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AUTHOR Keller, Edward L.
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ABSTRACT

This unit, which looks at applications of linear algebra to population studies, is designed to help pupils: (1) understand an application of matrix algebra to the study of populations; (2) see how knowledge of eigen values and eigen vectors is useful in studying powers of matrices; and (3) be briefly exposed to some difficult but interesting theorems of linear algebra, such as the Perron-Frobenius Theorem. The material examines mathematical models that are aids in analysis and prediction. The unit includes exercises, with answers to these problems provided near the conclusion of the document. The module contains two appendices. The first provides a program written in BASIC which computes powers of a matrix. A sample run is also shown. The second appendix goes into further examination of the powers of a Leslie matrix. (MP)

UMAP MODULE 345

UMAP: Modules and Monographs in Undergraduate Mathematics and Its Applications

Population Projection

by Edward L. Keller

"What will be the distribution of ages in the U.S. population 10 years from now, 50 years from now, or 100 years from now? Many things depend on the answer to that question--the future of the social security system, the demand for health care services, the enrollment in colleges and universities--to name a few. If we know the present age distribution of a population, can we predict the age distribution in 50 years?"

Applications of Linear Algebra to Population Studies

Birkhäuser Boston Inc.
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Title: POPULATION PROJECTION

Author: Edward L Keller
Department of Mathematics
California State University
Hayward, CA 94542

Review Stage/Date: IV 8/30/80

Classification: APPL LIN ALG/ POP STUDIES

Suggested Support Materials: Use of the computer is optional However, the subject is one which lends itself to computer analysis

Prerequisite Skills:
1. Matrix algebra (particularly matrix multiplication and inversion).
2. Ability to find eigenvalues and eigenvectors of a matrix

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1. Understand an application of matrix algebra to the study of populations.
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3. Be briefly exposed to some difficult but interesting theorems of linear algebra (such as the Perron-Frobenius Theorem).

The Project would like to thank Kenneth R Rebman of California State University at Hayward, and other members of the UMAP Finite Mathematics Panel for their reviews, and all others who assisted in the production of this unit.

This unit was field-tested and/or student reviewed in preliminary form by Deborah Frank Lockhart of Michigan Technological University, Russell Merris of California State University at Hayward, Charles Biles of Humboldt State University, Bruce Edwards of University of Florida at Gainsville, Martha Siegel of Towson State University, and Richard Melka of University of Pittsburgh at Bradford, and has been revised on the basis of data received from these sites.

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POPULATION PROJECTION

1. INTRODUCTION

What will be the distribution of ages in the U.S. population 10 years from now, 50 years from now, or 100 years from now? Many things depend on the answer to that question--the future of the social security system, the demand for health care services, the enrollment in colleges and universities--to name a few. If we know the present age distribution of a population (say, how many individuals are 0-4 years of age, 5-9 years of age, etc.) can we predict the age distribution in 50 years? It seems clear that any such prediction would be subject to lots of error. After all, we cannot foresee future changes in birth and death rates, nor can we estimate the effects of migration.

A somewhat simpler question which can be easily formulated and answered mathematically is this: If current birth rates and survival rates continue unchanged, what will the population distribution be at a future date (assuming the effects of migration are ignored)? Although this question is simpler, it is not without practical interest. For example, it is of value to explore mathematically the consequences of various birth and death rates without having to wait until the population actually experiences these consequences.

Several mathematical models have been developed for this simpler question. Here we shall examine one of these--a matrix model devised by P. H. Leslie and others. In this model, matrix multiplication is used to update the population from one time period to another. As we shall see, the study of such a process leads us to examine powers of matrices. Our object will be to learn something of the behavior of powers of matrices and to see how knowledge of eigenvalues and eigenvectors helps in understanding this behavior.

2. THE MODEL

2.1 Population Projection--An Example

Let's begin with a simple example. Consider a fictitious animal population consisting of 1000 young animals 0-1 year old, 800 individuals 1-2 years old, and 600 individuals 2-3 years old. We will assume that none of these animals lives longer than 3 years. We can record this data in an *age distribution vector*

$$\begin{pmatrix} 1000 \\ 800 \\ 600 \end{pmatrix}$$

For brevity we shall call the age groups Class I (0-1 year), Class II (1-2 years), and Class III (2-3 years).

If we want to know the age distribution vector one year from now we will need to know two things

1. the proportion of those animals currently alive that will survive until next year, and
2. how many offspring will be born and will survive long enough to be counted next year.

In our example, let us suppose that 1/2 of the individuals in Class I (i.e., 500 individuals) survive to be in Class II the next year, and let us also suppose that 1/2 of the individuals in Class II (i.e., 400 individuals) survive to be in Class III the next year. (Under our assumption, individuals currently in Class III will be dead by next year.) This process is indicated by the solid lines in Table 1. In addition let us suppose that individuals in Class I produce no offspring, that each individual in Class II produces 1 offspring on the average, and that each individual in Class III produces 2 offspring on the average. (Here we are including only those offspring who survive long enough to

TABLE 1

	NOW	NEXT YEAR	
Class	Number of Individuals	Number of Individuals	Class
I	1000	1(800) + 2(600) = 2000	I
II	800	½(1000) = 500	II
III	600	½(800) = 400	III

be counted the next year.) In Class I the next year we would expect to have 800 young (produced by last year's Class II individuals) and 2(600) = 1200 young (produced by last year's Class III individuals), for a total of 2000 new Class I individuals. This process is indicated by the dotted lines in Table 1.

The computations we have done can be summarized in this matrix computation:

$$\begin{pmatrix} 0 & 1 & 2 \\ \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{2} & 0 \end{pmatrix} \begin{pmatrix} 1000 \\ 800 \\ 600 \end{pmatrix} = \begin{pmatrix} 2000 \\ 500 \\ 400 \end{pmatrix} .$$

This computation is of the form $Ax_0 = x_1$, where A is the matrix containing the birth and survival parameters, x_0 is the initial age distribution vector, and x_1 is the age distribution vector after one year. We will call the matrix A a *Leslie matrix*.

