This document contains two units that examine integral transforms and series expansions. In the first module, the user is expected to learn how to use the unified method presented to obtain Laplace transforms, Fourier transforms, complex Fourier series, real Fourier series, and half-range sine series for given piecewise continuous functions. In the second unit, the student is expected to use the method presented to find a function when given the Laplace transform, the Fourier transform, the coefficient transform, or the Fourier series expansion of a function. Each module contains exercises and a model exam. Answers to all questions are provided. (MP)
A UNIFIED METHOD OF FINDING LAPLACE TRANSFORMS,
FOURIER TRANSFORMS, AND FOURIER SERIES

by

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TABLE OF CONTENTS

1. INTRODUCTION ........................................................................ 1

2. THE METHOD EXPLAINED .................................................. 1

3. THE METHOD IN ACTION .................................................. 5
   3.1 Example 1 .................................................................. 5
   3.2 Example 2 .................................................................. 6
   3.3 Example 3 .................................................................. 8
   3.4 Example 4 .................................................................. 9
   3.5 Example 5 .................................................................. 10
   3.6 Example 6 .................................................................. 12
   3.7 Example 7 .................................................................. 13
   3.8 Example 8 .................................................................. 14

4. ANSWERS TO EXERCISES .............................................. 17

5. MODEL EXAM .................................................................... 18

6. ANSWERS TO MODEL EXAM ......................................... 19

7. APPENDIX: THE METHOD DERIVED ............................... 20

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Title: A UNIFIED METHOD OF FINDING LAPLACE TRANSFORMS, FOURIER TRANSFORMS, AND FOURIER SERIES

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Classification: NMT TRANS & SERIES EXP

Suggested Support Materials: Standard textbooks on engineering mathematics, applied advanced calculus textbooks

Prerequisite Skills:
1. Differentiate and integrate elementary functions.
2. Sketch graphs of elementary functions.
3. Use Euler's formula.
4. Have a basic understanding of integral transforms and orthogonal functions.
5. Understand odd and even functions.
6. Understand piecewise continuous functions.
7. Sum geometric series.

Output Skills:
1. Use the unified method described in this unit to obtain
   a. Laplace transforms
   b. Fourier transforms
   c. Complex Fourier series
   d. Real Fourier series
   e. Half range sine series
   for given piecewise continuous functions.

Other Related Units:

<table>
<thead>
<tr>
<th>Module and Monographs in Undergraduate Mathematics and Its Applications Project (UMAP)</th>
</tr>
</thead>
<tbody>
<tr>
<td>The goal of UMAP is to develop, through a community of users and developers, a system of instructional modules in undergraduate mathematics and its applications which may be used to supplement existing courses and from which complete courses may eventually be built.</td>
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1. INTRODUCTION

Integral transforms and orthogonal functions provide the basis for widely used techniques in solving a large number of physical and engineering problems. In this unit we present a method which facilitates the finding of the Laplace transform, the Fourier transform, and the Fourier series when the given function is piecewise continuous (as are most functions that are encountered in practice).

The transforms and the series expansions that can be obtained by the method presented here may be obtained by various other techniques that are frequently used. Normally the two transforms and the series expansion are presented as three separate (but related) topics, and the techniques for handling piecewise continuous functions include integration, use of tables, and manipulation with unit step functions. The unified method has the following desirable characteristics:

- it avoids the use of tables;
- it avoids almost all integration;
- unit step functions are unnecessary;
- the method is quick;
- it provides a single, unified approach to all three problems;
- it employs graphical techniques.

2. THE METHOD EXPLAINED

We begin by recalling the basic definitions of the transforms and the series in question. The Fourier series expansion of a function \( f(t) \) of period \( 2p \) is given by

\[
f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[ a_n \cos \frac{n\pi t}{p} + b_n \sin \frac{n\pi t}{p} \right],
\]

where

\[
a_n = \frac{1}{p} \int_{0}^{2p} f(t) \cos \frac{n\pi t}{p} \, dt, \quad b_n = \frac{1}{p} \int_{0}^{2p} f(t) \sin \frac{n\pi t}{p} \, dt.
\]

The Laplace transform of a function \( f(t) \) is given by

\[
L\{f(t)\} = \int_{0}^{\infty} e^{-st} f(t) \, dt.
\]

Note that \( f(t) \) need be defined only for \( t > 0 \), however throughout this unit we shall regard \( f(t) \) as being identically zero for \( t < 0 \) when we are finding the Laplace transform. The Fourier transform of \( f(t) \) is given by

\[
F\{f(t)\} = \int_{-\infty}^{\infty} e^{-i\pi t/p} f(t) \, dt.
\]

There are, of course, basic questions that arise concerning conditions under which the series (1) and the improper integrals in (3) and (4) exist. There is also the question of whether the series in (1) actually represents \( f(t) \) and if so, in what sense. Answers to these questions can be found in textbooks on engineering mathematics and applied advanced calculus, and we shall limit our study to functions for which appropriate conditions are satisfied.

The similarity between expressions (3) and (4) is apparent. In addition, the integrals for \( a_n \) and \( b_n \) in (2) also resemble (3) and (4). In order to connect (2) with (3) and (4), we define the coefficient transform

\[
C\{f(t)\} = \int_{0}^{2p} e^{-i\pi t/p} f(t) \, dt.
\]

The structural similarity of (3), (4), and (5) is apparent and the connection with (2) can be seen from the Euler formula \( e^{i\theta} = \cos \theta + i \sin \theta \). The coefficient transform may be regarded as arising from the complex form of (1), which is

\[
f(t) = \sum_{n=-\infty}^{\infty} C_n e^{i\pi n t/p},
\]
where

\[(7) \quad c_n = \frac{1}{2p} \int_{-p}^{p} f(t) e^{-i\pi nt/p} \, dt.\]

Formulas for \(c_0\) and \(c_n\) may be obtained from (6) by substituting \(n = 0\) or by replacing \(n\) with \(-n\). Expressions (1) and (5) may each be obtained from the other via the relationships

\[(8) \quad c_n = \frac{1}{2}(a_n - ib_n), \quad c_0 = \frac{1}{2}a_0, \quad c_n = \frac{1}{2}(a_n + ib_n),\]

\[a_n = 2c_0, \quad a_n = c_n - c_{n-1}, \quad b_n = i(c_n - c_{n-1}).\]

(The Euler formula, of course, provides the basis for (8).)

Suppose now that \(f(t)\) is a piecewise continuous function. In Figures 1-3 we show functions of this type that are appropriate for the three transforms in question. The graphs also show the information needed to apply the unified method, namely:

- the location of the discontinuities of \(f(t)\) (indicated by \(a_i\));
- the "jump" in \(f(t)\) at a discontinuity (indicated by \(M_i\));
- the direction of the jump (indicated by an arrow up or down).

![Figure 1](image1.png)

**Figure 1.** One period of a 2p-periodic function \(f(t)\); this illustration is appropriate for the Fourier series expansion.

![Figure 2](image2.png)

**Figure 2.** An illustration that is appropriate for the Laplace transform. Note that \(f(t) = 0\) for \(t < 0\); however, \(f(t)\) need not be discontinuous at 0 in general.

![Figure 3](image3.png)

**Figure 3.** Illustration of \(f(t)\) appropriate for the Fourier transform.

Under the conventions that \(M_i\) will be positive if the jump at \(a_i\) is up and negative if the jump is down, we have the following unified formulas:

\[(9) \quad C(f(t)) = \frac{1}{1 + i\pi} \sum_{k=0}^{M} M_k e^{-i\pi n k/p} + \frac{1}{1 + i\pi} C(f'(t))\]

\[(10) \quad L(f(t)) = \frac{1}{s} \sum_{k=0}^{M} M_k e^{-a_k s} + \frac{1}{s} L(f'(t))\]
Apply formula (10) to \( f(t) \), \( f'(t) \), \( f''(t) \):

\[
L\{f(t)\} = \frac{1}{s} \left( e^{-2s} + e^{-4s} \right) + \frac{1}{s} \cdot \frac{1}{s^2} \cdot \frac{1}{s^2} \cdot \left( \frac{1}{s} \right)
\]

We note that Formulas (\( 9 \)), (\( 10 \)), (\( 11 \)) can be obtained by elementary methods. A derivation is carried out in detail for the coefficient transform in the appendix.

