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ABSTRACT

This is one of a series of geometry modules developed for use by secondary students in a laboratory setting. This module was conceived as an alternative approach to the usual practice of giving Euclid's parallel postulate and then mentioning that alternate postulates would lead to an alternate geometry or geometries. Instead, the student is led through an axiomatic development into a logical dead-end which requires a new postulate in order to allow further investigation. The student is then requested to take a postulate alien to his/her experience. Most high school students will not easily accept this, and a lot of student interest is generated. Units in the module are: (1) Existence of a Parallel: (2) A Parallel Postulate: (3) Hyperbolic Geometry: (4) The Poincare Model: and (5) Euclidean Geometry. (Author/MK)

GEOMETRY MODULE FOR USE IN A MATHEMATICS LABORATORY SETTING

PARALLELS, HOW MANY?

by

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The University of Denver
Mathematics Laboratory
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Dr. Ruth I. Hoffman, Director

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Objectives

1. The student will accept that a parallel postulate is needed in the axiomatic development of geometry.
2. The student will follow some of the axiomatic development that the hyperbolic postulate leads to and model some of the results.
3. The student will follow the axiomatic development that the Euclidean postulate leads to and be able to apply the results.

Overview

This module was conceived as an alternate approach to the usual practice of giving Euclid's parallel postulate and then mentioning that alternate postulates would lead us to an alternate geometry or geometries. Instead, the student is led through an axiomatic development into a logical dead end which requires a new postulate in order to allow further investigation. He is then requested to take a postulate alien to his experience. Most high school students will not easily accept this, and a lot of student interest is generated. At this point the teacher can mention some of the results of the postulate and then return to Euclidean development in unit IV, or continue Hyperbolic Geometry in unit III before returning to Euclidean Geometry.

Units in the module are:

I. Existence of a parallel. Picks up axiomatic development and follows it to existence of a parallel.

II. A Parallel Postulate. Leads students to a parallel postulate.

III. Hyperbolic Geometry. Develops the hyperbolic postulate to obtain results contrary to students previous experience. The new system is then modeled and physically embellished until it becomes for most students a viable system.

IV. The Poincaré Model. A geometric model for modeling plane hyperbolic geometry.

V. Euclidean Geometry. Axiomatic development from Euclid's parallel postulate.

Materials

Poincare's Model
(Instructions for building on page PT-3.)

"Shrinking" meter stick
(Instructions for building on page PT-3.)

I. EXISTENCE OF A PARALLEL

Teaching Suggestions

Teacher and students should share the work here. Perhaps the teacher could explain Theorem A, then ask the students either in groups or individually to prove the corollary. Someone might be selected to present the proof to the class. The rest of the work can be done the same way in an effort to keep students actively involved in the development.

Exercise Answers (Page P-3)

4. $\ell_1 \parallel \ell_2$, $\ell_4 \parallel \ell_5$

5. a) Yes
   b) No
   c) Yes
   d) Yes
   e) Yes
   f) No
   g) Yes
   h) Yes

II. THE PARALLEL POSTULATE

Teaching Suggestions

The teacher should play a very active roll in this development; not necessarily by transmitting the content but by guiding the students' investigation, and setting the tone. The tone becomes crucial as the student runs into the logical dead end of trying to prove the existence of, at most, one parallel through a given point. This problem can initiate more student enthusiasm, arguments and discussion than you might believe possible.

Exercise Answers (Pages P-6 and P-7)

1. a) $>$
   b) $>$
   c) $<$
   d) $=$

2. $\angle CBD$ and $\angle EDF$ are corresponding
   $\angle ABD$ and $\angle BDE$ are alternate interior angles.

3. a) $\angle FAB$ and $\angle AFG$
   b) $\angle DHB$
   c) $\angle BAF$ and $\angle CAE$
   d) $\triangle GAF$
   e) $\angle AFE$ and $\angle FEA$

4. a) T
   b) T
   c) T
   d) T

5. a) Yes
   b) No

6. a) $\overleftrightarrow{PS} \parallel \overleftrightarrow{QR}$
   b) None
   c) $\overleftrightarrow{PS} \parallel \overleftrightarrow{QR}$
   d) $\overleftrightarrow{PQ} \parallel \overleftrightarrow{RS}$
   e) $\overleftrightarrow{PQ} \parallel \overleftrightarrow{RS}$
   f) None
   g) $\overleftrightarrow{PQ}$ $\overleftrightarrow{RS}$

III. HYPERBOLIC GEOMETRY

Teaching Suggestions

The Definitions and Theorems can best be presented as overhead transparencies, allowing faster and easier presentation. You may wish to present the proofs as shown, or merely discuss with the students what can be proven. Notes for the teacher are included for each set of transparencies.

Materials

Poincaré Board

30 cm radius circular region cut of cardboard or wood.

Hyperbolic Lines

Arcs cut of plastic sheets, 2 mm - 3 mm thickness. Cut 10 arcs of 1/2 cm width to these specifications for the inside curve.

Radius	Chord Length
10 cm	19 cm
15 cm	26.8 cm
20 cm	33.3 cm
30 cm	42.4 cm
40 cm	48 cm
50 cm	51 cm
60 cm	53.7 cm
70 cm	55.1 cm
80 cm	56.2 cm
90 cm	56.9 cm

Following is the basic program if you wish to generate the measurements for more hyperbolic lines.

```
LET R = .30
PRINT "R 1 = "; R
PRINT "L    R 2"
FOR J = 0 to .9  Step .01
LET L = (2 * R * J) / (SQR(R ↑ 2 + J ↑ 2)
PRINT L , J
NEXT J
END
```

"Shrinking" meter sticks

Cut four cardboard strips 2 cm wide and 1 meter, 1/2 meter, 1/4 meter and 1/8 meter in length respectively. Label each as shown.

0 m	METER	$\frac{1}{2}$ m	STICK	1 m

IV. THE POINCARÉ MODEL

This section is very dependent on the teacher for guidance, motivation and timing. The student materials exist to give you and the student somewhere to start. However, this section may be done just as effectively without distributing student information sheets and instead, just guiding the student through the ideas of the section by yourself. To this end, we put forth here a statement of purpose for each individual activity, any pertinent notes, and urge you to use your own procedures to introduce and guide the activity.

A description of a Poincaré Model Board is given in the beginning of the Teacher's Guide in the materials section. The purpose of including a physical model here is four-fold.

1) It allows the student to do the modeling making the whole idea seem somewhat less esoteric.
2) The student can show himself that all the theorems and postulates preceeding Postulate H are consistant with this model.
3) It allows the student to view two parallels to a line, open triangles, Saccheri quadrilaterals, etc. in their proper perspective. That is in a hyperbolic plane, not a Euclidean plane.
4) To give the student a mental picture of a hyperbolic plane to which the physical world can be related.

It is strongly suggested that you try modeling the given statement before going into the classroom with the modeling board. This will give you an idea of where problems wi[illegible] encountered.

The purpose of Statement 1 is obviously to determine if a pair of points determines exactly one line. Statement 2 is included to check the plane separation postulate. Statement 3 points one toward the problems of measuring angles. The student may be able to rely on his intuition for this. It might help that intuition if the measure of an angle formed by tangents to the two curves is demonstrated. This will also assist them in Statement #'s 6, 7, and 8. The purpose for modeling Statement #'s 4, 5, 7, and 8 is apparent while #6 is included to again indicate the difference between parallels and ultra parallels.

<u>Class Activity 1</u>: Do you really know which way is straight?

The purpose of this activity is to indicate that determination of straightness is visual. Therefore, only if light travels in straight lines can one be confident that what he perceives to be straight is actually straight.

The Activity: Request volunteers from the class who feel they definitely know how to walk a straight line. Have them do so by selecting an object across the room, field, gym, etc. and walking a straight line to that object. Now have them each return and try the same thing again, blindfolded. Observe the difference in paths between trials. This should convince them that in the physical world their determination of what straight is depends on light. Note that the bigger the room, the more convincing will be the results.

Class Activity 2: First one out the door is extremely small.

The purpose of this activity is to help the students understand how a plane that is bounded can still seem infinite.

The Activity: You should have a set of the meter sticks described in the materials section of the Teacher's Guide hidden somewhere in the room. You must now explain to the students the predicament in which they now find themselves. Their predicament is:

Since the class has entered the classroom, you have inadvertently stepped on the hidden button that changes the room into a hyperbolic room. This makes it very difficult to leave the room. In fact, so difficult that you are willing to give a reward to anyone who can follow the rules imposed by the hyperbolic plane and still leave the room. Those rules are:

1) One can never change size. If one is a meter 70 tall, he must remain a meter 70 tall.
2) Between each step taken one must be measured to be sure he has not grown.

Start the student in the middle of the room. Take an initial height measurement with the full sized meter stick. Allow him his first step. Take the next height measurement with the second meter stick (one-half meter in length). Observe that the student has grown. Therefore, we must shrink the student. Shrink student. Allow the student to take another step in his shrunken state. Measure his height with the third meter stick. He has again grown and must, therefore, shrink some more.

The students might complain that the meter stick is shrinking rather than the student growing. Kindly explain that if the room were Euclidean instead of hyperbolic, that would be true. However, in a hyperbolic room the size of the meter stick varies with location. It should be apparent to the student and the rest of the class that the student will never make it out of the room. He may, therefore, consider the room to be infinite.

The last part of this section is included for the purpose of convincing the student that the universe might be hyperbolic. If you have an alternate way of doing this, use it. Or read the given information and present it to the class verbally. The important thing is that the students realize there exists a viable alternative to Euclidean Geometry.

V. Euclidean Geometry

Teaching Suggestions

This section begins with Euclid's Parallel Postulate and proceeds to the more important results. The proofs are left for students to do or the teacher can do them for the class.

The teacher ought to make a statement to the class explaining why we choose to continue with Euclidean geometry. (1) It is simpler - we have gone about as far as we can go in hyperbolic geometry with the mathematical background of the class. (2) Euclidean geometry is a better description of our experience.

EXISTENCE OF A PARALLEL

We pick up the axiomatic story line of geometry assuming that you are familiar with the line and distance postulates, the angle postulates, and Theorem Z, which gives the existence of one and only one line perpendicular to a given line at a given point on the given line. The intention now is to take this development into an investigation of parallel lines.

The central question in any development of parallel lines is:

> Given a line ℓ and a point P not on line ℓ, how many lines exist which contain P and are parallel to ℓ?

$\cdot$ P

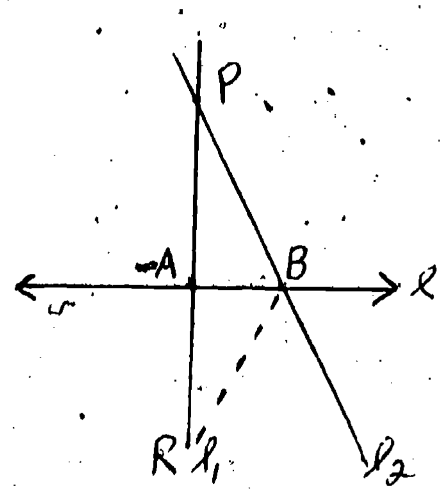
ℓ ————————————————

Immediately there appear three answers to be considered. There are none, there is exactly one, or there are at least two would cover all possibilities. Surely most of you already know the correct answer. But we are ahead of ourselves. We wish to put all of this on a firm axiomatic basis.

Let us proceed and share the work. The writers will provide some of the proofs and you will be requested to provide others. Our first objective will be to eliminate the first alternative - there are no lines through P parallel to ℓ. That is, we shall try to prove that there exists at least one line through P parallel to ℓ.

Theorem A: Given a line ℓ and a point P not on the line, there is at most one line that contains P and is perpendicular to ℓ.

Proof: Assume there are at least two lines, ℓ_1 and ℓ_2, perpendicular to ℓ at points A and B respectively. There exists a point R on the ray opposite $\overrightarrow{AP}$ such that $AP = AR$. Since $\ell_1 \perp \ell$, $\angle PAB \cong \angle RAB$. $\overline{AB} \cong \overline{AB}$ now implies that $\triangle PAB \cong \triangle RAB$ by SAS. Hence, $\angle PBA \cong \angle RBA$. Since $\ell_2 \perp \ell$, both $\angle PBA$ and $\angle RBA$ must be right angles. Thus, both ℓ_2 and $\overleftrightarrow{RB}$ are perpendicular to ℓ at B. But this contradicts Theorem Z. Hence, there must be at most one line through P perpendicular to ℓ.

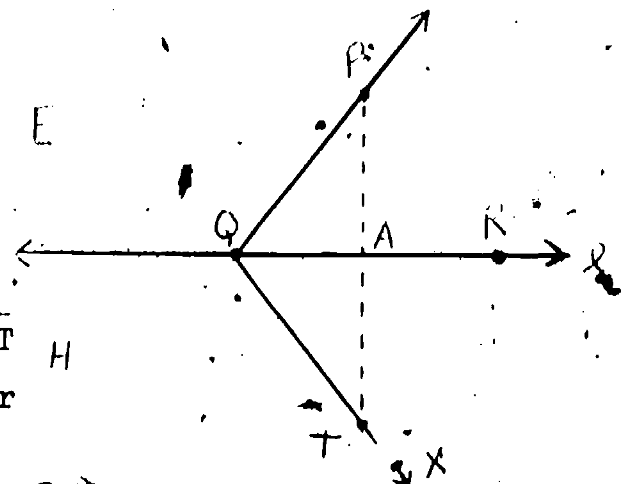


Corollary: No triangle has two right angles.

Proof: Proof is left to you.

Theorem B: Given a line ℓ and a point P not on ℓ, there is at least one line that contains the given point and is perpendicular to ℓ.

Proof: Line ℓ contains two points Q and R. E and H are the half planes defined by line ℓ. E contains point P. Construct ray $\overrightarrow{QX}$ in H such that $\angle RQX \cong \angle RQP$. There exists a point T on $\overrightarrow{QX}$ such that QP = QT. Since P and T are in opposite half planes, $\overline{PT}$ intersects ℓ at some point A. Either



Case 1 A = Q

Case 2 A is in the interior of $\overrightarrow{QR}$

Case 3 Q is between R and A.

Case 1: If A = Q, then A, Q, and T are collinear. Hence $m\angle PQR = m\angle XQR = 90$ and $\overleftrightarrow{PT}$ is the desired perpendicular.

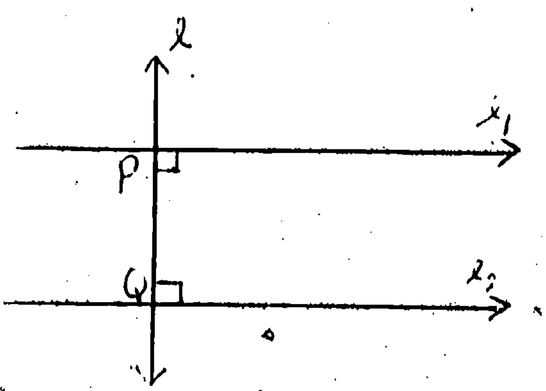
Case 2: Since $\overline{QA} = \overline{QA}$, $\triangle PAQ \cong \triangle TAQ$. Thus, $m\angle PAQ = m\angle TAQ = 90$ and PT is the desired perpendicular.

Case 3: Proof is left to you.

Definition: Two lines are parallel iff they are coplanar and do not intersect.

Theorem C: Given three coplanar lines. If two of those lines are both perpendicular to the third, those two are parallel.

Proof: Let ℓ_1 and ℓ_2 both be perpendicular to ℓ at points P and Q respectively. Theorem Z guarantees that that $P \neq Q$.



Assume ℓ_1 is not parallel to ℓ_2. Then ℓ_1 intersects ℓ_2 at some point H. But $\triangle PHQ$ contains two right angles which contradicts the corollary to Theorem A.

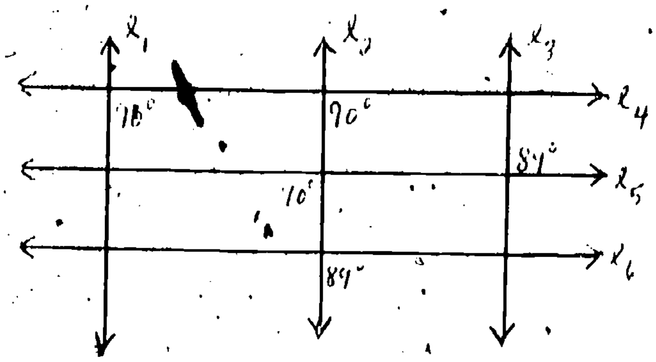
$\therefore \ell_1 \parallel \ell_2$.

Theorem D: Given a line ℓ and a point P not on ℓ, then there exists at least one line through P parallel to ℓ.

Proof: Proof is left to you.

Exercises:

1. Prove the corollary to Theorem A.
2. Prove Case 3 of Theorem B
3. Prove Theorem D.
4. Name two pairs of parallel lines.



5. Consider the following definitions:

A vertical line is one containing the center of the earth.

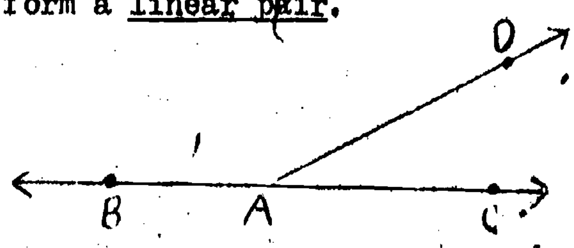
A horizontal line is one that is perpendicular to some vertical line.

a) Could two horizontal lines be parallel?
b) Could two vertical lines be parallel?
c) Could two horizontal lines be perpendicular?
d) Could two vertical lines be perpendicular?
e) Would every vertical line be horizontal?
f) Would every horizontal line be vertical?
g) Could a horizontal line be parallel to a vertical line?
h) Would every line be horizontal?

THE PARALLEL POSTULATE

Having so successfully disposed of the no parallel option, we throw caution to the wind and proceed at break-neck speed to select the correct answer from the remaining two. First we attack a question about the exterior angle of a triangle, then define angles useful in work with parallel lines and then we are ready.

