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ABSTRACT

This module considers ordinary linear differential equations with constant coefficients. The "complex method" used to find solutions is discussed, with numerous examples. The unit includes both problem sets and an exam, with answers provided for both. (MP)

umap

UNIT 497

MODULES AND MONOGRAPHS IN UNDERGRADUATE
MATHEMATICS AND ITS APPLICATIONS PROJECT

A (NOT REALLY) COMPLEX METHOD FOR FINDING SOLUTIONS TO LINEAR DIFFERENTIAL EQUATIONS

by James W. Uebelacker

This module describes a method for finding a particular solution to a large class of ordinary linear differential equations with constant coefficients, which involves complex arithmetic. Furthermore, belying its name "complex method," it is relatively simple to use. The ability to evaluate and differentiate polynomials, and to manipulate complex numbers algebraically, is basically all that is required.

DIFFERENTIAL EQUATIONS

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SOLUTIONS TO LINEAR DIFFERENTIAL EQUATIONS

by

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Intermodular Description Sheet: UMAP Unit 497

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Classification: DIFF EQUATIONS

Prerequisite Skills:
1. Ability to identify linear differential equations with constant coefficients.
2. Familiarity with the auxiliary equation (which involves the characteristic polynomial) and its roots for these equations.
3. Recognition that the general solution to a nonhomogeneous linear differential equation is the sum of the general solution to the associated homogeneous equation and any one particular solution.

Output Skills:
1. Ability to understand and use the complex method to find a particular solution to linear differential equations with constant coefficients for which the nonhomogeneous part is $e^{\alpha t} \cos \beta t$, $e^{\alpha t} \sin \beta t$, or a linear combination of these functions.

The Project would like to thank the members of UMAP's Analysis and Computation Panel for their reviews, and all others who assisted in the production of this unit. Members of the Analysis and Computation Panel include: Carroll O. Wilde of the Naval Postgraduate School, Panel Chair; Richard J. Allen of St. Olaf College; Louis C. Barrett of Montana State University; G. Robert Blakley of Texas A & M University; Roy B. Leipnik of the University of California at Santa Barbara; and Maurice D. Weir of the Naval Postgraduate School.

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1. INTRODUCTION

This module considers ordinary linear differential equations with constant coefficients such as

$$y'' + y = 10e^{2t},$$

$$y''' - 7y'' + 16y' - 12y = 5e^{3t},$$

$$y''' - y'' = 3e^{t} + \sin t.$$

The standard procedure for finding the general solution to such equations involves two main steps.

a. First find the general solution to the associated homogeneous equation; the resulting solution is called the homogeneous or complementary solution to the original differential equation. For equations with constant coefficients, the roots of the associated characteristic polynomial may be used to find linearly independent solutions involving polynomial functions, exponential functions and/or sine and cosine functions.

b. The second step is to find a particular solution to the given equation. The methods of variation of parameters and undetermined coefficients are most commonly used. While the former method is more generally applicable, the indefinite integration involved may present severe or unconquerable difficulties. While the latter method applies to a smaller class of equations, it is somewhat more tractable because it involves differentiating elementary functions and solving systems of linear algebraic equations.

1.1 Purpose of this Module

This module describes a third method for finding a particular solution, which in general, involves complex arithmetic. Although the class of equations to which this method applies is smaller, it nonetheless contains functions that appear quite frequently. Furthermore, belying its name "complex method", it is relatively simple to use. The ability to evaluate and differentiate polynomials, and to manipulate complex numbers algebraically, is basically all that is required. The method is described in Sections 2, 3, and 4. The basic method used in all three sections is the same, although complex numbers appear only in Sections 3 and 4.

The key idea of the method as it appears in Section 3, namely, the solution technique which employs the substitution of a complex exponential function for a sinosoidal function, is presented in Ordinary Differential Equations, by Garrett Birkhoff and Gian-Carlo Rota, third edition, Wiley, 1978, on pp. 70-73. It is also illustrated in Advanced Engineering Mathematics, by Erwin Kreyszig, fourth edition, Wiley, 1979, on pp. 125-128.

I would like to express my appreciation to Professors Maurice D. Weir and G.R. Blakley for their detailed, extensive and helpful comments and especially to Professor Carroll O. Wilde for his encouragement and meticulous consideration in improving this manuscript.

1.2 Terminology

An ordinary linear differential equation of order n with constant coefficients has the general form:

$$\text{(1)}\quad p(D)y = (a_nD^n + a_{n-1}D^{n-1} + \ldots + a_2D^2 + a_1D + a_0)y = f(t),$$

where each a_i is a real constant, and $a_n \neq 0$. The function $f(t)$ is called the nonhomogeneous part of the differential equation. The auxiliary or characteristic polynomial associated with Eq. (1) is:

$$\text{(2)}\quad p(\lambda) = a_n\lambda^n + a_{n-1}\lambda^{n-1} + \ldots + a_2\lambda^2 + a_1\lambda + a_0.$$

The zeros of $p(\lambda)$ are called the eigenvalues or characteristic roots of the given differential equation.

The general solution to the associated homogeneous differential equation, $p(D)y = 0$, is called the complementary or homogeneous solution of Eq. (1), although, strictly speaking, it is not a solution of Eq. (1) at all. Any specific solution of Eq. (1) is called a particular solution.

