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ABSTRACT

This document provides an overview of a National Science Foundation (NSF) funded project, Rational Number Ideas and the Role of Representational Systems. The rational number project consists of interacting instructional, evaluation, and diagnostic/remedial components. General project goals are: (1) to describe the development of the progressively complex systems of relations and operations that children in grades two through eight use to make judgments involving rational numbers; and (2) to describe the role that various representational systems (e.g., pictures, manipulative materials, spoken language, written symbols) play in the acquisition and use of rational number concepts. The project aims to develop a psychological "map" focusing on several aspects of the learning process. The project is concerned not only with what children can do naturally, but also with what they can do accompanied by minimal guidance or following theory based instruction. A list of five studies currently planned as future activities for the rational number project staff is included at the conclusion of this document. (MP)

Rational Number Ideas

and the Role of Representational Systems

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This paper provides an overview of an NSF funded project, Rational Number Ideas and the Role of Representational Systems (Note 1). The rational number project consists of three interacting components: (a) an instructional component in which 18 fourth and fifth graders were observed, interviewed, and tested frequently during sixteen weeks of theory based instruction, (b) an evaluation component in which more than 1600 second through eighth grade children were tested using a battery of written tests, instruction mediated tests, and clinical interviews, and (c) a diagnostic/remedial component in which children who were experiencing difficulties were identified and their misunderstandings were isolated using materials borrowed from the testing component--with remedial activities borrowed from the instructional component.

General goals of the rational number project are: (a) to describe the development of the progressively complex systems of relations and operations that children in grades 2-8 use to make judgments involving rational numbers, and (b) to describe the role that various representational systems (e.g., pictures, manipulative materials, spoken language, written symbols) play in the acquisition and use of rational number concepts. The project aims to develop a psychological "map" describing (1) how various rational number subconcepts (e.g., fractions, ratios, indicated quotients, etc.) gradually become differentiated and integrated to form a mature understanding of rational numbers, (2) how various representational systems interact during the gradual development

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of rational number ideas, and (3) how a variety of theory based interventions can alter this development.

The investigation of rational numbers is important in its own right, because so many of the "trouble spots" in elementary school mathematics are related to rational number ideas. The development of rational number ideas is also viewed as an ideal context in which to investigate general mathematical concept acquisition processes because:

(a) It occurs on the threshold of a significant period of cognitive reorganization: the transition from concrete to formal operational thinking.

(b) Interesting qualitative transitions occur not only in the structure of the underlying concepts but also in the representational systems used to describe and model these structures. For example, multiplication no longer means repeated addition; $\frac{1}{2} \times \frac{1}{3} = \square$ may be read "$\frac{1}{2}$ of $\frac{1}{3}$" rather than "$\frac{1}{2}$ times $\frac{1}{3}$," etc.

(c) The roles of representational systems are quite differentiated and interact in psychologically interesting ways. Both figurative and operational aspects of tasks are critical. Piaget has focused on the operational aspects of tasks and concepts, using the term "decalage" to refer to tasks which are characterized by the same structure but differing in difficulty. However, for rational number concepts, it is the rule, not the exception, that models embodying the same concept vary radically, i.e., by several years, in the ease with which they are understood by children. Therefore, information about how figurative content influences task difficulty is important for those who must select or devise appropriate models to illustrate rational number concepts.

(d) The rational number concept involves a rich set of integrated sub-constructs and processes, many of which have been identified in previous research. These are related to a wide range of "elementary but deep" concepts (e.g., measurement, probability, coordinate systems graphing, etc.) which are presupposed by a variety of problem solving situations and are often taken to be "easy," when, in fact, many of these concepts developed rather late in the history of science and are exceedingly unobvious to those who have not assimilated them (Hawkins, 1979).