Definition: If A is between B and C, and D is not on $\overleftrightarrow{BC}$, then $\angle CAD$ and $\angle BAD$ form a linear pair.



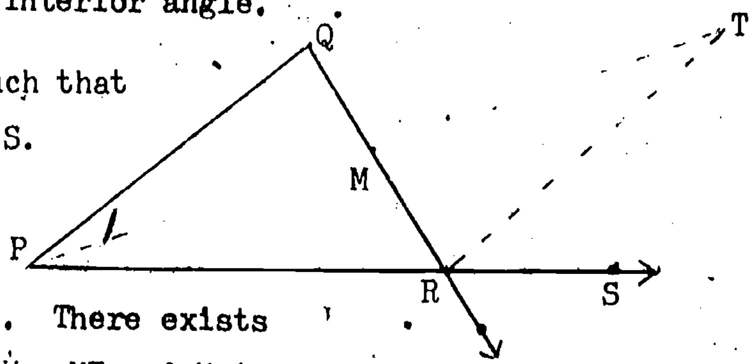
Definition: An exterior angle of a triangle is an angle that forms a linear pair with one of the angles of the triangles.

Definition: For any exterior angle of a triangle, the remote interior angles are those angles of the triangle which do not form a linear pair with the exterior angle.

Theorem E: The measure of an exterior angle of a triangle is greater than the measure of each remote interior angle.

Restatement: Given: $\triangle PQR$ and point S such that R is between P and S.

Prove: $m\angle QRS > m\angle Q$ and $m\angle QRS > m\angle P$



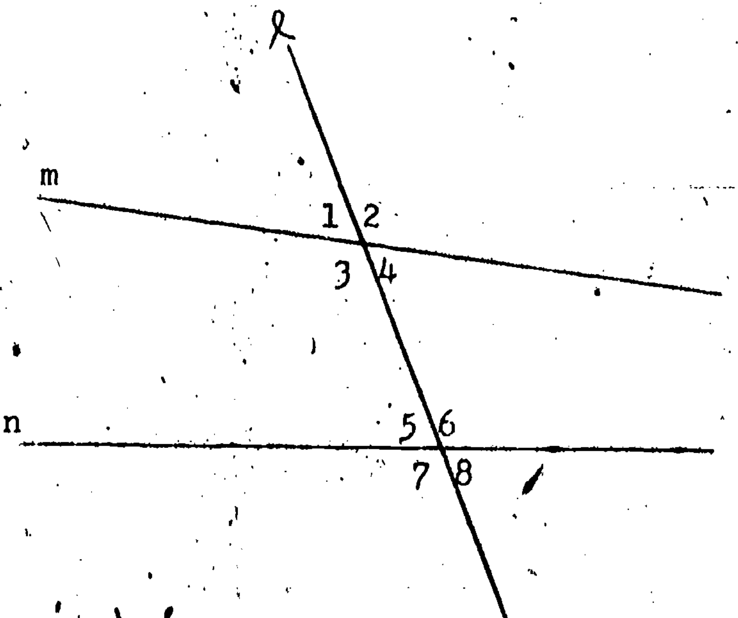
Proof: Let M be the midpoint of $\overline{QR}$. There exists a point T on $\overrightarrow{PM}$ such that $PM = MT$ and M is between P and T. Since $\angle QMP \cong \angle RMT$, $\triangle QMP \cong \triangle RMT$ by SAS. Therefore, $m\angle Q = m\angle QRT$. Since $\overrightarrow{RT}$ is between $\overrightarrow{RQ}$ and $\overrightarrow{RS}$, $m\angle QRS = m\angle QRT + m\angle TRS$. Hence, $m\angle QRS > m\angle QRT = m\angle Q$. The proof that $m\angle QRS > m\angle P$ is left to you.

Corollary: If a triangle contains one right angle, the other two angles are acute angles.

Proof: The proof is left to you.

We wish now to define pairs of angles which are useful in working with parallel or near parallel pair of lines. To avoid undue wordiness, we allude to definition by diagram.

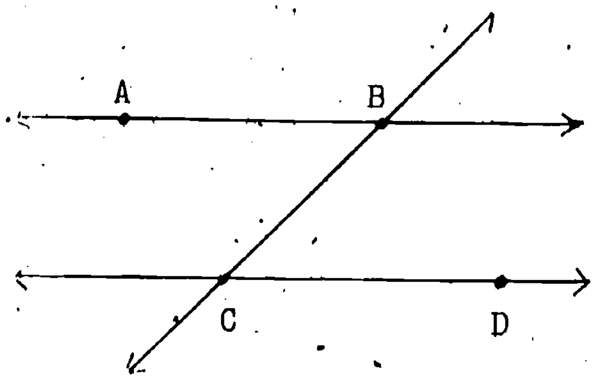
Definition: Line ℓ is called a <u>transversal</u> between m and n. (Note: m and n may or may not be parallel.)



Definitions: The pairs of angles 3 and 6, and 4 and 5 are called <u>alternate interior angles</u>.

Definition: The pairs of angles 1 and 5, 2 and 6, 3 and 7, and 4 and 8 are called <u>corresponding angles</u>.

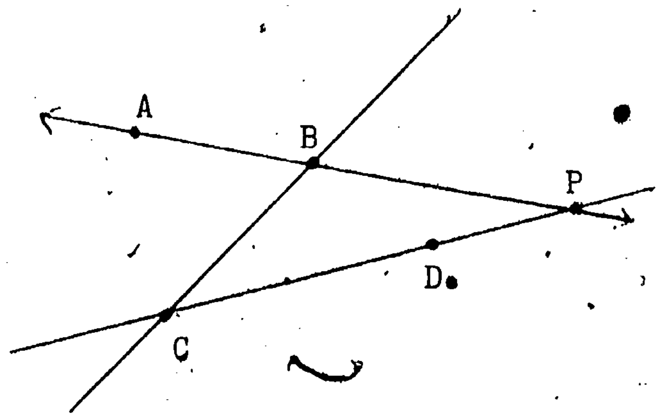
Theorem F: If two coplanar lines are cut by a transversal such that a pair of alternate interior angles are congruent, then the two lines are parallel.



Restatement: Given: $\angle ABC \cong \angle BCD$

Proof: $\overleftrightarrow{AB} \parallel \overleftrightarrow{CD}$

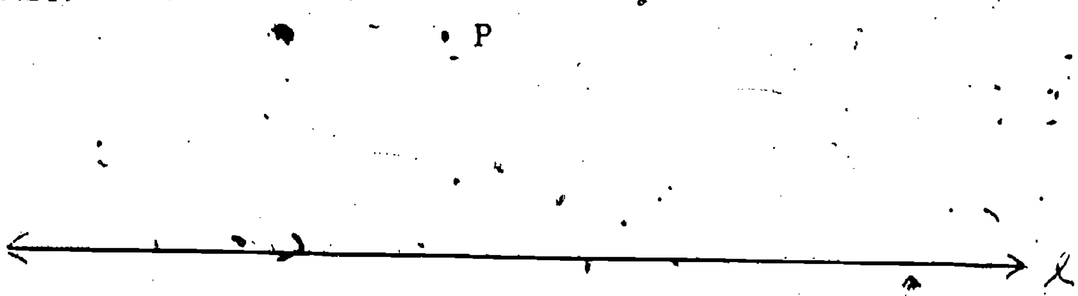
Proof: Assume $\overleftrightarrow{AB}$ is not parallel with $\overleftrightarrow{CD}$. Then $\overleftrightarrow{AB}$ and $\overleftrightarrow{CD}$ intersect at some point P. But the hypothesis that $m\angle ABC = m\angle BCD$ contradicts Theorem E. Therefore, $\overleftrightarrow{AB} \parallel \overleftrightarrow{CD}$.



Theorem G: If two coplanar lines are cut by a transversal such that a pair of corresponding angles are congruent, then the two lines are parallel.

Proof: The proof is left to you.

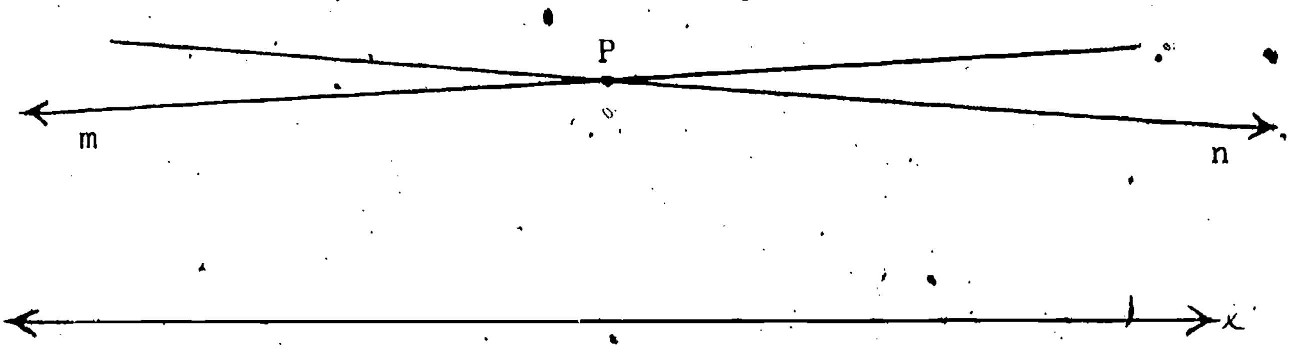
WE ARE READY!



We have already determined that there must be at least one line parallel to ℓ through P. In all honesty we must warn you that we have consulted with many mathematicians and read many books and articles about this. Everyone, to the last professor consulted, has advised us to make an assumption about which answer is correct and not try to prove that one is correct and the other is not. We mean to take heed of all this advice. So here comes our assumption in the form of Postulate H.

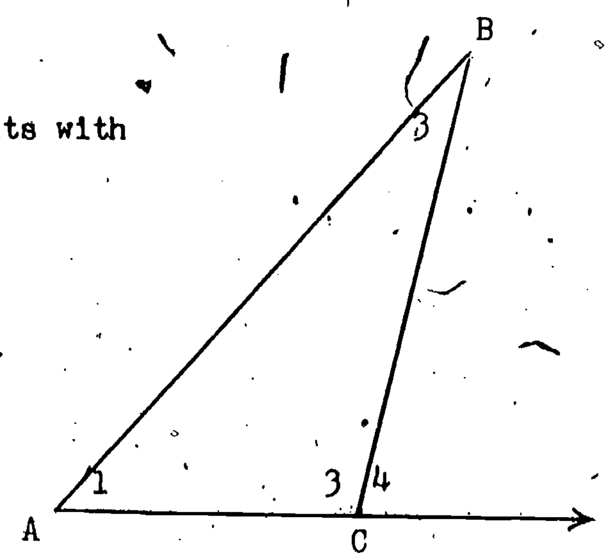
Postulate H: Given line ℓ and point P not on ℓ, then there exists at least two lines through P which are coplanar with ℓ and do not intersect ℓ.

There, we were sure most of you knew that was the correct answer long ago. We, therefore, illustrate this with a simple diagram and proceed with the development. In the diagram neither line m or line n intersect with ℓ.

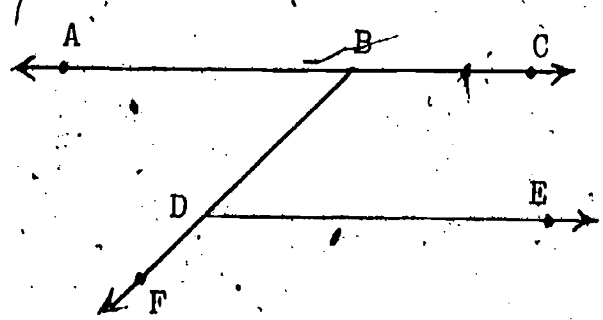


There may be a few of you who are not quite convinced that Postulate H is the correct choice. We invite you to try to prove that Postulate H is wrong or that the other choice is correct. We are sure that you will soon be convinced that we have made the proper choice.

Exercises

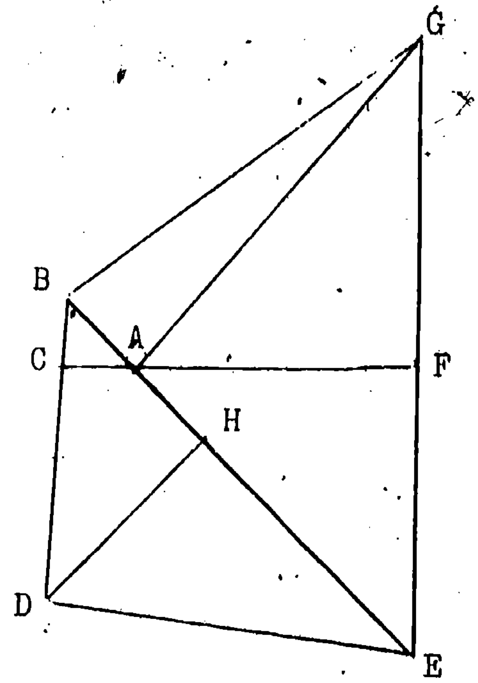


1. Complete each of the following statements with the symbol >, < or =.
   a) If $m\angle 1 = 40$ and $m\angle 2 = 30$, then $m\angle 4$ ___ 40.
   b) If $m\angle 1 = 72$ and $m\angle 2 = 73$, then $m\angle 4$ ___ 73.
   c) If $m\angle 4 = 112$, then $m\angle 1$ ___ 112.
   d) If $m\angle 4 = 150$, then $m\angle 3$ ___ 30.



2. From the four angles illustrated, find the following:
   a) A pair of corresponding angles.
   b) A pair of alternate interior angles.

3. Using only the points and segments shown in the diagram, complete the following:



a) Two exterior angles of $\triangle FAE$ are ___ and ___.

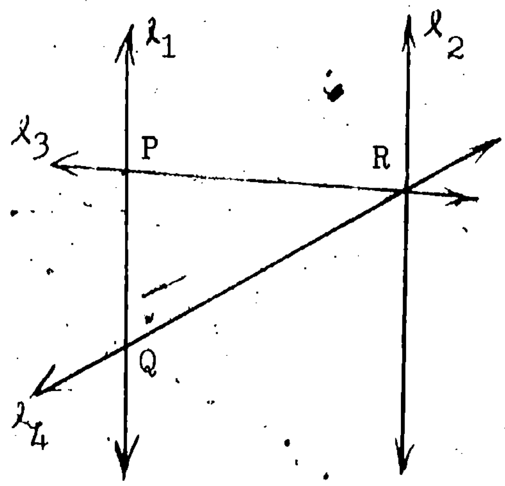
b) An exterior angle of $\triangle DHE$ is ___.

c) Two exterior angles of $\triangle BCA$ are ___ and ___.

d) $\angle CAG$ is an exterior angle of ___.

e) $\angle BAF$ is an exterior angle of $\triangle AEF$ with remote interior angles ___ and ___.

4. True or false



a) ℓ_1 is a transversal of ℓ_3 and ℓ_4.

b) ℓ_2 is a transversal of ℓ_3 and ℓ_4.

c) ℓ_4 is a transversal of ℓ_1 and ℓ_2.

d) ℓ_4 is a transversal of ℓ_1 and ℓ_3.

5. a) If $\angle Q \cong \angle S$, does it follow that $\overline{QP} \parallel \overline{TS}$?

b) If $\angle P \cong \angle S$, does it follow that $\overline{QP} \parallel \overline{TS}$?

