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ABSTRACT

Applied mathematics is seen often to involve the solution of single equations or systems of equations. A particular technique for solving equations in the context of functions on the real line is examined. This technique includes an iterative process, which is viewed as an algorithm that can be implemented readily on inexpensive and widely available calculators. Some examples of physical problems in which the equations of interest arise are presented. This module includes exercises and a model exam. Answers to problems in the exercises and on the test are provided in the material. (MP)

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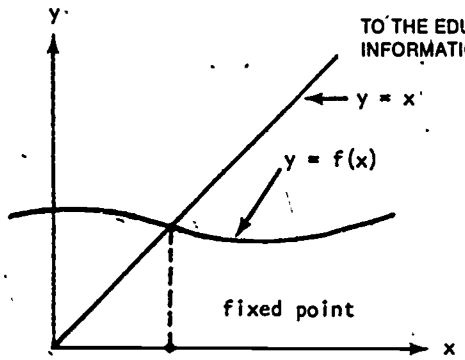
UNIT 326

MODULES AND MONOGRAPHS IN UNDERGRADUATE MATHEMATICS AND ITS APPLICATIONS PROJECT

THE CONTRACTION MAPPING PRINCIPLE

by Carroll O. Wilde



NUMERICAL METHODS IN ELEMENTARY ANALYSIS

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THE CONTRACTION MAPPING PRINCIPLE

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Intermodular Description Sheet: *UMAP Unit 326*

Title: THE CONTRACTION MAPPING PRINCIPLE

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Review Stage/Date: III 12/26/78

Classification: APPL NUM METH/ELEM ANALYSIS

Prerequisite Skills:
1. Differential calculus.
2. Introduction to differential equations helpful.

Output Skills:
1. Be able to recast the problem of solving a real equation as a fixed-point problem, and then to estimate the solution by the Picard method.
2. Be able to use the method in optimization problems that require finding zeros of derivatives as a first step.

Other Related Units:

4

MODULES AND MONOGRAPHS IN UNDERGRADUATE MATHEMATICS AND ITS APPLICATIONS PROJECT (UMAP)

The goal of UMAP is to develop, through a community of users and developers, a system of instructional modules in undergraduate mathematics and its applications which may be used to supplement existing courses and from which complete courses may eventually be built.

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1. INTRODUCTION

Applied mathematics often involves the solution of single equations or systems of equations. In this module we study a particular technique for solving equations in the context of functions on the real line. The technique includes an iterative process, i.e., an algorithm, that can be understood and interpreted with ease pictorially and can be implemented readily on inexpensive and widely available calculators (programming feature helpful but not necessary). We also present some examples of physical problems in which these equations arise.

2. THE PROBLEM

The basic problem here is to solve equations of the form

(1) $f(x) = x.$

A solution of Equation (1) is called a *fixed point* of f; geometrically, such a solution is a value of x for which the graph of the function $y = f(x)$ crosses the line $y = x$ (see Figure 1). For example, if $f(x) = x^2$,

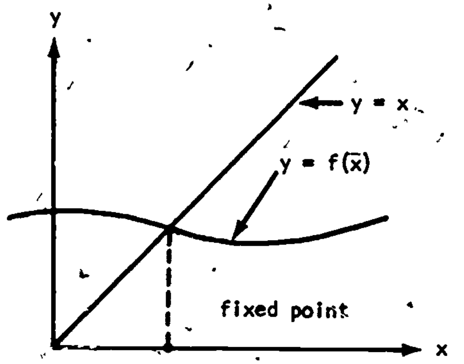

Figure 1. Geometric interpretation of a fixed point.

then Equation (1) takes the form

$$x^2 = x,$$

which has two solutions (fixed points), $x = 0$ and $x = 1$.

Equation (1) is more general than it may seem at first, because any equation of the form

(2) $g(x) = 0$

may be converted to the form (1) by adding x to both sides:

(3) $g(x) + x = x.$

and Equation (3), which is equivalent to (2), is now of the form (1), with $f(x) = g(x) + x$. The method presented here may therefore be applied to find roots of equations, as well as fixed points.

3. THE ALGORITHM

The iterative process that constitutes the basis of the contraction mapping principle is often called *Picard's algorithm* or simply *iteration*. The process is easy to describe: for a function f whose range is contained in its domain, start with a point x_0 in the domain, and then successively reapply the function. That is, let $\{x_n\}_{n=1}^{\infty}$ be the sequence

(4) $x_1 = f(x_0),\ x_2 = f(x_1),\ x_3 = f(x_2),\ \ldots$

Under some simple conditions that involve the concept of a *contraction* (to be described later), the sequence defined in (4) converges to a fixed point of f(x); that is, if we let $x = \lim_{n\to\infty} x_n$, then

$$f(x) = x.$$

We shall formulate the definition of a contraction and a statement of contraction mapping principle in Section 4, but first we illustrate the algorithm by some simple examples.

3.1. Example 1

Let us begin with an example that is easy to see and for which we know the outcome in advance. Let $f(x) = \frac{1}{3}x + 2$, and suppose we wish to solve the equation

$$\frac{1}{3}x + 2 = x$$

(see Figure 2). By elementary algebra, we know that

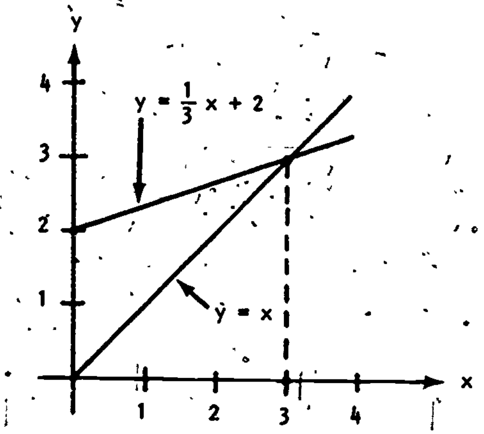

Figure 2. Geometrical interpretation of Example 1.

the solution is $x = 3$. To apply the Picard algorithm we choose any starting point, say $x_0 = -5$, and calculate successive values of the function according to (4). The results are indicated in Table 1; check them on your calculator.

The fixed point $x = 3$ is reached correct to four places in eleven steps. (Note that had we rounded to two places, we would have reached the solution to two places in only five steps.)

