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AUTHOR Brotherton, Sheila; And Others
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ABSTRACT

This is one of a series of geometry modules developed for use by secondary students in a laboratory setting. The purpose of this module is to teach solution of proportions, concepts and theorems of triangle similarity, solution of the Pythagorean Theorem, solution of the isosceles right triangle, and concepts involving "rep-tile" figures as well as pentominoes. Topics reviewed briefly include ratio and proportion, size transformations and a review of terminology covered in previous sections. These purposes are accomplished by the use of detailed examples, narrative explanations, and drawings. (Author/MK)

GEOMETRY MODULE FOR USE
IN A
MATHEMATICS LABORATORY SETTING

TRIANGLE SIMILARITY

by

Sheila Brotherton
Glenn Bruckhart
James Reed

Edited by

Harry Alderman

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The University of Denver
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Dr. Ruth I. Hoffman, Director

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OVERVIEW

The purpose of this module is to teach solution of proportions, concepts and theorems of triangle similarity, solution of the Pythagorean Theorem, solution of isoceles right triangles, and concepts involving "rep-tile" figures as well as pentominoes. Areas reviewed briefly include ratio and proportion, size transformations and a review of terminology covered in previous sections. These purposes are accomplished by the use of detailed examples, narrative explanations as well as drawings.

The student entering this modules should possess a good understanding of transformations as well as its related terminology.

The sections covered in this module are:

1. Proportions
2. Triangle similarity
3. Triangle similarity theorem
4. Pythagorean Theorem
5. Special Triangles
6. Similarity projects

Objectives

See objectives listed on next page.

Module Usage

This module is to be used as a reinforcement of classroom discussions. It should also provide the student with an adequate number of exercises to insure understanding of the stated objectives.

TRIANGLE SIMILARITY

Objectives

1. Given a proportion the students will be able to identify, algebraically manipulate, and solve the proportion for an unknown quantity.
2. Given two similar polygons the student will be able to formulate a correct proportion involving the two figures.
3. Given a right triangle the student will be able to solve proportions which relate the altitude to the hypotenuse of the right triangle.
4. Given a right triangle the student will be able to find the third side of the right triangle when given the lengths of the other two sides.
5. Given a special right triangle the student will be able to identify the special right triangle and find the lengths of two sides when given the length of the third side.

TESTING PROCEDURES

Pretest

Since successful completion of this module depends heavily upon a good understanding of the information covered in the previous module on transformations, the posttest for that module may be used as a pretest for this module. Students who were unsuccessful on the transformation posttest will have difficulty completing this module. These students should be referred to material within the transformation module to correct their deficiencies prior to attempting the work contained in this module.

Posttest

The posttest should be an accurate check to see if the student is able to succeed in accomplishing the objectives of the module.

Answers to set 1

1. x
2. 3
3. 5
4. y
5. 3 and 5
6. x and y
7. x, a and b
8. geometric mean
9. $\frac{OB}{OB'} = \frac{OA}{OA'}$
10. true if $x \neq 8$
11. true only if $a = o$
12. true only if $y = 1$ or -1
13. true only if $x = 2$
14. true only if $|AB| = |A'B'|$ and $AB \neq 0$
15. always true
16. 6
17. 10
18. $5\sqrt{2}$
19. $a = 9$
20. $b = -18$
21. $x = 10$
22. $t = 4$
23. $xy = 19/4$
24. $a = 11/7$
25. $m = 7$
26. $x = 5$ $x = -3$

27. $A'B' = 4$, $B'C' = 13\ 1/3$, $C'D' = 10\ 2/3$, $A'D' = 6\ 2/3$
28. $k = 5/2$, $y, a = 37.5$ $x'a' = 8$

Answers to Exercise Set 1

29. a. F

b. T, product property (a)

c. T, addition property (e)

d. F

e. F

f. T, product property (a)

g. T, inversion property (c)

h. T, inversion property (c) and denominator addition property (d)

i. F

j. F

30. a. $\frac{a}{x} = \frac{y}{b}$ e. $\frac{x}{a} = \frac{b}{y}$

b. $\frac{a}{y} = \frac{x}{b}$ f. $\frac{x}{b} = \frac{a}{y}$

c. $\frac{b}{x} = \frac{y}{a}$ g. $\frac{y}{a} = \frac{b}{x}$

d. $\frac{b}{y} = \frac{x}{a}$ h. $\frac{y}{b} = \frac{a}{x}$

31. a. $\frac{a}{b} = \frac{c}{d} \Rightarrow \frac{a}{b} \cdot b\,d = \frac{c}{d} \cdot b\,d$

$\Rightarrow a\,d = b\,c$

b. $\frac{a}{b} = \frac{c}{d} \Rightarrow a\,d = b\,c$

$\Rightarrow \frac{a\,d}{c\,d} = \frac{b\,c}{c\,d}$

$\Rightarrow \frac{a}{c} = \frac{b}{d}$

c. $\frac{a}{b} = \frac{c}{d} \Rightarrow a\,d = b\,c$

$\Rightarrow \frac{a\,d}{a\,c} = \frac{b\,c}{a\,c}$

$\Rightarrow \frac{d}{c} = \frac{b}{a}$

$\Rightarrow \frac{b}{a} = \frac{d}{c}$

Answers to Exercise Set 1, cont.

31. continued.

d. $\frac{a}{b} = \frac{c}{d} \Rightarrow \frac{a}{b} + \frac{b}{b} = \frac{c}{d} + \frac{d}{d}$

$\Rightarrow \frac{a+b}{b} = \frac{c+d}{d}$

e. $\frac{a}{b} = \frac{c}{d} \Rightarrow \frac{a}{c} = \frac{b}{d}$

$\Rightarrow \frac{a+c}{c} = \frac{b+d}{d}$

$\Rightarrow \frac{a+c}{b+d} = \frac{c}{d}$

$\Rightarrow \frac{a+c}{b+d} = \frac{a}{b}$

32. a. $a\,d = b\,c \Rightarrow a\,d \cdot \frac{1}{b\,d} = b\,c \cdot \frac{1}{b\,d}$

$\Rightarrow \frac{a}{b} = \frac{c}{d}$

b. $\frac{a}{c} = \frac{b}{d} \Rightarrow a\,d = b\,c$

$\Rightarrow \frac{a}{b} = \frac{c}{d}$

c. $\frac{b}{a} = \frac{c}{d} \Rightarrow a\,c = b\,d$

$\Rightarrow a\,c \cdot \frac{1}{b\,c} = b\,d \cdot \frac{1}{b\,c}$

$\Rightarrow \frac{a}{b} = \frac{d}{c}$

d. $\frac{a + b}{b} = \frac{c + d}{d} \Rightarrow \frac{a}{b} + \frac{b}{b} = \frac{c}{d} + \frac{d}{d}$

$$\Rightarrow \frac{a}{b} + 1 = \frac{c}{d} + 1$$

$$\Rightarrow \frac{a}{b} = \frac{c}{d}$$

e. $\frac{a + c}{b + d} = \frac{a}{b} \Rightarrow \frac{a + c}{b + d} = \frac{c}{d}$

$$\Rightarrow \frac{a + c}{c} = \frac{b + d}{d}$$

$$\Rightarrow \frac{a}{c} = \frac{b}{d}$$

$$\Rightarrow \frac{a}{b} = \frac{c}{d}$$

33. Proofs vary

34. Proofs vary

Answers to Exercises Set 2.

1. CE = 6

2. FJ = 28

3. T

4. T

5. T

6. T

7. PV = 14.4

8. MV = 6

9. PV = 8

10. MT = 5

11. TU = 8

12. TU = 18

13. ST = 2, TU = 12

14. ST = 4, TU = 16

15. CD = 2.25, DE = 1.92

16. OB = 20, AB = 10

17. $\frac{OC}{OD} = \frac{OA}{OB}$ and $\frac{OA}{OB} = \frac{OD}{OE}$

$\therefore \frac{OC}{OD} = \frac{OD}{OE}$

18. BC = 3

19. $\frac{9}{16} = \frac{W}{19}$, W = 10 11/16, Yes

20. $\frac{x-3}{3x-19} = \frac{4}{x-4}$, x = 12, x = 7

21. I - 80'
II - 160'
III - 120'

22. A. Yes, transitive property of ~

$\frac{AB}{GH} = \frac{8}{15}$

B. 1. 3/2
2. 5/4
3. 1/1
4. 15/8

23. a. $\angle P \cong \angle T$ $\quad \overline{PQ} \longleftrightarrow \overline{TS}$

$\angle Q \cong \angle S$ $\quad \overline{QR} \longleftrightarrow \overline{SR}$

$\angle R \cong \angle R$ $\quad \overline{PR} \longleftrightarrow \overline{TR}$

23\. b) $\angle A \cong \angle A$ $\quad \overline{AB} \longleftrightarrow \overline{AD}$

$\angle B \cong \angle D$ $\quad \overline{AC} \longleftrightarrow \overline{AE}$

$\angle C \cong \angle E$ $\quad \overline{BC} \longleftrightarrow \overline{DE}$

c. $\angle A \cong \angle C$ $\quad \overline{AC} \longleftrightarrow \overline{CB}$

$\angle C \cong \angle B$ $\quad \overline{CD} \longleftrightarrow \overline{BD}$

$\angle D \cong \angle D$ $\quad \overline{AD} \longleftrightarrow \overline{CD}$

Answers to Exercise Set 3

1\. $\angle 1 \cong \angle 2$ and

$\angle 2 \cong \angle 3$ and

$\angle 1 \cong \angle E$,

therefore $\angle E \cong \angle 3$ which implies $\overline{AB} \cong \overline{AE}$.

now, $\dfrac{AC}{CD} = \dfrac{AE}{BD}$ or $\dfrac{AC}{CD} = \dfrac{AB}{BD}$.

2\. $x = 4$
$y = 2\sqrt{5}$
$z = 4\sqrt{5}$

3\. $x = 2\sqrt{5}$
$y = 8$
$z = 4\sqrt{5}$

4\. $x = 4$
$y = 2\sqrt{21}$
$z = 21$

5\. $x = 12$
$y = 4\sqrt{3}$
$z = 8\sqrt{3}$

6\. $x = 12$
$y = 15$
$z = 20$

7\. $x = 8$
$y = 6\sqrt{5}$
$z = 4\sqrt{5}$

8. $\frac{x}{6} = \frac{6}{x+5}$

$x^2 + 5x - 36 = 0$
$(x + 9)(x - 4) = 0$
$x = -9$ or $\underline{\underline{x = 4}}$

9. $\frac{x}{x+2} = \frac{x+2}{8}$

$x^2 - 4x + 4 = 0$
$(x - 2)^2 = 0$
$\underline{\underline{x = 2}}$

10. a. $\triangle BCS \sim \triangle BDT$
$\triangle DCS \sim \triangle DBR$
$\triangle BSR \sim \triangle TSD$

b. $\frac{z}{y} = \frac{p}{p+q}$

c. $\frac{z}{x} = \frac{q}{q+p}$

d. $\frac{p+q}{x} = \frac{q}{z}$, $\frac{p+q}{y} = \frac{p}{z}$

$\frac{p+q}{x} + \frac{p+q}{y} = \frac{p+q}{z}$

$\frac{1}{x} + \frac{1}{y} = \frac{1}{z}$

Answers to Exercise Set 3

11. Consider the correspondence

$\triangle ACB \leftrightarrow \triangle ADC$. By AA, $\triangle ACB \sim \triangle ADC$.

$\therefore \frac{AB}{AC} = \frac{AC}{AD}$ Q.E.D.

12. $\frac{10}{15} = \frac{x}{30-x}$

$300 - 10x = 15x$
$25x = 300$
$x = 12$
$y = 18$

13. $\frac{4}{8} = \frac{6}{x}$

$4x = 48$
$x = 12$

*14. See part d. of #10.