6. Name the segments, if any, that are parallel if

a) $\angle 5 \cong \angle 9$

b) $\angle 2 \cong \angle 10$

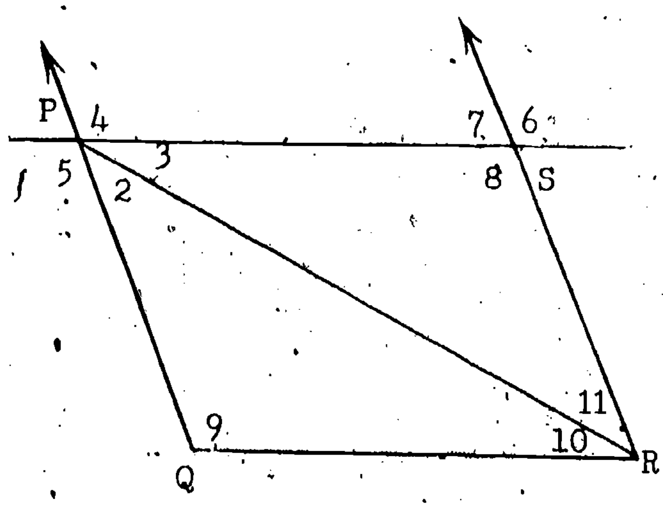
c) $\angle 3 \cong \angle 10$

d) $\angle 4 \cong \angle 6$

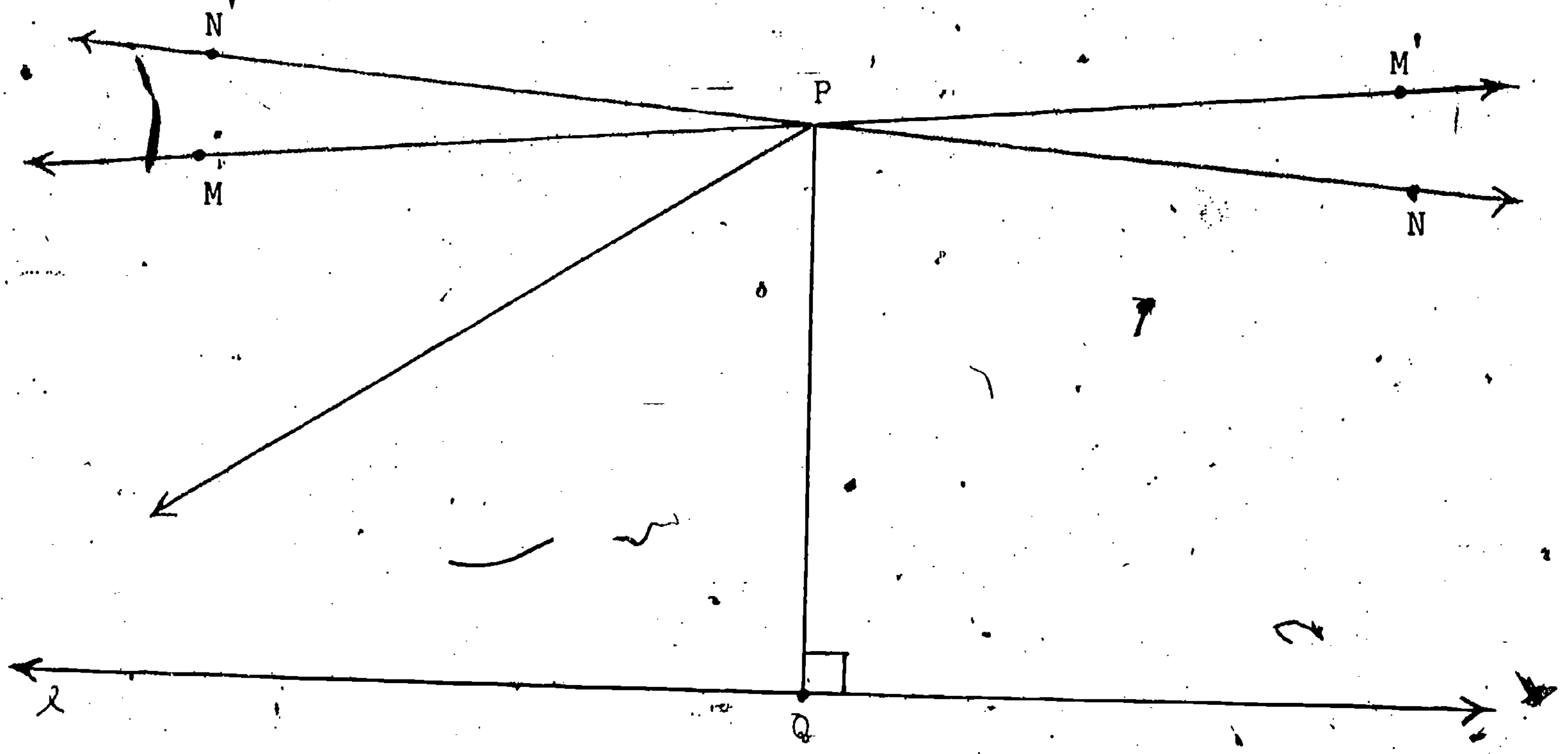
e) $\angle 2 \cong \angle 11$

f) $\angle 5 \cong \angle 7$

g) $\angle 5 \cong \angle 8$



Definitions for Hyperbolic Geometry



Definition: $\overleftrightarrow{MP}$ is a <u>parallel</u> to ℓ if and only if

a) $\overleftrightarrow{MP}$ and ℓ are coplanar,

b) $\overleftrightarrow{MP}$ and ℓ do not intersect,

c) All rays $\overrightarrow{PT}$ where T is interior to $\angle MPQ$ intersect with ℓ.

Definition: Any line ℓ' which satisfies conditions a) and b) but not c) is called an <u>ultra</u> <u>parallel</u>.

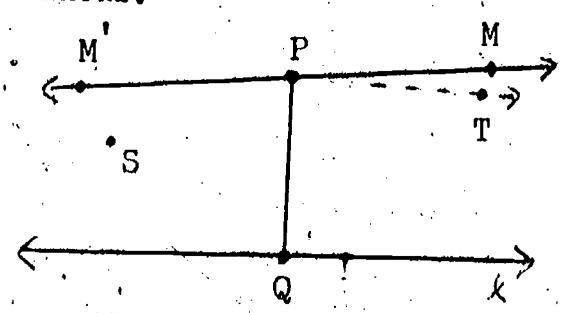
<u>Notes on Definition Sheet #1</u>

Postulate H confirms the existence of an infinite number of lines through P which fulfill conditions a) and b). There are, however, only two which meet condition c). Since these two lines are special, the word parallel is redefined to facilitate easy reference to this pair of lines.

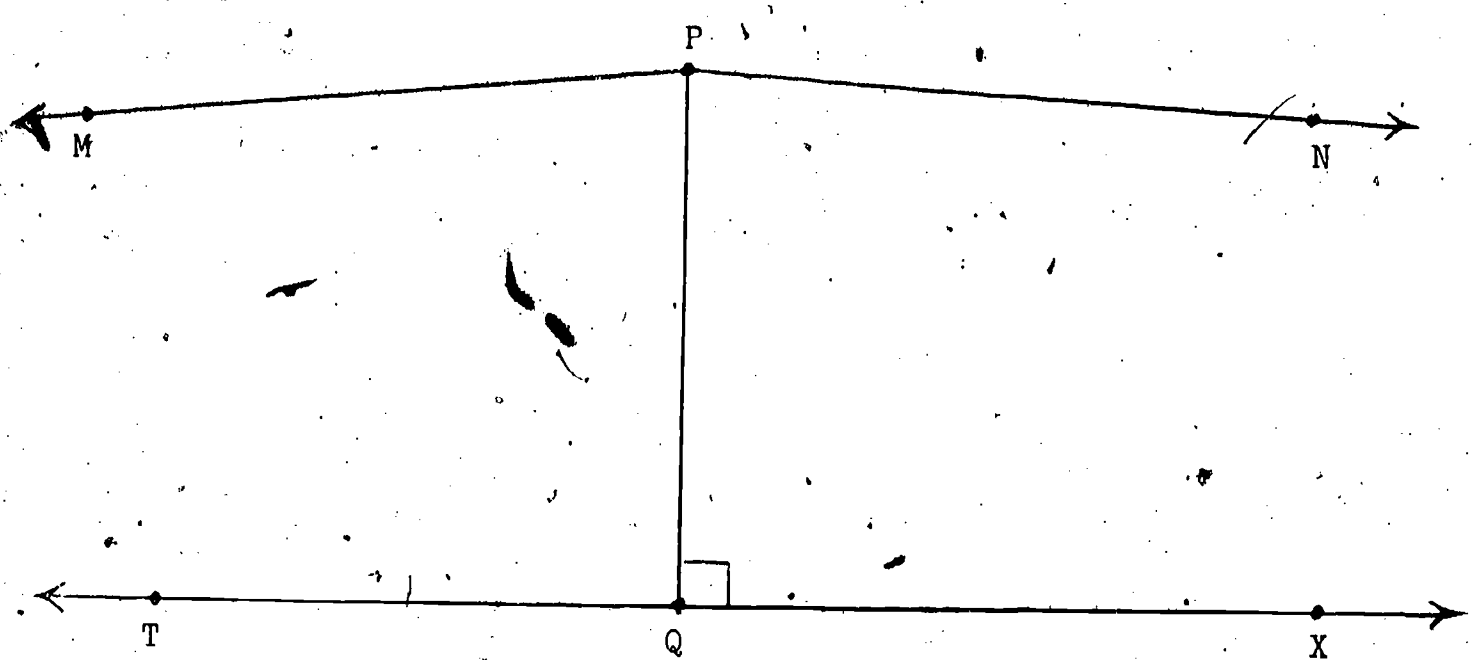
Usually parallel rays $\overrightarrow{PM}$ and $\overrightarrow{PN}$ will be referred to rather than parallels $\overleftrightarrow{PM}$ and $\overleftrightarrow{PN}$.

The difference between Postulate H and Euclid's postulate can be stated in terms of parallel rays $\overleftrightarrow{PM}$ and $\overleftrightarrow{PN}$. Euclid's postulate claims they are collinear and Postulate H claims they are non-collinear.

If one parallel is given in a diagram, it could be that $\overrightarrow{PT}$ does not intersect with ℓ. If $\overleftrightarrow{MP}$ is a parallel, then it must either be true that all rays $\overrightarrow{PS}$ intersect ℓ or all rays $\overrightarrow{PT}$ intersect ℓ.



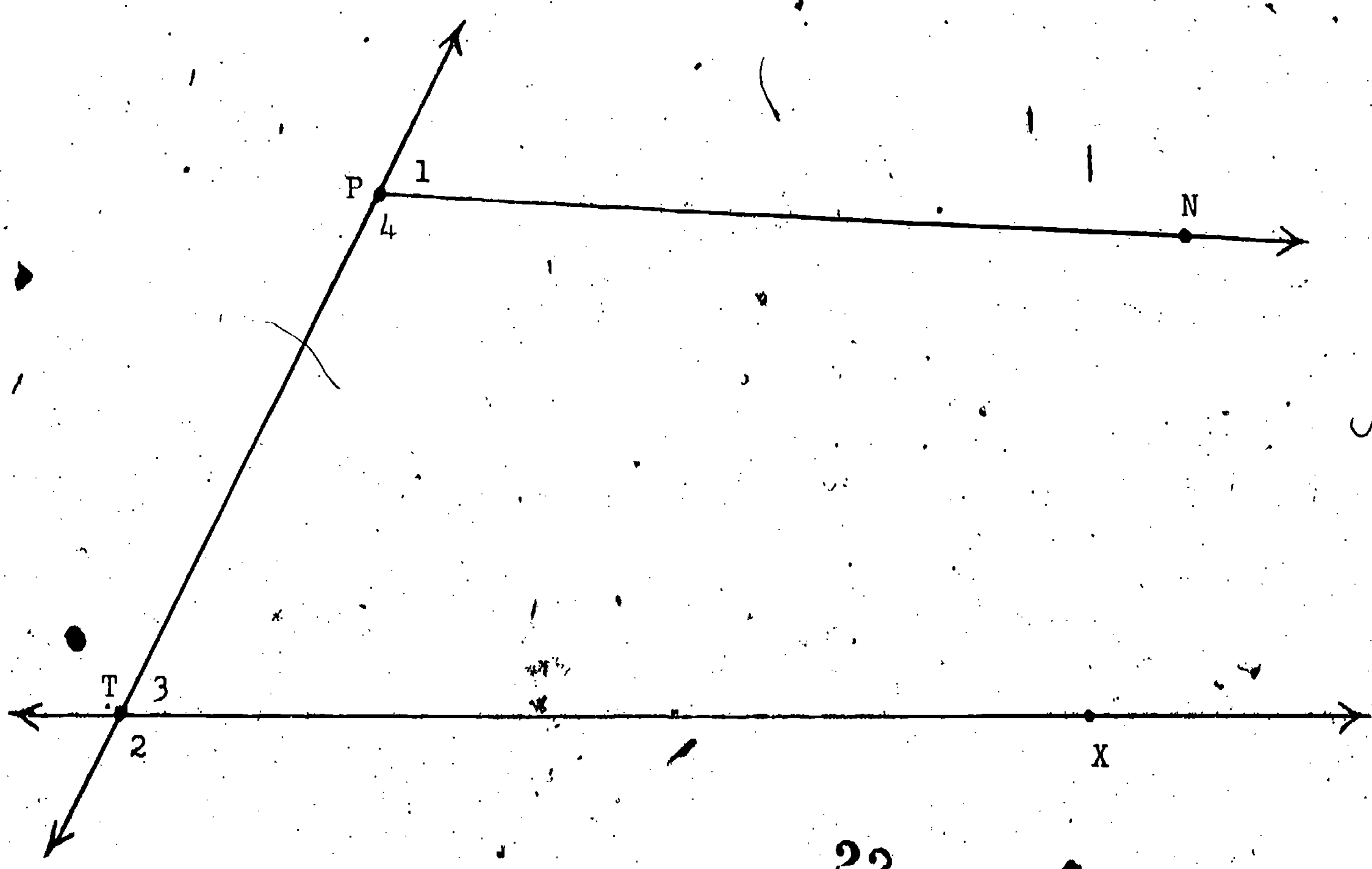
Definitions for Hyperbolic Geometry #2



Definition: Given line $\overleftrightarrow{TX}$, point P not on $\overleftrightarrow{TX}$, segment $\overline{PQ}$ perpendicular to $\overleftrightarrow{TX}$ at Q and rays $\overrightarrow{PM}$ and $\overrightarrow{PN}$ parallel to line $\overleftrightarrow{TX}$. Angles MPQ and NPQ are called the <u>angles of parallelism.</u>

Definition: The direction of ray $\overrightarrow{PM}$ is called the <u>direction of parallelism</u> of $\overleftrightarrow{PM}$.

Definitions for Hyperbolic Geometry #3



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Definition: Let $\overrightarrow{PN}$ be a parallel to $\overleftrightarrow{TX}$ through P. If $\overline{PT}$ is a transversal intersecting $\overleftrightarrow{TX}$ at T, then the union of $\overrightarrow{PN}$, $\overrightarrow{TX}$ and $\overline{PT}$ is called an <u>open triangle</u> and is denoted as $\triangle PT\Omega$.

Definition: An <u>exterior angle of an open triangle</u> is any angle which forms a linear pair with an angle of the open triangle.

Definition: An angle is a <u>remote inferior angle</u> of a given exterior angle if it is an angle of the open triangle which does not form a linear pair, the given exterior angle.

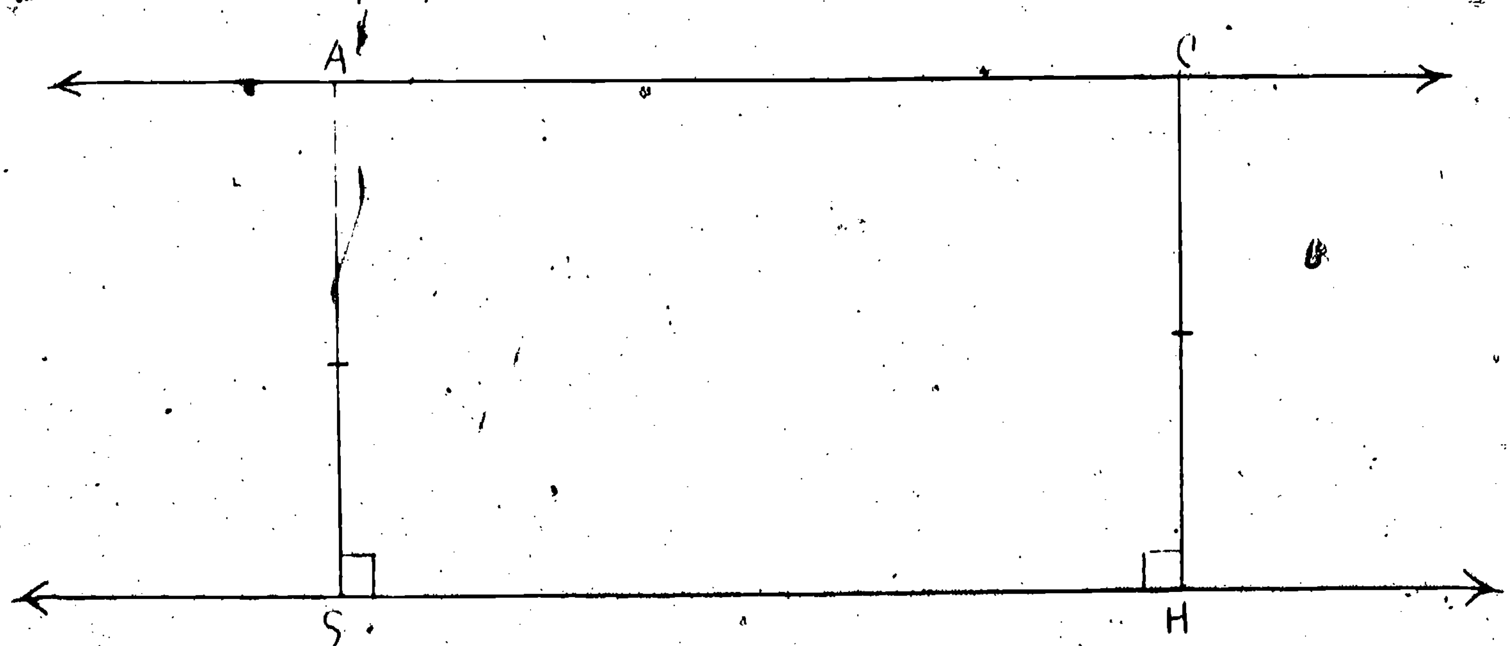
<u>Notes on Definition Sheet #3</u>

The letter omega (Ω) is used to indicate an ideal point at infinity. The symbol is introduced for notational purposes. It is not meant to be used to motivate a discussion of ideal points, what is at infinity, or any other similar esoteric subject matter. It should be noted, however, that it represents the fact that two of the sides of the triangle extend indefinitely.

Angles 1 and 2 are exterior angles of $\triangle PT\Omega$. There exist two other exterior angles to $\triangle PT\Omega$ which are not labeled.

Angle 3 is a remote interior angle of $\angle 1$. Angles 2 and 4 share the same relationship.

Definitions for Hyperbolic Geometry #4



Definition: A quadrilateral in which a pair of opposite sides are congruent and perpendicular to a third side is called a <u>saccheri quadrilateral</u>.

Definition: In Saccheri quadrilateral SACH, $\overline{AC}$ is called the <u>summit</u>, $\overline{SH}$ is called the <u>base</u>, $\angle ASH$ and $\angle SHC$ are called <u>base angles</u>, and $\angle SAC$ and $\angle ACH$ are called <u>summit angles</u>.

