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ABSTRACT

This is one of a series of geometry modules developed for use by secondary students in a laboratory setting. The thrust of this module is to introduce the student to transformations by having the student physically transform sets of points. Heavy use of manipulatives is made to aid the student in the transforming activity. Individual sections included in this module are: (1) What is a Transformation: (2) Paper Folding Machine: (3) Transparent Mirror Machine: (4) Symmetry (optional): (5) Reflecting a Reflection (composition of reflections): (6) The Rotor: (7) The Translator: (8) Isometrics: (9) Size Transformations: and (10) Similarity Transformations. (Author/MK)

GEOMETRY MODULE FOR USE

IN A

MATHEMATICS LABORATORY SETTING

TRANSFORMATIONS

by

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Overview

The thrust of this module is to introduce the student to transformations by having the student physically transform sets of points. Heavy use of manipulatives is made to aid him in the transforming activity. Perhaps one word of caution is appropriate here. A student should not be allowed to become dependent on the transformation machines. That is, he should be able to perform any of the transformations used without the aid of a machine if asked to do so.

The individual sections included in this module are:

I. What is a Transformation?
II. Paper Folding Machine
III. Transparent Mirror Machine
IV. Symmetry (optional)
V. Reflecting a Reflection (composition of reflections)
VI. The Rotor
VII. The Translator
VIII. Isometries
IX. Size Transformations
X. Similarity Transformations.

The initial activity of the module is aimed at reviewing the concept of function thus providing a basis for understanding transformations. The student is asked to model some functions and transformations to help him become comfortable with these concepts. In the next two activities, two devices are introduced for the purpose of assisting the student in performing line reflections on a set of points. The second of these is used repeatedly throughout the remainder of the module. For example: the activity on symmetry depends on the use of MIRA or a similar device.

Before rotations or translations are introduced, an activity is provided to acquaint the student with the composition of reflections. He is thus prepared to view both a rotation and a translation as such a composition. This also allows an isometry to be defined as the composition of reflections and the use of a MIRA or its equivalent to determine if isometries exist between two given figures.

The last new concept of the module is that of a size transformation. There is no unique manipulative included for this activity. However, the knowledge that one can scale any figure up or down in size as one wishes generally provides sufficient motivation for the student. There remain the tasks of combining isometries and size transformations to form a similarity transformation and determining the existence of such transformations between given figures. Work in establishing and solving proportions derived from the similarity relation on two figures is left to a separate module on triangle similarity.

It is possible at the end of sections VIII and X that a question may arise about the congruence or similarity of two noncoplanar triangles. The answer to this question is not simple. All the transformations introduced in this module are for a plane. Therefore, it may not be possible to transform a triangle into a given congruent or similar triangle without the use of a three-dimensional transformation unless the triangles are coplanar.

The resolution of this problem lies in the expansion of the definition of isometry to include reflections about a plane. However, this also means investigating the properties of reflections about a plane, which involves checking for preservation of distance,

dihedral angle measure and betweeness of points in three-dimensions.

We suggest a more reasonable approach is to explain what is missing and what would be involved in filling in the gap. Suggest that it might be more valuable for the class as a whole to proceed through the given material. However, any student wishing to pursue this question independently should be encouraged to do so.

Objectives

- The student will be able to recognize and model a function and transformation.
- Given a set of points a student will be able to perform a reflection, rotation, translation or size transformation on that set.
- The student will be able to determine congruence of figures by exhibiting an appropriate isometry.
- The student will be able to determine similarity of figures by exhibiting an appropriate similarity transformation.

Materials

1. Ditto Sheet #1 and scissors
2. A class room set of MIRA's or the equivalent. (See catalog of Creative Publications
   P.O. Box 10328
   Palo Alto, California 94303)

3. Rulers
4. Protracters
5. Used computer cards or the equivalent.
6. Stapler
7. Graph paper

Introductory Activity

Teaching Suggestions:

1. This activity is best done in small groups, 3 or 4 people.
2. One of the secondary objectives of this section is to help the student read for understanding. Allow each group to read at least the first three definitions by themselves, and with only this background, make their models. The correctness of each group's models should then be discussed with that group.
3. The answers to the questions of Steps 2, 4, 6, 7, and 8 can be posted at some point in the room allowing groups to check them and continue work at their own pace.

Materials

1. Ditto sheet for each group
2. Scissors

Answers to questions included in the activity:

Step 2: a. No
b. Yes
c. 9

Step 4: a. 64 - You might explain to the student a method for finding this. For example, there is 1 relation with no elements, 6 relations with one element, 15 relations with 2 elements, etc. We thus arrive at the number of relations being:

$$1 + 6 + 15 + 20 + 15 + 6 + 1 = 64$$

b. 64

c. 2

Step 6: a. 2

b. 3

Step 7: 1. a, d, e

2. For part a, one element of T must be used twice to provide an image for each member of R.
For part d, one element of R will not be used as an image for an element of T.
For part e, both parts a and d failed, therefore, no function from R to T or T to R can be both one-one and onto.

Step 8: They must be the same.

What Is A Transformation?

Exercise Answers:

1. (Cat, Fight), (Cat, Food), (Mouse, Fight), (Mouse, Food)
2. (1,6) (1,4), (1,2), (5,6), (5,4), (5,2), (10,6), (10,4), (10,2)

3. (3,0), (6,0)
4. (0,0), (0,1), (1,0), (1,1)
5. A x B is an infinite set of ordered pairs containing elements of the type ($\frac{1}{2}$, 2/3), (5/16, 7/9), etc.
6. A x B is an infinite set of ordered pairs containing elements of the type (2,10), (8,4), (128,694), etc.
7. Domain = (0,1,2,3,4)
   Range = (0,1,4,9,16)
   This relation is a function.
8. Domain = (-4,-2,0,2,4)
   Range = (0,2,4)
   This relation is a function.
9. Domain = (0,2,4)
   Range = (-4,-2,0,2,4)
   This relation is not a function.
10. Domain = (x| x is any real number)
    Range = (y| y is any non negative real number)
    This relation is a function.
11. Domain = (x| x is a positive real number)
    Range = (y| any real number except 0)
    This is not a function.
12. D = (1,2,3,4,5)
    R = (0,1,2,3,4)
    Yes
13. D = (0,1,2,3,4)
    R = (1,3,5,7,9)
    Yes
14. D = (0,1,3,4,5,6,8,9)
    R = (1,2,3,6,7,8)
    No
15. D = (0,1,2,3,4,5)
    R = (1,3,5)
    No
16. D = (0,1,2)
    R = (0,1)
    No
17. D = (0,2,3)
    R = (0,1,2,3)
    Not even a function.

Paper Fo[illegible]g Machine

Teaching suggestions:

1. This activity is simple and should be done by each individual. The teacher might want to demonstrate finding an image point by paperfolding to save class time.
2. It is very important that the student understand why the paperfolding machine is a transformation machine. Exercise 2 should be discussed by teacher and class before going on to the next activity.
3. Instruct students to write down the answers required in the activity.

Materials
Activity card for each student.
Protractors and rulers.

Exercise Answers

1. Discuss this with the class.
2. The teacher could make a transparency of a student's work and display on the overhead or pass around the room to use as an overlay key.
Another alternative is to delay until the student can use the transparent mirror in the next activity and then come back and verify the results obtained here.

Transparent Mirror Machine

Teaching suggestions:

1. Some students will have trouble understanding how to operat[illegible] machine and need help getting started.

2. Students enjoy working with this machine and might benefit by being allowed time to explore its possibilities before beginning the activity. Suggest they try to find the perpendicular bisector of a line segment, an angle bisector, construct a 60° angle, etc.
3. Make a transparency of a student's finished activity to display on the overhead or pass around the room to use as an overlay key.
4. At the end of the exercises discuss with the class why the transparent mirror is a transformation machine.

Materials
Mira or the equivalent (class room set).
Protractors
Rulers

Transparent Mirror Machine

Exercise Answers

1. Yes (Suggest they use the transparent mirror on their paper folding activity to verify this).
2. Conclude the measures of a line segment and its image are equal. Distance is preserved.
3. Conclude the measures of an angle and its image are equal. Angle measure is preserved.
4. Yes
Yes
Yes
5. Draw or construct a ray from point P perpendicular to line l. Call the point of intersection point D. On $\overrightarrow{PD}$ find the point P' (different from P) such that PD = DP'.
6. On the line of reflection.
7. Yes
Yes
Discuss this question with the class.

Symmetry

Teaching suggestions:

1. This activity is not essential to understanding transformations and is therefore optional.
2. Answers to the activity:

1. 2	2. 0	3. 4	4. 1
5. 1	6. 2	7. infinitely many	8. 0
9. 0	10. 0	11. 1	12. 1	13. 1

Materials

Mira or equivalent (classroom set)
Rulers
Protractors

Exercise Answers

1. Construction
2. 1 ellipse, 3 square, 6 rhombus, 7 circle, 10 letter s.

Reflecting a Reflection (Composition of Reflections)

Teaching suggestions:

1. If enough transparent mirrors are available, this activity is best done individually or in pairs.
2. A student will often not look ahead to the hint. Therefore after the class has been working for awhile, it would be well to point out the existence of the hint.
3. It would be helpful to have transparencies made of the answers for Exercises 6 and 7. This will mean more to the student than a simple answer of no.

Materials

A class room set of MIRA or transparent mirrors.
straight edge

Exercise answers

1. Upright in stall 2.
2. Same as 1.
   $r_e(r_m(\text{Henry})) = \text{Henry}'$
3. Yes
4. a. Yes b. Yes
5. $\overleftrightarrow{BC}$
6. NO
7. NO

The Rotor

Teaching suggestions:

1. A teacher demonstration of how to use a rotor might save classtime. This can be done on the overhead.
2. A transparency of a completed activity would be useful for discussing the results of the activity. Emphasize that a transparent mirror can do the same job a rotor does.
3. Discuss why a rotor is a transformation machine.

Materials

Used computer cards or equivalent
Stapler
Protractors and rulers
Transparent mirrors (classroom set)

Exercise answers:

1. Yes
   Yes
   Yes
2. Construct a rotor with angle measure greater than 180°, or construct a rotor with angle measure supplementary to the given angle and rotate clockwise.
3. $R_{30,\ y}\ (\Delta XYZ)$
4. Yes
5. Yes, Yes,
6. Make a transparency of a students finished drawing to show the class.

The Translator

Teaching suggestions:

1. This is a "confer and try it" type of activity. Therefore, pairs or threesomes may be the best working arrangement for this one.
2. Students may need to be reminded that their machine needs to transform individual points. A whole figure may then be transformed by transforming strategic points.
3. Students may need help in finding a method of using the mirrors to perform a translation. Often a suggestion that the composition of two reflections about parallel lines may work will be sufficient for them to proceed successfully.
4. It has been found useful to assign this activity with 5 to 10 minutes of class time remaining. Students may then be asked to decide what materials they need, and to bring those materials the following class period.