3. THE METHOD IN ACTION

3.1 Example 1

To find \( L\{f(t)\} \) for the function

\[
f(t) = \begin{cases} t^2, & 0 \leq t < 2 \\ 3, & 2 \leq t < 4 \\ 4, & 4 \leq t \end{cases}
\]

we first graph \( f(t) \) and its distinct nonzero derivatives, indicating all jumps (see Figure 4).

3.2 Example 2

Suppose we wish to find \( L\{f(t)\} \) for the function which is defined graphically in Figure 5.

Since we need only the segment slopes (which are obtainable from the end point coordinates) in order to graph \( f'(t) \), we do not even need explicit formulas.

(See Figure 6.)
3.3 Example 3

Find the coefficient transform $C(f(t))$ for the periodic function whose formula over one period is

$$f(t) = \begin{cases} 
  t^2, & 0 \leq t < 0.5 \\
  t^3 + 1, & 0.5 \leq t < 1 \\
  0, & 1 \leq t \leq 2.
\end{cases}$$

Here we have $2p = 2$, so $p = 1$. We proceed as in Example 2, graphing $f(t)$ and its distinct nonzero derivatives (see Figure 7).

Using the data recorded in Figure 7, and applying Formula (9), we may write down immediately:

$$L\{f(t)\} = \frac{2}{s^3} \left( -1 + 3e^{-s} - \frac{7}{2}e^{-2s} + \frac{3}{2}e^{-4s} \right)$$

Exercise 1

Use the unified method (the abbreviated form illustrated in Example 2) to find $L\{f(t)\}$ for the following functions:

a. $f(t) = \begin{cases} 
 t, & 0 \leq t < 1 \\
 2-t, & 1 \leq t < 2 \\
 0, & 2 \leq t.
\end{cases}$

b. $f(t) = \begin{cases} 
 t-1, & 0 \leq t < 2 \\
 6-t^2, & 2 \leq t < 3 \\
 0, & 3 \leq t.
\end{cases}$

c. The function shown in the diagram to the right.
\[ C(f(t)) = \sum_{n=-\infty}^{\infty} \left( \frac{1}{2\pi} \int_{0}^{\pi} f(t)e^{-int} dt \right) e^{int} \]

\[ = \frac{1}{\pi} \left( \frac{1}{2\pi} + e^{-i} \right) - \frac{2}{\pi} e^{-i} \]

Since \( e^{-in\pi} = \cos n\pi = (-1)^n \), we may simplify:

\[ C(f(t)) = \frac{1}{\pi} + \frac{1}{\pi} e^{-i} + \frac{1}{\pi} e^{-2i} \left( \frac{1}{\pi^2} \right) + \frac{1}{\pi} e^{-3i} \left( \frac{1}{\pi^2} \right) \]

(\text{This result holds, of course, for } n \neq 0.)

3.4 Example 4

Find the real form of the Fourier series expansion for the function \( f(t) \) from Example 3. Since

\[ C_n = C(f(t)) \quad \text{[see (5) and (7)],} \]

and since \( p = 1 \) (see Example 3), we have for this example

\[ C_n = \frac{1}{2\pi} C(f(t))^2 \]

Thus,

\[ C_n = \frac{1}{2\pi} \left( \frac{1}{\pi} + \frac{1}{\pi} e^{-i} + \frac{1}{\pi} e^{-2i} \left( \frac{1}{\pi^2} \right) + \frac{1}{\pi} e^{-3i} \left( \frac{1}{\pi^2} \right) \right) \]

from which

\[ C_0 = \frac{1}{2\pi} \int_0^\pi (t^2 + 1) dt = \frac{2}{3}. \]

From (8) we have

\[ \frac{a_0}{\pi} = C_0 = 2/3, \]

\[ a_n = C_n + C_{-n} = \frac{2(-1)^n}{n^2\pi^2}, \quad \text{for } n \neq 0, \]

\[ b_n = i(C_n - C_{-n}) = \frac{1}{n\pi} + \frac{2}{n^3\pi^3} + 2(-1)^n \left( \frac{1}{n\pi} - \frac{1}{n^3\pi^3} \right) \]

The Fourier series expansion is

\[ f(t) = \frac{2}{3} + \sum_{n=1}^{\infty} a_n \cos n\pi t + \sum_{n=1}^{\infty} b_n \sin n\pi t, \]

with \( a_n, b_n \) as given above.

3.5 Example 5

Find the half range sine expansion for the function

\[ f(t) = t^2 - 2t, \quad 0 \leq t < 1. \]

We first make an odd extension of \( f(t) \) to include the interval \(-1 \leq t < 0.\) Since we need only information on the jumps, the extension may be carried out graphically with no formulas necessary. We save additional effort by noting that the derivative of an odd function is even and the derivative of an even function is odd. See Figure 8.

Applying formula (9) with the information displayed in Figure 8, and noting that for the extended function \( 2p = 2 \) so that \( p = 1, \) we obtain

\[ C(f(t)) = \frac{1}{\pi} \left( \frac{1}{n\pi} - \frac{2}{(n^2\pi^2)} \right) + \frac{4}{(n\pi)^3} + \frac{4}{(n\pi)^3} \left( \frac{1}{n\pi} - \frac{2}{(n^2\pi^2)} \right) \]

\[ = \frac{1}{n\pi} \left( \frac{2}{n^2\pi^3} - \frac{4}{n^3\pi^3} \right) + \frac{4}{n^3\pi^3}. \]
Then we have for the Fourier coefficients
\[ C_n = \frac{1}{2\pi} C(f(t)) = \frac{1}{2\pi} C(f(t)) \]
\[ = (-1)^n \left( \frac{1}{n\pi} - \frac{2}{n^3\pi^3} \right) \frac{2i}{n^3\pi^3} \]
and
\[ C_n = (-1)^n \left( \frac{1}{n\pi} + \frac{2i}{n^3\pi^3} \right) - \frac{2i}{n^3\pi^3} \]

In the half range sine expansion \( a_n = 0 \) for all \( n \), and we have
\[ f(t) = \sum_{n=1}^{\infty} b_n \sin n\pi t, \]
where
\[ b_n = i(C_n \cdot C_{-n}) = (-1)^n \left[ \frac{2}{n\pi} + \frac{4}{n^3\pi^3} \right] - \frac{4}{n^3\pi^3} \]
or
\[ b_n = (-1)^n \left[ \frac{2}{n\pi} + \left((-1)^n - 1\right) \frac{4}{n^3\pi^3} \right] \]

Exercise 2

Find the complex Fourier series expansion for the following periodic functions, where the definition over one period is given by:

a. \( f(t) = \begin{cases} 1 & 0 \leq t < 3 \\ -2 & 3 \leq t < 6 \end{cases} \)

b. \( f(t) = \begin{cases} \frac{1}{2} t & 0 \leq t < 2 \\ 2 - \frac{1}{2} t & 2 \leq t < 4 \end{cases} \)

Exercise 3

Find the half range sine series expansion for the function
\[ f(t) = \frac{1}{2} t, \quad 0 \leq t < 5. \]

3.6 Example 6

Find the Fourier transform of the function
\[ f(t) = \begin{cases} 1 - t^2 & |t| \leq 1 \\ 0 & |t| > 1 \end{cases} \]

The process is the same as before: sketch the function and its distinct nonzero derivatives, recording the relevant data on all jumps (see Figure 9).
Applying Formula (11) and the information displayed in Figure 9, we obtain

\[ F\{f(t)\} = \frac{1}{(ia)^2} \left[ 2e^{ia} + 2e^{-ia} \right] + \frac{1}{(ia)^3} \left[ -2e^{ia} + 2e^{-ia} \right] \]
\[ = \frac{2}{a^2} \left( e^{ia} + e^{-ia} \right) - \frac{2}{a^3} \left( e^{ia} - e^{-ia} \right) \]
\[ = \frac{4 \cos a}{a^2} + \frac{4 \sin a}{a^3} \]

**Exercise 4**

For the following functions \( f(t) \) find the Fourier transform \( F\{f(t)\} \):

a. \( f(t) = \begin{cases} 1, & |t| \leq 2 \\ 0, & |t| > 2 \end{cases} \)

b. \( f(t) = \begin{cases} 1+t, & -1 \leq t \leq 0 \\ 1-t, & 0 < t < 1 \\ 0, & |t| > 1 \end{cases} \)

**3.7 Example 7**

Find the Fourier cosine transform of the function \( f(t) = e^{-mt} \). We first note that by definition the Fourier cosine transform of the given function is

\[ \int_0^\infty e^{-mt} \cos \omega t \, dt = R \int_0^\infty e^{-iat} e^{-mt} \, dt = R \, F\{f(t)\} \]

where \( f(t) \) in the latter expression is redefined as

\[ f(t) = \begin{cases} e^{-mt}, & t > 0 \\ 0, & t < 0 \end{cases} \]

and \( R \) denotes the real part of the transform. (See Figure 10.)