<u>Notes on Definition Sheet #4</u>

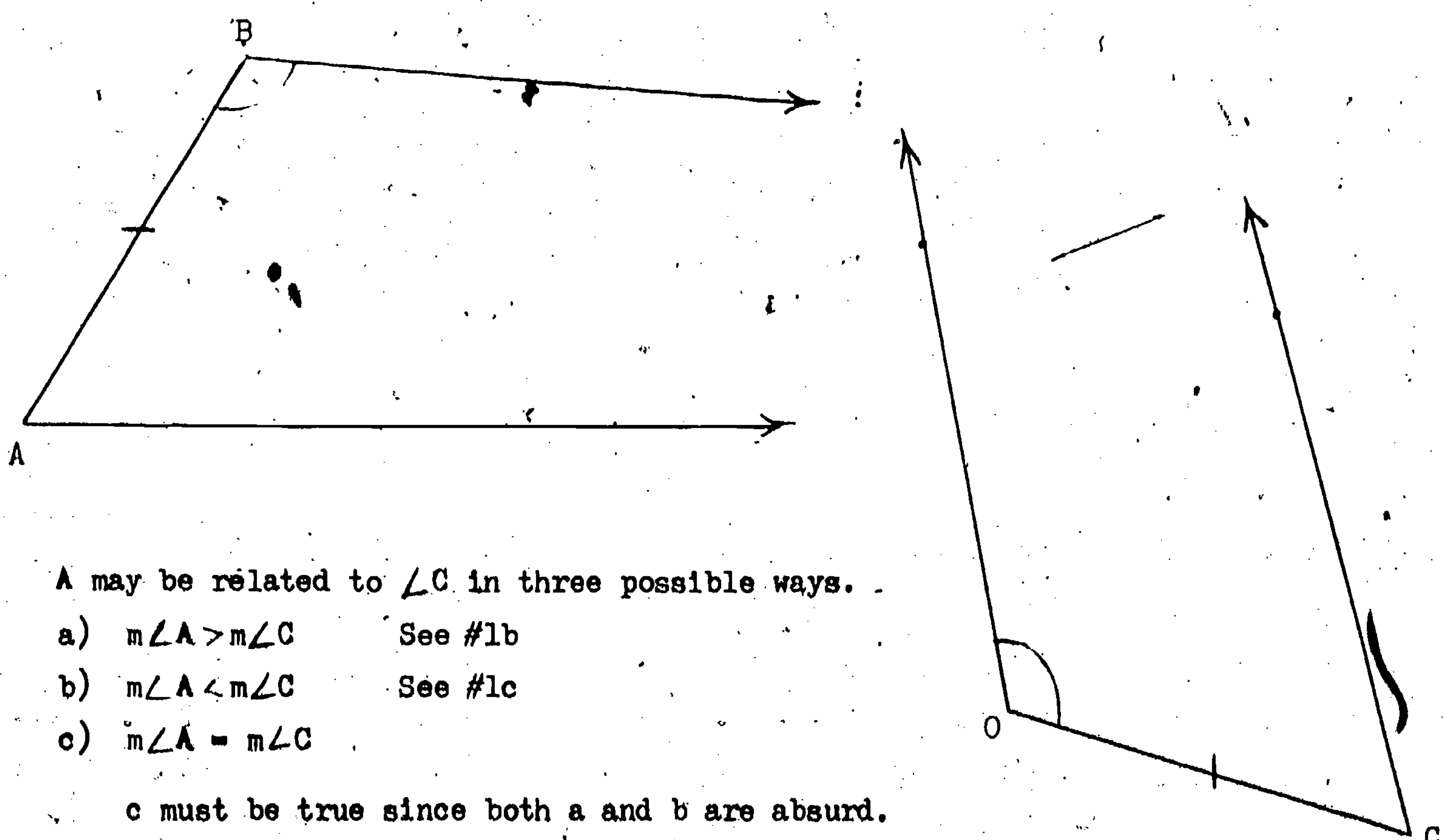
This is a good place to include some historical notes about Saccheri the mathematician and his life-long quest for a proof of Euclid's parallel postulate. We refer you here to any of the following sources.

1. Howard Eves
2. Moise

A discussion of some of the history would lead to an observation that the summit angles cannot be proven to be right angles.

Theorems of Hyperbolic Geometry #1a

Theorem I: If two open triangles $AB\Omega$ and $CO\Omega$ have $\angle B \cong \angle O$ and $AB = CO$, then $\angle A \cong \angle C$.



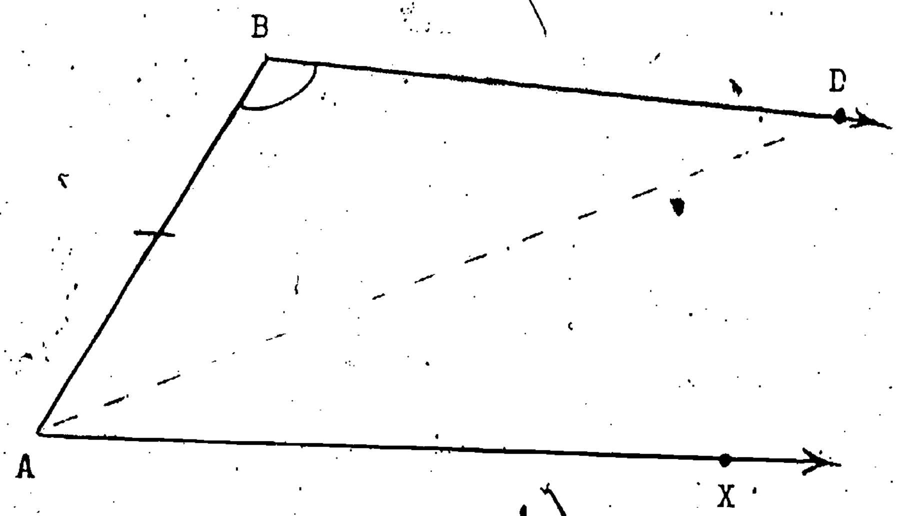
A may be related to $\angle C$ in three possible ways.

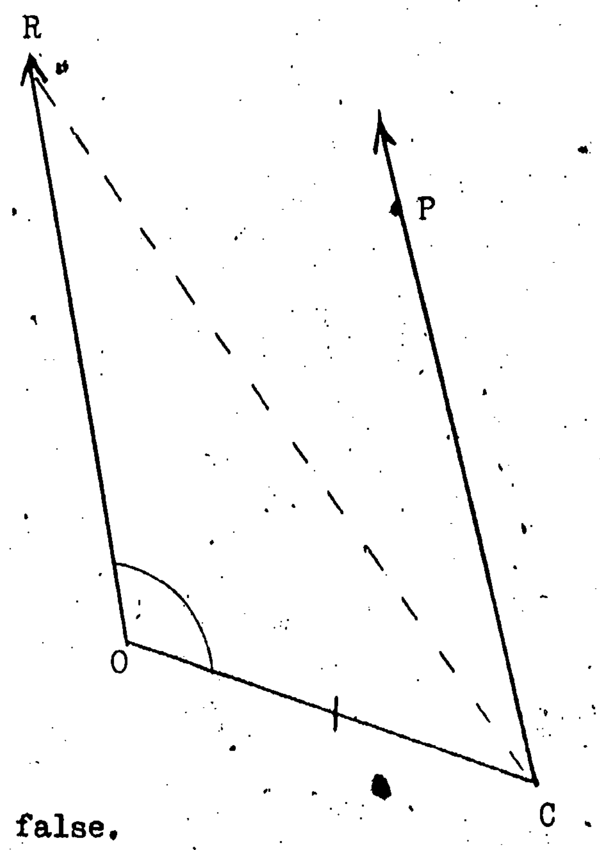
a) $m\angle A > m\angle C$ See #1b
b) $m\angle A < m\angle C$ See #1c
c) $m\angle A = m\angle C$

c must be true since both a and b are absurd.

Theorems of Hyperbolic Geometry #1b

If $m\angle A(\angle BAX) > m\ C(\angle PCO)$, then





$\overrightarrow{AD}$ exists such that $\angle DAB \cong \angle PCO$

R exists such that $BD = OR$

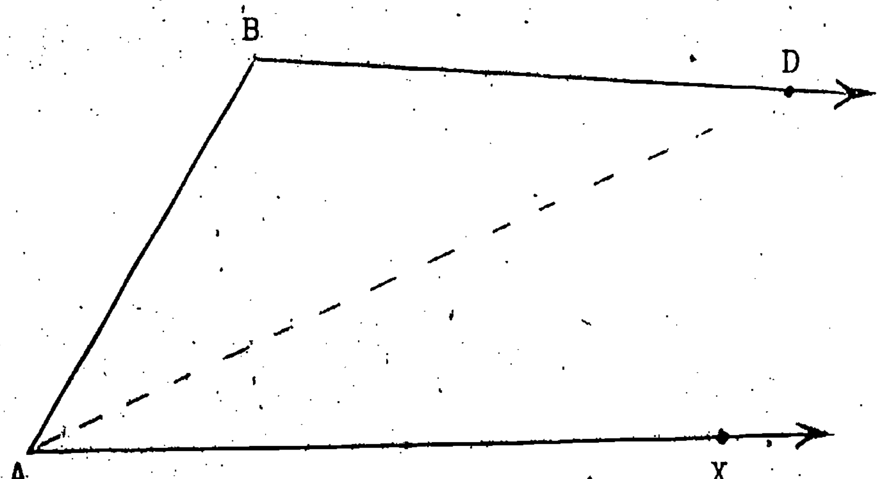
$\triangle ROC \cong \triangle DBA$ by SAS

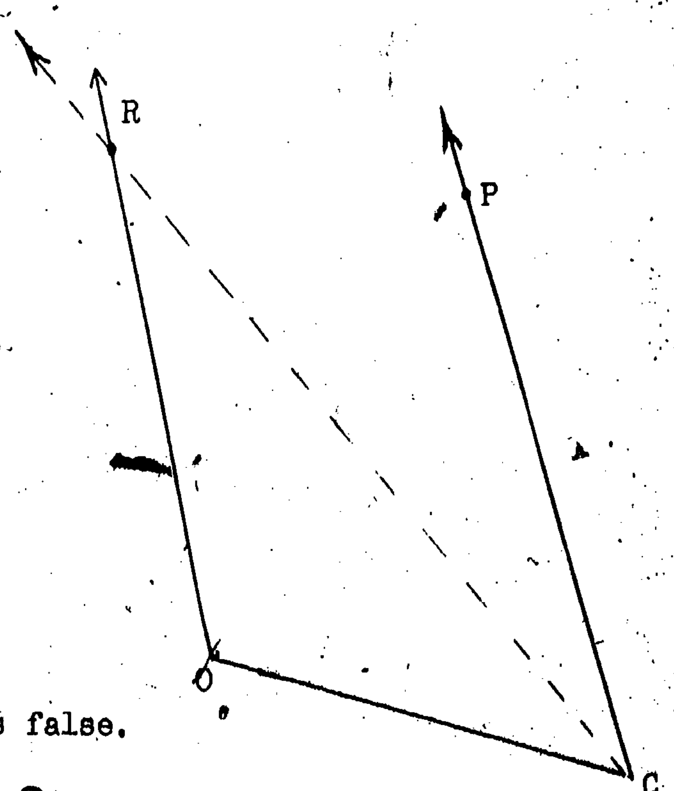
$\angle RCO \cong \angle DAB \cong \angle PCO$

But $\angle RCO \cong \angle PCO$ is absurd! So, $m\angle A > m\angle C$ is false.

Theorems of Hyperbolic Geometry #1c

If $m\angle A\ (\angle BAX) < m\angle C\ (\angle PCO)$, then





CR exists such that $\angle RCO \cong \angle BAX$

D exists such that $BD = RO$

$\triangle ROC \cong \triangle DBA$ by SAS $\therefore \angle BAD \cong \angle RCO \cong \angle BAX$

But $\angle RCO \cong \angle PCO$ is absurd! So, $m\angle A < m\angle C$ is false.

Notes on Theorem I

If the students are willing to accept Theorem I without proof, it is not necessary to expose them to transparencies 1b and 1c.

If the students need to be convinced, the transparencies do not give a complete proof. Included here is a more complete proof for transparency 1b, which may help in answering any questions which may arise.

If $m\angle BAX > m\angle PCO$, then there is a ray $\overrightarrow{AT}$ in the interior of $\angle BAX$ such that $m\angle BAT = m\angle PCO$. Since $AB\Omega$ is an open triangle $\overrightarrow{B\Omega}$ is a parallel to $A\Omega$ (not ultra-parallel). Therefore, $\overrightarrow{AT}$ intersects with $\overrightarrow{B\Omega}$ at some point D. By the point plotting theorem there exists a point R on $\overrightarrow{O\Omega}$ such that $OR \cong BD$. Now $\triangle ROC \cong \triangle DBA$ by SAS which leads directly to the absurd statement that $m\angle RCO = m\angle PCO$.

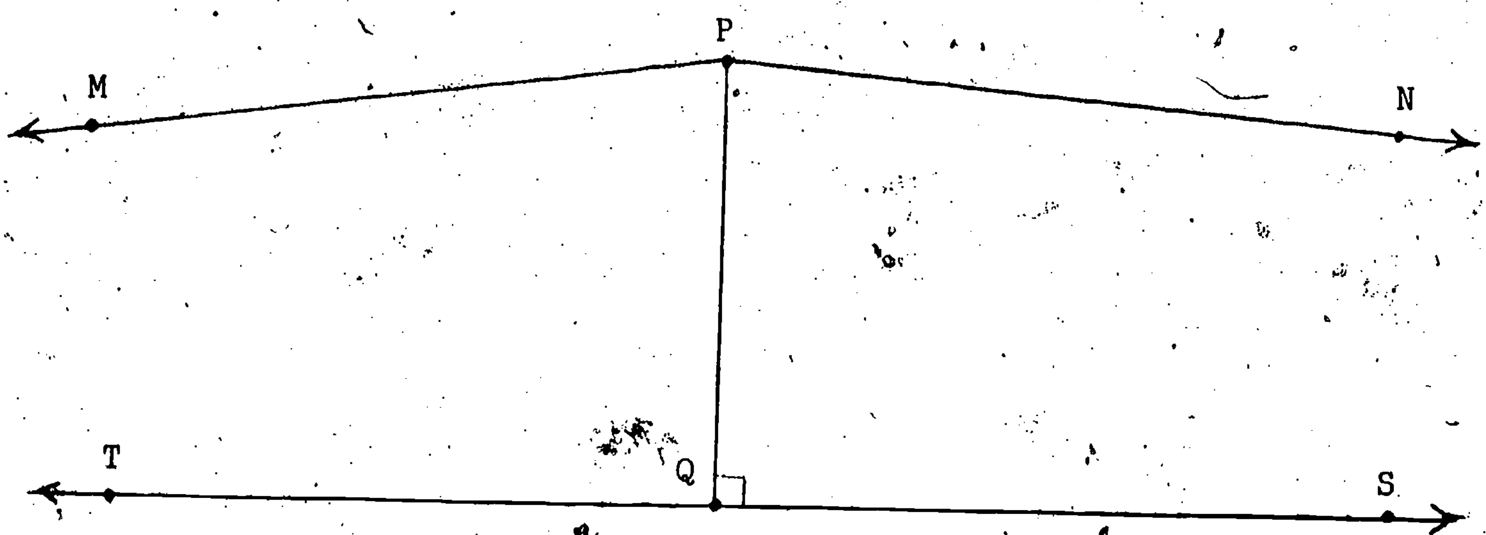
The proof that $m\angle A < m\angle C$ is absurd is similar.

Students should be shown the definition transparency #1 before attacking Theorem I.

Students should be shown definition transparency #2 before exposure to the corollary to Theorem I.

Theorems of Hyperbolic Geometry #1d

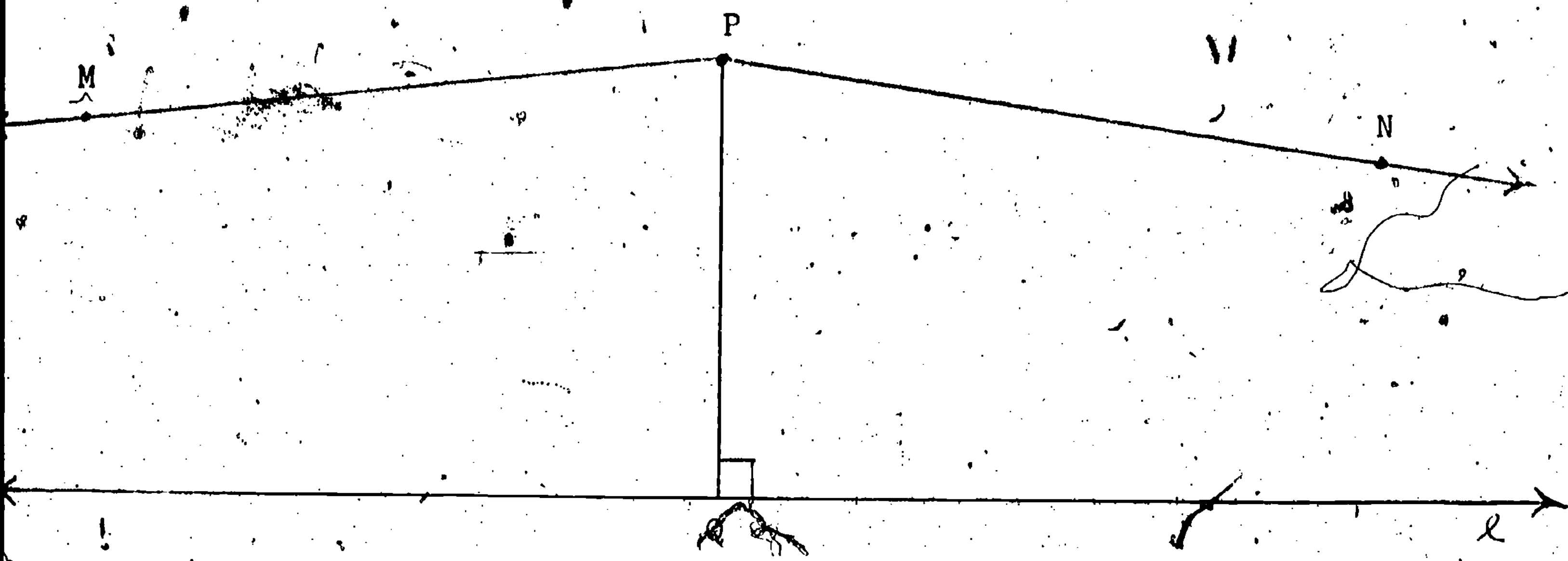
Corollary to Theorem I: Angles of parallelism are congruent.



In $\triangle PQ\Omega$ containing M and $\triangle PQ\Omega$ containing N, $\angle PQT \cong \angle PQS$ and $\overline{PQ} \cong \overline{PQ}$. Therefore, angles of parallelism $\angle MPQ$ and $\angle NPQ$ are congruent.

Theorems of Hyperbolic Geometry #2a

Theorem II: Angles of parallelism are acute.



