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ABSTRACT

This is one of a series of geometry modules developed for use by secondary students in a laboratory setting. This module includes: (1) Pythagorean Theorem (with review of radicals): (2) Basic Coordinate Geometry (distance and midpoint, slope, slope of parallels and perpendiculars, and equation of a line): (3) Selecting Coordinates: (4) Coordinate Proofs: and (5) Linear Inequalities, Non-negative Constraints, Linear Programming (optional). In addition to the activity sheets, a statement of objectives, teaching suggestions, and exercise answers are included. (MK)

GEOMETRY MODULE FOR USE IN A MATHEMATICS LABORATORY SETTING

TOPIC: Coordinate Geometry

by

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COORDINATE GEOMETRY

Teacher's Guide

Objectives

1. The student should know and be able to use the Pythagorean Theorem and its converse; distance, midpoint and slope formulas.
2. The student will be able to show line segments congruent, parallel, or perpendicular when coordinates are known.
3. The student will be able to select coordinates and prove theorems for triangles and quadrilaterals.

Overview

This module begins with a demonstration but not a proof of the Pythagorean Theorem. The proof can be added to or used to replace the demonstration if the class has the background. An extensive review of radicals is included and is usually needed. Exercise A or B could be used as a pretest to determine whether part, all, or none of the review should be used.

This module includes:

I. Pythagorean Theorem (with review of radicals)

II. Basic Coordinate Geometry
   1. Distance and Midpoint
   2. Slope
   3. Slope of parallels and perpendiculars
   4. Equation of a line

III. Selecting Coordinates

IV. Coordinate Proofs

V. Linear Inequalities, Non-negative Constraints, Linear Programming

(Section V is optional.)

Materials

Graph paper
Straightedge
Compass
Scissors

Teaching Suggestions

1. Have students work in small groups on this construction and demonstration of the Pythagorean Theorem. You might want to use colored paper for either the triangles or the square.
2. Square is defined later in the module. For this demonstration, however, the student's own definition of a square is sufficient.
3. We have included here a review of radicals if you feel it is needed.
4. The exercises introduce calculating the length of a line segment as preparation for developing the distance formula later.

Materials

Compass
Straightedge
Scissors

Exercise Answers (Simplifying Radicals)

1. 6
2. $2\sqrt{3}$
3. $8\sqrt{2}$
4. $3x\sqrt{3}$
5. $\frac{1}{3}\sqrt{2}$ or $\frac{\sqrt{2}}{3}$
6. $\frac{1}{4}\sqrt{10}$ or $\frac{\sqrt{10}}{4}$
7. $\frac{3}{2}\sqrt{2}$ or $\frac{3\sqrt{2}}{2}$
8. $\frac{1}{6}\sqrt{15}$ or $\frac{\sqrt{15}}{6}$
9. $4\sqrt{3}$
10. $9x\sqrt{x}$
11. $2\sqrt{10}$
12. $\frac{2}{3}\sqrt{5}$ or $\frac{2\sqrt{5}}{3}$
13. $2\sqrt{6}$
14. $\frac{1}{2}\sqrt{2}$ or $\frac{\sqrt{2}}{2}$
15. $4\sqrt{2}$
16. $\sqrt{3}$
17. $\sqrt{13}$
18. $6\sqrt{2}$
19. $\sqrt{7}$
20. $\frac{2}{5}\sqrt{5}$ or $\frac{2\sqrt{5}}{5}$
21. $5\sqrt{2}$
22. $\sqrt{2}$
23. $15\sqrt{2}$
24. $\sqrt{26}$
25. $\frac{1}{4}\sqrt{10y}$ or $\frac{\sqrt{10y}}{4}$
26. $18\sqrt{5}$
27. $6\sqrt{2}$
28. $\frac{1}{3}$
29. $\frac{7}{4}$
30. $\frac{1}{2}\sqrt{2}$ or $\frac{\sqrt{2}}{2}$

Exercise Answers (Multiplying & Dividing Radicals)

1. $\sqrt{10}$
2. $\sqrt{5}$
3. $12\sqrt{14}$
4. $\frac{9}{2}\sqrt{2}$ or $\frac{9\sqrt{2}}{2}$
5. $\frac{3}{2}\sqrt{30}$ or $\frac{3\sqrt{30}}{2}$
6. 3
7. $\frac{4}{5}$
8. 448
9. 336
10. 14

Exercise Answers (Adding Radicals)

1. $13\sqrt{5} + 3\sqrt{3}$
2. $15\sqrt{2} + 5\sqrt{10}$
3. $-13\sqrt{11}$
4. 100
5. $12\sqrt{3} - 11\sqrt{2}$
6. $-3\sqrt{13}$
7. $\sqrt{13} + 3$
8. -1
9. $-2\sqrt{3}$
10. 28

Exercise Answers (A. Review of Radicals)

1. $4\sqrt{2} - \sqrt{3}$
2. $\frac{6}{5}\sqrt{5}$ or $\frac{6\sqrt{5}}{5}$
3. $\sqrt{6}$
4. 3
5. $\sqrt{6}$
6. $\sqrt{6}$
7. $\frac{1}{9}\sqrt{3}$ or $\frac{\sqrt{3}}{9}$
8. 31
9. $40\sqrt{3}$
10. $\sqrt{10}$

Exercise Answers (B. Review of Radicals)

1. $3\sqrt{2} - \sqrt{6}$
2. $\frac{10}{3}\sqrt{2}$ or $\frac{10\sqrt{2}}{3}$
3. $2\sqrt{3} + \sqrt{6}$
4. 3
5. $\sqrt{5}$
6. $\sqrt{5}$
7. $\frac{1}{4}\sqrt{3}$ or $\frac{\sqrt{3}}{4}$
8. 28
9. $30\sqrt{2}$
10. $2\sqrt{3}$

Exercise Answers (C. Review of Radicals)

1. $3\sqrt{2}$
2. $\sqrt{2}$
3. $5\sqrt{3}$
4. $\sqrt{6}$
5. $\sqrt{3}$
6. $\sqrt{7}$
7. 1
8. $6\sqrt{10}$
9. $\frac{1}{2}\sqrt{3}$ or $\frac{\sqrt{3}}{2}$
10. $24\sqrt{2}$
11. $13\sqrt{2}$
12. $2\sqrt{3}$
13. $5\sqrt{5} - 5$
14. $5\sqrt{2}$
15. $\frac{1}{6}\sqrt{3}$ or $\frac{\sqrt{3}}{6}$
16. $4\sqrt{5} + 5\sqrt{3}$
17. $\frac{5}{4}\sqrt{2} + \frac{1}{2}\sqrt{10}$ or $\frac{5\sqrt{2}}{4} + \frac{\sqrt{10}}{2}$
18. $12\sqrt{3} - 72$
19. $-49\sqrt{3}$
20. $\sqrt{3} + \frac{2}{3}\sqrt{6}$ or $\sqrt{3} + \frac{2\sqrt{6}}{3}$

Exercise Answers

1. 10
2. $4\sqrt{2}$
3. 15
4. 2
5. $\sqrt{22}$
6. $2\sqrt{7}$
7. $3\sqrt{3}$
8. $3\sqrt{5}$
9. $10\sqrt{2}$
10. $6\sqrt{2}$
11. $\sqrt{6}$
12. 5
13. 15
14. 5
15. 5
16. $5\sqrt{2}$
17. $2\sqrt{10}$
18. a) Yes
    b) Yes
    c) No
    d) Yes
    e) No

Teaching Suggestions

1. This unit consists of four parts that establish the skills needed in coordinate geometry: Finding Distance and Midpoint; Finding Slope; Comparing Slope of Parallels and Perpendiculars; and Writing Equations of Lines. The activity in each part develops the main idea but will require teacher involvement for many students, either to reinforce the idea when the activity is completed, or to introduce the idea before students begin the activity.

2. Answers (Distance and Midpoint)

1. a) 3
   b) (3, 2)
   c) -6
   d) (8, -2)
2. A (4, 3)
   B (-1, -5)
   C (7, -2)
3. 3
   3
   7
   3
   8
   5
4. Yes
5. Hypotenuse
   AP_1, P_1Q_1
   $y_2 - y_1$
   4, 6
   18
   $3\sqrt{2}$
6. $\sqrt{58}$
   $\sqrt{265}$
   $\sqrt{89}$

7. $(-6, 2)$

$(4, 6)$

$(-1, 4)$

x, $x_2 - x$

$\frac{x_2 + x_1}{2}$

-1

4

$(-1, 4)$

8. -2

$-2\frac{1}{2}$

$\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$

9. -5

$\frac{y_1 + y_2}{2}$, $2y$

13

$(-5, 13)$

3. There are several important ideas in the exercises on slope. Discuss these with the class.

4. Showing the product of slopes of perpendiculars is -1 depends on similarity and so is not shown here.

Materials

Graph paper

Exercise Answers (Distance and Midpoint)

1. $RS = 5$

$ST = 5$

$TR = \sqrt{2}$

2. $PY = \sqrt{130}$

$YT = \sqrt{26}$

$TP = \sqrt{104}$

$(YT)^2 + (TP)^2 = (PY)^2$

3. $QA = 13$

$UD = \sqrt{89}$

4. a) $5\sqrt{17}$

b) $5\sqrt{5}$

5. $AB = 2\sqrt{2}$

$BC = 6\sqrt{2}$

$CA = 8\sqrt{2}$

$AB + BC = AC$

6. $AB = \sqrt{4s^2 + 4r^2}$

$AQ = \sqrt{s^2 + r^2}$

$\sqrt{4s^2 + 4r^2} = \sqrt{4\ (s^2+r^2)}$

$= \sqrt{4}\ \sqrt{s^2 + r^2}$

$= 2\ \sqrt{s^2 + r^2}$

7. $AB = 2\sqrt{2}$

$BC = 3\ \sqrt{2}$

$CA = 5\ \sqrt{2}$

$AB + BC = CA$

8. $d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (Z_2 - Z_1)^2}$

9. a) 13

b) 5

10. $AB = \sqrt{14}$

$BC = \sqrt{20}$

$CA = \sqrt{34}$

$(AB)^2 + (BC)^2 = (CA)^2$

11. $(-\frac{1}{2}, \frac{7}{2})$

$(-5, 2)$

$(\frac{11}{2}, -\frac{3}{2})$

12. $(4, 2)$

13. $\sqrt{10}$, 5, 5

14. Midpoint $\overline{QA} = (2\frac{1}{2}, 2)$

Midpoint $\overline{UD} = (2\frac{1}{2}, 2)$

15. $AC = \sqrt{68}$ $BD = \sqrt{68}$

Midpoint $\overline{AC} = M(3, 5)$ Midpoint $\overline{BD} = X(3, 5)$

$CM = \sqrt{17}$ $DM = \sqrt{17}$ $CD = \sqrt{34}$

$(CM)^2 + (DM)^2 = (CD)^2$

<u>Exercise Answers</u> (Slope)

1. a) $\frac{2}{5}$

b) $-\frac{3}{3} = -1$

c) $-\frac{2}{3}$

d) $-\frac{2}{2} = -1$

2. −, +, −, −, −

3. All slopes = −2

Equal

Segments

Segments

4. a) 3

b) $-\frac{2}{5}$

c) $\frac{4}{7}$

5. a) 0

   b) 0

   c) 0

   d) 0

6. a) $\frac{5}{0}$ = ? slope undefined

   b) $\frac{4}{0}$ = ? slope undefined

7. a) (6, 8). Other answers possible

   b) (-6, 3). Other answers possible

8. a) $\frac{3}{4}$

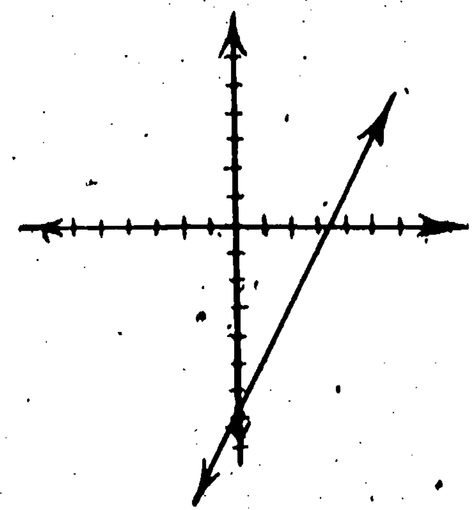
   b) -4

Exercise Answers (Slope of Parallels and Perpendiculars)

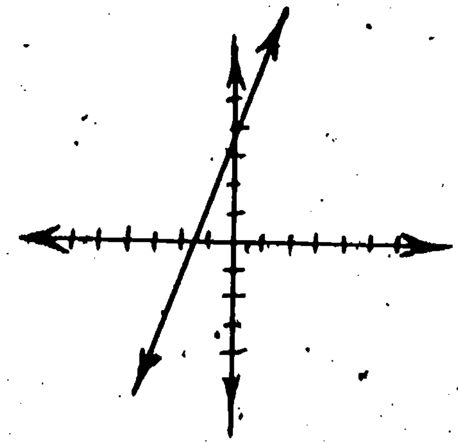
1. 0
2. 4
3. 0
4. 4
5. $\overline{AB} \parallel \overline{DC}$ or $\overline{BD} \parallel \overline{AC}$
6. $\overline{WX} \parallel \overline{YZ}$ or $\overline{XY} \parallel \overline{ZW}$
7. $\overline{XY} \perp \overline{WZ}$
8. $\overline{NM} \perp \overline{PM}$

Exercise Answers (Equation of a Line)

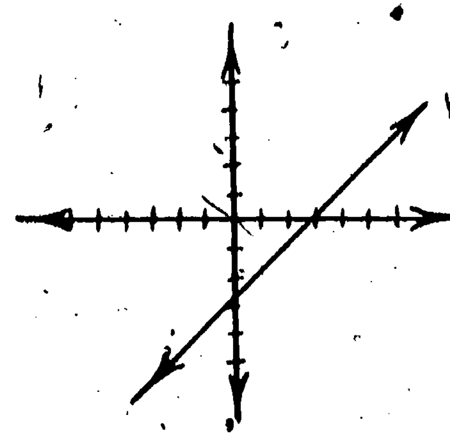
1. $y = 2x - 7$



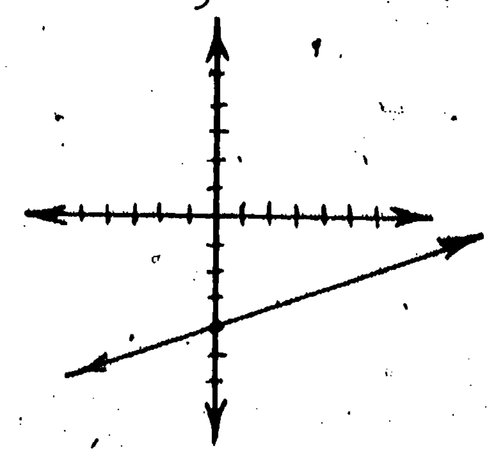
3. $y = 3x + 4$



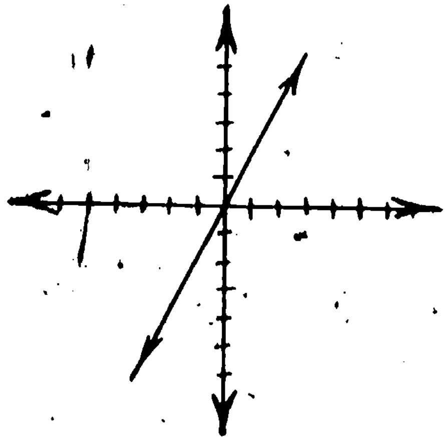
2. $y = x - 3$



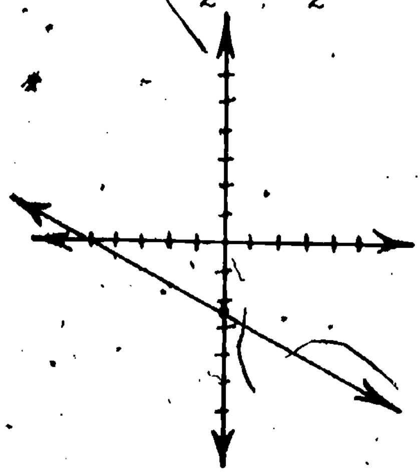
4. $y = \frac{1}{3}x - 4$



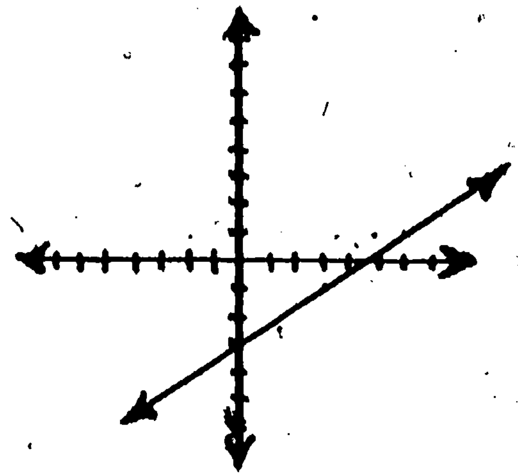
5. $y = 2x$



6. $y = \frac{1}{2}x - \frac{5}{2}$



7. $y = \frac{2}{3}x - 3$



8. $4y + 3x + 22 = 0$

9. $y + 2x + 8 = 0$

10. $3y + x - 11 = 0$

11. $3y - 5x - 15 = 0$

12. $y - x + 5 = 0$

13. $y + x = 0$

14. $y - 2x = 0$

15. $3y + 4x - 68 = 0$

16. $y - 6x = 0$
    $y - x - 1 = 0$

Selecting Coordinates

Teaching Suggestions

1. Work through the two examples with the class. Point out distance formula used to show line segments congruent, and slope used to show lines parallel.
2. The eight figures can be shown by overhead transparency for the whole class or by individual copies.
3. Instruct students to support each answer to a "check list" question, using distance formula, slope formula, etc.
4. Answers to the activity:

1.	Equilateral	5.	Parallelogram
2.	Rectangle	6.	Parallelogram
3.	Scalene	7.	Right
4.	Quadrilatera	8.	Square

5. You may want to use Exercises 1-6 as a class discussion. Exercises 7-15 can be assigned to individuals or small groups. Answers for the exercises will vary.