**3.8 Example 8**

As a final example we expand the function \( f(t) = \cos t, \quad 0 \leq t \leq 2\pi \)
in a half-range sine series. The graphs of the odd extension of \( f(t) \) and its first derivative are shown in Figure 11.
Figure 11. Graphs of the odd extension of \( f(t) \) and its first derivative.

In this example, \( 2p = 4\pi, \) \( p = 2\pi, \) hence \( \frac{p}{\ln p} = \frac{2}{\ln 2} \).

Since \( f''(t) = -f(t), \) we have by Formula (9)

\[
C(f(t)) = \frac{2}{\ln 2} \left( e^{2\pi n} - e^{-2\pi n} \right) + \frac{2}{(2\pi n)} C(-f(t)).
\]

Therefore,

\[
C\left(1 - \frac{4}{n^2}\right)f(t) = (-1)^n \frac{4i}{n} - \frac{4i}{n},
\]

from which

\[
C(f(t)) = \begin{cases} 
\frac{8ni}{n^2 - 4}, & n \text{ odd} \\
0, & n \text{ even}
\end{cases}
\]

Since \( 2p = 4\pi, \)

we have for \( n \) odd

\[
C_n = \frac{2ni}{(n^2 - 4)\pi},
\]

\[
C_{-n} = \frac{2ni}{(n^2 - 4)\pi},
\]

so that

Examples 5 and 8 illustrate that it is not necessary that \( f(t) \) have some derivative which vanishes in order to apply the unified method -- it is possible to use this method also when there is an algebraic relationship between \( f(t) \) and its first few derivatives. This fact will be useful in some of the following exercises.

The next exercises will conclude the application portion of this unit. For those who wish to learn how the unified method can be derived, we carry out the derivation in the Appendix for the coefficient transform. Derivation of Formulas (9) and (11) can be carried out in a similar way.

**Exercise 5**

Use the unified method to find the Laplace transform of \( f(t) = e^{2t} \).

**Exercise 6**

Find the Laplace transform of the periodic square wave shown to the right. This problem will require an extension of the unified method to a case where the number of jumps discontinuities in \( f(t) \) is countably infinite.
4. ANSWERS TO EXERCISES

1. a. \[ \frac{1}{s^2} - 2 \frac{e^{-s}}{s} + \frac{e^{-2s}}{s^2} \]

b. \[ \frac{1}{s^2} + \frac{1}{s} e^{-s} \left( \frac{1}{s} - \frac{1}{s^2} \right) + e^{-2s} \left( \frac{3}{s^2} - \frac{6}{s} + \frac{2}{s^3} \right) \]

c. \[ \frac{1}{s^2} + \frac{2e^{-s}}{s^2} + \frac{e^{-3s}}{s^2} \]

2. a. \[ f(t) = \sum_{n=-\infty}^{\infty} C_n e^{\gamma_n t/3} \]
   \[ C_n = \frac{1}{\gamma_n} \left[ 1 - (-1)^n \right] \]
   \[ C_0 = \frac{1}{\gamma_0} \]

b. \[ f(t) = \sum_{n=-\infty}^{\infty} C_n e^{\gamma_n t/2} \]
   \[ C_n = \frac{1}{\gamma_n} \left[ 1 - (-1)^n \right] \]
   \[ C_0 = \frac{1}{\gamma_0} \]

3. \[ f(t) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi t}{3} \]
   \[ b_n = \frac{(-1)^n}{n\pi} \]

4. a. \[ \frac{2\sin 2\alpha}{\alpha} \]
   b. \[ \frac{2}{\alpha^2} (1 - \cos \alpha) \]

5. \[ L(e^{-2t}) = \frac{1}{s} + \frac{1}{s} L(-2e^{-2t}) \]
   \[ L(-2e^{-2t}) = \frac{1}{s^2} \]

6. \[ L(f(t)) = \frac{1}{s} \left( 1 - 2e^{-s} + 2e^{-2s} - 2e^{-3s} + \ldots \right) \]
   \[ = \frac{1}{s} \left( 1 + 2 \frac{e^{-s}}{1 + e^{-s}} \right) \]
   \[ = \frac{1}{s} \tanh \frac{s}{2} \]

5. MODEL EXAM

Find the following transforms and series expansions using the unified method.

1. Find the Laplace transform \( L(f(t)) \) for the functions:
   a. \[ f(t) = \begin{cases} 
   t^2, & 0 < t \leq 1 \\
   1, & 1 < t.
   \end{cases} \]
   b. \[ f(t) = \cos t \]

2. Find the Fourier transform \( F(f(t)) \) for the functions:
   a. \[ f(t) = \begin{cases} 
   4 - t^2, & |t| \leq 2 \\
   0, & |t| > 2.
   \end{cases} \]
   b. \[ f(t) = e^{-2|t|}, \quad \text{all } t \]

3. Find the half range sine series for function \[ f(t) = 1 - \frac{1}{2}t, \quad 0 \leq t \leq 1. \]
6. ANSWERS TO MODEL EXAM

1. \[ \frac{2}{s^3} - 2e^{-s} \left( \frac{1}{s^2} + \frac{1}{s} \right) \]

b. \[ L(\cos t) = \frac{1}{s} + \frac{1}{s} \left( \frac{1}{s} L(-\cos t) \right) \]

Thus \[ L(\cos t) = \frac{s}{s^2 + 1} \]

2. a. \[ \frac{4}{a^3} \sin 2a - \frac{8}{a^3} \cos 2a \]

6. \[ F(e^{-2|t|}) = \frac{-4}{(ia)^2} + \frac{1}{(ia)^2} F(e^{-2|t|}) \]

Thus \[ F(e^{-2|t|}) = \frac{1}{4 + a^2} \]

3. \[ f(t) = \sum_{n=1}^{\infty} b_n \sin nt, \text{ where} \]

\[ b_n = \frac{1}{\pi} \left[ 2 + (-1)^n + 1 \right] \]

7. APPENDIX: THE METHOD DERIVED

In this section we carry out the derivation of formula (9). The formal proof for a function with a finite number of jump discontinuities requires an induction argument, but the idea can be seen by considering a function \( f(t) \) with jumps only at \( t = 0, t = 2p \) and one intermediate point \( t = a \) (see Figure A1). Thus we take \( f(t) \) of the form

\[ f(t) = \begin{cases} f_1(t), & 0 \leq t < a \\ f_2(t), & a \leq t < 2p \end{cases} \]

Figure A1. Graph of \( f(t) \) with one intermediate discontinuity.

