DOCUMENT RESUME

ED 218 081 SE 037 819

AUTHOR	Goldston, J. W.
TITLE	Solving Differential Equations Analytically. Elementary Differential Equations. Modules and Monographs in Undergraduate Mathematics and Its Applications Project. UMAP Unit 335.
INSTITUTION	Education Development Center, Inc., Newton, Mass.
SPONS AGENCY	National Science Foundation, Washington, D.C.
PUB DATE	79
GRANT	SED-76-19615-A02
NOTE	52p.
EDRS PRICE	MF01 Plus Postage. PC Not Available from EDRS.
DESCRIPTORS	Answer Keys; Biology; Chemistry; *College Mathematics; *College Science; Higher Education; Instructional Materials; *Learning Modules; Mathematical Applications; *Mathematical Concepts; *Problem Solving; *Supplementary Reading Materials; Tests
IDENTIFIERS	*Differential Equations

ABSTRACT

This unit introduces analytic solutions of ordinary differential equations. The objective is to enable the student to decide whether a given function solves a given differential equation. Examples of problems from biology and chemistry are covered. Problem sets, quizzes, and a model exam are included, and answers to all items are provided. The material is intended to amplify the examples of Units 81 through 83, titled "Graphical Solution of Differential Equations." Taken as a group, the four units are seen as a general introduction to a number of standard techniques for solving first-order ordinary differential equations. (MP)

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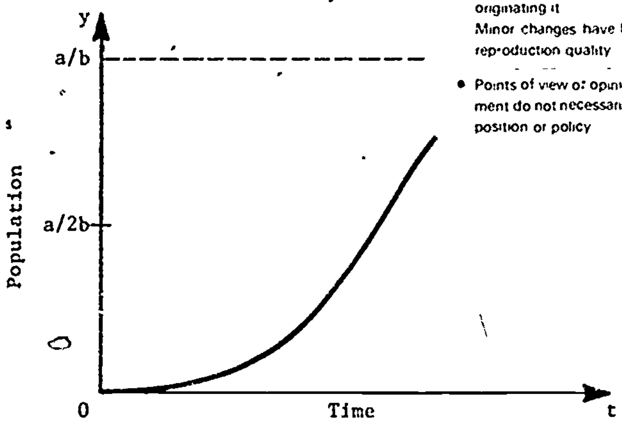
UNIT 335

MODULES AND MONOGRAPHS IN UNDERGRADUATE MATHEMATICS AND ITS APPLICATIONS PROJECT

SOLVING DIFFERENTIAL EQUATIONS ANALYTICALLY

by J.W. Goldston



ELEMENTARY DIFFERENTIAL EQUATIONS

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SOLVING DIFFERENTIAL EQUATIONS ANALYTICALLY

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Intermodular Description Sheet: UMAP Unit 86

Title: SOLVING DIFFERENTIAL EQUATIONS ANALYTICALLY

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Review Stage/Date: III 8/7/79

Classification: ELEM DIFF EQ

Prerequisite Skills:

1. To be able to differentiate and integrate simple rational, logarithmic, exponential and trigonometric functions.
2. UMAP Units 81-83.

Output Skills:

1. To be able to solve differential equations using antiderivatives, separation of variables and the general solution formula for first order linear differential equations.

Other Related Units:

Graphical and Numerical Solution of Differential Equations (Units 81-81)

MODULES AND MONOGRAPHS IN UNDERGRADUATE
MATHEMATICS AND ITS APPLICATIONS PROJECT (UMAP)

The goal of UMAP is to develop, through a community of users and developers, a system of instructional modules in undergraduate mathematics and its applications which may be used to supplement existing courses and from which complete courses may eventually be built.

The Project is guided by a National Steering Committee of mathematicians, scientists, and educators. UMAP is funded by a grant from the National Science Foundation to Education Development Center, Inc., a publicly supported, nonprofit corporation engaged in educational research in the U.S. and abroad.

The Project would like to thank Frances M. Thompson, Robert L. Baker, Jr., L.M. Larsen, and James Stasheff for their reviews, and all others who assisted in the production of this unit.

This material was prepared with the support of National Science Foundation Grant No. SED76-19615 A02. Recommendations expressed are those of the author and do not necessarily reflect the views of the NSF, nor of the National Steering Committee.

PREFACE

These units introduce analytic solutions of ordinary differential equations. They are intended to amplify the examples of Paul Calter's *Graphical and Numerical Solution of Differential Equations*, UMAP Units 81-83, and to provide, with Calter's units, a general introduction to a number of standard techniques for solving first-order ordinary differential equations. Examples have been included from physics, biology, and chemistry.

SOLVING DIFFERENTIAL EQUATIONS ANALYTICALLY

1. ANALYTIC SOLUTIONS

Objective: To be able to decide whether a given function solves a given differential equation.

The mathematical analysis of physical problems can lead to differential equations, as we saw in Units 81-83, and we often wish to gain information about the problems by examining the functions which satisfy these differential equations. For example, in the optical filter problem (Unit 81, Chapter 1) we want to know how many filters are needed to decrease a light's intensity to ten percent of its unfiltered intensity. Letting $L(t)$ represent the intensity when t filters are used, we would like to find the smallest t for which $L(t)$ is less than ten percent of the unfiltered intensity, $L(0)$. A graphical solution enabled us to find the required value of t. If we could find a formula that expressed $L(t)$ in terms of t, we might then find the same value of t by algebraic techniques alone. Thus, we are led to ask what functions $L(t)$ satisfy the differential equation

$$\frac{dL}{dt} = -kL(t),$$

where k is a positive constant.

Let us show that $L(t) = e^{-kt}$ is a solution of

$$(1) \qquad \frac{dL}{dt} = -kL.$$

To see this we differentiate $L(t)$, and find, indeed, that

$$\frac{dL}{dt} = \frac{d(e^{-kt})}{dt} = -ke^{-kt} = -kL(t).$$

Hence, $L(t) = e^{-kt}$ is a solution of (1).

If we are to profit from this solution, then we need to find k from our data. When t = 1 filter then the light intensity is eighty-five percent of $L(0) = 1$. In other words, $L(1) = 0.85L(0) = 0.85$. This means that

$$
\begin{aligned}
e^{-k(1)} &= 0.85, \\
\ln(e^{-k}) &= \ln(0.85), \\
-k &= \ln(0.85), \\
k &= -\ln(0.85).
\end{aligned}
\tag{2}
$$

Hence $k \approx 0.16$. This value of k is different from the value k that was obtained in the graphical solution in Unit 82. The graphical method is not as accurate as our analytic method because it uses an approximate tangent line. To find t so that the value of L(t) is no more than ten percent of $L(0) = 1$, we must find t so that

(3) $$L(t) = e^{-0.16t} \leq \frac{1}{10} \cdot 1.$$

Then,

$$
\begin{aligned}
\ln(e^{-0.16t}) &\leq \ln(\tfrac{1}{10}) \\
-0.16t &\leq -\ln(10) \\
-t &\leq -\frac{\ln(10)}{0.16} \\
t &\geq \frac{\ln(10)}{0.16} \approx 14.39.
\end{aligned}
\tag{4}
$$

This is in close agreement with the estimate of $t \geq 14.4$ that was obtained graphically. Both methods entail a certain amount of inaccuracy; the graphical method in using an approximate tangent and in reading the graph, the

analytic technique in using a log table which is accurate only to a certain number of digits.

We can see three important points from this example. First, if we can find a formula for a solution to a differential equation we can, many times, solve our original problem without turning to graphical or numerical techniques. We mention, however, that some problems can be solved *only* by using numerical techniques, so these techniques are important and necessary. But, if an analytic solution can be obtained, it is usually easier to work with. Second, to obtain an analytic solution we must either have enough knowledge to "guess" an answer, or we must develop techniques which will enable us to derive a solution. In the optical filter, we were able to guess the answer from known analytical and graphical properties of the exponential function. Third, we can always determine whether a guess is correct by seeing if it and its derivatives satisfy the equation. We give another example to amplify this last point.

Let us show that $y = \frac{cx^3}{6}$ is a solution for the differential equation

(5) $$\frac{d^2y}{dx^2} = \frac{d\left(\frac{dy}{dx}\right)}{dx} = cx.$$

Now, if

$$y = \frac{cx^3}{6},$$

then

$$\frac{dy}{dx} = \frac{3cx^2}{6} = \frac{cx^2}{2},$$

and

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{cx^2}{2}\right) = \frac{2cx}{2} = cx.$$

Hence, $y = cx^3/6$ satisfies Equation (5).