TABLE 1

Results of the Picard Process Applied to the Function in Example 1, with $x_0 = -5$.

x	Formula for $f(x)$	Numerical Approximation of $f(x)$
$x_0 = -5$	$\frac{1}{3}(-5) + 2$	$0.3333 \doteq x_1$
x_1	$\frac{1}{3}(.3333) + 2$	$2.1111 = x_2$
x_2	$\frac{1}{3}(2.1111) + 2$	$2.7037 = x_3$
x_3	$\frac{1}{3}(2.7037) + 2$	2.9012
x_4	$\frac{1}{3}(2.9012) + 2$	2.9671
x_5	$\frac{1}{3}(2.9671) + 2$	2.9890
x_6	$\frac{1}{3}(2.9890) + 2$	2.9963
x_7	$\frac{1}{3}(2.9963) + 2$	2.9988
x_8	$\frac{1}{3}(2.9988) + 2$	2.9960
x_9	$\frac{1}{3}(2.9960) + 2$	2.9999
x_{10}	$\frac{1}{3}(2.9999) + 2$	3.0000

It is important to understand the geometry of the Picard procedure. When we let $x_1 = f(x_0)$ and then calculate $f(x_1)$, the first y-value becomes the new x-value, and then we move to a new y-value. This process can be represented by a path in which we start at $(x_0, 0)$, move vertically to $(x_0, f(x_0))$ or (x_0, x_1), then move horizontally to (x_1, x_1), and then vertically to $(x_1, f(x_1))$, and so on. A picture of this procedure is shown in Figure 3.

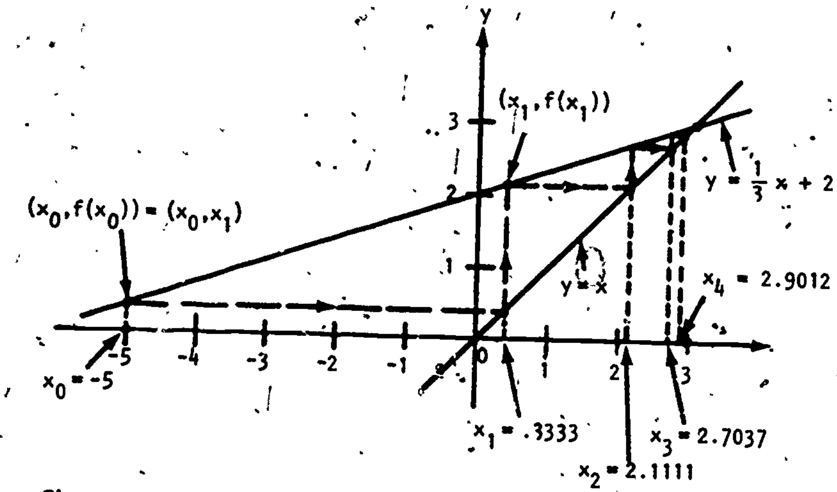

Figure 3. Geometric interpretation of the Picard procedure for the function in Example 1, with $x_0 = -5$.

Exercise 1

Use the Picard procedure to solve the equation $\frac{1}{2}x - 1 = x$.

3.2 Example 2

Suppose we wish to solve the equation

$$\cos x = x.$$

In this case, the equation is transcendental, and algebraic manipulations do not yield a solution. A reasonable approximation to the solution can be obtained graphically. Or, if you have tables for trigonometric functions handy, you may simply scan for the point where $\cos x = x$ (radians). The CRC tables show that $\cos(.73) \doteq .7452$ and $\cos(.74) \doteq .7385$, so we would expect the solution to be close to .74.

The Picard procedure for the equation $\cos x = x$ can be implemented easily on many calculators. With the calculator in the radian mode, start with $x_0 = 0$, and repeatedly calculate the cosine (see Table 2).

TABLE 2

Numerical Solution of the Equation $\cos x = x$.

x	Formula for $f(x)$	Numerical Approximation of $f(x)$
$x_0 = 0$	cos 0	$1.0000 = x_1$
$x_1 = 1$	cos (1)	$.5403 = x_2$
x_2	cos (.5403)	$.8576 = x_3$
x_3	cos (.8576)	.6543
x_4	cos (.6543)	.7935
x_5	cos (.7935)	.7014
x_6	cos (.7014)	.7640
x_7	cos (.7640)	.7221
x_8	cos (.7221)	.7504
x_9	cos (.7504)	.7314
.		
.		
.		
x_{23}	cos (.7391)	.7391

The convergence for the equation of Example 2 was much slower than for Example 1; this time 23 iterations were required for four place accuracy (11 for two place accuracy). You may be able to see the reason for the slow convergence by studying the geometric picture -- here we "wind around" the fixed point, as shown in Figure 4.

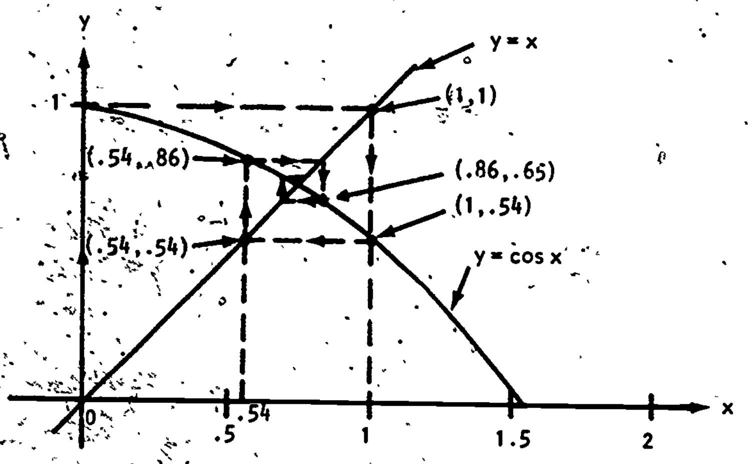

Figure 4. Graph showing the Picard procedure for the equation $\cos x = x$.