By the definition of \( C(f(t)) \) (see Formula (5)), we have

\[ C(f(t)) = \int_0^a e^{-int/p} f_1(t) dt + \int_a^{2p} e^{-int/p} f_2(t) \]

We integrate by parts, with \( u = f_1(t), f_2(t) \) and \( dv = e^{-int/p} \), so that \( du = f_1'(t) dt, f_2'(t) dt \) and

\[ v = -\frac{p}{in} e^{-int/p}. \]

Thus
\[
C(f(t)) = \frac{p}{\text{In} \pi} \left[ f(t) e^{-\text{In} \pi t/p} \right]_0^a - f(t) e^{-\text{In} \pi t/p} \right]_a^{2\pi} \\
+ \frac{p}{\text{In} \pi} \left[ \frac{a}{\text{In} \pi} f_1(t) e^{-\text{In} \pi t/p} f_2(t) e^{-\text{In} \pi t/p} \right]_0^a \\
+ \frac{p}{\text{In} \pi} \left[ \frac{\text{In} \pi}{\text{In} \pi} f_1(t) e^{-\text{In} \pi t/p} f_2(t) e^{-\text{In} \pi t/p} \right]_0^a 
\]

Collecting terms we find:

\[
C(f(t)) = \frac{p}{\text{In} \pi} \left[ f_1(0) + [f_2(a) - f_1(a)] e^{-\text{In} \pi a/p} + f_2(2\pi) \right] \\
+ \frac{p}{\text{In} \pi} \left[ \frac{\text{In} \pi}{\text{In} \pi} f_1(t) e^{-\text{In} \pi t/p} f_2(t) e^{-\text{In} \pi t/p} \right]_0^a 
\]

(A1)

since \( e^{-\text{In} \pi t/p} = 1 \) for \( t=\pi \). Let \( M_0, M_a, M_2\pi \) denote the "jumps" in \( f(t) \) as shown in Figure A1; moreover we assume that the value is positive when the jump is up and negative when down. (Thus, for the function pictured in Figure A1, \( M_0 > 0, M_a > 0, M_2\pi < 0 \). We may therefore write the expression (A1) as:

\[
(A2) \ C(f(t)) = \frac{p}{\text{In} \pi} \left[ M_0 e^{-\text{In} \pi 0/p} + M_a e^{-\text{In} \pi a/p} + M_2\pi e^{-\text{In} \pi 2\pi/p} \right] + \frac{p}{\text{In} \pi} C(f'(t)) 
\]

The first term in (A2) is more systematic that it appears, since it can be written as:

\[
M_0 e^{-\text{In} \pi 0/p} + M_a e^{-\text{In} \pi a/p} + M_2\pi e^{-\text{In} \pi 2\pi/p} 
\]

Thus, in actuality, each signed jump is multiplied by the exponential \( e^{-\text{In} \pi t/p} \) evaluated at the value of \( t \) where the jump is made, and the resulting products are summed. Finally, as may be verified by an easy induction argument, when \( a_0 = 0, a_m = 2\pi \), and the function \( f(t) \) has \( m \) jump discontinuities in between, at \( a_1, \ldots, a_{m-1} \), we have
STUDENT FORM 1
Request for Help

Return to:
EDC/UMAP
55 Chapel St.
Newton, MA 02160

Student: If you have trouble with a specific part of this unit, please fill out this form and take it to your instructor for assistance. The information you give will help the author to revise the unit.

Your Name

Page
OR
Middle
OR
Lower

Section
OR
Paragraph

Description of Difficulty: (Please be specific)

Instructor: Please indicate your resolution of the difficulty in this box.

Corrected errors in materials. List corrections here:

Gave student better explanation, example, or procedure than in unit. Give brief outline of your addition here:

Assisted student in acquiring general learning and problem-solving skills (not using examples from this unit.)

Instructor's Signature

Please use reverse if necessary.
STUDENT FORM 2
Unit Questionnaire

55 Chapel St.
Newton, MA 02160

Name ___________________ Unit No. _______ Date ________

Institution ___________________ Course No. ___________________

Check the choice for each question that comes closest to your personal opinion.

1. How useful was the amount of detail in the unit?
   (Circle one)
   ______ Not enough detail to understand the unit
   ______ Unit would have been clearer with more detail
   ______ Appropriate amount of detail
   ______ Unit was occasionally too detailed, but this was not distracting
   ______ Too much detail; I was often distracted

2. How helpful were the problem answers?
   (Circle one)
   ______ Sample solutions were too brief; I could not do the intermediate steps
   ______ Sufficient information was given to solve the problems.
   ______ Sample solutions were too detailed; I didn't need them

3. Except for fulfilling the prerequisites, how much did you use other sources (for example, instructor, friends, or other books) in order to understand the unit?
   (Circle one)
   ______ A Lot
   ______ Somewhat
   ______ A Little
   ______ Not at all

4. How long was this unit in comparison to the amount of time you generally spend on a lesson (lecture and homework assignments) in a typical math or science course?
   (Circle one)
   ______ Much Longer
   ______ Somewhat Longer
   ______ About
   ______ Somewhat
   ______ Much Shorter
   ______ Longer
   ______ the Same
   ______ Shorter
   ______ Much Shorter

5. Were any of the following parts of the unit confusing or distracting? (Check as many as apply.)
   (Circle or write)
   ______ Prerequisites
   ______ Statement of skills and concepts (objectives)
   ______ Paragraph headings
   ______ Examples
   ______ Special Assistance Supplement (if present)
   ______ Other, please explain

6. Were any of the following parts of the unit particularly helpful? (Check as many as apply.)
   (Circle or write)
   ______ Prerequisites
   ______ Statement of skills and concepts (objectives)
   ______ Examples
   ______ Problems
   ______ Paragraph headings
   ______ Table of Contents
   ______ Special Assistance Supplement (if present)
   ______ Other, please explain

Please describe anything in the unit that you did not particularly like.

Please describe anything that you found particularly helpful. (Please use the back of this sheet if you need more space.)
AN INVERSION METHOD FOR LAPLACE TRANSFORMS,
FOURIER TRANSFORMS, AND FOURIER SERIES

by C.A. Grimm

Department of Mathematics
South Dakota School of Mines and Technology
Rapid City, South Dakota 57701

TABLE OF CONTENTS

1. INTRODUCTION ............................................. 1
2. INVERSE LAPLACE TRANSFORMS ............................ 2
3. INVERSE FOURIER TRANSFORMS ............................ 6
4. INVERSE COEFFICIENT TRANSFORMS ..................... 8
5. MODEL EXAM ............................................... 22
6. ANSWERS TO EXERCISES .................................. 23
7. ANSWERS TO MODEL EXAM ................................. 24
Title: AN INVERSION METHOD FOR LAPLACE TRANSFORMS, FOURIER TRANSFORMS, AND FOURIER SERIES

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Classification: INT TRANS & SERIES EXP

Prerequisite Skills:
1. Differentiate and integrate elementary functions.
2. Sketch graphs of elementary functions.
3. Use Euler's formula.
4. Have a basic understanding of integral transforms and orthogonal functions.
5. Identify odd and even functions.
6. Identify piecewise continuous functions.
7. Apply basic theorems on uniformly convergent series of functions.

Output Skills:
1. Use the method described in this unit to find a function f(t) when given
   a) the Laplace transform L{f(t)}
   b) the Fourier transform F{f(t)}
   c) the coefficient transform C{f(t)}
   d) the Fourier series expansion of f(t).

Other Related Units:
A UNIFIED METHOD OF FINDING LAPLACE TRANSFORMS, FOURIER TRANSFORMS, AND FOURIER SERIES (Unit 324)
1. INTRODUCTION

In Unit 324 we presented a unified method of finding Laplace transforms, Fourier transforms, and Fourier series. The present unit is a sequel to Unit 324, and we shall freely carry over the notation and terminology used there. In particular, we are concerned primarily with the transforms \( L(f(t)) \), \( F(f(t)) \), and \( C(f(t)) \).

Some applied problems require only the use of the forward transforms. In such problems, the calculation of the transform represents passage from time domain to the frequency domain, and the information obtained by studying frequency-related properties is all that is required. Other problems (such as solution of differential equations by transform techniques), however, require determination of inverse transforms; that is, recovery of \( f(t) \) from \( L(f(t)) \) or from \( F(f(t)) \).

In this unit we build upon the ideas presented in Unit 324 to attack the problem of finding a function from its given transform or series expansion. Given a transform \( L(f(t)) \), \( F(f(t)) \) or \( C(f(t)) \) we attempt to reconstruct first the derivatives of \( f(t) \), and finally \( f(t) \) itself, by reversing the process used to find the forward transform. Since the forward process is based on integration by parts, the method is generally applicable. Hence the inverse methods presented here are also generally applicable — theoretically!! The problem is primarily that in finding a forward transform we may cancel terms which, if present, would provide clues to the nature of the derivative. For this reason, the inverse process does require patience and practice; nevertheless it does offer most of the same advantages listed in Unit 324 for the unified (forward) method.

