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ABSTRACT

This report is concerned with the observation that many teenagers and adults are unable to apply arithmetic even when they know how to perform the operations. It is speculated that this is probably due to not having been taught when to use the fundamental operations. Examples of types of "real" problems which require the various operations for their solution are presented. The applications of each operation are grouped by addition, subtraction, multiplication, and division and a model example is included with each category. The use of models for operations is considered important because they: (1) help to show when to use a given operation: (2) show important relationships between arithmetic studied and the real world: (3) can be used to develop and illustrate properties of operations: (4) can be used to check answers: (5) help to classify the applications of arithmetic: and (6) help to explain concepts in higher mathematics. (MP)

USES OF THE FUNDAMENTAL OPERATIONS[1]

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As every teacher knows, there is a vast difference between being able to multiply and being able to apply multiplication to a real situation. Performing multiplication is the means; applying multiplication is the end. The National Assessment of Educational Progress results (NAEP, 1975) showed that many adults can do multiplication and other fundamental operations accurately but these same adults often cannot apply the same operations in real situations.

Calculators make it even easier for a person to have the ability to get the right answer. So there is pressure on teachers of arithmetic to teach students <u>when</u> (i.e., to what situation) a particular operation should be applied. Given the increasing mathematical sophistication of every day life, this is a welcome trend. However, textbooks usually do not teach when to apply an operation except by inference from a very few examples in a very few places.

The purpose of this article is to offer a few examples of types of real problems which require the various operations for their solution. A scheme is given which hopefully will help the teacher to create further examples.

Uses of Numbers

Numbers are used for indicating results of counting, ordering, identification, locating, scoring, measuring, comparison, and change.

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These uses are described in a book by Bell (1972) which unfortunately is out of print. Negative numbers are appropriate whenever a situation has two directions. (An easy to read related article is by Schultz (1973).) Of all the uses, three types seem to be most important:

counting

measurement

comparison

These uses are related. One often compares measurements. One often counts to measure. But counting only requires the whole numbers. Measurement normally requires all positive numbers. Comparison leads one often to fractions or percentages. If situations have two directions (money coming in or out, temperature going up or down, profit or loss), then negative numbers are appropriate.

Uses of Operations

Every operation has many many uses. It is because of the uses that it is so important to learn them. The basic task for the teacher is to have some sort of scheme which can help learning of these uses.

What is done here is to group the applications of each operation into categories. In each category a model example of that category is given. Although the model is described here by variables, it is not necessary to do that for students.

Addition

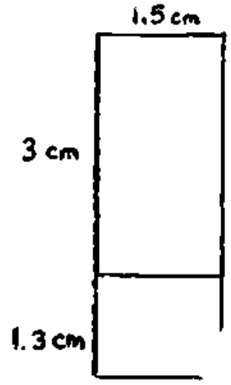
The uses of addition are the best taught of any operation. All of the models given here are in the standard curriculum.

Model 1: Union. If set A has a elements and set B has b elements, and A and B have no elements in common, then there are a+b elements altogether.

Example: If there are 12 boys and 15 girls in a class, then there are 12 + 15 students in all.

Model 2: Joining. If something with measure a is joined to something with measure b, then the total (union) has measure a+b.

Example: If the area of the top rectangle at left is 4.5 sq cm and the area of the bottom rectangle is 1.95 sq cm, then the area of the entire figure is 4.5 + 1.95 sq cm. (Other examples could involve angle measure, perimeter, partial and total scores on tests.)



Model 3: Sliding. If something is slid a units and then slid b units, the result is a slide of a+b units.

Example: A football team gains 6 yards on one play and loses 8 yards on the next. The net gain is 6 + -8 or -2 yards. (Other examples could involve changes in stock prices, weight gained and lost, or money in and out. In each case, the "+" stands for "followed by.")

The reader may recognize that Model 3 can be thought of as a generalization of Model 2, which is in turn a generalization of Model 1. So one might argue that only Model 3 needs to be taught. But to students these models "feel" different and different numbers are characteristically involved. So it is pedagogically helpful to separate them.

Subtraction

The major uses of subtraction are not easily considered as generalizations of each other. Only the first is commonly taught in the standard curriculum.

Model 1: Take-away. If a set has $\underline{a}$ elements (or measure $\underline{a}$) and $\underline{b}$ elements (or part with measure $\underline{b}$) are taken away, the result is a set with $\underline{a-b}$ elements (measure $\underline{a-b}$).