2.2 Extending the Projection

By performing a matrix multiplication we have found the age distribution after one year. What if we wanted to know the age distribution after 2 years? If we believe that birth and survival rates will remain unchanged, then we can again multiply by the Leslie matrix A, this time using x_1 as our starting distribution. Thus we compute

$$Ax_1 = \begin{pmatrix} 0 & 1 & 2 \\ \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{2} & 0 \end{pmatrix} \begin{pmatrix} 2000 \\ 500 \\ 400 \end{pmatrix} = \begin{pmatrix} 1300 \\ 1000 \\ 250 \end{pmatrix} = x_2 .$$

Notice that $x_2 = Ax_1 = A(Ax_0) = A^2x_0$. We needn't stop with 2 years of course: we could compute

$$x_3 = Ax_2 = A(A^2x_0) = A^3x_0, \text{ etc.}$$

More generally, if the birth and survival parameters remain constant (or if we wish to know what would happen if they did remain constant), we could find the age distribution vector after k years (let's call it x_k by computing $x_k = A^k x_0$.

Exercise 1. Continuing the example of this section, compute x_3, x_4, x_5 and x_6. (Note: You will probably find it easier to obtain x_3 by computing Ax_2, rather than first computing A^3 and then computing A^3x_0.) Do you notice any qualitative trends (e.g., does one class have consistently more or fewer individuals than others)? Try the process again using the same Leslie matrix but a different starting vector x_0. Does the choice of starting vector seem to affect the trend?

2.3 Powers of the Leslie Matrix

The exercise raises two questions:

1. If the multiplicative process is repeated again and again does the distribution of ages change randomly or is there some recognizable pattern in the successive age distribution vectors?
2. Does the ultimate behavior of the age distribution vector depend on the initial distribution?

If there are any patterns they should show up as we compute higher and higher powers of the Leslie matrix. Let's continue our example by computing some powers of A. You can do this by hand (a bit tedious, of course) or by computer (a program for doing this is shown in Appendix A). Here are some results (numbers are rounded to 6 decimals):

$$A = \begin{pmatrix} 0 & 1 & 2 \\ .5 & 0 & 0 \\ 0 & .5 & 0 \end{pmatrix} \qquad A^2 = \begin{pmatrix} .5 & 1 & 0 \\ 0 & .5 & 1 \\ .25 & 0 & 0 \end{pmatrix}.$$

$$A^4 = \begin{pmatrix} .25 & 1 & 1 \\ .25 & .25 & .5 \\ .125 & .25 & 0 \end{pmatrix} \qquad A^8 = \begin{pmatrix} .4375 & .75 & .75 \\ .1875 & .4375 & .375 \\ .09375 & .1875 & .25 \end{pmatrix}$$

$$A^{16} = \begin{pmatrix} .402344 & .796875 & .796875 \\ .199219 & .402344 & .398438 \\ .099609 & .199219 & .203125 \end{pmatrix} \qquad A^{32} = \begin{pmatrix} .400009 & .799988 & .799988 \\ .199997 & .400009 & .399994 \\ .099999 & .199997 & .200012 \end{pmatrix}$$

$$A^{64} = \begin{pmatrix} .400000 & .800000 & .800000 \\ .200000 & .400000 & .400000 \\ .100000 & .200000 & .200000 \end{pmatrix}.$$

We see that powers of A are much alike for "large" powers (at least this is true in our example--we will come to a more general case later). What does this observation imply about x_k, the age distribution vector? To see, let's suppose the initial age distribution vector is

$$x_0 = \begin{pmatrix} a \\ b \\ c \end{pmatrix}.$$

Then if k is large, x_k ($=A^k x_0$) is approximately

$$\begin{pmatrix} .4 & .8 & .8 \\ .2 & .4 & .4 \\ .1 & .2 & .2 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} .4a + .8b + .8c \\ .2a + .4b + .4c \\ .1a + .2b + .2c \end{pmatrix} = \alpha \begin{pmatrix} .4 \\ .2 \\ .1 \end{pmatrix}$$

where $\alpha = a + 2b + 2c$.

The results are rather surprising: Although the total size of the population depends on the initial values a, b and c, the relative proportions of individuals in the three age classes approach fixed ratios 4:2:1 as $k \to \infty$, and these proportions do *not* depend on the initial age distribution of the population. (This is sometimes described by saying that the population "forgets" its initial age structure.)

Although the ratios 4:2:1 are never quite reached in a finite length of time, it is interesting to notice that if the population *did* achieve these proportions at some time m, then the age distribution vector would not change during subsequent time periods. For example, if

$$x_m = \begin{pmatrix} 40 \\ 20 \\ 10 \end{pmatrix}$$

then

$$x_{m+1} = Ax_m = \begin{pmatrix} 0 & 1 & 2 \\ \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{2} & 0 \end{pmatrix} \begin{pmatrix} 40 \\ 20 \\ 10 \end{pmatrix} = \begin{pmatrix} 40 \\ 20 \\ 10 \end{pmatrix} = x_m .$$

The fact that $Ax_m = x_m$ can be expressed mathematically by saying that x_m is an eigenvector corresponding to the eigenvalue 1. (Recall that a nonzero column vector v is an *eigenvector* of a square matrix A if there is a scalar λ, called an *eigenvalue*, such that $Av = \lambda v$.)

In case it escaped your notice, go back and observe that the ratios 4:2:1 which occur in the eigenvector also occur approximately in the columns of A^k when k is a large number (see A^{64}, for instance).

2.4 Another Example

Based on the example of the preceding section we have several hunches about possible theorems. Before exploring these hunches it would be wise to look at one more example. We need to know whether the behavior shown in the previous example was typical of population growth using Leslie matrices. Consider the Leslie matrix

$$B = \begin{pmatrix} 1 & 4 \\ \frac{1}{2} & 0 \end{pmatrix}.$$

(Can you interpret the entries of this matrix using "population" language?). Some powers of B are shown:

$$B^2 = \begin{pmatrix} 3 & 4 \\ .5 & 2 \end{pmatrix} \qquad B^4 = \begin{pmatrix} 11 & 20 \\ 2.5 & 6 \end{pmatrix}$$

$$B^8 = \begin{pmatrix} 171 & 340 \\ 42.5 & 86 \end{pmatrix} \qquad B^{16} = \begin{pmatrix} 43691 & 87380 \\ 10922.5 & 21846 \end{pmatrix}.$$

Here we have a real population explosion! At first glance, the clear-cut patterns observed in our earlier example seem to be missing. But if you look closely at B^8 and B^{16} you will see that the ratio of each first-row number to the corresponding second-row number is approximately 4 to 1. Taking our cue from the example of Section 2.3, we might wonder whether a vector having a 4 to 1 ratio is an eigenvector of B. By computing

$$\begin{pmatrix} 1 & 4 \\ \frac{1}{2} & 0 \end{pmatrix}\begin{pmatrix} 4 \\ 1 \end{pmatrix} = \begin{pmatrix} 8 \\ 2 \end{pmatrix} = 2\begin{pmatrix} 4 \\ 1 \end{pmatrix}.$$

we see that the answer is yes. However, here the eigenvalue is 2, and that gives a clue to the reason for the population explosion: even if the ratio in the age distribution vector stabilized at 4 to 1, the population would continue to grow and would, in fact, *double* every time period.