Materials
See #4 in teaching suggestions
Transparent mirrors (classroom set)

Exercise Answers:

1. a. yes b. yes c. yes
2. $\overline{AA'} = \overline{BB'} = \overline{CC'}$
   $\overleftrightarrow{AA'} \parallel \overleftrightarrow{BB'} \parallel \overleftrightarrow{CC'}$
3. Yes
4. a. yes b. yes

Isometries

Teaching suggestions:

1. This is a good place for a class discussion summarizing what has been learned so far.
2. A careful explanation of what is to be done in this activity will be helpful.
3. Transparencies of the activity on the overhead can be used for discussing possible answers when the activity is completed.

Materials
Transparent mirrors (classroom set)
Used cards and stapler
Ruler and protractor

Exercise answers:

1. Reflection, rotation, translation, compositions of these.
2. Yes
3. Two figures are congruent if and only if there exists an isometry from one to the other.

Size Transformations

Teaching suggestions:

1. This is a "confer and try it" type of activity. Therefore, pairs or threesomes may be the best working arrangement for it.
2. A student will often not look ahead for hints. Therefore, after the class has been working for awhile, it would be well to point out the existence of the hint for making Henry three times his size.
3. Discuss why size transformation satisfies the test for a transformation.

Materials
- Rulers
- Graph paper

Answers to the activities 1 through 7.

1. $\Delta A'$ (8,12) B' (12,28) C'(32,16)
2. $AB = \sqrt{17}$ $A'B' = \sqrt{272} = 4\sqrt{17}$

   $BC = \sqrt{34}$ $B'C' = \sqrt{544} = 4\sqrt{34}$

   $AC = \sqrt{37}$ $A'C' = \sqrt{592} = 4\sqrt{37}$

   Corresponding sides are proportional.

3. slope $AB = 4$ slope $A'B' = 4$
   slope $BC = -3/5$ slope $B'C' = -3/5$
   slope $AC = 1/6$ slope $A'C' = 1/6$
   Corresponding sides are parallel.

4. A' (-1,1) B' (14,1) C (14,-9) D' (-1,-9)
5. A' (5/2,5/2) B' (5/2,13/2) C' (11/2,5/2)
6. A" (-12,4) B" (-6,-4) C" (0,6)
7. A" (12,14) B" (3,2) C" (-6,17)

Exercise answers:

1. a. no b. yes c. yes
2. No
3. Yes

Similarity Transformations

Teacher suggestions:

1. This second activity could be followed by a class discussion using an overhead transparency.

Materials
- Rulers and Protractors

Exercise answers:

1. Yes
2. Yes
3. No
4. Yes
5. No
6. No
7. No
8. Yes

WHAT IS A TRANSFORMATION?

Each workgroup should have a set R containing 3 rectangles and a set T containing 2 triangles. Observe that each set contains many duplicates. This does not add to the number of elements in the set, but instead allows you to represent the same object in more than one place at one time. Set R has only three elements and set T has only 2 elements.

Step 1: Read the definition given on the following pages of cartesian product and use your rectangles and triangles to make a physical model of R x T. Have your instructor check to make sure your model of R x T is correct.

Step 2: Make a model of T x R and answer the following questions.
- a. Is R x T the same as T x R?
- b. Does R x T have the same number of ordered pairs as T x R?
- c. If a set S contained three elements, how many ordered pairs are contained in S x R?

Step 3: Read the definition given on the following pages of relation and make a model of a relation from R to T.

Step 4: Make a model of a relation from T to R and answer the following questions.
- a. How many correct models are there for Step 3?
- b. How many are there for Step 4.
- c. How many ordered pairs does the relation " has the same numeral as " from R to T contain?

Step 5: Read the definition given on the following pages of function and model a function from T into R.

Step 6: Model a function from R into T and answer the questions.
- a. How many ordered pairs must be in your function from Step 5.
- b. How many ordered pairs must be in your function for Step 6.

Step 7: Make models for each of the following if possible.
- a. A one-one function from R into T.
- b. A one-one function from T into R.
- c. A function from R onto T.
- d. A function from T onto R.
- e. A one to one correspondence between R and T.

Questions for Step 7:

1. Which parts are not possible?
2. Why is each of these not possible?

Step 8: Using S, the set of squares given on the ditto, repeat Step 7 for sets S and R.

Question for Step 8:

If a one to one correspondence exists between A and B, how must the number of elements in A compare with the number of elements in B?

Step 9: Draw three noncollinear points on a piece of paper and label them A, B and C. Repeat this process for three other points E, F and G. Model a transformation between ABC and EFG.

Question for Step 9:

Can you model a transformation between ΔABC and ΔEFG? If you can, do it. If you can not, explain why you can not.

Definitions

Cartesian Product: The cartesian product of sets A and B (A x B) is the set of all ordered pairs (a,b) where a is an element of A and b is an element of B.

Relation: Any subset of A x B is called a relation from A to B.

Function: A function from A into B is a relation from A to B in which each element of A appears once and only once as the first element of an ordered pair.

Domain: If there is a function from A to B, set A is called the domain of the function.

Domain Element and Image: If (a,b) is an element of the function, a is called a domain element; b is called the image of a.

One-one Function: A function is a one-one function from A into B iff no element of set B appears more than once as the second element of an ordered pair.

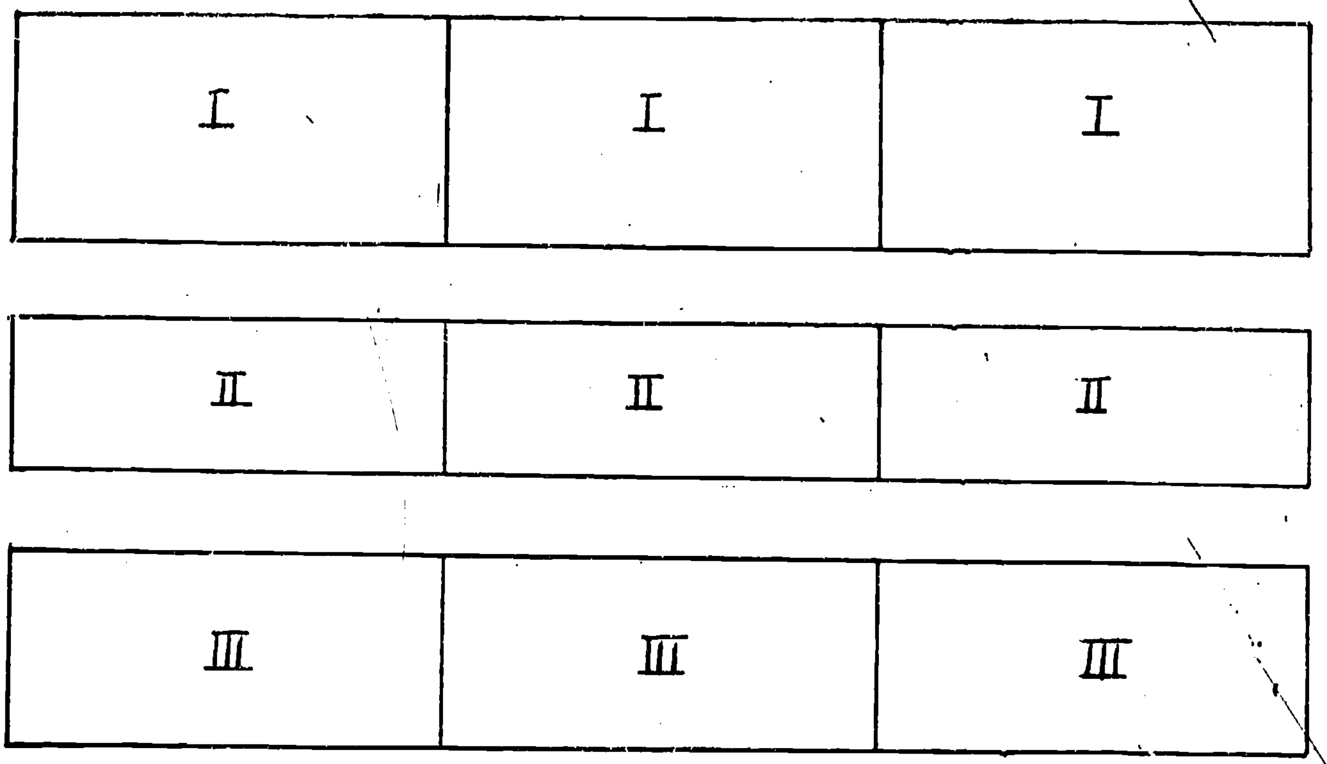
Onto Function: A function is from A onto B iff every element of B is used at least once as a second element of an ordered pair.

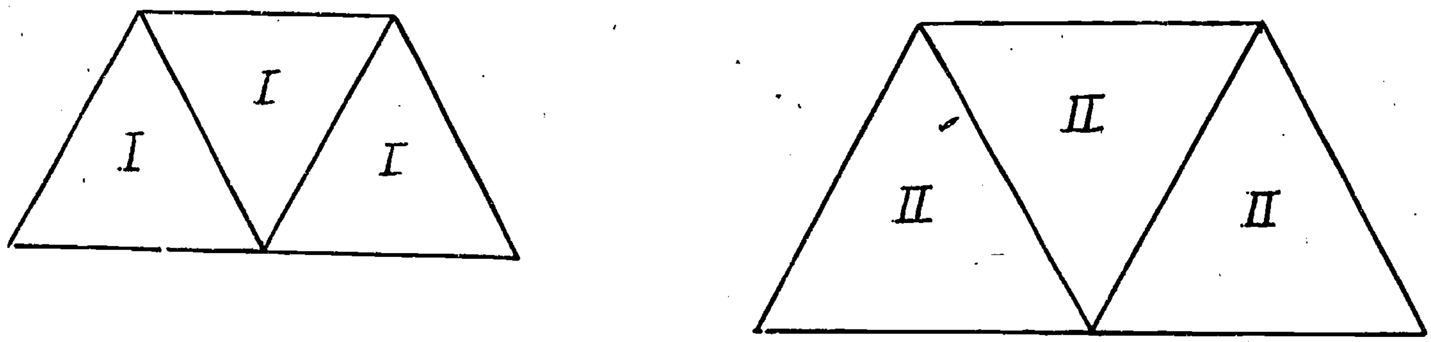
One to One Correspondence: A function is a one to one correspondence between A and B iff it is a one-one function from A onto B.

