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ABSTRACT

The dimensions of the broad social-cognitive and metacognitive matrix within which pure cognitions reside are examined. Tangible cognitive actions are the cross products of beliefs held about a task, the social environment within which the task takes place, and the problem solvers' perceptions of self and their relation to the task and environment. Students' verbal behavior in cognitive performance can be analyzed at three levels: analysis of tactical knowledge of facts, procedures, domain-specific knowledge and local heuristics; analysis of control knowledge, strategic behavior and conscious metacognition knowledge; and analysis of consciously and unconsciously held belief systems that drive problem solving behavior. The relationship of the analysis levels, and three sample protocols of implementing problem solving heuristics are discussed. The degree to which students are aware of their knowledge and belief systems is shown to affect the monitoring and assessment of their cognitive strategies and educational growth. (Author/CM)

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Beyond the Purely Cognitive:

Metacognition and Social Cognition as Driving Forces

in Intellectual Performance

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1. Overview

This paper is one of a pair which, together, try to delineate some of the psychological and methodological issues related to the use of certain "verbal methods" (clinical interviews and protocol analyses) for research into human problem solving processes. The two papers share the same foundation, the premise that "purely cognitive" behavior is extremely rare, and that what is often taken for pure cognition is actually shaped -- if not distorted -- by a variety of factors. The companion paper, "On the Analysis of Two-Person Problem Solving Protocols," (note 1) discusses the aims. rationales, and details of a particular protocol analysis scheme. That framework was designed to elucidate the nature of certain strategic (and for the most part "purely cognitive") decisions made by college students in the process of solving difficult mathematics problems. It explores the role that those strategic decisions play in the students' success or failure. This paper tries to place such methodologies in a much broader context, in an attempt to explicate some of the "driving forces" that generate the behaviors that we see. To put it briefly, the idea considered here is that the cognitive behaviors customarily studied in experimental fashion take place within, and are shaped by, a broad social-cognitive and metacognitive matrix. That is, the tangible cognitive actions that we observe are often the result of consciously or unconsciously held beliefs about (a) the task at hand, (b) the social environment within which the task takes place, and (c) the individual problem solver's perception of self and his or her relation to the task and the environment. It is argued that the behaviors we see must be interpreted in that light.

This is an exploratory discussion, an attempt to characterize some of the dimensions of the matrix within which pure cognitions reside. The discussion takes place in two parts. In the first part I shall try to outline the three levels of analysis I think may be necessary to fully make sense of verbal data, even when one's intentions are "purely cognitive." These levels are described in section 2, and a brief analysis of some protocols from that perspective is then given in section 3. In the second part, I shall try to broaden the discussion and flesh out some of the dimensions of the matrix. Much of what follows is highly speculative, and a good deal of the "evidence" anecdotal. The idea is to point out some of the pitfalls in current lines of inquiry, and to map out some (one hopes) useful directions for future inquiry.

2. Background; a Framework

I wish to suggest here that three separate levels or types of analysis may be necessary in order to obtain an accurate interpretation of students' problem solving performance from the analysis of "verbal data" that they produce while solving problems.* There are:

A. An analysis of "tactical knowledge, including facts, procedures, domain-specific knowledge, and "local" heuristics;

B. An analysis of "control" knowledge and behavior, including "strategic" or "executive" behavior and conscious metacognitive knowledge; ar

*There are, of course, many levels of analysis beyond those discussed here. At the microscopic level, see Monsell's [1981] review of what he calls the "nuts and bolts of cognition:" representations, processes, and memory mechanisms. At the very macroscopic level, there is the broad set of social cooperative behaviors within which "real" problem solving actions often take place. These too are beyond the scope of this study. Here we shall focus on analyzing the protocols obtained from students under relatively ideal laboratory situations.

C. An analysis of consciously and unconsciously held belief systems, and the way that they "drive" problem solving behavior.

Each of these categories is described below. As background, however, it is important to characterize some of the defining properties of the first two categories, "tactical" and "strategic" knowledge and decisions. Roughly, the distinction is as follows. A strategic decision is a global choice, one that in a substantive way affects the direction of a problem solution and the allocation of resources (including time) to be used in a solution. These "control" decisions include selecting goals and deciding to pursue or abandon particular (large-scale) courses of action. In short, they are decisions about <u>what</u> to do in a solution. In contrast, tactical knowledge and procedures are used to implement the strategic decisions. They deal with <u>how</u> to do what has been decided at the strategic level. Suppose, for example, that a student working on a problem decides to calculate the area of a particular region, or to "look at an easier related problem." If doing so will occupy, say, five or more of the allotted twenty minutes for solving the problem, that decision is strategic: it, alone, may "make or break" the solution. On the other hand, the decisions regarding how to implement that choice -- for example, whether to calculate the dimensions of the region by trigonometry or analytic geometry, or which easier related problem to explore -- are tactical. Note that in the latter case, the implementation of a problem solving heuristic is considered a tactical matter. This is non-standard.Some elaboration of the three categories follows.

<u>A. On Tactical Knowledge</u>

As suggested above, this category is quite broad. It includes a number of subcategories covering the range of facts and procedures that are available to the individual for implementation in a problem solution. A characterization of many of the relevant issues is given by Simon in his (1979) review article, "Information processing models of cognition." Simon is primarily concerned with psychological and AI simulations of expert problem solving performance in semantically rich domains. He describes the key issues as follows. "The central research questions are two: (a) how much knowledge does an expert or professional in the domain have stored in LTM [long term memory], and (b) how is that knowledge organized and accessed so that it can be brought to bear on specific problems?" The focus here is somewhat different since we are interested in analyzing students' performance to determine sources of both success and failure. But many of the issues are the same.

To begin with, one needs to know what domain-specific knowledge is accessible to the problem solver. If a student is solving a straightedge-and-compass construction problem from plane geometry, for example, (see protocols 1 and 2) does he or she know that the radius of a circle is perpendicular to the tangent line at the point of tangency? Whether the student chooses to use that fact is another matter, to be discussed later. But (obviously) a solution that depends on that particular piece of knowledge may evolve in radically different ways if the student does or does not have it, and an evaluation of the solution depends on an adequate characterization of the knowledge base. Similar comments apply to procedures relevant for the solution of a problem. In the example just cited, does the student know how to construct a perpendicular to a given line through a given point? If the student does not recall the construction, does he or she know that it can be done, so that deriving the construction is a possibility? Or must that too be discovered? These factors determine the potential evolution, and characterization, of a problem solving session.

After the question of the posession of factual and procedural knowledge comes the question of access to it. The student may know that similar triangles have certain properites, for example, but will the student "see" or even look for similar triangles in a particular circumstance? Much "expert" performance in given domains is attributed to the posession of certain problem solving schemata; this is, indeed, the foundation of much AI research. Questions of how to represent such "compiled" knowledge are open. Among the approaches to representation "particularly worth describing [are] the predicate calculus, production systems, semantic networks, and frames" (Walker, 1981). All of these approaches take as given that there are certain regularities in experts' perceptions of problem situations, and of appropriate behavior in them. This perspective is substantiated in various ways in the literature, for example with experimental results that experts in physics (Chi, Feltovich, and Glaser, 198X) and mathematics (Schoenfeld and Herrmann, in press) see through the "surface structure" of problems to perceive "deep structure" similarities and approach the problems accordingly. Moreover, students develop problem schemata that may or may not be consistent with those of experts (Hinsley, Hayes, and Simon, 1977; Silver, 1979), and these schemata change with experience (Schoenfeld and Herrmann, in press). For a characterization of the role of schemata in students' mathematical problem solving performance, see Silver (in press).

There is yet one more level of tactical behavior, that of implementing certain problem solving heuristics. Examples of these will be seen in protocols 1 and 2. In a sense, these are nearly on a par with domain-specific schemata. For example, "it is useful to assume that one has the desired object and then to determine the properties it must have" is a heuristic typically valuable in straightedge-and-compass constructions. Its domain-specific implementation (draw the figure and see what properties it has) is quite similar to the implementation of domain-specific schemata, such as "look for congruent triangles when faced with a problem of this nature." These heuristics, like the other categories of knowledge described above, fall into the category of <u>tools</u> potentially accessible to the problem solver. An inventory of these tools provides a characterization of what the problem solver <u>might</u> be able to use in approaching a problem. Which of these tools are selected or discarded, how such decisions are made, and what the impact of such decisions on the problem solving process is, is the next level of analysis.