Answers to Exercise Set 4

1. $a^2 + 15^2 = 17^2$

   $a^2 + 225 = 289$

   $a^2 = 64$

   $a = 8$

2. $6^2 + 8^2 = c^2$

   $c^2 = 36 + 64$

   $c^2 = 100$

   $c = 10$

3. $b^2 + 12^2 = 13^2$

   $b^2 + 144 = 169$

   $b^2 = 25$

   $b = 5$

4. $x^2 = 5^2 + 5^2$

   $x^2 = 50$

   $x = 5\sqrt{2}$

5. $y^2 + 1^2 = 2^2$

   $y^2 = 3$

   $y = \sqrt{3}$

6. $z^2 = 1^2 + 2^2$

   $z^2 = 1 + 4$

   $z = \sqrt{5}$

7. $h^2 + 6^2 = 10^2$

   $h^2 + 36 = 100$

   $h^2 = 64$

   $h = 8$

8. $90^2 + 90^2 = d^2$

   $d^2 = 90\sqrt{2} = 126.90$

   distance saved =

   $180 - 126.9 = 53.1$

9. let the hypotenuse = 2x

   1st leg = x

   2nd leg = y,

   then $x^2 + y^2 = (2x)^2$

   $y^2 = 3x^2$

   $x = \frac{y\sqrt{3}}{3}$, $2x = \frac{2y\sqrt{3}}{3}$

9a. hyp $= 6\sqrt{3}$

9b. hyp $= 12$

9c. hyp $= \frac{16\sqrt{3}}{3}$

9d. hyp $= \frac{2t\sqrt{3}}{3}$

10. a. $ED^2 = AB^2 + AD^2 + BE^2$

$= 2^2 + 2^2 + 1^2$

$ED^2 = 9$

$ED = 3$

b. $ED^2 = 11^2 + 10^2 + 2^2$

$= 121 + 100 + 4$

$ED^2 = 225$

$ED = 15$

c. $ED^2 = 3^2 + 4^2 + 12^2$

$= 9 + 16 + 144$

$ED^2 = 169$

$ED = 13$

11. a. $EB^2 = ED^2 - (AD^2 + AB^2)$

$= 10^2 (5^2 + 5^2)$

$EB^2 = 50$

$EB = 5\sqrt{2}$

b. $EB^2 = 17^2 - (2^2 + 11^2)$

$= 289 - (4 + 121)$

$EB^2 = 144$

$EB = 12$

c. $EB^2 = 6^2 - (4^2 + 3^2)$

$= 36 - (16 + 9)$

$EB^2 = 11$

$EB = \sqrt{11}$

12. $x_1^2 + 15^2 = 25^2$

$x_1^2 + 225 = 625$

$x_1^2 = 400$

$x_1 = 20$

$x_2^2 + 15^2 = 17^2$

$x_2^2 + 225 = 289$

$x_2^2 = 64$

$x_2 = 8$

$x = x_1 + x_2 = 28$

13. $(x + 17)^2 + x^2 = (x + 18)^2$

$x^2 + 34x + 289 + x^2 = x^2 + 36x + 324$

$x^2 - 2x - 35 = 0$

$(x - 7)(x + 5) = 0$

$x = 7$ or $x = -5$

$\underline{\underline{x = 7}}$

14. $8^2 + y^2 = 17^2$

$y^2 = 289 - 64$

$y^2 = 225$

$y = 15$

$(x + 8)^2 + 15^2 = 25^2$

$x^2 + 16x + 64 + 225 = 625$

$x^2 + 16x - 336 = 0$

$(x - 12)(x + 28) = 0$

$x = 12$ or $x = -28$

$\underline{\underline{x = 12}}$

15. $10\sqrt{5}$

16. $\sqrt{5}$

17. No

18. Yes

19. No

20. Yes

Answers to Exercise Set 5

1. $2\sqrt{2}$ cm

2. $\sqrt{2}$ m

3. $x\sqrt{2}$ cm

4. 2 cm

5. hyp = 8 ft
   leg = $4\sqrt{3}$ ft

6. hyp = 21.5 cm
   leg = $\frac{21\sqrt{3}}{2}$ cm

7. hyp = 2y
   leg = $y\sqrt{3}$

8. hyp = $2\sqrt{3}$
   leg = 3

9. $6\sqrt{3}$

10. $6\sqrt{2}$

11. AB = 6

12. BC = $6\sqrt{3}$

13. AD = $6\sqrt{2}$

14. CD = 12

15. BE = 4
    AB = $4\sqrt{2}$
    AE = 4
    EC = $4\sqrt{3}$

16. $x = \frac{20\sqrt{3}}{3}$

17. MP = $3\sqrt{3} - 3$
    PR = $3\sqrt{2}$
    PQ = 3

18. MR = 20

Triangle Similarity - Applications

I. Proportions:

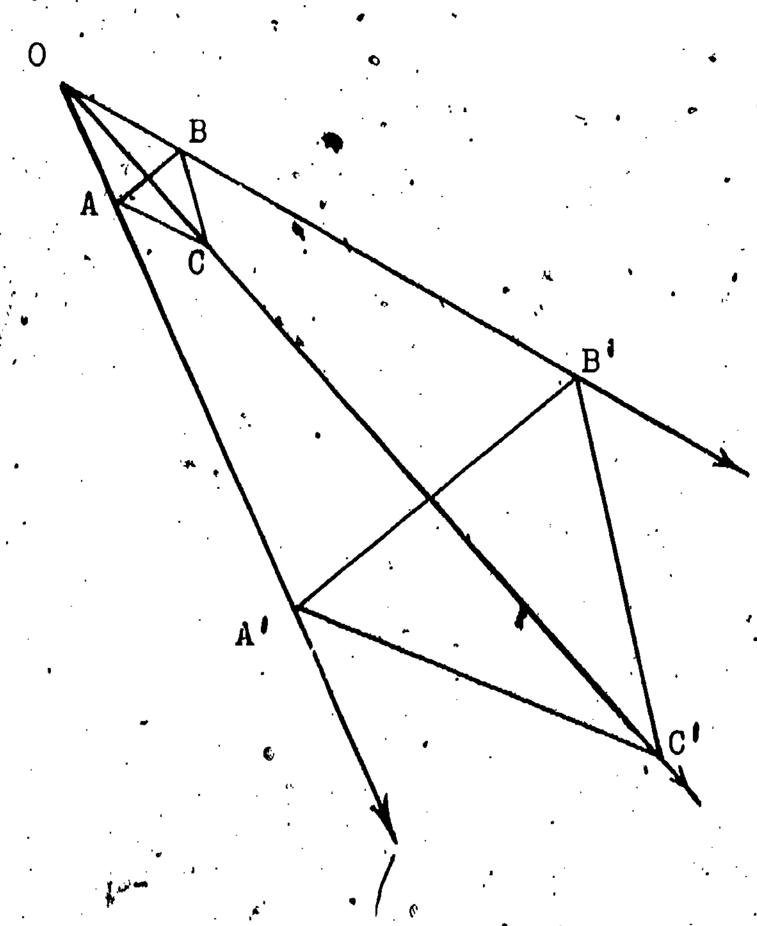
When one number x is divided by another number y ($y \neq 0$) the quotient $\frac{x}{y}$ is called the ratio of x to y.

Clearly the ratio $\frac{6}{2}$ equals the ratio $\frac{3}{1}$. Such a statement about the equality of two ratios is called a proportion.

The equations below are proportions.

$$\frac{AB}{A'B'} = \frac{BC}{B'C'}, \quad \frac{3}{x} = \frac{4}{9}, \quad \frac{x+4}{5} = \frac{2}{x}$$

Consider a size transformation with center O and a scale factor 4.



The ratio of the length of a line segment to the length of its image is 1 to 4, the scale factor of its size transformation.

That is:

$$\frac{AB}{A'B'} = \frac{BC}{B'C'} = \frac{AC}{A'C'} = \frac{1}{4}$$

Because certain procedures are used often in solving proportions, special names are used for the terms in a proportion.

In the proportion $\frac{a}{b} = \frac{c}{d}$

a, b, c, d are the first, second, thrid, and fourth terms respectively. a and d are called the extremes; b and c are called the means. Multiplying both sides of the proportion above by bd gives $ad = bc$. Since a and d are the extremes while b and d are the means, we make the following statement concerning proportions: "In a proportion, the product of the means equals the product of the extremes."

Example: Solve for y in $\frac{y + 3}{5} = \frac{y}{4}$

Solution: By the means - extremes property

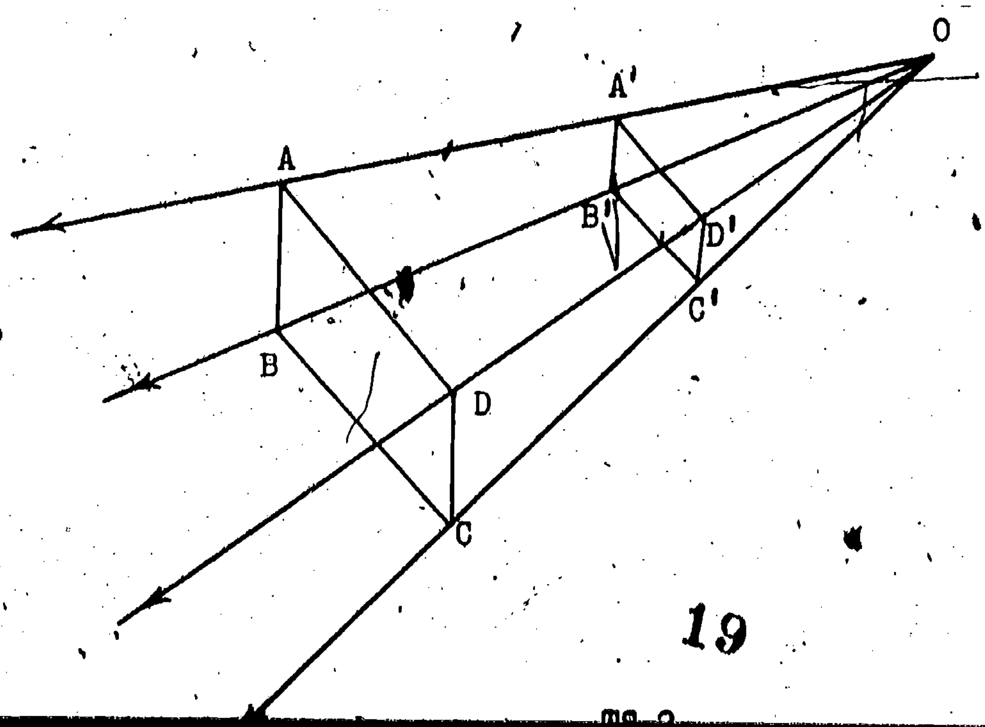
$$\frac{y + 3}{5} = \frac{y}{4} \Rightarrow 4(y + 3) = 5y$$

$$\Rightarrow 4y + 12 = 5y$$

$$12 = y$$

If the means of a proportion are identical as in: $\frac{a}{x} = \frac{x}{b}$

Then x is a geometric mean or mean proportional of a and b.



Let $S_{P,K}(ABCD) = A'B'C'D'$ with $BC = 9$, $B'C' = AB$, $A'B' = 4$.

What is the length of AB and the magnitude or scale factor K of S?

Solution:

$$\frac{AB}{A'B'} = \frac{BC}{B'C'}$$, substituting

the given lengths $$\frac{AB}{4} = \frac{9}{B'C'}$$

Since $B'C' = AB$ this becomes,

$$\frac{AB}{4} = \frac{9}{AB}$$

or $$(AB)^2 = 36$$

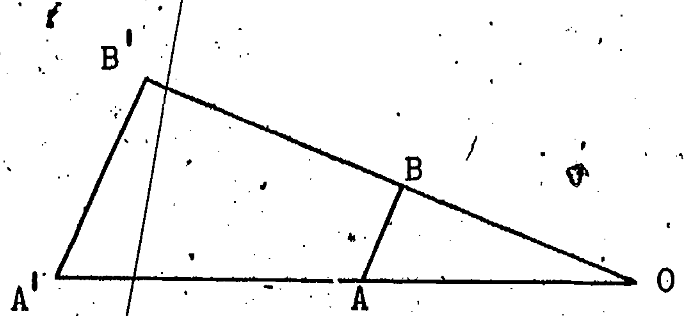
and $$AB = 6$$

therefore, $$K = \frac{AB}{A'B'} = \frac{6}{4} = 3/2$$

Exercises Set 1

Given the proportion $\frac{x}{3} = \frac{5}{y}$, name the following:

1. first term
2. second term
3. third term
4. fourth term
5. means
6. extremes
7. If $\frac{a}{x} = \frac{x}{b}$, then ____ is a mean proportional of ___ and ___
8. Another name for mean proportional is ______________.
9. Under a size transformation $S(A) = A'$ and $S(B) = B'$. The center is O. Name a proportion involving lengths of segments in this figure.



In problems 10-12
which proportions are always true? Assume denominators are not zero.

10. $\frac{3x - 24}{x-8} = \frac{3}{1}$

11. $\frac{a + 2}{a + 1} = \frac{2}{1}$

12. $\frac{y^8}{y^2} = y^4$

12. $\frac{y^8}{y^2} = y^4$

13. $\frac{5x^2}{5x} = \frac{2}{1}$

14. $\frac{AB}{A'B'} = \frac{A'B'}{AB}$

15. $\frac{12}{9} = \frac{16}{12}$

In problems 16-18
give a mean proportional of each pair of numbers.

16. 4 and 9

17. 5 and 20

18. 25 and 2

In problems 19-26
solve each proportion for the unknown term.

19. $\frac{a}{12} = \frac{3}{4}$

20. $\frac{12}{b} = \frac{-2}{3}$

21. $\frac{2}{x} = \frac{x}{50}$

22. $\frac{t + 3}{t} = \frac{14}{8}$

23. $\frac{xy - 2}{3} = \frac{9}{4}$

24. $\frac{a + 1}{2} = \frac{a - 1}{9}$

25. $\frac{4}{m-3} = \frac{9}{2+m}$

26. $\frac{5}{x} = \frac{x - 2}{3}$

27. S is a size transformation of magnitude 4/3. $S_{P,4/3}(ABCD) = A'B'C'D'$. If AB = 3, BC = 10, CD = 8 and AD = 5, find the lengths of the sides of A'B'C'D'.