3.3 Example 3

Consider a rigid bar that is hinged at one end, with the other end free to swing. The bar has length ℓ and uniform mass distribution, with total mass m. Let $\theta = \theta(t)$ be the angle the bar makes with the vertical at time t. At the hinged end of the bar, there is a torsion spring that exerts a force of magnitude $k\theta$, where k is a constant, to restore the bar to its vertical rest position. We suppose also that a downward force F acts at the free end of the bar. The bar is given an initial

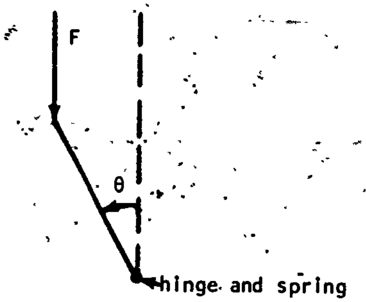

Figure 5. Illustration for Example 3.

velocity θ_0, and started into motion at time $t = 0$. (See Figure 5.)

If we neglect the gravitational force, the equation of motion can be obtained by considering (angular) forces; this equation is

$$\frac{1}{3} m \ell^2 \ddot{\theta} - k\theta + \ell F \sin\theta = 0. \tag{5}$$

(The factor $\frac{1}{3} m \ell^2$ is the moment of inertia of the bar about the hinged end.) If $k < \ell F$, then the system has exactly one equilibrium position between 0 and π. By Equation (5), with $\ddot{\theta}$ set equal to zero, the equilibrium position is characterized as the solution (between 0 and π) of the equation

$$-k\theta + \ell F \sin\theta = 0.$$

Thus, the value of θ that yields equilibrium is the positive solution of the equation

$$\frac{\ell F}{k} \sin x = x.$$

For example, if the values of ℓ, F and k are such that $\ell F/k = 1.2$, then we can find the equilibrium position by the Picard algorithm, starting at $x_0 = \pi/6$.

TABLE 3

Numerical Results of the Picard Sequence for $1.2 \sin x = x$, Starting at $x_0 = \pi/6$.

x	Formula for $f(x)$	Numerical Approximation of $f(x)$
$x_0 = \pi/6$	$1.2 \sin(\pi/6)$	$0.6000 = x_1$
$x_1 = .6$	$1.2 \sin(.6)$	$.6776 = x_2$
x_2	$1.2 \sin(.6776)$	.7523
x_3	$1.2 \sin(.7523)$	.8200
x_4	$1.2 \sin(.8200)$	.8773
.		
.		
.		
x_{22}	$1.2 \sin(1.0267)$	1.0267

Thus, in this case, the equilibrium position occurs near 1.0267 radians, or around 59°.

Exercise 2

Find the equilibrium position for the system in Example 3 when $\ell F/k = 1.4$.

---

4. CONTRACTION MAPPINGS

In this section we consider conditions under which the Picard algorithm can be used to solve equations.

4.1 Example 4

Consider the function $f(x) = 2x + 1$. We know by simple algebra that $x = 1$ is the solution of the equation $2x - 1 = x$. However, when we try the algorithm with $x_0 > 1$ or with $x_0 < 1$, we move *away* from the solution (see Figure 6, and compare Example 1 and Figure 3).

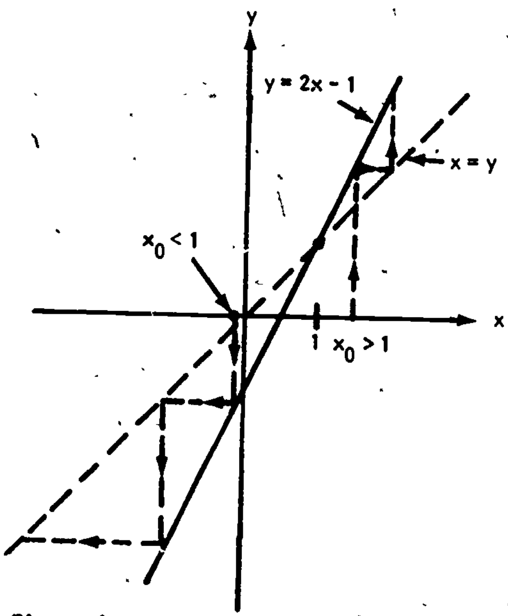

Figure 6: Illustration for Example 4.

It turns our that this behavior is related to the fact that the slope of the line $y = \frac{1}{3}x + 2$ is less than 1 and the slope of the line $y = 2x - 1$ is greater than 1, as we see in the discussion below.

Let f be a function that is defined on an interval I which is of any of the following types:

- $\{x: a \leq x \leq b\}$, where $a < b$ are finite numbers;
- $\{x: x \leq b\}$, where b is finite;
- $\{x: x \geq a\}$, where a is finite;
- the entire real line.

Thus, we consider the interval I as the domain of f (even though it may be that the formula for f can be applied to a larger set). When the values $f(x)$ are also contained in the interval I for x in I, we say that "f maps I into I" and write $f: I \to I$. (For example, let $f(x) = \frac{1}{2}x^2$, and consider $I = \{x: 0 \leq x \leq 1\}$. If $x \in I$, then $f(x) \in I$ also, and so $f: I \to I$. In this example, we note that the natural domain of f is the whole line, but we simply want to consider the behavior of $f(x)$ for values of x from 0 to 1.)

Let I be a real interval as described above, and let $f: I \to I$. Then f is called a *contraction* on I if there is a constant r such that $0 < r < 1$ and for all x,t in I we have

$$|f(x) - f(t)| \leq r\,|x - t|.$$

4.2 Example 5

Let $f(x) = \frac{1}{3}x + 2$, I the whole line. Then for all x,t we have

$$|f(x) - f(t)| = \left|\left(\frac{1}{3}x + 2\right) - \left(\frac{1}{3}t + 2\right)\right| = \frac{1}{3}|x - t|.$$

Thus, f is a contraction on I, with $r = \frac{1}{3}$.

4.3 Example 6

Let $f(x) = 2x - 1$, I the whole line. Then for any x,t we have

$$|f(x) - f(t)| = |(2x - 1) - (2t - 1)| = 2|x - t|.$$

Thus, f is *not* a contraction on I.

> *The contraction mapping principle for the real line.* Let I be a real interval of a type indicated above, and $f: I \to I$. If f is a contraction on I, then the equation
>
> $$f(x) = x$$
>
> has a unique solution in I, and this solution may be obtained by choosing any point x_0 in I, forming the sequence
>
> $$x_1 = f(x_0), \quad x_2 = f(x_1), \quad x_3 = f(x_2) \ldots,$$
>
> and passing to the limit. The solution is
>
> $$\lim_{n\to\infty} x_n.$$

A proof of the contraction mapping principle is presented in the Appendix.