The analysis of the powers of B might have been clearer had we accounted for the doubling tendency of the population by dividing entries of B^2 by 2^2, of B^4 by 2^4, and so on. Here are the results:

$$\frac{1}{2^2}B^2 = \begin{pmatrix} .75 & 1 \\ .125 & .5 \end{pmatrix} \qquad \frac{1}{2^4}B^4 = \begin{pmatrix} .6875 & 1.25 \\ .15625 & .375 \end{pmatrix}$$

$$\frac{1}{2^8}B^8 = \begin{pmatrix} .667969 & 1.328125 \\ .166016 & .335938 \end{pmatrix} \qquad \frac{1}{2^{16}}B^{16} = \begin{pmatrix} .666672 & 1.333313 \\ .166664 & .333344 \end{pmatrix}$$

(Computations are rounded to six decimals.) The pattern now is much clearer, and our computation would lead us to guess that[1]

$$\lim_{k\to\infty} \frac{1}{2^k}B^k = \begin{pmatrix} 2/3 & 4/3 \\ 1/6 & 1/3 \end{pmatrix}.$$

Of course, computation, however useful, is no substitute for understanding. How can we analyze the behavior of A^k? Can we predict the ultimate form of A^k without actually computing the powers?

---

[1]By $\lim_{k\to\infty} C^k$ we mean the matrix, if it exists, whose (i,j)-th entry is $\lim_{k\to\infty} c_{ij}^{(k)}$, where $c_{ij}^{(k)}$ denotes the (i,j)-th entry of C^k.

Our observations up to this point suggest that a knowledge of eigenvalues and eigenvectors would be valuable in our analysis. We already know one eigenvalue and a corresponding eigenvector for the matrix B. Let's find the other eigenvalue. Recall that the eigenvalues of B will be roots of the characteristic polynomial

$$\det(\lambda I - B) = \det\left[\lambda\begin{pmatrix}1 & 0\\0 & 1\end{pmatrix} - \begin{pmatrix}1 & 4\\ \frac{1}{2} & 0\end{pmatrix}\right] = \det\begin{pmatrix}\lambda-1 & -4\\ -\frac{1}{2} & \lambda\end{pmatrix}$$

$$= \lambda^2 - \lambda - 2 = (\lambda - 2)(\lambda + 1) .$$

There are two eigenvalues, $\lambda_1 = 2$ and $\lambda_2 = -1$. An eigenvector v_j corresponding to eigenvalue λ_j can be found by solving the equation $(\lambda_j I - B)v_j = 0$. For $\lambda_1 = 2$, this matrix equation is

$$\begin{pmatrix}1 & -4\\ -\frac{1}{2} & 2\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}0\\0\end{pmatrix} \quad \text{or} \quad \begin{cases} x - 4y = 0\\ -\frac{1}{2}x + 2y = 0\end{cases}, \text{ where } v_1 = \begin{pmatrix}x\\y\end{pmatrix} .$$

One solution for v_1 is

$$v_1 = \begin{pmatrix}4\\1\end{pmatrix}.$$

(Any multiple of this vector is also an eigenvector corresponding to λ_1--the eigenspace is one-dimensional.) An eigenvector corresponding to $\lambda_2 = -1$ can be found in a similar manner

$$v_2 = \begin{pmatrix}2\\-1\end{pmatrix} .$$

The general theory of eigenvalues tells us that if we let v_1 and v_2 be the columns of a matrix P, then P will be invertible and $PDP^{-1} = B$, where

$$P = \begin{pmatrix}4 & 2\\1 & -1\end{pmatrix} \quad \text{and} \quad D = \begin{pmatrix}\lambda_1 & 0\\0 & \lambda_2\end{pmatrix} = \begin{pmatrix}2 & 0\\0 & -1\end{pmatrix}.$$

Of what use is this in computing powers of B? We see that

$$B^2 = PDP^{-1}PDP^{-1} = PD^2P^{-1}$$

$$B^3 = PD^3P^{-1}$$

$$\vdots$$

$$B^k = PD^kP^{-1}.$$

This observation is useful because powers of a diagonal matrix are very easy to compute. In our example

$$D^k = \begin{pmatrix} 2^k & 0 \\ 0 & (-1)^k \end{pmatrix}.$$

It follows that

$$B^k = \begin{pmatrix} 4 & 2 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 2^k & 0 \\ 0 & (-1)^k \end{pmatrix} \begin{pmatrix} 4 & 2 \\ 1 & -1 \end{pmatrix}^{-1}$$

$$= \begin{pmatrix} 4 & 2 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 2^k & 0 \\ 0 & (-1)^k \end{pmatrix} \begin{pmatrix} \frac{1}{6} & \frac{1}{3} \\ \frac{1}{6} & -\frac{2}{3} \end{pmatrix}.$$

It is helpful to rewrite this in the form

$$\frac{1}{2^k}B^k = \begin{pmatrix} 4 & 2 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & (-\frac{1}{2})^k \end{pmatrix} \begin{pmatrix} \frac{1}{6} & \frac{1}{3} \\ \frac{1}{6} & -\frac{2}{3} \end{pmatrix}.$$

Since $\lim_{k\to\infty} (-\frac{1}{2})^k = 0$, we see that

$$\lim_{k\to\infty} \frac{1}{2^k} B^k = \begin{pmatrix} 4 & 2 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \frac{1}{6} & \frac{1}{3} \\ \frac{1}{6} & -\frac{2}{3} \end{pmatrix} = \begin{pmatrix} \frac{2}{3} & \frac{4}{3} \\ \frac{1}{6} & \frac{1}{3} \end{pmatrix},$$

which checks with our initial computations.