2. THE COMPLEX METHOD

2.1 Discussion of the Method

The nonhomogeneous part of the first differential equation listed in the introduction,

$$y'' + y' = 10e^{2t},$$

is $f(t) = 10e^{2t}$. This observation suggests a multiple of e^{2t} as a solution since derivatives of this exponential function are also such multiples, so try $y = Ae^{2t}$. For this trial solution

$$y'' = 2^2Ae^{2t},$$

and so

$$y'' + y = 2^2Ae^{2t} + Ae^{2t} = (2^2+1)\,Ae^{2t} = 5Ae^{2t}.$$

Thus, we need $A = 2$, because $y'' + y = 5 \cdot 2e^{2t} = 10e^{2t}$, as desired.

Although this procedure yields the correct result, it does not explicitly reveal a relationship between the final coefficient 2 and the original coefficient 10. However, by trying the general expression

$$y = Ae^{\alpha t},$$

as a candidate for a particular solution, a different light is shed on the problem. In this case,

$$y'' = \alpha^2 Ae^{\alpha t},$$

and so

$$y'' + y = \alpha^2 Ae^{\alpha t} + Ae^{\alpha t} = (\alpha^2+1)\ Ae^{\alpha t}.$$

Note that $\alpha^2 + 1 = p(\alpha)$, where $p(\lambda)$ is the characteristic polynomial for the given differential equation, so the last equation becomes

$$y'' + y = p(\alpha)\ Ae^{\alpha t}.$$

Comparing this equation with the original equation, which has the form

$$y'' + y = Ke^{\alpha t},$$

where $K = 10$, and $\alpha = 2$, we see that

$$p(\alpha)\ A = K$$

or

$$A = \frac{1}{p(\alpha)}\ K.$$

Thus, the particular solution has the form

$$y = \frac{1}{p(\alpha)}\ Ke^{\alpha t}.$$

2.2 Theoretical Statement of the Method

This illustration leads to the first theorem, which generalizes the solution given above. Note however, that $p(\alpha)$ cannot be 0.

Theorem 1: Let

$$p(D)y = Ke^{\alpha t}$$

be a linear differential equation with constant coefficients. Let $p(\lambda)$ be its characteristic polynomial. If $p(\alpha) \neq 0$, then the function

$$y = \frac{1}{p(\alpha)}\ Ke^{\alpha t}$$

is a particular solution to the given equation.

Proof: To verify the proposed function

$$y = \frac{1}{p(\alpha)}\ Ke^{\alpha t},$$

first note that its first n derivatives are given by

$$a_i D^i y = a_i \frac{1}{p(\alpha)}\ K\alpha^i e^{\alpha t},$$

where $i = 1, 2, \ldots, n$.

Then substitute y and its derivatives into the left-hand side of the given equation and simplify:

$$p(D)y = p(D)\ \left[\frac{1}{p(\alpha)}\ Ke^{\alpha t}\right]$$

$$= a_n D^n \left[\frac{1}{p(\alpha)}\ Ke^{\alpha t}\right] + a_{n-1} D^{n-1} \left[\frac{1}{p(\alpha)}\ Ke^{\alpha t}\right] + \ldots$$

$$+ a_1 D \left[\frac{1}{p(\alpha)}\ Ke^{\alpha t}\right] + a_0 \frac{1}{p(\alpha)}\ Ke^{\alpha t}$$

$$= a_n \frac{1}{p(\alpha)}\ K\alpha^n e^{\alpha t} + a_{n-1} \frac{1}{p(\alpha)}\ K\alpha^{n-1} e^{\alpha t} + \ldots$$

$$+ a_1 \frac{1}{p(\alpha)}\ K\alpha e^{\alpha t} + a_0 \frac{1}{p(\alpha)}\ Ke^{\alpha t}$$

$$= \frac{1}{p(\alpha)}\ Ke^{\alpha t}\ (a_n\alpha^n + a_{n-1}\alpha^{n-1} + \ldots + a_1\alpha + a_0)$$

$$= \frac{1}{p(\alpha)}\ Ke^{\alpha t} \cdot p(\alpha)$$

$$= Ke^{\alpha t}.$$

Thus, if $p(\alpha) \neq 0$, then the function

$$y = \frac{1}{p(\alpha)}\ Ke^{\alpha t}$$

does indeed satisfy the differential equation

$$p(D)y = Ke^{\alpha t}.$$

Before proceeding, use this theorem to find particular solutions to the following differential equations.

Exercise 1. $y'' - y = -6e^{2t}$.

Exercise 2. $3y'' + 4y' - 2y = 4e^{-t}$.

Theorem 1 can be extended to handle equations for which $p(\alpha) = 0$.

Theorem 2: Consider the same differential equation as in Theorem 1,

$$p(D)y = Ke^{\alpha t}.$$

If $p(\alpha) = 0$ and $p'(\alpha) \neq 0$, then the function

$$y = \frac{1}{p'(\alpha)} Kte^{\alpha t}$$

is a particular solution. More generally, if m is an integer such that $m \leq n$, $p^{(m)}(\alpha) \neq 0$ and $p^{(m-1)}(\alpha) = p^{(m-2)}(\alpha) = \ldots p'(\alpha) = p(\alpha) = 0$, then the function

$$y = \frac{1}{p^{(m)}(\alpha)} Kt^m e^{\alpha t}$$

is a particular solution to the given equation.