2. INVERSE LAPLACE TRANSFORMS

The process will be developed by examples. Again, it is assumed that you are familiar with the unified method presented in Unit 324.

Example 1

Suppose we wish to find \( f(t) \) if

\[ L(f(t)) = \frac{2}{5} e^{-3s} - \frac{2}{5} e^{-4s} \]

We first note that the right side can be written in slightly more revealing form as

\[ 2(\frac{1}{5}) e^{-3s} - 2(\frac{1}{5}) e^{-4s} \]

The factor \( \frac{1}{5} \) in each term indicates a jump in the function \( f(t) \), the multipliers \( 2 \) and \( -2 \) show the magnitude and the direction of each jump and, finally, the factors \( e^{-3s} \) and \( e^{-4s} \) show that the jumps occur at \( t = 3 \) and \( t = 4 \). Therefore the function must be given by

\[ f(t) = \begin{cases} 2, & 3 < t < 4 \\ 0, & \text{elsewhere.} \end{cases} \]

(See Figure 1.)

Figure 1. The graph of \( f(t) \) for Example 1.
Example 2:

Find \( f(t) \), if

\[
L(f(t)) = e^{-s} \left( \frac{2}{s^3} + \frac{2}{s^2} \right) + e^{-2s} \left( \frac{2}{s^3} \right) + e^{-4s} \frac{3}{s^2}
\]

The exponentials \( e^{-s} \), \( e^{-2s} \), and \( e^{-4s} \) tell us to look for jumps at \( t = 1, 2, \) and \( 4 \), so we must watch these positions. However, we begin construction of the function with the terms \( 2(1/s^3)e^{-s} \) and \( -2(2/s^3)e^{-2s} \), which indicate jumps of \( +2 \) at \( t=1 \) and \( +2 \) at \( t=2 \) in the second derivative. Hence, we have

\[
f''(t) = \begin{cases} 
2, & 1 < t < 2 \\
0, & \text{elsewhere}
\end{cases}
\]

as shown in Figure 2.

![Figure 2. Graph of \( f''(t) \) for Example 2.](image)

By integrating \( f''(t) \) we obtain the following expression:

\[
f'(t) = \begin{cases} 
C_1, & 0 < t < 1 \\
2t + C_2, & 1 < t < 2 \\
C_3, & 2 < t
\end{cases}
\]

where \( C_1, C_2, C_3 \) are constants to be determined.

Now the detective story begins! We must look to \( L(f(t)) \) to evaluate these constants. Since there is no term of the form \( \frac{1}{sk} \),

\[
\frac{1}{sk} = \frac{1}{sk} e^{-0s}
\]

there can be no jump in \( f(t) \) or any of its derivatives at the origin, so \( C_1 = 0 \). The term \( 2(1/s^2)e^{-s} \) reveals a jump of \( +2 \) in \( f'(t) \) at \( t=1 \) and since \( C_1 = 0 \), we must have

\[
2 = f'(1) = 2 + C_2,
\]

from which \( C_2 = 0 \). Next, the term \( -5(1/s^2)e^{-2s} \) shows a jump of \( +5 \) at \( t=2 \). But \( f'(t) = 2t \) to the left of \( t=2 \), and \( f'(t) = C_3 \) to the right of \( t=2 \). Hence at \( t=2 \) we jump \( 5 \) units from \( 4 \) down to \( C_3 \), so that \( C_3 = -1 \). Finally, the term \( 1(1/s^2)e^{-4s} \) is consistent with the value \( C_3 = -1 \), since it shows a unit jump back to the \( t \)-axis at \( t=4 \). Therefore

\[
f'(t) = \begin{cases} 
2t, & 1 < t < 2 \\
-1, & 2 < t < 4 \\
0, & \text{elsewhere}
\end{cases}
\]

(Figure 3).

![Figure 3. Graph of \( f'(t) \) for Example 3.](image)

While the above explanation of how to find \( f'(t) \) may seem complicated, in actuality by observing the transform carefully and proceeding from left to right, we can (after a little practice) sketch \( f'(t) \) section by section rather quickly. We illustrate by obtaining \( f(t) \) from \( f'(t) \) graphically. The result is (See Figure 4):
The result was obtained section by section, as follows. First, we have previously observed that \( f(t) = 0 \) for \( 0 < t < 1 \). Then, from the graph of \( f'(t) \) we obtain \( f(t) = t^2 + b_1 \) for \( 1 < t < 2 \), but the term \( 2(1/s)e^{-s} \) indicates a jump of \( +2 \) at \( t=1 \). Hence, \( 2 = f(1^+) = 1 + b_1 \) so that \( b_1 = 1 \), and \( f(t) = t^2 + 1 \). Now for \( 2 < t < 4 \), the graph of \( f'(t) \) yields \( f(t) = t^2 + b_2 \), and the term \( -3(1/s)e^{-2s} \) yields a jump of \( +3 \) at \( t=2 \). Since \( f(2^-) = 5 \) (from \( t^2 + 1 \)), and \( f(2^+) = -2 + b_2 \), we have \( 3 = 5 - (-2 + b_2) \), from which \( b_2 = 4 \) and \( f(t) = t^2 + 4 \) for \( 2 < t < 4 \). Since there is no jump in \( f(t) \) at \( t=4 \) we have \( f(t) = 0 \) for \( t > 4 \).

Exercise 1

For each of the following, find \( f(t) \) from the given expression for \( L\{f(t)\} \):

a) \( \frac{1}{s^2} e^{-s} \);

b) \( \frac{2}{s^5} - \frac{1}{5} e^{-3s} \);

c) \( \frac{2}{s^2} + e^{-2s} \left[ \frac{1}{2s^2} - \frac{4}{s} \right] \);

d) \( e^{-2s} \left[ \frac{3}{s^5} + \frac{2}{s^3} \right] + e^{-6s} \left[ \frac{2}{s^5} + \frac{1}{s} \right] + \frac{1}{s} + \frac{2}{s^3} \).

3. INVERSE FOURIER TRANSFORMS

We illustrate the procedure for finding the inverse Fourier transform by an example. Again, a familiarity with the forward transform from Unit 324 is assumed.

Example 3

Suppose we wish to find \( f(t) \), if

\[
F\{f(t)\} = \frac{2}{a^2} (\cos a) + \frac{2}{a} \sin a .
\]

We begin by converting \( F\{f(t)\} \) to exponential form so that we can identify the location, magnitude and direction of all jumps. Hence we have

\[
F\{f(t)\} = \frac{1}{a^2} \left[ e^{ia} + e^{-ia} \right] - \frac{2}{a^2} + \frac{1}{ia} \left[ e^{ia} - e^{-ia} \right] .
\]

We now recall that information on jumps for the forward transform is recorded in terms of powers of \( ia \); hence we must make a further adjustment to obtain

\[
F\{f(t)\} = e^{ia} \left[ \frac{1}{(ia)^2} + \frac{1}{(ia)^2} \right] + e^{-ia} \left[ \frac{1}{(ia)^2} - \frac{1}{(ia)^2} \right] .
\]

The terms

\[
\frac{1}{(ia)^2} e^{ia}, \frac{2}{(ia)^2}, \frac{1}{(ia)^2} e^{-ia}
\]

indicate jumps of \( +1 \) at \( t = -1 \), \( +2 \) at \( t = 0 \), and \( +1 \).
at $t = 1$. Hence we may now sketch the graph of $f'(t)$ (Figure 5). Then working from left to right as before,

Figure 5. Graph of $f'(t)$ for Example 3.

we find first that $f(t) = -t + C_1$, $-1 < t < 0$. But the term $\frac{1}{i}e^{i\alpha}$ shows a jump of +1 in $f(t)$ at $t = -1$; hence $f(-1) = 1 + C_1$, from which $C_1 = 0$. For $0 < t < 1$, $f(t) = t + C_2$; but the term $\frac{1}{i}e^{-i\alpha}$ shows a jump of +1 in $f(t)$ at $t = 1$, and in addition, no jump in $f(t)$ is indicated at $t = 0$. Therefore, $C_2 = 0$; hence we obtain

$$f(t) = \begin{cases} |t|, & |t| < 1 \\ 0, & |t| > 1. \end{cases}$$

(See Figure 6.)