B. On "Control" Knowledge and Behavior

Two students, trying to determine the characteristics of the largest triangle that can be inscribed in a given circle, guess that the equilateral is the desired triangle and set out to calculate its area. They get enmeshed in calculations and, when the 20-minute videocassette recording their performance runs out of tape, are still calculating. Asked what good the answer will do them, they cannot say. This is an extreme (although not atypical) example of what might be called an "executive" or "control" malfunction: one bad decision, unmonitored and unchecked, dooms an entire solution to failure. What the students actually kn[illegible], and what they might have done given the opportunity to employ that knowledge, becomes a moot question. In contrast an expert working on an unfamiliar problem generates a dozen potential "wild goose chases," but rejects all of them after brief consideration. With some clumsiness, he solves a problem the students did not -- although he began working on the problem with much less domain-specific knowledge than the students "objectively" had at their disposal. It can be argued that the expert's success and the students' failure were due respectively to the presence and absence of productive "metacognitive" behaviors (Schoenfeld, in press).

One of the early researchers to stress the importance of metacognition as a major factor in cognitive performance, Flavell (1976, p. 232) characterized it as follows:

> I am engaging in metacognition...if I notice that I am having more trouble learning A than B; if it strikes me that I should double-check C before accepting it as a fact...metacognition refers, among other things, to the active monitoring and consequent regulation and organization of these processes to the cognitive objects on which they bear.

For the most part, research in artificial intelligence has not dealt directly with issues of metacognition as they are characterized here. This is a subtle point, since many of the terms used in metacognition overlap with those used in AI (see Brown's definition, below). But the usages differ. Consider, for example, skilled problem solving in physics as modeled by production systems (Larkin, McDermott, Simon, and Simon, 1980). The idea is to model competent behavior in sufficient detail to be able to select the "appropriate" behavior, a certainly enormous task. But issues of the type that humans encounter when working on such problems -- "I've been doing this for five minutes and it doesn't seem to be getting me anywhere; should I perhaps take an entirely different perspective?" -- are not the focus of such programs. They model behavior where such problematic performance is not a "problem."

Likewise, there are difficult issues of strategy selection in any reasonably sophisticated program. But the use, for example, of "conflict resolution strategies" to determine precisely which production will "fire" when the conditions for more than one production have been met, still operates at a very different level than the one under consideration here. Few programs deal with planning and monitoring at that level, although there are many "planning" programs. One that does, and is worth singling out for special notice, is the Hayes-Roths' (1979) "opportunistic" model. Typical planning procedures call for leaving sequences of actions unspecified until one is constrained to specify their order, and checking for conflicts when one does so. A standard example is Sacerdoti's (1977) task, "paint the ladder and the ceiling." If one tries to proceed in that order, painting the ladder precludes painting the ceiling. "Planning" means specifying actions in efficient temporal order. Sacerdoti's "nets of actions hierarchies" are designed to allow for fleshing out plans in such a way that such impasses are avoided. This whole perspective, however, assumes that one works in domains where plans are there to be "fleshed out" -- certainly not a universal condition in problem solving. In contrast, the Hayes-Roths' model is many-leveled and, if it is appropriate, shifts rapidly from considerations at one level (do B before A, instead of the other way around) to another (revising the entire plan structure because of an unforseen major difficulty). This "opportunistic" model is highly structured, but also highly data-driven. It is open to the idea that one piece of new information may cause one to see everything that came before in a new light, and call for major revisions; that each piece of information, and the current state(s) of affairs must be continually evaluated and acted upon. To my knowledge, few other programs deal directly with this kind of issue.

There are, however, some programs that specifically separate what have been called "knowledge" and "tactics" here. For example, Bundy and Welham (1981) describe a technique called meta-level inference, in which

> ...inference is conducted at two levels simultaneously...The object level encodes knowledge about the facts of the domain...while the meta-level encodes control or strategic knowledge...What are the advantages of this technique?
> -The separation of factual and control information enhances the clarity of the program and makes it more modular.
> -All the power and flexibility of inference is available for controlling search (p. 189).

This perspective is at least sympathetic to the separation of "tactics" and "strategies" described above. It is an alternative to the production model system, where the decision-making resides in the nature of the productions.

There has recently been much discussion of metacognitive issues in the psychological literature. The bulk of such research has focused on metamemory (one's awareness of how he or she stores and retrieves information), and much of that work has been developmental. See Brown (1978) for an overview of the relevant literature. Research suggests that the use of self-regulation is a large component of older children's successful memory performance (Brown and DeLoache, 1978). Speaking in general, Brown (1978) describes metacognitive behaviors as "those attributed to the executive in many theories of human and machine intelligence: predicting, checking, monitoring, reality testing, and coordination and control of deliberate attempts to solve problems. I believe that these are the basic characteristics of thinking efficiently in a wide range of learning situations." One can hardly disagree. Moreover, the converse must be stressed. Just as the presence of such behaviors may promote efficient problem solving, the absence of them may doom problem solvers to failure. Discussions of metacognition in broad problem solving domains are rare. See Schoenfeld (in press) for one attempt in that direction, and Silver (note 2) for an overview of the issue.

C. On Belief Systems

Ulric Neisser begins the article "General, Academic, and Artificial Intelligence" (1976) with the following dialogue. It was taken from Cole, Gay, Glick, and Sharp's (1971) study of cognition in a Liberian people called the Kpelle.

> Experimenter: Flumo and Yakpalo always drink cane juice (rum) together. Flumo is drinking cane juice. Is Yakpalo drinking cane juice?
>
> Subject: Flumo and Yakpalo drink cane juice together, but the time Flumo was drinking the first one Yakpalo was not there on that day.
>
> Experimenter: But I told you that Flumo and Yakpalo always drink cane juice together. One day Flumo was drinking cane juice. Was Yakpalo drinking cane juice that day?
>
> Subject: The day Flumo was drinking the cane juice Yakpalo was not there on that day.
>
> Experimenter: What is the reason?
>
> Subject: The reason is that Yakpalo went to his farm on that day and Flumo remained in town that day (Cole et. al., 1971, pp 187-188).

The point Neisser wishes to stress is that the subject's answers <u>are</u> intelligent, although they are not directly responsive. "The respondents do not accept a ground rule that is virtually automatic with us: 'base your answer on the terms defined by the questioner.' People who go to school (in Kpelleland or elsewhere) learn to work within the fixed limitations of this ground rule, because of the particular nature of school experience" (p. 136). There are, Neisser argues, many dimensions to "intelligence" beyond the types of (academic) intelligence measured by IQ tests, the (artificial) intelligence modeled in computer programs, and the "purely cognitive" intelligence studied in psychological laboratories. Of course anthropologists take that as given (see, e.g. Cole, et. al., 1971, or Lave, 1980) and some cognitive scientists have urged that the range of cognitive investigations be substantially broadened (e.g. Norman, 1979).

The dialogue quoted above serves to make another point as well, one that bears directly on current methodological issues. In the dialogue we see a clash of belief systems, where the participants see the "ground rules" for their exchange in rather different ways. Were the experimenter to declare the subject "unintelligent" because he did not answer the questions as they were posed, we would argue that he missed the point: the responses must be interpreted in the context of the social environment that generated them, and not simply evaluated as "pure cognitions." I shall argue here that the same point holds in many of our methodologically "clean" laboratory studies, and that much of what we take to be "pure cognition" is often shaped by a variety of subtle but powerful factors. These factors may include the subject's response to the pressure of being recorded (resulting in a need to produce <u>something</u> for the microphone), his or her beliefs about the nature of the experimental setting (certain methods are considered "legitimate" for solving problems in a formal setting, others not), and the subject's beliefs about the nature of the discipline itself (is mathematical proof useful, for example, or a waste of time?). This network of beliefs provides the context within which verbal data are produced, and an understanding of that context is essential for the accurate interpretation of those data.

It should be clear that these comments are not meant as a blanket <u>a postiori</u> challenge to the accuracy of studies that have relied upon the interpretation of verbal data. It may well be that the issue of belief systems is moot in a number of contexts -- for example, in the analysis of experts' verbal protocols for purposes of constructing artificial intelligence programs. Experimenters tend to find their subjects among their colleagues, who are generally familiar with and sympathetic to the methodologies being used for protocol collection. It is unlikely, therefore, that an unsuspected difference in belief systems between experimenter and subject will result in the misinterpretation of the verbal data. The situation may be quite different, however, when students are the source of that data and the task at hand is to interpret (in the large) what they have produced. A miscellany of examples that document this point are offered in section 4. Some less "impressive" but more typical protocols are discussed, from the perspectives at all three levels, in the next section.