Coordinate Proofs

Teaching Suggestions

1. The activity can be done individually or in small groups.

2. Answer to Exercise 1

✓	✓	✓	✓
✓	✓	✓	✓
✓	✓	✓	✓
✓	✓	✓	✓
	✓		✓
		✓	✓
		✓	✓
		✓	✓
	✓		✓
		✓	✓
			✓

3. Answer to Exercise 2

			✓
			✓
			✓
			✓
	✓		✓
	✓		✓
		✓	✓
	✓		✓
✓	✓	✓	✓

Linear Inequalities
Non-negative Constraints
Linear Programming

These three sections wind up the coordinate geometry with three exercise sets which may help answer the old, most often asked question, "How am I ever going to use this stuff?"

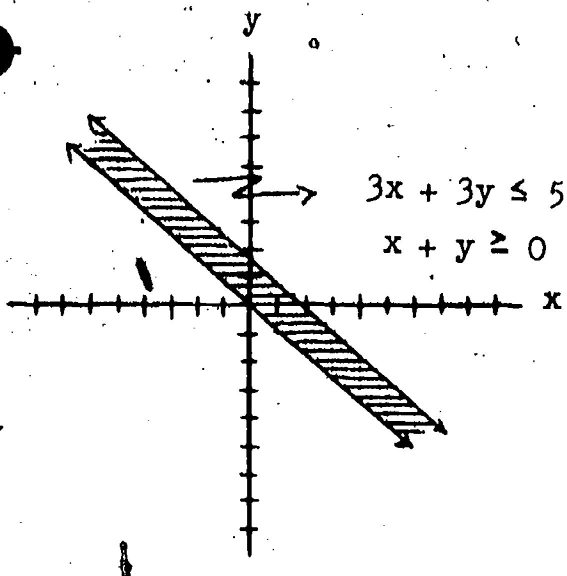
Linear programming has many applications as aptly pointed out by the exercise references.

You might want to make up a set of transparencies to illustrate a step-by-step procedure for a couple of examples.

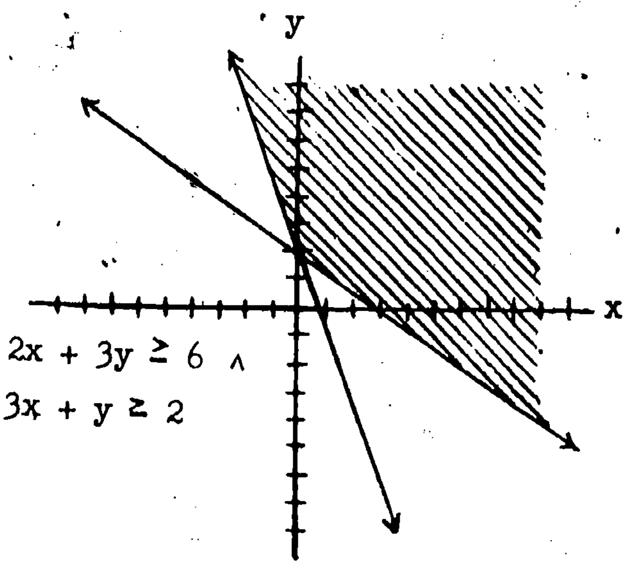
Graph paper is a must.

Insist that students use straight edges for drawing lines, and label their illustrations clearly.

Exercise Answers



2.



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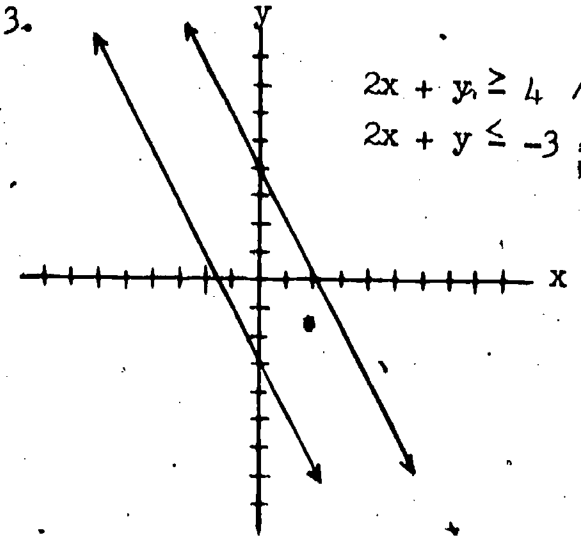

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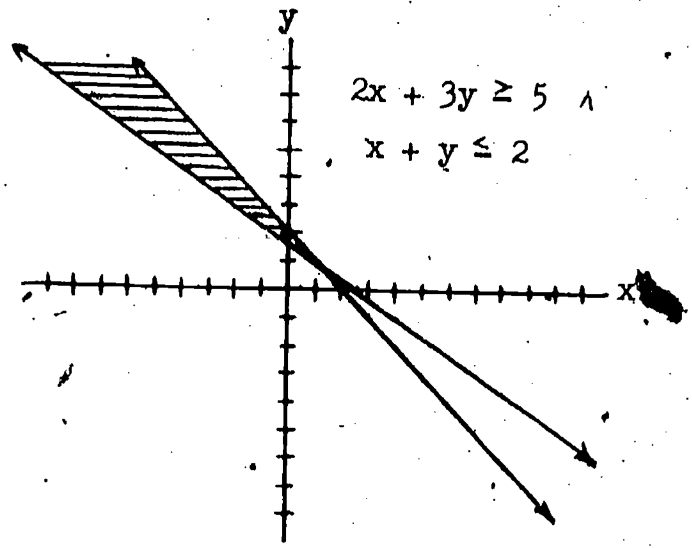

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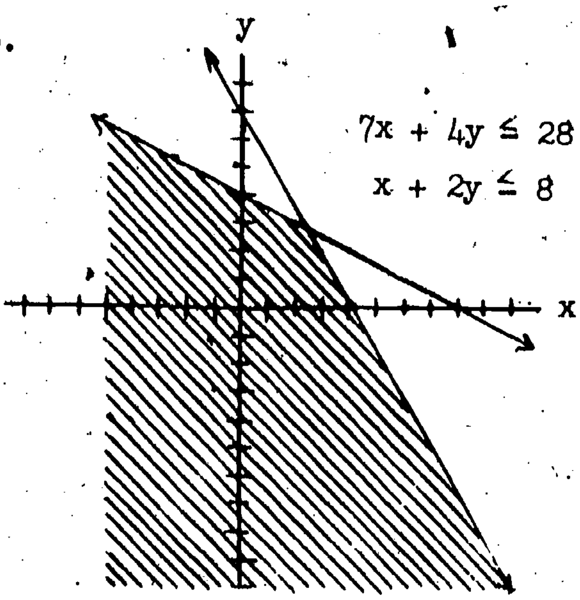

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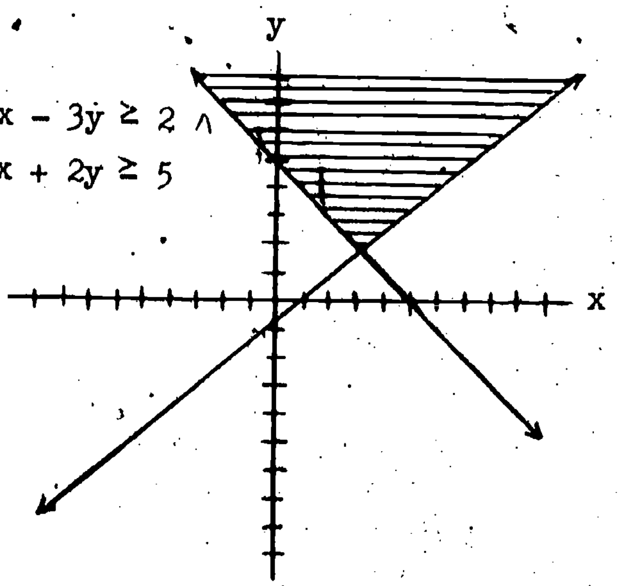

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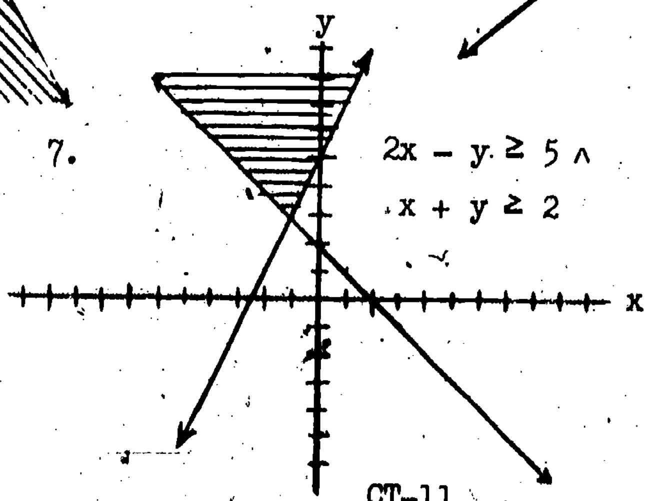

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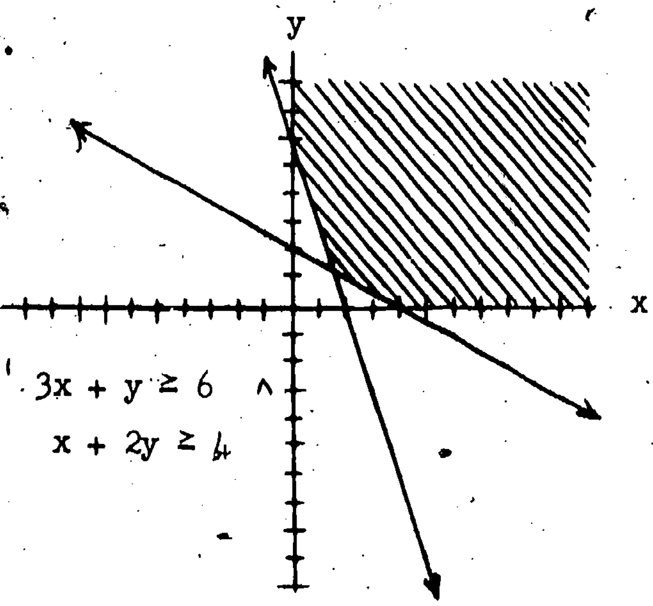
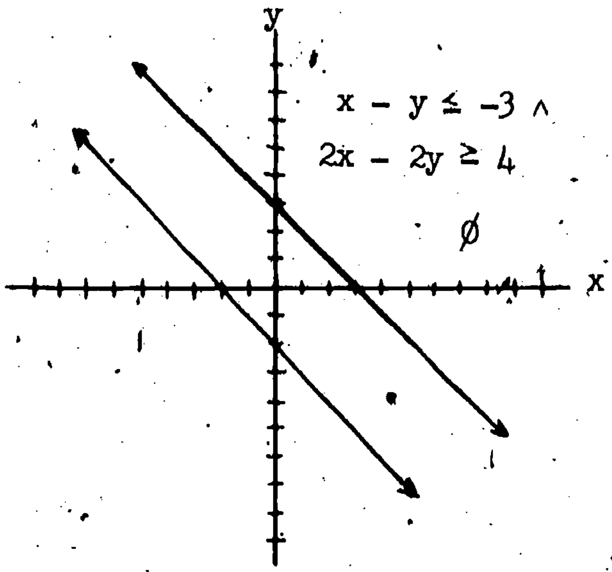
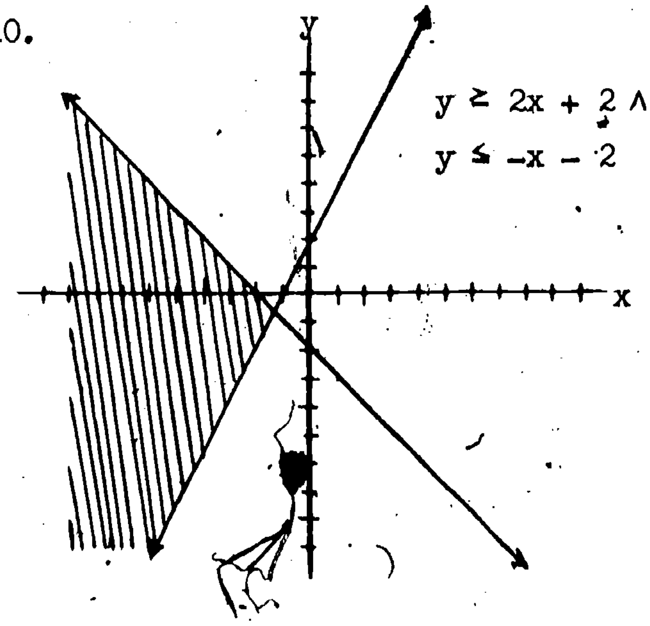

y
x
$3x + y \geq 6 \wedge$
$x + 2y \geq 4$

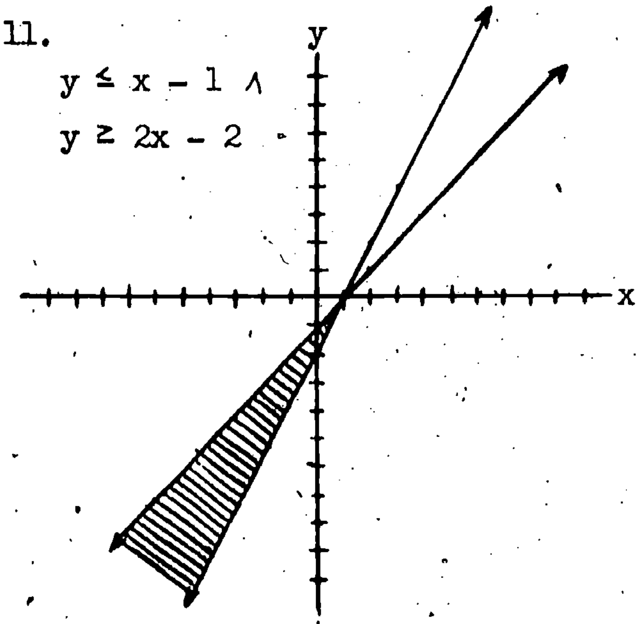
9.