There are three possibilities for $\angle MPQ$ and $\angle NPQ$.

a) $m\angle MPQ = m\angle NPQ = 90$

b) $m\angle MPQ = m\angle NPQ > 90$

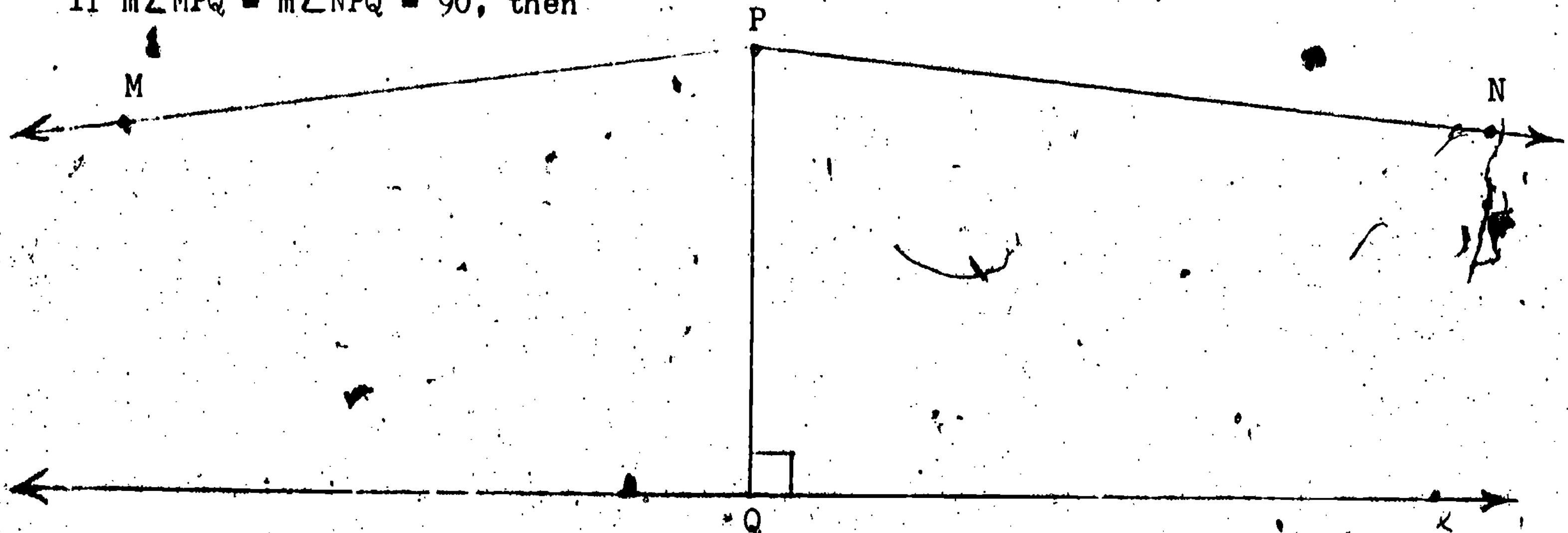
c) $m\angle MPQ = m\angle NPQ < 90$.

Both a) and b) lead to contradictions of previously accepted theorems and postulates.

Therefore, c) is the proper choice.

Theorems of Hyperbolic Geometry #2b

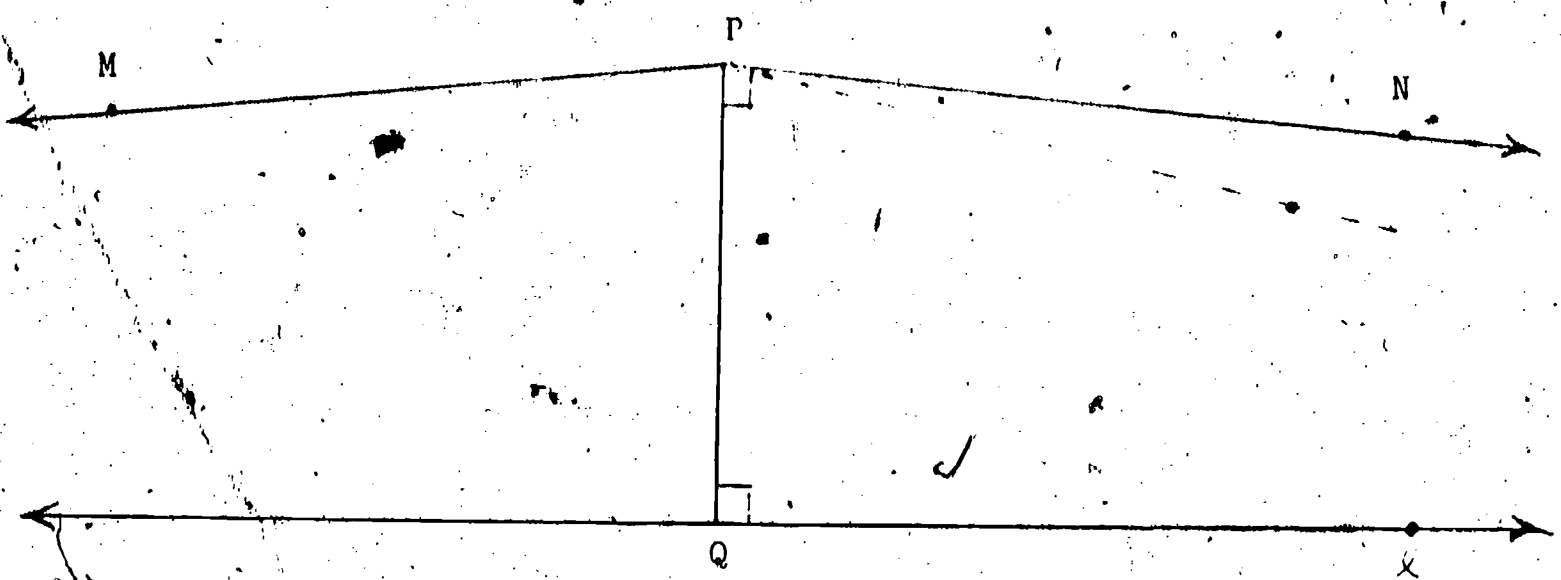
If $m\angle MPQ = m\angle NPQ = 90$, then



M, P and N are collinear. There is only one line through P parallel to ℓ. This contradicts Postulate H.

Theorems of Hyperbolic Geometry #2c

If $m\angle MPQ = m\angle NPQ > 90$, then



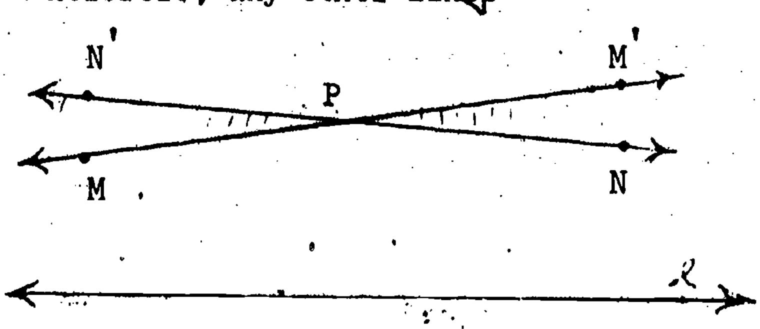
$\overrightarrow{PR}$ exists in the interior of $\angle NPQ$ such that $m\angle RPQ = 90$. But $\overrightarrow{PR}$ intersects with ℓ.

This contradicts Theorem C!

Notes on Theorem II

Students should be shown definition transparency #3 before exposure to Theorem II.

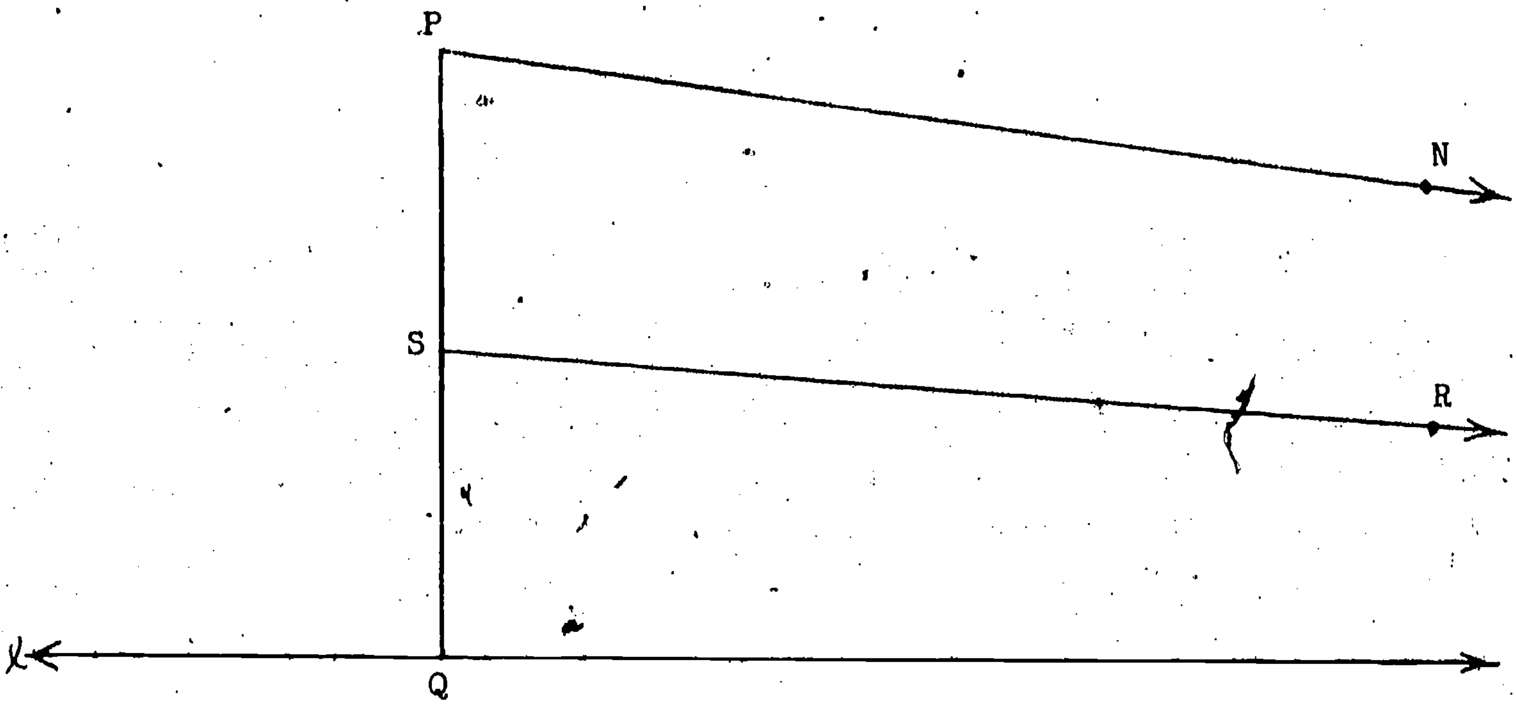
For transparency #2b, remind students that $\overrightarrow{PM}$ and $\overrightarrow{PN}$ are the two parallels. They are not ultra parallels! Therefore, any other lines coplanar to and non intersecting with ℓ must lie in the interiors of vertical angles MPN' and NPM'. It is then clear why M, N and P collinear implies at most one parallel to ℓ through P.



For transparency #2c, $\overrightarrow{PR}$ must intersect with ℓ since $\overrightarrow{PN}$ is a parallel and not an ultra parallel. Shown is the fact that $\angle NPQ$ cannot be obtuse. Since $\angle NPQ = \angle MPQ$ it follows that $\angle MPQ$ cannot be obtuse.

Theorems of Hyperbolic Geometry #3a

Theorem III: Two ~~rays~~ parallel to the same line in the same direction are parallel to each other.



Restatement: If $\overrightarrow{PN}$ is a parallel to ℓ and $\overrightarrow{SR}$ is a parallel to ℓ, then $\overrightarrow{SR}$ is a parallel to $\overrightarrow{PN}$.

Possibilities: a) $\overrightarrow{PN}$ and $\overrightarrow{SR}$ intersect

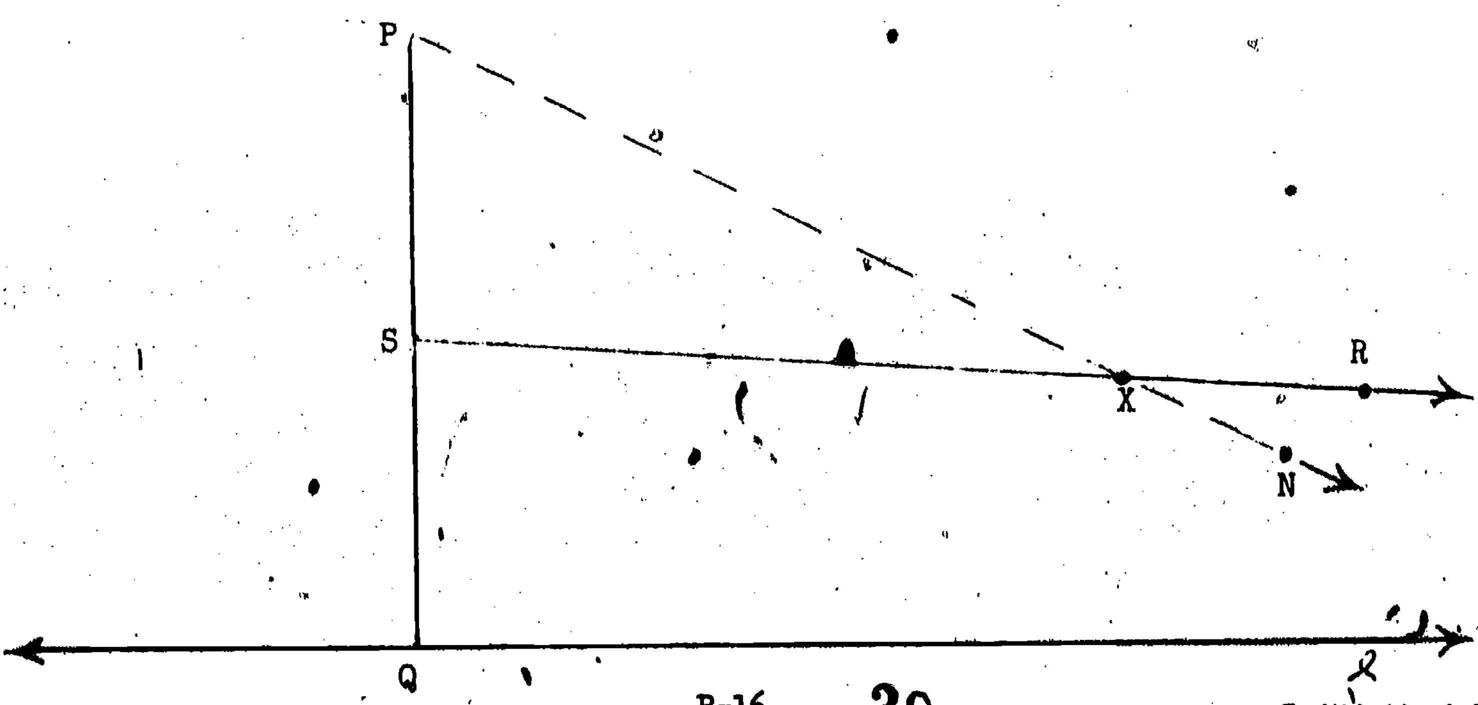
b) $\overrightarrow{PN}$ and $\overrightarrow{SR}$ are ultra parallel

c) $\overrightarrow{PN}$ and $\overrightarrow{SR}$ are parallel.

Again c) proves to be the only non-contradictory possibility.

Theorems of Hyperbolic Geometry #3b

If $\overrightarrow{PN}$ and $\overrightarrow{SR}$ intersect, then

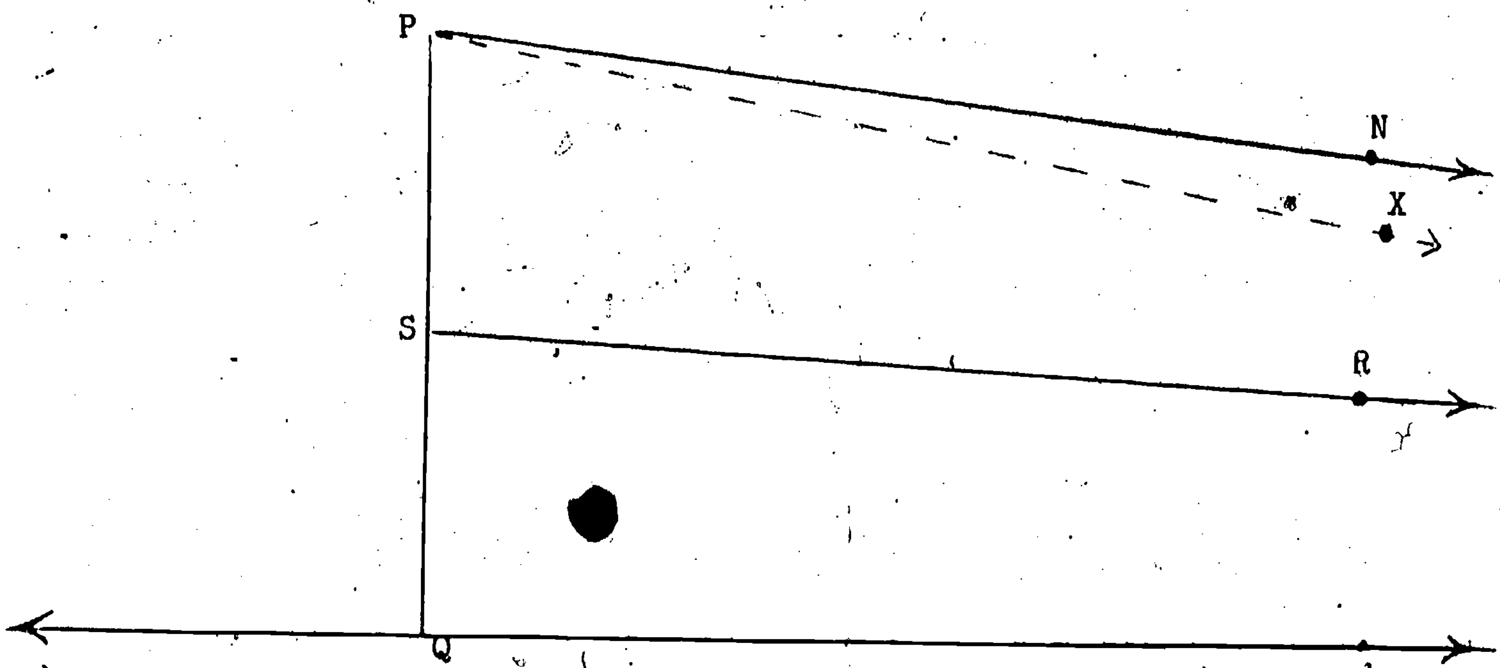


X is their point of intersection
$\overrightarrow{XR}$ is parallel to ℓ. Hence,
$\overrightarrow{XN}$ intersects ℓ.