The contraction mapping principle may seem to be restrictive, but there are two important techniques for extending the applicability. First, we note that if f is defined on I, if f is an *expansion*, i.e. for all x,t we have

$$|f(x) - f(t)| \geq k|x - t|,$$

where $k > 1$, and if the range R of f contains I, then we must have the following:

- f is one-to one, so that f^{-1} is defined on R;
- $f^{-1}: R \to I$, and $I \subseteq R$, so $f^{-1}: R \to R$;
- f^{-1} is a contraction on R, with $r = 1/k$.

Hence there must be a unique x in R such that

$$f^{-1}(x) = x.$$

But since the graphs of f and f^{-1} are symmetric in the line $y = x$, x must also be in I and we must have

$$f(x) = x.$$

4.4 Example 7

Let $f(x) = 2x - 1$, I the whole line. We saw in Example 6 that f is an expansion, with $k = 2$. The inverse function is defined also on I, and is given by the formula

$$f^{-1}(x) = \frac{1}{2}x + \frac{1}{2}.$$

Then f^{-1} is a contraction on I, with constant $r = \frac{1}{2} = \frac{1}{k}$. The Picard algorithm now yields the solution of the equation

$$\frac{1}{2}x + \frac{1}{2} = x,$$

which is $x = 1$, as shown in Figure 7. (Compare and

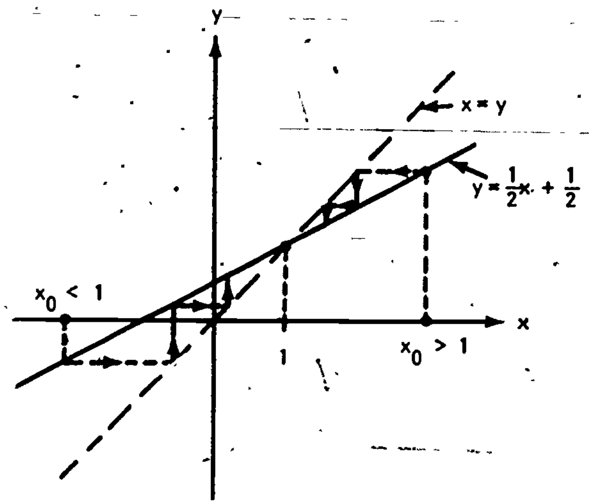

Figure 7. The Picard algorithm for Example 7.

contrast with Figure 6.) Since the equation

$$\frac{1}{2}x + \frac{1}{2} = x$$

may be rewritten

$$2x - 1 = x,$$

we have the solution of the original problem posed in Example 4.

The second way in which we can extend the applicability of the Picard algorithm is by considering the *iterates* of f rather than f itself. For $f: I \to I$ we let

$$f_1(x) = f(x),\ f_2(x) = f\left[f_1(x)\right],\ f_3(x) = f\left[f_2(x)\right], \ldots$$

It turns out that if there exists a positive integer n such that f_n is a contraction on I, then the fixed point of f_n is also the fixed point of f in I, and the fixed point can be found, as before, by choosing any starting point x_0 and iterating the function f, even though f itself is not a contraction. (It would be a challenging exercise for an advanced student who can handle the proof of the contraction mapping principle in the Appendix to construct a proof for this extension of the principle.) This technique may also be combined with the use of inverses, as the following example will illustrate.

4.5 Example 8

Suppose we wish to solve the equation

$$\frac{1}{4}e^x = x.$$

A rough sketch shows that we should expect to find two solutions, one between 0 and 1, the other greater than two. In Figure 8 we can see that if we take x_0 to be any point to the left of the larger solution, the Picard sequence will converge to the smaller solution. The approximate values obtained by starting at $x_0 = 0$ are: 0, .25, .3210, .3446, .3529, .3558, .3568, .3572, .3573, .3574.

The reason that the above sequence converges can be can be seen by choosing our interval I carefully and observing that f is a contraction on I. For example,

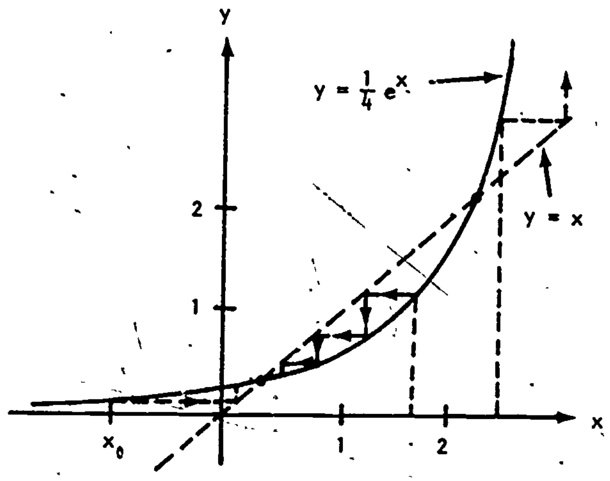

Figure 8. The graph for Example 8.

if we let

$$I = \{x: x \le 1\},$$

then $f: I \to I$, and for x in I we have

$$f'(x) = \frac{1}{4}e^x.$$

Thus, by the law of the mean, for $x, t \le 1$, we have

$$|f(x) - f(t)| = \left|\left(\frac{1}{4}e^\xi\right)(x - t)\right|,$$

where ξ is between x and t. Thus,

$$|f(x) - f(t)| = \frac{1}{4}e^\xi\ |x - t| \le \frac{e}{4}\ |x - t|.$$

Thus f is a contraction on I with contraction constant $r = e/4$.

If we try to reach the larger solution of $\frac{1}{4}e^x = x$ by taking x_0 between the solutions, we must fail, as shown also in Figure 8. This behavior can also be understood in terms of contraction mappings. This time we take as an example $I = \{x: x \le 2\}$. If we iterate f, letting $f_1(x) = f(x)$, $f_2(x) = f(f_1(x))$, $f_3(x) = f(f_2(x)), \ldots$, as discussed above, then it turns out that f_4 is a contraction on I, and in fact for $x, t \le 2$, it can be shown that

$$|f_4(x) - f_4(t)| \le .9|x - t|.$$

Thus, there is a unique fixed point for $\frac{1}{4}e^x$ in I and, since we have already found one (approximately), the Picard algorithm cannot yield the fixed point to the right of $x = 2$.