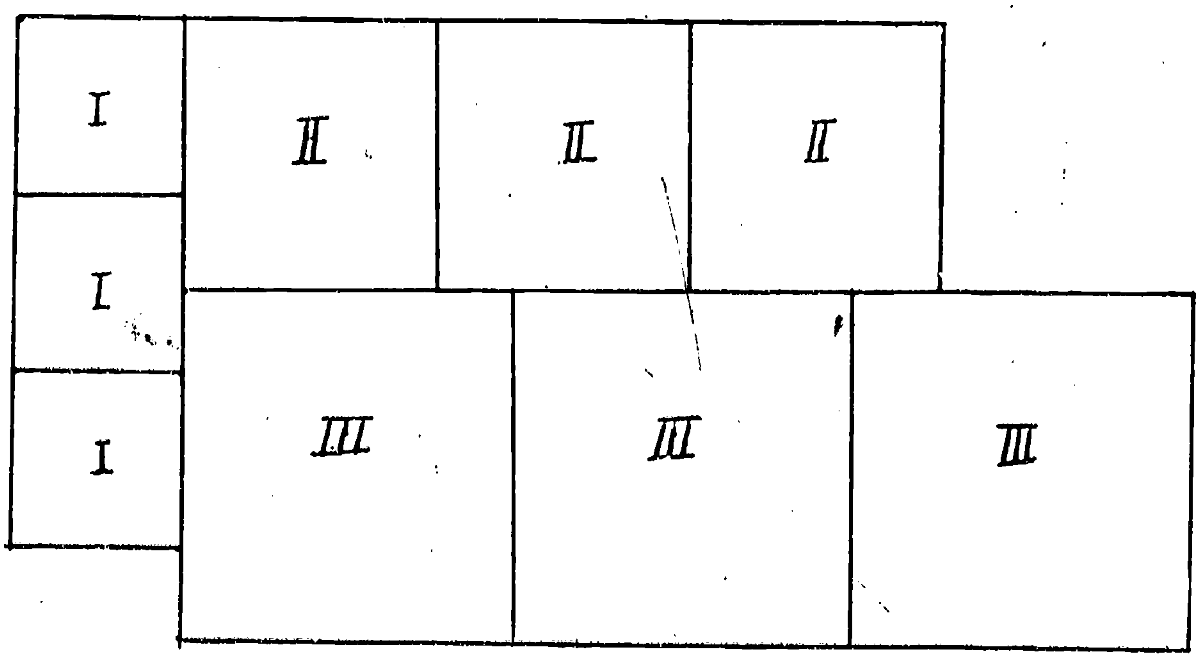
Transformation: A transformation is a function for which each member of the domain is a point of the plane and for which each element of the range is also a point in the plane.

WHAT IS A TRANSFORMATION?

DITTO SHEET







What is a Transformation?

Exercises:

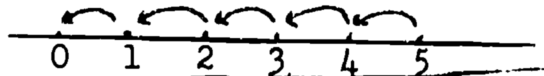
Find A x B in the following:

1. A = (Cat, Mouse); B = (Fight, Food)
2. A = (1,5,10) ; B = (6,4,2)
3. A = (3,6) ; B = (0)
4. A = B = (0,1)
5. A = B = $(x / 0 < x < 1$ and x is a fraction)
6. A = B = (All even counting numbers)

Often the image set of a relation or function is referred to as the range of the relation or function. What is the domain and what is the range for each of the following relations? Which relations are also functions on those domains?

7. (0,0), (1,1), (2,4), (3,9), (4,16)
8. (-4,4), (-2,2), (0,0), (2,2), (4,4)
9. (4,-4), (2,-2), (0,0), (2,2), (4,4)
10. (x,y) / $y = x^2$ and x is any real number)
11. (x,y) / $y^2 = x$ and x is any positive real number)

For each of the following find its domain, range and indicate if it is a one to one correspondence between the domain and the range.

12.


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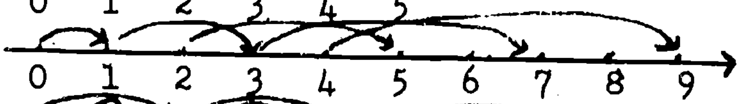

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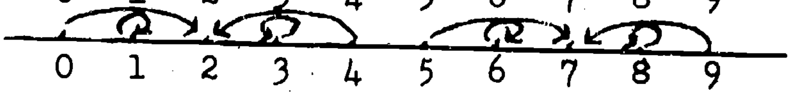

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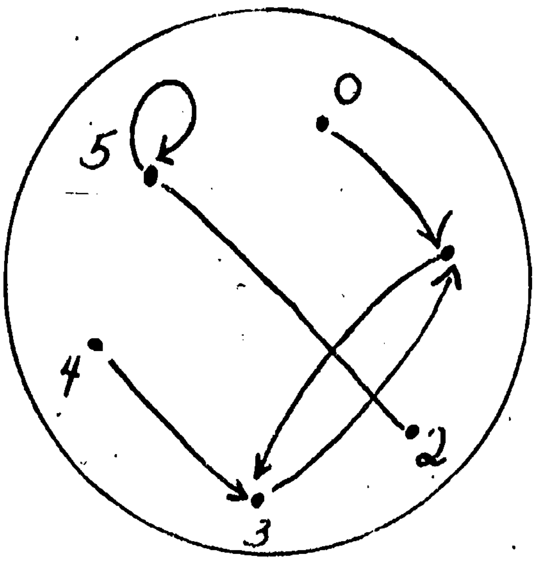

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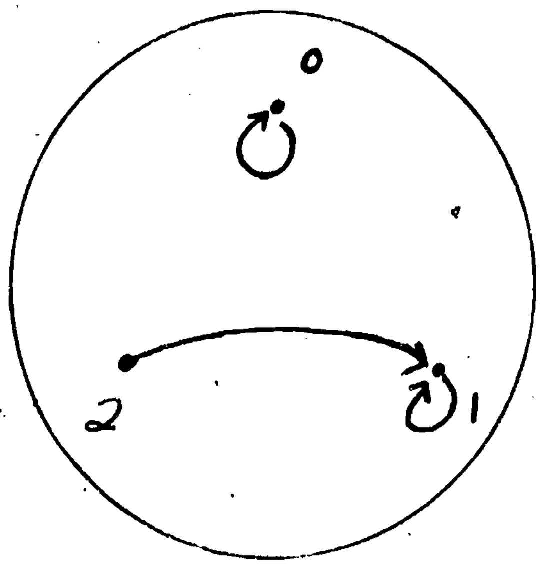

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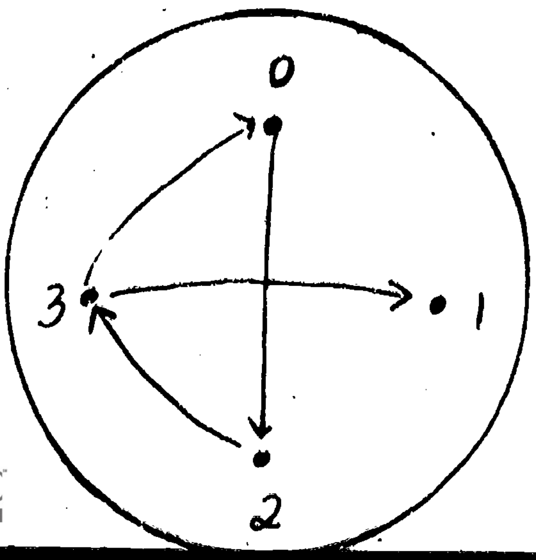


Paper Folding Machine

You may feel that whenever a teacher was talking about a function machine, he was really putting you on. You drew a picture of a machine but it never did anything. Have we got news for you! You can make a transformation machine that will do the work for you. In fact, there are several and we shall investigate the nature of certain transformations through the use of these machines. But do not take our word for it. Prove to yourself that each proposed machine does in fact perform a transformation on any set of points.

The first machine is a piece of paper. With every machine, there is a set of operating instructions:

1. Fold the paper along any line you choose, then unfold and lay it flat. The paper should have one sharp crease. Call the fold line ℓ.
2. Locate a domain point A anywhere on the paper.
3. Fold the paper along the crease. The image point is the unmarked point which touches point A. Mark that point.
4. Unfold the paper and label the image point A'.

I. On your paper draw $\overline{AA'}$. What is the angle formed by $\overline{AA'}$ and ℓ? Call the point where $\overline{AA'}$ and ℓ intersect point W. Measure $\overline{AW}$ and $\overline{WA'}$. What do you conclude?

II. On your paperfolding machine draw any line segment $\overline{PQ}$. Find the image of $\overline{PQ}$. Label it $\overline{P'Q'}$. Let R be any point on $\overline{PQ}$ between P and Q. From the betweenness property we know PR + RQ = PQ. Now operate the folding machine to find the image of R. Call it R'. Where is R' in relation to P' and Q'? Is P'R' + R'Q' = P'Q'? If it is we say that betweenness of points is preserved.

III. On your paper folding machine draw $\triangle XYZ$ such that at least one fold intersects the fold ℓ. Call a point of intersection of a side and the fold point D. Find the image of $\triangle XYZ$. Call it $\triangle X'Y'Z'$. What is the image of D? Is there a set of points for which each point is its own image? If so describe the set.

IV. On your paper folding machine draw any foursided figure STUV. Find its image.

Exercises:

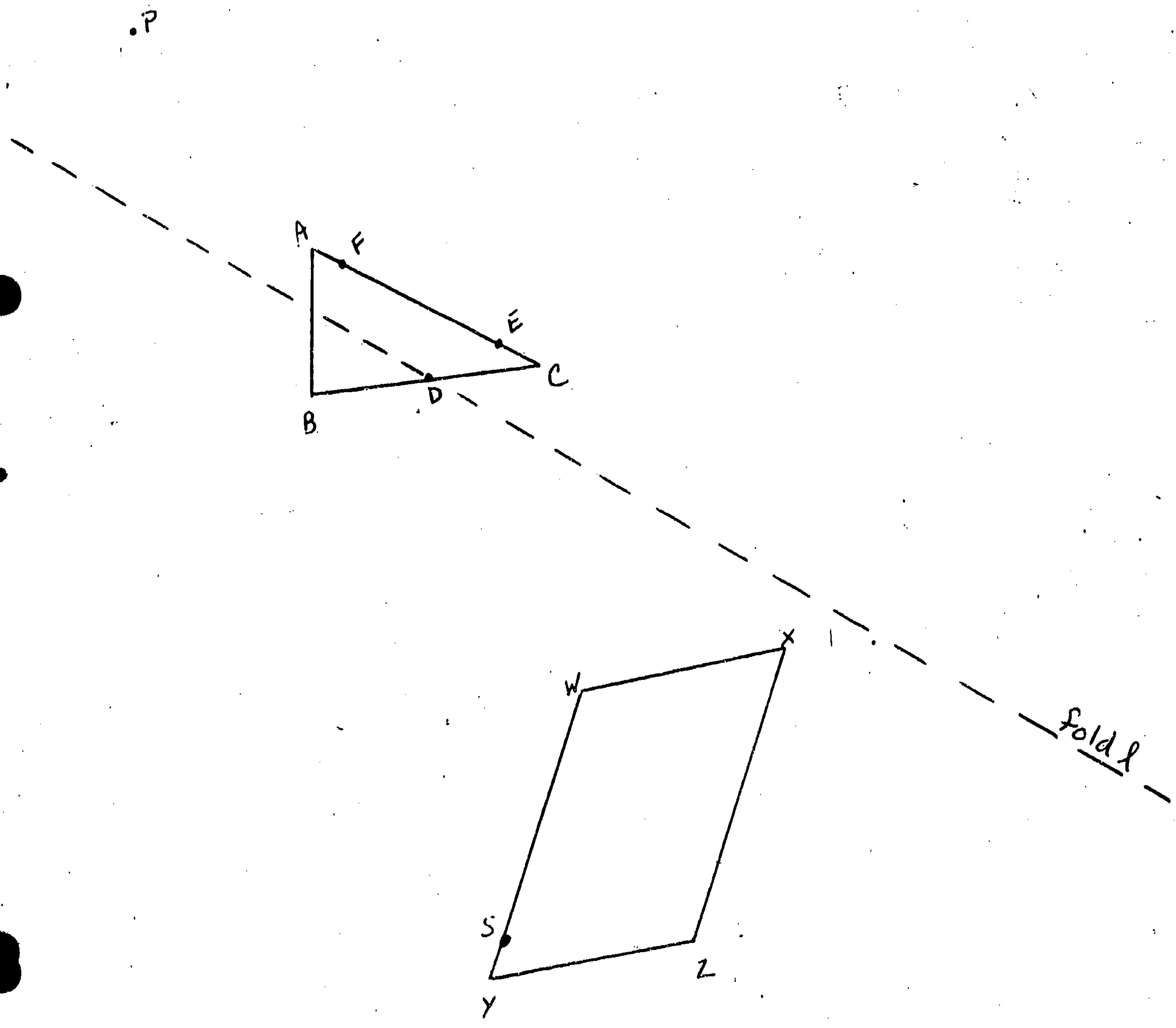
1. There are two properties a machine must have to be a transformation machine.