Notice that $y = \frac{cx^3}{6} + 1$ also satisfies Equation (5), for

$$\text{(6)}\qquad \begin{aligned}\frac{d^2y}{dx^2} &= \frac{d\left(\frac{cx^2}{2} + 0\right)}{dx}\\ &= \frac{d\left(\frac{cx^2}{2}\right)}{dx} = cx.\end{aligned}$$

Thus, Equation (5) has at least two solutions. In fact, it has an infinite number of solutions. (What are they?)

In general, when we are given a differential equation, our objective will be to find a class of solutions for it by using analytical techniques, i.e., methods of calculus.

Exercises

1. Show that $y = e^{kx}$ solves $\frac{dy}{dx} = ky$ where k is any constant.

2. Show that both $y = \sqrt{\frac{2}{3}}x^{\frac{3}{2}}$ and $y = -\sqrt{\frac{2}{3}}x^{\frac{3}{2}}$ are solutions for $\frac{dy}{dx} = \frac{x^2}{y}$.

3. Show that the functions

   $y(x) = \sin\sqrt{k}\,x$,

   $y(x) = \cos\sqrt{k}\,x$, and

   $y(x) = \sin\sqrt{k}\,x + \cos\sqrt{k}\,x$,

   all satisfy the differential equation

   $$\frac{d^2y}{dx^2} = -ky,$$

   where $k > 0$.

4. Show that the function $y(t) = (y_0-M)e^{-kt} + M$ satisfies the equation $\frac{dy}{dt} = -k(y-M)$, where $y_0 = y(0)$, and k and M are constants.

---

2. ANTIDERIVATIVES AND DIFFERENTIAL EQUATIONS

Objectives: (1) To use antiderivatives to solve some differential equations, and (2) to find the constants of integration using boundary conditions.

Many differential equations which arise naturally in physical sciences are easy to solve by finding antiderivatives. This should not be surprising, for differential equations result from combining derivatives of functions. Hence, we may hope to recover the functions by taking appropriate antiderivatives.

Consider a penny falling from the top of the Empire State Building. We measure time, t, in seconds, with $t = 0$ at the instant the penny is released, and let $s(t)$ be the height of the penny in feet *above the ground* at time t. Suppose the Empire State Building is h feet tall. Isaac Newton's second law of dynamics states that if we neglect the effect of the air on the penny then the acceleration of the penny is a constant, say k. Now, its acceleration is also the second derivative, $s''(t)$. Therefore,

$$\text{(7)}\qquad \frac{d^2s}{dt^2} = k.$$

To solve (7) analytically, we notice that $\frac{d^2s}{dt^2}$ is the derivative of $\frac{ds}{dt}$. So $\frac{ds}{dt}$ is the antiderivative of the constant k, in symbols

$$\text{(8)}\qquad \frac{ds}{dt} = \int \frac{d^2s}{dt^2}dt = \int k\,dt.$$

But

$$\int k\,dt = kt + c,$$

where c is some constant. So we have a new differential equation

(9) $\frac{ds}{dt} = kt + c.$

Before we solve (9), let us see if we can find a value for c. The first derivative, $\frac{ds}{dt}$, is the *velocity* of the penny at time t. When t = 0, at the start of the drop, the velocity will be 0, since the penny is stationary. Hence, when t = 0, we have $\frac{ds}{dt} = 0$. So, from (9),

$$0 = k(0) + c,$$

and therefore c = 0. Thus, (9) can be simplified to

(10) $\frac{ds}{dt} = kt.$

Let us reflect a minute on what we have done. We have solved the differential equation (7) by first taking the antiderivative which gave us a class of functions of the form kt + c. We then found that by using more information we could show that c = 0, i.e., that only one of these functions satisfied our added condition. This added condition, that $\frac{ds}{dt} = 0$ when t = 0, is called a *boundary condition.*

Exercises

5. Show that $\frac{ds}{dt} = kt + c$ satisfies Equation (7) for any value of c.

We now take the antiderivative of $\frac{ds}{dt}$ to get s(t). Thus, we have

$$s(t) = \int \frac{ds}{dt}\,dt$$
$$= \int kt\,dt$$
$$= \frac{kt^2}{2} + d,$$

where d is another constant. Can we also find what d must be? Since

$$s(t) = \frac{kt^2}{2} + d,$$

by Equation (11), we note that $s(0) = \frac{k(0)^2}{2} + d = d$. Thus, d is the height of the penny above the ground when t = 0. That is, d is equal to the height of the building, h. We then have

(12) $s(t) = \frac{kt^2}{2} + h.$

We now have found a function s(t) which describes the height of the penny for any time t. The only thing we have not done is to find the value of k. This must be determined by physical experiment, and physics texts give the value of k = -32.2 feet per second. The negative sign expresses the fact that the penny is falling down. Thus, we have

(13)
$$s(t) = \frac{-32.2t^2}{2} + 1250 \quad \text{or}$$
$$s(t) = -16.1t^2 + 1250,$$

for the height of the Empire State Building is about 1250 feet.

Exercises

6. a. Find how long the penny takes to hit the ground. It will hit the ground when s(t) = 0, so we can use Equation (13) to find the value of t > 0 for which s(t) = 0.

   b. Now find out how fast the penny will be falling when it hits. To do this note that Equation (10) gives the velocity at any time t. It will give us the velocity when the penny hits the ground, if we use the value of t found in part (a). Express your answer in miles per hour as well as feet per second.

7. Show that $s(t) = -16.1t^2 + 1250$ is a solution of Equations (10) and (7).

Let's summarize what we have done in the problem above. We were given a differential equation of the form $\frac{dy}{dt} = f(t)$. In Equation (7) $y = \frac{ds}{dt}$, $f(t) = k$ and

$$\frac{dy}{dt} = \frac{d\left[\frac{ds}{dt}\right]}{dt} = \frac{d^2s}{dt^2} = k = f(t).$$

In Equation (10), $y = s$ and $f(t) = kt$, for

$$\frac{dy}{dt} = \frac{ds}{dt} = kt = f(t).$$

In other words, we had to solve an equation of the form $\frac{dy}{dt} = f(t)$. To do it we were able to antidifferentiate to get

$$y = \int \frac{dy}{dt}\,dt = \int f(t)dt.$$

Thus, we could obtain y if we could find $\int f(t)dt$, the antiderivative of $f(t)$. We then obtained a class of solutions that involved a constant of integration. By using the boundary conditions we could then find the constant and get the *one* solution which solved our differential equation *and* satisfied the boundary condition. This procedure is the basis for most analytic solutions of differential equations. The difficulty arises in getting the equation in a form from which we can readily find the antiderivative.

Exercises

Solve the following differential equations, using the boundary conditions to find the constants of integration. Show that each answer satisfies its differential equation.

8. $\frac{ds}{dx} = k$; $s(0) = 1$.

9. $\frac{d^2y}{dx^2} = cx$; $\frac{dy}{dx} = 1$ when $x = 0$, and $y = 1$ when $x = 0$.

Let us look at the sagging beam problem again, this time analytically. We know from Unit 83, page 58, that

$$\frac{dy}{dx} = (8 \times 10^{-7})\frac{x^2}{2} + c,$$
$$(14) \qquad = 4 \times 10^{-7}x^2 + c,$$

where y is the vertical beam deflection. The problem was to find c. If we had some conditions on dy/dx we would be able to find c as in the previous examples. Since we only know conditions on y, we must proceed with (14) not knowing the value of c. We then get the following by antiderivatives.

$$y = \int \frac{dy}{dx}\,dx = \int (4 \cdot 10^{-7}x^2 + c)dx$$

or

$$y(x) = 4 \times 10^{-7}\,\frac{x^3}{3} + cx + d,$$

where d is a constant. Now $y(0) = 0$ and $y(30) = 0$ since the beam is fixed at both ends. From $y(0) = 0$ we get $0 = y(0) = d$. Hence $d = 0$. From $y(30) = 0$ we get

$$0 = y(30) = 4 \times 10^{-7}\,\frac{(30)^3}{3} + c(30).$$

Solving for c, we get

$$30c = \frac{-4 \times 10^{-7}(30)^3}{3}$$

$$30c = -(4 \times 10^{-7})(9000)$$

(15)
$$c = \frac{-(4 \times 10^{-7})(9000)}{30}$$

$$= -12 \times 10^{-5}.$$

Hence (14) becomes

$$\frac{dy}{dx} = 4 \times 10^{-7}x^2 - 12 \times 10^{-5}.$$

We know that the maximum deflection occurs for the value of x at which y(x) is the least, which can be found by letting dy/dx = 0. Thus solving

$$\frac{dy}{dx} = 4 \times 10^{-7}x^2 - 12 \times 10^{-5} = 0,$$

we get

$$x^2 = \frac{12 \times 10^{-5}}{4 \times 10^{-7}} = 3 \times 10^2,$$

or

$$x = \pm\sqrt{300} \approx \pm 17.3.$$

Note that -17.3 is not possible physically, so the maximum deflection occurs when x is approximately 17.3 ft. A quick calculation gives the maximum deflection as $y(17.3) \approx -138 \times 10^{-5}$ using

$$y(x) = \frac{4 \times 10^{-7}x^3}{3} - 12 \times 10^{-5}x.$$

Exercises

Solve each of the following differential equations. Check your answers by substitution.