28. M is a size transformation of triangle xyz such that $M_{P,K}(\Delta XYZ) = \Delta X'Y'Z'$. If X'Y' = 10, Y'Z' = 15, XZ = 20, and XY = 25 find K, YZ and X'Z'.

Additional properties of a proportion are useful in computations involving similar figures.

Each of the statements below is true if $\frac{a}{b} = \frac{c}{d}$.

a. $a\,d = b\,c$ (product property)

b. $\frac{a}{c} = \frac{b}{d}$ (equivalent - alternation property)

c. $\frac{b}{a} = \frac{d}{c}$ (inversion property)

d. $\frac{a+b}{b} = \frac{c+d}{d}$ (denominator addition property)

e. $\frac{a+c}{b+d} = \frac{a}{b}$ (addition property)

29. If x, y, z, and w are positive numbers and $\frac{x}{y} = \frac{z}{w}$, then which of the following equations are true? Which property does each true statement illustrate?

a. $xz = yw$

b. $xw = yz$

c. $\frac{x+z}{y+w} = \frac{z}{w}$

d. $\frac{x}{w} = \frac{z}{y}$

e. $\frac{x+w}{y+z} = \frac{x}{y}$

f. $yz = xw$

g. $\frac{w}{z} = \frac{y}{x}$

h. $\frac{y}{x+y} = \frac{w}{z+w}$

i. $\frac{y+w}{x} = \frac{x+z}{w}$

j. $\frac{x}{z} = \frac{w}{y}$

30. Starting with $ab = xy$, write a proportion whose left member is:

a. $\frac{a}{x}$ b. $\frac{a}{y}$ c. $\frac{b}{x}$ d. $\frac{b}{y}$

e. $\frac{x}{a}$ f. $\frac{x}{b}$ g. $\frac{y}{a}$ h. $\frac{y}{b}$

31. Verify the if portion of the five properties of a proportion stated previously.

32. Verify the "only if" portion of the five properties of a proportion. That is, beginning with each of the following equations derive the proportion $\frac{a}{b} = \frac{c}{d}$.

a. $a\,d = b\,c$

b. $\frac{a}{c} = \frac{b}{d}$

c. $\frac{b}{a} = \frac{d}{c}$

d. $\frac{a+b}{b} = \frac{c+d}{d}$

e. $\frac{a+c}{b+d} = \frac{a}{b}$

Optional:

33. Given: $\frac{a}{b} = \frac{c}{d}$ and a, b, c and d are positive.

Show that:

a. If $a > c$, then $b > d$

b. $a = c$ iff $b = d$

c. If $a < c$, then $b < d$

d. If $a < b$, then $c < d$

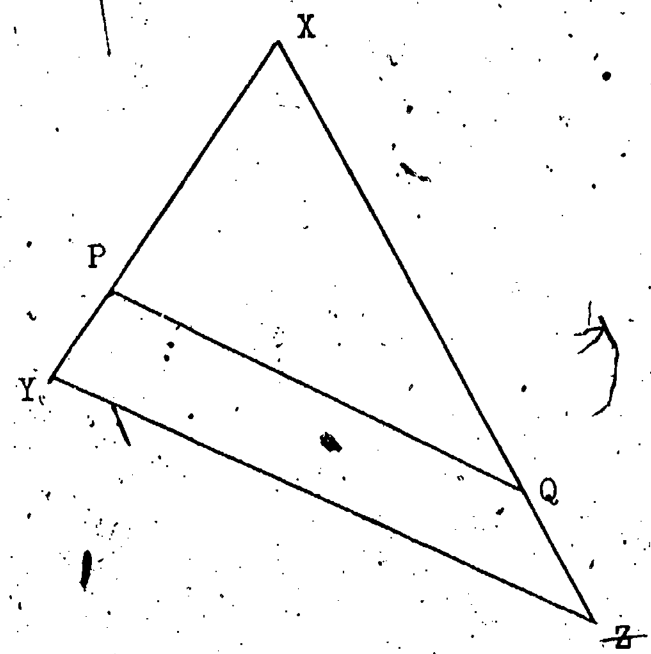
34. Show: If $\frac{a}{b} = \frac{c}{d}$, then

a. $\frac{a-b}{b} = \frac{c-d}{d}$

b. $\frac{a+2b}{b} = \frac{c+2d}{d}$

II. Triangle Similarity

Consider a size transformation S with center X and a magnitude (scale factor) equal to $\frac{XP}{XY}$ with $\overline{PQ} \parallel \overline{YZ}$



$$S_{X,\frac{XP}{XY}}(\triangle XPQ) = \triangle XYZ$$

In the above figure corresponding segments are proportional.

$$\frac{XY}{XP} = \frac{XZ}{XQ} = \frac{YZ}{PQ}$$

Select the two ratios. $\frac{XY}{XP} = \frac{XZ}{XQ}$

The betweenness property and substitution property yields:

$$\frac{XP + PY}{XP} = \frac{XQ + QZ}{XQ} \quad \text{and}$$

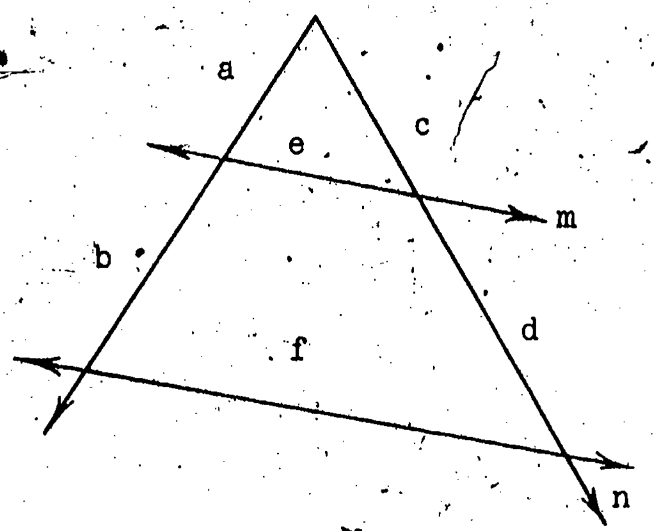
$$\frac{XP}{XP} + \frac{PY}{XP} = \frac{XQ}{XQ} + \frac{QZ}{XQ}$$

Subtract 1 from both members of the equation.

$$\frac{PY}{XP} = \frac{QZ}{XQ}$$

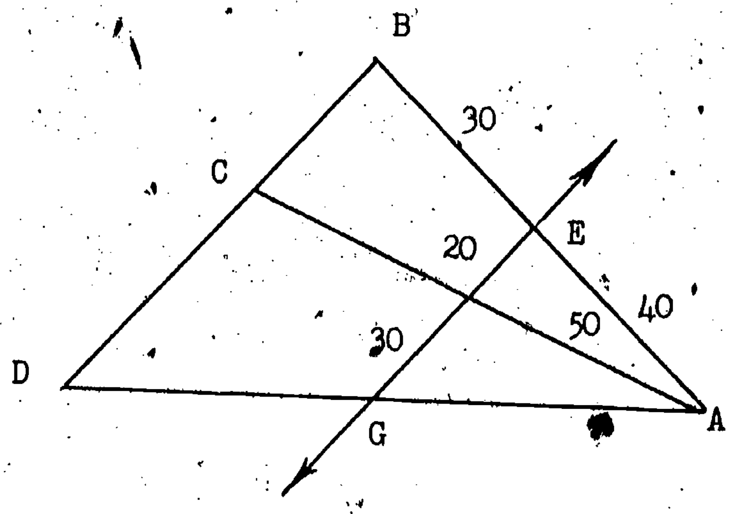
or $$\frac{XP}{PY} = \frac{XQ}{QZ}$$ (inversion property)

In short, if in the figure $m \parallel n$, then $\frac{a}{b} = \frac{c}{d}$



Example 1. Given: $\overleftrightarrow{EG} \parallel \overleftrightarrow{BD}$, lengths as shown.

Find: C F and C D



$\frac{AE}{BE} = \frac{AF}{CF} \Rightarrow \frac{40}{30} = \frac{50}{CF}$

Solving, $40\,(CF) = 1500$

$CF = 37.5$

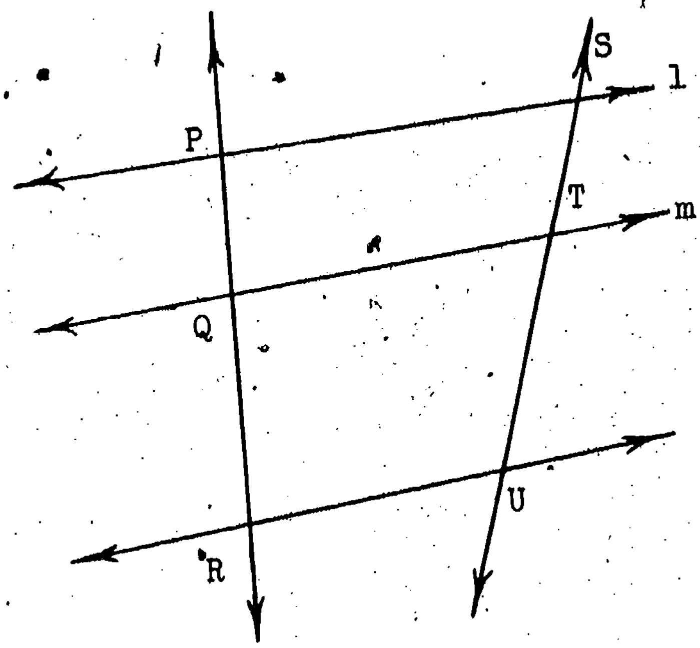
Also, $\frac{AF}{AC} = \frac{FG}{CD} \Rightarrow \frac{50}{87.5} = \frac{30}{CD}$

Solving, $50\,(CD) = (87.5)\,(30)$

$CD = 52.5$

The proceeding statement concerning a line parallel to a side of a triangle and intersecting the other two sides may be generalized to form the following.

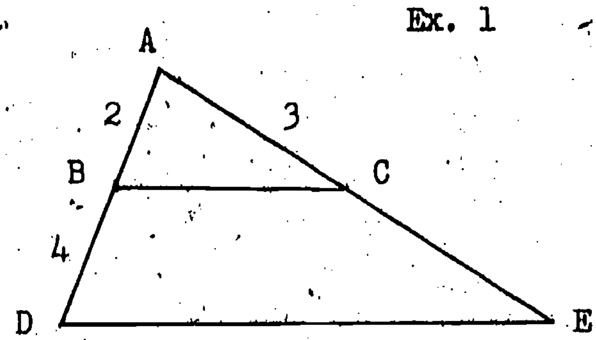
If $e \parallel m \parallel n$, then $\frac{PQ}{QR} = \frac{ST}{TU}$



Exercises Set 2

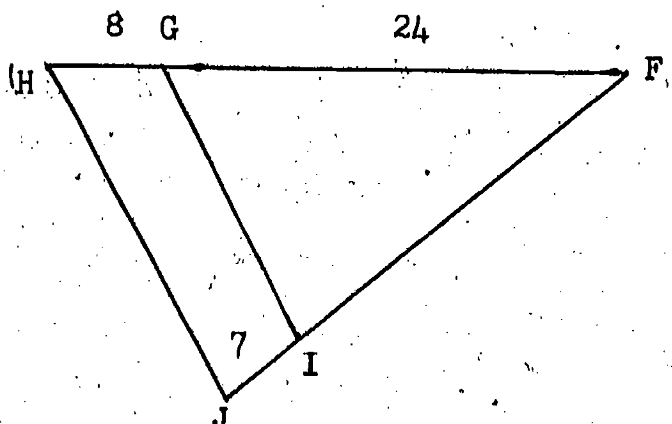
1. $\overline{BC} \parallel \overline{DE}$, find CE.

Ex. 1



2. $\overleftrightarrow{GI} \parallel \overleftrightarrow{HJ}$, find FJ.

Ex. 2



Given $\overline{LM} \parallel \overline{NO}$, answer questions 3-6 telling whether or not each statement is true.