3. A discussion of three problem solving protocols

Appendix 1 gives a protocol obtained from two students working on a straightedge-and-compass construction problem in plane geometry, recorded the second day of a problem solving course. The students were friends, and felt comfortable working with each other. They were both college freshmen, and had both just completed a course in first-semester calculus. They had taken the "standard" geometry courses in high school. Appendix 2 gives a protocol recorded by the same pair of students a month later, after the intensive problem-solving course. (See Schoenfeld [1982] for a brief description.) Geometric constructions were one of the topics discussed in the course. The students had read chapter 1 of Pólya's Mathematical Discovery (1962), and worked perhaps a dozen construction problems. Appendix 3 gives a protocol obtained from a professional mathematician who had not "done" any plane geometry for a number of years. The protocols are themselves quite eloquent. The discussion is brief, serving to illustrate some of the points made in section 2. Each of the comments made here needs to be elaborated in far greater detail.

I would like to begin with a general discussion of students' behavior on problems like the one given in appendix 1. From my perspective, the most telling information regarding their behavior is derived at the level of belief systems. Students' actions are shaped by their beliefs about the way that one solves geometric construction problems, and about the role of "proof" in mathematical problem solving. In my experience, the following collection of beliefs is nearly universal among the "typical" college freshmen who have studied geometry in high school and studied at least one semester of calculus:

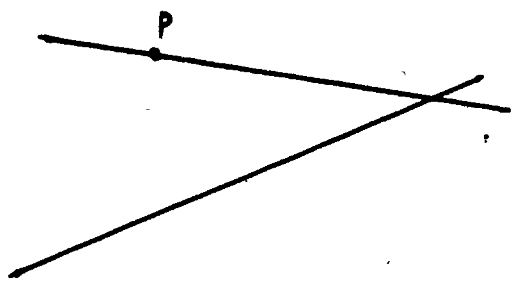
a. One gains "insight" into a problem situation in geometry by having a <u>very</u> accurate picture of it,

b. Verification is purely empirical. Hypotheses about constructions are tested by performing the indicated constructions. If the construction appears to work, it is correct.

c. "Proof" is irrelevant to discovery and verification. If absolutely necessary (i.e. the teacher asks for it) one can probably prove that constructions work. But this is simply "playing by the rules of the game," verifying formally what one already <u>knows</u> (empirically) to be correct.

d. Candidates for solutions are tested <u>seriatum</u>. Hypothesis 1 is tested until it is accepted or rejected, then hypothesis 2, and so on. Simple (intuitively apprehensible) hypotheses are tested first.

If one accepts these as the "ground rules" for constructions, one can predict stereotypical performance. Consider the problem given in protocol 1: Construct the circle that is tangent to the two lines in the figure below, and has the point P as its point of tangency to one of them.



Among the features in the problem likely to catch the student's attention are:

F1: the radius of the desired circle is perpendicular to the top line at P.

F2: by some sort of perceived symmetry, the point of tangency on the bottom line is probably directly opposite P.

F3: any "reasonable looking" line segment joining the top and bottom lines, and passing through P, is likely to be the diameter of the given circle.

F4: the center of the circle seems to be halfway between the two lines.

Combining F1 and F2 respectively with F3, we obtain the two "intuitively apprehensible" hypotheses regarding the construction:

H1. The line segment between the two lines, and perpendicular to the top line at P, is the diameter of the desired circle. The center of the circle is the midpoint of that line.

H2. The line segment between P and its "opposite," P', is the diameter of the desired circle.

The two hypotheses that are less intuitively apprehensible (but correct) are combinations of F1 and F2, and F1 and F4, respectively.

H3. The center of the circle lies on the intersection of the perpendiculars to P and its opposite, P'.

H4. The center of the circle lies on the intersection of the perpendicular to P and the bisector of the angle made by the two lines.

This set of hypotheses, combined with the four beliefs described above, allows for the following general predictions regarding the evolution of students' solutions to this problem. If F1 or F2 is first heeded, that will give rise to H1 or H2, which will be "tested" with straightedge and compass. When that one fails, the other (if the appropriate feature was noticed) will be tested and rejected in the same way. Only after the "apprehensible" hypotheses that were perceived have been tried will either of H3 or H4 be tried, if those are seen as possibilities. If they are not, the students will report being "stuck." If they are tried, and seem to work, the students will report having succeeded. Although they may feel uncomfortable about not being able to explain why it works, they will not doubt that it does. Whether they have been successful or not, half of their time will have been spent with straightedge and compass in hand. No active mathematical derivations (proof) will have been undertaken.

At this coarse level of detail, the predictions made above are remarkably robust. I could offer any number of protocols in which the students slavishly adhere to the outline just described. With one or two pairs of students and the same methodology,* the reader can generate his or her own. Instead I have chosen examine a more complex protocol, one that is much richer than most of the stereotypical ones. This protocol is better than average (!) in a number of ways. It is relatively free of the types of pathologies described in section 4. The students work well together, and concentrate on the problem for the full twenty minutes allotted for it. Most importantly, these students demonstrate much better awareness and control of their own problem solving processes than most (see in contrast

*The protocols were obtained by recording students in pairs. They were asked to work together as a team, and instructed not to "explain" what they were doing for the recorder. The underlying rationale for this methodology is given in note 1.

protocols 1 and 2 in Schoenfeld, in press). Their strategic and metacognitive behaviors work reasonably well -- but working within the context generated by the belief systems, these behaviors can only work to limited effect. The following is a brief running commentary.

T begins by sketching in the desired circle (Item 1), and there is a clear attempt to make sure that she and L understand the problem statement. This deliberateness in guaranteeing that they "understand" is respectable "control" behavior, in contrast to the impulsive actions taken by many students in similar circumstances.

By item 4, the sketched-in circle is erased: it was "legitimate" as an aid to understanding, but (according to their belief systems) does not belong in the figure as a proper part of working the problem. In item 5 feature F3 and the associated conjecture are introduced.

Here the dialogue is unusual in two ways. First, the students do not attend to F2, and are thus deprived of the opportunity to verify their conjecture empirically. Second, T actually raises plausible objections to the conjecture (items 6 and 8), and a meta-level dialogue ensues. This is certainly respectable executive behavior. But then the students spend 2½ minutes with straightedge and compass trying to resolve the dilemma.

Their construction "looks right" (item 11) but they again recognize that this one example does not guarantee validity in general. There is an attempt to exploit a related problem in items 14-24, again indicating some sophistication. Then five minutes (items 25-41) are spent in empirical work, resulting (finally) in the rejection of the initial hypothesis. The rejection, is, however, substantiated theoretically (the tangents to the endpoints of a diameter must be parallel).

In item 43 comes the belated recognition of F1, which again is combined with F3 to generate H1. The enthusiastic jump into implementation (items 45-50) may be in part a result of desperation, as well as the declaration that using a ruler to draw a right angle is "legal" (items 62-63). Yet items 56-57 and 61-63 say a great deal about students' perceptions of the nature of "being mathematical." Contrast this with protocol 3.

Conjecture H1 is again evaluated empirically, and the control functions are again relegated to performing post mortems: e.g. items 80-83. There is again a reference to the related problem (item 89), and -- as if we need any more evidence -- an indication that their approach to that problem was also purely empirical.* The solution degenerates from there. I wish to stress here that (a) the students did, as determined later, have an adequate factual knowledge to be able to solve the problem, and (b) their meta-level behaviors, as indicated in items 1, 6-8, 12, 14, 40-41, 80-83 and 89, are generally most respectable. The major "difficulty" is the very approach they take.

In contrast let us look briefly at protocol 3, where a mathematician works on the problem the students alluded to in item 14. It is, essentially, the same problem. A number of factors may contribute to the mathematician's success: better control behavior, more reliable recall of relevant facts, and (not to be underestimated) more confidence. But most important is the basic approach that the mathematician takes: he

*That comment is important in the following sense. It indicates that their behavior in this experimental environment is similar to their behavior when working on the problems in their own rooms. In view of some of the examples in section 4, this is non-trivial.

derives the information he needs through the use of proof-like procedures. Note that he is looking for congruence ("there've got to be congruent triangles in here.") long before there is a conjecture to "verify." Rather than being an afterthought or a method of verification, proof is a means of discovery for him.* The non-empirical nature of his approach is made emphatically clear the last line of the protocols, where performing the construction is the operation that is relegated to the status of an afterthought. He is certain the construction will work.