In addition, if we try to find the second solution of the equation $\frac{1}{4}e^x = x$ by choosing x_0 to the right of this root (try it for yourself, with $x_0 = 3$, for example), the sequence of iterates diverges, increasing without bound exponentially. This behavior, too, is indicated in Figure 8.

Thus, if we apply the Picard algorithm to any value x_0 (other than the second root itself exactly!), the sequence of iterates will not converge to the second, or larger, root of $\frac{1}{4}e^x = x$. However, we can find this second

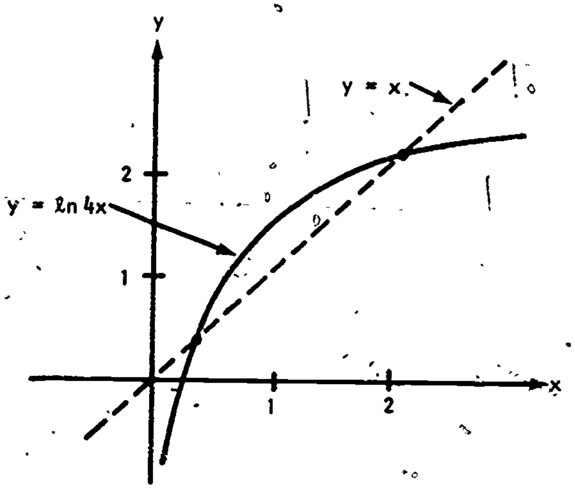

Figure 9. The graph of $y = \ln 4x$.

root by applying the Picard algorithm to the inverse function, $\ln 4x$, as described on pp. 11-12. For example, if we take $I = \{x: x \ge 1.4\}$ (1.4 is just a little larger than $\ln 4$, where the slope of $\frac{1}{4}e^x$ is 1), then f is an expansion on I, and the set of values assumed by $\frac{1}{4}e^x$ for x in I contains I. Thus, the inverse should be a contraction, and is, as can be seen in Figure 9.

Now if we take $I = \{x: x \ge 1.5\}$, for example, then $\ln 4x$ is a contraction on I, and we may solve the equation

$$\ln 4x = x$$

by the Picard algorithm, starting at any point to the right of $x = 1.5$. If we start at $x = 2$, we obtain the sequence 2, 2.0794, 2.1184, 2.1370, 2.1457, 2.1497, 2.1516, 2.1525, 2.1529, 2.1531, 2.1532, 2.1533, 2.1533, ... Thus,

$$\ln[4(2.1533)] \doteq 2.1533,$$

so that

$$e^{2.1533} \doteq 4(2.1533),$$

$$\frac{1}{4}e^{2.1533} \doteq 2.1533.$$

Thus, the larger fixed point of $\frac{1}{4}e^x$ is approximately 2.1533.

Note that in choosing a value for the left endpoint of the interval I indicated above, it was not really necessary to ensure that $\ln 4x$ would be a contraction on I. We chose 1.5, which is to the right of the point $x = 1$ where the curve $\ln 4x$ has slope one, and since the slope is decreasing the mean value theorem shows that $\ln 4x$ is a contraction on I for the choice we made. However, we could have chosen instead any value to the right of 0.3575, which is slightly larger than the first root of the equation $\ln 4x = x$. (Recall that our approximation for this root, which we obtained using the original equation

$\frac{1}{4}e^x = x$, was 0.3574 to four decimal places.) For example, if we had taken $I = \{x: x \geq .4\}$, then the iterate f_5 would be a contraction on I, so the iteration procedure would still yield the desired fixed point; this example illustrates the combining of the two extensions of the basic contraction mapping principle, as promised in the final sentence immediately preceding Example 8. (Try iterating the function $\ln 4x$ starting with $x_0 = .4$, and follow the iteration both on the calculator and geometrically on a graph of the function--you will see the points coming closer together starting with the fifth application of the function.)

Exercise 3

Solve the equation $\frac{1}{2}\cosh x = x$. (Hint: This problem is similar to the problem in Example 8. You may need to know that for $f(x) = \frac{1}{2}\cosh x$ the appropriate inverse function for this problem is given by the formula $f^{-1}(x) = \ln\,(2x + \sqrt{4x^2 - 1})$ in order to find the larger solution. On some calculators the inverse hyperbolic cosine is available directly.)

5. FURTHER EXAMPLES AND APPLICATIONS

5.1 Example 9

When a beam of light passes through a narrow slit, it spreads out in the shadow region. This effect is called *diffraction*, and a diagram representing a simple model, called Fraunhofer diffraction is shown in Figure 10.

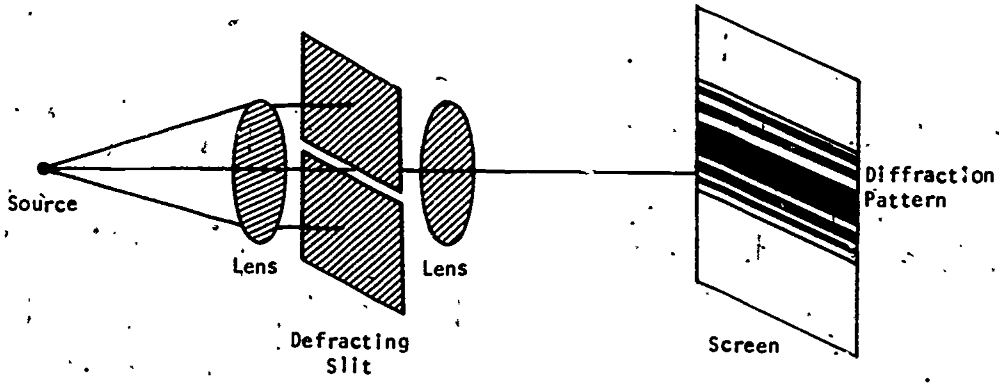

Figure 10. Experimental arrangement for obtaining the Fraunhofer diffraction pattern of a single slit.