Now that we have determined the limiting matrix for $(\frac{1}{2}^k)B^k$, what can we say about x_k, the age distribution vector? If

$$x_0 = \begin{pmatrix} a \\ b \end{pmatrix}$$

is the initial age distribution vector, then $(\frac{1}{2}^k)x_k$, which equals $(\frac{1}{2}^k)(B^kx_0)$, approaches

$$\begin{pmatrix} \frac{2}{3} & \frac{4}{3} \\ \frac{1}{6} & \frac{1}{3} \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = (a + 2b) \begin{pmatrix} \frac{2}{3} \\ \frac{1}{6} \end{pmatrix}$$

as $k \to \infty$. Thus x_k is approximately $2^k(a + 2b)\begin{pmatrix} 4/6 \\ 1/6 \end{pmatrix}$ when k is large. Notice again that the relative proportions in the age classes are eventually almost constant (80% in Class I, 20% in Class II--ratio of 4 to 1). Also notice that this 4 to 1 ratio is found in the eigenvector corresponding to $\lambda_1 = 2$. The difference between this example and the example of Section 2.3 is that now the total population grows by a factor of approximately 2 each time k increases by 1.

Exercise 2 Let

$$A = \begin{pmatrix} 1 & 3 \\ \frac{1}{4} & 0 \end{pmatrix}.$$

a. Interpret the entries of A in terms of births and survivals.

b. Compute the eigenvalues and eigenvectors of A.

c. Determine $\lim_{k \to \infty} A^k/\lambda_1^k$, where λ_1 is the larger of the eigenvalues of A.

d. Approximately what percent of the population is in Class I and what percent in Class II eventually?

e. How fast does the total population grow each time period when k is large?

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3. THEORETICAL BACKGROUND

3.1 Some Observations

In all the examples and exercises of the preceding sections there were certain common characteristics:

1. The age distribution vector eventually behaved like a multiple of some fixed vector.

2. This fixed vector was an eigenvector corresponding to the largest eigenvalue.
3. This largest eigenvalue was real and positive.
4. The population eventually tended to grow at a rate equal to this largest eigenvalue.

The natural question now is: "How typical were these examples?" Could we expect this behavior to hold in general (probably too much to expect) or under what conditions *would* it hold?

3.2 The Perron-Frobenius Theorem

Since everything seems to depend on having a positive eigenvalue which is larger than the other eigenvalues, our first question might logically be: "Is there always such an eigenvalue?" Also, can we find a nonnegative[2] eigenvector corresponding to this eigenvalue? (After all, to be realistic, age distribution vectors must have nonnegative components.) It is surprising that with only some mild assumptions the answer to these questions is "yes." A famous theorem, known as the *Perron-Frobenius Theorem* goes a long way toward answering these questions. The proof of this theorem is beyond what we intend to do here,[3] but we shall at least see how the theorem applies to Leslie matrices.

Recall that a Leslie matrix has the form

$$A = \begin{pmatrix} a_1 & a_2 & a_3 & \dots & a_{n-1} & a_n \\ b_1 & 0 & 0 & \dots & 0 & 0 \\ 0 & b_2 & 0 & \dots & 0 & 0 \\ 0 & 0 & b_3 & \dots & 0 & 0 \\ 0 & 0 & 0 & \dots & b_{n-1} & 0 \end{pmatrix} \qquad (L)$$

[2] A *nonnegative* vector is one for which all components are nonnegative. If all components are positive, the vector is called a *positive* vector.

[3] If you want to see more about this theorem you can consult the book by Varga (see the references).

where a_i ($i = 1, \ldots, n$) gives the average number of offspring born to an individual in the ith class and b_i ($i = 1, \ldots, n-1$) gives the probability that an individual in the ith class will survive to be counted in the (i+1)st class during the next time period. Naturally we assume that $a_i \geq 0$ for $i = 1, \ldots, n$. We will assume that the survival probabilities are positive, i.e., $b_i > 0$ ($i = 1, \ldots, n-1$). To satisfy the hypotheses of the Perron-Frobenius Theorem, we must also assume that a_n is strictly positive. This assures that the Leslie matrix is irreducible--a concept we shall not explore here. Under these assumptions, the Perron-Frobenius Theorem guarantees the following:

1. The matrix A has a positive eigenvalue, call it λ_1, such that $\lambda_1 \geq |\lambda_i|$ for all other eigenvalues λ_i (λ_i can be real or complex).
2. Corresponding to λ_1 there exists an eigenvector, call it v_1 having all positive components.
3. The eigenspace corresponding to λ_1 is one-dimensional, i.e., any eigenvector corresponding to λ_1 is a multiple of v_1.

Exercise 3 The structure of a Leslie matrix is rather simple and makes it possible to make some conclusions about eigenvalues and eigenvectors without appealing to the Perron-Frobenius Theorem

a Write out the characteristic equation for an arbitrary 3 x 3 Leslie matrix of the form (L). Using Descartes' rule of signs,[4] what can you say about the number of positive eigenvalues? What about an arbitrary n x n Leslie matrix?

b If $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$ is to be an eigenvector of Leslie matrix A corresponding to eigenvalue λ_1, it must satisfy

$$\begin{pmatrix} a_1 & a_2 & a_3 \\ b_1 & 0 & 0 \\ 0 & b_2 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \lambda_1 \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

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[4] If you are not familiar with Descartes' rule of signs, you may look it up in Uspensky, J. V., Theory of Equations, McGraw-Hill, 1948

i.e., $a_1x + a_2y + a_3z = \lambda_1 x$

$$b_1 x = \lambda_1 y$$

$$b_2 y = \lambda_1 z.$$

Using the second two equations solve for y and z in terms of x (x may be assigned any value, say $x = 1$), and in this way obtain a *formula* for a positive eigenvector corresponding to λ_1. Explain why the values of x, y and z which you found automatically satisfy the remaining equation, $a_1x + a_2y + a_3z = \lambda_1 x$? Can you write a formula for a positive eigenvector in the $n \times n$ case?

3.3 An Example with Oscillations

As you can see, the Perron-Frobenius Theorem gives us *almost* what we want. However, in analyzing the examples of Sections 2.3 and 2.4 it was important that for each $i \neq 1$, $(\lambda_i/\lambda_1)^k \to 0$ as $k \to \infty$. This required that λ_i/λ_1 be strictly less than 1, i.e., that λ_1 be *strictly* greater than the modulus of any other eigenvalue. (When $\lambda_1 > |\lambda_i|$ for all other eigenvalues λ_i we will say that λ_1 is *strictly dominant*.) Unfortunately, the Perron-Frobenius Theorem guarantees only that $\lambda_1 \geq |\lambda_i|$, and in fact a little experimentation shows that the assumptions we made for that theorem are not sufficient to justify strict inequality. Try this exercise.