Proof for the case $p(\alpha) = 0$, $p'(\alpha) \neq 0$.

We verify the proposed solution as we did in Theorem 1, by substituting the function and its derivatives into the original equation. We have

$$a_0 y = a_0 \frac{1}{p'(\alpha)} Kte^{\alpha t},$$

and

$$a_i D^i y = a_i \frac{1}{p'(\alpha)} Ki\alpha^{i-1} e^{\alpha t} + a_i \frac{1}{p'(\alpha)} K\alpha^i te^{\alpha t}$$

for $i = 1, 2, \ldots, n$.

Upon substitution of these expressions into the formula for $p(D)y$, the resulting terms can be rearranged into the two natural groups suggested by the terms of each $a_i D^i y$. The two groups can be summed individually to yield

$$\frac{1}{p'(\alpha)} Ke^{\alpha t}(a_1 + 2a_2\alpha + 3a_3\alpha^2 + \ldots + na_n\alpha^{n-1})$$

$$= \frac{1}{p'(\alpha)} Ke^{\alpha t}(p'(\alpha)),$$

and

$$\frac{1}{p'(\alpha)} Kte^{\alpha t}(a_0 + a_1\alpha + a_2\alpha^2 + \ldots + a_n\alpha^n)$$

$$= \frac{1}{p'(\alpha)} Kte^{\alpha t}(p(\alpha)).$$

Since $p(\alpha) = 0$, the overall sum for $p(D)y$ reduces to

$$\frac{1}{p'(\alpha)} Ke^{\alpha t}(p'(\alpha)) = Ke^{\alpha t}.$$

Therefore, the function

$$y = \frac{1}{p'(\alpha)} Kte^{\alpha t}$$

satisfies the given equation, and hence is a particular solution.

The reason for the name "complex method" is that Theorems 1 and 2 remain valid when y is replaced by a complex function z of the form

$$z(t) = u(t) + i\,v(t)$$

and α by a complex number of the form $a + ib$, and K is allowed to represent a complex number. The derivative of the function z is defined by the natural relation

$$z'(t) = u'(t) + i\,v'(t).$$

Using this definition and the Euler identity (see Section 3), we can show that complex exponential functions obey the usual rule for real functions:

$$\frac{d}{dt} e^{(a+ib)t} = (a+ib)e^{(a+ib)t}.$$

The proof is straightforward, involving only differentiation of products of real exponential and trigonometric functions, but we omit the details here.

In Section 2 we consider the case in which all functions and constants are real. Applications of Theorems 1 and 2 in the complex case are given in Section 3.

Exercise 3. Modify the proof given for the case $p(\alpha) = 0$ and $p'(\alpha) = 0$ to obtain a proof for the case $p'(\alpha) = p(\alpha) = 0$, $p''(\alpha) \neq 0$. Compare your work with the outline of the proof for the general case, which is given in the Appendix.

2.3 Illustrations

Now consider some applications of Theorem 2. If $Ke^{\alpha t}$ is the nonhomogeneous part, calculate $p(\alpha)$, $p'(\alpha)$, $p''(\alpha)$, ... until the first nonzero number is obtained. Then find the appropriate solution, as illustrated in the following examples.

Example 1: $y''' - 7y'' + 16y' - 12y = 5e^{3t}$.

The characteristic polynomial $p(\lambda)$ is given by:

$$p(\lambda) = \lambda^3 - 7\lambda^2 + 16\lambda - 12 = (\lambda-2)^2(\lambda-3),$$

with

$$p'(\lambda) = 3\lambda^2 - 14\lambda + 16$$

and

$$p''(\lambda) = 6\lambda - 14.$$

Since $5e^{3t} = Ke^{\alpha t}$, we have $K = 5$ and $\alpha = 3$. Now $p(3) = 0$ and $p'(3) = 3(9) - 14(3) + 16 = 1 \neq 0$, so by Theorem 2 the function

$$y = \frac{1}{p'(3)}\ 5te^{3t} = 5te^{3t}$$

is a particular solution.

Example 2: $y''' - 7y'' + 16y' - 12y = 5e^{2t}$.

Using the same auxiliary polynomial, we find that

$$p(2) = 0,\ p'(2) = 0, \text{ and } p''(2) = 12 - 14 = -2 \neq 0.$$

Thus, the function

$$y = \frac{1}{p''(2)}\ 5t^2e^{2t} = \frac{1}{-2}\ 5t^2e^{2t} = \frac{-5}{2}\ t^2e^{2t}$$

is a particular solution.

Exercise 4. Find a particular solution to each of the following differential equations:

a. $y'' - y = 6e^{-t}$

b. $y'' - 5y' + 6y = 5e^{-3t}$

c. $y'' - 5y' + 6y = 5e^{3t}$

d. $y''' - y'' = 2e^{t}$

e. $y''' - y'' = 2.$

Exercise 5. Construct a linear differential equation with constant coefficients such that

$$y = \frac{t^2e^{3t}}{8}$$

is a particular solution. (Hint: use properties of characteristic polynomials.)