Example: Twenty dollars discount is given on an item which sells for $79.95. The resulting cost is $\underline{79.95 - 20}$ dollars. (Cost is a very common measure of the value of an object; other examples could deal with cutting off a length from a straw or an area from a region.)

Model 2: Fill-up. In order to increase an amount $\underline{b}$ to an amount $\underline{a}$, something of amount $\underline{a-b}$ is needed.

Example: In the House of Representatives, 218 members constitute a majority. If you know you have 173 votes, how many more are needed to get a majority? (Answer: 218 - 173.)

Model 3: Slide Comparison. If two things measure a and b, the first is $a-b$ "bigger than" the second.

Example: The elevations of Mt. Whitney and Death Valley (the highest and lowest points in California) are 4418 and -86 meters respectively. Mt. Whitney is thus 4418 - -86 meters higher than Death Valley. In turn, Death Valley is -86 - 4418 meters "higher" than Mt. Whitney; the negative result here means the opposite of "higher," namely "lower." (Other examples could deal with comparing prices from one month or year to the next, comparing a prediction of a score with the actual score, or comparing heights.)

Because Model 3 for subtraction is not normally exemplified in the curriculum, subtractions with negative numbers must often be contrived by extending the take-away model in rather phony ways. One of the advantages of having models is that they make it easier to find realistic uses of numbers other than the counting numbers.

Multiplication

Multiplication is harder to deal with than either addition or subtraction. The applications are more varied and not as easy to describe. Floyd Vest (1969) distinguished 20 different models! Many of these are combined below so that the list contains 4 models. The only one normally taught to students is the repeated addition model.

Model 1: Cartesian Product (or Ordered Pair or Filling Blanks). If set

A has a elements and set B has b elements, then there are ab ways of connecting an element of A to an element of B.

Example: With 3 blouses and 5 pairs of slacks, there are

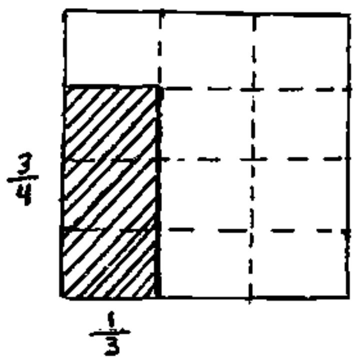
3 X 5 possible outfits.

Some books break this model up into various sub-models suggested by the different pictures given here for 3 X 5.

. . .	11	21	31
. . .	12	22	32
. . .	13	23	33
. . .	14	24	34
. . .	15	25	35

Model 2: Area. If a rectangle has length a and width b, then its area is ab.

Example: The shaded rectangle at left is 1/3 unit by 3/4 unit. Its area is seen to be 3/12 or 1/4 of the square unit. (The area model can be generalized to other measures. One can multiply 150 watts by 60 hours to get 9000 watt-hours or 9 kilowatt-hours. 40 feet times 3 pounds gives 120 foot-pounds. 6 hours at 20 kilometers per hour gives 120 kilometers. The product is always in a different unit than either of the two factors.)



Model 3: Size Change. If something measures b and a size change factor a is applied, the result has measure ab.

Example: Suppose a person makes \$2.30 an hour and gets time-and-a-half for overtime. The "time-and-a-half" signified a size change or scale factor of 1.5 to calculate overtime pay. So the overtime pay is 1.5 X \$2.30 per hour, or \$3.45 per hour. (A 20% discount implies

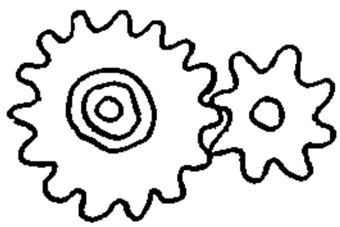
a scale factor of .80; almost all uses of percentages involve scale factors. In size change applications, one factor always has no unit - it is the scalar - while the product has the same unit as the other factor. In "their apartment is twice as big as ours" - the "twice" indicates a scale factor of two.)

Model 4: Repeated addition. $\underbrace{b + b + \cdots + b}_{a \text{ terms}} = ab$

Example: If 6 cans of orange juice are bought for 49¢ apiece, the total cost is .49 + .49 + .49 + .49 + .49 + .49 or 6 X .49.