By an application of basic principles of optics, it can be shown that the light intensities on the screen can be (approximately) expressed in the form

$$I = A_0^2 \frac{\sin^2\beta}{\beta^2},$$

where β is a suitably chosen spatial variable. (See D. Halliday and R. Resnick, Physics.) The quantity

$$A = A_0 \frac{\sin\beta}{\beta}$$

is called the *amplitude* of the vibration. A problem of interest in optics is to determine the location of the maximum intensities, thus we wish to optimize the function

$$A = A_0 \frac{\sin\beta}{\beta}.$$

We have

$$A' = A_0 \frac{\beta\cos\beta - \sin\beta}{\beta^2}$$

and we find initial points by setting $A' = 0$:

$$A_0 \frac{\beta\cos\beta - \sin\beta}{\beta^2} = 0.$$

The critical values can be obtained by solving the equation

$\tan\beta = \beta$, or $\tan x = x$.

The geometric picture is shown in Figure 11.

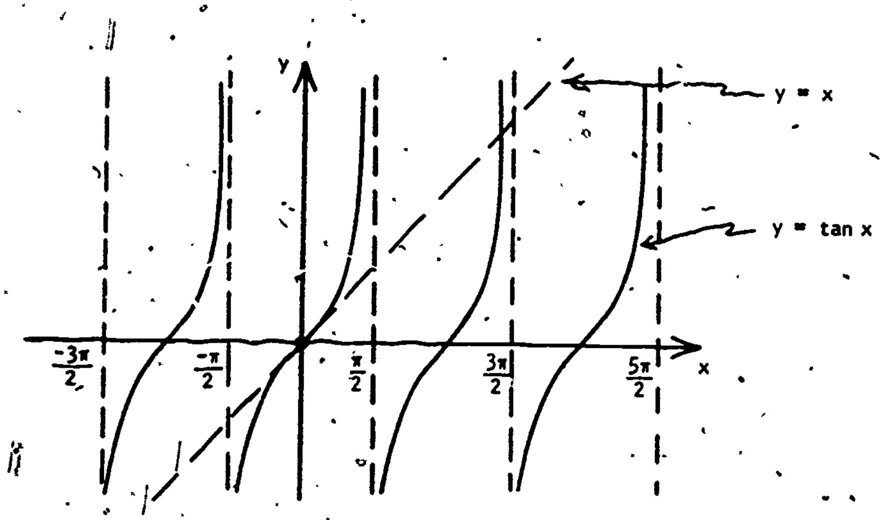

Figure 11. The graphs of $y = \tan x$ and $y = x$.

We see the first maximum intensity at $x = 0$. For larger values of x the fixed points will be near odd multiples of $\pi/2$, such as $7\pi/2$, $9\pi/2$, ... We find the location of the first nonzero intensity here, using the Picard algorithm applied to the function $y = \pi + \arctan x$, starting at $x_0 = 0$ (see Figure 12).

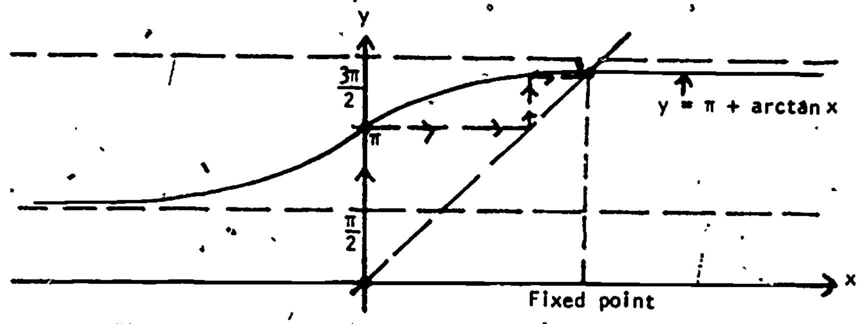

Figure 12. The Picard algorithm for finding the smallest positive fixed point of $\tan x$, using the inverse.

The resulting sequence of approximation is 0, 3.1416, 4.4042, 4.4891, 4.4932, 4.4934, 4.4934, ... The location of the first maximum intensity to the right 0 corresponds to $\beta \doteq 4.4934$.

Exercise 4

Find the fixed point of $\tan x$ that lies between $3\pi/2$ and $5\pi/2$ (i.e., the location of the second maximum intensity to the right of 0) in the Fraunhofer diffraction pattern.

Exercise 5*

In cavity resonators used in traveling-wave tubes the energy storage ratio W_1'/W_2 has an expression of the form

$$\frac{W_1}{W_2} = \frac{2\sin^2(\theta/2)}{\theta_t(\theta/2)} \quad (\theta \text{ in radians})$$

where θ_t is constant. Use the Picard algorithm to find the

---

* See H.M. Nodelman and F. W. Smith, Mathematics for Electronics.

maximum value of $\frac{\sin^2 x}{x}$ (and then multiply the critical value of x by 2) to find the value of θ that maximizes the energy storage ratio.

5.2 Example 10

In submarine location problems, it is often important to find the submarine's closest point of approach (CPA) to a sonobuoy in the water. Suppose that a sonobuoy is located at $(2, -\frac{1}{2})$ on a rectangular system and that a submarine travels on a parabolic path, along the curve $y = x^2$. (See Figure 13.) For any point (x,x^2) on the parabola, the distance to the sonobuoy is

$$D = [(x - 2)^2 + (x^2 + \tfrac{1}{2})^2]^{\frac{1}{2}}$$

$$= (x^4 + 2x^2 - 4x + \tfrac{17}{4})^{\frac{1}{2}}$$

We wish to find the critical x that minimizes D; this

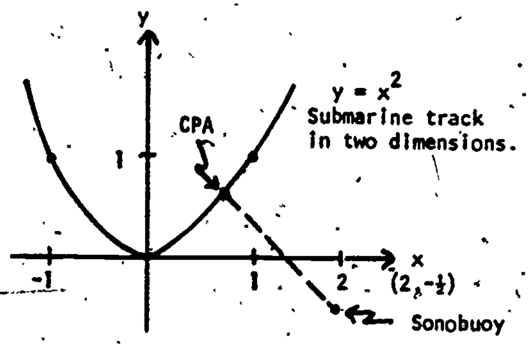

Figure 13. Diagram for Example 11.