The project is concerned not only with what children can do naturally, but also with what they can do accompanied by minimal guidance or following theory based instruction. The interest is in exploring the "zone of proximal development" of children's rational number concepts (Vygotsky, 1976), not only to describe the "schemas" children typically use to process rational number information and interpret rational number situations, but also to describe how these schemas change as a result of theory based instruction. Rather than seeking only to accelerate rational number understandings along narrow conceptual paths, the project is interested in studying the results of broadening and strengthening deficient conceptual systems.

Theoretical Foundations:

Theoretical foundations for the project were derived from four separate but mutually supportive theoretical bases:

(a) Kieren's (1976) mathematical analysis of rational number into sub-constructs;

(b) Post and Reys (1979) interpretation of Dienes' perceptual and mathematical variability principles;

(c) Lesh's (1979) analysis of modes of representation related to mathematical concept acquisition and use;

(d) An analysis of memory structures developed by the learner (Behr, Lesh, & Post, Note 1).

(a) The Analysis of Rational Number Subconstructs.

Kieren (1976) provided a logical analysis of the rational number construct into five subconstructs. Our work has resulted in a redefinition of some of Kieren's categories and a subdivision of others. The resulting scheme includes the following seven subconstructs.

The fractional measure subconstruct of rational number represents a reconceptualization of the part-whole notion of fraction. It addresses the question of how much there is of a quantity relative to a specified unit of that quantity.

The ratio subconstruct of rational number expresses a relationship between two quantities; for example, a relationship between the number of boys and girls in a room.

The rate subconstruct of rational number defines a new quantity as a relationship between two other quantities. For example, speed is defined as a relationship between distance and time. We observe here that one adds rates in such a context as computing average speed but seldom adds ratios.

The quotient subconstruct of rational number interprets a rational number as an indicated quotient. That is, a/b is interpreted as a ÷ b. For a curricular context this subconstruct is exemplified by the following problem situation.

> There are 4 cookies and 3 children. If the cookies are shared equally by the three children, how many cookies does each child get?

The linear coordinate subconstruct of rational number is similar to Kieren's notion of a measure interpretation. It emphasizes properties associated with the metric topology of the rational number line such as betweenness, density, distance, and (non) completeness. Rational numbers are interpreted as points on a number line and suggests the rational numbers as a subset of the real numbers.

The <u>decimal</u> subconstruct of rational number emphasizes properties associated with the base ten numeration system.

The <u>operator</u> subconstruct of rational number imposes on rational number a function concept; a rational number is a transformation. The stretcher-shrinker notion developed by UICSM, CSMP, and Z. P. Dienes represents physical embodiments of this construct.

(b) <u>The Analysis of Concrete Models Related to Rational Number Concepts</u>.

Post and Reys (1979) interpreted the mathematical and perceptual variability principles of Dienes using a matrix model. Kieren's rational number subconstructs constituted the mathematical variability dimension of the matrix. The perceptual variability dimension included discrete objects, length models, area models, and written symbolic models.

The rational number project has refined Post and Reys' matrix to include the categories shown in Figure 1.

Mathematical Variability

Perceptual Variability	Fractional Measure	Ratio	Rate	Quotient	Decimals	Linear Coordinates	Operators
Discrete							
Countable Continuous							
Continuous							

Figure 1

Operational definition of the concept of rational number from which instructional routines are developed.

Discrete materials used in the project included counters, egg cartons, and other sets of objects. Continuous materials involved some quantity such as length or area, and included Cuisenaire rods, number lines, and sheets of paper. Countable continuous materials involved a continuous quantity which had been partitioned into countable units. Unlike the discrete materials, however, the countable continuous units had to be the same "size" (but not necessarily the same shape). Examples of countable continuous materials included tiles and graph paper.

(c) An Interactive Representational Systems Model.

Lesh (1979) reconceptualized Bruner's (1966) enactive mode, partitioned Bruner's iconic mode into manipulative materials and static figural models (i.e., pictures), and partitioned Bruner's symbolic mode into spoken language and written symbols. Furthermore, these systems of representation were interpreted as interactive rather than linear, and translations within and between modes were given as much emphasis as manipulation of single representational systems. Figure 2 shows the rational number project's modified version of Lesh's model.