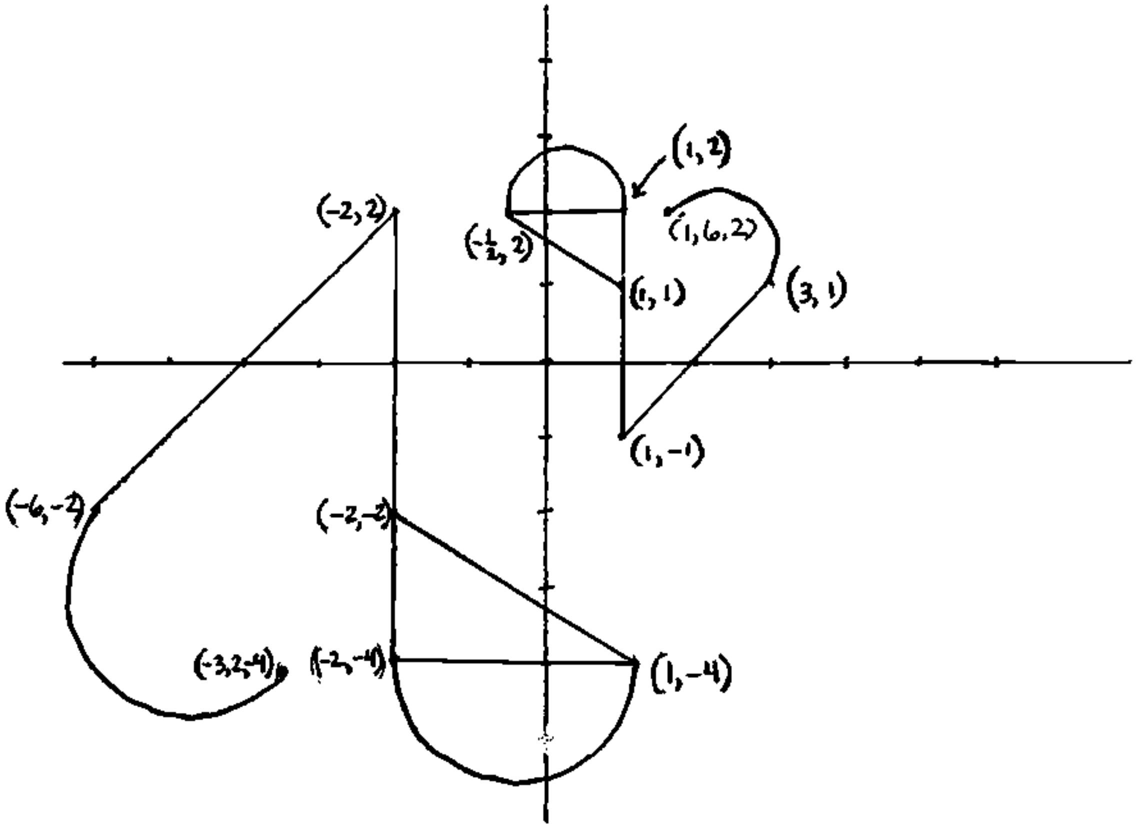
It is difficult to judge which model for multiplication appears most often in the real world. What is certain is that repeated addition is not enough to explain uses of multiplication of fractions or of decimals.

The size change model can be extended to include negative scale factors. A negative scale factor means a change in direction from profit to loss, in to out, up to down, etc. For example, pictured below is a gear system which pictures a scale factor of -2. That is, one revolution of the left gear yields 2 revolutions of the right gear in the opposite direction.



It is customary to represent counterclockwise turns by positive numbers, clockwise turns by negative numbers. If the left gear turns -5 (that is, 5 revolutions clockwise), then the right gear will turn -5 X -2 or 10 (that is, 10 revolutions counterclockwise).

The size change model gets its name from its application to pictures. Here is another picture of a scale factor of -2. Coordinates of points in the smaller upright figure have been multiplied by -2 to yield the larger upside-down figure. The larger figure has linear dimensions twice those of the smaller and it faces the opposite direction.



In geometry, one would call the above figures similar with a ratio of similitude of 2:1 (or, most simply, 2). That ratio is calculated by dividing lengths and is an example of the third model for division dis-

cussed below.