y
x
$x - y \leq -3 \wedge$
$2x - 2y \geq 4$
$\emptyset$

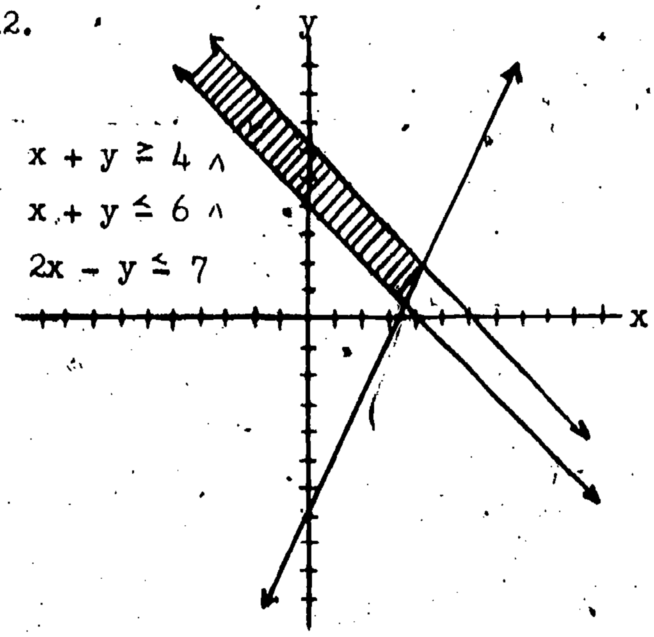
10.


y
x
$y \geq 2x + 2 \wedge$
$y \leq -x - 2$

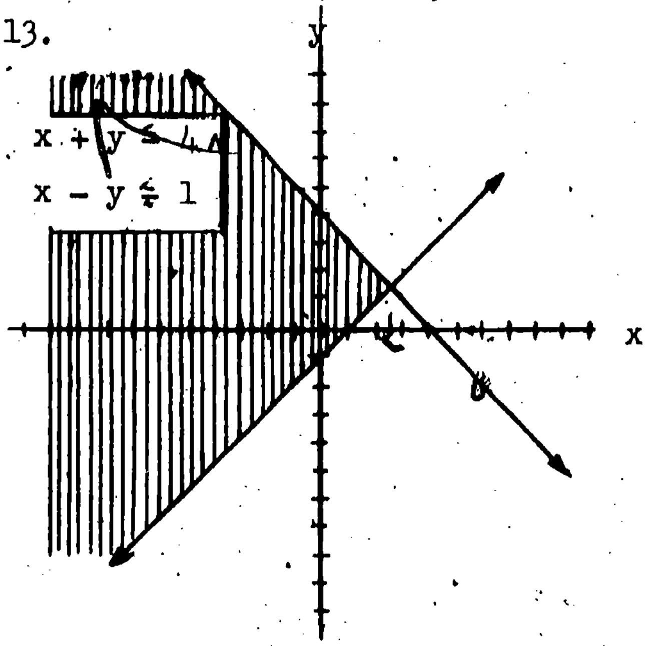
11.


y
x
$y \leq x - 1 \wedge$
$y \geq 2x - 2$

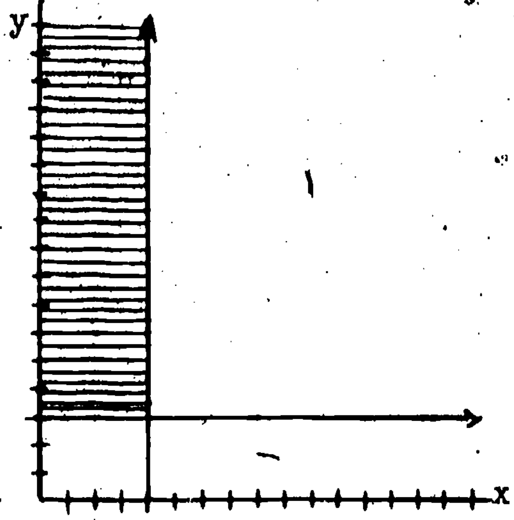
12.


y
x
$x + y \geq 4 \wedge$
$x + y \leq 6 \wedge$
$2x - y \leq 7$

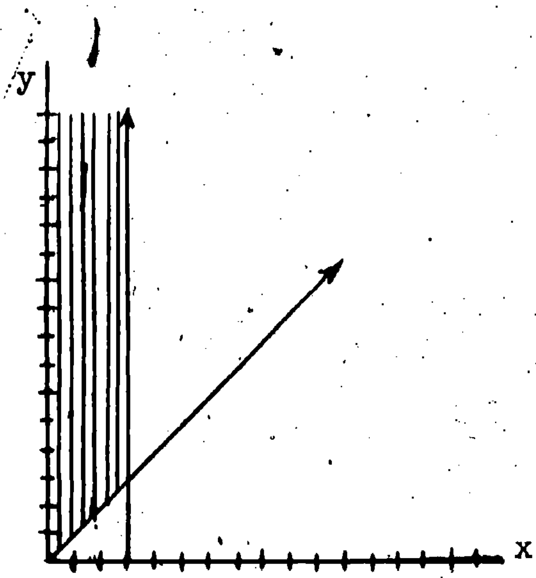
13.


y
x
$x + y \leq 4 \wedge$
$x - y \leq 1$

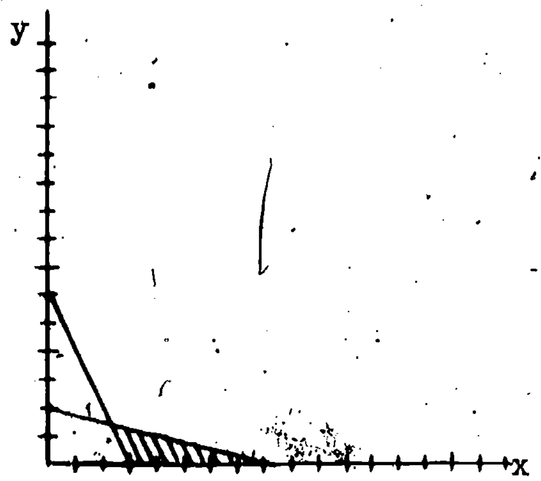
Exercise Answers



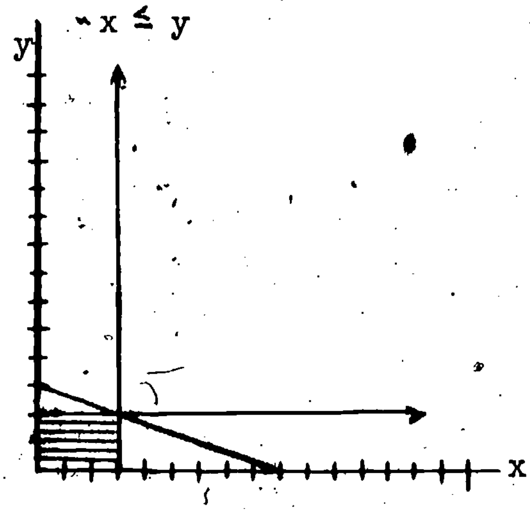
1. $x \geq 0$
   $y \geq 0$
   $y \geq 3$
   $x \leq 4$



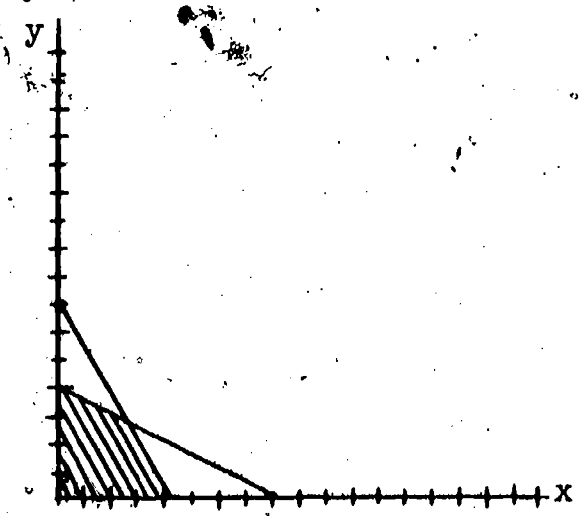
2. $x \geq 0$
   $y \geq 0$
   $x \leq 3$
   $x \leq y$



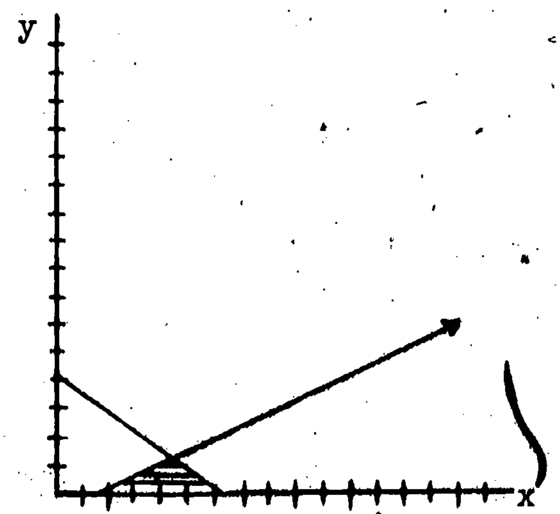
3. $x \geq 0$
   $y \geq 0$
   $2x + y \geq 6$
   $x + 4y \leq 8$



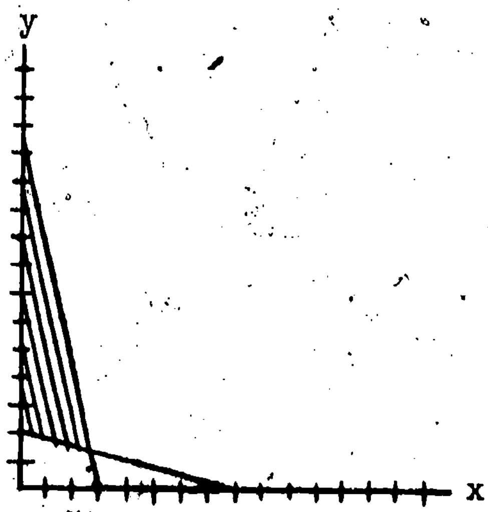
4. $x \leq 3$ $\quad$ $x \geq 0$
   $y \leq 2$ $\quad$ $y \geq 0$
   $x + 3y \leq 9$