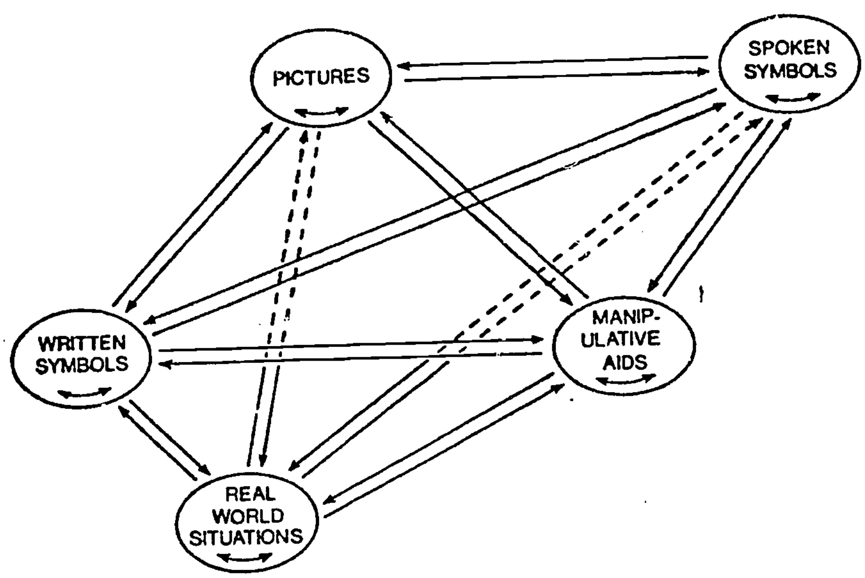

Figure 2

An interactive model for using representational systems.

A major focus of the project is the role of manipulative materials in facilitating the acquisition and use of rational number concepts as the child's understanding moves from concrete to abstract. Psychological analyses show that manipulatives are just one component in the development of representational systems and that other modes of representation also play a role in the acquisition and use of concepts (Lesh, Landau, & Hamilton, 1980). A major hypothesis of the project is that it is the ability to make translations among and between these several modes of representation that makes ideas meaningful to learners.

Two triads in Figure 2 are of particular interest in our research. One involves the translations between manipulative aids and mathematical symbols, with the oral mode serving as a mediator. The other involves real world situations, manipulative aids, and written symbols; the concern is how to use manipulative aids to facilitate the mathematical modeling required in problem solving.

Figure 2 is intended to suggest that realistic mathematical problems are frequently solved by: (a) translating from the real situation to some system of representation, (b) transforming or operating on the representational system to produce some result or prediction, and (c) translating the result back into the real situation (see Figure 3).

Figure 2 is also intended to imply that many problems are solved using a sequence of partial mappings involving several representational systems. That is, pictures or concrete materials may be used as an intermediary between a real situation and written symbols, and spoken language may function as an intermediary between the real situation and the pictures, or the pictures and the written symbols.

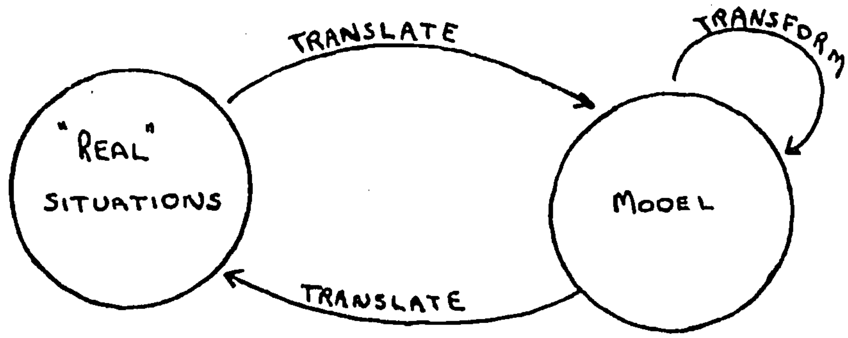

Figure 3

Modeling processes needed to use
rational number ideas in real situations.