value will also minimize D^2, and we proceed

$$\frac{d}{dx}(D^2) = 4x^3 + 4x - 4,$$

so we wish to solve the equation

$$x^3 + x - 1 = 0.$$

We can arrange this as a fixed point problem:

$$x^3 + x = 1$$

$$x(x^2 + 1) = 1$$

$$\frac{1}{1 + x^2} = x.$$

A rough sketch of the graph of $f(x) = \frac{1}{1 + x^2}$ (see Figure 14) shows that f is a concentration mapping on the whole line, so we implement the Picard algorithm, starting at $x_0 = 0$, to obtain the sequence 0, 1, .5,

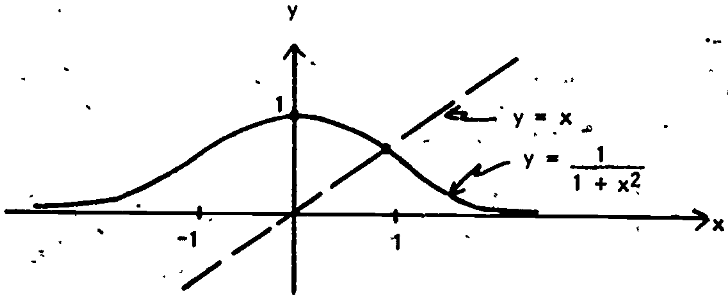

Figure 14. Graph of $y = \frac{1}{1 + x^2}$

.8, .6098, .7290, .6530, .7011, .6705, .6899, .6775, .6854, .6804, ..., .6823, .6823, ... Thus the CPA of the submarine to the sonobuoy is approximately the point $(.6823, (.6823)^2)$ or $(.6823, .4656)$.

We conclude this section with an example that is presented in Wylie's Advanced Engineering Mathematics.

5.3 Example 11

A slender rod of length L has its curved surface perfectly insulated against heat flow. The rod is located along the x-axis with its left end at $x = 0$, the right at $x = L$. The left end is maintained at temperature $u(0,t) = 0$ for all time $t \geq 0$, the right end radiates heat freely into air of constant temperature 0. If the initial temperature distribution in the rod is given by

$$u(x,0) = g(x),$$

the problem is to find the temperature $u(x,t)$ at any point $0 \leq x \leq L$ and $t \geq 0$. By an application of Stefan's law and a "separation of variables" argument, Wylie shows that the solution can be expressed in the form

$$u(x,t) = \sum_{n=1}^{\infty} B_n e^{-\lambda_n^2 t/a^2} \sin\lambda_n x. \tag{6}$$

What is important for us is the fact that the values λ_n are given by

$$\lambda_n = \frac{z_n}{L},$$

where the $z_1, z_2, \ldots$ are the positive roots of an equation of the form

$$\tan z = -\alpha z$$

where α is a positive contant (see Figure 15). For example, if $\alpha = .25$, then we must solve the equation of the form

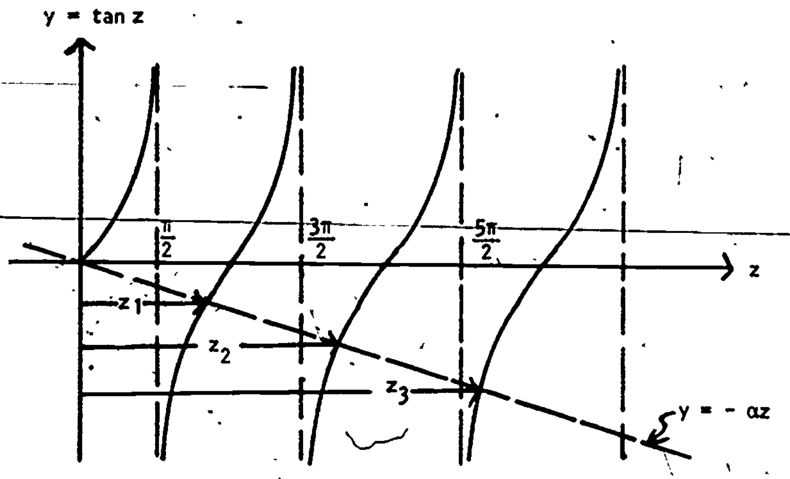

Figure 15. Sketch showing z_1, z_2, z_3 for Example 12.

$$-\frac{1}{.25}\tan x = x$$

in order to find the eigenvalues $\lambda_1, \lambda_2, \ldots$. We obtain the Picard sequence from the algorithm

$$y = \pi + \arctan(-.25x):$$

0, 3.1416, 2.4758, 2.5873, 2.5674, 2.5710, 2.5703, 2.5704, 2.5704, ... Thus $z_1 \doteq 2.5704$, and the first eigenvalue can be obtained from the relation

$$\lambda_1 = \frac{2.5704}{L}.$$

When we have found several values of λ_n, we may substitute them into Equation (6) and thereby find series approximations for the temperature distribution $u(x,t)$.

Exercise 6

Find the second root z_2 in Example 11 with $\alpha = .25$.

6. COMPARISON WITH NEWTON'S METHOD

There are many numerical methods of solving real equations of the form

$$g(x) = 0.$$

(This is equivalent to the problem of solving

$$f(x) = x,$$

since we may write the latter as

$$f(x) - x = 0$$

and simply let $g(x) = f(x) - x$.) One popular such method is called *Newton's Method*, which is carried out by choosing a starting point x_0 carefully and then applying the algorithm

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}.$$

Newton's method enjoys many advantages; for example, the resulting numerical sequence often converges faster than that obtained from the contraction mapping principle. There is rather general agreement that Newton's method is the superior of the two. (A comparison of these methods can be found in M.L. Dertouzos, et al., Systems, Networks, and Computation: Basic Concepts.)

If Newton's method *is* better, you may well ask, "Why bother with Picard at all?" There are two primary reasons. First, there is really not a dichotomy between these methods, i.e. you need not choose between them. Learn both! Because it is simpler and often easier to implement on a (programmable or nonprogrammable) calculator, the contraction mapping principle serves as an excellent introduction to numerical methods. The simplicity and ease of implementation, coupled with its range of applicability make the Picard

algorithm also a viable alternative to Newton's method in many problems.

The second reason for learning this method is that the contraction mapping principle can be formulated in more general terms to solve a much wider class of problems than real equations. For example, it can be used to solve systems of linear equations, nonlinear equations in higher dimensions, differential equations, and integral equations. (See, for example A. W. Naylor and G. R. Sell, Linear Operator Theory in Science and Engineering on differential equations; they also have a nice discussion of the relation of the principle to closed loop feedback systems.) Concrete applications on the real line serve as a solid foundation for exploring interesting applications in more advanced settings.