5. $x \geq 0$
   $y \geq 0$
   $7x + 4y \leq 28$
   $x + 2y \leq 8$



6. $x \geq 0$
   $y \geq 0$
   $2x + 3y \leq 12$
   $x - 2y \geq 2$

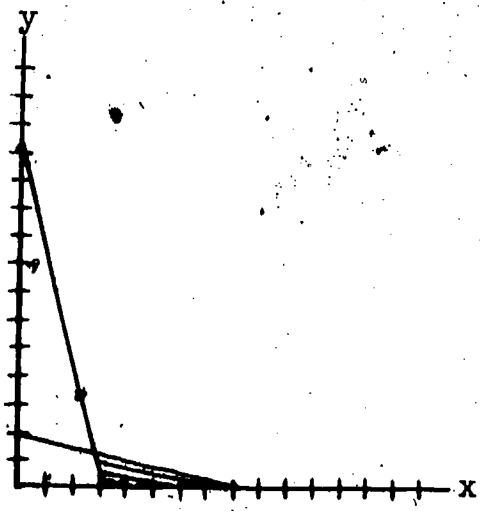


7. $x \geq 0$

$y \geq 0$

$4x + y \leq 12$

$x + 4y \geq 8$

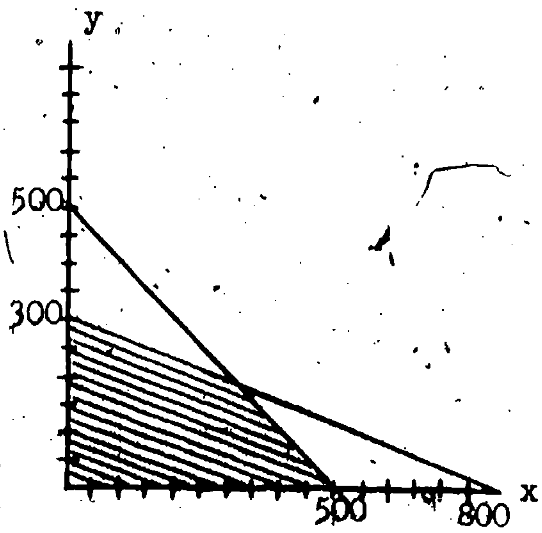


8. $x \geq 0$

$y \geq 0$

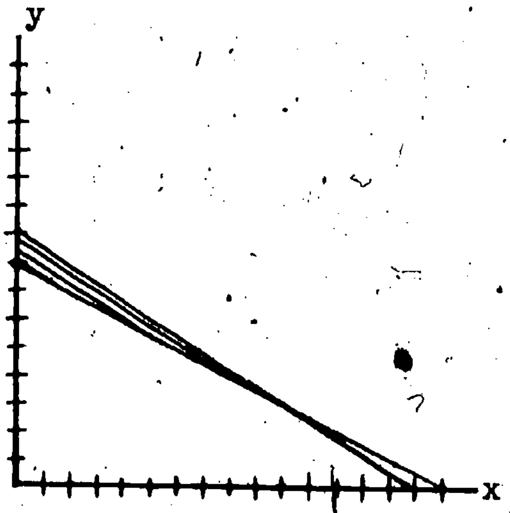
$4x + y \geq 12$

$x + 4y \leq 8$



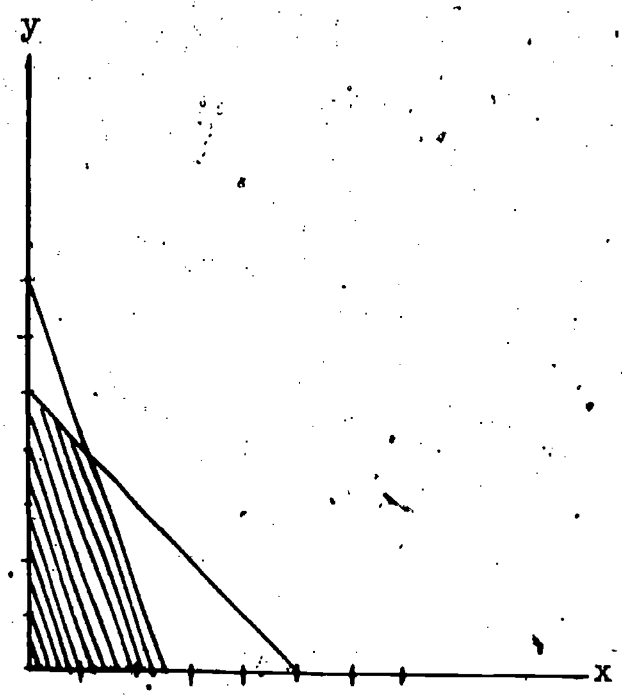
9. $x + y \leq 500$

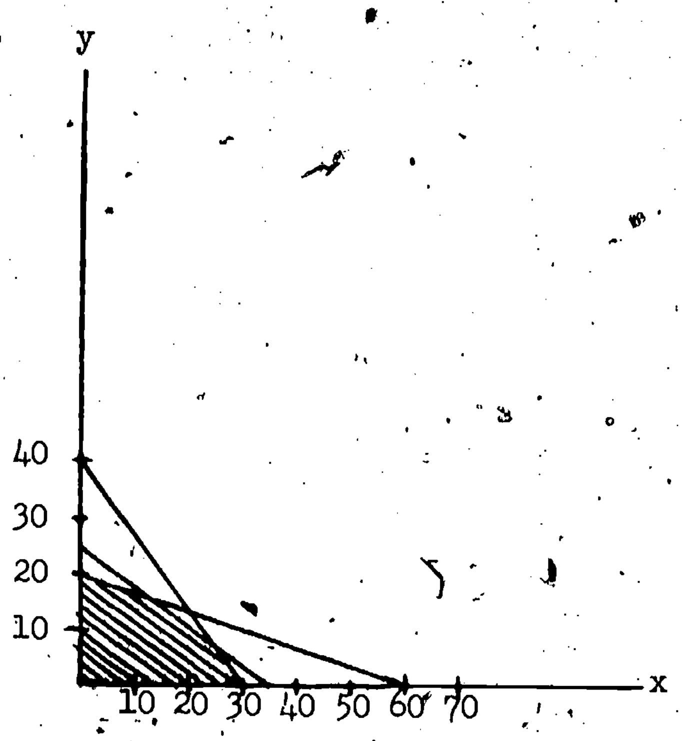
$3x + 8y \leq 2400$



10. $.5x + y \geq 8$

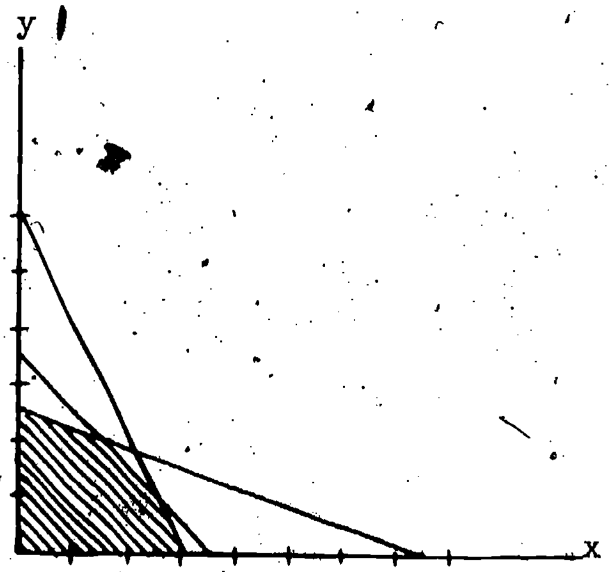
$3x + 5y \leq 45$

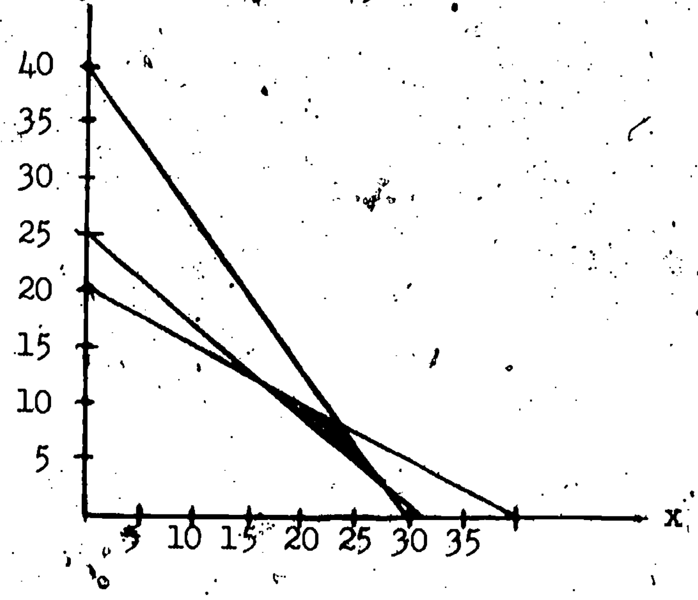




11. $x + y \leq 5$
$2x + y \leq 7$

12. $4x + 3y \leq 120$
$x + 3y \leq 60$
$7x + 5y \leq 175$





13. $2x + y \leq 12$
$x + y \leq 7$
$x + 3y \leq 15$

14. $4x + 3y \leq 120$
$x + 2y \leq 40$
$3x + 4y \leq 100$

REVIEW RADICALS

Simplifying Radicals

I. Never leave a perfect square factor under a radical.

Example: $\sqrt{50} = \sqrt{25 \cdot 2} = \sqrt{25}\ \sqrt{2} = 5\sqrt{2}$

II. Never leave a fraction under a radical.

Example: $\sqrt{\frac{1}{8}} = \sqrt{\frac{1}{8} \cdot \frac{2}{2}} = \sqrt{\frac{2}{16}} = \frac{\sqrt{2}}{\sqrt{16}} = \frac{\sqrt{2}}{4}$ or $\frac{1}{4}\sqrt{2}$

III. Never leave a radical in a denominator.

Example: $\frac{3}{\sqrt{2}} = \frac{3}{\sqrt{2}} \cdot \frac{\sqrt{2}}{\sqrt{2}} = \frac{3\sqrt{2}}{\sqrt{4}} = \frac{3\sqrt{2}}{2}$ or $\frac{3}{2}\sqrt{2}$

Exercises

Simplify:

1. $\sqrt{36} =$
2. $\sqrt{12} =$
3. $\sqrt{128} =$
4. $\sqrt{27x^2} =$
5. $\sqrt{\frac{2}{9}} =$
6. $\sqrt{\frac{2}{8}} =$
7. $\sqrt{4\frac{1}{2}} =$
8. $\sqrt{\frac{5}{12}} =$
9. $\sqrt{48} =$
10. $\sqrt{81x^3} =$
11. $\sqrt{40} =$
12. $\sqrt{\frac{20}{9}} =$
13. $\sqrt{24} =$
14. $\sqrt{\frac{1}{2}} =$
15. $\sqrt{32} =$
16. $\frac{3}{\sqrt{3}} =$
17. $\sqrt{13} =$
18. $3\sqrt{8} =$
19. $\frac{7}{\sqrt{7}} =$
20. $\frac{2}{\sqrt{5}} =$
21. $\frac{10}{\sqrt{2}} =$
22. $\sqrt{\frac{30}{15}} =$
23. $5\sqrt{18} =$
24. $\sqrt{26} =$
25. $\sqrt{\frac{5}{8}}\, y =$
26. $9\sqrt{20} =$
27. $\sqrt{72} =$
28. $\sqrt{\frac{1}{9}} =$
29. $7\sqrt{\frac{1}{16}}$
30. $\frac{1}{\sqrt{2}}$

Multiplying and Dividing Radicals

Rule: $\sqrt{a} \cdot \sqrt{b} = \sqrt{ab}$ Example: $\sqrt{2} \cdot \sqrt{3} = \sqrt{6}$

Rule: $\frac{\sqrt{a}}{\sqrt{b}} = \sqrt{\frac{a}{b}}$ Example: $\frac{\sqrt{15}}{\sqrt{3}} = \sqrt{5}$

Exercises

1. $\sqrt{2} \cdot \sqrt{5} =$
2. $\sqrt{10} \div \sqrt{2} =$
3. $3\sqrt{7} \cdot 4\sqrt{2} =$
4. $(9\sqrt{2} \cdot 3\sqrt{5}) \div 6\sqrt{5} =$
5. $\frac{1}{2}\sqrt{5} \cdot 3\sqrt{6} =$
6. $\sqrt{12} \cdot \sqrt{\frac{3}{4}} =$
7. $\frac{1}{2}\sqrt{8} \cdot \frac{2}{5}\sqrt{2} =$
8. $(8\sqrt{7})^2 =$
9. $(11\sqrt{3})^2 =$
10. $(\sqrt{14})^2 =$

Adding Radicals

Rule: $x\sqrt{a} + y\sqrt{a} = (x + y)\sqrt{a}$

(If radicals are not equal, addition is not defined.)

Example: $3\sqrt{2} - 5\sqrt{7} + 1\sqrt{2} = 4\sqrt{2} - 5\sqrt{7}$

1. $8\sqrt{5} - 2\sqrt{3} + 5\sqrt{5} + 5\sqrt{3} =$
2. $7\sqrt{2} + 8\sqrt{2} + 5\sqrt{10} =$
3. $-4\sqrt{11} - 9\sqrt{11} =$
4. $19\sqrt{25} + \sqrt{25} =$
5. $12\sqrt{3} - 11\sqrt{2} =$
6. $5\sqrt{13} - 8\sqrt{13} =$
7. $\sqrt{13} + \sqrt{9} =$
8. $2\sqrt{16} - 9 =$
9. $2\sqrt{3} + 5\sqrt{3} - 9\sqrt{3} =$
10. $8\sqrt{16} - \sqrt{16} =$

A. Review of Radicals

1. $\sqrt{32} + \sqrt{27} - \sqrt{48} =$

2. $\frac{10}{\sqrt{5}} - \frac{6}{\sqrt{20}} - \sqrt{\frac{1}{5}} =$

3. $\sqrt{\frac{2}{3}} + \sqrt{\frac{3}{2}} + 2\sqrt{\frac{1}{24}} =$

4. $\sqrt{54} \div \sqrt{6} =$

5. $\sqrt{36} \div \sqrt{6} =$

6. $6 \div \sqrt{6} =$

7. $2\sqrt{2} \div 6\sqrt{6} =$

8. $\sqrt{31} \cdot \sqrt{31} =$

9. $4\sqrt{2} \cdot 5\sqrt{6} =$

10. $2 \div \sqrt{\frac{2}{5}} =$

B. Review of Radicals

1. $\sqrt{24} + \sqrt{18} - \sqrt{54} =$

2. $\frac{1}{\sqrt{2}} + \sqrt{\frac{25}{2}} + \sqrt{\frac{2}{9}} =$

3. $8\sqrt{\frac{3}{4}} - \frac{1}{2}\sqrt{12} + 2\sqrt{1\frac{1}{2}} =$

4. $\sqrt{45} \div \sqrt{5} =$

5. $\sqrt{25} \div \sqrt{5} =$

6. $5 \div \sqrt{5} =$

7. $3\sqrt{2} - 4\sqrt{6} =$

8. $\sqrt{28} \cdot \sqrt{28} =$

9. $2\sqrt{3} \cdot 5\sqrt{6} =$

10. $3 \div \sqrt{\frac{3}{4}} =$

C. Review of Radicals

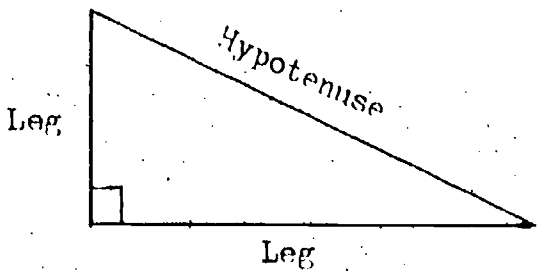
1. $\sqrt{3} \cdot \sqrt{6} =$
2. $\sqrt{\frac{3}{2}} \cdot \sqrt{\frac{4}{3}} =$
3. $\sqrt{5} \cdot \sqrt{15} =$
4. $\sqrt{12} \cdot \sqrt{\frac{2}{3}} \cdot \sqrt{\frac{3}{4}} =$
5. $6 \div 2\sqrt{3} =$
6. $7 \div \sqrt{7} =$
7. $\sqrt{\frac{3}{5}} \cdot \sqrt{1\frac{2}{3}} =$
8. $2\sqrt{5} \cdot 3\sqrt{2} =$
9. $3 \div \sqrt{12} =$
10. $8 \cdot 3\sqrt{2} =$
11. $\sqrt{13} \cdot \sqrt{26} =$
12. $\sqrt{27} - \sqrt{75} + \sqrt{48} =$
13. $\sqrt{20} + \sqrt{45} - \sqrt{25} =$
14. $2\sqrt{50} - 3\sqrt{8} + 4\sqrt{\frac{1}{8}} =$
15. $\sqrt{3} - \sqrt{\frac{3}{4}} - \sqrt{\frac{1}{3}} =$
16. $\sqrt{80} + \sqrt{75} =$
17. $\sqrt{\frac{1}{2}} + \sqrt{2\frac{1}{2}} + \sqrt{4\frac{1}{2}} =$
18. $4\sqrt{27} - 8\sqrt{81} =$
19. $-5\sqrt{75} - 8\sqrt{27} =$
20. $\sqrt{\frac{1}{3}} + \sqrt{1\frac{1}{3}} + \sqrt{2\frac{2}{3}} =$

PYTHAGOREAN THEOREM

Activity I

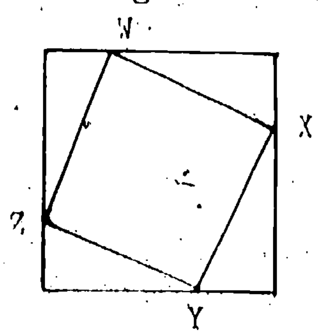
Review

A right triangle is a triangle with one right angle. The sides that form the right angle are called legs. The side opposite the right angle is the hypotenuse.



Construct any right triangle. Let a and b be the lengths of the legs and c be the length of the hypotenuse. Construct three more triangles congruent to the first. Construct a square with side a + b. Cut out the four triangles.

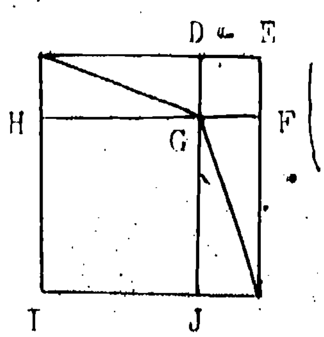
I. Arrange the triangles in this pattern on the square.



1. What is the measure of $\angle$WXY? Why?
2. How do you know WXYZ is a square?
3. What is the area of WXYZ?
   (Record your answer in the box provided.)

There are many different non-overlapping patterns in which you could place your four triangles and cover part of the square region, yet in each case shouldn't the areas of the uncovered regions be equal?

II. Now try this pattern:



1. What is the measure of $\angle$ DGF? Why?
2. Is DEFG a square?
3. What is the measure of $\angle$ HGJ? Why?
4. Is GHIJ a square?
5. Find area of DEFG. (Record answer below.)
6. Find area of GHIJ. (Record answer below.)

Area WXYZ = Area DEFG + Area GHIJ. Why?

Record answers here $\square$ = $\square$ + $\square$

This demonstration was to show you that for any right triangle with legs length a and b and hypotenuse length c, $a^2 + b^2 = c^2$. This is known as the Pythagorean Theorem. There are many ways to prove this famous theorem: algebraically, one attributed to Pythagoras, even one done by President Garfield.

Write the converse of the Pythagorean Theorem:

Exercises

Example: Given a right triangle with legs measuring 3 cm and 4 cm, find the length of the hypotenuse.

By Pythagorean Theorem $a^2 + b^2 = c^2$

In our triangle $3^2 + 4^2 = c^2$

$9 + 16 = c^2$

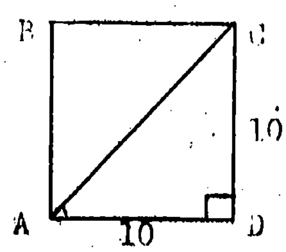
$25 = c^2$

The hypotenuse is $5 = c$.

In the following table, a and b are lengths of the legs and c is the length of the hypotenuse of a right triangle. Complete the table. Simplify all radicals.

	a	b	c
1.	6	8	
2.	4	4	
3.		8	17
4.	1	$\sqrt{3}$	
5.	$\sqrt{3}$		5
6.	$2\sqrt{2}$		6
7.	3		6
8.	3	6	

9. In the square shown, $\overline{AC}$ is a diagonal. Find the length of the diagonal if the square has sides of length 10.

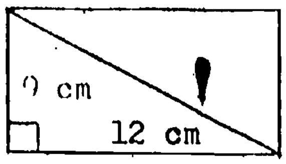


10. Find the length of the diagonal of a square if the square has sides 6.

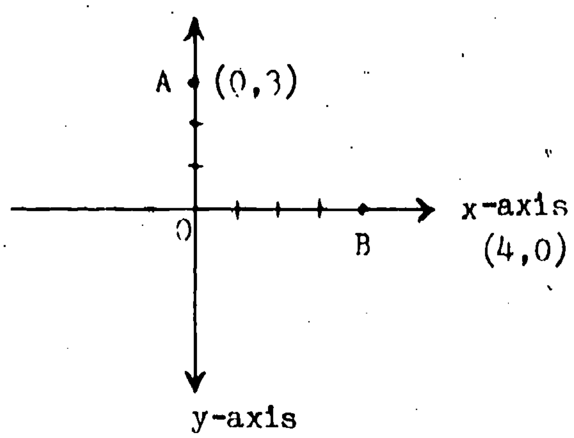
11. Find the length of the sides of a square if the diagonal has length $2\sqrt{3}$.

12. Find the length of the sides of a square if the diagonal has length $5\sqrt{2}$.

13. Find the length of the diagonal of a rectangle of dimensions 9 cm x 12 cm.



Example: Recall from algebra that you can locate a point with respect to coordinate axes by an ordered pair of real numbers. For example: (0, 3) has x-coordinate 0 and y-coordinate 3. This is point A shown below.