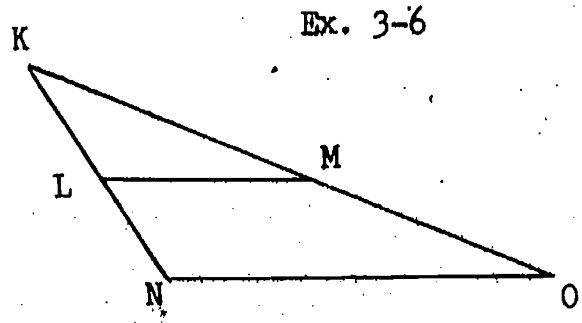
3. $\frac{KL}{KN} = \frac{KM}{KO}$

4. $\frac{KL}{LN} = \frac{KM}{MO}$

5. $\frac{KN}{LN} = \frac{KO}{MO}$

6. $\frac{KL}{KN} = \frac{LM}{NO}$

Ex. 3-6



Given: $\overline{MT} \parallel \overline{QP}$

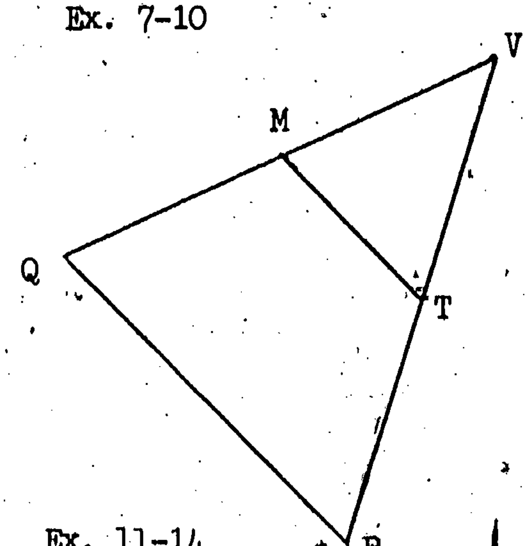
7. If QM = 10, MV = 6, PT = 9, find PV.

8. If PQ = 12, MT = 8, QM = 3, find MV.

9. If PT = 6, PQ = 20, MT = 15, find PV.

10. If TV = 5, PT = 1, PQ = 6, find MT.

Ex. 7-10



Given $l \parallel m \parallel n$

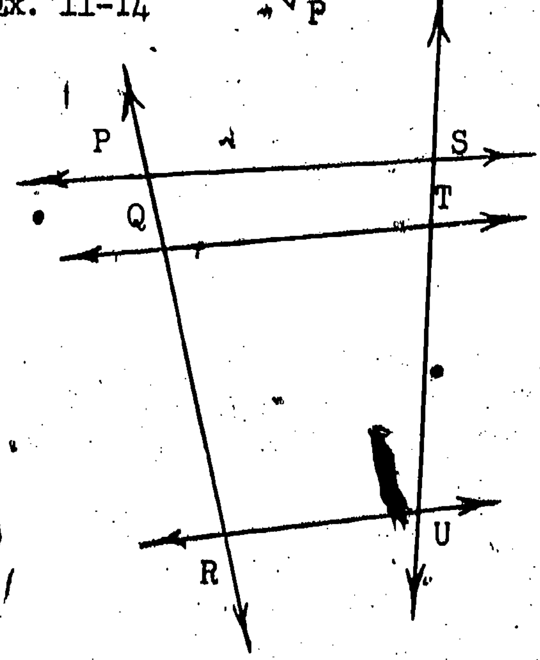
11. If PQ = 5, QR = 10, ST = 4, find TU.

12. If PQ = 5, PR = 20, ST = 6, find TU.

13. If PQ = 3, QR = 15, SU = 12, find ST and TU.

14. If PR = 25, QR = 15, SU = 20, find ST and TU.

Ex. 11-14

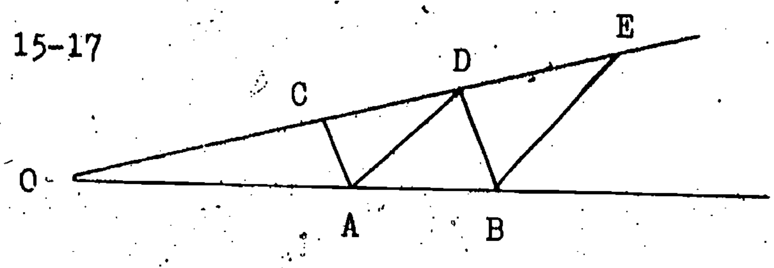


Given: $S_{O,K}(\triangle ACD) = \triangle BDE$.

15. If $OA = 4$, $AB = 3$, $OC = 3$, find CD and DE.

16. If $OA = 10$, $AC = 7$, $BD = 14$, find the lengths of as many other segments as you can.

Ex. 15-17

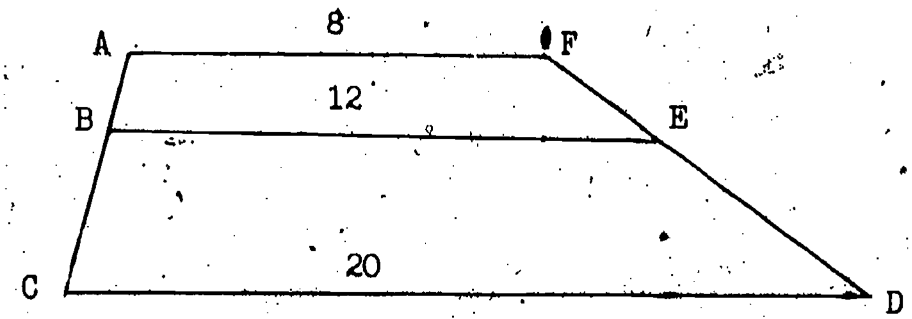


17. Show that OD is a mean proportional of OC and OE.

18. Given: Trapezoid ACDF with $\overline{AF} \parallel \overline{BE} \parallel \overline{CD}$.

$AB = 2$, $AF = 8$, $BE = 12$, $CD = 20$. Find BC

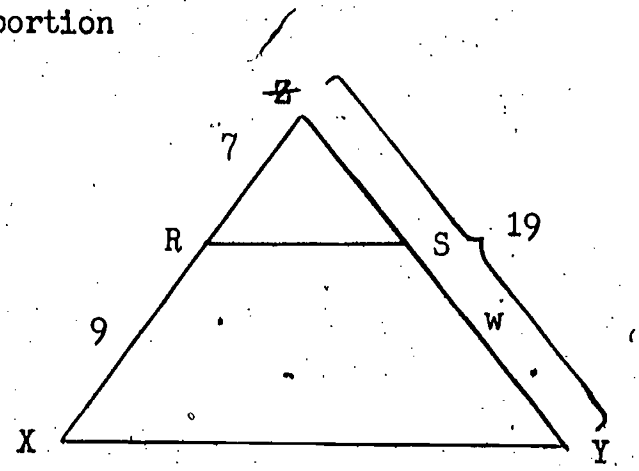
Ex. 18



19. Given: $\overline{xy} \parallel \overline{rs}$, segments as indicated.

One person suggested the following proportion to find w.

$$\frac{7}{9} = \frac{19-w}{w}$$

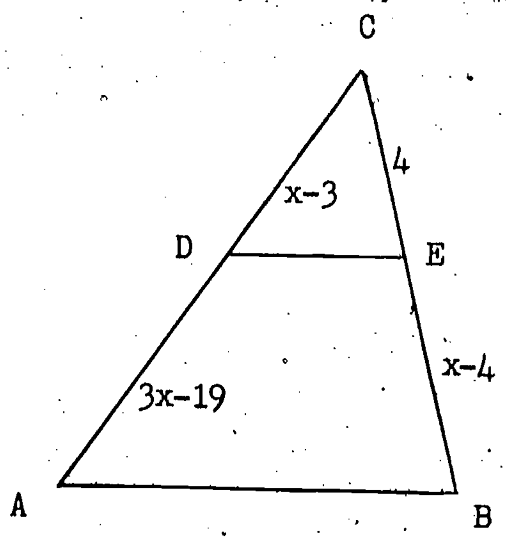


Propose a more convenient proportion.

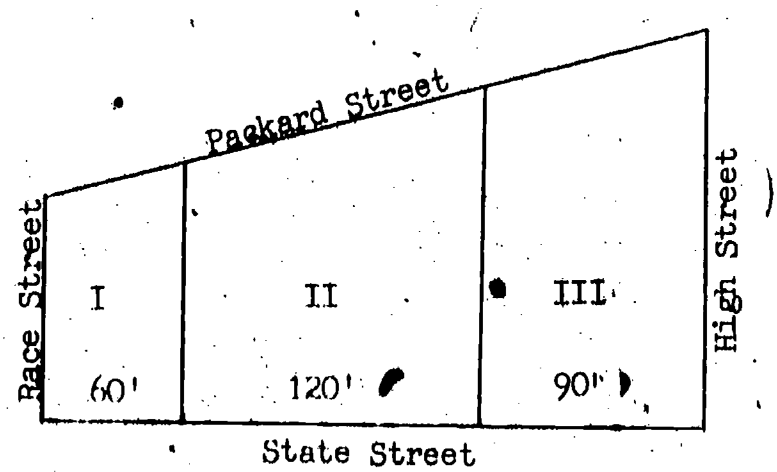
Do you get the same result?

20. $\overline{DE} \parallel \overline{AB}$ with $CD = x - 3$, $DA = 3x - 19$, $CE = 4$, and $EB = x - 4$

Find value (s) of x.



21. Three lots extend from Packard Street to State Street as shown. The side boundaries of each lot make right angles with State Street. The total frontage on Packard Street is 360'. Find the frontage of each lot on Packard Street.



22. Suppose $\triangle ABC \sim \triangle DEF$ with $\frac{AB}{DE} = \frac{2}{3}$ and

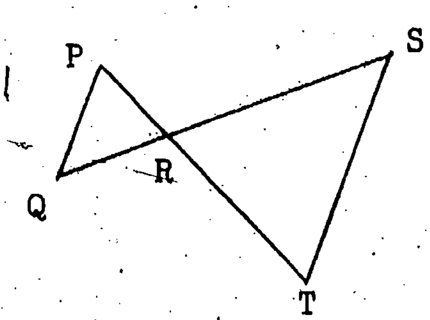
$\triangle DEF \sim \triangle GHK$ with $\frac{DE}{GH} = \frac{4}{5}$

a. Is $\triangle ABC \sim \triangle GHK$? Why? What is the value of $\frac{AB}{GH}$?

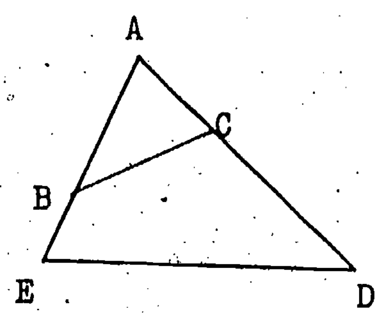
b. For each of the following, tell why it is true and find the ratio of a side of the first triangle to the corresponding side of the second triangle.

1. $\triangle DEF \sim \triangle ABC$
2. $\triangle GHK \sim \triangle DEF$
3. $\triangle DEF \sim \triangle DEF$
4. $\triangle GHK \sim \triangle ABC$

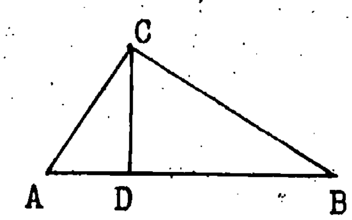
23. Assume that the correspondences indicated below are similarities. State which angles must be congruent and which sides must be proportional.



a. $PQR \longleftrightarrow TSR$



b. $ABC \longleftrightarrow ADE$

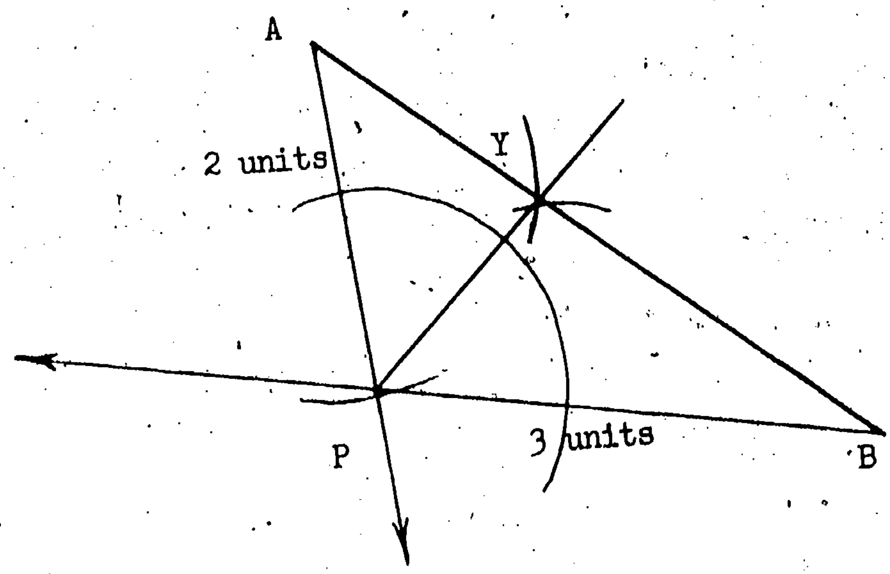


c. $ACD \longleftrightarrow CBD$

Triangle Similarity Theorems

Suppose that you would like to divide a line segment into two segments which have a ratio of two to three. There exists a standard construction procedure for doing this. But, we now show a slightly different algorithm.