In protocol 2 we see an indication of the "intermediate" status of the students after a month of problem-solving instruction. The course focused on heuristic and executive problem solving strategies. Some of these are evident in the protocol; some were present before the course. Proof was often discussed in the course, but in the usual way: "Yes it seems that way, but how do you know it will always be true?"

Objectively the students' behavior in this protocol compares favorably with their behavior in protocol 1, along all three of the dimensions outlined in section 2. Their recall of relevant facts (e.g. that the radius of a circle is perpendicular to any tangent at the point of tangency, item 69) is more assured, and called into play at appropriate times. Domain-specific procedural knowledge is also more accurate, and they are confident about their abilities to perform the appropriate constructions. However, these were not disabling factors in protocol 1 and only tell a small part of the story.

*It was Pólya, I believe, who defined geometry as the art of "right reasoning on wrong figures" -- clearly the mathematician's perspective, and antithetical to the students' belief systems.

There is a telling difference in their performance at the heuristic level. A few years ago that difference would have tempted me to attribute their success to the heuristics that they had learned. They draw a picture of the goal state to determine what properties it has (items 14ff.), look at extreme cases (items 34-46), consider only obtaining partial fulfillment of the conditions (item 52), and so on. The first of these heuristics alone might have guaranteed success in problem 1. However, there is a good deal more.

Their strategic (meta-level) behavior is quite good, as it was in protocol 1. They monitor and assess both the state of their knowledge and the state of the solution with some regularity (e.g. item 71), and avoid the kinds of "wild goose chases" that often guarantee failure for less sophisticated students. Here, in fact, control behaviors become a positive force in the evolution of the solution. At the very beginning (item 20), empiricism is put in its place. Time constraints are taken into account: in item 63 the expedient of using the markings on a ruler is acknowledged as "illegal" but used anyway -- they could bisect the line if they had to. They know that they are supposed to prove that their constructions "work," and predict early on that they can "do it with similar triangles and things" (item 72). In this context proof is still regarded as a means of verification, to be used after one is convinced he or she knows the answer. The convincing comes by means of good sketches and "gut feeling," however, not by perfect constructions. "Proof by construction" is clearly put to rest in item 78.

It is tempting, then, to argue that the control strategies serve as enabling factors, allowing the students to employ their tactical knowledge with some success. Certainly the absence of efficient control behaviors would have sabotaged their attempts (Schoenfeld, in press). But, the discussion in the last paragraph indicates that the control behaviors were operating within the context of new beliefs regarding proof and empiricism. Had those belief systems not changed, the control strategies could not have operated the way that they did. One can conjecture that without this change in belief systems their behavior would still resemble their behavior in protocol 1 -- even if, say, they had been given a review of basic facts and procedures, and taken a course that stressed meta-level problem solving skills.

This brief discussion serves merely to raise a host of questions. It is not meant to minimize the importance of tactical or strategic knowledge, but to indicate that a third and often hidden level of analysis must also be taken into account when one analyzes problem solving behavior. As indicated in section 2C, there may well be contexts in which one level of behavior predominates: the tactical in AI "expert" simulations, the strategic in "wild goose chase" solutions, and belief systems in protocol 1. Even in this "purely cognitive" kind of investigation, other than pure cognitions must be taken into account. But this is only the beginning, as the next section indicates.

4. The Matrix Within Which Pure Cognition Resides

While the previous section raises some questions about the interpretation of verbal data, it does not at all challenge their legitimacy. That is, the discussion was predicated on the assumptions that (1) protocols like those in appendices 1 through 3 provide an accurate reflection of the cognitions and behaviors of the people who produced them, and (2) in turn, models of behavior based on such protocols (for example, the model outlined at the beginning of section 3) thus reflect the subjects' behavior with some accuracy. In the case of the particular protocols discussed, I am reasonably confident that this is the case. In general, I am much less sanguine about the "legitimacy" of verbal data, even of some data obtained in methodologically "clean" settings.

Of course this issue is not new. Methodological battles were waged, for example, over the legitimacy of introspection as a means of characterizing cognitive processes. "We have also long known, both from experiments and everyday experience, how subjects' behaviors are affected by expectation, context, and measurement procedures. The notion that there can be 'neutral' methods for gathering data has been refuted decisively" (Ericsson and Simon, 1981, p. 17). That point granted, the question then becomes one of the intrusiveness of various experimental methods. For example, it is generally acknowledged that asking subjects to analyze their problem solving processes while they work on problems does have measurable effects on performance. However, the current literature indicates that sufficiently "bland" instructions may not have a measurable effect on data gathered in the laboratory: subjects who are instructed simply to "talk out loud" as they solve problems, and <u>not</u> to interpret or explain, will yield essentially the same performance that they would have if they were not speaking out loud (Ericcson and Simon, 1980).

There is, in that last sentence, a very subtle but powerful disclaimer. It is revealed by the following.* In 1978 I made a series of recordings

*Other aspects of this issue, and the complete protocol from which the excerpts below are taken, are given in Note 1.

of students solving the following problem "out loud."

> Estimate, as accurately as you can, how many cells might be in an average-sized adult human body. What is a reasonable upper estimate? A reasonable lower estimate? How much faith do you have in your figures?

The problem is a particular favorite of mine, an excellent task to use for examining cognitive strategies and memory searches. It can, actually, be solved without any special technical information. One wants good estimates for "average human body volume" and "average cell volume," under the assumption that there are such things. Since there will be a huge amount of guesswork on cell volume, body volume can be roughly approximated: a box with dimensions 6' x 6" x 18" will be close enough (probably within a factor of two) to the actual average.* With regard to cell size, we can see the markings of a ruler down to 1/32" so perhaps 1/50" is a lower limit to what we can see clearly without "help." Cells were discovered with early microscopes, which must have been greater than 10 power (magnifying glasses probably give about 5 power) and less than 100 power. So a "canonical cell" (say a cube) must be between 1/500" and 1/5000" on a side. The rest is arithmetic.

My first set of subjects were junior and senior college mathematics majors. The students knew me reasonably well and were familiar with my work. Some had done protocol recording themselves, as parts of senior projects. I took all of the appropriate precautions to set them at ease for the recording sessions, and recorded them working on the problem one at a time. See appendix 1 of Note 1 for a representative protocol.

*A more accurate figure can be obtained by taking an estimate of average body weight (say 150 pounds) and converting it to volume. Since the human body (barely) floats, its density is close to 1. However, the point is that there is no need to be so precise: this degree of specificity is an indulgence.

Typically, students would quickly choose volume as the quantity to compute. After brief consideration they would decide to compute body volume first, and would then begin extraordinarily detailed computations. Generally an "average" body (most often their own) would be approximated by a series of geometric solids whose volume was rigorously calculated. For example:

> and now a leg...a cone might be more appropriate. And the base of my leg is approximately 6 or 7 inches in diameter so you would have $(3\frac{1}{2})^2 \times \pi$ and the height would be...what is my inseam size, about 32 or 34. So you've got to have a 34 and it's a cone so you've got to multiply it by one third.

In sharp contrast to their meticulous calculations of body volumes, the students' estimates of cell size were (1) crude and (2) not accompanied by estimates of how accurate they might be. For example: "All right, I know I can see 1/16 of an inch on a ruler, so say a cell is 1/100 of an inch on a side." The students spent the great majority of their time making estimates of body volume. These results, though puzzling, were remarkably consistent.

Later in the year I began making recordings with pairs of students solving problems together. I recorded perhaps two dozen pairs of students, who solved the same problem after receiving nearly identical instructions. Not once did a pair of students demonstrate the kind of behavior I have just described. With hindsight, it became apparent that the behavior in the single-student protocols was not a reflection of their "typical" cognitions. Rather, their behavior was pathological -- and the pathology was induced by the experimental setting itself. This problem upset the students, because they had no idea of how to approach it. Feeling "on trial" to produce something for a mathematics professor, they responded to the pressure by doing the only mathematics they could think of under the circumstances:

computing volumes of solids. This, at least, was demonstrating mathematical behavior! (The students in two-person protocols manage to dissipate the environmental pressure between themselves, and thus to avoid extreme manifestations of pathology.)