(You will have a complete review of working with coordinate axes later.)

Point B with coordinate (4, 0) is also shown. Draw AB, and you have a right triangle AOB. What is the length of leg OA? of leg OB?

14. Use the Pythagorean Theorem to find the length of hypotenuse AB.

15. Find the distance from point E (0, -4) to point F (3, 0).

16. Find the distance from point C (0, 5) to point D (5, 0).

17. Find the distance from G (0, 6) to H (2, 0).

18. Given are the measures of three line segments. Use the converse of the Pythagorean Theorem to determine whether or not the segments could form a right triangle.

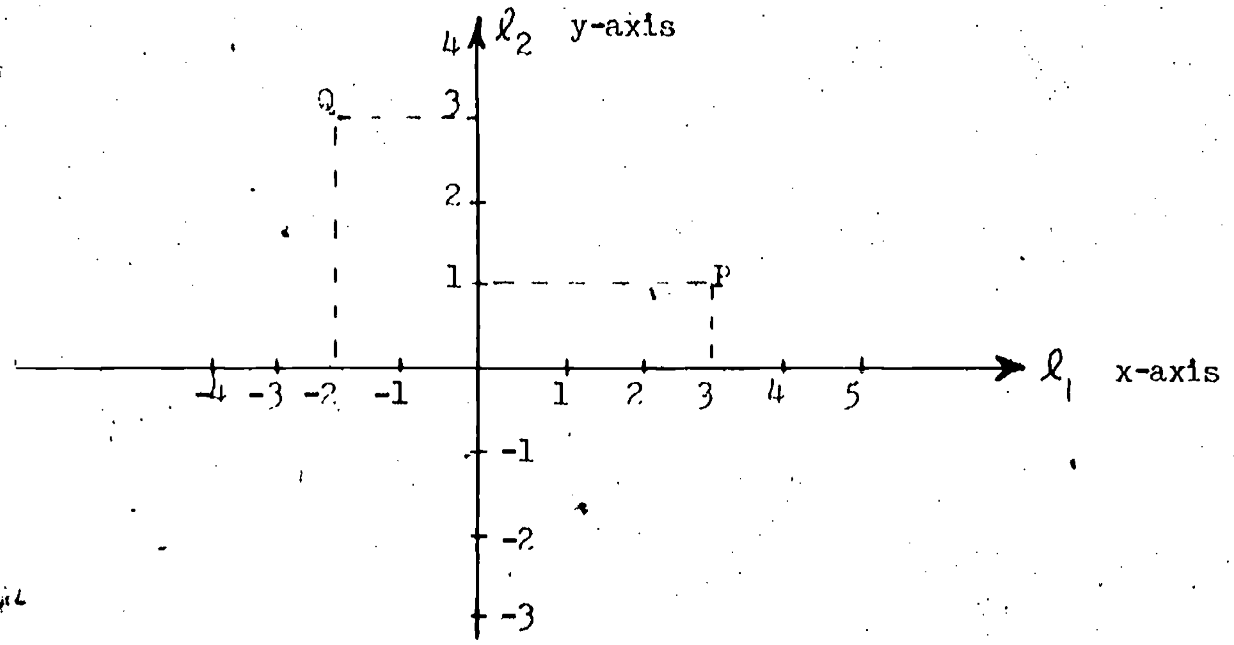
(a) 5, 13, 12 (c) 4, 5, 6 (e) 2, 4, $2\sqrt{3}$

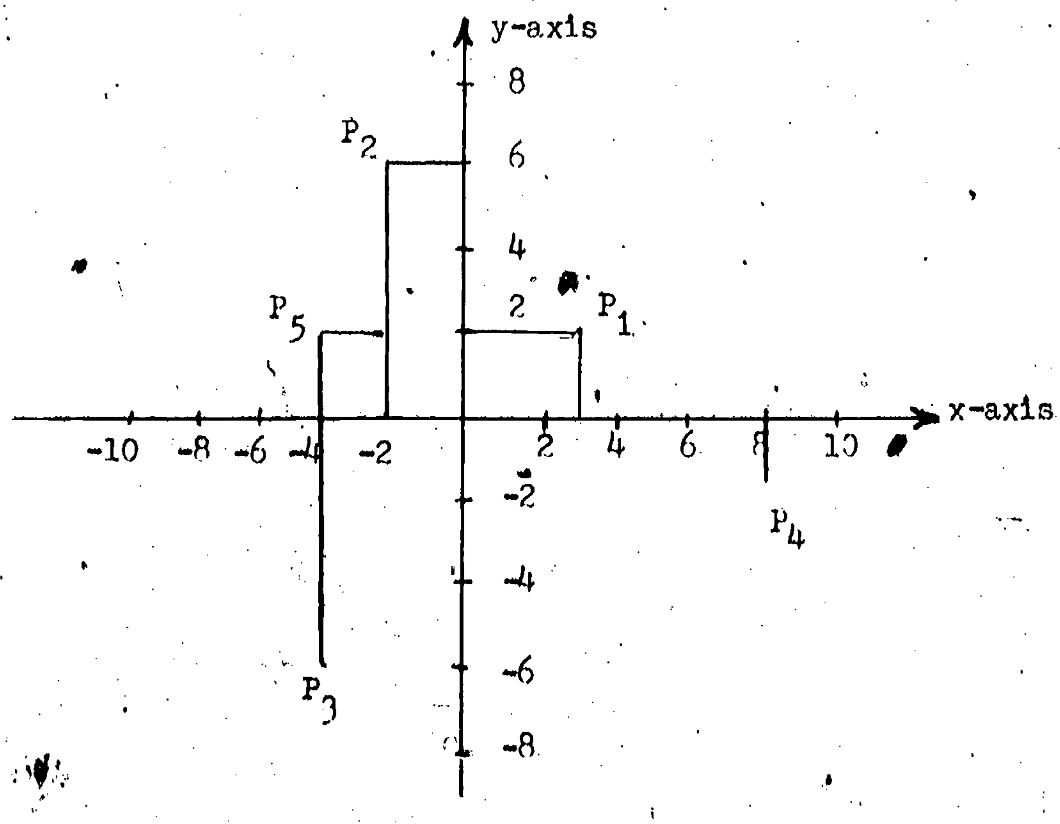
(b) 3, 3, $3\sqrt{2}$ (d) 7, 14, $7\sqrt{3}$

DISTANCE AND MIDPOINT

Activity II

Let ℓ_1 and ℓ_2 be perpendicular lines in a plane. The point of intersection we shall call the origin. ℓ_1 and ℓ_2 in the figure below will be the x-axis and y-axis respectively. The x-coordinate of a point P is the coordinate on ℓ_1 (x-axis) of the foot of the $\perp$ from P to the x-axis. The y-coordinate of a point P is the coordinate on ℓ_2 (y-axis) of the foot of the $\perp$ from P to the y-axis. The coordinates of P in the figure are the set of ordered pair of numbers (x, y) or (3, 1). Note that the x-coordinate is always written first. The coordinates of Q are (-2, 3).





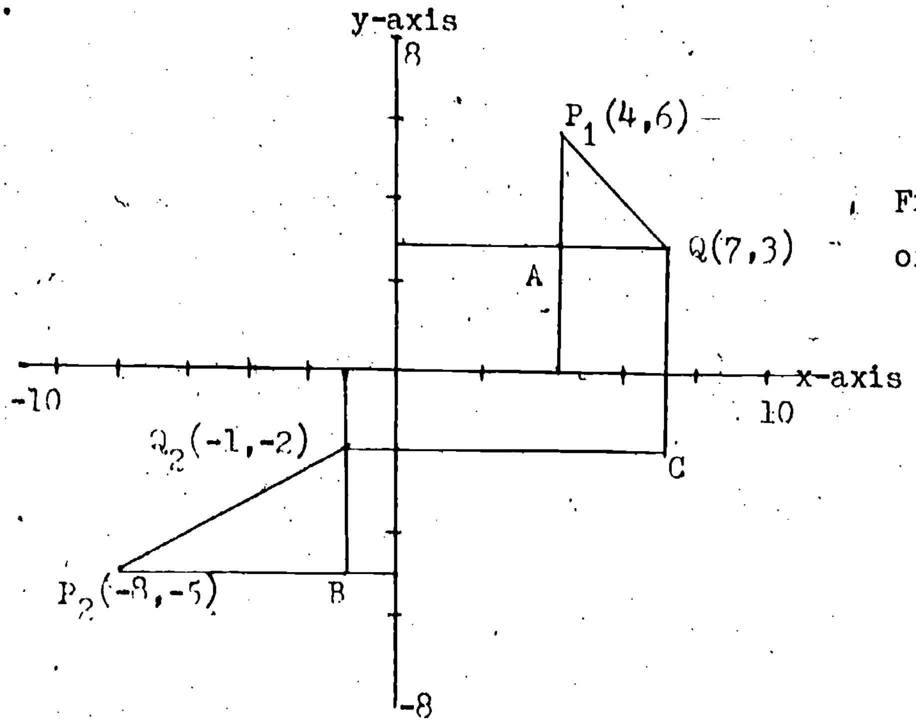
1. a) The x-coordinate of P_1 is ______.

   b) The coordinates of P_1 are ______.

   c) The y-coordinate of P_3 is ______.

   d) The coordinates of P_4' are ______.

2.



Find the coordinates of A, B, and C.

A ______

B ______

C ______

3. Use the figure in Exercise 2 to find the distance from A to Q_1.

AQ_1 = ______

AP_1 = ______

BP_2 = ______

BQ_2 = ______

CQ_2 = ______

CQ_1 = ______

4. If you identify $P_1(4, 6)$ as $P_1(x_1, y_1)$ and $Q_1(7, 3)$ as $P_2(x_2, y_2)$ is

$AQ_1 = |x_2 - x_1|$ and $AP_1 = |y_2 - y_1|$? ______

5. Find the distance between P_1 and Q_1. Since $\overline{P_1Q_1}$ is the ______ of a right triangle, $(AQ_1)^2 + (\_\_\_)^2 = (\_\_\_)^2$.

$$(x_2 - x_1)^2 + (\_\_\_ - \_\_\_) = (P_1Q_1)^2$$

$$(7 - \_\_\_)^2 + (3 - \_\_\_)^2 = (P_1Q_1)^2$$

$$3^2 + (-3)^2 = (P_1Q_1)^2$$

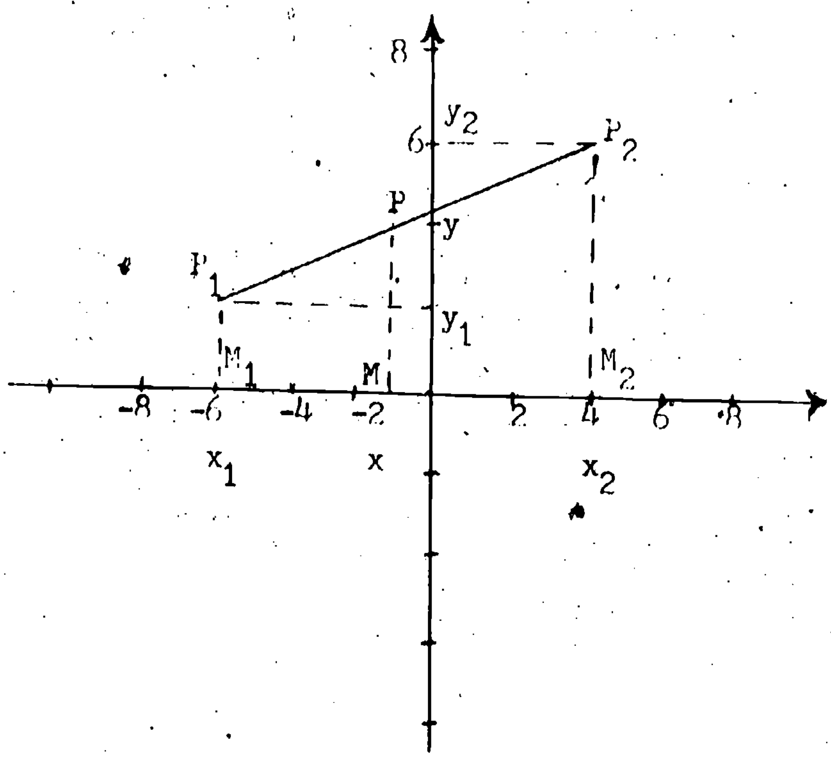
$$P_1Q_1 = \sqrt{\_\_\_}$$

$$P_1Q_1 = \_\_\_$$

6. Find the distance between P_2 and Q_2; between P_1 and P_2; between Q_1 and Q_2.

________; ________; ________

7.



The coordinates of P_1 are ________.
The coordinates of P_2 are ________.
The coordinates of P are ________.
Let M be the midpoint of M_1M_2. $(M_1M = MM_2)$

Then $M_1M = |x - x_1| = x - x_1$, and

$MM_2 = |x_2 - \_\_\_| = \_\_\_ - \_\_\_$.

$x - x_1 = x_2 - x$ or $x = (\_\_\_\_\_)$

The x-coordinate of P is ________.

In the same way, the y-coordinate of P is ______. Therefore, the midpoint of $\overline{P_1P_2}$ (Point P) has coordinates (________, ________)

8. a) P_1 has coordinates of (1, -1), P_2 has coordinates (-5, -4). The x-coordinate of the midpoint is _____. The y-coordinate of the midpoint is ________.

b) $P_1(x_1, y)$ and $P_2(x_2, y_2)$ are the endpoints of segment P_1P_2. P is the midpoint of $\overline{P_1P_2}$. P has coordinates of (______, _______).

9. The midpoint of a segment is (-1, 4). One endpoint of the segment is (3, -5). The coordinates of the other endpoint are to be found.

Since $x = \dfrac{x_1 + x_2}{2}$, $-1 = \dfrac{\_\_\_ + 3}{2}$

or $x_1 = \_\_\_$.

$y = (\_\_\_\_\_\_)$, $y_1 = \_\_\_\_\_\_ - y_2$

$y_1 = 8 - (\_\_\_)$

$y_1 = \_\_\_\_$

Coordinates of $P_1(x_1, y_1)$ are (_____, _____).

Exercises - Distance and Midpoint

1. Show that a triangle with vertices R(0,0), S(3,4) and T(-1,1) is an isosceles triangle.
2. Use the converse of the Pythagorean Theorem to prove that the triangle P(-6,2) Y(5,-1), T(4,4) is a right triangle.
3. The vertices of QUAD are Q(4,-3), U(7,10), A(-8,2) and D(-1,5). Find the lengths of the diagonals.

4. Find the distance between these two points:

   a) (8,11), (15,35)

   b) (-6,3), (4,-2)

5. A(-1,6), B(1,4), C(7,-2). Show that A,B and C are collinear.

6. For A(r,s), B(r + 2s, s+ 2r) and Q(0,0), show that AB = 2(AQ)

7. Use the distance formula to show that A(1,1+b), B(3,3+b) and C(6,6+b) are collinear.

8. State a formula for the distance between points in space — $P_1(x_1,y_1,z_1)$ $P_2(x_2,y_2,z_2)$.

9. Find the distance between these two points:

   a) (4, -1, -5), (7, 3, 7)

   b) (3, 0, 7), (-1, 3, 7)

10. Show that a triangle with vertices A(2, 4, 1), B(1, 2, -2) and C(5, 0, -2) is a right triangle.

11. Find the midpoint of $\overline{PQ}$.

    a) P(-2, 3) Q(1, 4)

    b) P(-5, -2) Q(-5, 6)

    c) P(5, 7), Q(6, -10)

12. The coordinates of one endpoint of a segment are (4, 0), the coordinates of the midpoint are (4, 1), find the coordinates of the other endpoint.

13. A(-2, 1), B(0, 5) and C(2, -1). Find the length of each median.

14. The vertices of QUAD are Q(0, 0), U(5, 0) A(5, 4) and D(0, 4). Use the midpoint formula to show that $\overline{QA}$ and $\overline{UD}$ bisect each other.