7. MODEL EXAM

1. Solve the following equations using the contraction mapping principle (Picard's algorithm):

   a. $e^{-x} = x$;

   b. $e^{-x^2/2} = x$;

   c. $\sin 2x = x$.

2. Use the contraction mapping principle, together with a little manipulation, to solve the equations:

   a. $\sin x = x^2$;

   b. $\cot x = x$ (find the solution that lies between 0 and π).

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8. ANSWERS TO EXERCISES

1. -2.0000.
2. 1.3720.
3. .5894; 2.1268.
4. .7253.
5. 2.3311.
6. 5.3540.

9. SOLUTIONS FOR MODEL EXAM

1. a. .5671;
   b. .7531;
   c. .9477.

2. a. Use $\frac{\sin x}{x} = x$; then $x = .8767$.
   b. Use $\arctan \frac{1}{x} = x$; then $x = .8603$.

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10. APPENDIX

Proof of the basic contraction mapping principle. For definiteness, suppose that I is an interval of the form $\{x: a \le x \le b\}$, that $f: I \to I$, and that there exists a positive constant such that for all x, t in I we have

$$|f(x) - f(t)| \le r|x - t|.$$

Let x_0 be any point in I, and let

$$x_1 = f(x_0), \quad x_2 = f(x_1), \quad x_3 = f(x_2) \ \ldots$$

Then,

$$|x_2 - x_1| = |f(x_1) - f(x_0)| \le r|x_1 - x_0|,$$
$$|x_3 - x_2| = |f(x_2) - f(x_1)| \le r|x_2 - x_1| \le r^2|x_1 - x_0|,$$
$$|x_4 - x_3| = |f(x_3) - f(x_2)| \le r|x_3 - x_2| \le r^3|x_1 - x_0|,$$
$$\ldots$$

It can be shown by mathematical induction that for every n,

$$|x_{n+1} - x_n| \le r^n|x_1 - x_0|.$$

Thus, for all positive integers n and k we have

$$\begin{aligned} |x_{n+k} - x_n| &\le |x_{n+1} - x_n| + \ldots + |x_{n+k} - x_{n+k-1}| \\ &\le (r^n + \ldots + r^{n+k-1})\,|x_1 - x_0| . \\ &\le |x_1 - x_0| \sum_{i=n}^{n+k-1} r^i \\ &\le |x_1 - x_0|(S_{n+k-1} - S_{n-1}) \end{aligned}$$

where $\{S_m\}_{m=0}^{\infty}$ is the sequence of partial sums for the series

$$\sum_{i=0}^{\infty} r^i .$$

Since this sequence is a Cauchy sequence (remember $0 < r < 1$, so the geometric series converges), it follows that the sequence

$$\{x_n\}_{n=1}^{\infty}$$

is also Cauchy, and hence converges. Let

$$\lim_{n\to\infty} x_n = x.$$

Now the condition that f be a contraction also implies that f is continuous (for a given $\varepsilon > 0$ take $\delta = \varepsilon$ in applying the definition of continuity). Therefore,

$$f(x) = f(\lim_{n\to\infty} x_n) = \lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} x_{n+1} = x,$$

so x is a fixed point of f. Moreover the fixed point of f (in I) is unique, because if x' is any fixed point of f in I, then

$$|x - x'| = |f(x) - f(x')| \le r|x - x'|,$$

from which

$$(1 - r)|x - x'| \le 0.$$

But $1 - r > 0$, and since $|x - x'| \ge 0$, we must have

$$0 \le (1 - r)|x - x'| \le 0,$$

from which

$$(1 - r)|x - x'| = 0$$

and finally

$$|x - x'| = 0$$

so that $x = x'$, and the fixed point is unique.

STUDENT FORM 1

Request for Help

Return to:
EDC/UMAP
55 Chapel St.
Newton, MA 02160

Student: If you have trouble with a specific part of this unit, please fill out this form and take it to your instructor for assistance. The information you give will help the author to revise the unit.

Your Name_______________________ Unit No.________

Page________ O Upper O Middle O Lower	OR	Section________ Paragraph________	OR	Model Exam Problem No.______ Text Problem No.______

Description of Difficulty: (Please be specific)

Instructor: Please indicate your resolution of the difficulty in this box.

O Corrected errors in materials. List corrections here:

O Gave student better explanation, example, or procedure than in unit. Give brief outline of your addition here:

O Assisted student in acquiring general learning and problem-solving skills (not using examples from this unit.)

Instructor's Signature_______________________

Please use reverse if necessary.

STUDENT FORM 2

Unit Questionnaire

Return to:
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Name____________________________Unit No.______________Date______________

Institution____________________________Course No.______________________

Check the choice for each question that comes closest to your personal opinion.

1. How useful was the amount of detail in the unit?

____Not enough detail to understand the unit
____Unit would have been clearer with more detail
____Appropriate amount of detail
____Unit was occasionally too detailed, but this was not distracting
____Too much detail; I was often distracted

2. How helpful were the problem answers?

____Sample solutions were too brief; I could not do the intermediate steps
____Sufficient information was given to solve the problems
____Sample solutions were too detailed; I didn't need them

3. Except for fulfilling the prerequisites, how much did you use other sources (for example, instructor, friends, or other books) in order to understand the unit?

____A Lot ____Somewhat ____A Little ____Not at all

4. How long was this unit in comparison to the amount of time you generally spend on a lesson (lecture and homework assignment) in a typical math or science course?

____Much Longer ____Somewhat Longer ____About the Same ____Somewhat Shorter ____Much Shorter

5. Were any of the following parts of the unit confusing or distracting? (Check as many as apply.)

____Prerequisites
____Statement of skills and concepts (objectives)
____Paragraph headings
____Examples
____Special Assistance Supplement (if present)
____Other, please explain____________________________________

6. Were any of the following parts of the unit particularly helpful? (Check as many as apply.)

____Prerequisites
____Statement of skills and concepts (objectives)
____Examples
____Problems
____Paragraph headings
____Table of Contents
____Special Assistance Supplement (if present)
____Other, please explain____________________________________

Please describe anything in the unit that you did not particularly like.

Please describe anything that you found particularly helpful. (Please use the back of this sheet if you need more space.)