   I. Every domain point must be associated with exactly one image point. We of course can not check every domain point to see if this is true, but we can check enough points to draw a conclusion.
   Are points A, P, X, U each associated with exactly one image point? Name them.
   Choose any point G on $\overline{ST}$. Does it have exactly one image point?

II. Every image point is associated with exactly one domain point. Choose several image points to check, for example, Q', Z', S'. Operating the machine on exactly the same fold could these points be associated with any other domain points besides Q, Z, S respectively?

If conditions I and II are met the machine is a transformation machine.

2. Given the drawing below find the image of point P, triangle ABC, and parallelogram WXYZ. Do this without folding the paper - use a ruler and and protractor instead.



Transparent Mirror Machine

This machine is a mirror (you can see your reflection in it) and is also transparent (you can see through it). It reflects part of the light and part of the light passes through it.

Operating instructions:

1. Draw a line segment on your paper. Call it $\overline{AB}$.
2. Place the transparent mirror along the line $\overleftrightarrow{AB}$ and perpendicular to the paper.
3. Mark a point P anywhere on your paper.
4. Note the reflection of P in the mirror. Place pencil on paper behind the mirror and mark the reflected image of point P. Label it P'.

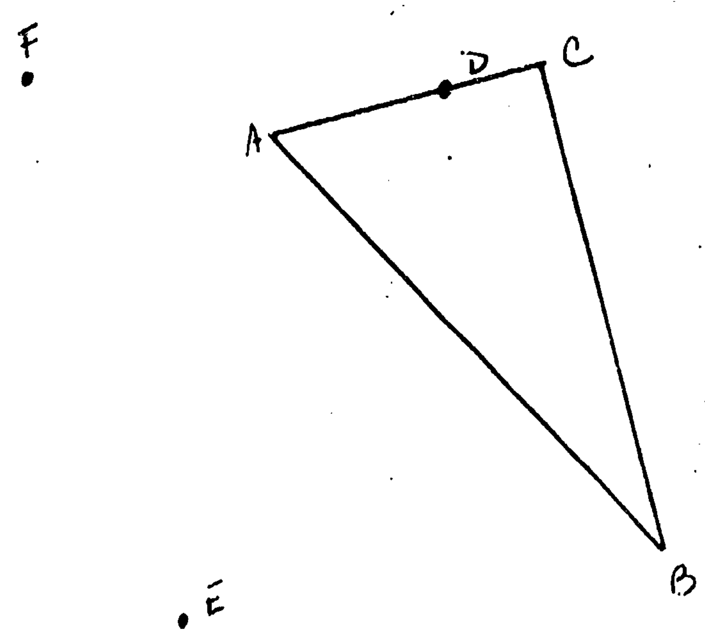
   We call the image found by this machine a reflection. We say the reflection of P about $\overline{AB}$ is P', or $r_{\overleftrightarrow{AB}}(P) = P'$. $\overleftrightarrow{AB}$ is called the line of reflection. Now that you can operate this machine, let us put it to use.

Find $r_{\overleftrightarrow{FE}}(\triangle ABC)$

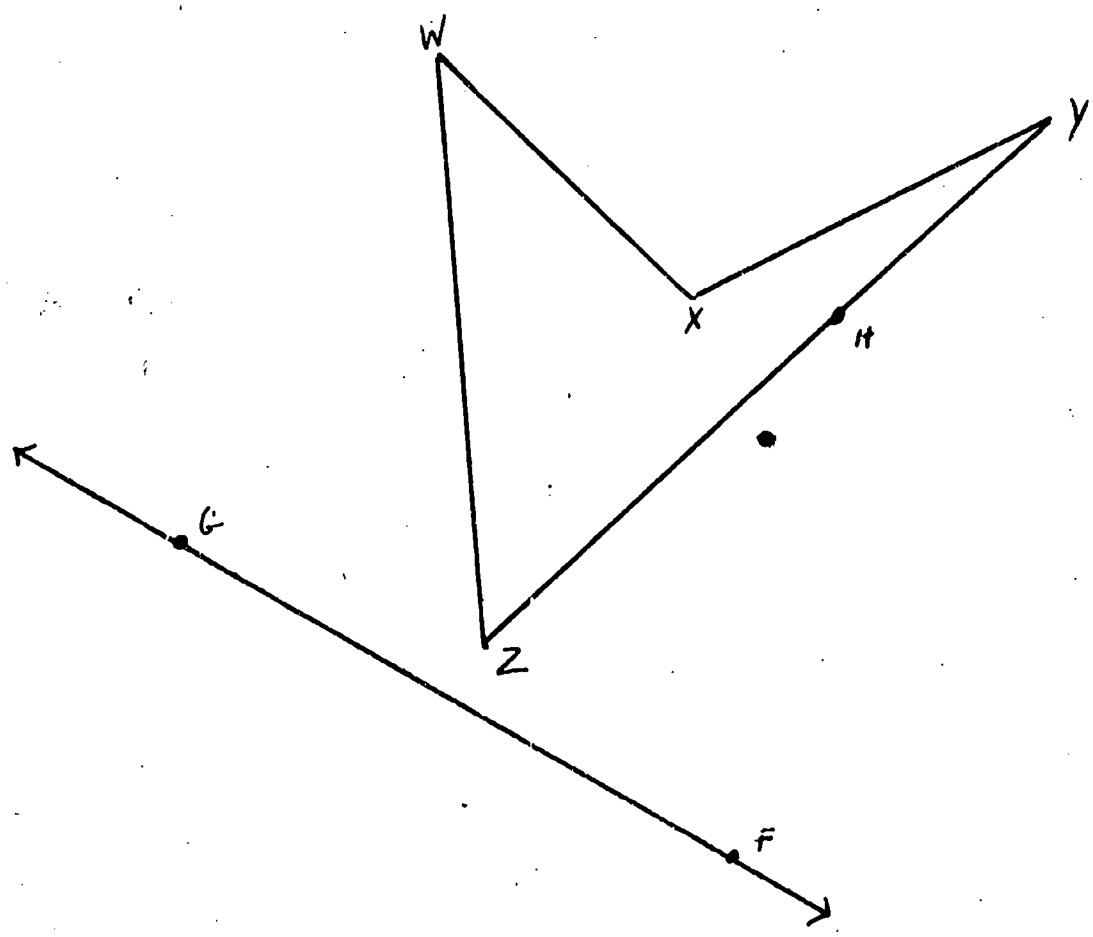
Find $r_{\overleftrightarrow{BC}}(\triangle ABC)$

Find $r_{\overleftrightarrow{ED}}(\triangle ABC)$

Find $r_{\overleftrightarrow{EB}}(\triangle ABC)$



Find $r_{\overleftrightarrow{FG}}$ (WXYZ)



Transparent Mirror Machine

Exercises:

1. Do the paper folding machine and the transparent mirror do the same job?
2. Use a ruler to measure the following segments from the previous activity.

WX=	W'X' =
ZY=	Z'Y' =
ZH=	Z'H' =
HY=	H'Y' =

What do you conclude?

3. Use a protractor to measure the following angles on the activity card.

$m\angle W$ =	$m\angle W'$ =
$m\angle Z$ =	$m\angle Z'$ =
$m\angle Y$ =	$m\angle Y'$ =

What do you conclude?

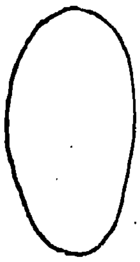
4. In number 2 we concluded distance is preserved under reflection and in number 3 that angle measure is preserved.

   Is size preserved?
   Is shape preserved?
   Is betweenness of points preserved?

5. Suppose you have a line ℓ and a point P not on line ℓ. Describe how you would locate the reflection of point P about line ℓ using ruler and protractor.
6. In using a transparent mirror where is the set of points for which each is its own image?
7. Test the transparent mirror to see if it is really a transformation machine.
   1. Is every domain point associated with exactly one image point?
   2. Is every image point associated with exactly one domain point?

Symmetry

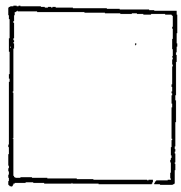
Artists often classify an object as having beauty because it has symmetry. Mathematicians, of course, like to define a term precisely. We say a figure has line symmetry if there exists a line about which one half of the figure is a reflection of the other. Use your transparent mirror to locate lines of symmetry (if they exist) for each figure. Tell how many lines of symmetry you think each figure has.

1. 

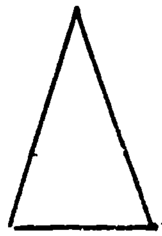
elipse

2. 

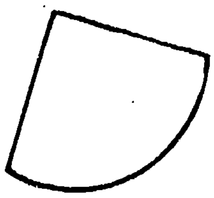
right triangle

3. 

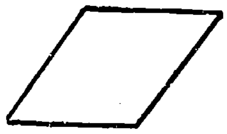
square

4. 

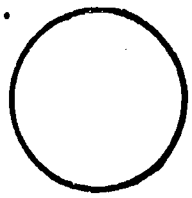
isosceles triangle

5. 

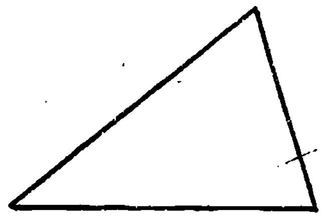
sector of circle

6. 

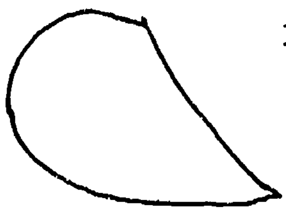
rhombus

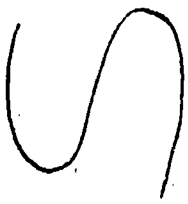
7. 

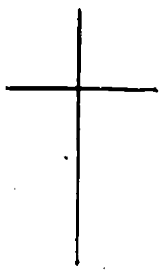
circle

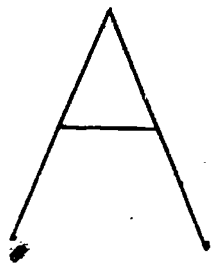
8. 