Figure 6. Graph of $f(t)$ for Example 3.

4. INVERSE COEFFICIENT TRANSFORMS

For our final examples we shall find the "inverse" of three Fourier series. That is, the problem we solve is the following: given a Fourier series, find the function to which the series converges. Those of you who are familiar with Fourier series may well be surprised to learn that this problem may have a reasonable solution.

The problem of recovering a function from its Fourier series representation may well require considerable ingenuity, insight and, perhaps, even some experimentation. The reason for this is three-fold. The first reason is the nature of the expansion itself -- very simple functions may yield expansions with coefficients of considerable complexity. The second difficulty arises from the terms of the form

$$e^{i\alpha \pi / p}$$

[see formula (9) of Unit 324]. The problem is that for $a_k = 0$ and for $a_k = 2p$, we have

$$e^{i\alpha \pi / p} = 1.$$  

Therefore, we may not know whether the jump is at $t = 0$.

Exercise 2

For each of the following, find $f(t)$ from the given expression for $F(f(t))$:

a. $\frac{1}{\alpha} (\sin 2\alpha - \sin 2\alpha - 1)$;

b. $\frac{2i}{\alpha^2} (\sin \alpha - \sin 2\alpha)$;

c. $\frac{2}{\alpha^3} (\alpha^2 \sin \alpha + 2\alpha \cos \alpha - 2\sin \alpha)$.
or at $t = 2p$, or perhaps both. Similarly, if terms of the form $(-1)^n$ occur in the expansion, we may have either $a_k = -p$ or $a_k = p$, since $(-1)^n = \cos n\pi = e^{in\pi} = e^{-in\pi}$. Therefore, in this case the actual integration in $C\{f(t)\}$ would have been from $t = -p$ to $t = p$ but again, we may not know whether the jump was at $-p$ or at $p$.

The third difficulty is related to the second. Since the exponentials involved are equal at the end points of the interval in question, it follows that if the corresponding coefficients are equal in magnitude but opposite in sign, then the sum of these terms will vanish! Hence, we may be looking at a situation in which there is actually a jump present, but no indication of it. It may well require some patience to overcome these difficulties!

**Example 4**

Suppose we wish to find the function $f(t)$ whose Fourier series expansion is

$$
\sum_{n=1}^{\infty} \left[ \frac{3}{n^2 \pi^2} \left( \cos \frac{4\pi n}{3} - 1 \right) \cos \frac{2n\pi t}{3} + \left( \frac{4}{n\pi} + \frac{3}{n^2 \pi^2} \sin \frac{4\pi n}{3} \right) \sin \frac{2n\pi t}{3} \right].
$$

We observe immediately that

$$
a_0 = \frac{16}{3},
$$

and for $n = 1, 2, 3, \ldots$ we have

$$
a_n = \frac{3}{n^2 \pi^2} \left( \cos \frac{4\pi n}{3} - 1 \right),
$$

$$
b_n = \left( \frac{4}{n\pi} + \frac{3}{n^2 \pi^2} \sin \frac{4\pi n}{3} \right).
$$

The approach we use is: first find $C_n$, next find $C\{f(t)\}$, and then recover the function $f(t)$. Since,

$$
C_n = \left( a_n - ib_n \right)/2 \quad \text{(formula (8) from Unit 324)}
$$

we have

$$
a_n = \frac{1}{2} \left[ -\frac{3}{n^2 \pi^2} \left( \cos \frac{4\pi n}{3} - i \sin \frac{4\pi n}{3} \right) \frac{5}{n^2 \pi^2} + \frac{4i}{n\pi} \right]
$$

$$
= \frac{1}{2} \left[ -\frac{3}{n^2 \pi^2} e^{-4\pi i/3} + \frac{3}{n^2 \pi^2} + \frac{4i}{n\pi} \right].
$$

From the general form of the Fourier expansion we obtain that for this example

$$
\cos \frac{n\pi t}{p} = \cos \frac{2\pi n t}{p}
$$

from which

$$
p = \frac{3}{2}.
$$

Since

$$
C\{f(t)\} = 2pC_n = 3C_n,
$$

we have

$$
C\{f(t)\} = \frac{9}{2n^2 \pi^2} e^{-4\pi i/3} - \frac{9}{2n^2 \pi^2} + \frac{6i}{n\pi}.
$$

Because the coefficients in the $C$-transform involve powers of $\frac{p}{2\pi n} = \frac{3}{2\pi n}$ and exponentials of the form $e^{-in\pi k/p}$, we write

$$
C\{f(t)\} = -2 \left( \frac{3}{2\pi n} \right)^2 e^{-2i\pi} + \left( \frac{3}{2\pi n} \right)^2 = 4 \left( \frac{3}{2\pi n} \right).
$$

The first term indicates a jump of $-2$ in the first derivative, $f'(t)$, at $t = 2$. The second term is the bothersome one -- it could indicate a jump of $+2$ at $t = 0$ or a jump of $+2$ at $t = 2p = 3$, or it could be the result of a combination of jumps at both places.
In order to allow for the various possibilities, we write for one period:

\[
    f'(t) = \begin{cases} 
        2-a, & 0 < t < 2 \\
        -a, & 2 < t < 3 
    \end{cases}
\]

Figure 7. Graph of f'(t) for Example 4.

From the expression for f'(t) we obtain for one period:

\[
    f'(t) = \begin{cases} 
        (2-a)t + b, & 0 < t < 2 \\
        -at + c, & 2 < t < 3 
    \end{cases}
\]

The expression for C(f(t)) above shows no jump in f(t) at t = 2, and therefore the left and right sections of f(t) agree at t = 2. Thus,

\[
    (2-a)(2) + b = (-a)(2) + c,
\]

from which

\[
    c = b + 4.
\]

The last term in the expression for C(f(t)) could arise from a jump of 4 at t = 0 or at t = 3, or from a combination of jumps at both ends. Now observe that the jump at t = 0 is f(0+) and the jump at t = 3 is f(3-). Since these jumps must combine to produce the value -4, we have

\[
    -4 = f(0) - f(3) = b - (3a + c) = (b - c) + 3a = -4 + 3a,
\]

from which

\[
    a = 0.
\]

So far we have for one period

\[
    f(t) = \begin{cases} 
        2t + b, & 0 < t < 2 \\
        b + 4, & 2 < t < 3 
    \end{cases}
\]

Finally, since

\[
    a_0 = \frac{16}{3} = \frac{1}{p} \int_0^2 f(t) \, dt,
\]

we have

\[
    \frac{16}{3} = \frac{2}{3} \left[ \int_0^2 (2t + b) \, dt + \int_2^3 (b + 4) \, dt \right],
\]

from which

\[
    8 = \left[ t^2 + bt \right]_0^2 + \left[ (b + 4)t \right]_2^3 = 8 + 3b,
\]

so that

\[
    b = 0.
\]

Hence the given series converges to a function of period 3 whose definition for one period is

\[
    f(t) = \begin{cases} 
        2t, & 0 < t < 2 \\
        4, & 2 < t < 3 
    \end{cases}
\]

(See Figure 8.)
Example 5

Suppose we wish to find the function \( f(t) \) whose Fourier series expansion is given by

\[
f(t) = \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2 \pi^2} \sin n\pi t.
\]

We first note that from the given expansion, \( a_n = 0 \) for \( n = 0, 1, 2, \ldots \), so that \( f(t) \) is an odd function. In addition, from the terms \( \sin n\pi t \) we have \( p = 1 \), hence \( f(t) \) has period 2, and we must find an expression for \( f(t) \) over any interval of length 2; we choose \(-1 < t < 1\).