Division

Division is the most difficult of operations for students to apply. Many students who know how to divide whole numbers, decimals, and fractions cannot think of one instance in which division is used.

Model 1: Splitting-up. If a set with measure a is split evenly into b parts, each part will have measure a/b.

Example: If 50 cookies are divided among 5 children, each child can have 50/5 cookies. (If one number does not divide evenly into the other, the quotient often still has meaning. For example, if $7\frac{1}{2}$ cookies are divided among 5 children, each child could get $1\frac{1}{2}$ cookies. It is also possible to split up lengths, areas, or scores.)

Model 2: Rate. Let a first quantity be measured in a unit (unit 1), a second quantity in a different unit (unit 2).

$\frac{a \text{ unit 1's}}{b \text{ unit 2's}} = \frac{a}{b}$ unit 1's per unit 2, the mean rate of unit 1's per unit 2.

Example: If there are 62 students in 3 classes, then the mean rate of students per class is $\frac{62 \text{ students}}{3 \text{ classes}}$ or $\frac{62}{3}$ students per class. (The most common rate is distance divided by time, represented as mph, kph, or feet per second. There are many other rates, such as cost per ounce, pressure (i.e., force per unit area), words per minute, people per square mile, telephones

per house, and so on.)

Model 3: Size comparison. If two things measure a and b, the first is a/b times the size of the second.

Example: If a company makes \$800,000 profits one year and \$1,000,000 in profits the second, then the first year's profits were $\frac{800,000}{1,000,000}$ or .8 times or 80% of the second year's profits. Dividing the other way, the second year's profits were 10/8 or 1.25 or 125% of the first year's profits. (Other examples could deal with weights, lengths of time to do an assignment, areas or lengths in geometric figures.)

Model 4: Repeated Subtraction. a/b is the number of times that b can be subtracted from a without arriving at a negative number.

Example: If there are 100 cookies and children march by, each child taking 4 cookies, then you will be able to serve 100/4 children. (One can also interpret the answer if b does not divide evenly into a.)

The long division algorithm is based upon repeated subtraction; otherwise that model for division is the least used of any.

Size comparison differs from rate in that the measures being compared in size comparison must be of the same ilk. It is possible to compare miles to kilometers in size; the result will be a scalar - a conversion factor - of the type found in the size change model of multiplication.

There is a tendency to ignore the rate model because the units "get in the way." However, the units can help in understanding. The

word "per" can be interpreted to mean "divided by," and so "miles per hour" becomes "miles divided by hours," exactly what is done to calculate the rate. That "per cent" then means "divided by 100" is a wonderful bonus.

Both subtraction and division are used for comparison. This can result in great confusion because the English is the same whereas the mathematics is different. Compare these two sentences:

Prices of coffee are down 25%.

Prices of coffee are down $1.00 a pound.

Being mathematically precise, 25% does not equal $1.00 a pound. 25% = .25 and has no associated unit. The first sentence is <u>size</u> comparison, comparing the amount of decrease to the original price by division. The second sentence is <u>slide</u> comparison, comparing the original price to the new price by subtraction. The potentially confusing English occurs in many data reports. If the number of cases per year of a rare disease in the U.S. increase from 4 to 8, those who wish government funding for research will speak of the 100% increase. Those wishing to alleviate fears will speak of the increase of 4.

Summary of Models

Just as there are comparison models in both division and subtraction, the other models can be related to each other. The following table displays some of the relationships. Three of the four column headings relate to fundamental uses of numbers. In this way the uses of numbers and uses of operations are associated.

Operation	Counting model	Measure model	Transformation/ Comparison model	Repetition model
Addition	union	joining	slide	
Subtraction	take-away or fill-up		slide comparison	
Multiplication	cartesian product	area	size change	repeated addition
Division	splitting up	rate	size comparison	repeated subtraction

Table: Uses of Numbers

Why Models for Operations?

This article makes a great deal of fuss about models for operations. At the beginning, one reason for this was given.

1. Models help to show when to use a given operation.

In telling when to use an operation, models motivate why so much time is spent studying the operation.

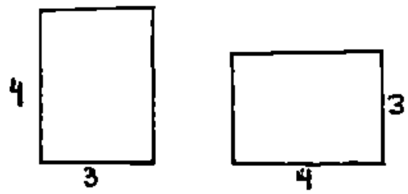
2. Models show the important relationships between the arithmetic the child studies and the real world.