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3. HOW TO HANDLE NONHOMOGENEOUS PARTS INVOLVING $K \cos \beta t$, $K \sin \beta t$

3.1 Euler's Identity

The complex method can be applied to find a particular solution when the nonhomogeneous part of the given differential equation is either $K \cos \beta t$ or $K \sin \beta t$. By Euler's identity,

$$e^{i\theta} = \cos \theta + i \sin \theta,$$

we have

$$Ke^{i\beta t} = K \cos \beta t + i\ (K \sin \beta t).$$

Since $K \cos \beta t$ is the real part and $K \sin \beta t$ the imaginary part of $Ke^{i\beta t}$, this exponential form can be used as a first step to obtain a complex function (one of the form $u_1(t) + i\ v_1(t)$). Then the real part $u_1(t)$ of this function is a particular solution to the original differential equation if $K \cos \beta t$ is the nonhomogeneous part of the given equation, and the imaginary part $v_1(t)$ is a particular solution if $K \sin \beta t$ is the given nonhomogeneous part.

3.2 Illustrations and Further Explanation

The following examples demonstrate this use of Euler's identity.

Example 3: $y'' + y = 3 \cos 2t.$

Here $K = 3$ and $\beta = 2$, so we form the corresponding complex differential equation

$$z'' + z = 3e^{i2t},$$

and solve it by applying Theorem 1. Using the notation of Theorem 1, we have $K = 3$, $\alpha = i2$, and $p(\lambda) = \lambda^2 + 1$. Thus

$$p(\alpha) = p(i2) = (i2)^2 + 1 = -4 + 1 = -3.$$

So by Theorem 1,

$$z = \frac{1}{-3} \cdot 3e^{i2t} = -e^{i2t}$$

is a solution to the complex equation we formed. Rewrite this solution using Euler's identity:

$$-e^{i2t} = -\cos 2t - i \sin 2t.$$

Since the nonhomogeneous part of the given differential equation is $3 \cos 2t$, the real part of the expression for $-e^{i2t}$ is taken as the solution of the given equation:

$$y = -\cos 2t.$$

(On the other hand, had the nonhomogeneous part been $3 \sin 2t$, the solution would have been $y = -\sin 2t$.)

To see why this use of Euler's identity is valid, note that we are starting with a differential equation of the form

$$p(D)y = \begin{cases} u(t) \\ \text{or} \\ v(t) \end{cases}$$

where $u(t) = K \cos \beta t$ and $v(t) = K \sin \beta t$. We then form the associated complex differential equation

$$p(D)\ z = u(t) + i\ v(t).$$

This equation may be solved using the complex method of Section 2 because

$$u(t) + i\ v(t) = K \cos \beta t + i\ K \sin \beta t = Ke^{i\beta t}$$

and because Theorems 1 and 2 remain valid when α represents any complex number. So if z_1 is a complex function that is a solution of the associated complex equation, then

$$p(D)z_1 = u(t) + i\ v(t).$$

If $z_1 = u_1(t) + i\ v_1(t)$, then the linearity allows us to write

$$p(D)z_1 = p(D)u_1 + i\ p(D)v_1.$$

By first equating these two expressions for $p(D)z_1$ and then equating the real and imaginary parts of the resulting equation, we obtain

$$p(D)u_1 = u(t)$$

and

$$p(D)v_1 = v(t).$$

Thus, the final step is to take as our solution either the real or the imaginary part of the solution function in z, whichever is appropriate.

The use of Theorem 2 is illustrated in the following example.

Example 4: $y'' + y = 3 \cos t$.

This equation is similar to the equation in Example 3, and the first step is to form the corresponding complex equation

$$z'' + z = 3e^{it}.$$

In the notation of Theorems 1 and 2 we have $\alpha = i$. Since

$$p(i) = i^2 + 1 = -1 + 1 = 0,$$

we must use Theorem 2, with $p'(\lambda) = 2\lambda$. Then

$$p'(i) = 2i,$$

and

$$z = \frac{3te^{it}}{2i} = -\frac{3}{2}\ ite^{it}$$

$$= -\frac{3}{2}\ it \cos t + \frac{3}{2} t \sin t.$$

Thus, the solution in y is the real part of z, namely,

$$y = \frac{3}{2}\ t \sin t.$$

In the next example we show how to handle equations for which $p(\alpha)$ is strictly complex, i.e., of the form $a + bi$, where $a \neq 0$ and $b \neq 0$.

Example 5: $y''' - y'' = \sin t$.