15. The vertices of quadrilateral ABCD are A(2, 1), B(7, 4), C(4, 9) and D(-1, 6). Show that $\overline{AC}$ and $\overline{BD}$

    a) are congruent

    b) are perpendicular

    c) bisect each other.

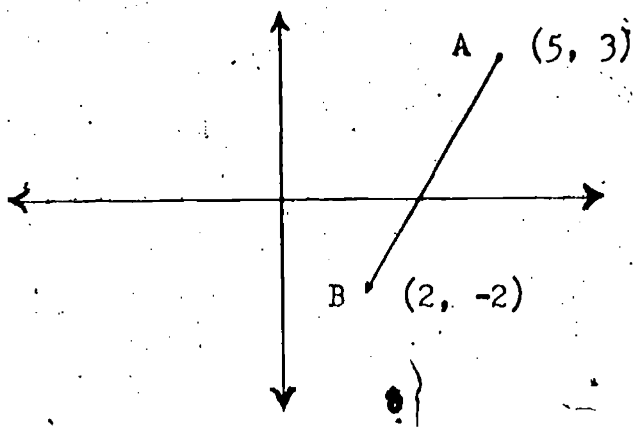
SLOPE

Activity III

The slope of a line segment $\overline{P_1 P_2}$ is defined in terms of the coordinates of the two points P_1 and P_2. Suppose the coordinates of P_1 are (x_1, y_1) and coordinates of P_2 are (x_2, y_2).

$$\text{Slope} = \frac{y_2 - y_1}{x_2 - x_1}$$

Example:



$$\text{Slope } \overline{AB} = \frac{(3) - (-2)}{(5) - (2)} = \frac{5}{3}$$

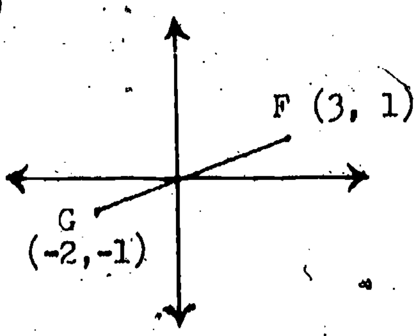
We found the slope letting P_1 be B and P_2 be A. If we let P_1 be A and P_2 be B, slope becomes:

$$\text{Slope } \overline{BA} = \frac{(-2) - (3)}{(2) - (5)} =$$

Finish calculating slope $\overline{BA}$ and compare to slope $\overline{AB}$. State a conclusion.

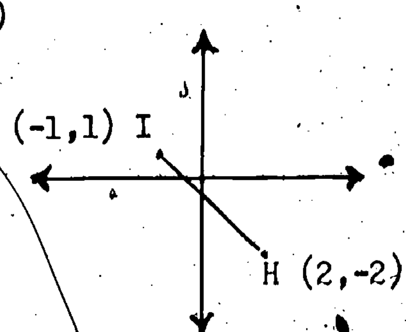
Exercises

1. a)



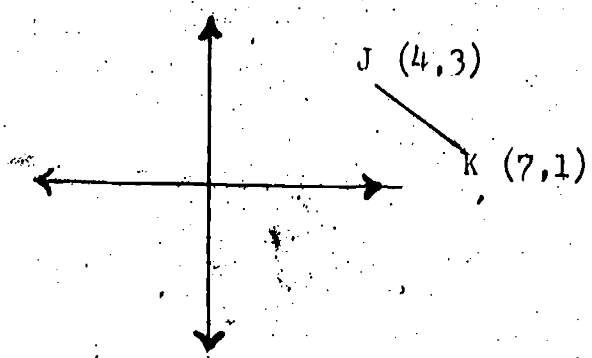
Find slope of $\overline{GF}$

b)



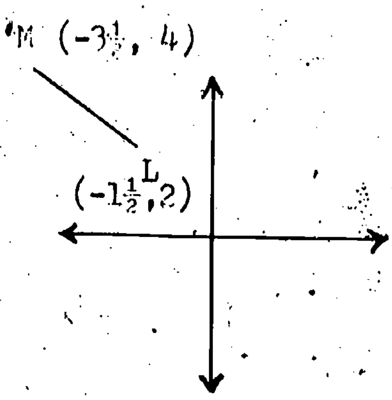
Find slope of $\overline{HI}$

c)



Find slope of $\overline{JK}$

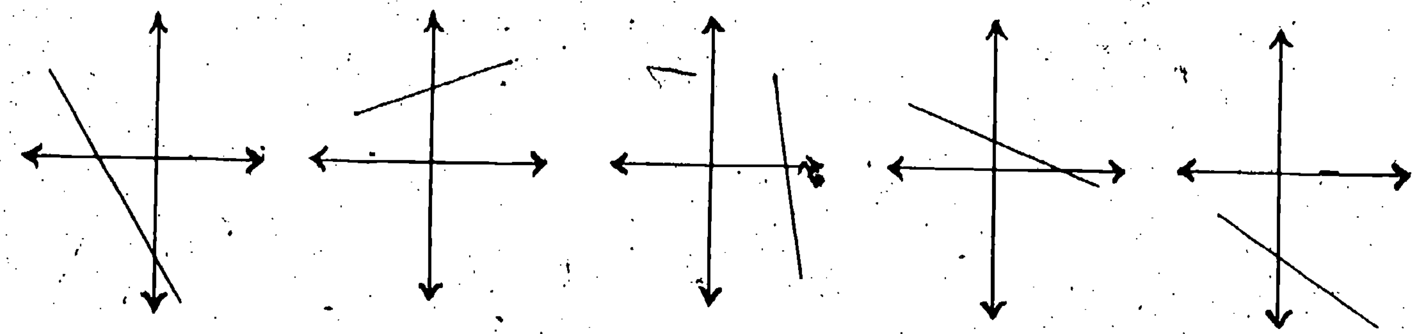
d)



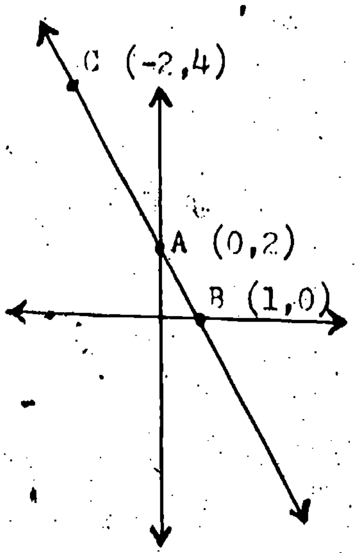
Find slope of $\overline{LM}$

Study these four problems. Can you tell by looking at a line segment whether a line segment has positive or negative slope?

2. In each example, tell whether the line segment has + or - slope.



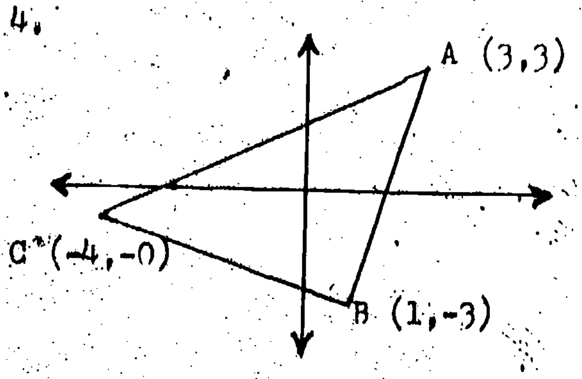
3.



a) Find slope of $\overline{AB}$.
b) Find slope of $\overline{AC}$.
c) Find slope of $\overline{BC}$.
d) Find slope of $\overleftrightarrow{AB}$.
e) Find slope of $\overrightarrow{CB}$.

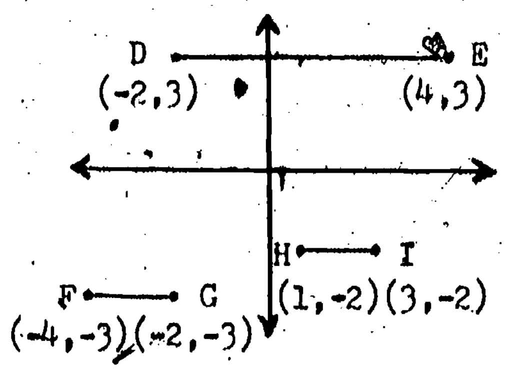
The slopes of any two segments of a given line are __________.
The slope of a line is the slope of any of its __________.
The slope of a ray is the slope of any of its __________.

4.



a) Find slope of $\overline{AB}$.
b) Find slope of $\overline{BC}$.
c) Find slope of $\overline{CA}$.

5.



a) Slope $\overline{DE}$ =

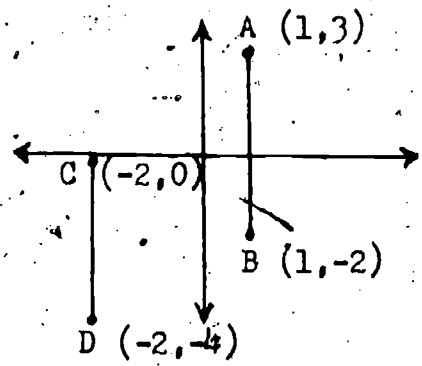
b) Slope $\overline{FG}$ =

c) Slope $\overline{HI}$ =

A segment parallel to the x-axis is said to be horizontal $\overline{DE}$, $\overline{FG}$, and $\overline{HI}$ are horizontal.

d) Horizontal segments always have slope = ________.

6.



a) Slope $\overline{AB}$ =

b) Slope $\overline{CD}$ =

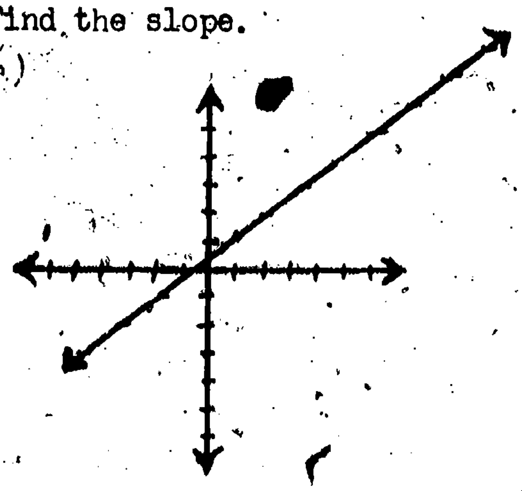
A segment parallel to the y-axis is said to be vertical. All points on a vertical line have the same x-coordinate so slope = $\frac{y_2 - y_1}{0}$. Since division by zero is undefined for the set of real numbers, we cannot express the slope of a vertical line with a real number. We say the slope of a vertical line is undefined.

7. a) (2, 5) is a point on $\overleftrightarrow{AB}$ which has slope 3/4. Name the coordinates of another point on $\overleftrightarrow{AB}$.

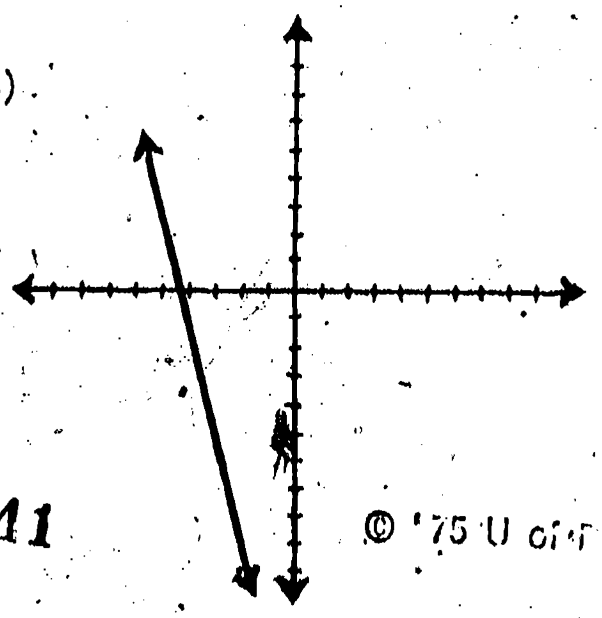
b) (-1, -1) is a point on $\overleftrightarrow{xy}$ which has slope - 4/5. Name the coordinates of another point on $\overleftrightarrow{xy}$.

8. Find the slope.

a)

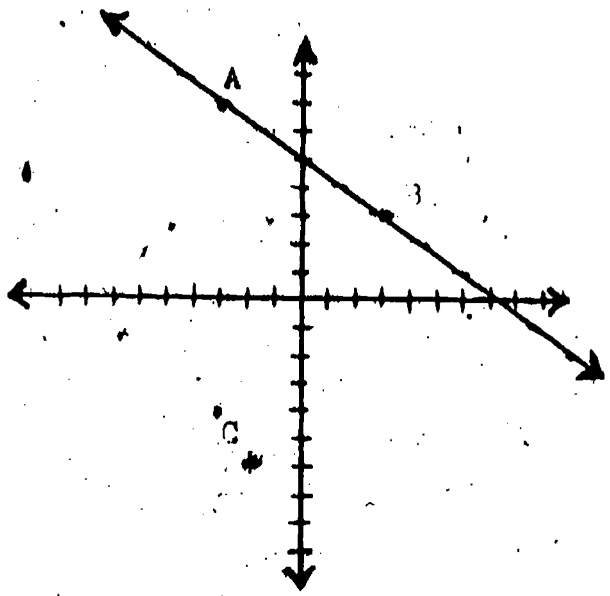


b)



SLOPE OF PARALLELS AND PERPENDICULARS

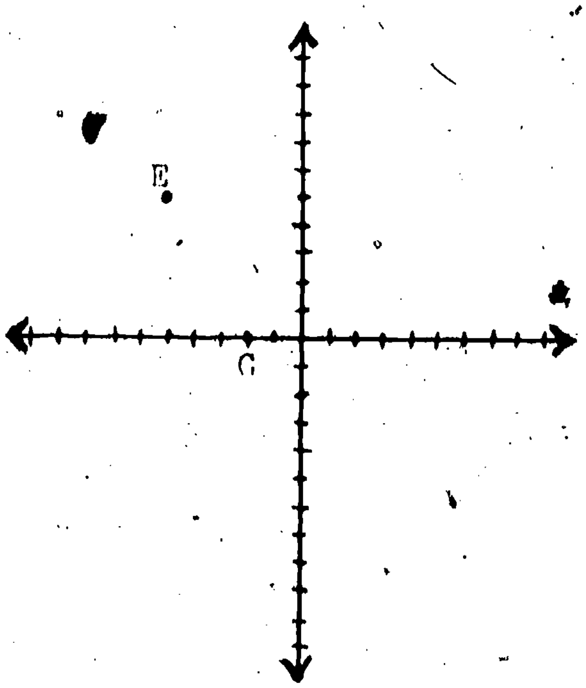
Activity IV



Find slope of $\overleftrightarrow{AB}$.

Through C draw a line $\overleftrightarrow{CD}$ parallel to $\overleftrightarrow{AB}$.

Find the slope of $\overleftrightarrow{CD}$.

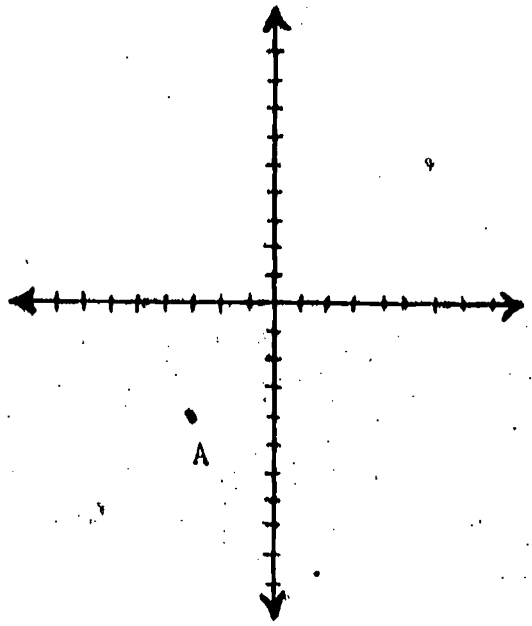


Through E draw a line $\overleftrightarrow{EF}$ with slope 3/2.

Through G draw aline GH with slope 3/2.

Is $\overleftrightarrow{EF}$ // $\overleftrightarrow{GH}$?