(d) Memory Structures.

Various writers discuss categorizations of memory. For example, Gagné and White (1978) consider the relationship between memory structures and learner performance. Tulving (1972) discusses functions of different types of memory structures and the interrelations among them. Of interest to this project are memory structures called episodic, imaginal, semantic, and intellectual skills. This theory serves two functions: (1) to guide the development of the interview protocols for assessment of student learning, and (2) to provide the theoretical rationale for the proposed oral mode study.

(e) Internal Versus External Representations.

The rational number project has been especially interested in interactions between internal and external representations of problem situations. Frequently, when children solve problems, an internal "interpretation" of the problem influences the selection (or generation) of an external representation. This external representation may involve a picture, concrete materials, or written symbols. Often the external representation models only part of the problem. For example, if the problem involved "addition of fractions," then the child's first picture might represent the fractions without any attempt to represent the addition process. Or, an even more primitive drawing might more closely resemble a photograph of the problem situation than a schematic diagram depicting the underlying mathematical relationships. In either case, however, the external representation typically allows the child to refine his internal representation/interpretation--which may lead to the generation (or selection) of a more refined external representation, or to a solution.

External representations play a number of important roles in the acquisition and use of rational number concepts. These roles include: reducing memory load or increasing storage capacity, coding information in a form that is more manipulable, simplifying complex relationships, and so on.

A Brief Description of the Major Project Components:

(a) The Instructional Component.

A. Materials Development - Twenty weeks of student instructional materials (over 600 pages) have been developed by project staff, piloted, and used in teaching experiments with fourth and fifth grade children. Extensive observation guides and interview protocols have been produced to collect data about children's cognitive behavior on a lesson-by-lesson basis. The materials were written and piloted during the project's first year (1979-1980), revised during Summer 1980, and used in small group, teaching experiment type settings at Minneapolis and DeKalb during the current year (1980-1981).

The instructional materials reflect the project's underlying theoretical foundations. Part-whole, quotient, measure, and ratio interpretations of rational number, and translations within and between five representational modes are emphasized.

Each twenty-week teaching experiment (one in DeKalb, two in Minneapolis) involved groups of six students and utilized audio and video taping and extensive interviewing and pre- and post-achievement testing. Control students were identified for each group for comparison purposes.

Normally a minimum of three project staff members were present for each lesson, one teacher and two or more observer/interviewers. Observational data, frequent interviews, and audio or video taping of many lessons resulted in a vast amount of anecdotal data. Post-tests were administered to assess the impact of the instructional program. Spring and Summer 1981 will be used for data analysis and reporting of results.

The first phase of the project's instructional program has provided numerous insights into the way students interact with project materials and has suggested areas in need of further refinement. Most importantly, these teaching experiments have begun to uncover the nature of the difficulties which impede effective learning of these concepts.

One example will be given here. Rational number concepts appear to be deceptively simple. For example, in the part-whole interpretation, any rational number $\frac{a}{b} < 1$, $b \neq 0$, has at least three different quantitative dimensions: (1) the size of each piece (part), (2) the number of subdivisions into which the whole has been divided, and (3) the overall magnitude of the fraction, which is dependent upon the relationship between these. We have found that children, even after 12 weeks of instruction, continue to confuse these variables. This confusion is further confounded by a language problem in which words such as "amount," "size," and "value" are interpreted differently within individual children and differentially across children.