Here $K = 1$, $\beta = 1$, $p(\lambda) = \lambda^3 - \lambda^2$, and we form the complex differential equation

$$z''' - z'' = e^{it}.$$

Then $\alpha = i$, and

$$p(\alpha) = p(i) = i^3 - i^2 = -i + 1,$$

so

$$z = \frac{1}{1-i}\ e^{it}.$$

To use Euler's identity we must first rationalize the denominator as follows:

$$z = \frac{(1+i)}{(1+i)}\ \frac{1}{1-i}\ e^{it}$$

$$= \frac{1+i}{1+1}\ e^{it}$$

$$= \frac{1}{2}\ e^{it} + \frac{i}{2}\ e^{it}$$

$$= \left(\frac{1}{2} \cos t + \frac{1}{2}\ i \sin t\right) + \left(\frac{i}{2} \cos t + \frac{i}{2}\ i \sin t\right)$$

$$= \left(\frac{1}{2}\cos t - \frac{1}{2}\sin t\right) + i\left(\frac{1}{2}\cos t + \frac{1}{2}\sin t\right).$$

The imaginary part of z yields a particular solution to the given equation:

$$y = \frac{1}{2}\cos t + \frac{1}{2}\sin t.$$

Exercise 6. Convert each of the following expressions to the form $u(t) + i\,v(t)$.

a. $\dfrac{-5e^{2it}}{3+4i}$

b. $\dfrac{te^{2it}}{4i}$

Exercise 7. Find a particular solution to each of the following differential equations.

a. $y'' - 2y' + y = 5\sin 2t$

b. $y'' - 2y' + y = 3\cos 2t$

c. $y'' + 4y = \cos 2t.$

3.3 Further Extension

If the nonhomogeneous part of the given equation is of the form $Ke^{\alpha t}\cos \beta t$ or $Ke^{\alpha t}\sin \beta t$, we also can find a particular solution using the complex method. Note that

$$Ke^{\alpha t}\cos \beta t + iKe^{\alpha t}\sin \beta t = Ke^{\alpha t}e^{i\beta t} = Ke^{(\alpha + i\beta)t}.$$

Thus, $Ke^{\alpha t}\cos \beta t$ is the real part and $Ke^{\alpha t}\sin \beta t$ the imaginary part of the complex function $Ke^{(\alpha+\beta i)t}$. The following example illustrates the way we can use this observation by building on our previous technique.

Example 6: $y'' - y = e^{2t}\cos t.$

Here $p(\lambda) = \lambda^2 - 1$, $\alpha = 2$, $\beta = 1$, and $K = 1$. We first form the complex equation

$$z'' - z = e^{(2+i)t}.$$

We have

$$p(2+i) = (2+i)^2 - 1 = (4+4i-1) - 1 = 2 + 4i,$$

from which we obtain the complex solution

$$z = \frac{1}{2+4i}\,e^{(2+i)t}.$$

Now express z in terms of its real and imaginary parts by first rationalizing the demoninator and then applying Euler's identity:

$$z = \frac{1}{2+4i}\cdot\frac{2-4i}{2-4i}\,e^{(2+i)t}$$

$$= \frac{(2-4i)e^{(2+i)t}}{20}$$

$$= \frac{e^{2t}}{10}(1-2i)(\cos t + i\sin t)$$

$$= \frac{e^{2t}}{10}(\cos t + i\sin t - 2i\cos t + 2\sin t)$$

$$= \left(\frac{1}{5}e^{2t}\sin t + \frac{1}{10}e^{2t}\cos t\right)$$

$$+ i\left(-\frac{1}{5}e^{2t}\cos t + e^{2t}\sin t\right).$$

Since the real part is desired, we have

$$y = \frac{1}{5}e^{2t}\sin t + \frac{1}{10}e^{2t}\cos t.$$

Exercise 8. Find a particular solution to each of the following differential equations:

a. $y'' - y = e^{t}\sin 2t$

b. $y''' + 2y' - 4y = e^{-t}\cos t$

c. $y''' + 2y' = e^{-t}\sin t.$

Exercise 9. Consider the equation

$$a_n y^{(n)} + a_{n-1}y^{(n-1)} + \dots + a_1 y' + a_0 y = K,$$

where K is a constant. Use Theorems 1 and 2 to show that for some integer m such that $0 \le m \le n$ we have

$$a_0 = a_1 = \dots = a_{m-1} = 0 \text{ and } a_m \neq 0;$$

and the function

$$y = \frac{Kt^m}{m!\,a_m}$$

is a particular solution of the given equation.

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4. A FINAL EXTENSION OF THE COMPLEX METHOD

4.1 The Nonhomogeneous Part as a Linear Combination

Our final extension of the complex method obtains when the nonhomogeneous part is of the form

$$f(t) = \sum_{j=1}^{m} K_j f_j(t),$$

where K_j is a constant, and $f_j(t)$ is a function to which the method already applies. Examples include

$$e^t + 2\cos t - e^{2t}\sin 3t ,$$

$$2 + e^{2t} - 3\sin 4t + \cos 2t .$$

This extension is a direct consequence of the following general result, which is often called the "superposition principle".

Theorem 3: If y_1 is a solution to the differential equation

$$p(D)y = f(t),$$

y_2 is a solution to the equation

$$p(D)y = g(t),$$

and a and b are constants, then the function $ay_1 + by_2$ is a solution to the equation

$$p(D)y = af(t) + bg(t).$$

Proof: By linearity of differentiation,

$$p(D)(ay_1 + by_2) = ap(D)y_1 + bp(D)y_2 .$$

But by hypothesis,

$$p(D)y_1 = f(t) \text{ and } p(D)y_2 = g(t) .$$

Hence,

$$p(D)(ay_1 + by_2) = af(t) + bg(t) ,$$

which means that $ay_1 + by_2$ satisfies the required equation.