Given: Segment AB. Divide $\overline{AB}$ into two segments which have a ratio of 2 to 3.

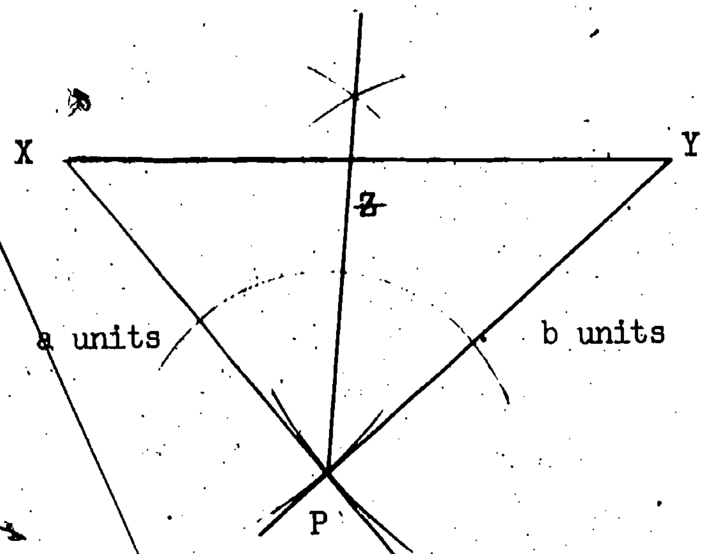


Solution:

1. With A as a center, construct an arc of two units as shown.
2. Use B as a center, construct an arc of 3 units which will intersect the previous arc. Label the point of intersection P.
3. Draw $\overline{AP}$ and $\overline{BP}$.
4. Bisect P. Label the point where the angle bisector intersects $\overline{AB}$, Y.
5. AY and YB are the two segments.

This procedure may be generalized as follows:

Given: Segment x y. Divide $\overline{x\,y}$ into two segments which have the ratio of a: b.

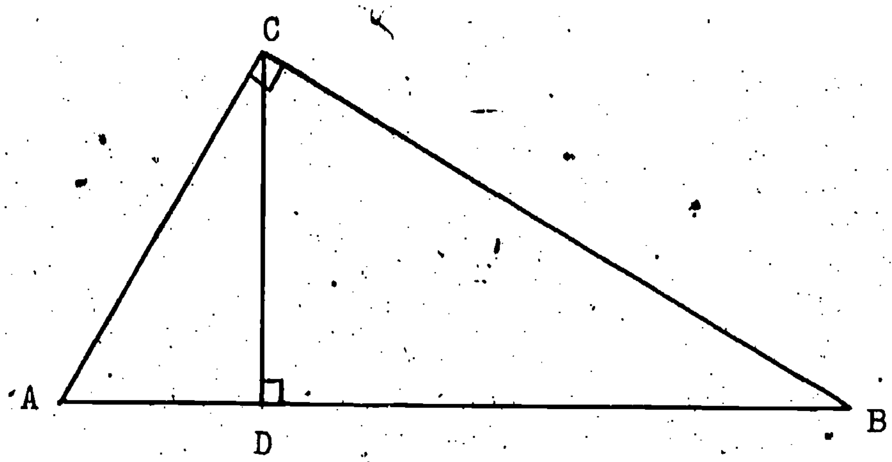


Construct arcs of length a and length b. Draw XP and YP. Bisect angle P. X Z and Z Y are the two segments.

Geometrically, this property may be stated as follows:

> "The bisector of an angle of a triangle divides the opposite side into segments proportional to the adjacent sides."

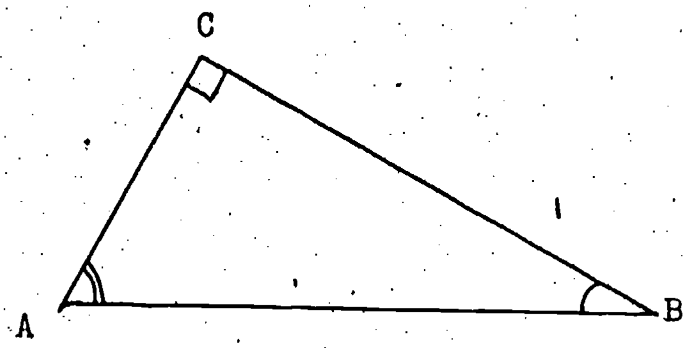
Consider now a right triangle ABC with the altitude $\overline{CD}$. <u>This</u> altitude separates the triangle into two triangles which are similar to each other and to the original triangle.

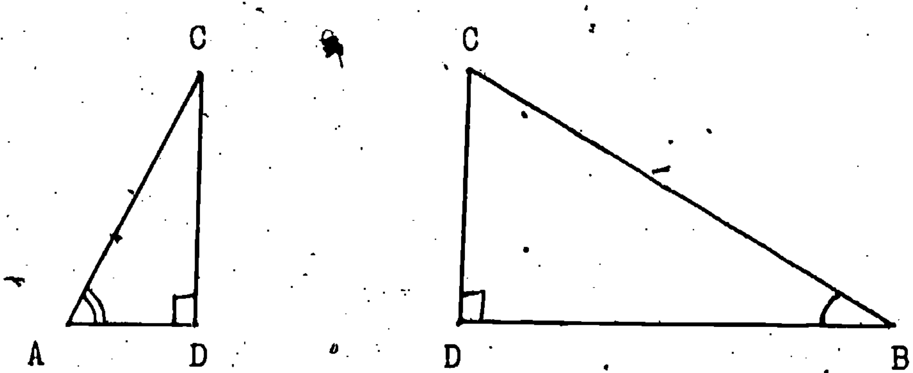


In this figure, $\triangle ABC$ is the original right triangle with the altitude to the hypotenuse $\overline{CD}$. Then

$$\triangle ACD \sim \triangle ABC \sim \triangle CBD$$

By separating the triangles, congruent angles may be marked.





By using the triangle similarities above, three conclusions may be reached immediately.

I. The altitude is the geometric mean of the segments into which it separates the hypotenuse.

In terms of the figure

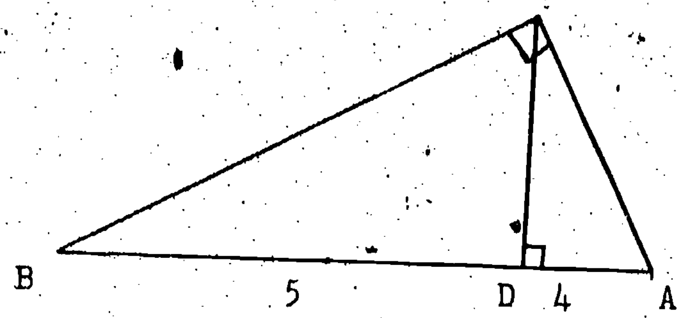
$$\frac{AD}{CD} = \frac{CD}{DB}$$

II. Either leg is the geometric mean of the hypotenuse and the segment of the hypotenuse adjacent to the leg.

$$\frac{AD}{AC} = \frac{AC}{AB} \qquad \text{or} \qquad \frac{BD}{AC} = \frac{AC}{BC}$$

III. The product of the two legs is equal to the product of the hypotenuse and the altitude to the hypotenuse.

$$AC \cdot BC = AB \cdot CD$$



Example 1. Given: Right angles as shown;

$AD = 4$, $DB = 5$.

Find: CD, CA, and CB.

Solution:

$\frac{AD}{CD} = \frac{CD}{DB} \Rightarrow (CD)^2 = AD \cdot DB \Rightarrow (CD)^2 = (4 \times 5) = 20 \Rightarrow CD = 2\sqrt{5}$

$\frac{AD}{CA} = \frac{CA}{AB} \Rightarrow (CA)^2 = AD \cdot AB \Rightarrow (CA)^2 = (4 \times 9) = 36 \Rightarrow CA = 6$

$\frac{DB}{CB} = \frac{CB}{AB} \Rightarrow (CB)^2 = DB \cdot AB \Rightarrow (CB)^2 = (5 \times 9) = 45 \Rightarrow CB = 3\sqrt{5}$

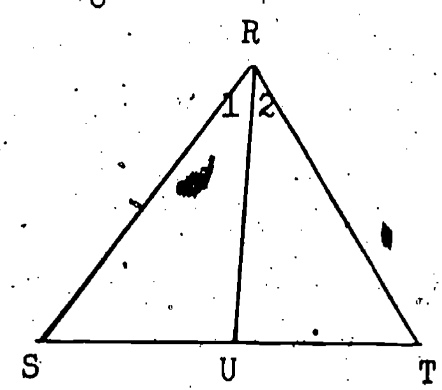
An alternate method of finding CB is as follows:

$$AC \cdot BC = AB \cdot CD \Rightarrow BC = \frac{AB \cdot CD}{AC} \Rightarrow BC = \frac{9 \cdot 2\sqrt{5}}{6} \Rightarrow BC = 3\sqrt{5}$$

Example 2. Given: $\angle 1 \cong \angle 2$

$RS = 7$, $RT = 5$, $ST = 10$

Find: SU and UT



Solution:

Let $s = SU$. Now, since $\frac{SU}{UT} = \frac{RS}{RT}$ we have $\frac{s}{10-s} = \frac{7}{5}$.

$5s = 7(10-s)$, $5s = 70 - 7s$, $12s = 70$, $s = 5\ 5/6$.

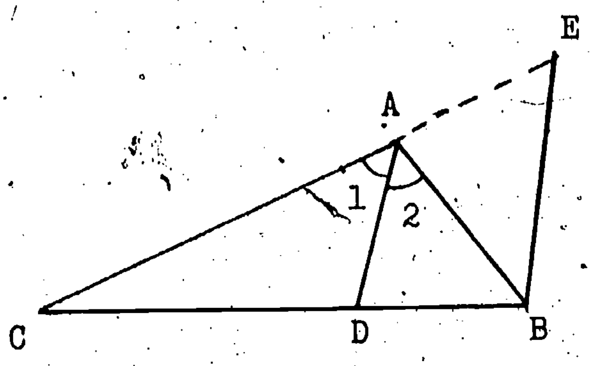
$\therefore SU = 5\ 5/6$, $UT = 10 - SU$, $UT = 4\ 1/6$

Exercise Set 3

1. Given: $\triangle ABC$, $\overline{AD}$ the bisector of $\angle A$ as shown.

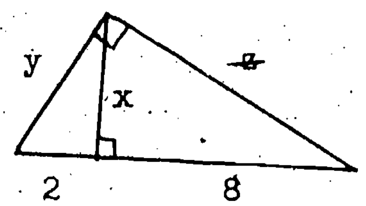
Show: $\dfrac{AB}{AC} = \dfrac{BD}{CD}$

(Hint: Draw $\overline{BE} \parallel \overline{AD}$)

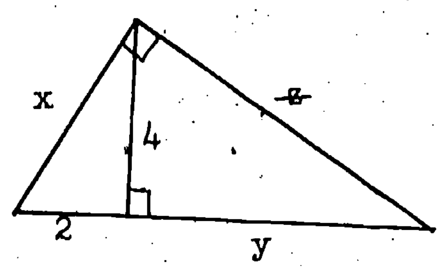


In exercises 2 - 9 find values for x, y, and z.

2.



3.