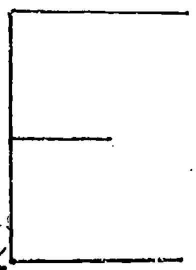
scalene triangle

9. 

10. 

11. 

12. 

13. 

Symmetry

Exercises:

Line symmetry is easy to determine using a transparent mirror. Point symmetry is more difficult. We have found a reflection about a line using a ruler and protractor. We can find the reflection about a point using a ruler.

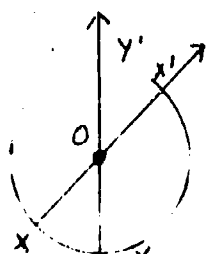
1. Find r_x (P) x

P

   A. Draw ray $\overrightarrow{Px}$
   B. Measure $\overline{Px}$
   C. Locate a point P' (different from P) on ray Px, so that Px = xP'.

   r_x (P) = P'

2. We say a figure has point symmetry if there is a point P such that any point on the figure reflected about P has an image also on the figure.
   Example: a circle has point symmetry with its center as the point of symmetry.

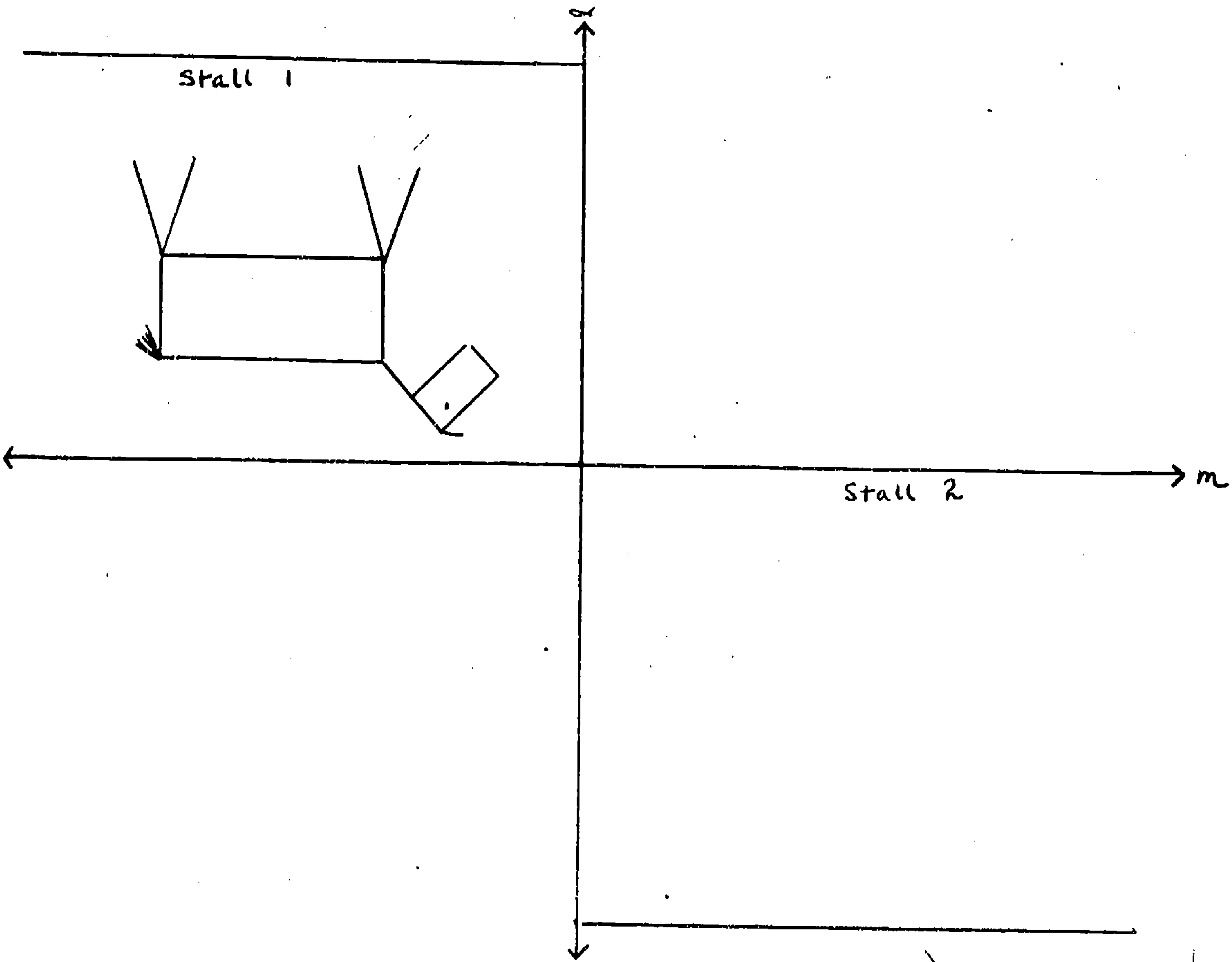


r_o (x) = x' xo = ox'
r_o (y) = y' yo = oy'

On the activity card five of the figures have point symmetry. Put a check beside each one and locate the point of symmetry.

Reflecting a Reflection (Composition of Reflections)

Use a transparent mirror to transfer Henry the Horse from stall 1 to stall 2 and leave him in an upright position.



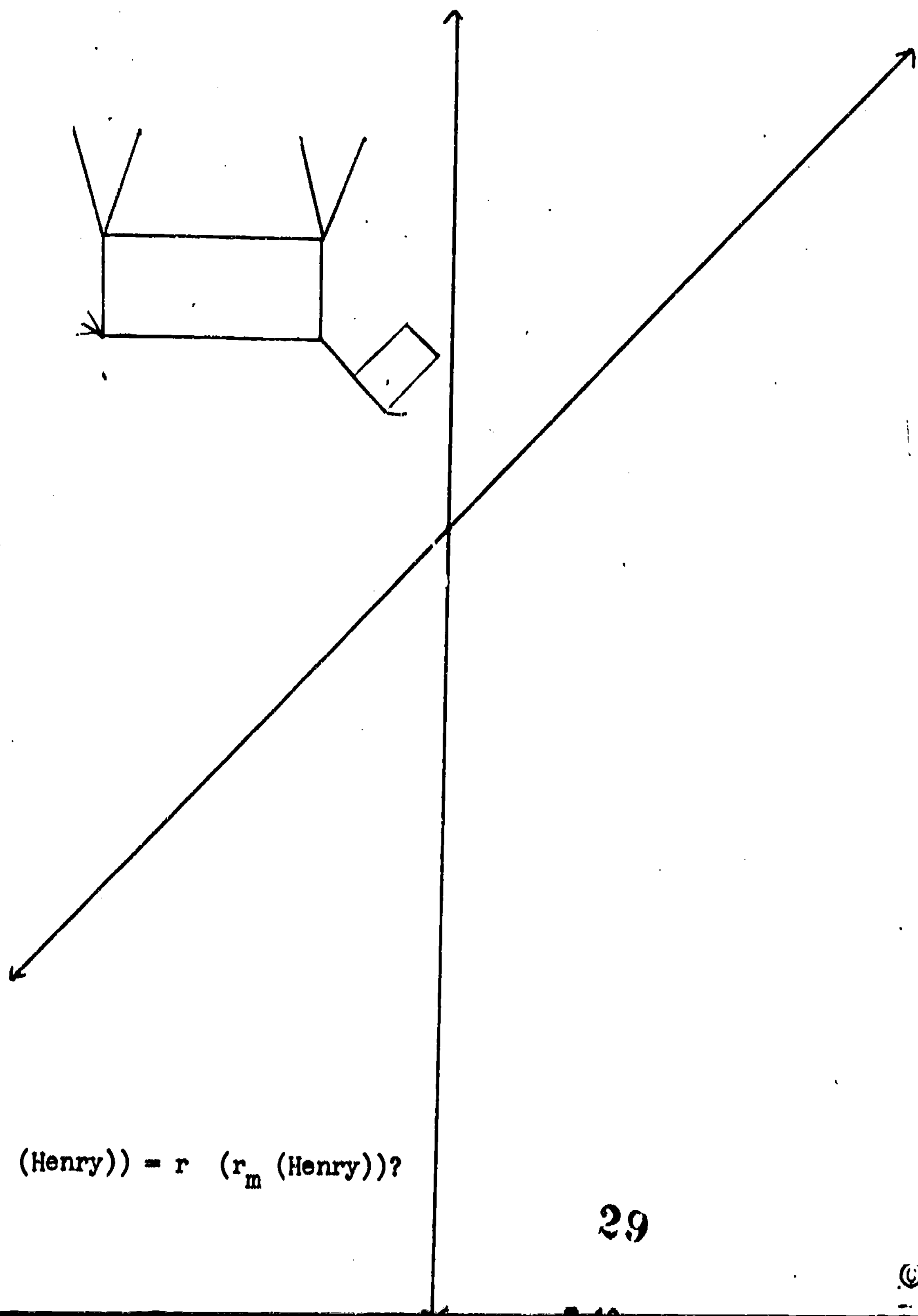
Do not conclude it can not be done. It just can not be done in one reflection. Try reflecting Henry about a line and then reflecting that image again about another line.

Reflecting a Reflection (Composition of Reflections)

Exercises:

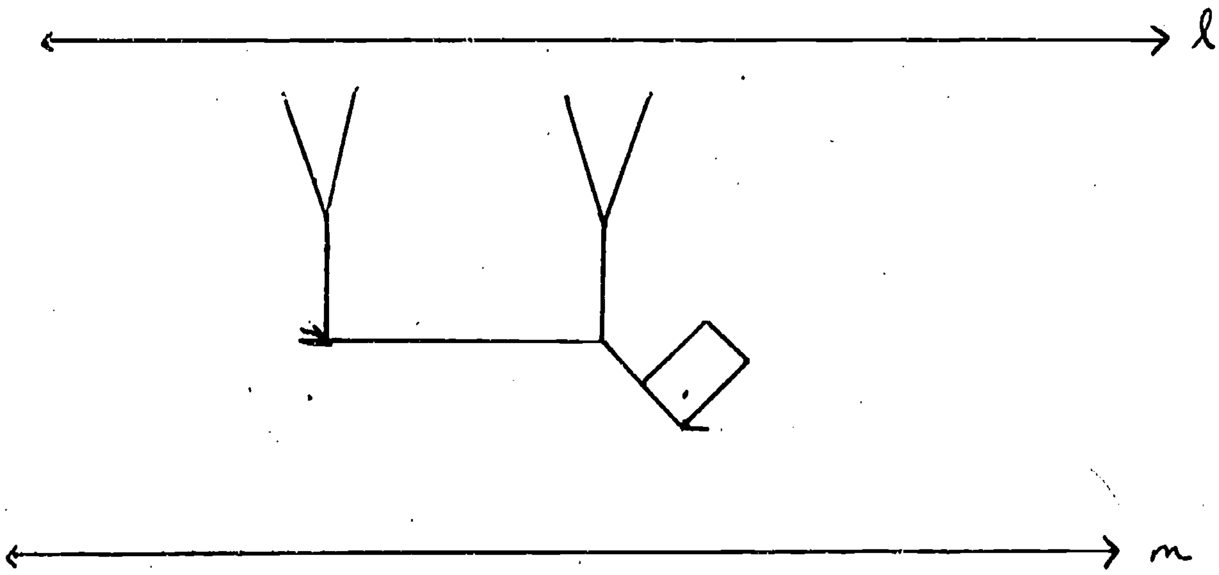
1. If you reflected Henry first about ℓ and then about m, where would the last image be and in what position?
   Using two reflections, one after the other on a set of points is called a composition of reflections. We write $r_m(r_\ell(\text{Henry})) = \text{Henry}'$.
2. If you reflected Henry first about m and then about ℓ where would the last image be and in what position?
   How would you write this composition?
3. Is $r_m(r_\ell(\text{Henry})) = r_\ell(r_m(\text{Henry}))$?
4. Test composition of reflections to see if it is really a transformation.
   a. Is every domain point associated with exactly one point on the final image?
   b. Is every point of the final image associated with exactly one point in the domain?
5. In the reflection $r_{\overleftrightarrow{AB}}(r_{\overleftrightarrow{BC}}(\triangle XYZ))$ which line is the triangle reflected about first?