Example 7: $y''' - y'' = 3e^t + \sin t$.

Note that

$$p(\lambda) = \lambda^3 - \lambda^2, \; p'(\lambda) = 3\lambda^2 - 2\lambda .$$

a) First find a particular solution to the equation

$$y''' - y'' = 3e^t .$$

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With

$$K_1 = 3, \; \alpha = 1, \; p(1) = 1 - 1 = 0, \; p'(1) = 3 - 2 = 1,$$

we have

$$y_1 = \frac{3}{1} te^t = 3te^t .$$

b) Next, find a particular solution to the equation

$$y''' - y'' = \sin t .$$

This equation was solved in Example 4, and we have

$$y_2 = \frac{1}{2}\cos t + \frac{1}{2}\sin t .$$

c) Then superpose the solutions from a) and b):

$$y = y_1 + y_2 + 3te^t + \frac{1}{2}\cos t + \frac{1}{2}\sin t .$$

Example 8: $y''' - 6y'' + 11y' - 6y = -7e^{2t} + 2e^{3t}\sin t$.

We have

$$p(\lambda) = \lambda^3 - 6\lambda^2 + 11\lambda - 6$$

and

$$p'(\lambda) = 3\lambda^2 - 12\lambda + 11$$

a) Consider

$$y''' - 6y'' - 11y' - 6y = -7e^{2t}.$$

Since

$$K_1 = -7; \quad \alpha = 2, \; p(2) = 0, \text{ and } p'(2) = -1,$$

we have

$$y_1 = \frac{-7}{-1} te^{2t} = 7te^{2t}.$$

b) Next, let

$$z''' - 6z'' + 11z' - 6z = 2e^{(3+i)t}.$$

Since

$$K_2 = 2, \; \alpha = 3 + i, \text{ and } p(3+i) = i - 3 ,$$

we have

$$z = \frac{2}{i-3} e^{(3+i)t} = \frac{2}{(i-3)} \frac{(i+3)}{(i+3)} e^{(3+i)t}$$

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$$= \left(-\frac{1}{5} i - \frac{3}{5}\right)\left(e^{3t} \cos t + i\, e^{3t} \sin t\right)$$

$$= \left(-\frac{3}{5} e^{3t} \cos t + \frac{1}{5} e^{3t} \sin t\right)$$

$$+ i \left(-\frac{1}{5} e^{3t} \cos t - \frac{3}{5} e^{3t} \sin t\right).$$

Thus,

$$y_2 = -\frac{1}{5} e^{3t} \cos t - \frac{3}{5} e^{3t} \sin t.$$

c) By superposition,

$$y = y_1 + y_2 = 7te^{2t} - \frac{1}{5} e^{3t} \cos t - \frac{3}{5} e^{3t} \sin t.$$

Exercise 10. Find a particular solution to each of the following differential equations.

a. $y'' - 5y' + 6y = 5e^{-3t} + 5e^{3t}$

b. $y'' - 2y' + y = 5 \sin 2t - 3e^{t}$

c. $y''' - 9y'' + 27y' - 27y = 4e^{3t} + e^{4t} \cos(-t)$

d. $y^{(4)} + 8y'' + 16 = -\sin t + \cos 2t + e^{t}$

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5. UNIT EXAM

Find a particular solution to each of the following differential equations:

1. $y'' + 16y = \sin 4t$
2. $y'' - 4y' + 5y = 3 \sin t$
3. $y'' - 3y' + 4y = \cos 2t + 3e^{-4t}$
4. $y''' - y'' = 3$
5. $y'' + 2y' - 3y = 2e^{-3t} \sin 4t$
6. $y''' + y'' - 8y' - 12y = 3e^{-2t} - 5 \cos 3t$.

6. ANSWERS TO EXERCISES

1. $p(\lambda) = \lambda^2 - 1$, $K = -6$, $\alpha = 2$.

Thus,

$$p(\alpha) = p(2) = 2^2 - 1 = 3,$$

and so

$$y = \frac{Ke^{\alpha t}}{p(\alpha)} = \frac{-6e^{2t}}{3} = -2e^{2t}.$$

2. $K = 4$, $\alpha = -1$, $p(-1) = -3$;

$$y = \frac{4e^{-t}}{-3}.$$

4.a. $K = 6$, $\alpha = -1$, $p(-1) = 0$, $p'(-1) = -2$;

$$y = \frac{6te^{-t}}{-2} = -3te^{-t}.$$

b. $K = 5$, $\alpha = -3$, $p(\lambda) = \lambda^2 - 5\lambda + 6$,

so $p(\alpha) = p(-3) = 30$ and $y = \frac{5e^{-3t}}{30} = \frac{1}{6} e^{-3t}$.

c. $p(3) = 0$, and $p'(3) = 1$,

so $y = \frac{5te^{3t}}{1} = 5te^{3t}$.

d. $y = 2te^{t}$.

e. $y = -t^2$.