I have dwelled on this example at length because it indicates the subtle difficulties inherent in protocol analysis. When I discovered the social causes that I now believe explain the students' behavior, I was on the verge of writing a paper describing (a) their surprising inability to make "order of magnitude" calculations, and (b) their poor allocation of strategic resources in problem solving. In hindsight, this "purely cognitive" explanation of their verbal data would make no more sense than "objectively" assigning a low IQ score to the Kpellan native quoted in section 2C on the basis of his responses to the experimenter's questions. We need not travel to Liberia; clashes in belief systems between experimenter and subject occur here in our own laboratories.

Since the length of this paper has already grown out of hand, the rest of the discussion will be very brief. My intention is to sketch out some of the dimensions of the matrix within which "pure cognition" resides. A broad outline of it, given in the form of a mathematical cross product, is given below.

{SETTING} × {KNOWLEDGE, BELIEF AND VALUE (KBV) SYSTEMS} × {DEGREE OF AWARENESS}

SETTING	×	KNOWLEDGE, BELIEF AND VALUE (KBV) SYSTEMS	×	DEGREE OF AWARENESS
Individual (Self)		KBV about Self		Unaware
Cognitive Structures: access to facts, to procedures, and to strategies		KBV about facts KBV about procedures KBV about strategies		Aware but non-reflective
Task		KBV about task		locally aware and reflective (monitoring and assessment)
Environment		KBV about environment		Reflexive Abstraction

--figure 1--

The column on the left of the figure represents an "objective" description of the problem setting, the product of the two columns on the right the set of "driving forces" that operate in and on the setting. We take one column at a time.

The first column is familiar. In the best of circumstances, this is all that one need be concerned with. "Task variables" can be described objectively, and the environment as well. "Cognitive structures" are the focus of customary laboratory investigations: facts, procedures, and strategies. Under the assumption that laboratory investigations provide an accurate reflection of problem solving behavior, the investigator's focus can be on the overt manifestations of these cognitive structures. In this context the issue is more delicate: one must (somehow) ascertain the set of facts, procedures, and strategies that are potentially accessible to the problem solver.

The second column deals with belief systems. Some ideas about belief systems have reached the level of folk wisdom: for example, the notion that, through perseverance, a person will turn the belief in his or her ultimate success into self-fulfilling prophecy. A student's belief in his or her ultimate failure will affect the verbal data one obtains as well: I have videotapes of students who never seriously engaged themselves with a problem, in order to later rationalize what they saw as their inevitable failure. (This has been admitted to me, long after taping, by more than one student.) Beliefs about the very nature of facts and procedures will determine students' performance. The student who believes that mathematical knowledge must be remembered will be stymied when a particular object (say a procedure for constructing a line parallel to a given line) is forgotten, while another who believes that the procedure can be derived will act rather differently. The effects of strategic and task-related beliefs (one approaches constructions empirically, etc.) were considered in section 3. And the effect of beliefs about the environment (one must produce mathematics when one is solving problems for a mathematics professor!) were the causes of the pathological examples that began this section. These examples barely scratch the surface, of course. But the point is that if we wish to describe behavior as it occurs, we must worry about such things.

The third column reflects the degree to which the individual is aware of his or her knowledge and belief systems. This column is important for the following reason: one can only act upon those beliefs that he is aware of. As long as the students in protocol 1 believed that discovery and proof in geometry are purely empirical, they would continue to approach problems that way. Once they were made aware of that belief (and that other possibilities exist) they could change their behavior. Similarly, students who are aware that they can monitor and assess their own cognitive strategies can, then, serve as active agents in their own growth. Making students aware of their own (and competing) beliefs may be one of the most valuable functions we can perform as educators.

5. Discussion

This paper covered a huge amount of territory, much of it at breakneck speed. First, let me highlight some of the methodological issues.

A. There are at least three qualitatively different levels at which one can analyze verbal data. Depending on circumstances, one level or another may provide the "key" to understanding what happens in a given protocol. Examples of primarily "tactical" protocols are those gathered from experts working on routine tasks in familiar domains, e.g. those in Larkin, McDermott, Simon, and Simon (1980). Examples of primarily "strategic" or executive protocols are those where students go off on "wild goose chases," e.g. those in Schoenfeld (in press). An example where belief systems provide the primary level of analysis (protocol 1) was discussed in section 3. A comprehensive discussion of verbal data requires the consideration of all three levels.*

B. Belief systems can be modeled. Such models exist, for example, in decision theory. Kahneman and Tversky's (1979) prospect theory includes computational models of decision-making that take into account subjects' belief systems. The gain or loss of the same dollar amount (say $1000) are not viewed in the same subjective terms: generally, loss is more traumatic. Similarly, winning $2000 may not have twice the emotional value of winning $1000. Prospect theory assigns to each of the dollar amounts above its <u>subjective</u> value (say, for example, -1200 for the loss of $1000, +800 for the gain of $1000, and +14[illegible]0 for the gain of $2000). These figures are used to make computations of "subjective expected utility," which have reasonably good predictive power.

I believe that rigorous models characterizing the effects of belief systems on problem solving behavior can be made, and that these models will have both ecological validity and predictive power. The discussion

*This is oversimplified, of course. Belief systems may have served to "explain" most of protocol 1, but protocol 2 provided a (perhaps more typical) example of the dynamic interplay among the different levels. The "real" question, as I see it, is: what accounts for the differences in problem solving performance between the two tapes? This question is of nearly overwhelming complexity. This framework offers, I hope, a first step towards unraveling it.

of "typical" student behavior on geometry constructions that began section 3 is, in essence, a prospectus for that kind of model.

C. Great care must be taken in the interpretation of verbal data. It may well be true that, with sufficiently bland instructions, students' performance in the laboratory may not be measurably changed by speaking "out loud" as they solve problems. But the behavior that they produce may be completely abnormal -- even if it is consistent enough to model with great accuracy. Under such circumstances, we may simply be modeling abnormal pathology in the name of cognition. Again, the issue may be moot where the belief systems of the people on both sides of the microphone coincide (with experts generating protocols for their colleagues' simulations). But the more alien the setting for the subject, the more likely it is that the data will be "driven" by covert beliefs that skew its meaning (see Note 1).

The second set of issues deals with applications of cognitive research to educational research and development. Here the potential for the misunderstanding and misapplication of basic cognitive research is enormous. There are dangers in adapting both the methods and results of much current research to educational settings.

D. Researchers in education increasingly rely on "verbal methods" such as protocol analysis for their research, using for their analyses the successful analytical tools and perspectives derived from AI and information processing research. Yet the goals and the contexts of such studies can be substantially different. In much AI work the goal is to model idealized, purely cognitive behavior. Both the subjects and the tasks are selected to facilitate this kind of modeling, and a "purely cognitive" approach appears to be sufficient. In educational work, characterizing "idealized" intellectual behavior is only one component of a much larger enterprise. If one wishes to affect students' behavior, one must be able to describe it accurately and to characterize what causes it -- and it would appear that belief systems are a major driving force in students' behavior. Any framework that ignores them -- regardless of how accurate it is in other contexts -- can result in the severe distortion and misinterpretation of the data.

E. The applications of cognitive research to schooling must take into account the context in which cognitions are embedded. The brief discussion of figure 1 in section 4 is an attempt to sketch out the range of issues that must be taken into account if our increasing knowledge about cognition is to be employed usefully in the schools. There are any number of examples regarding that context. Jean Lave (Note 3) reports that people's use of arithmetic in everyday situations does not correlate well with their scores on paper-and-pencil tests of it. Dick Lesh (Note 4) reports that students' problem solving behavior when dealing with "real" problems bears little or no relation to their "academic" problem solving behavior. Neisser (1976) argues the point in general.

I think that a broad attempt to deal with cognition in its "real world" context can have a strong positive effect on schooling. The three dimensions that appear most critical to me are represented in the three columns of figure 1. It goes without saying that knowledge of the basic facts, procedures, and strategies (the first column) is essential. Most of this paper has argued for the importance of the second column, and I will not labor the point further. The third, "awareness," has only been alluded to, and is worth discussing a bit. I would assume that the purpose of schooling is to prepare students for life after school: to help them develop the mechanisms they will use throughout life to adapt to new situations. Yet virtually all of the college freshmen in my problem solving courses enter the course completely unaware of the fact that they can observe, evaluate, and change their own cognitive behavior! It is as if their minds are autonomous, independently functioning entities, with the students as passive (oftimes frustrated) spectators. So long as this remains the case, the students are slaves to their own behavior. Once this belief, or any other, is made conscious, it can be acted upon and changed. Providing students with the potential for this kind of adaptation may be the greatest service we can render them.