A child, focusing on the size of each piece will order 1/4 and 1/3 differently than the child who is concerned with the number of pieces into which the unit has been divided. Furthermore, the supposedly clarifying teacher-question, "Which is the larger amount, 1/3 or 1/4?" does not always clarify the issue since some children interpret "amount" to mean size and others have confounded it with the concept of number. Past usages and experiences have no doubt had powerful effects on students' interpretations of quantitatively oriented language. Rational number concepts and their associated meta-languages continually require restructuring and reformulating of past interpretations and experiences with numbers. Traditional programs pay little if any attention to this aspect of concept development.

Phase I of the rational number project has already unearthed a number of problems of this nature. These have a profound influence on student conceptual development. Others are sure to follow when the large amount of existing data is more carefully scrutinized.

B. Data Collection and Analysis - Four major types of instruments have been used at the DeKalb and Minneapolis sites to identify and assess the development of children's rational number concepts within the theory based instruction.

(1) The Rational Number Test was used as a pre- and post-measure with both experimental and randomly selected non-experimental students. This test, mainly concerned with content mastery, identified levels of student achievement in three areas: rational number concepts, relations, and operations. This instrument was also used with classroom groups in grades two through eight across five geographic locations ($n > 1600$).

(2) Class Observation Guides were developed for each of the 12

lessons. Each lesson spanned from two to six instructional days. These guides were designed to provide staff with insights into the cognitive processes employed by students when dealing with situations involving rational number concepts. Since the amount of information called for was extensive, the guides were often supplemented by audio or video tapes.

(3) Interview Protocols. The individual interview, conducted with each student after each lesson, is considered a crucial source of project data. These interviews, lasting from 15 to 50 minutes, were designed for both specificity and flexibility. They provide extensive information on the mental processes, memory structures (inferred), thought patterns, and understandings gained and utilized. These data will provide detailed longitudinal information on the development of rational number concepts in individual students. Interview data is examined on a lesson-by-lesson basis to assess the impact of specific instructional "moves" on conceptual development. Either audio or video tapes were always used to provide a record of these interviews.

(4) Translation Coding System. This instrument was designed to provide specific information as to the types of translations which students used, the relative frequency of each type, and the identification of those which proved particularly troublesome. This observational scheme is important to the goals of the present project since an underlying hypothesis has been that "translations within and between modes contribute to the meaningfulness of ideas in children."

(5) Data Analysis. Our observation and data analysis to date have provided clear directions for future work. More extensive and refined data analysis will be undertaken following completion of the teaching experiments.

These refined analyses will give detailed information about the mathematical and psychological variables which affect the development of rational number concepts in children.

(b) The Evaluation Component.

There are three components to the testing program: paper and pencil tests, instruction mediated tests, and clinical interviews.

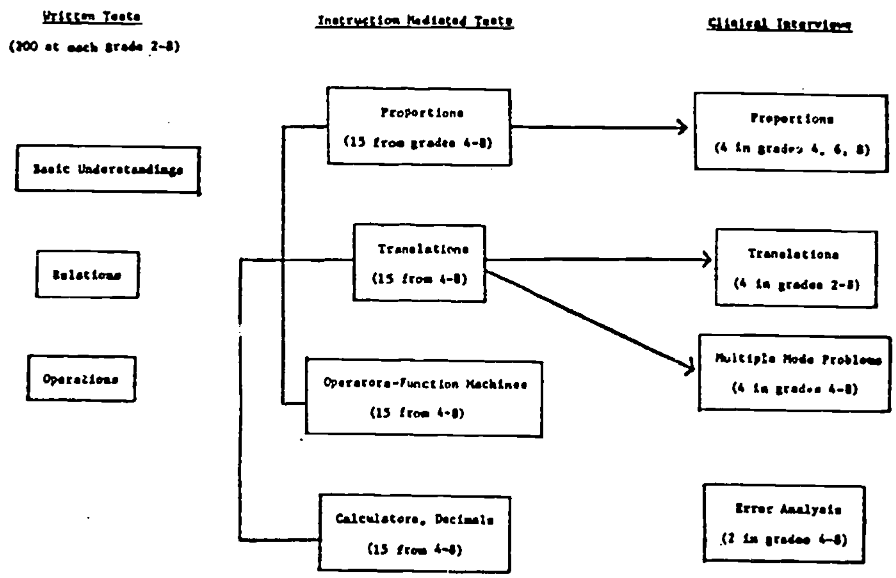

Figure 4

Components of the evaluation program.