6.



Is $r_m(r\ \ (\text{Henry})) = r\ \ (r_m(\text{Henry}))$?

7.



Is $r_m\ (r_\ell(\text{Henry})) = r_\ell(r_m(\text{Henry}))$?

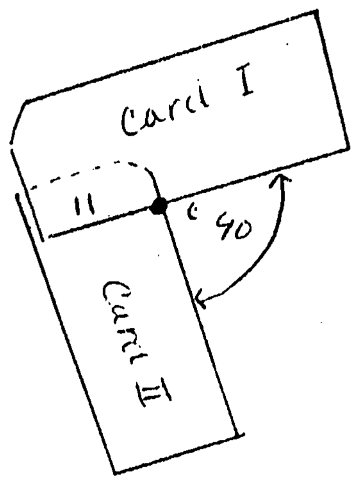
The Rotor

This machine can be easily constructed of two used computer cards (or any cards) using a stapler and a protractor. The type of transformation the rotor will perform must be specified by a real number between 0 and 180. This number must be known before the machine is actually constructed.

Construction of a 90-rotor:
Staple two computer cards together so that the angle formed by their edges is the given number 90.

Directions for use of 90-rotor:

1. Let A be a domain point and P be the point of rotation.

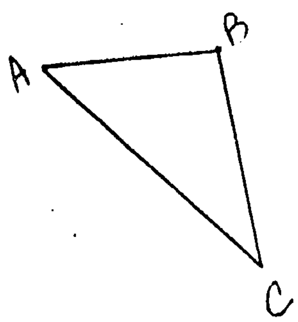


.P

.A

2. Place the 90-rotor so that C coincides with P and A is on the inside edge of card I.
3. Mark on the edge of card I at A (call this point Q)
4. Rotate the machine so that C still coincides with P but point A is on the inside edge of Card II. For convenience we will always rotate counter clockwise.
5. Mark the paper at point Q. This is the image of A. Call it A'.

Use your 90-rotor to find the image of $\triangle ABC$. Choose the point of rotation P to be any point in the exterior of $\triangle ABC$.

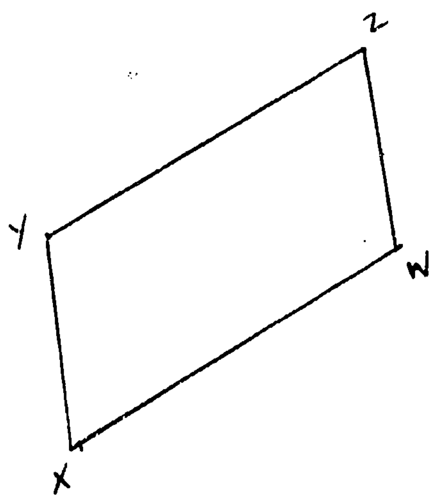


We use the following notation to indicate the rotation at $\triangle ABC$ 90 degrees about point P:

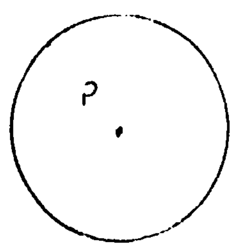
$$R_{90,P}\ (\triangle ABC) = (\triangle ABC)'$$

Point P is called the center of rotation.
Does the choice of point P make any difference in the resulting image of $\triangle ABC$?

Construct a 60-Rotor and find $R_{60,X}$ ($\square$ XYZW)



Construct a 170-Rotor and find $R_{170,P}$ (Circle)



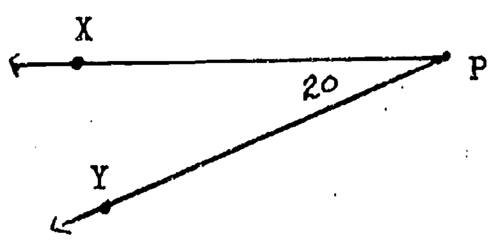
Construct a 20-Rotor. Find $R_{20,P}$ (C).

•C

•P

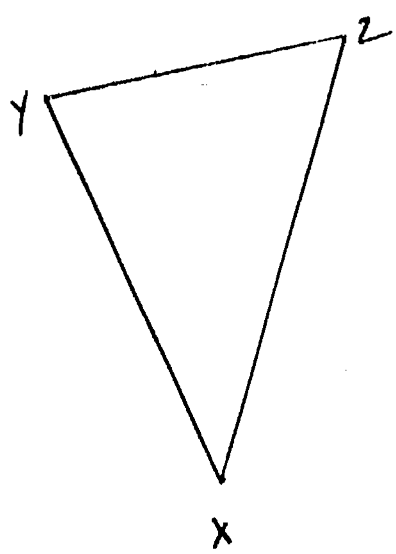
It is possible to use a transparent mirror to find $R_{20,P}(C)$. First it is necessary to draw a 20 degree angle with vertex at P.

•C



1. Using a transparent mirror reflect point C onto $\overrightarrow{PX}$. Call this point C'.
2. Reflect $\overrightarrow{PY}$ so that its image would intersect point C. Keep the transparent mirror in this position and find the image of C'. Call this point C".
3. Compare C" to the image of C above.

Find $R_{45,X}$ ($R_{30,Y}$(Δ XYZ))



The Rotor

Exercises:

1. Is distance preserved under rotation?
   Is angle measure preserved under rotation?
   Is betweenness of points preserved under rotation?

2. Can you construct a machine for any number X such that $180 \leq X < 360$? How?

3. In the composition of rotations $R_{60,X}(R_{30,Y}(\Delta XYZ))$
   Which rotation is preformed first?

4. Can you always perform a rotation using a transparent mirror?

5. Test the rotor to see if it is a transformation machine.
   1. Is every domain point associated with exactly one image point?
   2. Is every image point associated with exactly one one domain point?

6. Find $R_{55,P}$ (ABCD) using only a ruler and protractor.

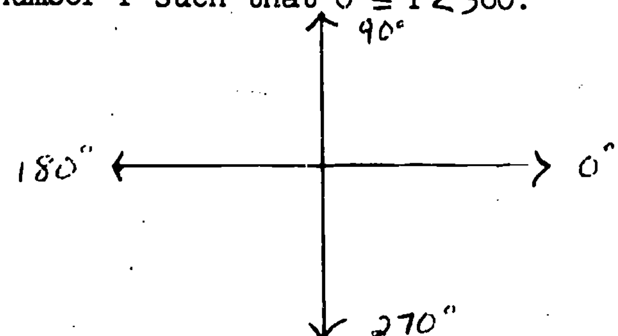
The Translator

We have developed three transformation machines. As it turned out the paper folding machine and the transparent mirror did the same job making one of them unnecessary. The rotor appeared to do a somewhat different job, but it was found a transparent mirror could do the same job only it has to try harder (do its job twice).

We would now like you to assume the role of engineer and design a machine to do a job. The job is to translate a set of points from one position to another. That means slide the points on the plane from their original position to their final position while making sure no rotation takes place. We need to know two things: what direction and how far shall we slide any point.

What direction?

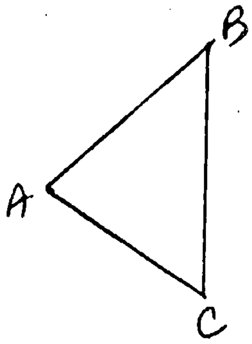
To help answer this question we will use coordinate axes as shown. Any direction can be indicated by a number r such that $o \leq r < 360$.



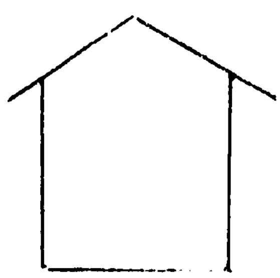
How far?
This can be answered by using standard units of measure such as centimeters, inches, etc. A particular translation may now be indicated by the notation. $T_{r,d}$ (A) where the direction is r and the distance is d.

Use the translator you have designed for the following activities:

Find $T_{90,2 \text{ in}}$ (Δ ABC)

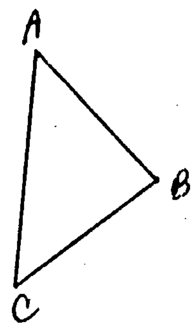


Find $T_{240,4 \text{ cm}}$ (⇧)

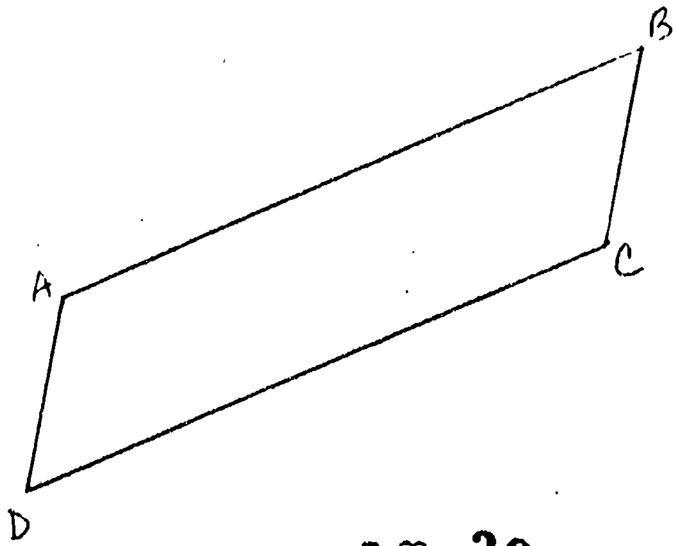


Draw Henry and find $T_{45,\ 6\ cm}$ (Henry)

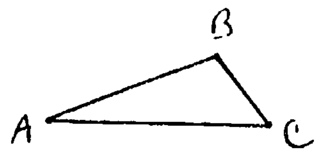
Now use a transparent mirror to do the following $T_{180,4\ in.}(\triangle ABC)$

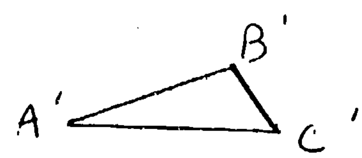


Use a transparent mirror to find $T_{280,\ 3\ cm.}(\square ABCD)$.