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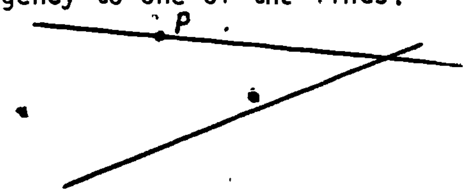
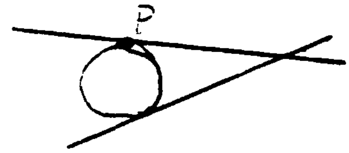
Appendix 1: Protocol 1

Problem worked the first week of instruction, by students L and T (college freshmen who had completed one semester of calculus).

You are given two intersecting straight lines, and a point P marked on one of them, as in the figure below. Show how to construct, using a straight-edge and compass, a circle which is tangent to both lines and has the point P as its point of tangency to one of the lines.





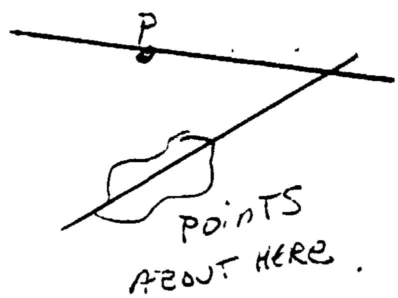
1. T: reads the problem. Oh, ok. What you want to do is that (sketches in a circle by hand), basically. Ok, how?

2. L: Now, ok, we have to find the center.

3. T: Of what?

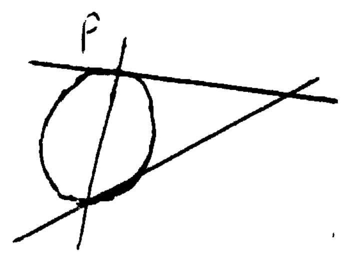
(erases circle)

4. L: Of the circle. We are trying to find the circle, right? If we did that then we could...oh, and the radius of course.



5. T: All right, well we know the point of tangency on this line is going to be right here (points to P). What we need to find is where the point of tangency is going to be on this other line, I think. So we can find the diameter in which case we can find the center.

6. L: Is that...that's not necessarily true, is it? Is it true that if you have a circle like that (see right), and then that (points with finger) would be the diameter. You know what I mean? Or maybe you couldn't have it that way...

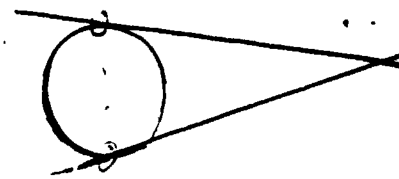


7. T: The circle has like...no, you don't have a diameter running up through there. No, we have to find the diameter from the point of tangency on this line to the point of tangency on this line, wherever it lies.

8. L: No, wait: the point of tangency, the point of tangency here, would the line connecting those two points be the diameter? It seems that you could maybe construct one where it wouldn't always work.

9. T: Wait, but see, I don't know, we're not drawing it (i.e. sketching it) the right way,

10. L: Wait, do you want to try drawing it (with the compass) and see...



(2½ minutes elapse in empirical work. A reasonably accurate drawing results.)

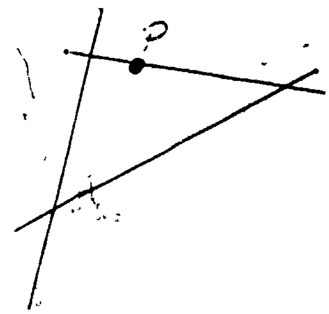
11. L: So, maybe it looks like it might be opposite, see?

12. T: But would that be true for any triangle? Oh, but see...

13. L: I'm confused. I don't think it would be. Let's say you had your radius over here and you went like that. I don't think that could be...ok, I think there could be, there is a possibility.

14. T: Remember on the first problem sheet we had to inscribe a circle on a triangle? Could you do that? I couldn't.

15. L: I couldn't either.



16. T: We're in pretty sad shape. But just say we draw a triangle even though we don't know how to do it. We will draw a triangle anyway.'

17. L: So how's that going to help?

18. T: Because we don't have to inscribe it actually. We just have to have something to help us (visualize it). (Draws an apparently arbitrary third line.)

19. L: Although...

20. T: Does that do anything?

21. L: Not at this point, I don't think. Maybe further along if we need a radius we could...but I don't think it does anything now.

22. T: We've gotta do something. With what we have, you just can't do it, right? We don't have enough lines or whatever there.

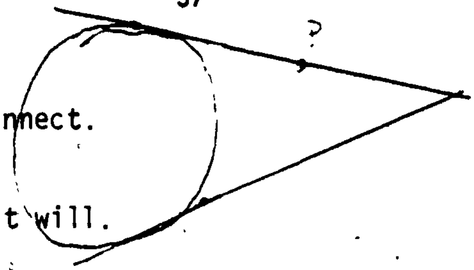
23. L: Ok, we need a center and a radius. So how do we locate the center? It has to do with, I think it has something to do with, could we do this?

24. T: No, maybe you have an equilateral triangle.

25. L: Wait, let me just try this. (Begins to expand compass.)

26. T: What are you doing?

27. L: Don't you want to see if it's true? If you have a



center way out there, because it may not connect. Don't you see? (sketch at right).

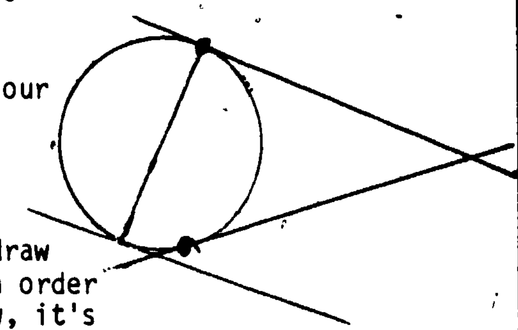
28. T: I'm pretty sure it won't. I don't think it will.

29. L: But if it won't make a circle, then that means this circle is ours (points back to earlier sketch). The one we have to deal with. You know what I mean?

30. T: I see what you mean. Like try to draw a circle out here like going through this point. See, it won't. It won't work because in order for it to work... (another few minutes with the compass. The dialogue has to do with their attempts to draw a very accurate figure, so that they can draw conclusions from it.)

31. L: Ok so that's what we're doing, right? We don't need it that big.

32. L: Yeah, wait, you couldn't because it is going to go through (the point P). I think it does have to be, right...



33. T: If we have these two points that's definitely our diameter going through it. Now we can draw...

34. L: But neither is it a tangent.

35. T: That's just what I was going to say. Can we draw these two lines so that...see you can't for in order for this to cut through this, it's too shallow, it's shallow...

36. T: Ok as soon as this...ok, make this a tangent.

37. L: In order for this to be...do you think it's going to be tangent to...

38. T: No because, because we know this one is not going to... I want to see if like we make this a tangent. You see what I mean? But that doesn't look like a diameter either. Well, I don't think that's it. Of course it couldn't be because a diameter is going to be when it's parallel, isn't it?

39. L: That's the diameter.

40. T: Ok. That's not going to help us (laughs).

41. L: You figured that out.

42. T: Right.

43. L: Can we construct one parallel to it? (Looks at original diagram.) But then we still don't know

the center.

(pause)

Could we just draw a perpendicular?

44. T: Yeah, that's what I was just going to say. If we draw a perpendicular line to this and just call that the diameter it will work from there. And then it should touch if it's perpendicular. It should be tangent at one point, shouldn't it?

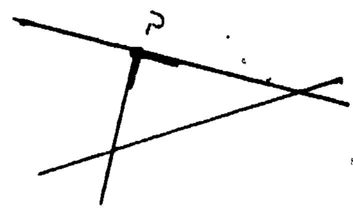
45. L: Right!