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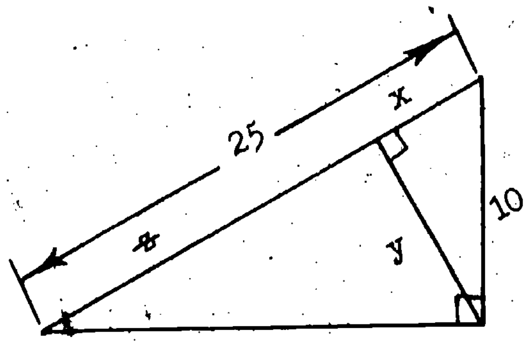

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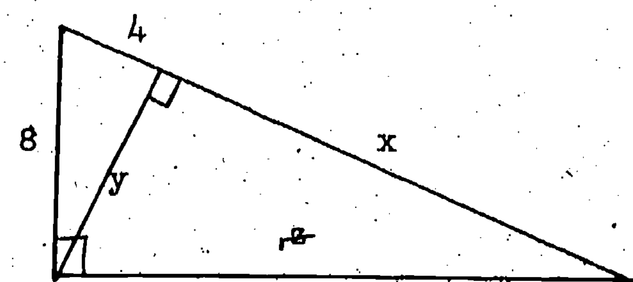

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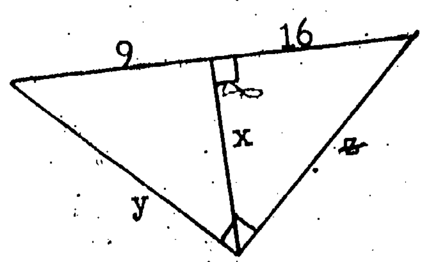

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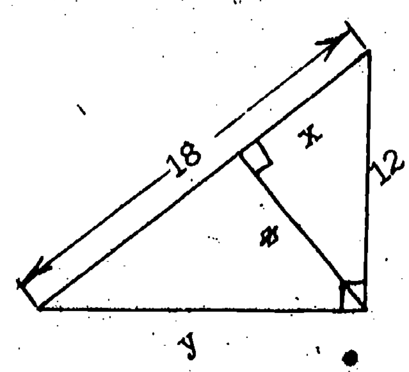

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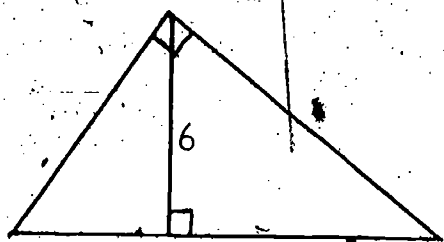

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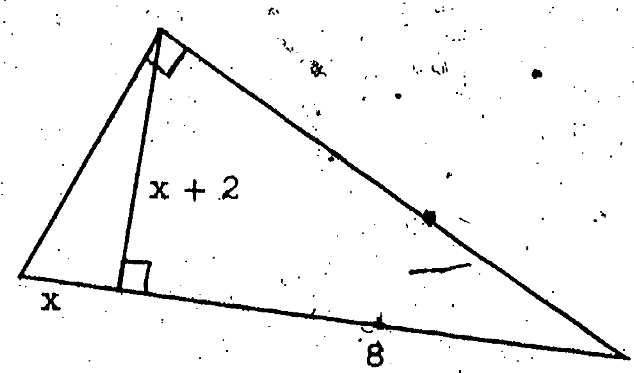
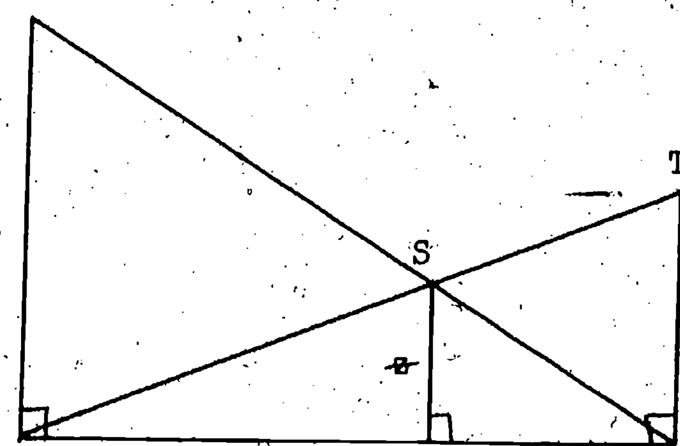


10. $\overline{BR}$, $\overline{CS}$ and $\overline{DT}$ are $\perp$ to $\overline{BD}$.



a. Name the pairs of similar triangles

b. Which is correct?

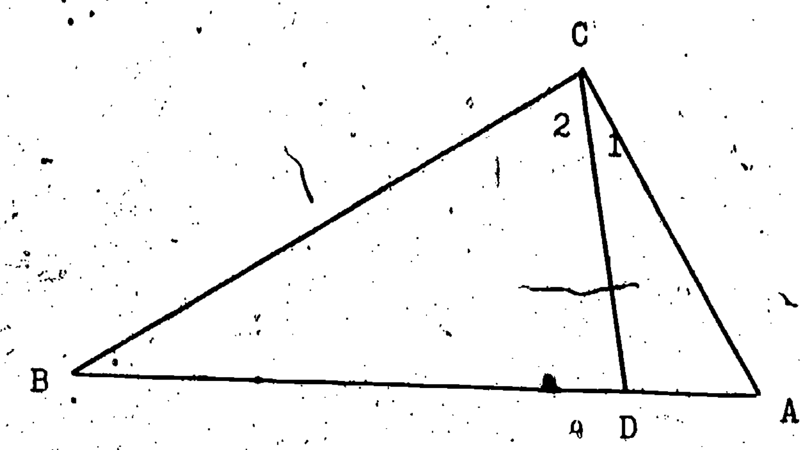
$\frac{z}{y} = \frac{p}{q}$ or $\frac{z}{y} = \frac{p}{p+q}$

c. Which is correct?

$\frac{z}{x} = \frac{q}{p}$ or $\frac{z}{x} = \frac{q}{p+q}$

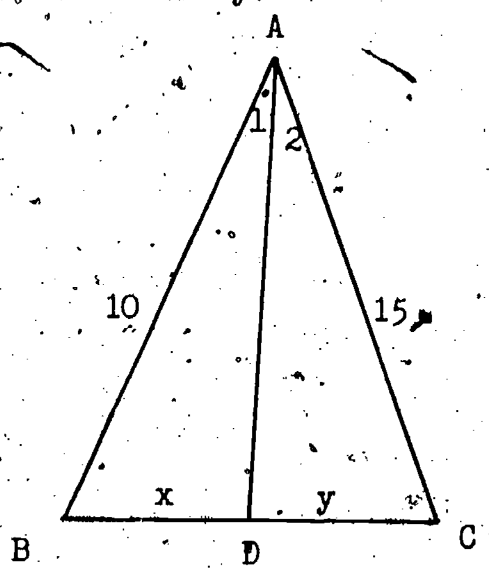
d. Show that $\frac{1}{x} + \frac{1}{y} = \frac{1}{z}$

11. Given; $\angle 1 \cong \angle B$. Show that AC is a mean proportional to AB and AD.



Find x and y.

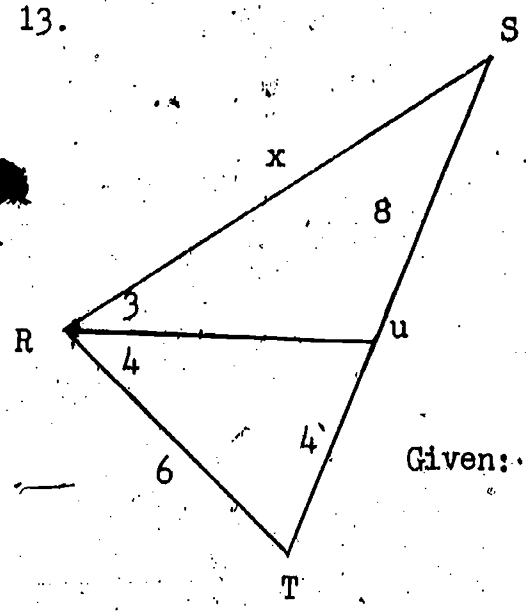
12.



Given: $\angle 1 \cong \angle 2$

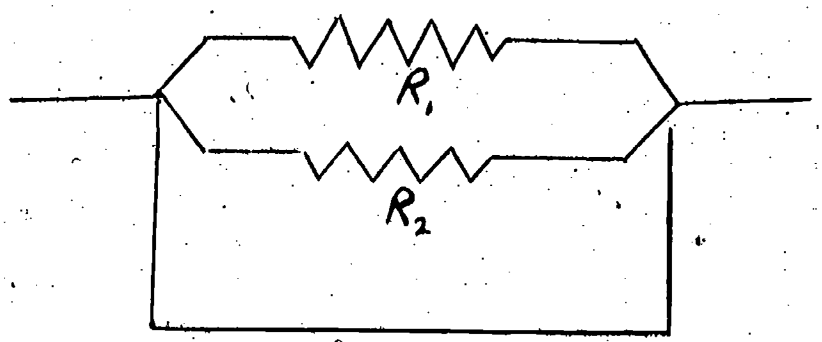
$x + y = 30$

13.



Given: $\angle 3 \cong \angle 4$

14. If we have an electrical circuit consisting of two wires in parallel, with resistances R_1 and R_2, then the resistance R of the circuit is given by $\frac{1}{R_T} = \frac{1}{R_1} + \frac{1}{R_2}$



The following scheme has been used to find R_T, given R_1 and R_2.

Numerical scales are marked off on three rays as shown in Figure 1 on the next page. A straight edge is placed so as to pass through R_1 and R_2 on the two outer scales, and R is read off on the third scale. Use the scales of the figure, select values for R_1 , R_2, find R_T from the figure and check your result to see that the equation is satisfied.

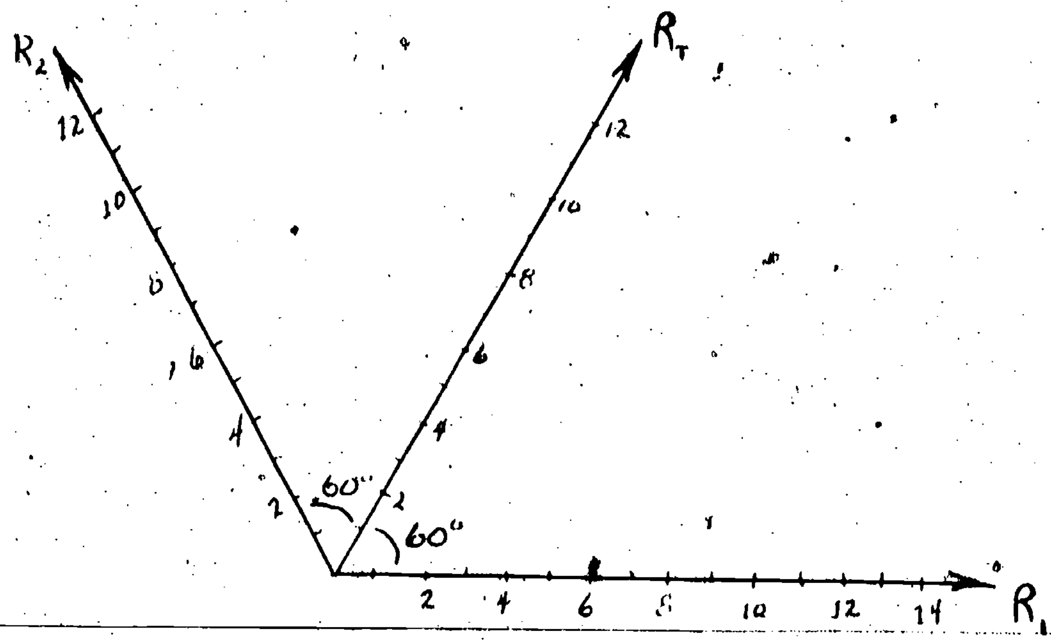
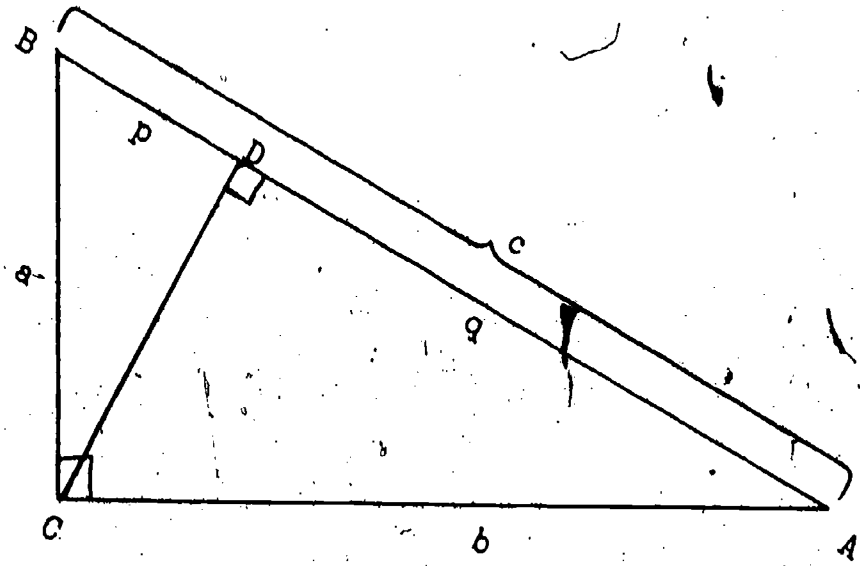


Fig. 1

Show that the method really works; could the same diagram be used to find R_T in the equation $\frac{1}{R_T} = \frac{1}{R_1} - \frac{1}{R_2}$

Pythagorean Theorem



In the right triangle above,

$$a^2 = p \cdot c \quad \text{and} \quad b^2 = q \cdot c \quad \text{and}$$

$$a^2 + b^2 = p \cdot c + q \cdot c$$

$$a^2 + b^2 = (p + q) \cdot c \qquad \text{But, } p + q = c$$

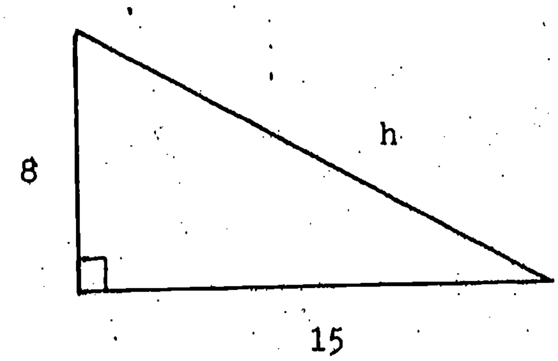
Therefore, $a^2 + b^2 = c^2$

This result is probably the most famous theorem in all mathematics. The first proof of this theorem is attributed to the Pythagoreans and bears their name.