10. $\frac{dx}{dt} = t$; $x(0) = -1$.

11. $\frac{d^2x}{dt^2} = -32$; $\frac{dx}{dt} = 100$ when $t = 0$, $x = 0$ when $t = 0$.

QUIZ #1

Use antiderivatives to find the solution to each of the following differential equations. Check your answer by substitution.

1. $\frac{d^2y}{dt^2} = t$; $\frac{dy}{dt} = 1$ and $y = 1$ when $t = 1$.

2. $\frac{d^2y}{dt^2} = k_1$; $\frac{dy}{dt} = k_2$ and $y = k_3$ when $t = 0$ where k_1, k_2, and k_3 are constants.

****

3. SEPARATION OF VARIABLES

Objective: To use the method of separation of variables to solve more differential equations.

In Unit 82 (Chapter 9) we encountered the differential equation

(16) $$\frac{dy}{dx} = \frac{x^2}{y}$$

with the boundary condition of $y = 1$ when $x = 1$. If we attempt to solve (16) with the technique of the last chapter we immediately run into a problem. When we try to take the antiderivative of the right side of (16) we get an integral which has both y and x in it, namely

(17) $$\int \frac{x^2}{y} dx.$$

To integrate (17) we need to know y as a function of x. But this is what we are trying to solve the differential equation for—to find y as a function of x! What can we do? Well, since y is "in the way" on the right side when we integrate, it may be a good idea to move it before we try to integrate. Thus, in (16), we must get y from the right side. We can do this if we multiply both sides of (16) by y to obtain

(18) $$y\frac{dy}{dx} = x^2.$$

Now we can integrate both sides of (18) to get

(19) $$\int y\frac{dy}{dx}dx = \int x^2 dx.$$

But the left side of (19) requires more work to integrate. However, we know that

$$\int y\frac{dy}{dx}dx = \frac{y^2}{2} + c_1,$$

where c_1 is a constant. So (19) becomes

(20) $$\frac{1}{2}y^2 + c_1 = \frac{1}{3}x^3 + c_2,$$

where c_2 is also a constant. If we let $c = c_2 - c_1$ then c is a constant and we can write (20) in the form

(21) $$\frac{1}{2}y^2 = \frac{1}{3}x^3 + c,$$

which is similar to the cubic we obtained in the beam-deflection problem. The difference lies in that (21) does not express y as a function of x directly. To get y we

must solve for y in (21). Now,

(22) $$y^2 = \frac{2}{3}x^3 + 2c,$$

or

(23) $$y = \pm\sqrt{\frac{2}{3}x^3 + 2c}.$$

Thus, we have two possible solutions to (16). However, from the boundary condition y(1) = 1 we obtain

(24)
$$1 = \pm\sqrt{\frac{2}{3}(1)^3 + 2c}$$
$$1 = \pm\sqrt{\frac{2}{3} + 2c}$$
$$(1)^2 = \left(\pm\sqrt{\frac{2}{3} + 2c}\right)^2$$
$$1 = \frac{2}{3} + 2c$$
$$1 - \frac{2}{3} = 2c$$
$$c = \frac{1}{2}\left(\frac{1}{3}\right) = \frac{1}{6}.$$

With this value of c, Equation (23) becomes

$$y = \pm\sqrt{\frac{2}{3}x^3 + \frac{1}{3}} = \pm\sqrt{\frac{2x^3 + 1}{3}}.$$

To decide which of the two possibilities satisfies the boundary condition y(1) = 1, we substitute x = 1 in (25) and find

$$y = \pm\sqrt{\frac{2(1)^3 + 1}{3}} = \pm\sqrt{\frac{3}{3}} = \pm 1.$$

This tells us to choose the plus sign, so that

(26) $$y = \sqrt{\frac{2x^3 + 1}{3}}$$

is our answer. To check that this function does indeed satisfy Equation (16) we may differentiate it:

$$
\begin{aligned}
\frac{dy}{dx} &= \frac{d}{dx}\sqrt{\frac{2x^3+1}{3}} \\
&= \frac{d}{dx}\left(\frac{2x^3+1}{3}\right)^{\frac{1}{2}} \\
&= \frac{1}{2}\left(\frac{2x^3+1}{3}\right)^{-\frac{1}{2}}\cdot\frac{d}{dx}\left(\frac{2x^3+1}{3}\right) \\
(27) \qquad &= \frac{1}{2}\cdot\frac{1}{\sqrt{\frac{2x^3+1}{3}}}\cdot\frac{d}{dx}\left(\frac{2x^3}{3}+\frac{1}{3}\right) \\
&= \frac{1}{2}\cdot\frac{1}{y}\cdot 2x^2 \\
&= \frac{x^2}{y}.
\end{aligned}
$$

Exercises

12. Show $y = -\sqrt{\frac{2x^3+1}{3}}$ satisfies Equation (16) but does *not* satisfy the boundary condition $y = 1$ when $x = 1$. What condition does it satisfy when $x = 1$?

The technique we used to solve $\frac{dy}{dx} = \frac{x^2}{y}$ is called *separation of variables*. In summary, if we are given a differential equation which has both variables appearing on the right-hand side of the equation, we move the dependent variable to the left side of the differential equation. We then integrate and solve the resulting equation for y.

Exercises

13. Solve the equation $\frac{dy}{dx} = ky$ by separation of variables (k is a constant not equal to zero). Solve for y after integrating and then compare your answer to the one obtained in Exercise 1.

Another phenomenon of interest is the way the temperature of a body changes when it is placed in a cooling medium. Let $y(t)$ represent the temperature of the object at any time t. Suppose we place the object in a cooler medium of sufficient quantity so that the temperature of the medium is not changed by the hotter object. (For example, a meteorite into an ocean.) Let M represent this constant temperature. Experimental data suggest that the *rate* of cooling of the object is directly proportional to the *difference* in the two temperatures at any time t. We can express this law of cooling by the differential equation

$$(28) \qquad \frac{dy}{dt} = -k(y-M),$$

where M is the temperature of the medium and k is a positive constant.* Note that since the temperature of the object is falling, y is decreasing and its derivative is negative. We can solve (28) by separation of variables to get

$$(29) \qquad y(t) = (y_0-M)e^{-kt} + M,$$

where $y_0 = y(0)$.

Exercises

14. Solve Equation (28) for $y(t)$ by carrying out the following steps: First, by using separation of variables, show that $\ln(y-M) = -kt + C$. Next, using the boundary condition $y(0) = y_0$, show that $C = \ln(y_0-M)$, and hence that $\ln\frac{y-M}{y_0-M} = -kt$. Then find y using the exponential function.

We would expect the temperature of the object to decrease until it reaches the temperature of the medium, M. To see if this happens from (29) we look at y as $t \to \infty$. But since $k > 0$, we know $e^{-kt} \to 0$ as $t \to \infty$.

*See Appendix 1 for a biological example which also uses a differential equation of this form.

(Recall the behavior of the graph of e^{-kt}.) Therefore, $y \to (y_0 - M)(0) + M = M$ as $t \to \infty$. Note that y is always greater than M, since $y_0 - M > 0$ and $e^{-kt} > 0$. Thus, the value of $y(t)$ approaches M but never reaches M. In practice, the temperature of the object eventually gets so close to the temperature of the medium that we cannot measure any difference between the two.

Exercises

15. A thermometer is removed from boiling water. The temperature decreases from 95° C to 80° C in half of a minute. If the room temperature is 20° C, about how long will it take the thermometer to get within one Celsius degree of the room temperature. Hint: Find k first.

Appendices 2 and 3 give two more differential equations used in the sciences which may be solved by the technique of separation of variables.

Exercises

Solve each of the following differential equations. Check your answer by substitution.

16. $\frac{dy}{dx} = y^2$; $y(0) = 1$.

17. $\frac{dy}{dx} = \frac{y}{x}$; $y(1) = 1$.