Next we find \( C_n \):

\[
C_n = \frac{1}{2} (a_n - ib_n) = \frac{(-1)^n}{n^2 \pi^2} \left( \frac{-1}{\sin n\pi} \right)
\]

Therefore,

\[
C(f(t)) = 2p C_n = \frac{12(-1)^n}{n \pi^3}
\]

In general, we must express \( C(f(t)) \) in powers of \( p/\sin \pi = 1/n \), but this task is already accomplished here. The single term in \( C(f(t)) \) indicates a jump of magnitude 12 in \( f''(t) \), and the factor \((-1)^n = e^{in\pi} = e^{-in\pi} = (-1)^n\) shows that the jump is at \( t = 1 \) or \( t = -1 \) or that, perhaps, the whole term results from a combination of jumps at both ends. However, a little reflection shows that we simply cannot have a positive jump at just one end of the interval \(-1 < t < 1\) or, for that matter, any combination of jumps at both ends with sum total positive if, after a jump is made the function remains constant until the next jump.

The following function is a possibility for \( f''(t) \) and we take it as our starting point:

\[
f''(t) = -6t + a, \quad -1 < t < 1.
\]

(See Figure 9.) If we try the above function for

\[
f''(t) = \frac{6 \cdot 1}{\sin \pi} e^{-i\pi(1)} + \frac{6 \cdot 1}{\sin \pi(1)} e^{-i\pi(-1)}
\]

which results from jumps of +6 at \( t = -1 \) and +6 at \( t = 1 \). However, since \( e^{in\pi} = e^{-in\pi} = (-1)^n \), the sum

\[
C(f(t)) = \frac{12(-1)^n}{n \pi^3}
\]


Figure 8. Graph of \( f(t) \) for Example 4.

Figure 9. Graph of a possible \( f''(t) \) for Example 5.
The series
\[
\sum_{n=1}^{\infty} (-1)^n \frac{12}{n^3} \sin n\pi t
\]
and the series of its derivatives with respect to \(t\)
\[
\sum_{n=1}^{\infty} (-1)^n \frac{12}{n^2} \cos n\pi t
\]
converge uniformly by the Weierstrass M-test, applied with the series of constants
\[
\frac{12}{n^2} \sum_{n=1}^{\infty} \frac{1}{n^3}, \quad \frac{12}{n^2} \sum_{n=1}^{\infty} \frac{1}{n^3},
\]
respectively, used for comparison. Since the sum of a uniformly convergent series of continuous functions is continuous, we have continuity of \(f(t)\). In addition, since the series for \(f(t)\) converges and the series of derivatives converges uniformly, we have that
\[
f'(t) = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{12}{n^2} \cos n\pi t,
\]
and hence \(f'(t)\) is also continuous.

Now, there are two easy arguments we can use to find the constant \(a\) in \(f'(t)\). First, since \(f'(t)\) is given by a cosine series, it is an even function, and hence \(a = 0\). An alternative argument which is also useful in general is that by continuity and periodicity we have
\[
f'(-1) = f'(1),
\]
so that
\[-3 - a + b = -3 + a + b,
\]
from which \(a = 0\). Hence, either way we find
\[
f(t) = -t^3 + bt + c.
\]
Similarly, to find \(c\) we may observe that \(f(t)\) is an odd function, so that \(c = 0\). We could also argue that since \(f(t)\) is a sine series, we must have \(f(0) = 0\), from which \(c = 0\). Finally, to find \(b\), we observe that by the series definition of \(f(t)\) and its continuity, which precludes jumps from one period to the next, we have \(f(1) = 0\), hence \(-1 + b = 0\), and \(b = 1\). Alternatively, by the continuity and the periodicity of \(f(t)\) we have \(f(-1) = f(1)\), so that
\[
1 - b = -1 + b, \quad \text{and} \quad b = 1.
\]
Thus, for one period,
\[
f(t) = -t^3 + bt, \quad -1 < t < 1.
\]
(See Figure 10.)
Example 6

Suppose we wish to find the function $f(t)$ whose Fourier series expansion is given by

$$f(t) = \frac{8}{\pi} \sum_{n=1,3,5,...} \frac{(-1)^{(n-1)/2}}{n^2} \sin nt.$$  

We first note that $n\pi t / p = nt$, from which $p = \pi$, $2p = 2\pi$. In addition, $a_n = 0$ for $n = -0, 1, 2, ...$, so that $f(t)$ is an odd function. The coefficients $b_n$ are given by

$$b_n = \begin{cases} \frac{8}{n^2 \pi^2} (-1)^{(n-1)/2}, & n \text{ odd} \\ 0, & n \text{ even} \end{cases}$$

Since

$$C_n = \frac{1}{2}(a_n - ib_n),$$

we have

$$C_n = \begin{cases} \frac{4i}{n^2 \pi^2} (-1)^{(n-1)/2}, & n \text{ odd} \\ 0, & n \text{ even} \end{cases}$$

from which

$$C(f(t)) = 2\pi C_n = \begin{cases} \frac{-8i}{\pi n^2} (-1)^{(n-1)/2}, & n \text{ odd} \\ 0, & n \text{ even} \end{cases}$$

Since the transform coefficients in formula (9) of Unit 324 are expressed in powers of $p/\pi n$, we rewrite the preceding expression in the form

$$C(f(t)) = \frac{\pi}{1n\pi} \begin{cases} \frac{8i}{\pi} (-1)^{(n-1)/2}, & n \text{ odd} \\ 0, & n \text{ even} \end{cases}$$

We now face the crucial problem of finding the way in which the two distinct expressions (for the odd and the even coefficients) in $C(f(t))$ can be unified into a single form. Your ability to make this step requires careful observation in working with the forward transform, and with trigonometric functions in general.

We simply note the result

$$(-1)^{(n-1)/2} \sin \frac{n\pi}{2}, \quad n = 1, 2, 3, ...$$

(Check out a few values of $n$ for yourself!) We therefore have

$$C(f(t)) = \frac{\pi}{1n\pi} \begin{cases} \frac{8i}{\pi} \sin \frac{n\pi}{2}, & n = 1, 2, 3, ... \end{cases}$$

We now convert to exponential form:

$$C(f(t)) = \frac{\pi}{1n\pi} \begin{cases} \frac{4}{\pi} e^{i\pi n/2} - e^{-i\pi n/2} \end{cases}$$

With $p = \pi$, we now obtain the coefficient transform in the form of (9) from Unit 324:

$$C(f(t)) = \frac{\pi}{1n\pi} \begin{cases} \frac{4}{\pi} e^{-i\pi n/2} - e^{-i\pi n/2} \end{cases}$$

This form reveals jumps of $+4/\pi$ at $-\pi/2$ and $+4/\pi$ at $\pi/2$ in the first derivative $f'(t)$. However, we must also be alert to possible cancellation of terms, especially at $-\pi$ and $\pi$. The simplest type of function whose behavior agrees with what we have so far is one with derivative of the form

$$f(t) = 5i$$
\[ f'(t) = \begin{cases} a, & -\pi < t < -\pi/2 \text{ and } \pi/2 < t < \pi \\ \frac{4}{\pi} + a, & -\pi/2 < t < \pi/2 \end{cases} \]

(See Figure 11. We note that because the period is \( \pi \), and contributions at \(-\pi\) and \(\pi\) would cancel:
\[ ae^{in\pi} - ae^{-in\pi} = 0. \]
Such cancellation does not occur at \(\pi/2\).)

---

**Figure 11.** A possible form of \(f'(t)\) for Example 6.

---

Proceeding from our point of departure, we have
\[ f(t) = \begin{cases} at + b, & -\pi < t < -\pi/2 \\ \frac{4}{\pi} + at + c, & -\pi/2 < t < \pi/2 \\ at + d, & \pi/2 < t < \pi \end{cases} \]

To evaluate the constants, we first apply the M-test (as in Example 5), using \(8/\pi^2\) Σ \(1/n^2\) for \(n=1,3,5\ldots\) comparison to see that \(f(t)\) is continuous everywhere. Again, we illustrate two alternative methods for determining the constants.

First, since \(f(t)\) is an odd function, \(f(-t) = -f(t)\), so that

\[ f(t) = \begin{cases} -at + b, & -\pi < t < -\pi/2, \text{i.e., } -\pi/2 < t < \pi/2 \\ -\left(\frac{4}{\pi} + a\right)t + c, & -\pi/2 < t < \pi/2 \\ -at + d, & \pi/2 < t < \pi \end{cases} \]

from which
\[ b = -d \text{ and } c = 0. \]

By continuity of \(f(t)\) at \(t = \pi/2\),
\[ an/2 + d = \left(\frac{4}{\pi} + a\right) \frac{\pi}{2}, \]
from which \(d = 2\) and hence \(b = -2\). From the series definition of \(f(t)\) and continuity, \(f(\pi) = 0\), hence \(a\pi + 2 = 0\), and \(a = -2/\pi\).