This contradicts the hypothesis that $\overrightarrow{PN}$ is a parallel to ℓ.

Theorems of Hyperbolic Geometry #3c

If $\overrightarrow{PN}$ and $\overrightarrow{SR}$ are ultra parallel, then



$\overrightarrow{PX}$ exists such that $\overrightarrow{PX}$ is parallel to $\overrightarrow{SR}$
$\overrightarrow{PX}$ and ℓ do not intersect.

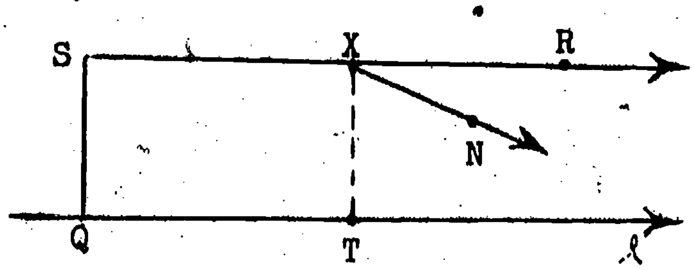
This contradicts the hypothesis that $\overrightarrow{PN}$ is parallel to ℓ.

Notes on Theorem III

The theorem establishes a transitive property for a special set of parallel rays; namely, those in the same direction.

Once again the difference between parallels and ultra parallels is very important to the understanding of the theorem.

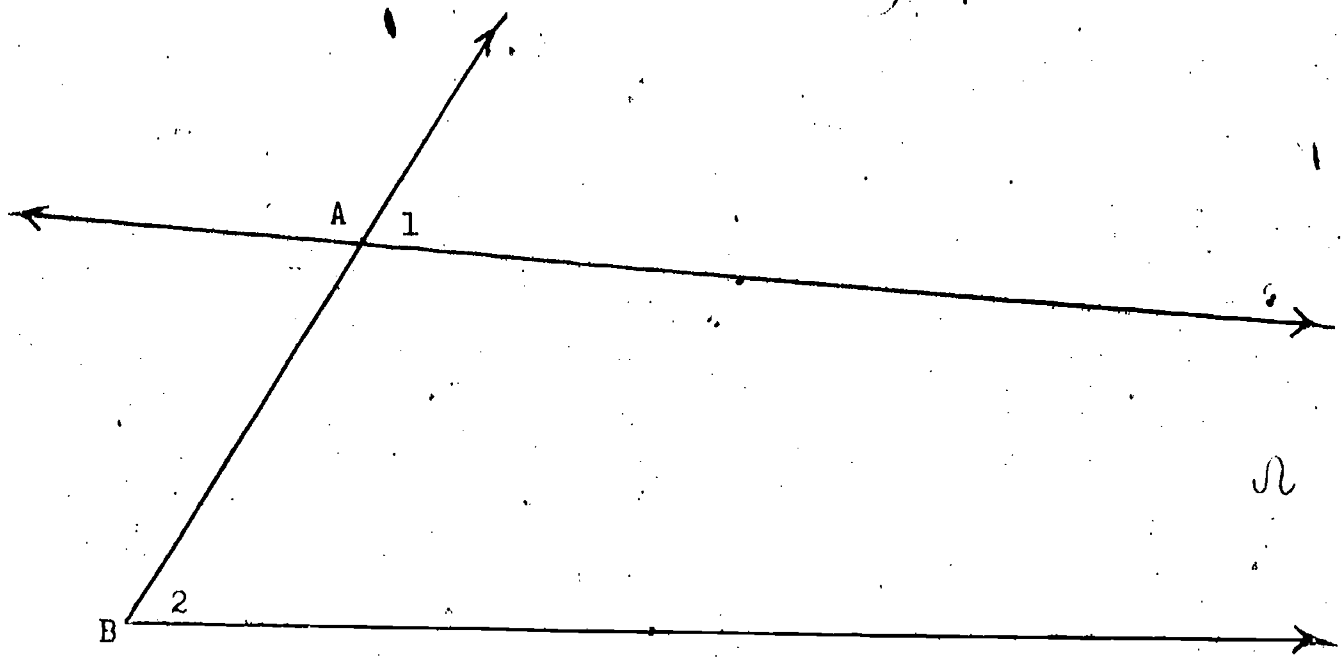
In transparency #3b, $\overrightarrow{XR}$ is claimed to be parallel to ℓ. It is not obvious that $\overrightarrow{XR}$ can not be ultra parallel to ℓ. A proof that $\overrightarrow{XR}$ cannot be ultra parallel to ℓ has the following basis. If $\overrightarrow{XR}$ is an ultra parallel, then a ray $\overrightarrow{XN}$ parallel to ℓ exists in the interior of $\angle RXT$. Ray $\overrightarrow{SN}$ must intersect ℓ at some point Z. By Pasch's theorem $\overrightarrow{XN}$ must intersect side TZ of ΔTZN. This contradicts the fact that $\overrightarrow{XN}$ is a parallel to ℓ.



In transparency #3c, $\overrightarrow{PX}$ and ℓ do not intersect since they are contained in opposite half planes with edge $\overleftrightarrow{SR}$.

Theorems of Hyperbolic Geometry 4a

Theorem IV: An exterior angle of an open triangle is greater in measure than the remote interior angle.



Restatement: Given $\triangle AB\Omega$ with exterior angle 1, prove $m\angle 1 > m\angle 2$.

Either a) $m\angle 1 < m\angle 2$

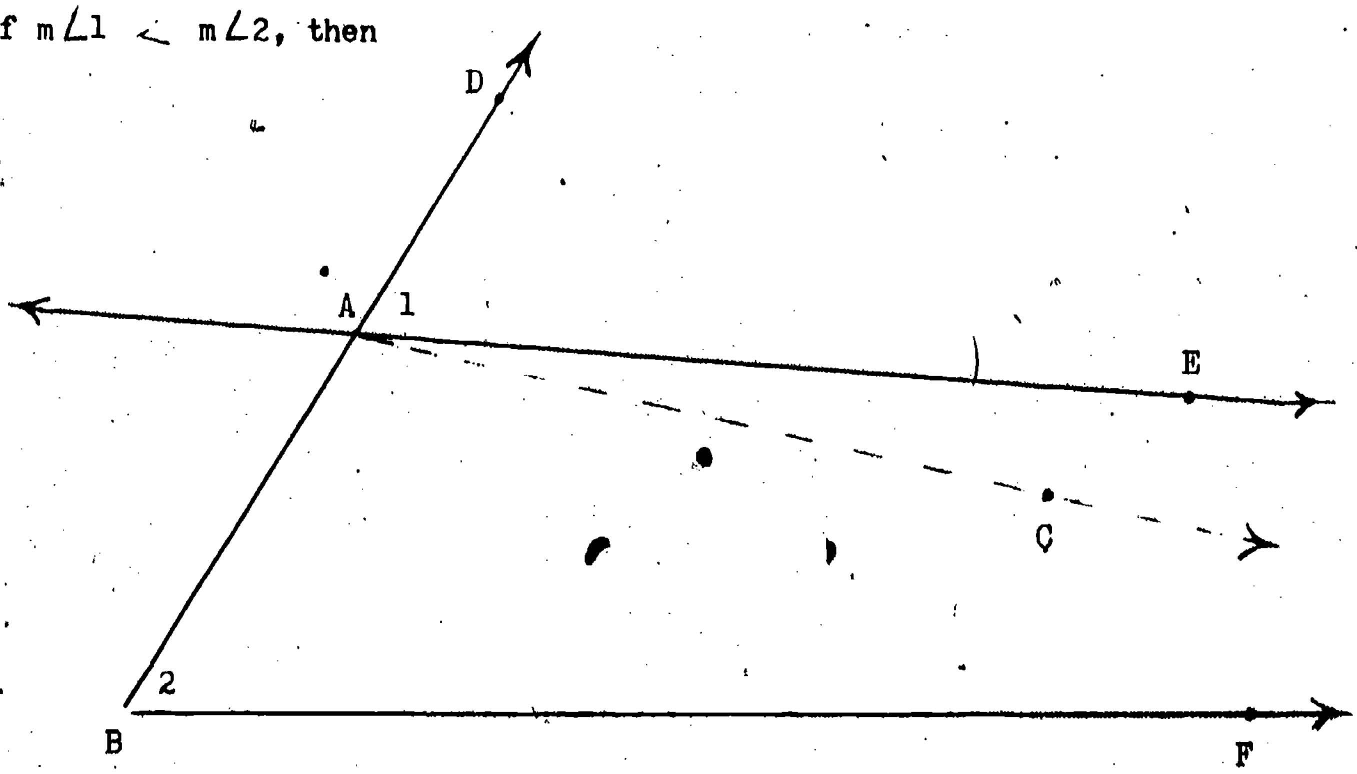
b) $m\angle 1 = m\angle 2$

c) $m\angle 1 > m\angle 2$

a) and b) lead to contradictions, leaving c) as the correct choice.

Theorems of Hyperbolic Geometry 4b

If $m\angle 1 < m\angle 2$, then

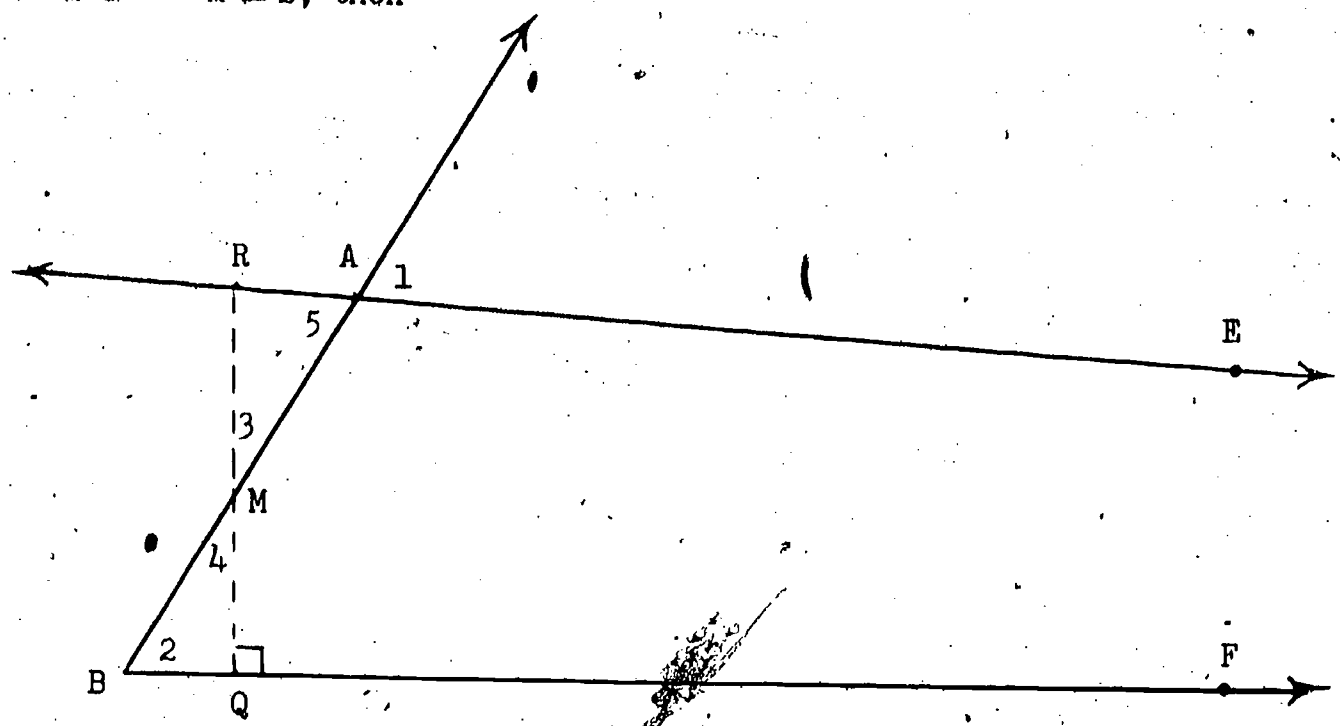


$\overrightarrow{AC}$ exists such that $m\angle DAC = m\angle 2$.
Since $\overrightarrow{AE}$ is a parallel to $\overrightarrow{BF}$, $\overrightarrow{AC}$ must intersect $\overrightarrow{BF}$.

This contradicts Theorem E.

Theorems of Hyperbolic Geometry 4c

If $m\angle 1 = m\angle 2$, then



With M as the midpoint of $\overline{AB}$, draw $\overrightarrow{QM}$ perpendicular to $\overrightarrow{BF}$.

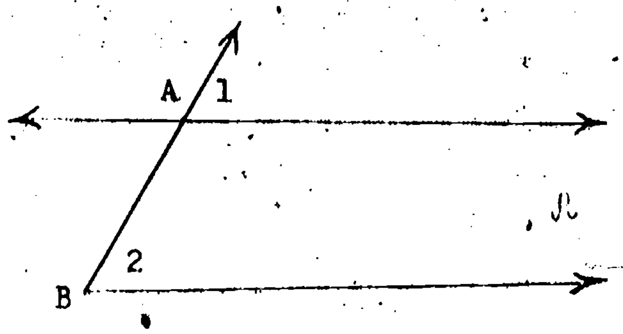
$\triangle RAM \cong \triangle QBM$ by ASA

$\angle ARM$ is a right angle

$\angle ARM$ is also an angle of parallelism.

This contradicts Theorem II.

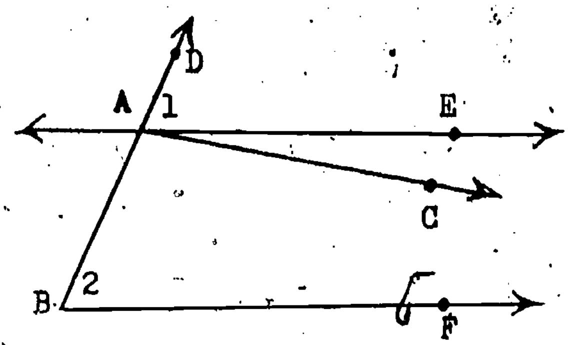
Notes on Theorem IV



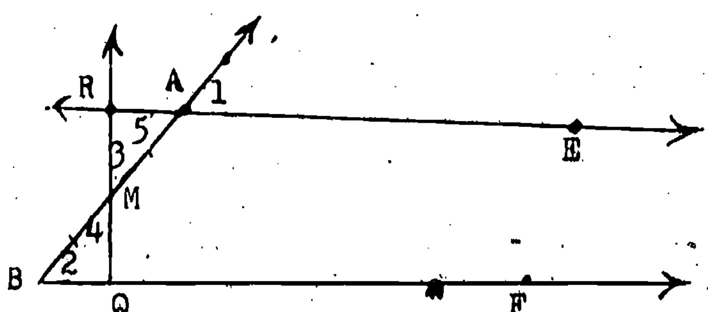
Theorem 4: An exterior angle of an open triangle is greater in measure than the remote interior angle.

Restatement: Given: $AB\Omega$ and exterior angle $\angle 1$.
Prove: $m\angle 1 > m\angle 2$

There are three possibilities. (1) $m\angle 1 < m\angle 2$ (2) $m\angle 1 = m\angle 2$ and (3) $m\angle 1 > m\angle 2$.

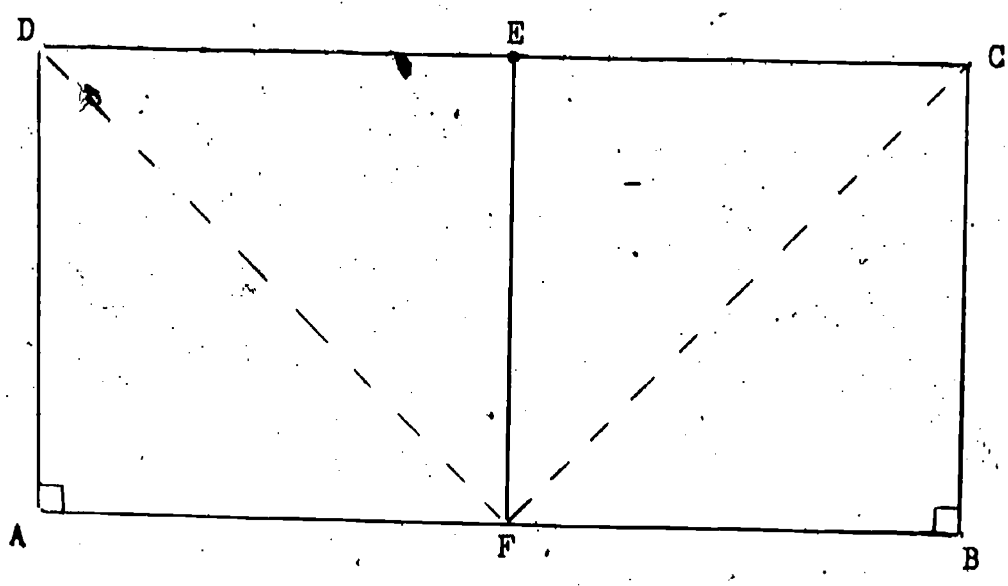


Case 1: Assume $m\angle 1 < m\angle 2$. Then there exists AC between $\overrightarrow{AE}$ and $\overrightarrow{AB}$ such that $m\angle DAC = m\angle 2$. Since $\overrightarrow{AE}$ is a parallel to $\overrightarrow{BF}$, $\overrightarrow{AC}$ must intersect with $\overrightarrow{BF}$ forming a triangle with exterior angle $\angle DAC$. This contradicts Theorem E. Hence $m\angle 1 < m\angle 2$ is false.