QUIZ #2

Solve each of the following differential equations by separation of variables. Check your answer by substitution.

(1) $\frac{dy}{dx} = xy$; $y(0) = 1$.

(2) $\frac{ds}{dt} + s + 1 = 0$; $s(0) = 0$.

****

4. FIRST ORDER LINEAR DIFFERENTIAL EQUATIONS

Objective: To solve first order linear differential equations.

4.1 The Fish Pond Revisited

The fish pond problem was solved in Unit 82 (Chapter 8) using tangent fields. We now study the problem analytically. We begin with the differential equation

$$(30) \qquad \frac{dQ}{dt} = 0.6 - \frac{2Q}{4320 + t},$$

where Q is the number of fish in the pond at time t. We first note that none of our previous techniques permit us to solve this equation. We cannot just use antiderivatives since the function Q appears in the right-hand side of (30). This suggests, of course, that we try to separate the variables t and Q. But if we divide both sides of (30) by Q to move Q, we are still left with Q on the right-hand side because of the 0.6, for we get

$$(31) \qquad \frac{1}{Q}\frac{dQ}{dt} = \frac{0.6}{Q} - \frac{2}{4320 + t}.$$

We have to come up with a new technique if we are to solve (30). Our first move is to change (30) to

$$(32) \qquad \frac{dQ}{dt} + \frac{2Q}{4320 + t} = 0.6.$$

To make things a little easier to write, let's replace $\frac{2}{4320 + t}$ by $f(t)$. Then (32) becomes

$$(33) \qquad Q' + f(t)Q = 0.6.$$

Now it would be nice if the left-hand side of (33) were the derivative of *something*, for we could then use antiderivatives. The form of the left side of (33) suggests the product rule, $yQ' + y'Q = (yQ)'$. In other words, if we could find a function $y(t)$, so that

$$Q' + f(t)Q = yQ' + y'Q,$$

then we could use antiderivatives to solve (33). But to have this we would need to have the product yQ' in (33). Since we do not have it, we put it there by multiplying both sides of Equation (33) by y. This gives

$$(34) \qquad yQ' + yf(t)Q = 0.6y.$$

Now we need y to satisfy

$$yQ' + yf(t)Q = yQ' + y'Q,$$

or

$$yf(t)Q = y'Q,$$

$$(35) \qquad yf(t) = y'.$$

Thus, a function which would permit us to use antiderivatives must satisfy (35). But (35) is just a differential equation! And we can solve it! For by separation of variables,

$$(36) \qquad \frac{1}{y}y' = f(t),$$

where

$$f(t) = \frac{2}{4320 + t},$$

from (32). A solution of (36) is

$$y(t) = (4320 + t)^2.$$

Exercises

18. Solve the equation $\frac{1}{y}y' = \frac{2}{4320 + t}$ by separation of variables.

When we substitute $(4320 + t)^2$ for y in Equation (34) we obtain

$$(4320 + t)^2Q' + \frac{2}{4320+t}(4320+t)^2Q = 0.6(4320+t)^2,$$

$$(38) \qquad (4320+t)^2Q' + 2(4320+t)Q = 0.6(4320+t)^2.$$

Now we see that the left side of (38) is just the derivative of $(4320+t)^2Q$.

Exercises

19. Show that the left side of Equation (38) is the derivative of $(4320 + t)^2Q$.

Thus, we can rewrite Equation (38) as

$$(39) \qquad \frac{d}{dt}\left[(4320+t^2)Q\right] = 0.6(4320+t)^2.$$

Now solve (39) for $(4320+t)^2Q$ by using antiderivatives to get

$$(4320+t)^2Q = \int 0.6(4320+t)^2dt,$$

or

$$(40) \qquad Q(t) = \frac{1}{(4320+t)^2}\int 0.6(4320+t)^2dt.$$

Thus, we can solve (30) by integrating in (40).

Exercises

20. Find $\int 0.6(4320 + t)^2dt$. Don't forget the constant of integration.

Using the result of Exercise 20, we have

$$(41) \qquad Q(t) = \frac{1}{(4320+t)^2}[0.2(4320+t)^3+c].$$

Now $Q(0) = 1864$, so we can find c from (41), as follows:

$$Q(0) = 1864 = \frac{1}{(4320)^2}[0.2(4320)^3+c],$$

$$c = 1864(4320)^2 - 0.2(4320)^3$$

$$= 18662400000.$$

Thus,

$$Q(t) = \frac{1}{(4320+t)^2}[0.2(4320+t)^3 + 18662400000].$$

For the original problem, $Q(12960) = 3519$, rounded to the nearest fish, in close agreement with the figure of 3500 that may be read from the graph on page 38 of Unit 82.

4.2 A Second Example

As a second example, we look at the equation $\frac{dy}{dx} + 2y - x = 0$ from Unit 82, Chapter 7. We put it in the same form

$$\frac{dy}{dx} + 2y = x, \tag{42}$$

to separate y and its derivative from x. To solve (42) we search for a function u(x) so that when we multiply both sides of Equation (42) by u, the left side is the derivative of uy. We first get

$$uy' + 2yu = xu. \tag{43}$$

Then, since $(uy)' = uy' + yu'$, we must have

$$yu' = 2yu,$$

or

$$u' = 2u. \tag{44}$$

But we know that

$$u(x) = e^{2x}$$

is a solution of (44). With u replaced by e^{2x}, Equation (43) becomes

$$e^{2x}y' + 2ye^{2x} = xe^{2x}, \tag{45}$$

or

$$\frac{d}{dx}(e^{2x}y) = xe^{2x}. \tag{46}$$

This means that

$$e^{2x}y = \int xe^{2x}dx,$$

so that

$$y = \frac{1}{e^{2x}}\int xe^{2x}dx. \tag{47}$$

To find y we must find $\int xe^{2x}dx$. We need a function whose derivative is xe^{2x}. Notice that $\frac{1}{2}xe^{2x}$ almost works, because

$$\frac{d}{dx}\left(\frac{1}{2}xe^{2x}\right) = xe^{2x} + \frac{1}{2}e^{2x}.$$

We need to add a term to the trial function $\frac{1}{2}xe^{2x}$ whose derivative will cancel the unwanted term $\frac{1}{2}e^{2x}$. That is, we need to add ~~the~~ antiderivative of $-\frac{1}{2}e^{2x}$, which is $-\frac{1}{4}e^{2x}$. The function

$$\frac{1}{2}xe^{2x} - \frac{1}{4}e^{2x}$$

now has the derivative we want, for

$$\frac{d}{dx}\left(\frac{1}{2}xe^{2x} - \frac{1}{4}e^{2x}\right) = xe^{2x} + \frac{1}{2}e^{2x} - \frac{1}{2}e^{2x}$$
$$= xe^{2x}.$$

Hence, the solution of Equation (47) is

$$y = \frac{1}{e^{2x}}\left[\frac{1}{2}xe^{2x} - \frac{1}{4}e^{2x} + c\right],$$

or

$$y = \frac{1}{2}x - \frac{1}{4} + ce^{-2x}. \tag{48}$$

Exercises

21. Show that xe^{2x} is the derivative of

$$\frac{1}{2}xe^{2x} - \frac{1}{4}e^{2x}.$$

22. Show (48) satisfies (42) for all values of c.

4.3 The General Procedure

We now formalize the procedure of the two previous examples. If we have a differential equation of the form

(49) $$y' + f(x)y = g(x),$$

where f and g are functions of x, then we find u(x) so that

(50) $$u'(x) = f(x)u(x).$$

We solve (50) to get $u(x) = e^{\int f(x)dx}$. Then (49) becomes, by multiplying by u(x),

$$uy' + ufy = gu,$$

or

$$uy' + u'y = gu$$

by (50). Thus,

$$\frac{d(uy)}{dx} = gu,$$

and

$$uy = \int gu\ dx + c$$

so that

$$y = \frac{1}{u}\left[\int gu\ dx + c\right],$$

where c is a constant.

We summarize: *The solution of the first order linear differential equation*

$$y'(x) + f(x)y(x) = g(x)$$

is

$$y(x) = \frac{1}{u(x)}\left[\int g(x)u(x)dx + c\right],$$

where

$$u(x) = e^{\int f(x)dx}.$$

Exercises

Use Formula (51) to solve the following differential equations.

23. $y' = ky;\ \ y(0) = N$

The solution of this equation describes exponential growth.