Alternatively, we could have used the series definition of \(f(t)\) to obtain \(f(t) = 0\), from which \(c = 0\). Continuity of \(f(t)\) at \(t = \pi/2\) now yields the equation
\[ an/2 + d = \left(\frac{4}{\pi} + a\right) \frac{\pi}{2}, \]
from which \(d = 2\). Similarly, from continuity of \(f(t)\) at \(-\pi/2\) we obtain \(b = -2\). The constant \(a\) is determined as above.

By either approach we obtain
\[ f(t) = \begin{cases} -at + 2, & -\pi < t < -\pi/2 \\ \frac{2}{\pi} t, & -\pi/2 < t < \pi/2 \\ -\frac{2}{\pi} t + 2, & \pi/2 < t < \pi \end{cases} \]

(See Figure 12.)
Exercise 3

For each of the following, find the function \( f(t) \) whose Fourier series expansion is given:

a. \( f(t) = \frac{4}{\pi} \sum_{n=1,3,5,\ldots} \frac{1}{n} \sin \frac{n\pi t}{2} \)

(but may be helpful to note that \( \cos n\pi = (-1)^n \) and that

\[ 1 - (-1)^n = \begin{cases} 2, & n \text{ odd} \\ 0, & n \text{ even} \end{cases} \]

b. \( f(t) = \frac{1}{3} + 4 \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \cos \frac{n\pi t}{2} + \frac{(-1)^{n+1}}{n\pi} \sin \frac{n\pi t}{2} \)

5. MODE EXAM

1. Find \( f(t) \) if

a. \( L(f(t)) = \frac{1}{2s^2} \left[ 3 + e^{-2s} - 4e^{-3s} + 2e^{-5s} \right] - \frac{1}{s} \left[ 1 + e^{-3s} \right] \)

b. \( L(f(t)) = e^{-3s} \left[ \frac{1}{2s^2} - \frac{1}{s} \right] + e^{-s} \left[ \frac{2}{s} + \frac{3}{s^2} + \frac{1}{s} \right] \)

2. Find \( f(t) \) if

\[ F(f(t)) = \frac{4}{\alpha} \sin \frac{\alpha}{\alpha} (\cos \alpha - 1) \]

3. Find \( f(t) \) if \( f(t) \) is periodic of period 3, \( f(3/2) = 0 \) and if

\[ C f(t) = \begin{cases} \frac{3i}{2n\pi} - \frac{9}{n^2 \pi^2}, & n \text{ odd} \\ \frac{3i}{2n\pi}, & n \text{ even} \end{cases} \]
6. ANSWERS TO EXERCISES

1. a. \( f(t) = \begin{cases} \ 0, & 0 < t < 1 \\ \ t, & 1 < t. \end{cases} \)
   b. \( f(t) = \begin{cases} \ 0, & 0 < t < 1 \\ \ 2, & 1 < t < 3 \\ \ 1, & 3 < t. \end{cases} \)
   c. \( f(t) = \begin{cases} \ 2, & 0 < t < 2 \\ \ \frac{1}{2}t - 3, & 2 < t. \end{cases} \)
   d. \( f(t) = \begin{cases} \ t^2 + 1, & 0 < t < 2 \\ \ -t + 4, & 2 < t < 6 \\ \ 0, & 6 < t. \end{cases} \)

2. a. \( f(t) = \begin{cases} \ -1, & -2\pi < t < 0 \\ \ 0, & \text{elsewhere}. \end{cases} \)
   b. \( f(t) = \begin{cases} \ t + 2, & -2 < t < -1 \\ \ 1, & -1 < t < 1 \\ \ 2 - t, & 1 < t < 2 \\ \ 0, & 2 < |t|. \end{cases} \)
   c. \( f(t) = \begin{cases} \ t^2, & |t| < 1 \\ \ 0, & |t| > 1. \end{cases} \)

3. a. \( f(t) = \begin{cases} \ -1, & -2 < t < 0 \\ \ 1, & 0 < t < 2. \end{cases} \)
   b. \( f(t) = t^2 + 2t, \)

7. ANSWERS TO MODEL EXAM

1. a. \( f(t) = \begin{cases} \ \frac{3}{2}t - 1, & 0 < t < 2 \\ \ t, & 2 < t < 3 \end{cases} \)
   b. \( f(t) = \begin{cases} \ 2, & 1 < t < 3 \\ \ \frac{1}{2}t - \frac{1}{2}, & 3 < t. \end{cases} \)

2. a. \( f(t) = \begin{cases} \ t^2 + t + 1, & 0 < t < 1 \\ \ 2 - t, & 1 < t < 2 \\ \ 0, & 2 < |t|. \end{cases} \)
   b. \( f(t) = \begin{cases} \ |t|, & -1 < t < 1 \\ \ 0, & 2 < |t|. \end{cases} \)

3. a. \( f(t) = \begin{cases} \ -1, & -2 < t < -1 \\ \ -\frac{4}{3}t + 2, & 0 < t < \frac{3}{2} \end{cases} \)
   b. \( f(t) = \begin{cases} \ \frac{3}{2}t + 1, & -\frac{3}{2} < t < 0 \\ \ \frac{1}{2}t + 2, & 0 < t < \frac{3}{2} \end{cases} \)
STUDENT FORM 1
Request for Help

Return to:
EDC/UMAP
55 Chapel St.
Newton, MA 02160

Student: If you have trouble with a specific part of this unit, please fill out this form and take it to your instructor for assistance. The information you give will help the author to revise the unit.

Your Name

Page
○ Upper
○ Middle
○ Lower

OR

Section

Paragraph

OR

Unit No.

Model Exam
Problem No.

Text
Problem No.

Description of Difficulty: (Please be specific)

Instructor: Please indicate your resolution of the difficulty in this box.

○ Corrected errors in materials. List corrections here:

○ Gave student better explanation, example, or procedure than in unit. Give brief outline of your addition here:

○ Assisted student in acquiring general learning and problem-solving skills (not using examples from this unit.)

Instructor's Signature

Please use reverse if necessary.
STUDENT FORM 2
Unit Questionnaire

Name ___________________________ Unit No. ___________ Date ___________
Institution __________________________ Course No. ___________

Check the choice for each question that comes closest to your personal opinion.

1. How useful was the amount of detail in the unit?
   ___ Not enough detail to understand the unit
   ___ Unit would have been clearer with more detail
   ___ Appropriate amount of detail
   ___ Unit was occasionally too detailed, but this was not distracting
   ___ Too much detail; I was often distracted

2. How helpful were the problem answers?
   ___ Sample solutions were too brief; I could not do the intermediate steps
   ___ Sufficient information was given to solve the problems
   ___ Sample solutions were too detailed; I didn't need them

3. Except for fulfilling the prerequisites, how much did you use other sources (for example, instructor, friends, or other books) in order to understand the unit?
   ___ A Lot
   ___ Somewhat
   ___ A Little
   ___ Not at all

4. How long was this unit in comparison to the amount of time you generally spend on a lesson (lecture and homework assignment) in a typical math or science course?
   ___ Much Longer
   ___ Somewhat Longer
   ___ About the Same
   ___ Somewhat Shorter
   ___ Much Shorter

5. Were any of the following parts of the unit confusing or distracting? (Check as many as apply.)
   ___ Prerequisites
   ___ Statement of skills and concepts (objectives)
   ___ Paragraph headings
   ___ Examples
   ___ Special Assistance Supplement (if present)
   ___ Other, please explain

6. Were any of the following parts of the unit particularly helpful? (Check as many as apply.)
   ___ Prerequisites
   ___ Statement of skills and concepts (objectives)
   ___ Examples
   ___ Problems
   ___ Paragraph headings
   ___ Table of Contents
   ___ Special Assistance Supplement (if present)
   ___ Other, please explain

Please describe anything in the unit that you did not particularly like.

Please describe anything that you found particularly helpful. (Please use the back of this sheet if you need more space.)