Given ΔABC and a translation T such that $T(\Delta ABC) = \Delta A'B'C'$. Find a pair of lines ℓ and m so that $T(\Delta ABC) = r_m(r_\ell(\Delta ABC)) = \Delta A'B'C'$.





Exercises:

1. a. Is distance preserved under translation?
   b. Is angle measure preserved under translation?
   c. Is betweenness of points preserved under translation?

2. In the last activity draw and compare AA', BB', CC'.

3. Can you always perform a translation using a transparent mirror?

4. Test the translator to see if it is a transformation machine.
   a. Is every domain point associated with exactly one image point?
   b. Is every image point associated with exactly one domain point?

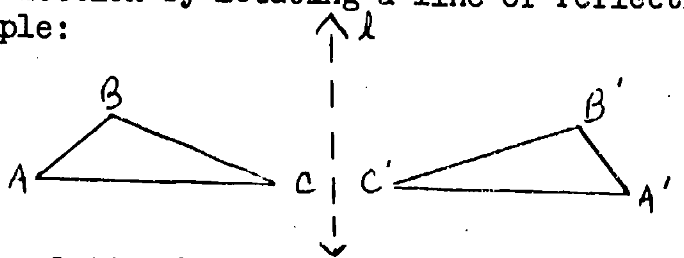
Isometries

We have looked at three transformations: reflection, rotation, translation. Each time we devised a machine to do one of these transformations we found a transparent mirror could also do the job. Each of these transformations is either a reflection or composition of reflections.

Definition: An Isometry is a transformation which is a reflection or composition of reflections.

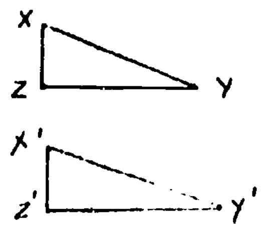
In this activity you will see pairs of congruent figures. For each pair you are to specify an isometry for which one figure is the domain and the other the image if possible.
You can specify:

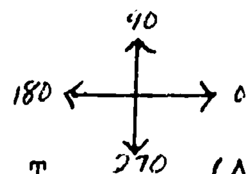
1. A reflection by locating a line of reflection.
   Example:



$r_\ell(\Delta ABC) = \Delta A'B'C'$

2. A translation by specifying a direction and distance of the translation.
   Example:

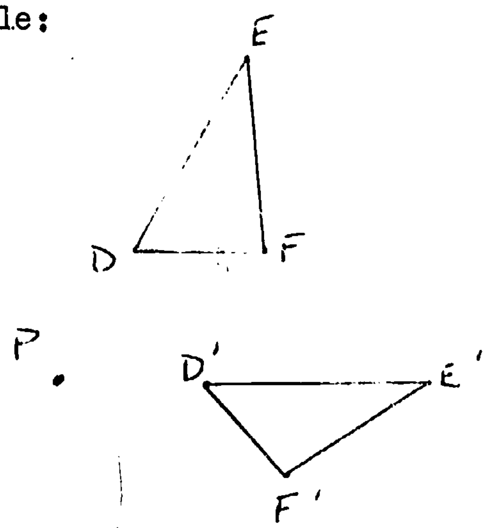




$T_{270,2\text{ cm}}(\Delta XYZ) = \Delta X'Y'Z'$

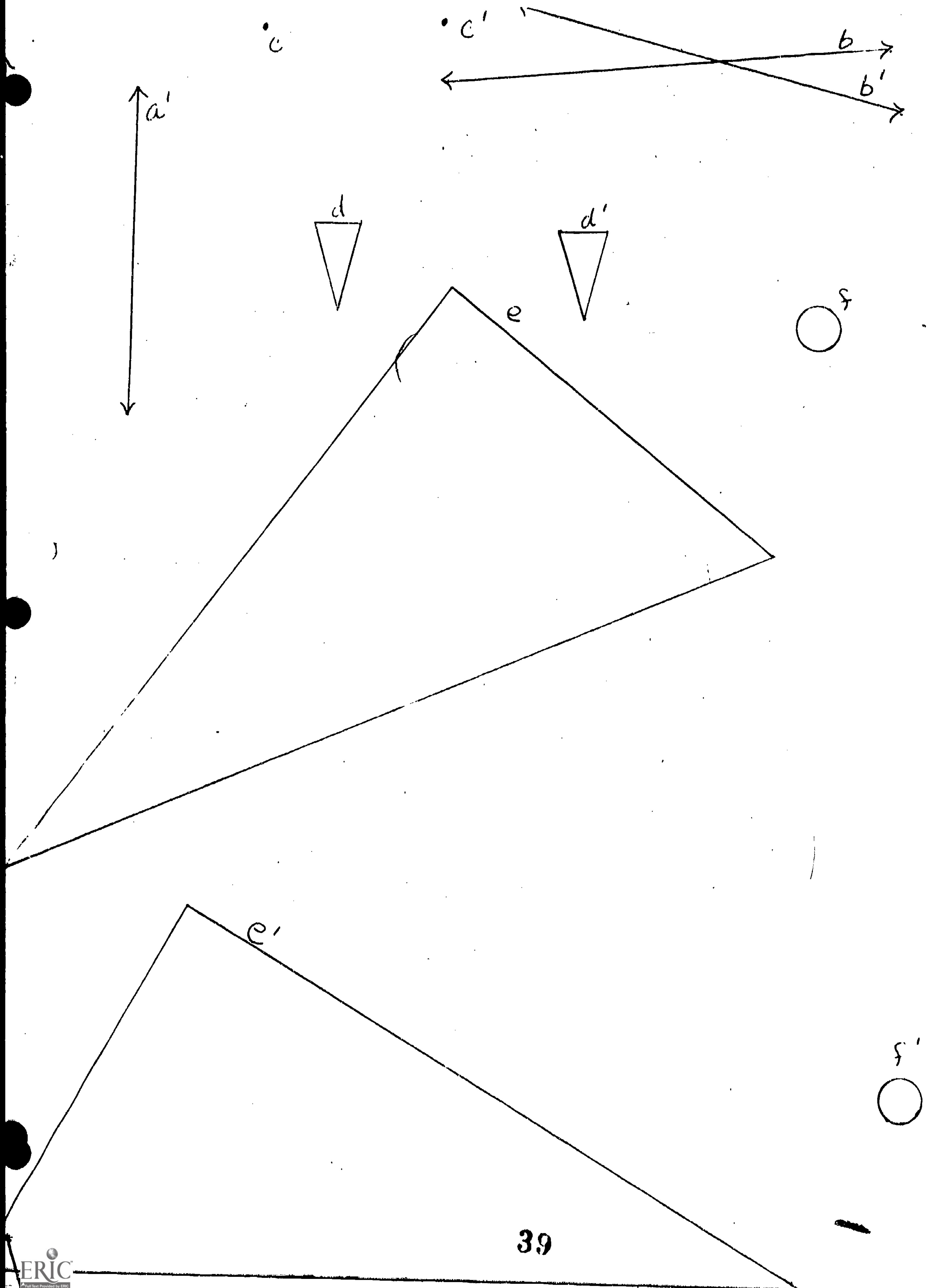
3. A rotation by specifying a point of rotation and anangle of rotation.

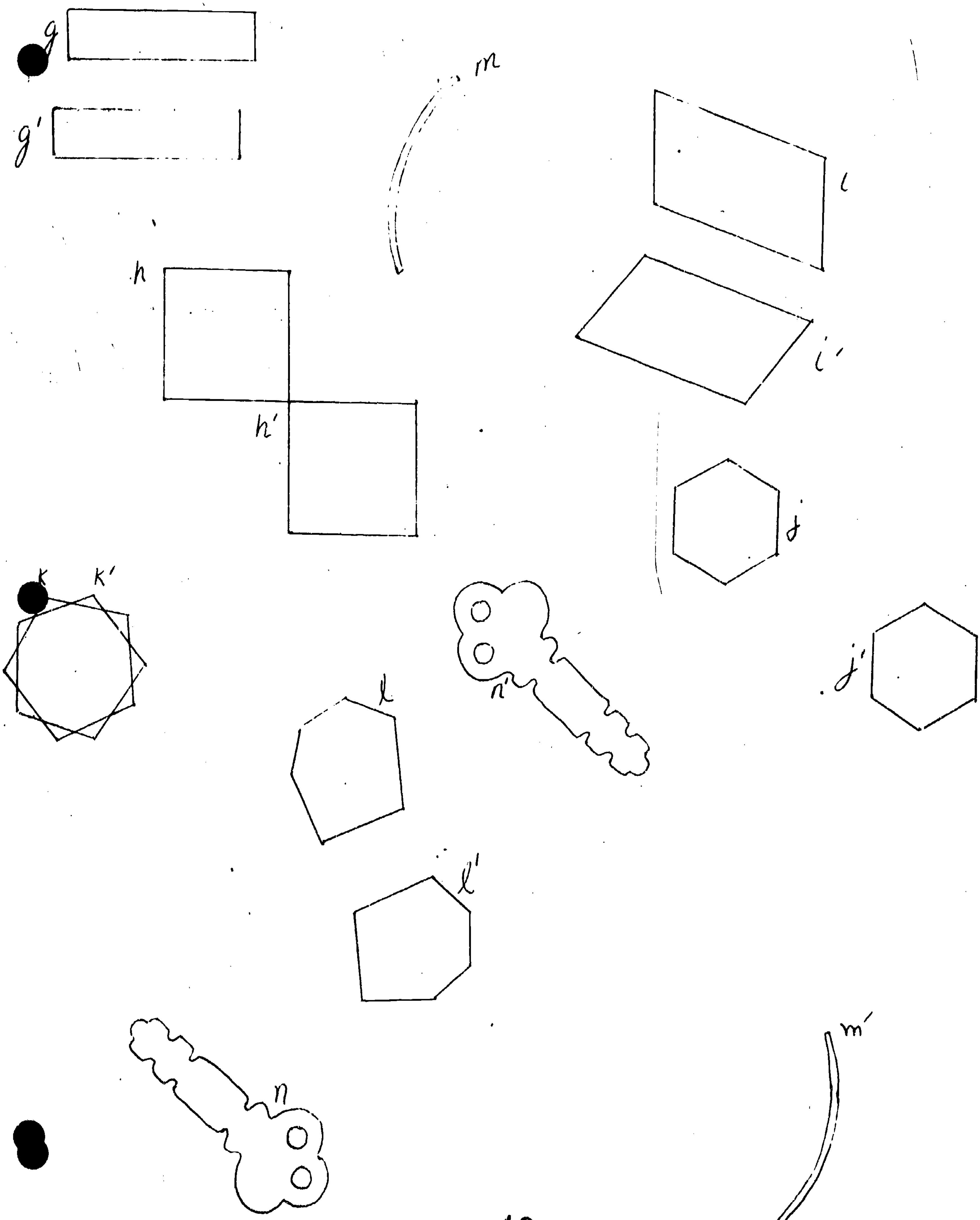
   Example:



$R_{P,315}(\ DEF) = D'E'F'$

Specify an isometry for each pair of congruent figures on the following pages. You may use any of our transformation machines.


c
c'
b
b'
a'
d
d'
e
f
e'
f'


g
g'
m
i
i'
h
h'
j
j'
k
k'
n'
l
l'
n
m'

Isometries

Exercises:

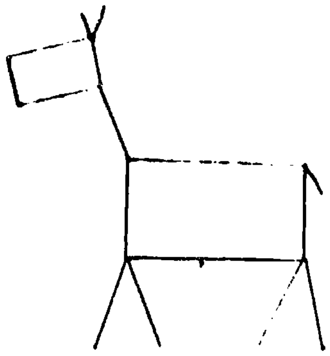
1. List all the transformations you know of that are isometries.