24. $y' + y + 1 = 0;\ \ y(0) = 0$

25. $y' = x$

To check Equation (51) we notice that, for any function h(x),

$$\frac{d\left(\int h(x)dx\right)}{dx} = h(x).$$

Then, from

$$y = [u(x)]^{-1}\left[\int g(x)u(x)dx + c\right]$$

we get

$$y' = -[u(x)]^{-2}\,\frac{d(u)}{dx}\left[\int g(x)u(x)dx + c\right]$$

$$+ [u(x)]^{-1}[g(x)u(x)]$$

$$= -[u(x)]^{-1}\,\frac{du}{dx}y + [u(x)]^{-1}[g(x)u(x)]$$

$$= \frac{-1}{u(x)}f(x)u(x)y + g(x) \quad \text{(using } u'(x) = f(x)u(x)\text{)}$$

$$= -f(x)y + g(x).$$

So

$$y' + f(x)y = g(x),$$

which is our original differential equation, Equation (49).

The linear differential equation is the last type of differential equation we will attempt to handle analytically in this unit. We note many other types appear in practice. The value of the analytic method is that, if it works, it is more tractable than numerical methods. However, numerical methods are sometimes more useful and faster, and at times they provide the only known way to get to a solution.

Exercises

26. If in the linear first-order differential equation (49) we have $g(x) \equiv 0$, show that the equation can be solved by separation of variables. Compare your answer with (51).

QUIZ #3

Solve each of the following differential equations by using Formula (51). Determine the behavior of the solution as $t \to \infty$.

27. $\frac{dy}{dt} + \frac{y}{t} = \frac{1}{t^2}$

28. $\frac{dy}{dt} + t^2 y = t^2$.

****

5. MODEL EXAM

1. Solve each of the following differential equations by using *two* of the methods of the text; antiderivatives, separation of variables, or the linear formula.

   a. $y' + x = 0$; $y(1) = 1$

   b. $y' + \frac{1}{x} = 0$; $y(1) = 1$

   c. $y' + y = 0$; $y(0) = 1$

   d. $y' + y = 1$; $y(0) = 0$.

2. The velocity of a body falling in a resisting medium may be modeled by the equation

$$\frac{dv}{dt} + kv - g = 0,$$

where $v = v(t)$ is the velocity of the body, and k and g are constants. Assume $v(0) = 0$.

   a. Solve the differential equation for $v(t)$.

   b. Show that the body's speed approaches a limiting speed as $t \to \infty$, and find this speed. This speed is called the *terminal velocity* of the falling body.

6. ANSWERS TO EXERCISES

Chapter 1

1. $y' = ke^{kx} = ky$.

2. $y' = \frac{3}{2}\sqrt{\frac{2}{3}}\, x^{\frac{1}{2}} = \dfrac{\frac{3}{2}\sqrt{\frac{2}{3}}\, x^{\frac{1}{2}}\sqrt{\frac{2}{3}}\, x^{3/2}}{\sqrt{\frac{2}{3}}\, x^{3/2}}$

$$= \frac{\frac{3}{2}\cdot\frac{2}{3}x^2}{\sqrt{\frac{2}{3}}\, x^{3/2}} = \frac{x^2}{y}.$$

3. $y' = \sqrt{k}\cos(\sqrt{k}\, x)$

$y'' = \sqrt{k}\cdot\sqrt{k}\,(-\sin(\sqrt{k}\, x))$

$= -k\sin(\sqrt{k}\, x) = -ky.$

4. $y' = (y_0 - M)(-k)e^{-kt}$

$= -k[(y_0 - M)e^{-kt} + M] + kM$

$= -ky + kM = -k(y - M).$

Chapter 2

5. $s' = kt + c$; $s'' = k$.

6. a. $0 = s(t) = -16.1t^2 + 1250$

$$0 = -16.1t^2 + 1250$$

$$t^2 = \frac{1250}{16.1}$$

$$t \approx 8.81.$$

b. $v(8.81) = -32.2(8.81) = -283.73$ ft/sec

$= 193.45$ mi/hr.

7. $s' = -32.2t$; $s'' = -32.2$, where $k = -32.2$.

8. $s(x) = s = \int k\,dx = kx + c$

$$s(0) = 1 = k(0) + c$$

$$1 = c$$

$$s(x) = kx + 1.$$

9. $y' = \int cx\,dx = \frac{cx^2}{2} + c_1.$

$$y'(0) = 1 = \frac{c(0)^2}{2} + c_1$$

$$1 = c_1$$

$$y'(x) = \frac{c}{2}x^2 + 1$$

$$y(x) = \int\left(\frac{c}{2}x^2 + 1\right)dx = \frac{c}{6}x^3 + x + c_2$$

$$y(0) = 1 = c_2$$

$$y(x) = \frac{c}{6}x^3 + x + 1.$$

10. $x(t) = \int t\,dt = \frac{t^2}{2} + c$

$$x(0) = -1 = c$$

$$x(t) = \frac{t^2}{2} - 1.$$

11. $x'(t) = \int -32dt = -32t + c$

$$x'(0) = 100 = c$$

$$x'(t) = -32t + 100$$

$$x(t) = \int(-32t + 100)dt$$

$$= -16t^2 + 100t + c$$

$$x(0) = 0 = c$$

$$x(t) = -16t^2 + 100t.$$

Chapter 3

12. $y' = -\frac{1}{\sqrt{3}}\frac{1}{2}(2x^3+1)^{-\frac{1}{2}}(6x^2)$

$$= \frac{\frac{1}{6}(6x^2)}{-\frac{1}{\sqrt{3}}(2x^3+1)^{\frac{1}{2}}} = \frac{x^2}{y}$$

$$y(1) = -\frac{1}{\sqrt{3}}\sqrt{2+1} = -1.$$

13. $\int\frac{1}{y}dy = \int k\,dx$

$$\ln y = kx + c$$

$$y = e^{kx+c}.$$

14. $\int\frac{1}{y-M}dy = \int -kdt$

$$\ln(y-M) = -kt + c$$

$$\ln(y_0-M) = c$$

$$\ln(y-M) = -kt + \ln(y_0-M)$$

$$\ln(y-M) - \ln(y_0-M) = -kt$$

$$\ln\left(\frac{y-M}{y_0-M}\right) = -kt$$

$$\frac{y-M}{y_0-M} = e^{-kt}$$

$$y = (y_0-M)e^{-kt} + M.$$

15. $y = (y_0-M)e^{-kt} + M$

$y(\frac{1}{2}) = 80 = (95-20)e^{-k/2} + 20$

$60/75 = e^{-k/2}$

$-k/2 = \ln(0.8) = -0.223$

$k = 0.446.$

$y(t) = 75e^{-0.446t} + 20 \leq 21$

$75e^{-0.446t} \leq 1$

$e^{-0.446t} \leq 1/75$

$-0.446t \leq \ln(1/75)$

$t \geq 9.68$ min.

16. $\int \frac{dy}{y^2} = \int dx.$

$-y^{-1} = x + c$

$-1 = c$

$y^{-1} = 1 - x$

$y = \frac{1}{1-x}.$

17. $\int \frac{1}{y} dy = \int \frac{1}{x} dx$

$\ln y = \ln x + c$

$0 = c$

$y = x.$

Chapter 4

18. $\int \frac{1}{y} dy = \int \frac{2}{4320+t} dt$

$\ln y = 2 \ln (4320+t) + c$, with $c = 0$

$\ln y = \ln (4320+t)^2$

$y = (4320+t)^2.$

19. $2(4320+t)Q + (4320+t)^2 Q'$, by the product rule

20. $\int 0.6(4320+t)^2 dt$

$= \frac{0.6}{3}(4320+t)^3 + c$

$= 0.2(4320+t)^3 + c.$

21. By the product rule, the derivative of

$\frac{1}{2} xe^{2x} - \frac{1}{4}e^{2x}$ is

$\frac{1}{2}e^{2x} + \frac{1}{2}x(2e^{2x}) - 2 \cdot \frac{1}{4}e^{2x}$

$= \frac{1}{2}e^{2x} + xe^{2x} - \frac{1}{2}e^{2x}$

$= xe^{2x}.$

22. $y' = \frac{1}{2} - 2ce^{-2x}$

$y' + 2y = \frac{1}{2} - 2ce^{-2x} + x - \frac{1}{2} + 2ce^{-2x}$

$= x.$

23. $u(x) = e^{\int -k dx} = e^{-kx}$

$y(x) = e^{kx}(\int 0 \cdot e^{-kx} dx + c)$

$= ce^{kx}$

$c = N$

$y(x) = Ne^{kx}.$

24. $u(x) = e^{\int 1 dx} = e^x$

$y(x) = e^{-x}(\int -e^x dx + c)$

$= e^{-x}(-e^x + c)$

$= -1 + ce^{-x}$

$0 = -1 + c$

$1 = c$

$y(x) = e^{-x} - 1.$

25. $u(x) = e^{\int 0 \cdot dx} = e^0 = 1$

$$y(x) = 1 \cdot \left(\int 1 \cdot x\,dx + c\right)$$

$$= \frac{x^2}{2} + c.$$

26. $g(x) \equiv 0$ gives

$$\frac{dy}{dx} + f(x)y = 0$$

$$\frac{dy}{dx} = -f(x)y$$

$$\int \frac{1}{y}dy = \int -f(x)dx$$

$$\ln y = -\int f(x)dx + c'$$

$$y = ce^{-\int f(x)dx}, \quad \text{where } c = e^{c'}.$$

By (51)