Pythagorean Theorem: <u>In a right triangle</u>, the <u>square</u> of the hypotenuse is equal to the sum of the <u>squares</u> of its legs.

The applications of this theorem are many and varied. Foremost, the equation $a^2 + b^2 = c^2$ may be used to find the length of any side of a right triangle, given the lengths of the two other sides.

Example 1: The legs of a right triangle are 8 and 15. Find the length of the hypotenuse.

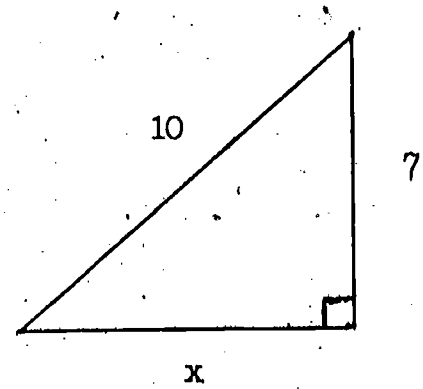


Solution: $8^2 + 15^2 = h^2$

$64 + 225 = h^2$

$h = \sqrt{289} = 17$

Example 2: One leg of a right triangle is 7, and the hypotenuse is 10. Find the length of the second leg.



Solution: $7^2 + x^2 = 10^2$

$49 + x^2 = 100$

$x^2 = 51$

$x = \sqrt{51} = 7.1$

The converse of the Pythagorean Theorem provides a way of showing whether or not a triangle is a right triangle.

Converse of the Pythagorean Theorem:

If a triangle has sides with measures a, b, and c and $a^2 + b^2 = c^2$, then the triangle is a right triangle.

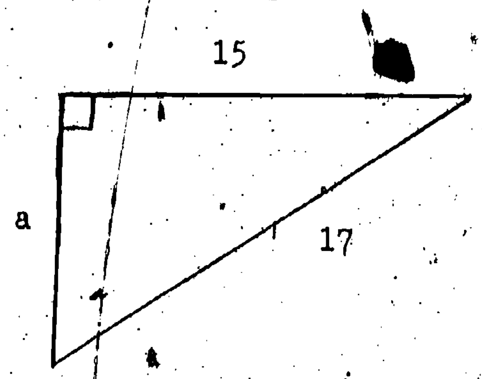
Example 3. A triangle has sides of lengths 2, 3, and 4. Is the triangle a right triangle?

Solution: If $2^2 + 3^2 = 4^2$ then the $\triangle$ will be a right $\triangle$.

$2^2 + 3^2 = 13$ and $13 \neq 4^2$. Therefore, the triangle is not a right triangle.

Exercise Set 4

Find the length of the third side of each right triangle.

1.


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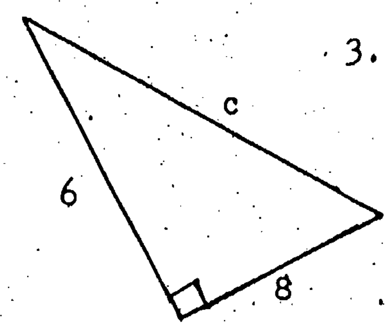

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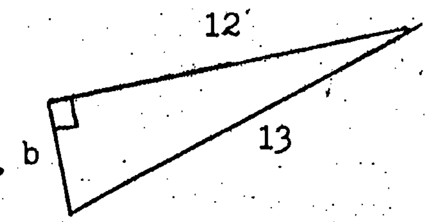

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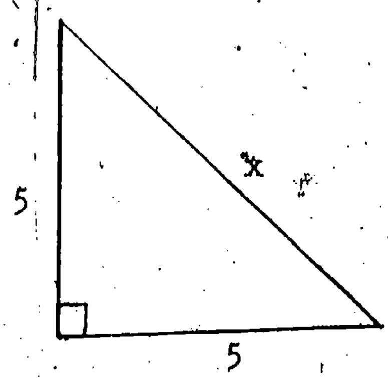
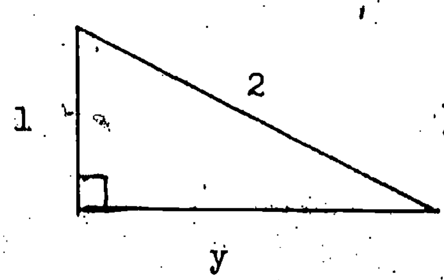


5.


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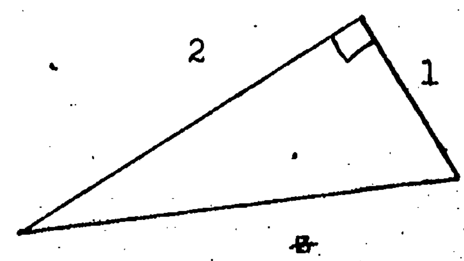


7. The sides of a triangle measure 10, 10, and 12. Find the length of the altitude to the longest side.

8. How many feet (to the nearest foot) would a person save by running from first base to third base instead of running from first to second to third? The distance between bases is 90'.

9. The hypotenuse of a right triangle is twice as long as one of its legs. Find the length of the hypotenuse if the other leg is

a. 9 b. $6\sqrt{3}$ c. 8 d. t

10. $\overline{AB} \perp \overline{AD}$, $\overline{EB} \perp \overline{BD}$, Find ED when

a. AB = 2, AD = 2, EB = 1.

b. AB = 11, AD = 10, EB = 2

c. AB = 3, AD = 4, EB = 12.

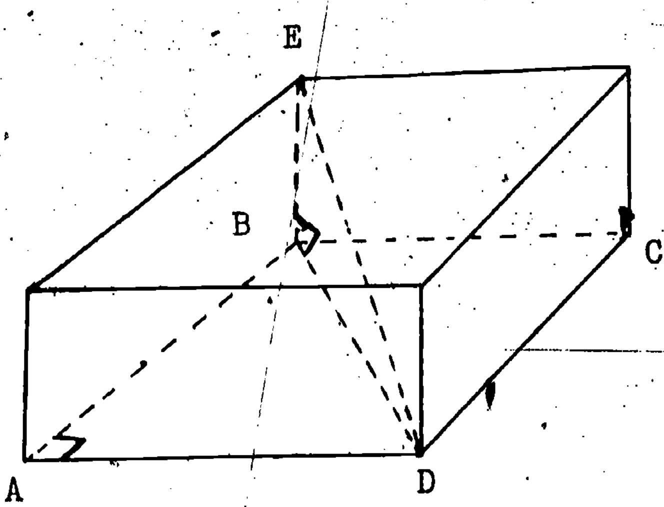


Fig. 1

11. For figure 1, find EB when

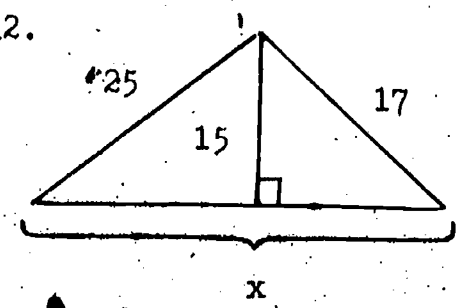
a. AB = 5, AD = 5, ED = 10.

b. AB = 11, AD = 2, ED = 17.

c. AB = 3, AD = 4, ED = 6.

For each of the following, 12-16, find x.

12.



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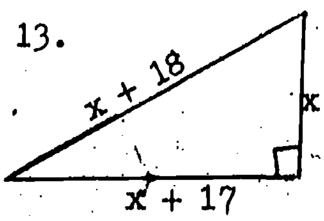

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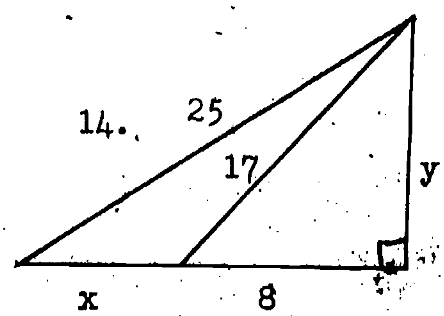

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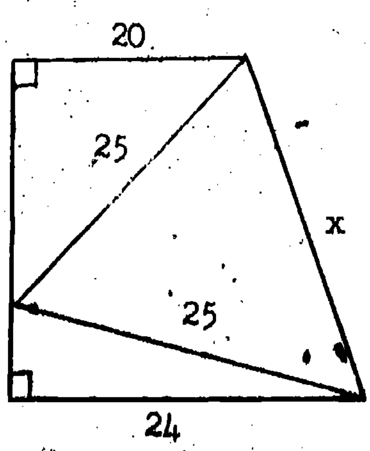
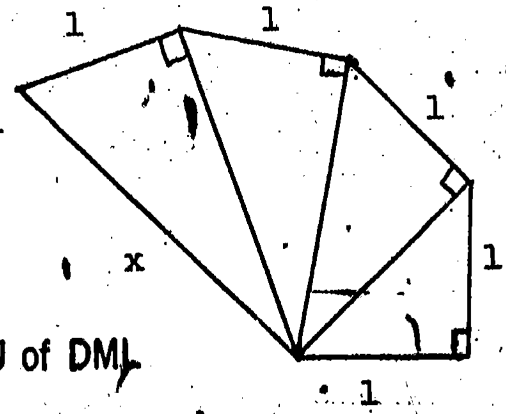


16.



Could the numbers given be the lengths of the three sides of a right triangle?

17. 3.5, 2.0, 2.5

18. $\sqrt{3}$, $\sqrt{4}$, $\sqrt{7}$

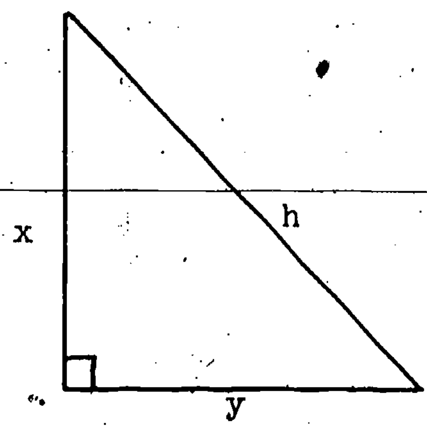
19. $\sqrt{2}$, 5, $\sqrt{21}$

20. 11, 36, 37

Special Triangles

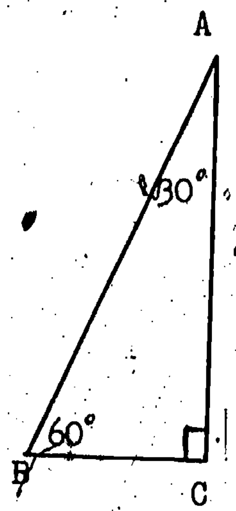
There are two special types of right triangles that are particularly useful in later work in mathematics. One is an isosceles right triangle. The other is a right triangle in which the acute angles are $30°$ and $60°$; an altitude of an equilateral triangle determines two such triangles.

We will consider the right isosceles triangle first. Both legs are the same length, say x. Find h.



By the Pythagorean relationship, $x^2 + x^2 = h^2$. So $h^2 = 2x^2$ or $h = x\sqrt{2}$. So, if the leg of an isosceles right triangle has length x, the hypotenuse has length $x\sqrt{2}$.

ΔABC is a right triangle with $m\angle A = 30°$, $BC = x$.



Now, let AB'C be the reflection of ΔABC over line $\overleftrightarrow{AC}$. Then triangle ABB' is an equilateral triangle with $BB' = 2x$ and $AB = 2x$.

~~Again, by the Pythagorean~~ relationship

$$(BC)^2 + (AC)^2 = (AB)^2 \quad \text{or} \quad x^2 + (AC)^2 = (2x)^2$$

$$x^2 + (AC)^2 = 4x^2$$

$$(AC)^2 = 3x^2$$

$$AC = x\sqrt{3}$$

So, if a right triangle has a 30° angle and the leg opposite that angle has length x, the other leg has length $x\sqrt{3}$ and the hypotenuse has length 2x.