Exercise 4. Consider the Leslie matrix

$$A = \begin{pmatrix} 0 & 8 \\ \frac{1}{2} & 0 \end{pmatrix}.$$

Starting with the initial age distribution vector $x_0 = \begin{pmatrix} 10 \\ 10 \end{pmatrix}$, compute $x_1, x_2, x_3, \ldots, x_6$. What do you observe? Find the eigenvalues and eigenvectors of A. Then try to carry out the limiting argument used in Section 2.4. What happens?

In the exercise we found that $|\lambda_1| = |\lambda_2|$ — something which did not happen in our previous examples. A quick review of the methods used in our other examples shows that the limiting argument that we used breaks down for the example in Exercise 4. As we see, the existence of a second eigenvalue whose modulus equals λ_1 leads to oscillating behavior in the age distribution vectors.

There are rather simple conditions which will guarantee that λ_1 is strictly dominant. One such condition is used in this theorem.

> If a Leslie matrix of the form (L) satisfies the conditions of Section 3.2 and if in addition there are two consecutive indices i and i+1 such that a_i and a_{i+1} are both positive, then λ_1 is strictly dominant.[5]

In terms of populations, this theorem requires that there be two consecutive age classes having positive fertility-- a requirement that will usually be met in practice.

Thinking again of the examples of Section 2.3 and 2.4, one final question might arise. In these examples we were able to diagonalize the Leslie matrix. What if we couldn't diagonalize? Fortunately, the conclusions about limiting behavior of the age distribution vector do not depend on our being able to diagonalize the matrix. If you are familiar with the Jordan canonical form, you can see how to extend our analysis to the more general case. We will not pursue the nondiagonalizable case here.

3.4 Summary

It's worthwhile to stop and pull together the theory which we have so far. First let's recall our assumptions.

> Assumptions: The population is governed by a Leslie matrix of the form (L) in which

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[5] If you are interested in the proof of this theorem, see the book by Pollard mentioned in the references. A fascinating (and more general) theorem requires only that the greatest common divisor of the set of indices i for which $a_i > 0$ be one.

1. $a_1, a_2, \ldots, a_{n-1}$ are nonnegative
2. a_n and $b_1, b_2, \ldots, b_{n-1}$ are positive, and
3. at least two consecutively indexed a_i's are positive.

Hopefully you are convinced by now (although we have given no formal proof) that the method of Section 2.4 can be carried out for any matrix satisfying our assumptions and that the following conclusions will hold:

> Conclusions: As $k \to \infty$, the proportions of individuals in the various age classes of x_k approach fixed values, and these values are determined by a positive eigenvector corresponding to the dominant eigenvalue. The dominant eigenvalue gives the eventual growth rate of the population.

3.5 A Simplification Using Left and Right Eigenvectors

In the examples we saw that A^k/λ_1^k equals

$$P \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & \left(\lambda_2/\lambda_1\right)^k & 0 & \cdots & 0 \\ 0 & 0 & \left(\lambda_3/\lambda_1\right)^k & \cdots & 0 \\ \vdots & & & \ddots & \\ 0 & 0 & 0 & \cdots & \left(\lambda_n/\lambda_1\right)^k \end{pmatrix} P^{-1}$$

which for k large, is approximately

$$P \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} P^{-1}$$

where 0's represent blocks of zeros of the appropriate size. In other words, the effect of the eigenvalues other than the dominant one, λ_1, is less and less as k grows larger. We might wonder whether we could approximate A^k without having to compute the smaller eigenvalues (or their eigenvectors) at all. It turns out that we *can* do this. We will state the facts that we need here. If you want to understand the justification for these facts, see Appendix B.

In order to state the desired conclusions we must first mention the eigenvalues and eigenvectors of A^T, the transpose of the Leslie matrix A. Perhaps you already know from your study of linear algebra that A and A^T have the same eigenvalues. If not, do this exercise.

Exercise 5. Let A be an n x n matrix. Prove that A and A^T have the same characteristic polynomial and hence the same eigenvalues. Give an example to show that A and A^T do not in general have the same eigenvectors.

Let's turn again to our analysis of powers of a Leslie matrix A. Suppose λ_1 is the dominant eigenvalue of A. Of course, by the exercise above, λ_1 is also the dominant eigenvalue of A^T. Let v_1 be a positive eigenvector of A corresponding to λ_1 and let u_1 be a positive eigenvector of A^T corresponding to λ_1.[6] The theorem we have is this

> Under the same assumptions stated in Section 3.4, <u>A^k/λ_1^k tends to $v_1u_1^T/u_1^Tv_1$ as $k \to \infty$.</u>

It is important to examine the dimensions of the products in $v_1u_1^T/u_1^Tv_1$. If A is n x n, then both u_1 and v_1 are n x 1 vectors. So $u_1^Tv_1$ is 1 x 1 and so can be treated as a scalar. However, $v_1u_1^T$ is the product of an n x 1 matrix by a 1 x n matrix and so has dimensions n x n (the same as A, as it should). Incidentally, you perhaps noticed in previous examples that A^k/λ_1^k tended to a matrix of rank one. Can you convince yourself that $v_1u_1^T$ always has rank one?

Now, what about x_k? We see that for k large,

---

[6] u_1 is sometimes called a *left eigenvector* of A. The reason: by definition, u_1 satisfies $A^Tu_1 = \lambda_1u_1$, but this equation is equivalent to $u_1^TA = \lambda_1u_1^T$, hence the reference to *left* eigenvector. As you might guess, v_1 is sometimes referred to as a *right* eigenvector of A.

$$x_k = A^k x_0 \approx \lambda_1^k \frac{(v_1 u_1^T)}{(u_1^T v_1)} x_0 = \lambda_1^k v_1 \frac{(u_1^T x_0)}{(u_1^T v_1)} .$$

Since $u_1^T x_0$ is also 1 x 1, you can see that x_k is approximately a scalar multiple of v_1, where the scalar is $\lambda_1^k u_1^T x_0 / u_1^T v_1$. Of course, we already knew that x_k behaved like a scalar multiple of the right eigenvector v_1 for large k, but now we have a formula for the scalar.