Case 2: Assume $m\angle 1 = m\angle 2$. Construct $\overrightarrow{QM}$ such that M is the midpoint of $\overline{AB}$ and $\overrightarrow{QM} \perp \overrightarrow{BF}$. $m\angle 5 = m\angle 2$; $AM = MB$ and $m\angle 4 = m\angle 3$. Hence, $\triangle BQM \cong \triangle ARM$ by ASA. This implies $\angle ARM$ is a right angle. By definition $\angle ARM$ is also an angle of parallelism and by Theorem J must be acute. Contradiction. Hence, $m\angle 1 = m\angle 2$ is false. Therefore, $m\angle 1 > m\angle 2$ is the only remaining possibility.

Theorem V: In a Saccheri quadrilateral, the line joining the midpoints of the base and summit is perpendicular to both and the summit angles are congruent.



Restatement: Given: $\overline{AD} \perp \overline{AB}$, $\overline{AB} \perp \overline{BC}$, AD = BC, ED = EC, and FA = FB.

Prove: $\overline{EF} \perp AB$, $EF \perp DC$; $m\angle ADE = m\angle BCE$.

By SAS $\triangle ADF \cong \triangle BCF \rightarrow \angle ADF \cong \angle BCF$, DF = CF

$\angle DFA \cong \angle CFB$

By SSS $\triangle DEF \cong \triangle CEF \rightarrow \angle EDF \cong \angle ECF$

$\angle DEF \cong \angle CEF$

$\angle DFE \cong \angle CFE$

By angle addition $\rightarrow \angle ADE \cong \angle BCE$

$\angle AFE \cong \angle BFE$

$\angle DEF \cong \angle CEF \rightarrow \overline{EF} \perp \overline{DC}$

$\angle AFE \cong \angle BFE \rightarrow \overline{EF} \perp \overline{AB}$

Notes on Theorem V

Expose students to definition transparency 4 before giving this theorem.

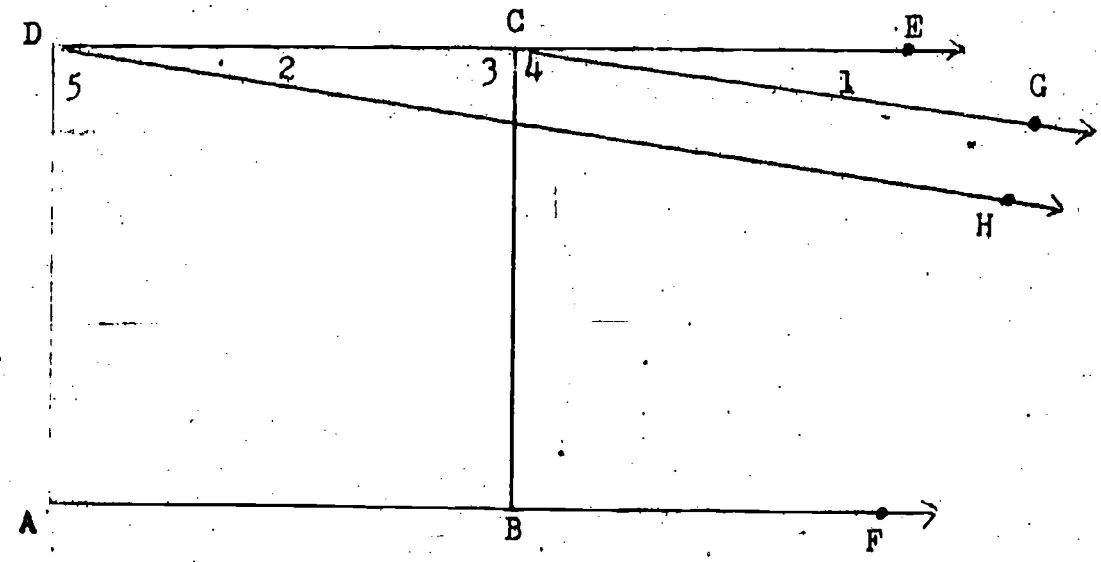
Most students should agree with the results of this theorem. They should also be able to prove it with little help.

The complete proof is included for your convenience. Construct $\overline{DF}$ and $\overline{FC}$. By SAS $\triangle DAF \cong \triangle CBF$. This implies that $m\angle ADF = m\angle BCF$ and DF = CF.

Since $EF = EF$, $\triangle DFE \cong \triangle CFE$ by SSS. Now $m\angle FDE = m\angle FCE$. But by the angle addition theorem $m\angle ADE = m\angle BCE$ which was to be proven. Also $m\angle DEF = m\angle CEF$ which implies that both are right angles. Hence, $\overline{EF} \perp \overline{DC}$. By angle addition theorem and definition of congruent triangles $m\angle AFE = m\angle BFE$. Therefore, both are right angles and $\overline{EF} \perp \overline{AB}$.

Corollary: The base and summit of a Saccheri quadrilateral are ultra parallel. The above corollary follows since $\angle DEF$ and $\angle CEF$ are not acute and, therefore, cannot be angles of parallelism.

Theorems of Hyperbolic Geometry #6b



Theorem I $\longrightarrow$ $\angle 4 \cong \angle 5$

Theorem III $\longrightarrow$ $\overrightarrow{CG} \parallel \overrightarrow{DH}$

CG, DH, and $\overline{CD}$ for open $\triangle CD\Omega$

Theorem IV $\longrightarrow$ $m\angle 1 > m\angle 2$

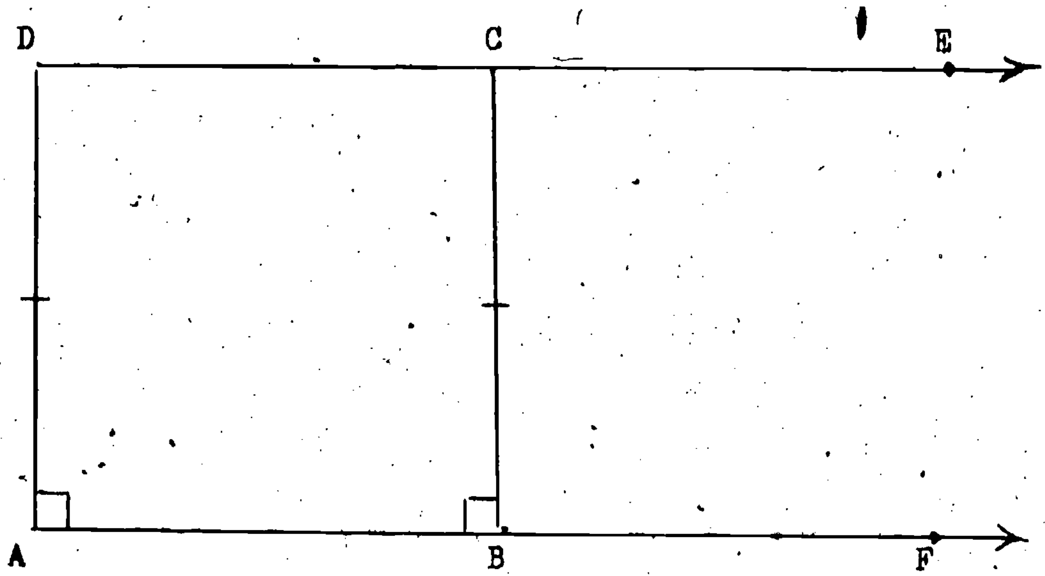
Therefore, $m\angle 1 + m\angle 4 > m\angle 2 + m\angle 5$

Theorem V $\longrightarrow$ $m\angle 3 = m\angle 2 + m\angle 5$

Hence, $m\angle 1 + m\angle 4 > m\angle 3$

And $m\angle 3 < 90$ or $\angle BCD$ is acute.

Theorem VI: The summit angles of a Saccheri quadrilateral/are acute.



Restatement: Given: $\overline{AD} \perp \overline{AB}$, $\overline{AB} \perp \overline{BC}$, $AD = BC$

Prove: $\angle BCD$ is acute (It then follows that $\angle ADC$ is also acute.)

Construct $\overrightarrow{CG}$ parallel to $\overrightarrow{AB}$ in the direction of $\overrightarrow{AB}$.
Construct $\overrightarrow{DH}$ parallel to $\overrightarrow{AB}$ in the same direction.

Notes on Theorem VI

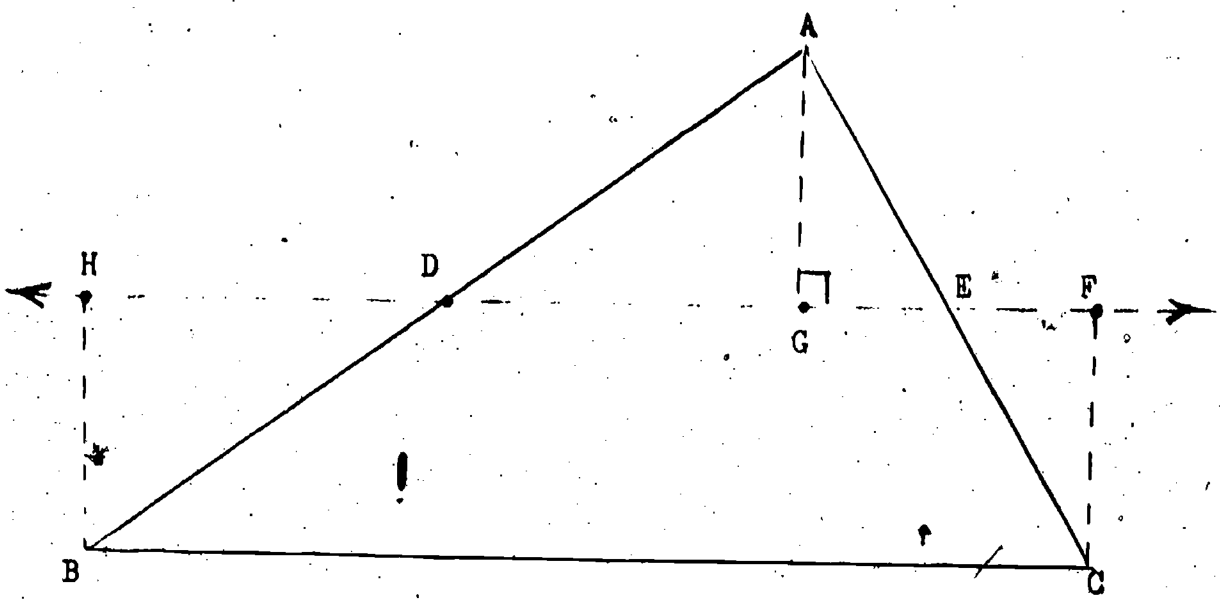
This is a result students will generally not be willing to accept. It will probably be useful to go over the proof with the students and indicate that all the other results covered lead to the indicated conclusion.

Not indicated in the transparency of the theorem is the fact that $\overrightarrow{CG}$ must be in the interior of $\angle BCE$ and $\overrightarrow{DA}$ must be interior to $\angle ADC$ since $\overrightarrow{DC}$ is an ultra parallel to $\overrightarrow{AB}$.

A result of this theorem that should be mentioned is: It is impossible to draw a rectangle. No matter how hard one tries, there will always be a slight defect in at least one of the proposed right angles.

Theorems of Hyperbolic Geometry #7a

Theorem VII: The sum of the measures of the angles of a triangle is less than 180.



Restatement: Given: $\triangle ABC$

Prove: $m\angle A + m\angle B + m\angle C < 180$

Construct $\overleftrightarrow{DE}$ where D and E are midpoints of $\overline{AB}$ and $\overline{AC}$ respectively.

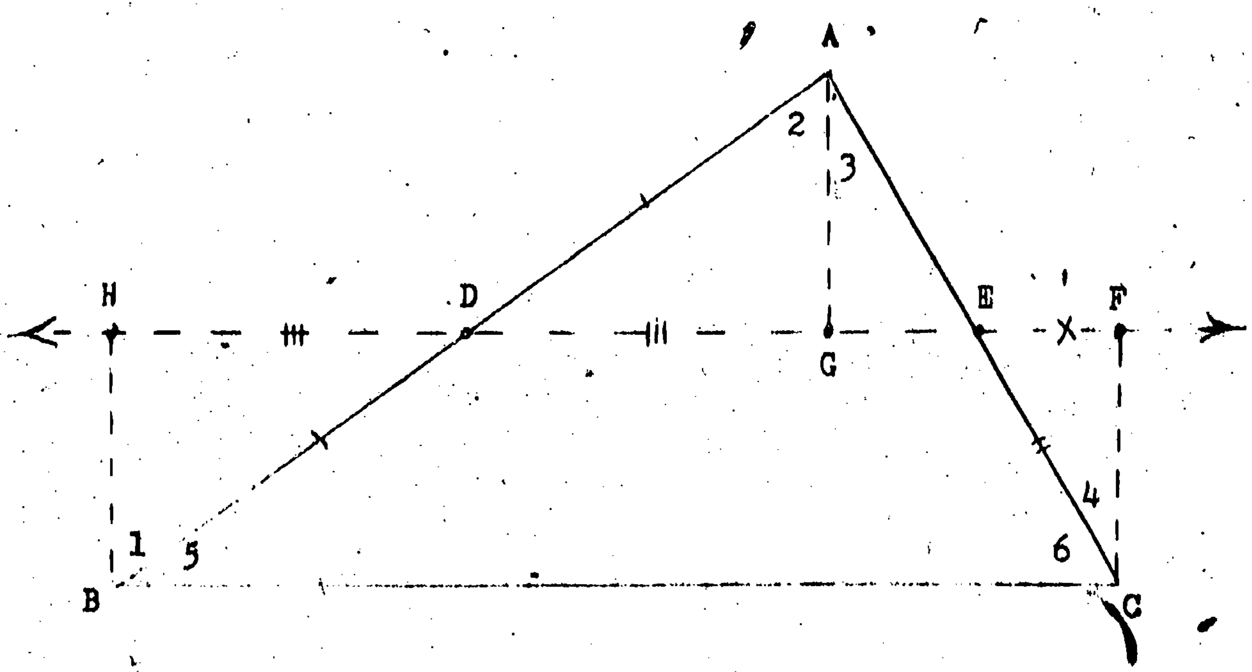
Construct $\overline{AG}$ such that $\overline{AG} \perp \overline{DE}$ at G.

Locate F such that EF = EG.

Locate H such that HD = DG.

Construct $\overline{HB}$ and $\overline{FC}$.

Theorems of Hyperbolic Geometry #7b



$\triangle HBD \cong \triangle GAD$ and $\triangle GAE \cong \triangle FCE$ by SAS

Therefore, $\angle 2 \cong \angle 1$, $\angle 3 \cong \angle 4$

And $FC = AG = HB$

And $m\angle BHD = 90 = m\angle CFE$.

HFCB is a Saccheri quadrilateral with summit BC.

Therefore, $m\angle 1 + m\angle 5 + m\angle 4 + m\angle 6 < 180$

And $m\angle 2 + m\angle 5 + m\angle 3 + m\angle 6 < 180$

That is, $m\angle A + m\angle B + m\angle C < 180$.

Notes on Theorem 7

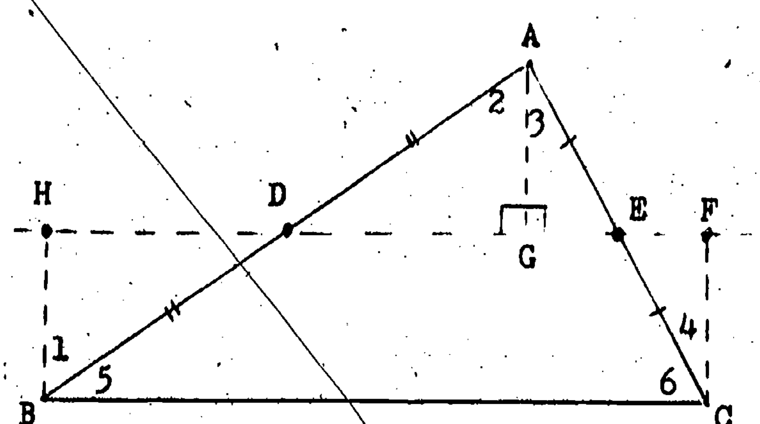
This theorem tells the student that something he has believed in for a long while is not true. It is time to remind him that this is a result of accepting Postulate H. That is if one cannot really accept this result, he must retreat and do away with Postulate H.

Indicate to the students that they should not throw in the towel on Postulate H until they have worked with a model of this geometry. Since every one is familiar with modeling in Euclidian Geometry, these techniques have been used instead of hyperbolic techniques. This may well be the reason for not being able to accept the results just proven.

You may not have to prove this theorem at all. After Theorem 6, most students will admit that Theorem 7 follows directly. The proof can be supplied if needed. There is, however, no need to force it on the students if they do not need to be convinced.

Theorem P: The sum of the measures of the angles of a triangle is less than $180°$.

Proof: Let D and E be the midpoints of $\overline{AB}$ and $\overline{AC}$ respectively. Construct $\overleftrightarrow{DE}$. Construct a ray through A $\perp$ to $\overleftrightarrow{DE}$. Call it $\overrightarrow{AG}$. Locate point F on $\overleftrightarrow{DE}$ such that E is between G and F and GE = FE. Likewise, locate point H on $\overleftrightarrow{DE}$ such that D is between G and H and HD = DG. Construct $\overline{FC}$ and $\overline{HB}$.



By SAS $\triangle HBD \cong \triangle GAD$ and $\triangle FEC \cong \triangle GEC$. Therefore, $\angle BHD$ and $\angle EFC$ are right angles. Also since $FC = AG = HB$, $FC = HB$. Hence, $\square HBCF$ is a Saccheri quadrilateral.

From the congruent triangles it follows that $m\angle 1 = m\angle 2$ and $m\angle 3 = m\angle 4$.