5. We compare the given expression with the general form

$$\frac{Kt^2 e^{\alpha t}}{p''(\alpha)}.$$

By Theorem 2, with $\alpha = 3$ and $m = 2$, we see that $p(\lambda) = (\lambda-3)^2$ is a reasonable choice for the characteristic polynomial. Then, $p''(\lambda) = 2$, and so $K = 1/4$. Thus, the given function is a solution of the equation $y'' - 6y' + 9y = 1/4\, e^{3t}$.

6.a. $\left(\frac{-3}{5} \cos 2t - \frac{4}{5} \sin 2t\right) + i\left(\frac{4}{5} \cos 2t - \frac{3}{5} \sin 2t\right)$

b. $\frac{1}{4} t \sin 2t + i\left(-\frac{1}{4} t \cos 2t\right)$.

7.a. Solve $z'' - 2z' + z = 5e^{2it}$ to obtain

$$z = \frac{-5e^{2it}}{3+4i} \text{ and hence } y = \frac{4}{5} \cos 2t - \frac{3}{5} \sin 2t.$$

b. $y = -\frac{9}{25} \cos 2t - \frac{12}{25} \sin 2t$

c. $y = \frac{1}{4} t \sin 2t$.

8.a. $p(\lambda) = \lambda^2 - 1$.

Solve $z'' - z = e^{(1+2i)t}$:

$K = 1$, $\alpha = 1 + 2i$, and

$$\begin{aligned} p(1+2i) &= (1+2i)^2 - 1 \\ &= 4i - 3 - 1 \\ &= 4i - 4. \end{aligned}$$

Thus,

$$\begin{aligned} z &= \frac{e^{(1+2i)t}}{4i-4} = \frac{e^{(1+2i)t}}{4(i-1)} \frac{(i+1)}{(i+1)} \\ &= \frac{e^{(1+2i)t}(i+1)}{-8} \\ &= -\frac{1}{8} e^t \cos 2t + \frac{1}{8} e^t \sin 2t \\ &\qquad + i\left(-\frac{1}{8} e^t \cos 2t - \frac{1}{8} e^t \sin 2t\right). \end{aligned}$$

We have

$$y = -\frac{1}{8} e^t \cos 2t - \frac{1}{8} e^t \sin 2t.$$

b. $y = \frac{1}{8} e^{-t} \sin t - \frac{1}{8} e^{-t} \cos t$

c. $y = -\frac{1}{4} e^{-t} \cos t$.

9. Note that $K = Ke^{0t}$, so let $\alpha = 0$ in Theorem 2. An application of Theorem 2 then yields the solution

$$y = \frac{Kt^m e^{0t}}{p^{(m)}(0)} = \frac{Kt^m}{p^{(m)}(0)}$$

if $p^{(m)}(0) \neq 0$, and $p^{(m-1)}(0) = 0 = \ldots = p'(0) = p(0)$. Since $q(0)$ is the constant term for any polynomial $q(t)$, the condition on the values at 0 forces $a_{m-1} = 0 = \ldots = a_1 = a_0$. Thus, $p(\lambda) = a_n\lambda^n + a_{n-1}\lambda^{n-1} + \ldots + a_m\lambda^m$, and hence the constant term of $p^{(k)}(\lambda)$ is 0, for $0 \leq k < m$, and the constant term of $p^{(m)}(\lambda) = m(m-1)(m-2) \ldots (2)(1)a_m = m!\, a_m$. Therefore,

$$y = \frac{Kt^m}{m!\,a_m}.$$

10.a. $y = \frac{1}{6} e^{-3t} + 5te^{3t}$,

since

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$$y_1 = \frac{1}{6} e^{-3t} \quad \text{and} \quad y_2 = 5te^{3t}.$$

b. $y = \frac{4}{5} \cos 2t - \frac{3}{5} \sin 2t - \frac{3}{2} t^2 e^t$.

c. $y = \frac{2}{3} t^3 e^{3t} - \frac{1}{4} e^{4t} \cos(-t) - \frac{1}{4} e^{4t} \sin(-t)$.

d. $y = \frac{1}{9} \sin t - \frac{1}{32} t^2 \cos 2t + \frac{1}{25} e^t$.

7. ANSWERS TO UNIT EXAM

5.1 $y = -\frac{1}{8} t \cos 4t$.

5.2 $y = \frac{3}{8} \cos t + \frac{3}{8} \sin t$.

5.3 $y = -\frac{1}{6} \sin 2t - \frac{3}{32} e^{-4t}$.

5.4 $y = -\frac{3}{2} t^2$.

5.5 $y = \frac{1}{16} e^{-3t} \cos 4t - \frac{1}{16} e^{-3t} \sin 4t$.

5.6 $y = \frac{35}{1014} \cos 3t + \frac{85}{1014} \sin 3t - \frac{3}{10} t^2 e^{-2t}$.

