objects and concepts. If they find this difficult, they may appreciate the difficulty elementary school students have with words like point, line, and plane. It is hoped that students will enjoy playing this game and that they may appreciate more fully the game of deductive geometry.

ANSWERS:

1. Theorem: If there are two hedges, then there are three hedges.
   Proof: Call the two hedges H_1 and H_2. By Axiom 1 each of these hedges contains at least two bushes. Let H_1 contain b_{11} and b_{12}; let H_2 contain b_{21} and b_{22}. By Axiom 2 (which guarantees the existence of at least three bushes) either (1) b_{11}, b_{12}, b_{21}, b_{22} are all distinct or (2) b_{1i} is the same bush as b_{2j} for some $i = 1, 2$ and $j = 1, 2$. If (1) then there exist six hedges, namely $H_1 = b_{11}, b_{12}$; $H_2 = b_{21}, b_{22}$; $H_3 = b_{11}, b_{21}$; $H_4 = b_{12}, b_{22}$; $H_5 = b_{11}, b_{22}$; $H_6 = b_{21}, b_{12}$. If (2) suppose the three distinct bushes are b_{11}, b_{22}, and b_{12} which is the same as b_{21}. Then the three hedges are $H_1 = b_{11}, b_{12}$; $H_2 = b_{22}, b_{12}$; and $H_3 = b_{11}, b_{22}$.

2. Theorem: If there are three hedges, then there are four hedges.
   Proof: In case (1) above, there are six hedges so, clearly, there are 4. So consider Case 2. As before let the three hedges be $H_1 = b_{11}, b_{12}$; $H_2 = b_{22}, b_{12}$; and $H_3 = b_{11}, b_{22}$. Take

$H_3 = b_{11}, b_{22}$; by Axiom 4 there exists one (and only one) hedge which contains b_{12} and does not touch H_3. This cannot be H_1 because H_1 touches H_3 at b_{11} nor can it be H_2 because H_2 touches H_3 at b_{22}; thus, it must be the required fourth hedge.

3. These theorems have nothing to do with real hedges and bushes. For one thing, two bushes do not a hedge make.

ACTIVITY 26
VERIFICATION FOR KIDS

MATERIALS PREPARATION:

None

COMMENTS AND SUGGESTED PROCEDURE:

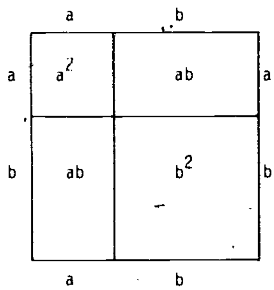
This activity provides a chance for students and the instructor to share lesson ideas related to geometry for children at several age levels. This activity could be expanded to several class periods searching for and exchanging ideas for novel ways of verifying or proving facts from geometry.

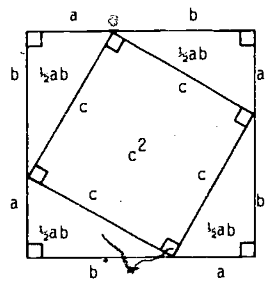
ANSWERS:

These questions are discussion questions in which the supporting arguments are at least as important as the answers proposed. Possible, although by no means unique, answers are given below.

1. a) experiences, possibly with paper folding or drawing and measuring activities.

   b) authority, perhaps; one could also try a deductive argument based on the equality of the areas of the following two figures.





Since the figures have equal areas $a^2 + 2ab + b^2 = c^2 + 2ab$, so $a^2 + b^2 = c^2$.

c) Experience with squares; authority cited to confirm that each square he is given is in fact a square.

d) Experience and plausibility, probably including experiments with strips of paper as in Activity 8.

2. This question solicits students' opinions of their high school geometry experiences. As moderator of the discussion, the instructor might ask whether students believe that high school geometry could be improved by incorporating any of the activities included in the unit.

BIBLIOGRAPHY

PERIODICAL ARTICLES:

The Arithmetic Teacher issues of October 1969 and October 1973 each contain several articles on geometry activities in the elementary school. Articles are both theoretical and practical. An extensive bibliography of journal articles on the teaching of geometry is included in Backman and Cromie (1971), listed below.

BOOKS:

Allendoerfer, Carl B. Arithmetic & Geometry for Elementary School Teachers. New York: Macmillan, 1971. (Part V)

Backman, Carl A., and Cromie, Robert G. Introduction to Concepts of Geometry. Englewood Cliffs, N.J.: Prentice-Hall, 1971. (This book contains an extensive bibliography of Arithmetic Teacher and Mathematics Teacher articles on geometry.)

Brumfiel, Charles F. and Vance, Irvin E. Algebra and Geometry for Teachers. Reading, Mass.: Addison-Wesley, 1970. (This book deals more with content than methods.)

Fehr, Howard F. and Phillips, Jo McK. Teaching Modern Mathematics in the Elementary School. Reading, Mass.: Addison-Wesley, 1967.

Hartung, Maurice L. and Welch, Ray. Geometry for Elementary Teachers. Glenview, Ill.: Scott, Foresman, 1970.

Heddens, James W. Today's Mathematics. Chicago: SRA, 1964, and other editions. (Units 19-21 of this well-known book deal with geometry. This book contains a wealth of ideas clearly explained and illustrated.)

Johnson, Paul B. and Kipps, Carol Herdina. Geometry for Teachers. Belmont, Calif.: Brooks/Cole, 1970. (This book approaches its topic with a strong "behavioral-objectives" outlook.)

Juster, Norton. The Dot and the Line: A Romance in Lower Mathematics. New York: Random House, 1963.
(This book, also available as a film, is a fable about straightness. It should be an entertaining introduction to Activity 1.)

Marks, John L., Purdy, C. Richard, and Kinney, Lucien B. Teaching Elementary School Mathematics for Understanding. New York: McGraw-Hill, 1965.

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