46. T: Shouldn't it?

47. L: Yes!

48. T: Won't it?

49. L: Yes!



50. T: Ok, draw a perpendicular, oh good.

51. L: Does one know how to do that with a compass? Do you?

52. T: This is a right angle, so...(uses the corner of the ruler).

53. L: Ok, that's perpendicular, ok. Doesn't look it but it is.

54. T: That's our diameter.

55. L: So if we say this is the point of tangency..

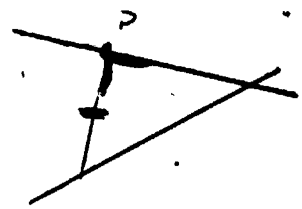
56. T: So we can bisect this to find the center, right? So call it center C. Maybe we should have done our steps.

57. L: That's all being unmathematical, completely disorganized.

58. T: Ok, back to the drawing board.

59. L: I don't know how.

60. T: Me either.



61. L: Ok, if we just use the ruler with the little numbers on it here.

62. T: Or isn't that legal?

63. L: Sure it's legal (does by hand).

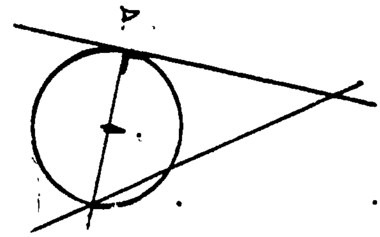
Now we have the radius, now we just draw it.

64. T: Uh, oh, do we know, we have to see if this is going to work. I know! Ugg.

65. L: My guess is, I think it's not. But we'll try.

66. T: I would think, though, it would have to, though, wouldn't it?

67. L: No.



68. T: The radius is shorter as...

69. L: I don't know. Well, let's see what happens when it goes through there.

70. T: Somehow it doesn't look perpendicular, though, doesn't it?

71. L: See this line isn't straight relative to the page which is why it doesn't look perpendicular.

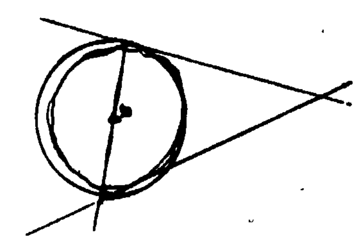
72. T: Oh right, but...

73. L: It looks good. Now we can tell something.

74. T: Maybe, I think this tells us the point of tangency has to be way more (points to right). I think.

(Three minutes of constructions)

75. L: What circle was this one? Yup, that was a right angle. Oh, darn it.



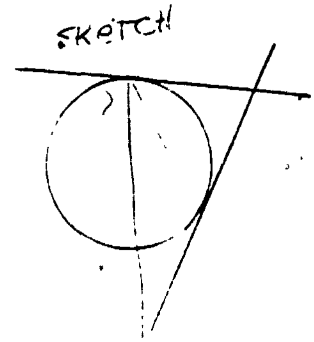
76. T: Ok so the radius has got to be smaller because it's going outside of this line. So it's got to be a little smaller and the center has got to be up and over, like here...

77. L: But how do we...

78. T: But I don't know how to do that, without doing it until it comes out right.

79. L: Yeah.

(pause and evaluation of prior failure)



80. T: That was dumb. By doing that we were saying that no matter what this line looked like, then it looked like this, if we dropped a perpendicular we could do it and we could get the diameter for that angle and still expect to do it. You know what I mean?

81. L: Yeah, I don't think it will work for any angle though.

82. T: I know, that's what I mean.

83. L: Yeah, well, we goofed again.

(pause)

84. T: Well the only thing I can think of to do is what we did in class the other...well, what we were supposed to do, you know. The triangle thing, trying to inscribe it.

85. L: Wait, we know...

86. T: I know, that's the problem. We don't know how to do it.

87. L: I don't know what to do.

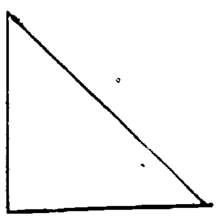
88. T: Alright, we are going to have to try something else.

89. L: Alright, what are we, what were those sort of things we tried with triangle one? Cause maybe we could... do the same thing with, on a smaller scale.

90. T: I got absolutely nowhere.

91. L: Yeah.

92. T: But I was trying to do things like, bisect this side.



93. L: Yeah, I did that.

94. T: It didn't work.

95. L: Yeah, let's see what we have here. We want to inscribe a circle in this right triangle.

96. T: Why do you want to do a right triangle?

97. L: I don't know. It just is one. Oh, I blew it now, no. The ends don't matter because we're, you see, we want to inscribe it. We're putting in the extra conditions, because it doesn't have to touch this line. It doesn't have to...oh, I don't know.

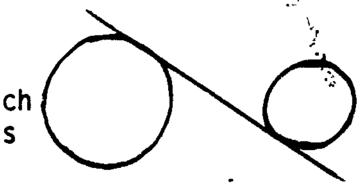
98. T: I don't think that will get us anywhere.

99. A: Ok, guys..

100. Both: We give up.

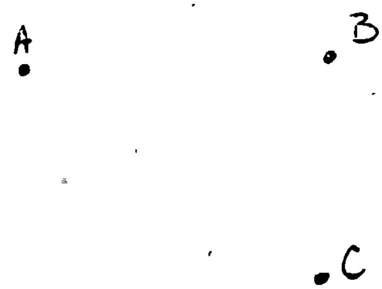
Appendix 2: Protocol 2

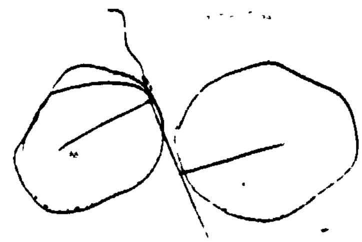
Problem worked after problem solving course.



The common internal tangent to two circles is the line which is tangent to both, but has one circle on each "side" of it, as in the picture to the right.

You are given three points A, B, and C as below. Using straightedge and compass, you wish to construct two circles <u>which have the same radius</u>, with centers A and B respectively, such that the common internal tangent to both circles passes through the point C. How do you do it? Justify.





1. T: Reads problem.

2. L: Wait, I have to read this. Ummm.

3. T: What we want basically is this, circles and a line something like this that is going to pass through here (makes sketch).

4. L: Right. Ummm.

5. T: Like that.

6. L: Except they have...where is it...have the same radius...

7. T: Uh huh

8. L: ...so it isn't going to look like that.

9. T: Right.

10. L: But, ok. Wait, I've got to think for a second.

(erasing to draw again.)

11. L: Ok, wouldn't it...no, maybe not.

12. T: What?

13. L: No, that was dumb. Let me think.

(pause)

14. L: Umm...should we try and draw it maybe, how it would be to see what the relationship of C is to the two circles, since that's not drawn.

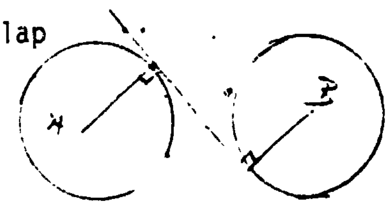
15. T: Right.

16. L: You know how I am with compasses...go ahead.

17. T: Well, how big am I supposed to draw it?

(draws with a compass)

18. L: I've made this too big because they're going to overlap one another with that radius.



19. T: Yeah.

20. L: Just draw (i.e. sketch) it...you don't have to use the compass. Just draw it...just draw...no, no, no.

21. T: Ok, and I'll make my circles better. (unclear). Ok.

C

(unclear)

(good sketch)

22. T: What are you going to do?

23. L: I just want to see what it would look like more accurately (draws with compass).

24. T: Why?

25. L: Just so I could see (unclear) but you can think out loud if you have an idea. Ok. Can you think of anything? (finishes sketch)

26. T: Umm. These two radii are the same, right?

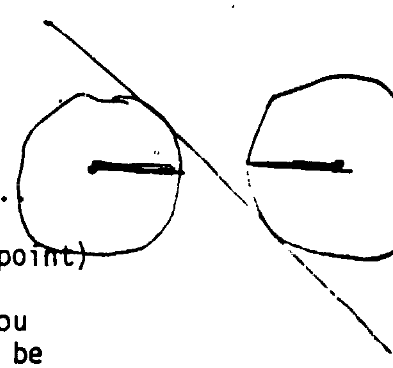
27. L: Yep. Except it doesn't look the same, does it?