By Theorem VI $\quad m\angle 1 + m\angle 5 < 90$ and

$\qquad m\angle 4 + m\angle 6 < 90$

Therefore, $\quad m\angle 1 + m\angle 4 + m\angle 5 + m\angle 6 < 180$

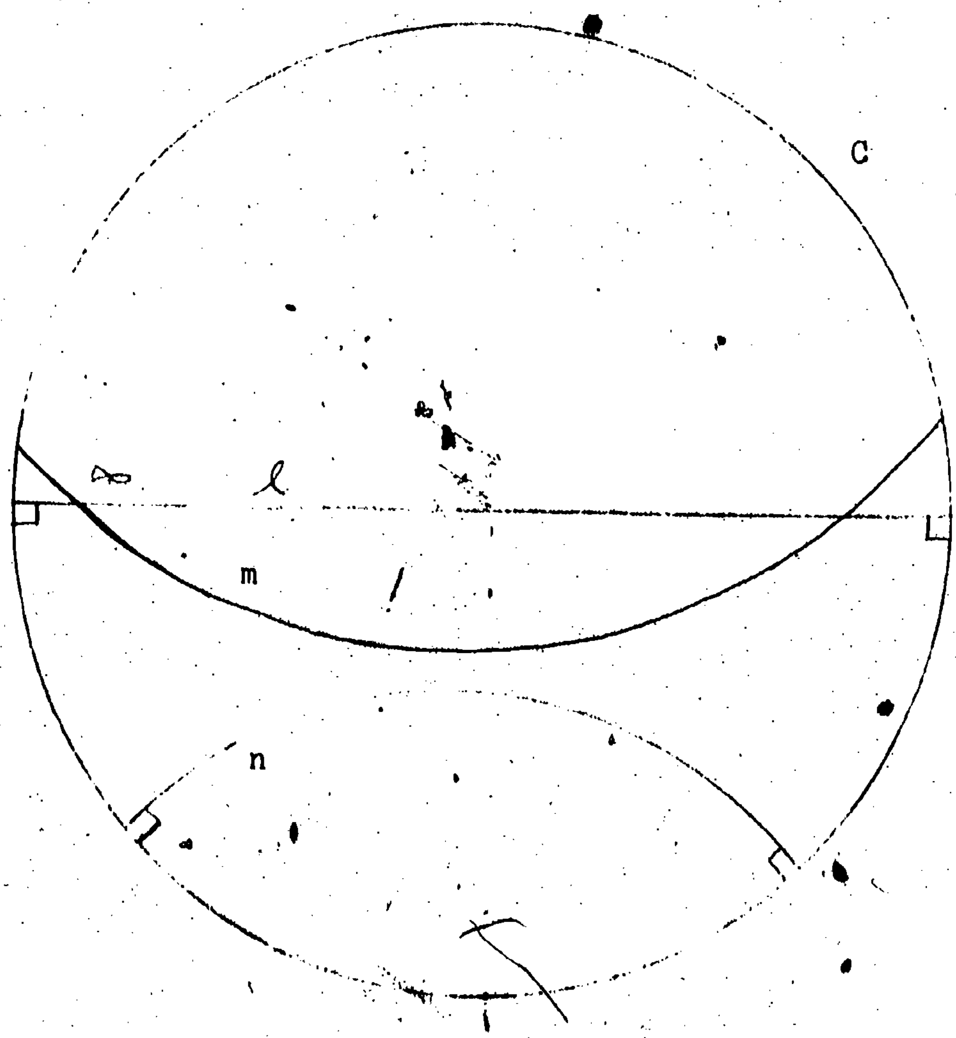
Substituting $\quad m\angle 2 + m\angle 3 + m\angle 5 + m\angle 6 < 180$

But $m\angle 2 + m\angle 3 + m\angle 5 + m\angle 6$ is the sum of the measure of the angles of $\triangle ABC$.

Q.E.D.

The Poincaré Model

The Poincaré model is a geometric model developed for the purposes of modeling plane hyperbolic geometry. It differs from the usual model in two significant ways. A plane, instead of being unbounded, is bounded by a circle. A line is a subset of a plane which is an arc of a circle that is perpendicular to the boundary circle of the plane at its two points of intersection.



In the above model, C is the bounding circle, ℓ and n are hyperbolic lines. m is not considered a line since it is not perpendicular to the bounding circle. Use the Poincaré Modeling board with included hyperbolic lines to investigate each of the following conjectures. (Note: Each of the plastic lines only represent a hyperbolic line when its ends are on the edge of the plane.)

Statement: Two points determine exactly one line.

Procedure: Select two random points on the plane. Go to the box containing the hyperbolic lines. Get a variety of arcs from the box. By trial and error, fit them over the two points until you find one which when covering the two points has its ends on the edge of the circle. (As in horseshoes and hand grenades, close counts because of the limited selection of lines.) Check to see if there is any other size line that fits the two points and has its ends on the edge of the plane. Assuming you have found one and only one such line, we consider that sufficient evidence and move on.

Statement: Given any line ℓ, the points that do not lie on line ℓ form two non-intersecting sets. Any line containing a point from each of these two sets must cross ℓ.

Procedure: Place any hyperbolic line on the plane and observe. Pick a point on either side of the line. Does a line containing both those points cross ℓ ?

Statement: Given a line ℓ there exists at least one line perpendicular to it.

Procedure: Place two lines on the plane. Change the position of one of the lines until it appears to be perpendicular to the other line. How did you determine that they were perpendicular? Could you use a protractor to determine if the angles were right angles?

Now that you are familiar with placing the lines, we will give you another set of statements and allow you to develop your own procedure to model these statements.

Statement: If two lines are perpendicular to the same line, they do not intersect.

Statement: Given a line ℓ and a point P not on ℓ, there exists at least two lines through P which are parallel to ℓ.

Statement: The measure of an exterior angle of an open triangle is greater than that of the remote interior angle. (Remember that two sides of an open triangle are parallel, not ultra parallel.)

Statement: The summit angles of a Saccheri quadrilateral are acute.

Statement: The sum of the measures of the angles of a triangle is less than 180.

What you have accomplished in the first part of this activity is that you can show with these funny circular lines, all the things you can do with straight lines and infinite planes. That is, all the postulates and theorems discussed thus far in the course can be modeled on the Poincaré plane. We admit to neglecting the set of postulates dealing with distance between points. This was done deliberately to spare you from working with the complicated distance formula of the Poincaré model and to spare us from trying to explain it.

Aside from this, we would guess that there may still be one minor detail that you find bothersome. That is, in the real world, lines do not curve and planes are not bounded, are they? At this point, your instructor has two class activities skillfully prepared to further confuse you on this issue. They are: (1) Do you really know which way is straight and (2) First one out the door is extremely small.

Take it away Teach! !

From these two activities you have hopefully learned the following. First, what is considered straight is determined by line of sight. You have no built-in system that guides you in a straight line, particularly over large distances and if there are no visual readings allowed. Second, if you tried to get to the end of a bounded line, you might never make it if both you and your measuring stick shrank as you approached the end of the line.

But this still has not answered the question, "Is the physical world really like this?" There is a noted set of physical theories which predict that lines in our universe are curved and that the size of an object changes with its speed as well as the object's mass and conception of time. They are the theories of general and special relativity as put forth by Dr. Albert Einstein.

The general theory of relativity predicts that light travels through the universe in curved lines which are distorted near large bodies of mass. The distortion the sun would produce in the path of the light from a distant star was computed and then actually observed during a solar eclipse. This simply means that a theory of space which claims that the shortest distance between two points is a curved line not a straight line, may accurately describe space.

The special theory of relativity states that if an object is moving at a very great speed relative to your own speed, you will observe it to be different than if it were simply sitting near you. At high speeds it would appear to you to be shorter, heavier and to age less quickly. This too has been verified in experiments. Very small particles called electrons have been observed to become heavier when given speeds close to the speed of light. Since, in order to travel to the edge of the universe, you would have to travel extremely fast, it is therefore possible that you and your measuring stick will shrink. Hence, you will never reach the edge of the universe and you will therefore conclude that it is infinite.

The theory of relativity is a theory used to describe the universe and not just our locality. If you look at a very small portion of a hyperbolic line, it will appear to you to be straight. Hence, maybe we believe that lines are straight and unbounded simply because they appear so in our restricted field of observation.

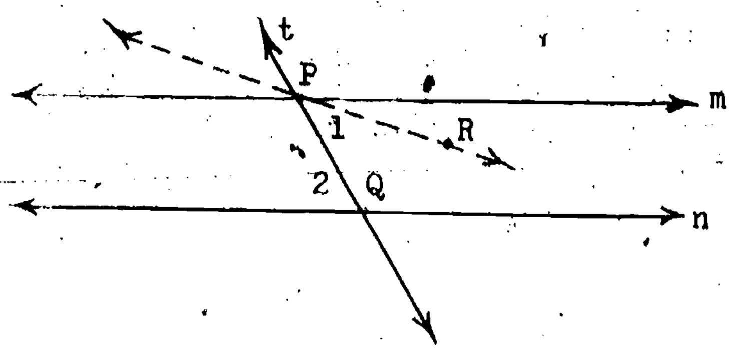
Quickly while you are still confused, let's go back, take the other possible parallel postulate, and see what results it leads to.

Euclidean Geometry

Euclid's Parallel Postulate: Through a point P not on line ℓ, there is at most one line parallel to ℓ.

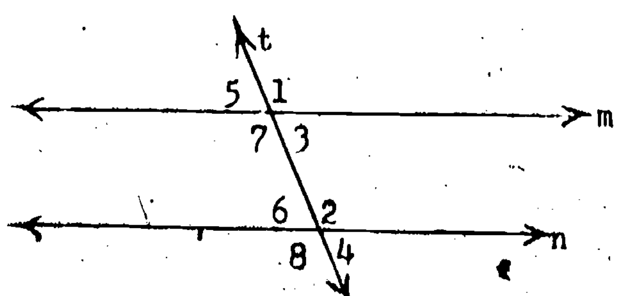
Euclid's postulate can now be used to prove some important theorems. We will state and illustrate the theorems here. After the first one, the proofs are left to you.

1. If two parallel lines are cut by a transversal, the alternate interior angles are congruent.



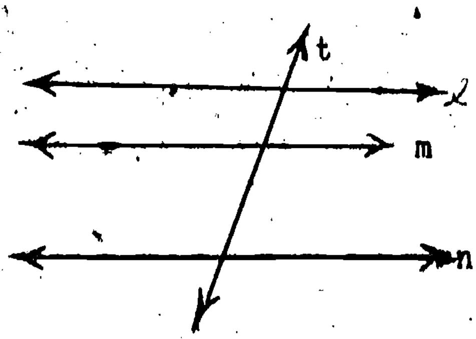
In the drawing, $m \parallel n$ with transversal t intersecting them at P and Q, and forming alternate interior angles 1 and 2. We know there is a line $\overleftrightarrow{RP}$ so that $\angle RPQ \cong \angle 2$. These congruent angles are also alternate interior angles so $\overleftrightarrow{RP} \parallel n$. According to Euclid's Postulate, there can be at most one parallel to n through P so $\overleftrightarrow{PR}$ must be m and $\angle RPQ \cong \angle 1$. Therefore, $\angle 1 \cong \angle 2$.

2. If two parallel lines are cut by a transversal, each pair of corresponding angles are congruent.



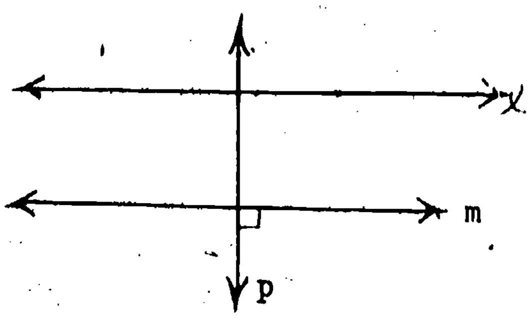
$m \parallel n$ and cut by transversal t. Show $\angle 1 \cong \angle 2$, $\angle 3 \cong \angle 4$, $\angle 5 \cong \angle 6$, $\angle 7 \cong \angle 8$.

3. In a plane, two lines parallel to the same line are parallel to each other.



$m \parallel l$ and $n \parallel l$. Show $m \parallel n$.

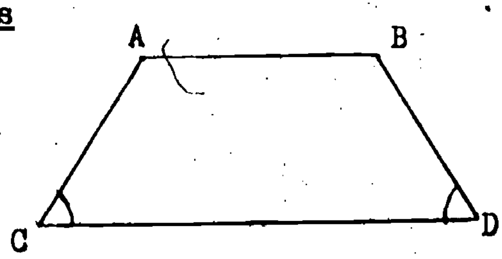
4. In a plane, if a line is perpendicular to one of two parallels, it is perpendicular to the other.



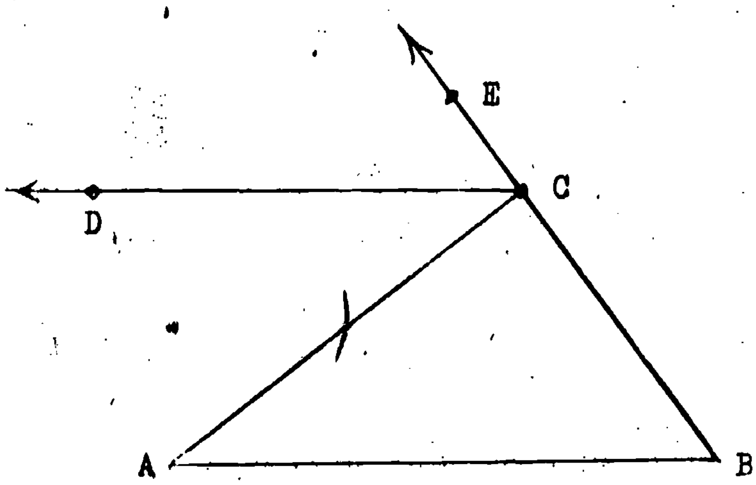
$\ell \parallel m$ and $p \perp m$. Show $p \perp \ell$.

Exercises

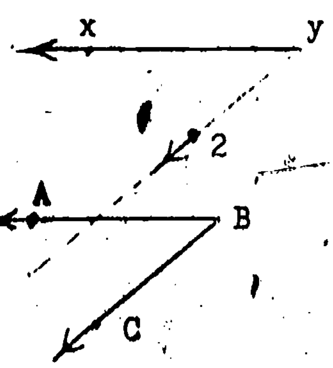
1. Given: $\overline{AB} \parallel \overline{CD}$, $\angle C \cong \angle D$
   Prove: $\angle A \cong \angle B$



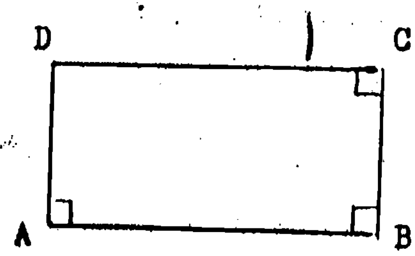
2. Given: $\overline{DE} \parallel \overline{AB}$
   $m\angle ACD = 2x + 14$
   $m\angle CAB = 5x - 1$
   $m\angle ACB = 9x + 10$
   Find: a) Value of x
   b) $m\angle BAC$
   c) $m\angle DCE$
   d) $m\angle ABC$
   e) $m\angle ACB$



3. Given: $\angle XYZ$ and $\angle ABC$ coplanar
   Acute angles
   $\overline{XY} \parallel \overline{AB}$, $\overline{YZ} \parallel \overline{BC}$
   Prove: $\angle XYZ \cong \angle ABC$



4. Given: $m\angle A = m\angle B = m\angle C = 90$
   Prove: $m\angle D = 90$



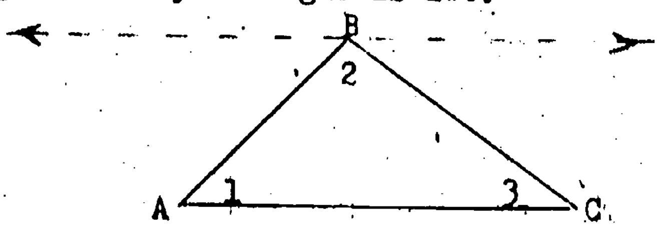
Euclidean Geometry

Using Euclid's Parallel Postulate we can prove an important theorem about triangles.

1. The sum of the measures of the angles of any triangle is 180.

Given: $\triangle ABC$

Prove: $m\angle 1 + m\angle 2 + m\angle 3 = 180$



By Euclid's Postulate, we know there is at most one line p through B so that $p \parallel \overline{AC}$. Let D, E be points on p with B between D and E.

The proof is left to you.

Exercises

1. Prove: The acute angles of a right triangle are complementary.

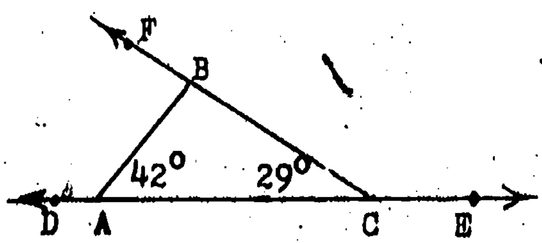
2. Prove: For any triangle, the measure of each exterior angle is equal to the sum of the measures of its two remote interior angles.

   $\angle BCD$ is an exterior angle of $\triangle ABC$. Show $m\angle 1 + m\angle 2 = m\angle BCD$.

3. Given: $\triangle ABC$. The measure of two angles is given. Supply the third.

   a) $m\angle A = 57$, $m\angle B = 32$, $m\angle C = ?$
   b) $m\angle A = 108$, $m\angle B = 19$, $m\angle C = ?$
   c) $m\angle A = 80$, $m\angle B = X$, $m\angle C = ?$
   d) $m\angle A = X$, $m\angle B = Y$, $m\angle C = ?$

4. a) $m\angle DAB =$
   b) $m\angle ABC =$
   c) $m\angle BCE =$
   d) $m\angle ABF =$



5. $m\angle ABC = 6x$
   $m\angle ACB = 7x + 3$
   $m\angle DAB = 11x + 17$

   a) Find x
   b) Find $m\angle DAB$
   c) Find $m\angle BAC$
   d) Find $m\angle BCA$
   e) Find $m\angle BAC$