State a conclusion:

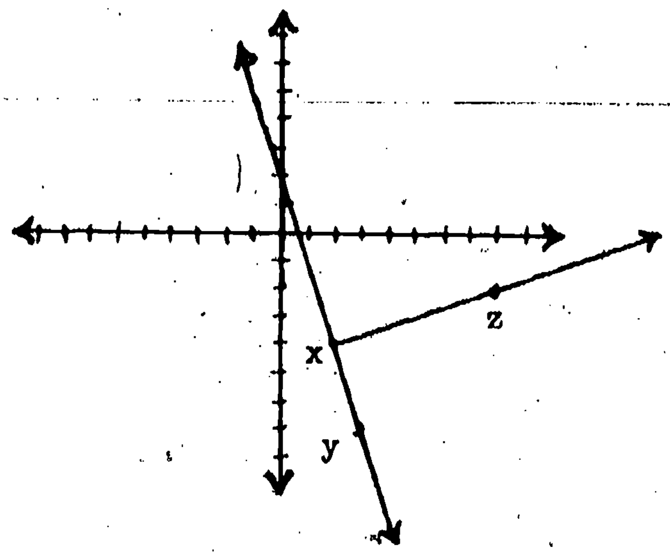


Through A draw a line $\overleftrightarrow{AB}$ with slope 2/5.

Through A draw a line $\overleftrightarrow{AC}$ such that $\overleftrightarrow{AC} \perp \overleftrightarrow{AB}$.

What is the slope of $\overleftrightarrow{AC}$?

(Slope $\overleftrightarrow{AB}$) x (slope $\overleftrightarrow{AC}$) = ________



What is the slope of $\overleftrightarrow{xy}$?

What is the slope of $\overleftrightarrow{xz}$?

What is the measure of $\angle zxy$?

(Slope $\overleftrightarrow{xy}$) x (slope $\overleftrightarrow{xz}$) = ________

State a conclusion:

Slope of Parallels and Perpendiculars

Exercises

Given: A(-4, 3); B(3, 3); C(-5, -1); D(2, -1)

1. Slope $\overline{AB}$ =
2. Slope $\overline{BD}$ =
3. Slope $\overline{DC}$ =
4. Slope $\overline{AC}$ =
5. Name a pair of parallel segments.

Given: A set of six line segments with end points
W(0, 5); x(4, 0); y(0, -5); Z(-4, 0)

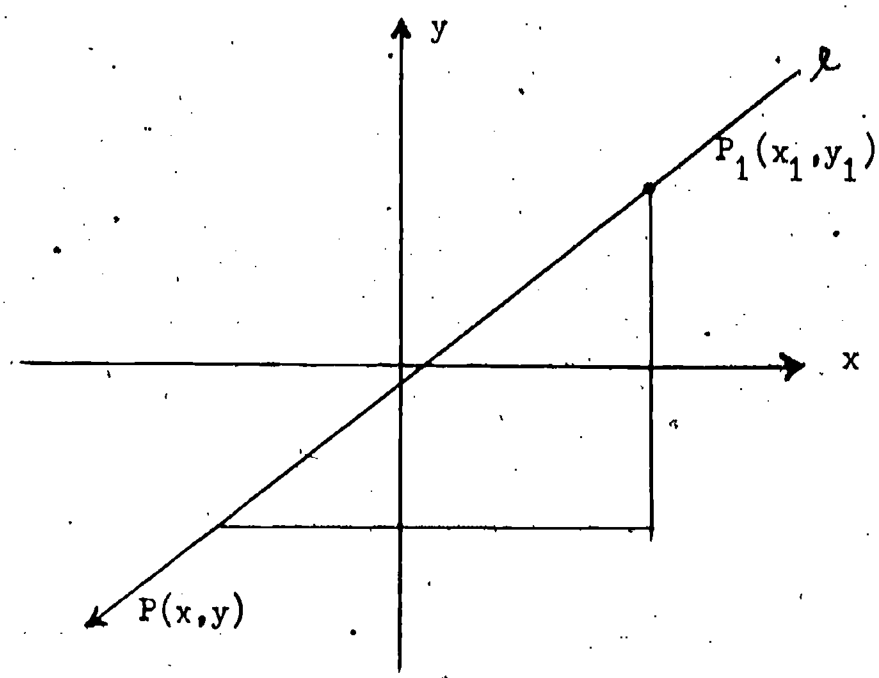
6. Name a pair of segments that are parallel.
7. Name a pair of segments that are perpendicular.

Given: A set of line segments with end points
M(4, -2); N(7, 0); P(6, 1)

8. Name a pair of segments that are perpendicular.

EQUATIONS OF LINES

Activity V



Suppose that you are given $P_1(x_1, y_1)$ and a slope m of some line. What is the linear equation for that set of points?

The slope, $m = \dfrac{y_2 - y_1}{x_2 - x_1}$

Since $P(x, y)$ is any point on ℓ, $m = \dfrac{y - y_1}{x - x_1}$. $\therefore$ $y - y_1 = m(x - x_1)$.

This result is called the "point-slope" equation of a line.

Example: A line passes through the point $(4, -2)$ and has a slope of $-\frac{3}{2}$, what is the equation of the line?

$$y - (-2) = -\frac{3}{2}(x - 4) \Rightarrow 2y + 4 = -3x + 12$$
$$\Rightarrow 3x + 2y - 8 = 0$$

If two points on a line are given, it is first necessary to find the slope of the line and use either point as the given point and use the "point-slope" equation. For example: $P_1(-2, 1)$, $P_2(4, 3)$,

$$\text{the slope } m = \frac{3 - 1}{4 - (-2)} = \frac{2}{6} = \frac{1}{3}$$

You may use either P_1 or P_2 as "the point".

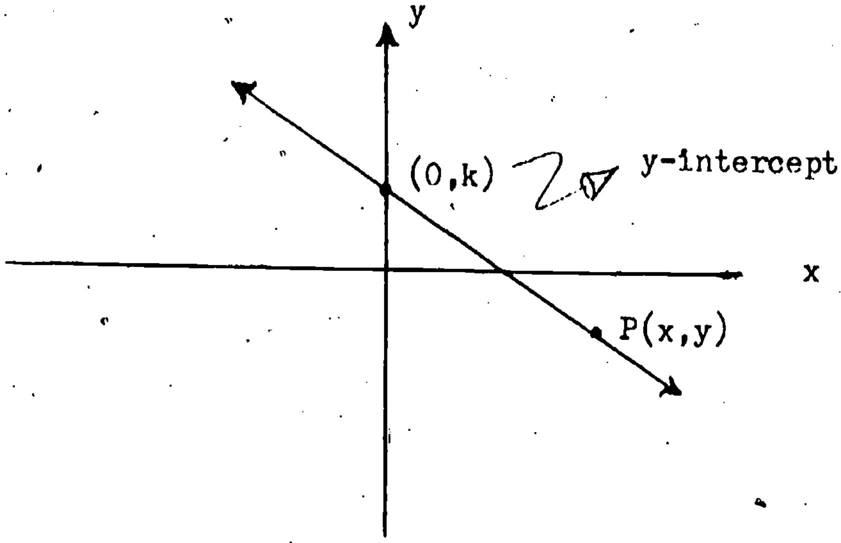
$$y - 1 = \frac{1}{3}(x + 2)$$
$$3y - 3 = x + 2$$
$$x - 3y + 5 = 0$$

By using P_2, $y - 3 = \frac{1}{3}(x - 4)$

$$3y - 9 = x - 4$$

$$x - 3y + 5 = 0$$

Quite often we would like to know the specific point where the line crosses the y-axis. This point is called the y-intercept. The coordinates of that point are (0, k).



Using the "point-slope" equation

$$y - k = m(x - 0) \text{ or}$$
$$y = mx + k.$$

This famous result is known as the "slope-intercept" form of a linear equation.

Exercises

Write each of the following equations in "slope-intercept" form and sketch the graph of each.

1. $1/2(y - 1) = x - 4$
2. $2y = 2x - 6$
3. $3(y + 5) = x + 3$
4. $x - 3y = 12$
5. $y - 2x = 0$
6. $x - 2y = 5$
7. $2x - 3y = 6$

Write an equation in point-slope form for the line that contains P and has the given slope.

8. $P(2, -7)$ $m = -\frac{3}{4}$

9. $P(-3, -2)$, $m = -2$

Write an equation in point-slope form for $\overleftrightarrow{AB}$.

10. $A(1, 4)$ $B(4, 3)$

11. $A(0, 5)$ $B(-3, 0)$

12. $A(2, -3)$ $B(4, -1)$

13. $A(-1, 1)$ $B(1, -1)$

14. Write an equation in slope-intercept form of the line that contains $P(0, 0)$ and is parallel to the line that contains $Q(2, 3)$ and $R(1, 1)$.

15. ℓ_1 has a slope of 3/4 and contains $P(8, 12)$. Write an equation of ℓ_2 such that ℓ_2 is perpendicular to ℓ_1 and passes through P.

16. A triangle has vertices $A(0, 0)$, $B(1, 6)$, $C(5, 2)$.
    a) Write an equation of $\overleftrightarrow{AB}$.
    b) Write an equation for the $\perp$ bisector of $\overline{BC}$.

SELECTING COORDINATES

Activity VI

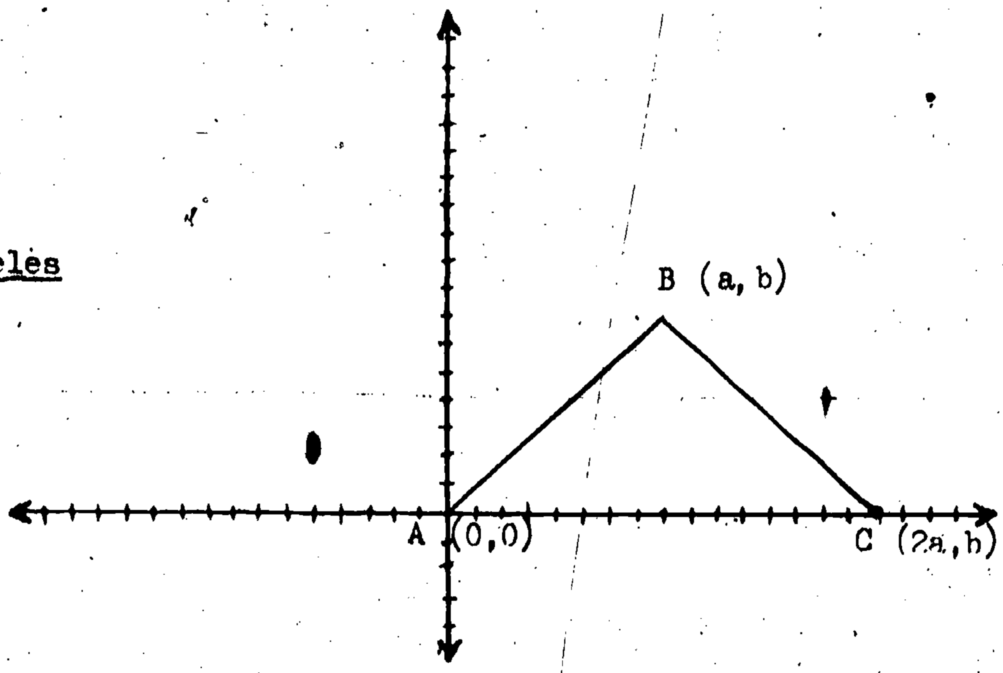
Your teacher will show eight diagrams containing triangles or quadrilaterals. Each of these will be imposed on a coordinate system with specific coordinates given for the vertices. From this you are to list all possible conditions that exist on the figure and from your list of definitions pick an appropriate name.

Example 1:

For $\triangle ABC$

$AB = BC = \sqrt{a^2 + b^2}$

$\therefore \triangle ABC$ is <u>isosceles</u>



Example 2:

For quadrilateral WXYZ

$WX = a$

$XY = \sqrt{(a+b-a)^2 + (\sqrt{a^2-b^2})^2}$

$= \sqrt{a^2} = a$

Slope $WX = 0$

Slope $YZ = 0$

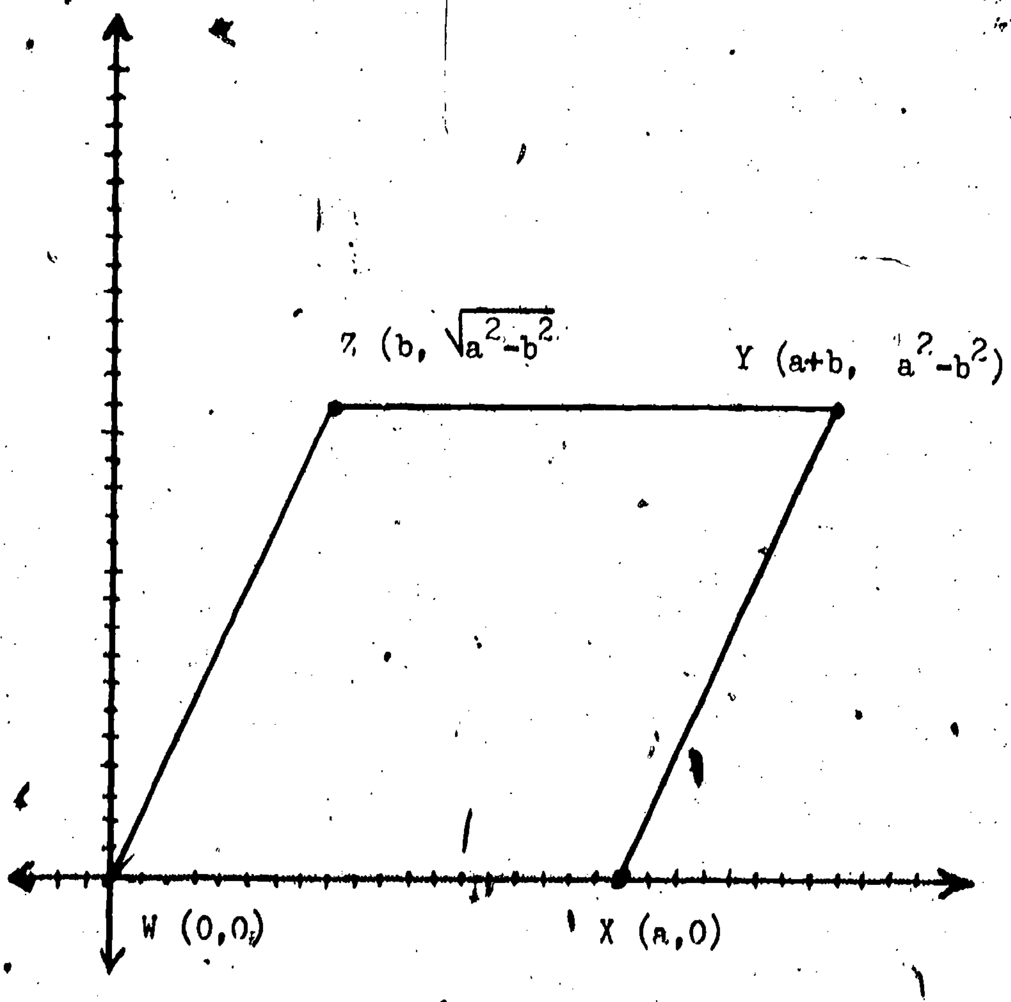
$\therefore \overline{WX} \parallel \overline{YZ}$

Slope $WZ = \dfrac{\sqrt{a^2-b^2}}{b}$

Slope $XY = \dfrac{\sqrt{a^2-b^2}}{a+b-a}$

$\therefore \overline{WZ} \parallel \overline{XY}$

Hence ▱ WXYZ is a rhombus.



Definitions

Polygon: If $P_1, P_2, \ldots P_n$ is a set of three or more distinct coplanar points, the union of $\overline{P_1 P_2}, \overline{P_2 P_3}, \ldots \overline{P_n P_1}$ is a polygon iff no two segments intersect except at their endpoints and no two intersecting segments are colinear.

Isosceles Triangle: An isosceles triangle is a triangle with two congruent sides.

Scalene Triangle: A scalene triangle is a triangle with no congruent sides.

Equilateral Triangle: An equilateral triangle is a triangle with three congruent sides.

Right Triangle: A right triangle is a triangle with one right angle.