Exercise 6. Return to the example of Section 2.3. Find an eigenvector of A^T corresponding to $\lambda_1 = 1$. Use this, together with the right eigenvector

$$v_1 = \begin{pmatrix} 4 \\ 2 \\ 1 \end{pmatrix}$$

to find $\lim_{k\to\infty} A^k/\lambda_1^k$

Exercise 7. Let

$$A = \begin{pmatrix} 1 & 6 \\ \frac{1}{3} & 0 \end{pmatrix} .$$

a. Compute the dominant eigenvalue λ_1 of A.

b. Find a positive eigenvector of A corresponding to λ_1.

c. Find a positive eigenvector of A^T corresponding to λ_1.

d. Find $\lim_{k\to\infty} A^k/\lambda_1^k$.

e. Describe the population distribution vector x_k as $k\to\infty$.

Exercise 8. Carry out the analysis of Exercise 7a-e for the matrix

$$A = \begin{pmatrix} 0 & \frac{1}{4} & \frac{1}{2} \\ \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{4} & 0 \end{pmatrix} .$$

What will eventually happen to a population whose growth is governed by such as matrix A?

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4. A HUMAN POPULATION EXAMPLE

Although we began this unit by referring to the U.S. population, all of the examples so far have dealt with fictitious populations whose matrices were concocted to make the computations simple. Let's end the unit by returning to the U.S. population. Specifically, we will consider the population of U.S. females in 1967. It is convenient to divide the population into 10 age classes 0-4, 5-9, 10-14, ..., 45-49. Since the number of births to females over 50 is negligible we will consider only females aged 0-49. The Leslie matrix consists largely of zeros. The only nonzero entries are in the first row (the a_i's) and in the sub-diagonal (the b_i's). These entries are given in Table 2.

TABLE 2

Start of Age Interval	First Row (a_i)	Sub-diagonal (b_i)
0	00000	.99694
5	.00105	99842
10	08203	99785
15	28849	.99671
20	.37780	.99614
25	.26478	.99496
30	.14055	99247
35	05857	.98875
40	.01344	98305
45	00081	

Source Keyfitz and Flieger (see References)

The dominant eigenvalue can be approximated using numerical methods. It is approximately 1.0376. (Recall that each time period lasts 5 years, so $\lambda_1 = 1.0376$ means that the total population size is multiplied by approximately

1.0376 every 5 years.) A corresponding positive eigenvector is

$$v_1 = \begin{pmatrix} 8.539 \\ 8.204 \\ 7.894 \\ 7.591 \\ 7.292 \\ 7.000 \\ 6.712 \\ 6.420 \\ 6.117 \\ 5.795 \end{pmatrix}.$$

Of course any positive scalar multiple of v_1 would do. The particular choice of v_1 given here is useful because the entries represent percentages in the various age classes. For example, if 1967 birth and survival rates persisted, then eventually approximately 8.539% of the population would be aged 0-4, 8.204% would be aged 5-9, etc. Naturally the percentages shown do not add up to 100%. The remaining 28% (approximately) of the population would have age 50 years or older. It is interesting to note that in 1967 only about 24% of the population was 50 years of age or older. It is well known that the average age of the U.S. population is continuing to increase. Demographers believe that this increase will continue well into the twenty-first century, hence the concern about pension plans, etc.

Exercise 9.

a. Explain how to use high powers of a Leslie matrix A to approximate left and right eigenvectors of A. (For a hint about finding right eigenvectors see the final sentence of Section 2.3.)

b. Based on your answer to part (a) show how to find positive left and right eigenvectors for the Leslie matrix of this section (U.S. females, 1967) using the computer program from Appendix A (or another program of your own writing). How could you use high powers of the Leslie matrix to find the dominant eigenvalue?

5. REFERENCES

In this unit we have given only a brief introduction to the Leslie model for population growth. There are many interesting extensions of the model and uses of the model in related applications. If you want to read more about this model and see some applications to harvesting, you might begin by examining

Rorres, C. and H. Anton. Applications of Linear Algebra. John Wiley and Sons, 1977.

(See particularly Chapters 6, 9 and 10.)

For a discussion of population models, including the Leslie model and others, you will find these helpful

Pielou, E. C. An Introduction to Mathematical Ecology. John Wiley and Sons, 1969.

Pollard, J. H. Mathematical Models for the Growth of Human Populations. Cambridge University Press, 1973.

A book which considers many interesting questions about human populations is

Keyfitz, N. Applied Mathematical Demography. John Wiley and Sons, 1977.

A good source of data on human populations is

Keyfitz, N. and W. Flieger. Population: Facts and Methods of Demography. W. H. Freeman and Co., 1971.

The study of nonnegative matrices is fascinating and you may wish to pursue it further. If so, you may look at Chapter 2 of

Varga, R. S. Matrix Iterative Analysis. Prentice-Hall, 1962.

In this unit we have generally dealt with small-sized matrices. For such matrices we could easily find the eigenvalues and eigenvectors and then use these to compute powers of the matrix. For large matrices hand computation is not practical. A number of numerical methods are available for computing eigenvalues and eigenvectors of large matrices. To see more about these methods consult

Fox, L. An Introduction to Numerical Linear Algebra. Oxford University Press, 1965.

6. ANSWERS TO EXERCISES

1. $\begin{pmatrix}1500\\650\\500\end{pmatrix}, \begin{pmatrix}1650\\750\\325\end{pmatrix}, \begin{pmatrix}1400\\825\\375\end{pmatrix}, \begin{pmatrix}1575\\700\\412.5\end{pmatrix}.$

There are more individuals in Class I than in Class II (roughly twice as many) and more in Class II than in Class III (again, about twice as many). If we begin with

$x_0 = \begin{pmatrix}100\\200\\400\end{pmatrix}$, we have for $x_1, x_2, \ldots, x_6$

$\begin{pmatrix}1000\\50\\100\end{pmatrix}, \begin{pmatrix}250\\500\\25\end{pmatrix}, \begin{pmatrix}550\\125\\250\end{pmatrix}, \begin{pmatrix}625\\275\\62.5\end{pmatrix}, \begin{pmatrix}400\\312.5\\137.5\end{pmatrix}, \begin{pmatrix}587.5\\200\\156.25\end{pmatrix}$

If you try several other starting vectors, you can probably convince yourself that the trend mentioned above seems to hold regardless of which starting vector is chosen.