The paper and pencil portion consists of three tests: Concepts, Relations, and Operations. The first assesses basic fraction and ratio concepts. The second assesses understanding of relationships between rational numbers, such as ordering, equivalent forms, and simple proportions. The third test assesses abilities to perform addition and multiplication with fractions, and various applications items. For purposes of pre- and post-testing, parallel versions of each test were constructed. The tests are modularized to accommodate children in grades two through eight.

The paper and pencil tests serve five purposes. One is as a pre- and post-test instrument for the instructional unit of the project. Another is to test a cross section of elementary and junior high school students. Base-line data now exist for over 1600 students dealing with understandings of rational number concepts for children in grades two through eight. Five geographic sites (Evanston, DeKalb, Minneapolis, San Diego, and Pittsburgh) were involved in the data collection. The third purpose is to identify students for the diagnostic/remedial and error analysis component of the project. The fourth purpose is to identify students who display particular types of error patterns for follow-up clinical interviews. The fifth purpose was to integr... whenever possible, results from previous research (e.g., Wagner, 1976; Karplus, 1980; Kieren, 1976; Carpenter, Coburn, Reys, & Wilson, 1978; Noelting, 1979; Klahr & Siegler, 1978) and to provide a basis for future research.

Some General Observations:

Past research indicates that instruction using manipulative aids is at least as effective, but perhaps less efficient, than other forms of instruction. Unfortunately, the learning outcomes that have been assessed nearly always have been restricted to initial learning or short term retention; less attention has been given to the transferability and usefulness in real problem solving situations of the learned concepts--the precise learning outcomes that concrete materials are expected to facilitate. The best manipulative aids are half-way between the real world of everyday mathematical situations and the world of abstract ideas and written mathematical symbols. They are symbols in that they can be used to act out (or represent) a variety of real world situations, and they are concrete in that they involve real materials (e.g., Cuisenaire rods, folded paper discs) that are similar to everyday objects like lumber, pies, and cakes. By helping youngsters move gradually from concrete to progressively more abstract understandings of mathematical ideas, concrete materials serve as a bridge from the real world into the world of mathematics. However, they can also serve as a bridge from the world of mathematics back into the real world, using the kinds of translation/modeling processes depicted in Figure 2 earlier in this paper.

The rational number project has shifted away from attempting to identify the "best" manipulative aid for illustrating (all) rational number concepts toward the realization that: (a) different materials are useful for modeling different real world situations or different rational number subconcepts (i.e., part-whole fractions, ratios, operators, proportions, etc.); (b) different materials may be useful at different points in the development of rational number concepts. For example, paper folding may be excellent for representing part-whole relationships or equivalent fractions, but may be misleading for representing addition of fractions. There is no single manipulative aid that is "best" for all children and for all rational number situations. A concrete model that is meaningful for one child in one situation may not be meaningful to another child--or to the same child in a different situation. The goal is to identify manipulative activities using concrete materials whose structure fits the structure of the particular rational number subconcept being taught. For this reason, current research is focusing heavily on analyzing the cognitive structures children use to perform various rational number tasks.

From the above kinds of task analyses, it has become clear that, even within the category of concrete materials, some are more concrete than others. Therefore, for teachers who are attempting to concretize abstract rational number concepts, it is important to begin instruction with those materials that are most concrete, least complex, and which draw upon useful intuitive understandings. For example, a balance scale can be used to illustrate certain facts about ratios and proportions, but the ideas required for understanding a balance scale are more complex and sophisticated than those required for understanding the underlying mathematical models (i.e., ratios and proportions). Ideas about ratios and proportions explain the principles underlying a balance scale rather than the balance scale explaining the principles underlying ratios and proportions. Therefore, a balance scale is a rather abstract apparatus for rational number concepts, and is not useful in early rational number instruction.