2. For any two congruent figures in the same plane does there always exist an isometry from one to the other?

3. Write a definition of congruent figures in terms of transformations.

Activity IX.

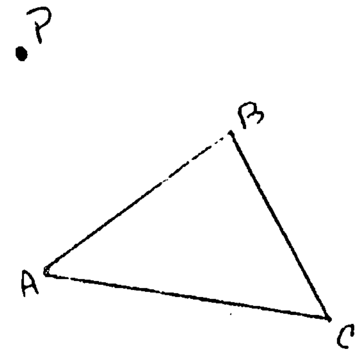
Size Transformation



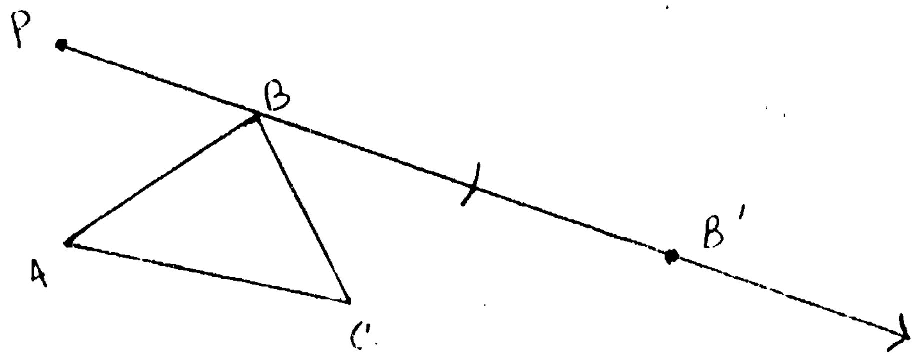
Henry

Draw a horse three times as large, with exactly the same shape and same orientation as Henry.

If you are having trouble making Henry grow, try this: Let us find a triangle three times as large as Δ ABC.
First select any point P.



Then draw $\overrightarrow{PB}$ and locate B' so that PB' = 3 PB.



We leave it up to you to locate A' and C' in the same way and draw Δ A'B'C'. This is known as a size transformation with center at P and scale factor 3. The notation for this size transformation is $S_{P,3}$ (ΔABC) = A'B'C'.

Now try this on Henry.

1. On graph paper draw Δ A(2,3) B(3,7) C(8,4).
   Find $S_{(0,0),4}$ (Δ ABC)

2. Use distance formula to find

AB =	A'B' =
BC =	B'C' =
AC =	A'C' =

   What do you conclude?

3. Use slope formula to find

   slope AB =   slope A'B' =

   slope BC =   slope B'C' =

   slope AC =   slope A'C' =

   What do you conclude?

4. On graph paper find $S_{A,5}$ (A(-1,1) B(2,1) C(2,-1), D(-1,-1))

5. On graph paper find $S_{(3,3),\frac{1}{2}}$ (A(2,2), B(2,10), C(8,2))

6. Find $r_{y\ axis}$ ($S_{(0,0),2}$ (A(6,2), B(3,-2), C(0,3)))

7. Find $T_{90,4\ units}$($S_{B,3}$ (A(6,2), B(3,-2), C(0,3)))

Exercises:

1. A. Is distance preserved under a size transformation?
   B. Is angle measure preserved under size transformation?
   C. Is betweenness of points preserved under size transformation?

2. Is size transformation an isometry?

3. Is size transformation really a transformation?
   Explain your answer.

Similarity Transformations

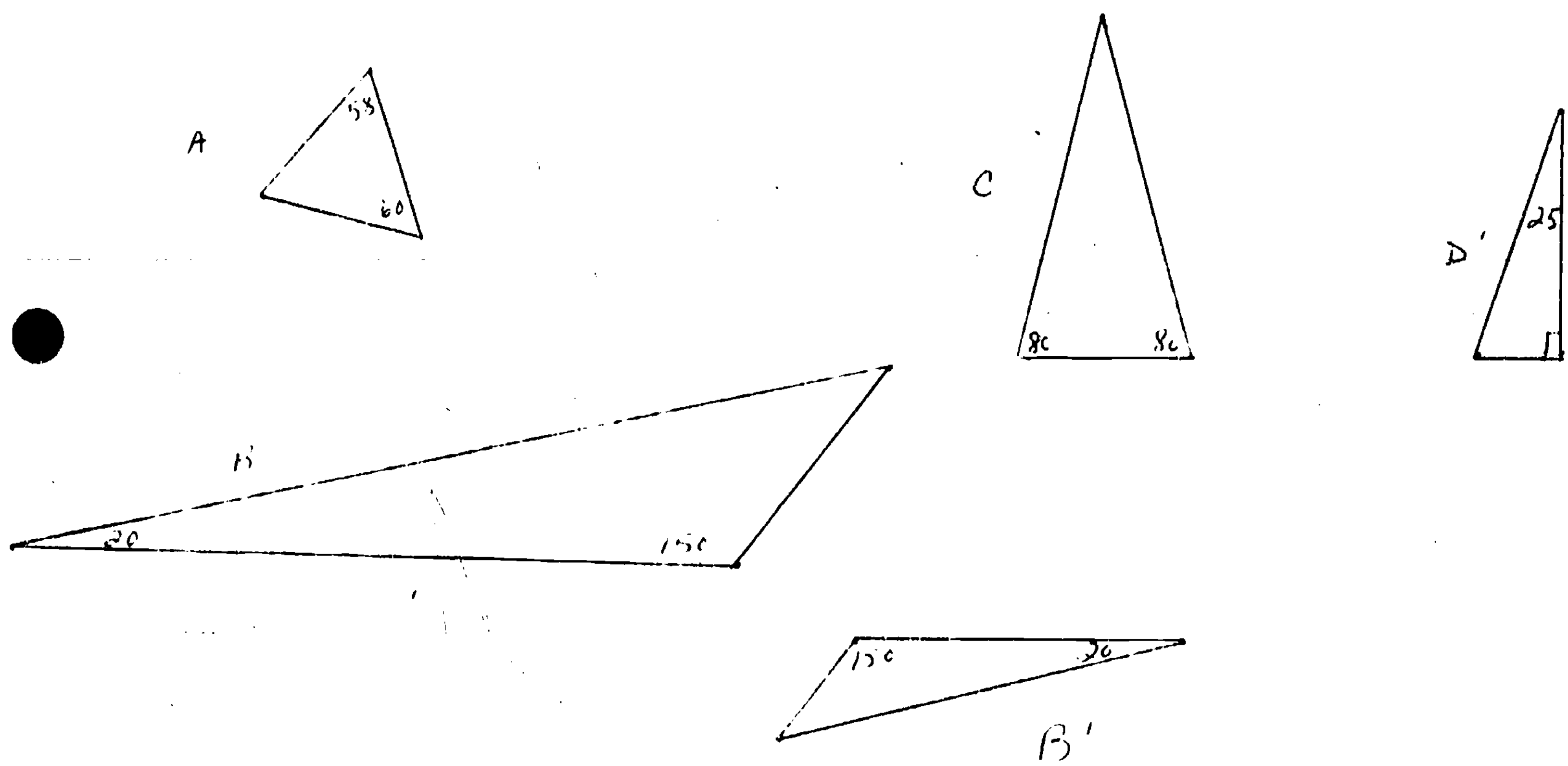
A transformation is a similarity transformation if and only if it is a size transformation or a composition of isometries and a size transformation.

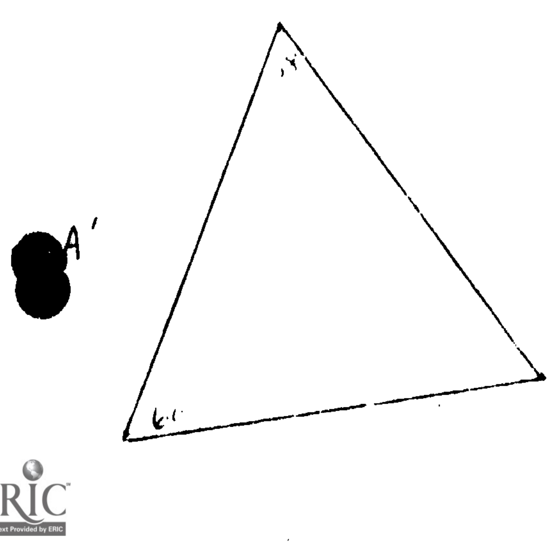
In the last activity all the transformations were similarity transformations.

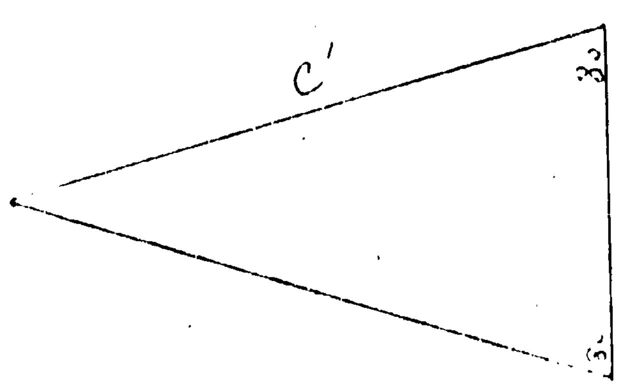
Definition: Two figures are similar if and only if there is a similarity transformation from one to the other.

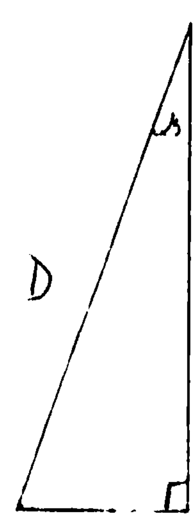
If $\triangle ABC$ is similar to $\triangle A'B'C'$ we write $\triangle ABC \sim \triangle A'B'C'$

In this activity specify a similarity transformation from triangle A to A', B to B', etc. if one exists.









Exercises:

1. If two angles of one triangle are congruent to two angles of another triangle must the triangles be similar?

2. Are any two squares similar?

3. Are any two rectangles similar?

4. Are any two circles similar?

5. Are any two right triangles similar?

6. Are any two parallelograms similar?

7. Are any two pentagons similar?

8. Are any two regular pentagons similar?

TRANSFORMATIONS

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Ladd, Kelly, Elements of Geometry; Scott, Foresman and Co. 1972. Ch. 11.

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