$$u(x) = e^{\int f(x)dx}$$

$$y(x) = e^{-\int f(x)dx}\left(\int 0 \cdot u(x)dx + c\right)$$

$$= ce^{-\int f(x)dx}.$$

7. ANSWERS TO QUIZZES AND MODEL EXAM

Quiz #1

1. $y' = \frac{t^2}{2} + c$

$$y'(1) = 1 = \frac{1}{2} + c$$

$$y' = \frac{t^2}{2} + \frac{1}{2}$$

$$y = \frac{t^3}{6} + \frac{t}{2} + c$$

$$y(1) = 1 = \frac{1}{6} + \frac{1}{2} + c$$

$$\frac{1}{3} = c$$

$$y(t) = \frac{t^3}{6} + \frac{t}{2} + \frac{1}{3}.$$

2. $y' = k_1 t + c$

$$k_2 = c$$

$$y = \frac{k_1 t^2}{2} + k_2 t + c$$

$$k_3 = c$$

$$y(t) = \frac{k_1 t^2}{2} + k_2 t + k_3.$$

Quiz #2

1. $\int \frac{1}{y}dy = \int x\,dx$

$$\ln y = \frac{x^2}{2} + c$$

$$0 = c$$

$$y = e^{x^2/2}.$$

2. $s' = -(1+s)$

$$\int \frac{ds}{1+s} = \int -dt$$

$$\ln(1+s) = -t + c$$

$$0 = c$$

$$1 + s = e^{-t}$$

$$s = e^{-t} - 1.$$

Quiz #3

1. $u(t) = e^{\int \frac{1}{t} dt} = e^{\ln t} = t$

$$y(t) = \frac{1}{t}\left(\int t \cdot \frac{1}{t^2} dt + c\right)$$

$$y(t) = \frac{c}{t} + \frac{\ln t}{t}$$

$y(t) \to 0$ as $t \to \infty$.

2. $u(t) = e^{\int t^2 dt} = e^{t^3/3}$

$$y(t) = e^{-t^3/3}\left(\int t^2 e^{t^3/3}\, dt + c\right)$$

$$= ce^{-t^3/3} + e^{-t^3/3}\left(e^{t^3/3}\right)$$

$$= ce^{-t^3/3} + 1$$

$y \not\to 0$ as $t \to \infty$.

Model Exam

1. a. By antiderivatives,

$$y' = -x$$

$$y = -\frac{x^2}{2} + c$$

$$1 = c$$

$$y = -\frac{x^2}{2} + 1.$$

By linear formula,

$$u(x) = e^{\int 0\, dx} = 1$$

$$y(x) = \int -x\, dx + c$$

$$= -\frac{x^2}{2} + c$$

$$y(x) = -\frac{x^2}{2} + 1.$$

b. By antiderivatives,

$$y' = -1/x$$

$$y = -\ln x + c$$

$$1 = c$$

$$y = 1 - \ln x.$$

By linear formula,

$$u(x) = e^0 = 1$$

$$y(x) = \int -\frac{1}{x} dx + c$$

$$= -\ln x + 1.$$

c. By separation of variables,

$$\frac{1}{y} y' = -1$$

$$\ln y = -x + c$$

$$0 = c$$

$$y = e^{-x}.$$

By linear formula,

$$u(x) = e^{\int 1\, dx} = e^x$$

$$y(x) = e^{-x}\left(\int 0 \cdot e^x dx + c\right)$$

$$= ce^{-x}$$

$$1 = c$$

$$y(x) = e^{-x}.$$

d. By separation of variables,

$$y' = 1-y$$

$$\frac{1}{1-y}y' = 1$$

$$-\ln(1-y) = x + c$$

$$0 = c$$

$$\ln(1-y) = -x$$

$$1-y = e^{-x}$$

$$y = 1 - e^{-x}.$$

By linear formula,

$$u(x) = e^{\int 1\,dx} = e^{x}$$

$$y(x) = e^{-x}\left(\int 1 \cdot e^{x}dx + c\right)$$

$$= e^{-x}(e^{x}+c)$$

$$= 1 + ce^{-x}$$

$$-1 = c$$

$$y(x) = 1 - e^{-x}.$$

2. a. $\frac{dv}{dt} + kv - g = 0$

By linear formula,

$$u(t) = e^{\int k\,dt} = e^{kt}$$

$$v(t) = e^{-kt}\left(\int ge^{kt}dt + c\right)$$

$$= e^{-kt}\left(\frac{ge^{kt}}{k} + c\right)$$

$$= g/k + ce^{-kt}$$

$$0 = g/k + c$$

$$-g/k = c$$

$$v(t) = g/k(1 - e^{-kt}).$$

b. As $t\to\infty$, $e^{-kt}\to 0$ so $v(t)\to g/k$.

APPENDIX I

AN EXAMPLE FROM BIOLOGY

Biologists use an equation of the kind that appears in the law of cooling (Chapter 3) to describe the diffusion of chemicals through the wall of a cell. The cell is assumed to have a constant volume, V, and to be immersed in a liquid. We consider the flow of a particular chemical or solute into and out of the cell. Assume that the solute concentration in the liquid is constant at a value of c_0. Let $c(t)$ represent the concentration of the solute *in the cell* at time t. The solute will diffuse into and out of the cell. We are interested in the *net* flow of the solute. Now if $m(t)$ represents the mass of the solute in the cell, then

(1) $\qquad m(t) = V \cdot c(t).$

The derivative, dm/dt, is the net flow rate. A differential equation, known as *Fick's law*, says that

(2) $\qquad \frac{dm}{dt} = kA(c_0-c),$

where k is a constant called the permeability of the membrane, and A is the fixed area of the cell's surface. By differentiating (1) we get

$$\frac{dm}{dt} = V\frac{dc}{dt}$$

which may be substituted in (2) to get

(3) $\qquad \frac{dc}{dt} = \frac{kA}{V}(c_0-c).$

Since kA/V and c_0 are constants we see (3) is like the differential equation

$$\frac{dy}{dt} = -k(y-M)$$

that describes the law of cooling in Chapter 3. We can solve (3) to get

(4) $$c(t) = Ke^{\frac{-kA}{V}t} + c_0$$

where K is the constant of integration. As before, $c(t)$ approaches c_0 as $t \to \infty$. The constant k must be determined by experiment.

Exercise

1. Solve Equation (3) to get Equation (4).

APPENDIX 2

A POPULATION MODEL

The long-run growth of populations is not exponential, because the environment does not permit unlimited growth. A differential equation which leads to a useful model of limited growth is

(1) $$\frac{dy}{dt} = ay - by^2,$$

where $y(t)$ is the number of organisms in the population at time t, and a and b are positive constants. The term $-by^2$ causes the growth to be smaller than ay, and provides a "limiting" factor. If b = 0, then (1) simplifies to the differential equation for exponential growth.

To solve (1) we first separate variables to get

(2) $$\int \frac{1}{ay - by^2}\frac{dy}{dt}dt = \int 1\,dt = t + c.$$

To integrate the left side of (2), we expand the quotient

$$\frac{1}{ay - by^2}$$

by the method of partial fractions. In other words, we find A and B so that

$$\frac{1}{ay - by^2} = \frac{1}{y(a - by)} = \frac{A}{y} + \frac{B}{a - by}.$$

Some algebra shows that A = 1/a and B = b/a will work.