Many teachers use models like these to explain properties of the operations. For example, the area model for multiplication helps to explain commutativity of those operations. This cannot be done easily if the student only knows multiplication as repeated addition.

For example, for $3 \times 4 = 4 \times 3$:

$4 + 4 + 4$ $\qquad$ $3 + 3 + 3 + 3$

Sums equal?
(not obvious without calculating)



Areas equal?
(obvious without calculating)

The models also tell what to do with negative numbers. Almost every book uses the slide model to illustrate addition of positive and negative numbers. To illustrate multiplication, recall that a negative scale factor switches direction. Thus a negative scale factor applied to a negative number makes the product positive; a negative scale factor applied to a positive number makes the product negative. To illustrate division of negative numbers, the rate model can be used. Here are two situations which lead to the same rate.

Situation 1: Twenty days from now I hope to weigh 5 pounds more.

$$\text{Rate of weight gain} = \frac{\text{5 pounds more}}{\text{20 days from now}} = \frac{\text{5 pounds}}{\text{20 days}} = \frac{1}{4}\text{ pound/day}$$

Situation 2: I weighed myself 20 days ago and weighed 5 pounds less.

$$\text{Rate of weight gain} = \frac{\text{5 pounds less}}{\text{20 days ago}} = \frac{\text{-5 pounds}}{\text{-20 days}} = \frac{1}{4}\text{ pound/day}$$

Since the situations yield the same rate, $\frac{5}{20} = \frac{-5}{-20}$. In this way:

3. Models can be used to develop and illustrate properties of operations.

A fourth reason for models is related to correcting errors. If a student makes an arithmetic error, the teacher can use a model to show that the student was wrong. For addition of decimals, it is common to use money. Suppose, in explaining that .30 + .70 should be 1.00, students are told to think of dollars and cents. This is an implicit use of the joining model for addition, for we assume the student knows that when you put money together ("join" it), the mathematical operation of addition gives you the total amount. Students who understand this application are helped to understand the mathematics.

4. Models can be used to check answers.

A fifth reason for using models is more subtle. There are myriads of applications of arithmetic. The twentieth century has seen a remarkable increase in the amount of arithmetic used by the average citizen: checking accounts; purchases of floor, window, or wall coverings; understanding of polls and odds and lotteries; mortgages, retirement plans, loans; and so on. Jobs have seen an even greater increase in mathematical sophistication, due very much to computers. With all of this, there now is a need to begin to classify the applications in order that they may be more easily assimilated and understood.

5. Models help to classify the applications of arithmetic.

There are operations other than the four fundamental operations. A fifth fundamental operation is "powering," found in expressions of the form a^b. There are models for powering just as for the operations described above. These models are detailed elsewhere (Usiskin, 1976).

In algebra, the arithmetic models help explain why the formula for slope involves both subtraction and division. (Since slope is rate of change, operations which describe change and which describe rate are needed.) In calculus, rate undoes area and vice-versa just as the rate model for division corresponds to the area model for multiplication, its inverse operation. This gives a further reason for having models.

6. Models help to explain concepts in higher mathematics.

Summary

Many teenagers and adults are unable to apply arithmetic even when they know how to perform the operations. This is probably due to not having been taught when to use the fundamental operations. Given here is a list of what seem to be the most basic types of applications for each of the fundamental operations. These applications are categorized in a way which may have many potential pedagogical benefits.

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15. Abstract

This paper is concerned with the observation that many teenagers and adults are unable to apply arithemetic even when they know how to perform the operations. This article offers examples of types of real problems which require the various operations for their solution. The applications of each operation are grouped into categories and a model example is included within the category. The use of models for operations is considered to be important for the following reasons: (1) They help to show when to use a given operation; (2) They show important relationships between arithmetic studied and the real world: (3) They can be used to develop and illustrate properties of operations; (4) They can be used to check answers; (5) They help to classify the applications of arithmetic; (6) They help to explain concepts in higher mathematics. It is anticipated that the categories of applications presented in this paper will have direct pedagogical benefits.

16. Descriptors

Mathematics Instruction
Mathematical Models
Mathematical Applications
Mathematical Concepts
Instructional Materials
Mathematics Curriculum
Adolescents
Adults

17. Identifiers

18. Field Mathematics, General / Mathematical Sciences	19. Target Audience Grades 1-12
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