Exercise Set 5

Find the length of the diagonal of a square with side lengths given as:

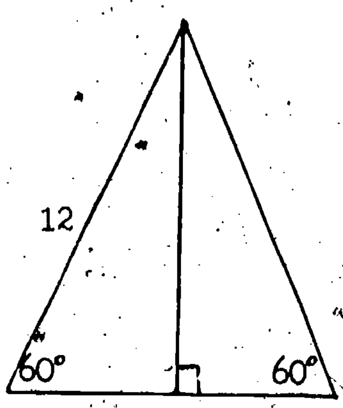
1. 2 cm 2. 1 m 3. x cm 4. $\sqrt{2}$ cm

What are the lengths of the other two sides of a 30-60-90 triangle whose shortest side is given as:

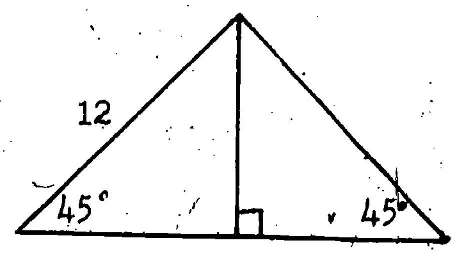
5. 4 ft. 6. 10.5 cm 7. y 8. $\sqrt{3}$

Find the length of the altitude drawn in each triangle.

9.



10.



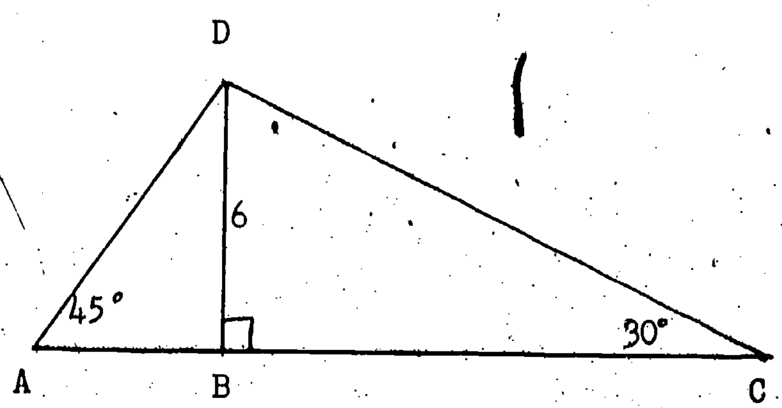
Find the length of each segment:

11. $\overline{AB}$

12. $\overline{BC}$

13. $\overline{AD}$

14. $\overline{CD}$



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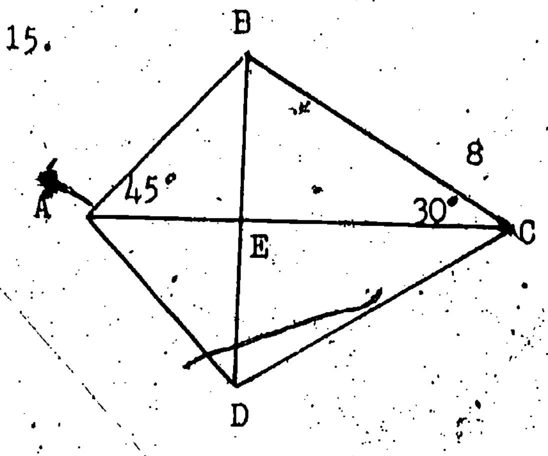
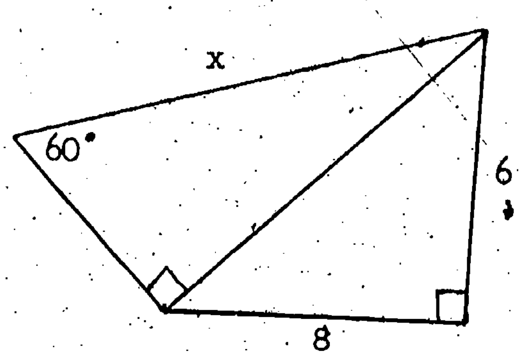
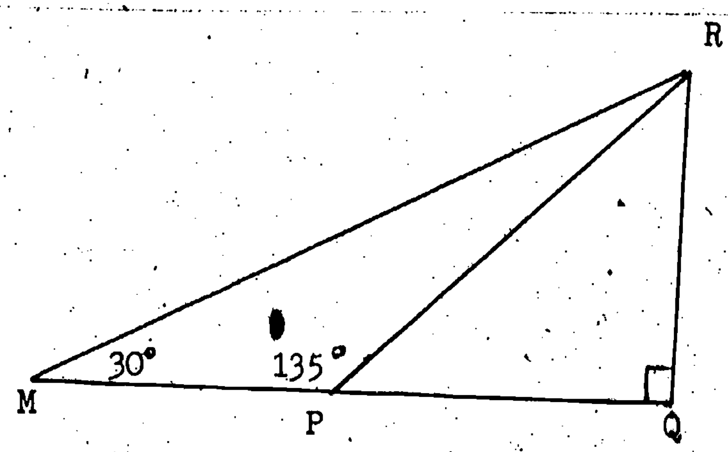


Given: ABCD is a kite; BC = 8

Find: BE, AB, AE, and EC.

16. Find x





Ex. 17-18

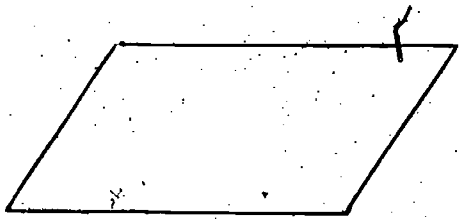
17. If MR = 6, find MP, PR, and PQ.

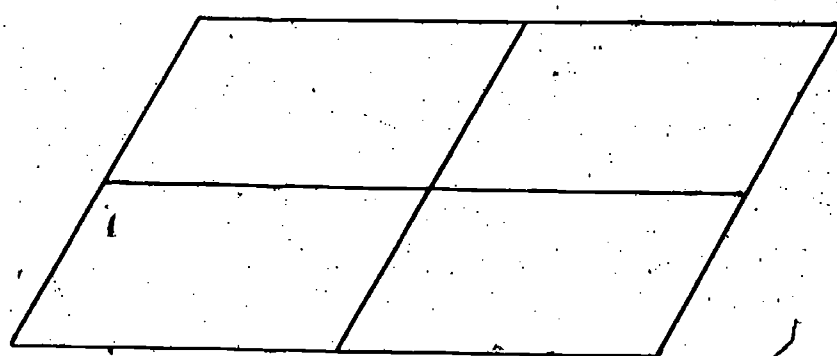
18. If PR = $10\sqrt{2}$, find MR.

VI. Similarity Projects.

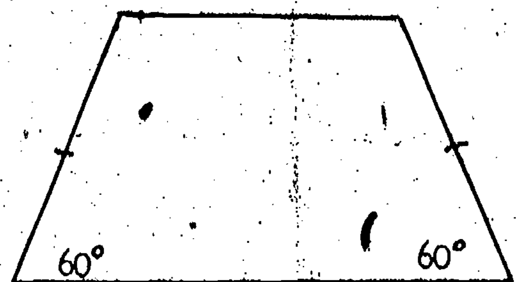
A figure is called a "Rep-tile" if copies of the figure fit together to form a larger similar figure. The "rep-" refers to the fact that the figure "repeats" or "replicates" itself in a larger similar figure. The "tile" refers to the fact that if copies of the larger figure are fitted together the same way and this is repeated over and over, we tile the plane.

For instance four copies of any parallelogram fit together to form a similar parallelogram.

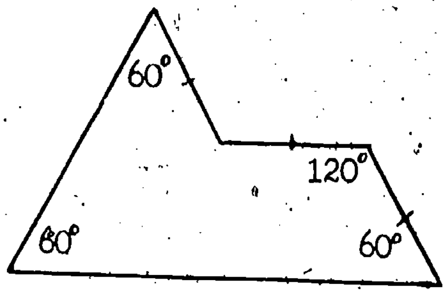




1. Show that four copies of the trapezoid below can be fitted together to form a similar trapezoid.



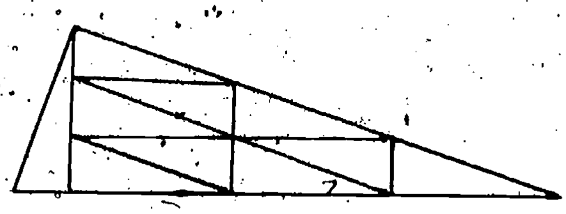
2. Show that four copies of the "Sphinx" pentagon below can be fitted together to form a similar pentagon.



3. Are there other figures which are "Rep-tiles"?

4. For each positive integer N, consider a right triangle with legs of lengths 1 and N. Fit $N^2 + 1$ copies of the right triangle to form a similar triangle.

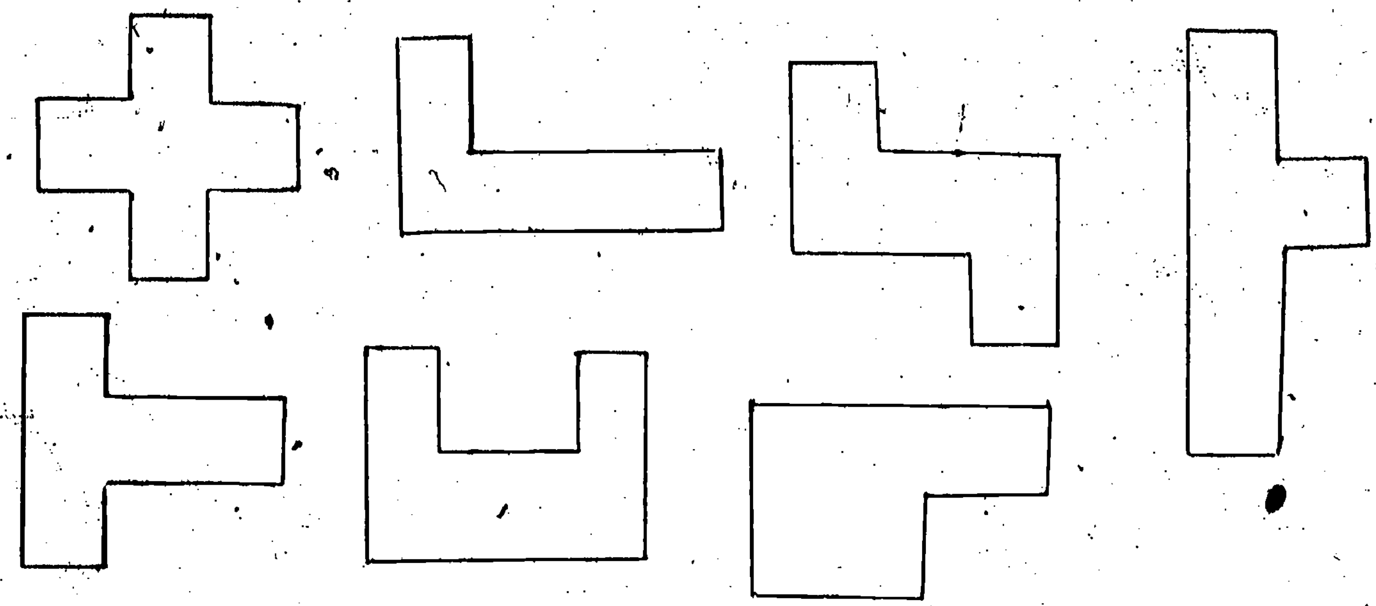
The figure below shows the case N = 3.



Show the cases for N = 2, N = 1, and N = 4.

Any figure obtained by taking five squares all the same size and fitting them together along <u>complete</u> edges is called a "pentomino".

Below are seven pentominoes.

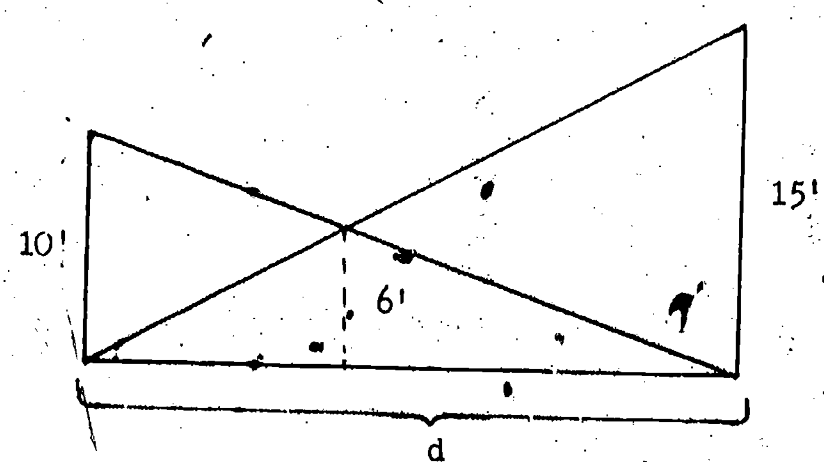


5. There are twelve different pentominoes.
   Find the other five.

6. Fit the twelve pentominoes together to form:
   a. a 5 by 12 rectangle
   b. a 4 by 15 rectangle
   c. a 6 by 10 rectangle
   d. two 5 by 6 rectangles

7. Select one pentomino. Now, fit together the remaining pentominoes together to form a larger similar pentomino to the pentomino first selected.

8. A 10 ft. pole and a 15 ft. pole are a certain distance apart. Ropes are attached from the top of each pole to the base of the other pole. The ropes intersect 6 ft. above the ground. What is the distance between the poles?



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