Exercises

1. Show that

$$\frac{\frac{1}{a}}{y} + \frac{\frac{b}{a}}{a - by} = \frac{1}{ay - by^2}.$$

Thus,

$$\int \frac{1}{ay - by^2}\frac{dy}{dt}dt = \int \frac{\frac{1}{a}}{y}\frac{dy}{dt}dt + \int \frac{\frac{b}{a}}{a - by}\frac{dy}{dt}dt.$$

2. Show

$$\int \frac{\frac{1}{a}\frac{dy}{dt}}{y}dt = \frac{1}{a}\ln y + c_1, \quad \text{for } y > 0.$$

3. Show

$$\int \frac{\frac{b}{a}}{a - by}\frac{dy}{dt}dt = \frac{-1}{a}\ln(a - by) + c_2, \quad \text{for } y < \frac{a}{b}.$$

Using the results of Exercises 2 and 3, we may rewrite Equation (2) as

(3) $$\frac{1}{a}\ln(y) - \frac{1}{a}\ln(a - by) = t + c, \quad \text{for } 0 < y < \frac{a}{b}.$$

To find c in this equation, we let t = 0. Then

$$\frac{1}{a}\ln[y(0)] - \frac{1}{a}\ln[a - by(0)] = c.$$

If we let $y_0 = y(0)$, then

$$c = \frac{1}{a}\ln y_0 - \frac{1}{a}\ln(a - by_0),$$

and Equation (3) may be rewritten as

$$t = \frac{1}{a}\ln y - \frac{1}{a}\ln(a - by) - \frac{1}{a}\ln y_0 + \frac{1}{a}\ln(a - by_0).$$

We multiply both sides of this equation by a, to get

$$\begin{aligned} at &= \ln y - \ln(a-by) - \ln y_0 + \ln(a-by_0) \\ &= \ln \frac{y(a-by_0)}{y_0(a-by)}. \end{aligned}$$

Then,

$$\frac{y(a-by_0)}{y_0(a-by)} = e^{at},$$

$$ay - by_0 y = ay_0 e^{at} - by_0 y e^{at},$$

$$y(a-by_0+by_0e^{at}) = ay_0e^{at},$$

and, finally,

$$y = \frac{ay_0e^{at}}{a - by_0 + by_0e^{at}},$$

or

$$(4) \qquad y = y(t) = \frac{ay_0}{by_0 + (a-by_0)e^{-at}}, \quad 0 < y < \frac{a}{b}.$$

Notice that $e^{-at} \to 0$ as $t \to \infty$, because $a > 0$. Therefore,

$$y \to \frac{ay_0}{by_0 + (a-by_0)\cdot 0} = \frac{a}{b}$$

as $t \to \infty$. The limiting value of the population is $\frac{a}{b}$.

A routine investigation of y and its first two derivatives reveals the graph of y to have a point of inflection at $y = a/2b$. The graph is concave up for values of y between 0 and $a/2b$, and concave down for values of y greater than $a/2b$. The graph approaches the horizontal asymptote $y = a/b$ from below as y gets large. See Figure 1.

The curve in Figure 1 is called a *logistic curve*. Note that for $y < a/2b$, the curve resembles an exponential curve. In other words, at the beginning, the population

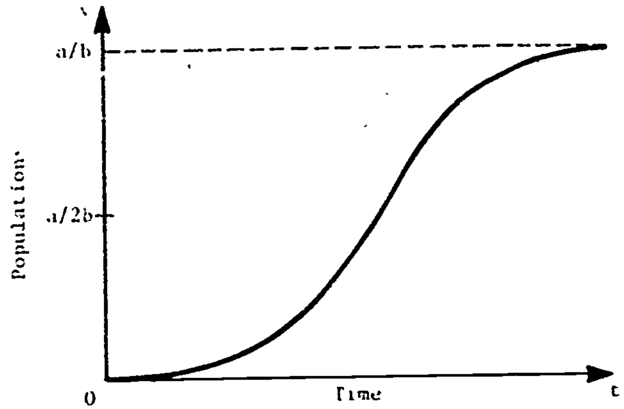

Figure 1. The graph of $y = \dfrac{ay_0}{by_0 + (a-by_0)e^{-at}}$, for y_0 close to 0.

grows almost exponentially. But, as y increases, the limiting term $-by^2$ in the first derivative has more effect and causes the population growth to taper off.

Exercises

4. Experimental data suggest that the value of a in Equation (1) for the human population of the earth is about $a = 0.03$. Suppose we have the following data for the earth's population:

Year	1960	1970	1975
Population (in billions)	3.01	3.59	3.92

If we let the year 1960 be our starting point then $t = 0$ in 1960. Then $y_0 = y(0) = 3.01$. Find $y(10)$ from the table. Then solve Equation (4) for b, using $a = 0.03$ and $t = 10$. Now use Equation (4) with this value of b to find $y(15)$, and compare it to the table value. What is the limiting value of the earth's population under this equation? In what year will $y = a/2b$?

APPENDIX 3

AN EXAMPLE FROM CHEMISTRY

When a chemical C is formed by combining the molecules of two chemicals A and B, it is sometimes reasonable to assume that the more of A and B present the faster the reaction could take place. In other words, the *rate* of formation of C is proportional to the amounts of A *and* B present at any time.

To develop a differential equation that describes the formation of C, let $y(t)$ represent the concentration of C at time t. We can assume $y(0) = 0$. Let a be the concentration of the chemical A and b the concentration of chemical B when $t = 0$. Now if the formation of a molecule of C requires one molecule of A and one molecule of B, then $a - y(t)$ and $b - y(t)$ is the concentration of A and B at time t. Then the statement that the rate of formation of C is proportional to the amounts of A and B present becomes

$$(1) \qquad \frac{dy}{dt} = k(a - y)(b - y),$$

for some positive constant k. To solve (1) we consider two cases, (i) $a = b$ and (ii) $a \neq b$.

Suppose that $a = b$. Then by separation of variables (1) becomes

$$(2) \qquad \int \frac{1}{(a-y)^2}\frac{dy}{dt}\,dt = \int k\,dt.$$

Substituting u for $a - y$ we have

$$\int \frac{1}{(a-y)^2}\frac{dy}{dt}\,dt = -\int \frac{1}{u^2}\frac{du}{dt}\,dt$$

$$= -\left(-\frac{1}{u}\right) = \frac{1}{u} = \frac{1}{a - y}.$$

Then (2) becomes

$$\frac{1}{a - y} = kt + c.$$

But $y(0) = 0$, so

$$\frac{1}{a} = c$$

and

$$\frac{1}{a - y} = kt + \frac{1}{a}$$

$$= \frac{akt + 1}{a}.$$

So

$$a - y = \frac{a}{akt + 1}$$

$$y = a - \frac{a}{akt + 1}$$

$$y = \frac{a^2kt}{akt + 1}.$$

Note that as $t \to \infty$

$$y(t) = \frac{a^2k}{ak + \frac{1}{t}} \to \frac{a^2k}{ak} = a.$$

Thus, the concentration of C approaches the common concentration of A and B.

Now suppose that $a \neq b$. A different technique is then required to solve (1). By separation of variables, (1) becomes

$$(3) \qquad \int \frac{1}{(a-y)(b-y)}\frac{dy}{dt}\,dt = \int k\,dt.$$

To integrate $\frac{1}{(a-y)(b-y)}$ we notice that

$$\frac{1}{(a-y)(b-y)} = \frac{\frac{1}{(a-b)}}{b-y} + \frac{\frac{1}{(b-a)}}{a-y}.$$

Exercises

1. Show
$$\frac{1}{(a-y)(b-y)} = \frac{\frac{1}{(a-b)}}{b-y} + \frac{\frac{1}{(b-a)}}{a-y}.$$

Then
$$\int \frac{1}{(a-y)(b-y)} \frac{dy}{dt} dt$$
$$= \frac{1}{a-b}\int \frac{1}{b-y} \frac{dy}{dt} dt + \frac{1}{b-a}\int \frac{1}{a-y} \frac{dy}{dt} dt.$$

Exercises

2. Show
$$\int \frac{1}{a-y} \frac{dy}{dt} dt = -\ln(a-y) + c_1, \quad \text{for } y < a.$$
and
$$\int \frac{1}{b-y} \frac{dy}{dt} dt = -\ln(b-y) + c_2, \quad \text{for } y < b.$$

Then, combining the results of Exercise 2 with Equation (3), we have

(4) $$\frac{-1}{a-b}\ln(b-y) + \frac{-1}{b-a}\ln(a-y) = kt + c.$$

Since $y(0) = 0$, we get
$$c = \frac{-1}{a-b}\ln b + \frac{-1}{b-a}\ln a.$$

Equation (4) becomes
$$\frac{1}{a-b}\ln(a-y) - \frac{1}{a-b}\ln(b-y) = kt + \frac{1}{a-b}\ln a - \frac{1}{a-b}\ln b.$$

We can then get

$$\frac{1}{a-b}[\ln(a-y) - \ln(b-y) + \ln(b) - \ln(a)] = kt,$$

or
$$\ln\frac{b(a-y)}{a(b-y)} = k(a-b)t.$$

Then,
$$b(a-y) = a(b-y)e^{k(a-b)t}$$
$$ab - by + aye^{k(a-b)t} = abe^{k(a-b)t}$$
$$y = \frac{abe^{k(a-b)t} - ab}{ae^{k(a-b)t} - b}$$

(5) $$y = \frac{ab[e^{k(a-b)t} - 1]}{ae^{k(a-b)t} - b}.$$

Now if $a < b$ then $k(a-b) < 0$, so $e^{k(a-b)t} \to 0$ as $t \to \infty$. Then $y \to -ab/-b = a$ as $t \to \infty$. In other words, the concentration of C approaches the concentration of A, the lesser of the two concentrations. This is necessary for there must be enough of each chemical to form C and C could not form more than the smaller amount.