8. APPENDIX

Outline of the proof of Theorem 2 for the general case

$$p^{(m)}(\alpha) \neq 0,\ p^{(m-1)}(\alpha) = \ldots = p'(\alpha) = p(\alpha) = 0,\ 1 < m \leq n.$$

We wish to verify that

$$y = \frac{1}{p^{(m)}(\alpha)} Kt^m e^{\alpha t}$$

is a particular solution of the given equation. In a manner similar to the way in which we proved Theorem 2, we begin by finding the first n derivatives of y for subsequent substitution into the original differential equation. In these expressions,

$$\binom{m}{k}$$

denotes the binomial coefficient

$$\frac{m!}{k!(m-k)!}$$

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$$a_0 y = \frac{1}{p^{(m)}(\alpha)} K t^m e^{\alpha t}(a_0)$$

$$a_1 Dy = \frac{1}{p^{(m)}(\alpha)} K\binom{m}{1} t^{m-1} e^{\alpha t}(a_1) + \frac{1}{p^{(m)}(\alpha)} K t^m e^{\alpha t}(a_1\alpha)$$

$$a_2 D^2 y = \frac{1}{p^{(m)}(\alpha)} K\binom{m}{2} t^{m-2} e^{\alpha t}(2a_2) + \frac{1}{p^{(m)}(\alpha)} K\binom{m}{1} t^{m-1} e^{\alpha t}(2a_2\alpha)$$

$$+ \frac{1}{p^{(m)}(\alpha)} K t^m e^{\alpha t}(a_2\alpha^2)$$

$$a_n D^n y = \frac{1}{p^{(m)}(\alpha)} K e^{\alpha t} \frac{n!}{(n-m)!} a_n \alpha^{n-m} + \ldots$$

$$+ \frac{1}{p^{(m)}(\alpha)} K\binom{m}{2} t^{m-2} e^{\alpha t} n(n-1) a_n \alpha^{n-2}$$

$$+ \frac{1}{p^{(m)}(\alpha)} K\binom{m}{1} t^{m-1} e^{\alpha t} n a_n \alpha^{n-1} + \frac{1}{p^{(m)}(\alpha)} K t^m e^{\alpha t}(a_n \alpha^n) .$$

Substituting these derivatives into the original equation and rearranging into natural groups, we obtain

$$\frac{1}{p^{(m)}(\alpha)} K e^{\alpha t}(p^{(m)}(\alpha)) + \frac{1}{p^{(m)}(\alpha)} K\binom{m}{m-1} t e^{\alpha t}(p^{(m-1)}(\alpha))$$

$$+ \ldots + \frac{1}{p^{(m)}(\alpha)} K\binom{m}{2} t^{m-2} e^{\alpha t}(p''(\alpha))$$

$$+ \frac{1}{p^{(m)}(\alpha)} K\binom{m}{1} t^{m-1} e^{\alpha t}(p'(\alpha))$$

$$+ \frac{1}{p^{(m)}(\alpha)} K t^m e^{\alpha t}(p(\alpha)) .$$

Since the first term is $Ke^{\alpha t}$ and the rest are all 0, the proposed solution is verified.

STUDENT FORM 1

Request for Help

Return to:
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55 Chapel St.
Newton, MA 02160

Student: If you have trouble with a specific part of this unit, please fill out this form and take it to your instructor for assistance. The information you give will help the author to revise the unit.

Your Name____________________ Unit No.______

Page_____ ○ Upper ○ Middle ○ Lower	OR	Section_____ Paragraph_____	OR	Model Exam Problem No._____ Text Problem No._____

Description of Difficulty: (Please be specific)

Instructor: Please indicate your resolution of the difficulty in this box.

○ Corrected errors in materials. List corrections here:

○ Gave student better explanation, example, or procedure than in unit. Give brief outline of your addition here:

○ Assisted student in acquiring general learning and problem-solving skills (not using examples from this unit.)

Instructor's Signature____________________

Please use reverse if necessary.

STUDENT FORM 2

Unit Questionnaire

Return to:
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Name ____________ Unit No. ________ Date ________

Institution ____________ Course No. ____________

Check the choice for each question that comes closest to your personal opinion.

1. How useful was the amount of detail in the unit?

____Not enough detail to understand the unit
____Unit would have been clearer with more detail
____Appropriate amount of detail
____Unit was occasionally too detailed, but this was not distracting
____Too much detail; I was often distracted

2. How helpful were the problem answers?

____Sample solutions were too brief; I could not do the intermediate steps
____Sufficient information was given to solve the problems
____Sample solutions were too detailed; I didn't need them

3. Except for fulfilling the prerequisites, how much did you use other sources (for example, instructor, friends, or other books) in order to understand the unit?

____A Lot ____Somewhat ____A Little ____Not at all

4. How long was this unit in comparison to the amount of time you generally spend on a lesson (lecture and homework assignment) in a typical math or science course?

Much	Somewhat	About	Somewhat	Much
____Longer	____Longer	____the Same	____Shorter	____Shorter

5. Were any of the following parts of the unit confusing or distracting? (Check as many as apply.)

____Prerequisites
____Statement of skills and concepts (objectives)
____Paragraph headings
____Examples
____Special Assistance Supplement (if present)
____Other, please explain ____________

6. Were any of the following parts of the unit particularly helpful? (Check as many as apply.)

____Prerequisites
____Statement of skills and concepts (objectives)
____Examples
____Problems
____Paragraph headings
____Table of Contents
____Special Assistance Supplement (if present)
____Other, please explain ____________

Please describe anything in the unit that you did not particularly like.

Please describe anything that you found particularly helpful. (Please use the back of this sheet if you need more space.)