2a. Individuals in Class I produce 1 offspring on the average, individuals in Class II produce 3 offspring on the average. Of the individuals in Class I, one-fourth can be expected to survive until the next time period.

b. $\det(\lambda I - A) = \lambda^2 - \lambda - 3/4 = (\lambda - \frac{3}{2})(\lambda + \frac{1}{2})$, so the eigenvalues are $\frac{3}{2}$ and $-\frac{1}{2}$.

$v_1 = \begin{pmatrix}6\\1\end{pmatrix}$ and $v_2 = \begin{pmatrix}2\\-1\end{pmatrix}$ are corresponding eigenvectors

c. A^k/λ_1^k approaches

$$\begin{pmatrix}6 & 2\\1 & -1\end{pmatrix}\begin{pmatrix}1 & 0\\0 & 0\end{pmatrix}\begin{pmatrix}6 & 2\\1 & -1\end{pmatrix}^{-1} = \begin{pmatrix}3/4 & 3/2\\1/8 & 1/4\end{pmatrix}.$$

d. The percents are determined by v_1. 6/7 or about 85.7% will be in Class I, 1/7 or about 14.3% will be in Class II

e. If $\begin{pmatrix}a\\b\end{pmatrix}$ is the initial age distribution vector, x_k is approximately

$$(1.5)^k\begin{pmatrix}3/4 & 3/2\\ 1/8 & 1/4\end{pmatrix}\begin{pmatrix}a\\ b\end{pmatrix} = (1.5)^k(\tfrac{1}{8}a + \tfrac{1}{4}b)\begin{pmatrix}6\\ 1\end{pmatrix}.$$

Each time k increases by 1, every entry in the age distribution vector is multiplied approximately by 1.5, so the population grows by 50%.

3a. $\begin{vmatrix}\lambda-a_1 & -a_2 & -a_3\\ -b_1 & \lambda & 0\\ 0 & -b_2 & \lambda\end{vmatrix} = \lambda^3 - a_1\lambda^2 - b_1a_2\lambda - b_1b_2a_3 = 0.$

There is one variation in sign (remember that all the a_i's and b_i's are nonnegative). It follows from Descartes' rule of signs that there is exactly one positive root. You can show by induction that the characteristic polynomial for the n x n matrix is

$$\lambda^n - a_1\lambda^{n-1} - b_1a_2\lambda^{n-2} - b_1b_2a_3\lambda^{n-3} \cdots - b_1b_2\cdots b_{n-1}a_n .$$

Descartes' rule of signs again implies that there is one positive root.

b. $y = b_1x/\lambda_1$ (note that λ_1 is nonzero); $z = b_2y/\lambda_1 = b_1b_2x/\lambda_1^2$.

If we set x = 1, we see that

$$\begin{pmatrix}1\\ b_1/\lambda_1\\ b_1b_2/\lambda_1^2\end{pmatrix}$$

is a positive eigenvector corresponding to λ_1. Substituting these components into the equation $a_1x + a_2y + a_3z = \lambda_1x$, we have $a_1 + a_2b_1/\lambda_1 + a_3b_1b_2/\lambda_1^2 = \lambda_1$. This is equivalent to $\lambda_1^3 - a_1\lambda_1^2 - b_1a_2\lambda_1 - b_1b_2a_3 = 0$, which is just the result of substituting λ_1 into the characteristic equation.

4. $\begin{pmatrix}80\\ 5\end{pmatrix}, \begin{pmatrix}40\\ 40\end{pmatrix}, \begin{pmatrix}320\\ 20\end{pmatrix}, \begin{pmatrix}160\\ 160\end{pmatrix}, \begin{pmatrix}1280\\ 80\end{pmatrix}, \begin{pmatrix}640\\ 640\end{pmatrix} \cdots$

The distribution vectors alternate between ones with ratio 16:1 and those with ratio 1 1. Every two periods the total population size grows by a factor of 4. The eigenvalues are 2 and -2, with corresponding eigenvectors

$$\begin{pmatrix}4\\1\end{pmatrix} \text{ and } \begin{pmatrix}-4\\1\end{pmatrix}$$

thus

$$A^k/2^k = \begin{pmatrix}4 & -4\\1 & 1\end{pmatrix}\begin{pmatrix}1 & 0\\0 & (-1)^k\end{pmatrix}\begin{pmatrix}4 & -4\\1 & 1\end{pmatrix}^{-1}.$$

Here $(\lambda_2/\lambda_1)^k = (-1)^k$, which does not tend to zero but rather produces the oscillating effect.

5. $\det(\lambda I - A^T) = \det(\lambda I^T - A^T) = \det((\lambda I - A)^T) = \det(\lambda I - A)$. An example: for the matrix

$$A = \begin{pmatrix}0 & 8\\ \tfrac{1}{2} & 0\end{pmatrix}$$

an eigenvector corresponding to eigenvalue 2 is

$$\begin{pmatrix}4\\1\end{pmatrix}.$$

You can easily check that this vector is not an eigenvector of A^T.

6. $\begin{pmatrix}1\\2\\2\end{pmatrix}$ is an eigenvector of A^T corresponding to λ_1.

The limiting matrix is $v_1u_1^T/u_1^Tv_1 = \frac{1}{10}\begin{pmatrix}4 & 8 & 8\\2 & 4 & 4\\1 & 2 & 2\end{pmatrix}$.

7a. Eigenvalues are 2 and -1.

b. An eigenvector of A corresponding to 2 is $\begin{pmatrix}6\\1\end{pmatrix}$.

c. An eigenvector of A^T corresponding to 2 is $\begin{pmatrix}1\\3\end{pmatrix}$.

d. The limiting matrix for A^k/λ_1^k is $v_1u_1^T/u_1^Tv_1 = \frac{1}{9}\begin{pmatrix}6 & 18\\1 & 3\end{pmatrix}$.

e. x_k behaves like a multiple of $\begin{pmatrix}6\\1\end{pmatrix}$. Each time k increases by 1, the entries in the age distribution vector are multiplied by approximately 2.

8a. Eigenvalues are $\frac{1}{2}$, $\frac{1}{4}(-1 \pm i)$.

b. An eigenvector of A corresponding to ½ is $\begin{pmatrix}2\\2\\1\end{pmatrix}$.

c. An eigenvector of A^T corresponding to ½ is $\begin{pmatrix}1\\1\\1\end{pmatrix}$.

d The limiting matrix for A^k/λ_1^k is $\begin{pmatrix}.4 & .4 & .4\\.4 & .4 & .4\\.2 & .2 & .2\end{pmatrix}$.

e. x_k behaves like a multiple of $\begin{pmatrix}2\\2\\1\end{pmatrix}$. The entries in the age distribution vector are multiplied by approximately ½ each time k increases by 1. The population will eventually die out (practically, if not mathematically).