28. T: That's the way you put your centers in the center.

29. L: (unclear)

30. T: (unclear) Ok. These two centers have to like... do you know what I mean?

31. L: No. Wait, what am I looking for now?



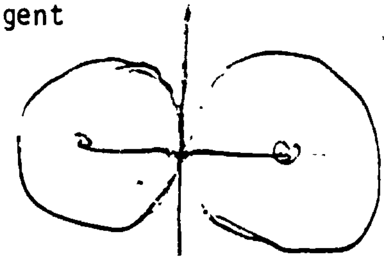
32. T: (rereads problem) Why don't we first just try to..

33. L: If we can find (unclear) (pencil placed at center point)

34. T: All right...if you just have the two centers and you go over...say the radius...the radius will have to be half way in between the centers. Alright, and then...

35. L: Say...wait...wha-wha-wha-what?

36. T: If we just try to draw the two circles and the tangent line without worrying about point C for right now.



37. L: Right.

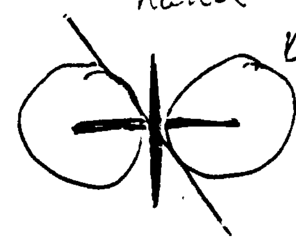
38. T: Ok. Since they have to be of equal radius...the radius will be half way between the two centers?* It's like the tangent line would be like this.

39. L: I don't get this about the radius being half way between two centers.

40. T: Me neither.

(Rapidly hand-drawn)

41. L: I don't get what you mean. How's the radius half way...I don't get what you mean.



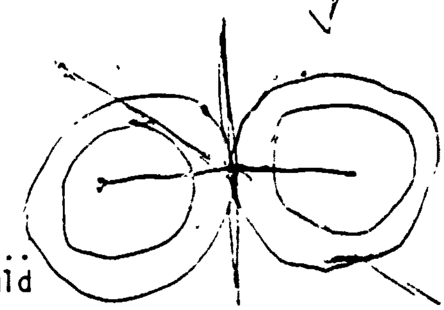
42. T: If it was like this and the tangent line would just be (unclear)

43. L: Ok, yeah.

44. T: Ok? These two have to be the same length.

45. L: Right.

46. T: And the thing that is going to determine how long they are is the angle on this line. What I mean like if they are exactly...half way in between the two centers then the line is vertical.



47. L: Right.

48. T: If we make it somehow shorter right here and here... the circles would be like this and the tangent would be on a slant like this.

49. L: Ok. Ummm.

50. T: We have to figure out how they go through point C. So...

51. L: I don't know either.

52. T: Can we just start with C and draw a line through it somewhere and then make the circles tangent to it?

53. L: No.

54. T: Or...

55. L: No we're given the centers.

*She meant to say that the length of the radius in this extreme case was half the distance between the centers of the two circles.

56. T: We're also given C.

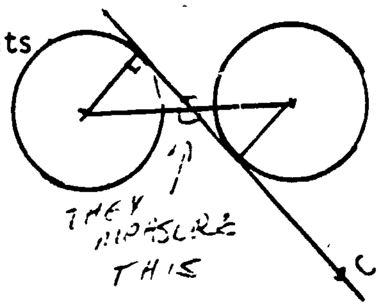
57. L: Uh huh. But just drawing the line can't guarantee you could end it with something like this if you just drew the line here. Ummm. Isn't there another way we can characterize the line? Find the locus.

58. T: Ummm.

59. L: This might not work for all of them, but, look here, doesn't this look like...that's just like the center?

60. T: That's just what I was going to measure.

61. L: Ummm. Because if we did that, we were given points A, B, and C.



62. T: Yes (looks at her sketch) that crosses it too. That's exactly what we're going to do.

63. L: Alright...wait, we're not allowed to use a ruler, but...yeah, divide it in half.

64. T: Yeah, bisect.

65. L: Why don't you actually do it...

66. T: Let's try it on here since we're not sure.

(Begins new sketch)

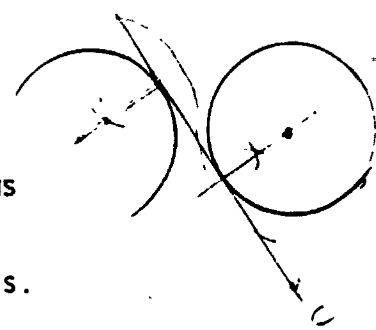
67. L: Wait, I think it was the other line. (unclear) Just connect point B. We're going to have to drop a perpendicular from B to the line.

68 T: What are you doing that for?

69. L: Because this is perpendicular and that's what the radius would be, a perpendicular and from A coming to the line also.

70. T: Right.

71. L: Ok. I don't know why this works, I mean, I just seem to see it, you know.



72. T: I think we can do it with similar triangles and things so let's just make sure it works (unclear).

73. L: We can do it here too...this isn't a very nice compass.

74. T: We're running out of time (whispering). Draw faster, draw faster.

(unfinished construction)

75. L: I can't...this is hard.

76. T: Draw faster anyway.

77. L: I didn't construct it right.

78. T: Well just draw it...it'll work.

79. L: Oh, wait, maybe I did actually. Ok, that's the radius then.

80. T: Right.

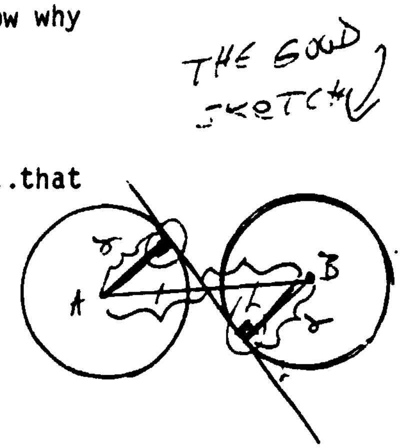
81. L: Perpendicular. Then we just have to draw...I think that's just the right thing.

82. T: That'll do it, that'll do it...wait, we've got to draw...ok, we did it. We've got to show why. We have to show that these...the reason that these are half way in between these two points is because.. angle side...we have to show that...what this side. .

83. L: Like we have an angle.

84. T: But what are we trying to show...we want to show why this is in between A and B.

85. L: Right.



86. T: So we want to show that this is equal to this...that they...

87. Both say: ...are congruent.

88. T: Ok, we have that. We have...

89. L: ...an angle and a side. How do we know...

90. T: And we need to show that this side is compared to that side. And...

91. L: (to A): Must we prove why something works or just show you the construction?

92. A: If you can justify it I would be happy.

93. L: Ok, let's try to justify it.

94. T: Now the angle...

95. L: Well, we know, I mean, r is equal to r so it is just like...

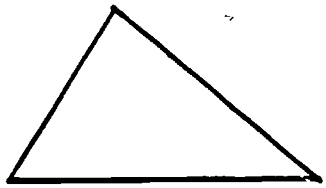
96. T: We have these angles, so this angle equals this one.

After a few minutes, and with some slight confusion, they prove that their construction has the desired properties.

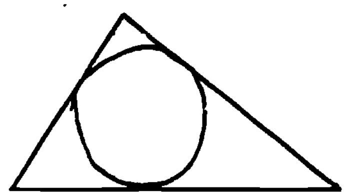
Appendix 3: Protocol 3

The subject is a professional mathematician.

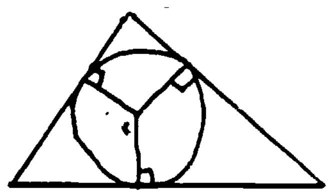
Using a straightedge and compass, inscribe a circle in the triangle below.*



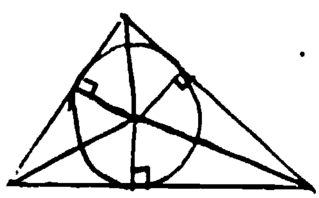
(reads problem) All right, so the picture's got to look like this (draws figure) and the problem is obviously to find the center of the circle...



Now what do I know about the center? We need some lines in here. Well, the radii are perpendicular at the points of tangency, so the picture's like this (draws figure)...



That doesn't look right, there's something missing...What if I draw in the lines from the vertices from the center? (draws figure)



That's better. There've got to be congruent triangles in here... let's see, all the radii are equal, and these are all right angles... (marks diagram) and with this, of course, this line is equal to itself (marks "x" on the figure), so these two triangles (at lower left vertex) are congruent. Great. Oops, it's angle-side-side, oh no, it's a right triangle and I can use Pythagoras or hypotenuse-leg or whatever it's called. I'm ok. So the center is on the

*The inscribed circle is a circle that lies inside the triangle and is tangent to all three sides of it.

bisectors. (Turns to investigator) I've solved it. Do you want me to do the construction?