Because teachers can illustrate ideas such as "addition of fractions" using folded paper, Cuisenaire rods, or other manipulative materials, they may underestimate the level of sophistication that is required for performing these tasks. It is one thing for a child to know how to illustrate fractions like 1/2 or 1/3 using Cuisenaire rods, and quite another to be able to illustrate 1/2 + 1/3 using the rods. Concrete materials that are useful for illustrating fractions may not be useful for illustrating <u>addition</u> of fractions. That is, the addition of fractions may be more meaningful if it is built on a strong concrete understanding of individual fractions, but this does not imply that learning to add Cuisenaire rods or folded paper will facilitate the child's understanding of addition of fractions.

Young learners do not work in a single representational mode throughout the solution of a problem. They may think about one part of the problem (say, the number) in a concrete way, but may think about another part of the problem using other representational systems (i.e., actions, spoken language rules, or written symbol procedures). It is only a mature student who becomes able to work through an entire complex problem using a single representational mode or model.

Not only do youngsters shift from one representational mode to another during different solution steps to a problem, but some problems inherently involve more than one mode at the start. For example, in real addition situations that involve fractions, the two items to be added may not always be two written symbols, two spoken symbols, or two pizzas; they may be one pizza and one written symbol, or one pizza and one spoken word. That is, the problem may involve showing the child half of a pizza and then asking how much he would have if he were given another 1/3 of a pizza. In such problems, which occur regularly in real situations, part of the difficulty is to represent both addends using a single representational system.

While a goal of the rational number project has been to trace the development of rational number ideas, our data have made us sensitive to the need to explain concept stability and concept deterioration--as well as concept development. For example, even though our study used criteria for "mastery" which were considerably more stringent than those typically used in school instruction, it was common to observe significant regression in concept understanding over two or three week periods. Not only must "mastered" concepts be remembered, they must be integrated into progressively more complex systems of ideas; and sometimes they must be reconceptualized when they are extended to new domains. Ideas which are "true" in restricted domains (e.g., "multiplication is like repeated addition" or "a fraction is part of a whole") are misleading, incorrect, or not useful when they are extended to new domains. Mathematical ideas usually exist at more than one level of sophistication. They do not simply go from "not understood" to "mastered." Therefore, as they develop they must be reconceptualized periodically, and they must be imbedded in progressively complex systems which often may significantly alter their original interpretation.

Directions for Future Research.

The following studies are currently planned as future activities for the rational number project staff.

(1) Longitudinal Study. The purpose of this three-year study is to investigate the effects of exposing children to the five rational number subconstructs on the development of their rational number understandings over an appropriate period of time and to extend the instructional group to a full-sized classroom.

(2) Unit Fraction Approach to Rational Number Study. The purpose of this study is to examine more closely the development of initial rational number concepts through iteration of unit fractions as contrasted to the more conventional part-whole interpretation.

(3) Oral Mode Study. The purpose of this study is to explore the hypothesis that the oral mode of representation is an important facilitating intermediary between mathematical ideas as embodied in manipulative materials or iconic representations and written symbols.

(4) Teacher Training, Classroom Implementation, and Selected Hypothesis Testing. This phase of the project will assess the feasibility of training classroom teachers to use project developed instructional materials, evaluate the impact of these materials in classroom size groups of fourth and fifth grade children, and experimentally validate selected project hypotheses.

(5) Longitudinal Testing Study. This effort will replicate and extend data gathering procedures on large numbers of students across seven grade levels (two through eight) and will assess the change in children's rational number thinking from grade to grade.

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