9. For large k, A^k is approximately a scalar multiple of $v_1u_1^T$. But in the matrix $v_1u_1^T$ the columns are all multiples of v_1 and the rows are multiples of u_1. Hence, any column of A^{64} is an approximation to a right eigenvector of A and any row of A^{64} is an approximation to a left eigenvector of A. Each time k increases by 1, the entries of A^k are multiplied by approximately λ_1. Therefore, to find λ_1 we can take any entry of A^{k+1} (e.g., the program of Appendix A could easily be modified to compute A^{65}) and divide by the corresponding entry of A^k (A^{64} in our program).

APPENDIX A

Here is a BASIC program which computes powers of a matrix A. A sample run is also shown.

```
LIST
POWERS
5 DIM A(10,10)
10 INPUT N
15 MAT INPUT A(N,N)
20 PRINT
25 PRINT "MATRIX A"
30 PRINT
35 MAT PRINT A;
40 FOR I = 1 TO 6
45 MAT B = A*A
50 PRINT "MATRIX A";2**I
55 PRINT
60 MAT PRINT B;
65 MAT A = B
70 NEXT I
75 END

READY

RUN
POWERS
? 3
? 0,1,2,.5,0,0,0,.5,0

MATRIX A

 0   1   2

 .5   0   0

 0   .5   0

MATRIX A 2

 .5   1   0

 0   .5   1

 .25   0   0

MATRIX A 4

 .25   1   1

 .25   .25   .5

 .125   .25   0
```

```
MATRIX A 8
 .4375  .75  .75
 .1875  .4375  .375
 .09375  .1875  .25
MATRIX A 16
 .402344  .796875  .796875
 .199219  .402344  .398438
 .996094E-1  .199219  .203125
MATRIX A 32
 .400009  .799988  .799988
 .199997  .400009  .399994
 .999985E-1  .199997  .200012
MATRIX A 64
 .4  .8  .8
 .2  .4  .4
 .1  .2  .2

READY
```

APPENDIX B

We will look once more at powers of a Leslie matrix A and show how the limiting behavior can be described using left and right eigenvectors. We will continue to make those assumptions about A which are stated in Section 3.4. As before, let u_1 be a positive eigenvector of A^T corresponding to the dominant eigenvalue λ_1, i.e., $A^T u_1 = \lambda_1 u_1$; also, let v_1 be a positive eigenvector of A corresponding to λ_1, i.e., $Av_1 = \lambda_1 v$. Let P be the square matrix whose columns are made up of the eigenvectors of A, with v_1 being the first column,[7] i.e.,

$$P = (v_1 \vdots v_2 \vdots v_3 \vdots \cdots \vdots v_n)$$

now

$$\begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \\ 0 & 0 & & 1 \end{pmatrix} = P^{-1}P = P^{-1}\left(v_1 \vdots v_2 \vdots \cdots \vdots v_n\right),$$

so

$$P^{-1}v_1 = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \qquad \text{(B-1)}$$

Also since $P^{-1}AP = \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix}$

and since a diagonal matrix is its own transpose,

$$P^{-1}AP = (P^{-1}AP)^T = P^T A^T (P^{-1})^T = \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix}.$$

---

[7] We will assume, as before, that A is diagonalizable, the final result is true without this assumption.

This means that $(P^{-1})^T = (\bar{u}_1\bar{u}_2\ldots\bar{u}_n)$ where the $\bar{u}_i$ are eigenvectors of A^T. u_1 need not be the same as $\bar{u}_1$, but it will be a multiple of $\bar{u}_1$, say $u_1 = \alpha\bar{u}_1$.

It follows that $$P^T(\bar{u}_1\bar{u}_2\ldots\bar{u}_n) = P^T(P^{-1})^T = \begin{pmatrix} 1 & 0 & \ldots & 0 \\ 0 & 1 & \ldots & 0 \\ \vdots & \vdots & \ddots & \\ 0 & 0 & & 1 \end{pmatrix}$$

so $$P^T\bar{u}_1 = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}.$$ Therefore $$P^Tu_1 = \alpha P^T\bar{u}_1 = \begin{pmatrix} \alpha \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

and this implies that $u_1^TP = (\alpha 0\ldots 0)$. (B-2)

Putting (B-1) and (B-2) together, we have

$$P^{-1}v_1u_1^TP = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}(\alpha\ldots 0) = \alpha\begin{pmatrix} 1 & 0 & \ldots \\ 0 & & \\ \vdots & & \end{pmatrix}$$

and $$u_1^Tv_1 = u_1^TPP^{-1}v_1 = (\alpha 0\ldots 0)\begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} = \alpha$$

so $$\frac{P^{-1}v_1u_1^TP}{u_1^Tv_1} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

and therefore, $$\frac{v_1u_1^T}{u_1^Tv_1} = P\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}P^{-1}.$$

Thus we have

$$A^k = P\begin{pmatrix} \lambda_1^k & 0 & \dots & 0 \\ 0 & \lambda_2^k & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \lambda_n^k \end{pmatrix}P^{-1} = P\begin{pmatrix} \lambda_1^k & 0 \\ 0 & 0 \end{pmatrix}P^{-1} + P\begin{pmatrix} 0 & 0 & \dots & 0 \\ 0 & \lambda_2^k & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \lambda_n^k \end{pmatrix}P^{-1}$$

and

$$A^k/\lambda_1^k = P\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}P^{-1} + P\begin{pmatrix} 0 & 0 & \dots & 0 \\ 0 & \lambda_2^k/\lambda_1^k & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \lambda_n^k/\lambda_1^k \end{pmatrix}P^{-1}$$

$$= \frac{v_1 u_1^T}{u_1^T v_1} + P\begin{pmatrix} 0 & 0 & \dots & 0 \\ 0 & \lambda_2^k/\lambda_1^k & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \lambda_n^k/\lambda_1^k \end{pmatrix}P^{-1}.$$

As $k \to \infty$, we see that A^k/λ_1^k approaches $v_1 u_1^T/u_1^T v_1$ which is what we wanted to see.