Exercises

3. Show that if $b < a$, then $y(t) \to b$ as $t \to \infty$.

ANSWERS TO EXERCISES

Appendix 1

1. First,

$$\frac{dc}{dt} = \frac{kA}{V}(c_0-c)$$

$$\int \frac{dc}{c_0-c} = \int \frac{kA}{V}dt$$

$$\int \frac{dc}{c-c_0} = -\int \frac{kA}{V}dt$$

$$\ln|c-c_0| = -\frac{kA}{V}t + c_1$$

$$|c-c_0| = e^{-\frac{kA}{V}t + c_1}$$

$$= e^{c_1} \cdot e^{-\frac{kA}{V}t}.$$

Then,

$$c-c_0 = \pm e^{c_1} \cdot e^{-\frac{kA}{V}t}.$$

If we now write K for $\pm e^{c_1}$, we have

$$c-c_0 = Ke^{-\frac{kA}{V}t},$$

or

$$c = Ke^{-\frac{kA}{V}t} + c_0,$$

which is Equation (4).

Appendix 2

1. Starting with the sum on the left, we may take the following algebraic steps:

$$\frac{\frac{1}{a}}{y} + \frac{\frac{b}{a}}{a-by} = \frac{1}{ay} + \frac{b}{a(a-by)}$$

$$= \frac{a(a-by) + aby}{a^2y(a-by)}$$

$$= \frac{a^2 - aby + aby}{a^2y(a-by)}$$

$$= \frac{a^2}{a^2y(a-by)}$$

$$= \frac{1}{y(a-by)}$$

$$= \frac{1}{ay-by^2}.$$

2. $$\int \frac{\frac{1}{a}\frac{dy}{dt}}{y}dt = \frac{1}{a}\int\frac{dy}{y}$$

$$= \frac{1}{a}\ln|y| + c_1$$

$$= \frac{1}{a}\ln y + c_1 \quad (\text{since } y > 0).$$

3. $$\int \frac{\frac{b}{a}}{a-by}\frac{dy}{dt}dt = \frac{1}{a}\int\frac{bdy}{a-by}$$

$$= -\frac{1}{a}\int\frac{bdy}{by-a}$$

$$= -\frac{1}{a}\ln|by-a| + c_2$$

$$= -\frac{1}{a}\ln(by-a) + c_2.$$

Absolute value bars are not necessary, because the given condition $y < a/b$ requires the quantity $by-a$ to be positive.

4. The value of y(10) is 3.59. To save work we rewrite Equation (4) as

$$y(t) = \frac{ay_0}{by_0(1-e^{-at}) + ae^{-at}}.$$

This groups the terms that involve b. Then, with $a = 0.03$, $t = 10$, and $y(10) = 3.59$, we have

$$3.59 = \frac{(0.03)(3.01)}{b(3.01)\left(1-e^{-(0.03)(10)}\right) + (0.03)e^{-(0.03)(10)}}$$

$$\approx \frac{0.09}{b(3.01)(0.26) + 0.02}$$

$$\approx \frac{0.09}{b(0.78) + 0.02},$$

$$2.80b + 0.07 \approx 0.09$$

$$b \approx 0.02/2.80 \sim 7.1 \times 10^{-3}.$$

Appendix 3

1. If we multiply both sides of the given equation by $(a-y)(b-y)$, and combine terms, we find

$$1 = (a-y)\frac{1}{(a-b)} + (b-y)\frac{1}{(b-a)}$$

$$= \frac{a-y}{a-b} + \frac{y-b}{a-b}$$

$$= \frac{a-y+y-b}{a-b}$$

$$= \frac{a-b}{a-b}$$

$$= 1.$$

2. $$\int \frac{1}{a-y}\frac{dy}{dt}\,dt = \int \frac{1}{a-y}\,dy$$

$$= -\int \frac{dy}{y-a}$$

$$= \ln|y-a| + c_1$$

$$= \ln(y-a) + c_1 \quad \text{(since } y-a > 0\text{)}.$$

Similarly,

$$\int \frac{1}{b-y}\frac{dy}{dt}\,dt = \int \frac{1}{b-y}\,dy$$

$$= -\int \frac{dy}{y-b}$$

$$= \ln|y-b| + c_2$$

$$= \ln(y-b) + c_2 \quad \text{(since } y-b > 0\text{)}.$$

3. Since $b < a$, the quantity $a-b$ is positive. The product $k(a-b)$ is then positive, and

$$\frac{1}{e^{k(a-b)t}} \to 0 \quad \text{as } t \to \infty.$$

To see that $y \to b$ as $t \to \infty$, first divide the numerator and denominator of the right side of Equation (5) by $ae^{k(a-b)t}$. The result is

$$y = \frac{b\left[1 - \frac{1}{e^{k(a-b)t}}\right]}{1 - \frac{b}{a}\cdot\frac{1}{e^{k(a-b)t}}}.$$

From this we see that, as $t \to 0$,

$$y \to \frac{b[1-0]}{1 - \frac{b}{a}\cdot 0} = \frac{b}{1} = b.$$

STUDENT FORM 1

Request for Help

Return to:
EDC/UMAP
55 Chapel St.
Newton, MA 02160

Student: If you have trouble with a specific part of this unit, please fill out this form and take it to your instructor for assistance. The information you give will help the author to revise the unit.

Your Name____________________ Unit No.______

Page______
- ○ Upper
- ○ Middle
- ○ Lower

OR

Section______

Paragraph______

OR

Model Exam Problem No.______

Text Problem No.______

Description of Difficulty: (Please be specific)

Instructor: Please indicate your resolution of the difficulty in this box.

○ Corrected errors in materials. List corrections here:

○ Gave student better explanation, example, or procedure than in unit. Give brief outline of your addition here:

○ Assisted student in acquiring general learning and problem-solving skills (not using examples from this unit.)

Instructor's Signature____________________

Please use reverse if necessary.

Return to:
EDC/UMAP
55 Chapel St.
Newton, MA 02160

STUDENT FORM 2

Unit Questionnaire

Name________________________________ Unit No.____________ Date______________

Institution____________________________ Course No.____________________________

Check the choice for each question that comes closest to your personal opinion.

1. How useful was the amount of detail in the unit?

____Not enough detail to understand the unit
____Unit would have been clearer with more detail
____Appropriate amount of detail
____Unit was occasionally too detailed, but this was not distracting
____Too much detail; I was often distracted

2. How helpful were the problem answers?

____Sample solutions were too brief; I could not do the intermediate steps
____Sufficient information was given to solve the problems
____Sample solutions were too detailed; I didn't need them

3. Except for fulfilling the prerequisites, how much did you use other sources (for example, instructor, friends, or other books) in order to understand the unit?

____A Lot ____Somewhat ____A Little ____Not at all

4. How long was this unit in comparison to the amount of time you generally spend on a lesson (lecture and homework assignment) in a typical math or science course?

Much Longer	Somewhat Longer	About the Same	Somewhat Shorter	Much Shorter
____	____	____	____	____

5. Were any of the following parts of the unit confusing or distracting? (Check as many as apply.)

____Prerequisites
____Statement of skills and concepts (objectives)
____Paragraph headings
____Examples
____Special Assistance Supplement (if present)
____Other, please explain________________________________

6. Were any of the following parts of the unit particularly helpful? (Check as many as apply.)

____Prerequisites
____Statement of skills and concepts (objectives)
____Examples
____Problems
____Paragraph headings
____Table of Contents
____Special Assistance Supplement (if present)
____Other, please explain________________________________

Please describe anything in the unit that you did not particularly like.

Please describe anything that you found particularly helpful. (Please use the back of this sheet if you need more space.)