Quadrilateral: A quadrilateral is a polygon that has four sides.

Parallelogram: A parallelogram is a quadrilateral with opposite sides parallel.

Rectangle: A rectangle is a parallelogram with one right angle.

Rhombus: A rhombus is a rectangle with a pair of adjacent sides parallel.

Square: A square is a rectangle and a rhombus.

Trapezoid: A trapezoid is a quadrilateral with one pair of sides parallel.

Diagonal of a polygon: A diagonal of a polygon is a segment whose endpoints are two non-consecutive vertices of the polygon.

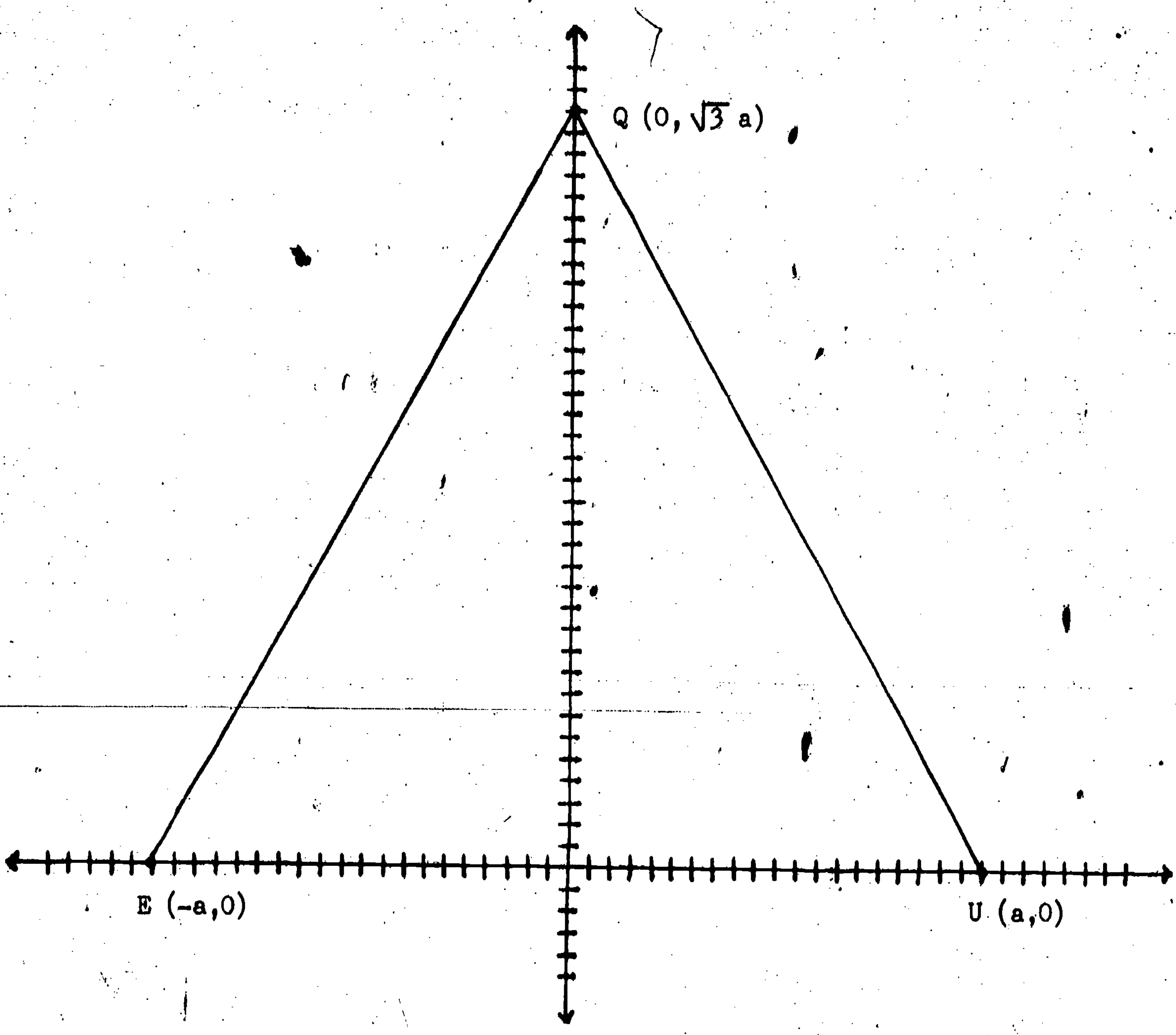


Figure 1

Check list:

Are there any congruent sides?

Are there any right angles?

ΔEQU is a(an) __________ triangle.

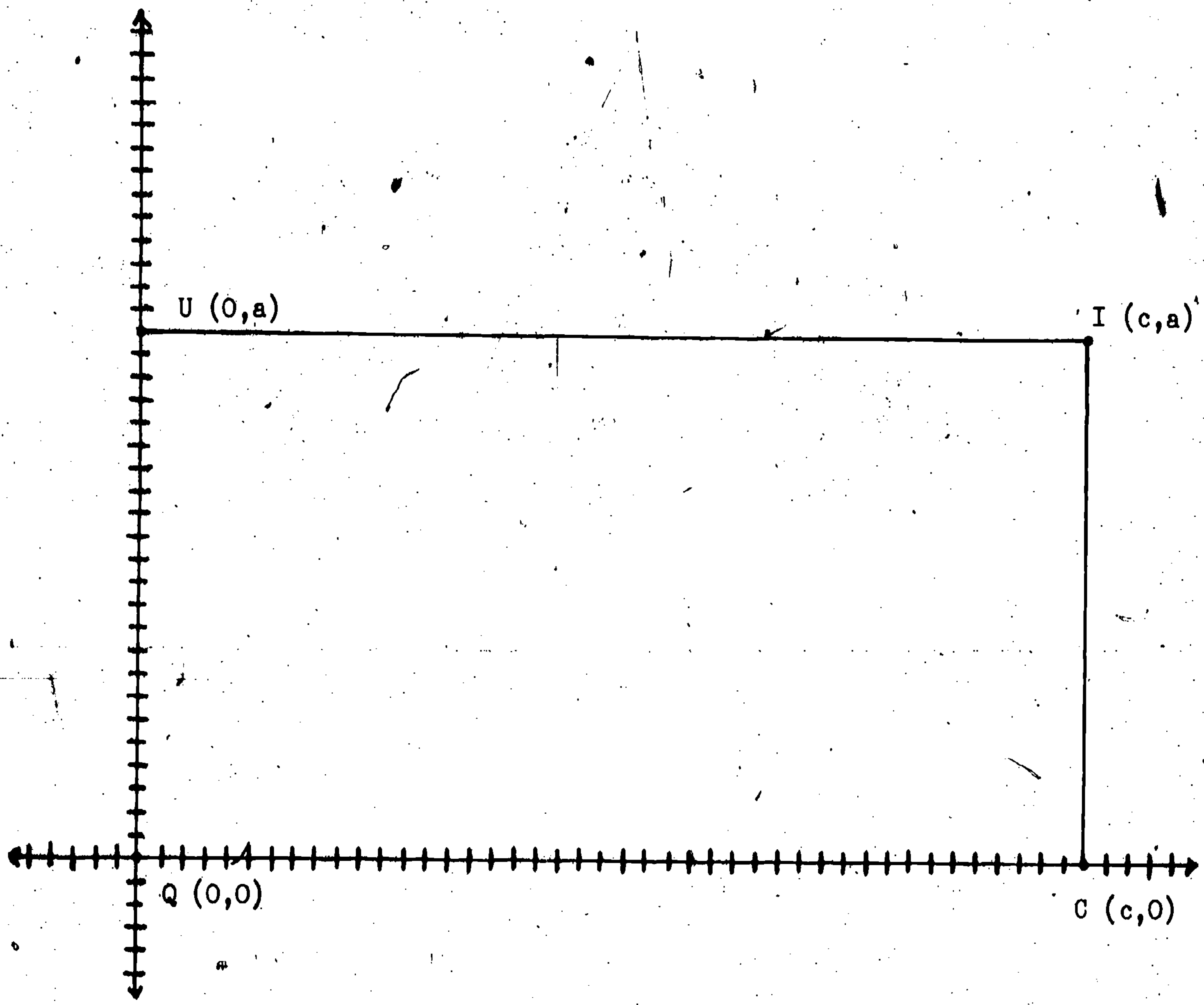


Figure 2

Check list:

Are there congruent sides?

Are there parallel sides?

QUIC is a ______________.

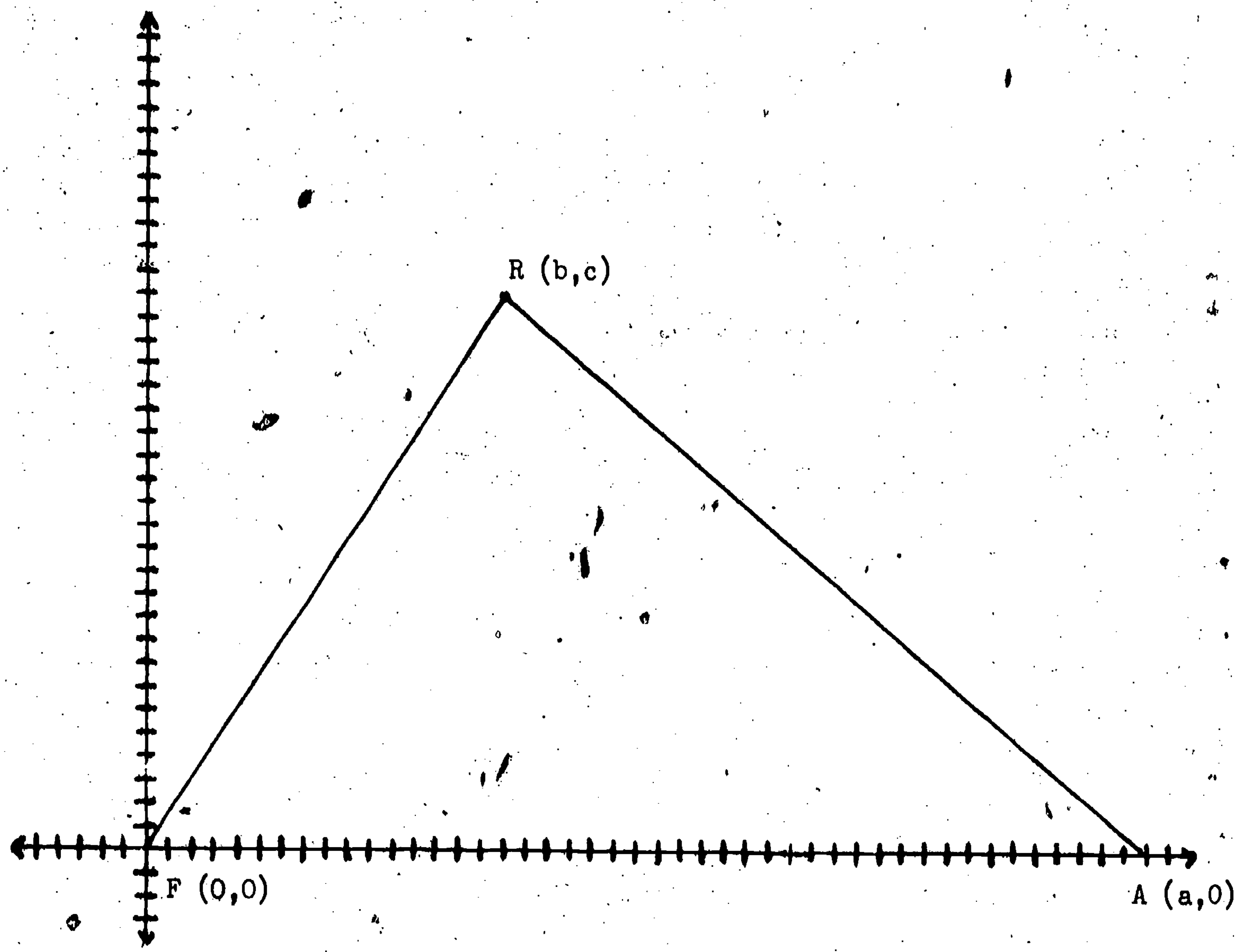


Figure 3

Check list:

Are there congruent sides?

Are there right angles?

$\triangle$FRA is a(an) ________________ triangle.

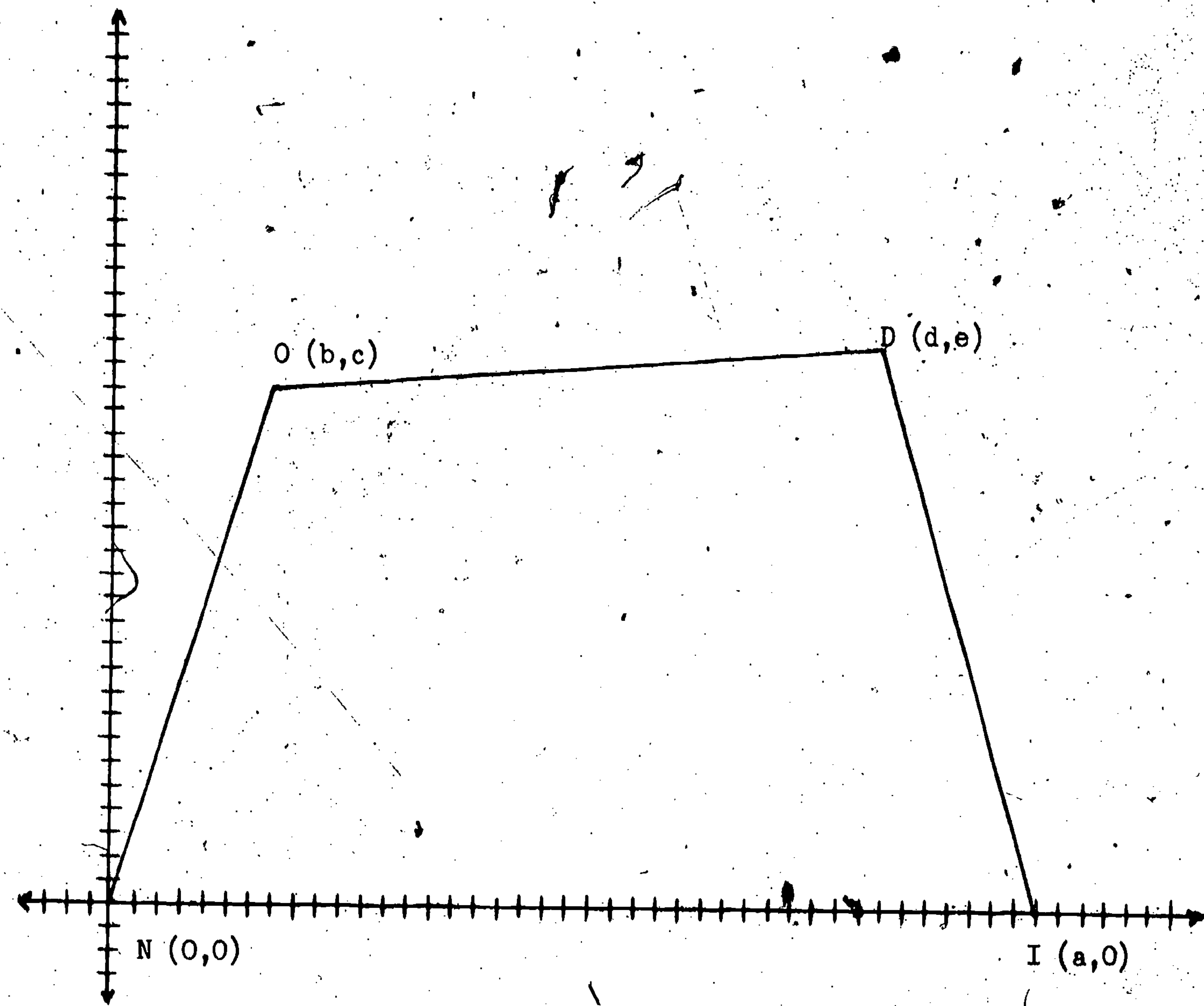


Figure 4

Check list:

Are there congruent sides?

Are there parallel sides?

Are there right angles?

NODI is a ______________ .

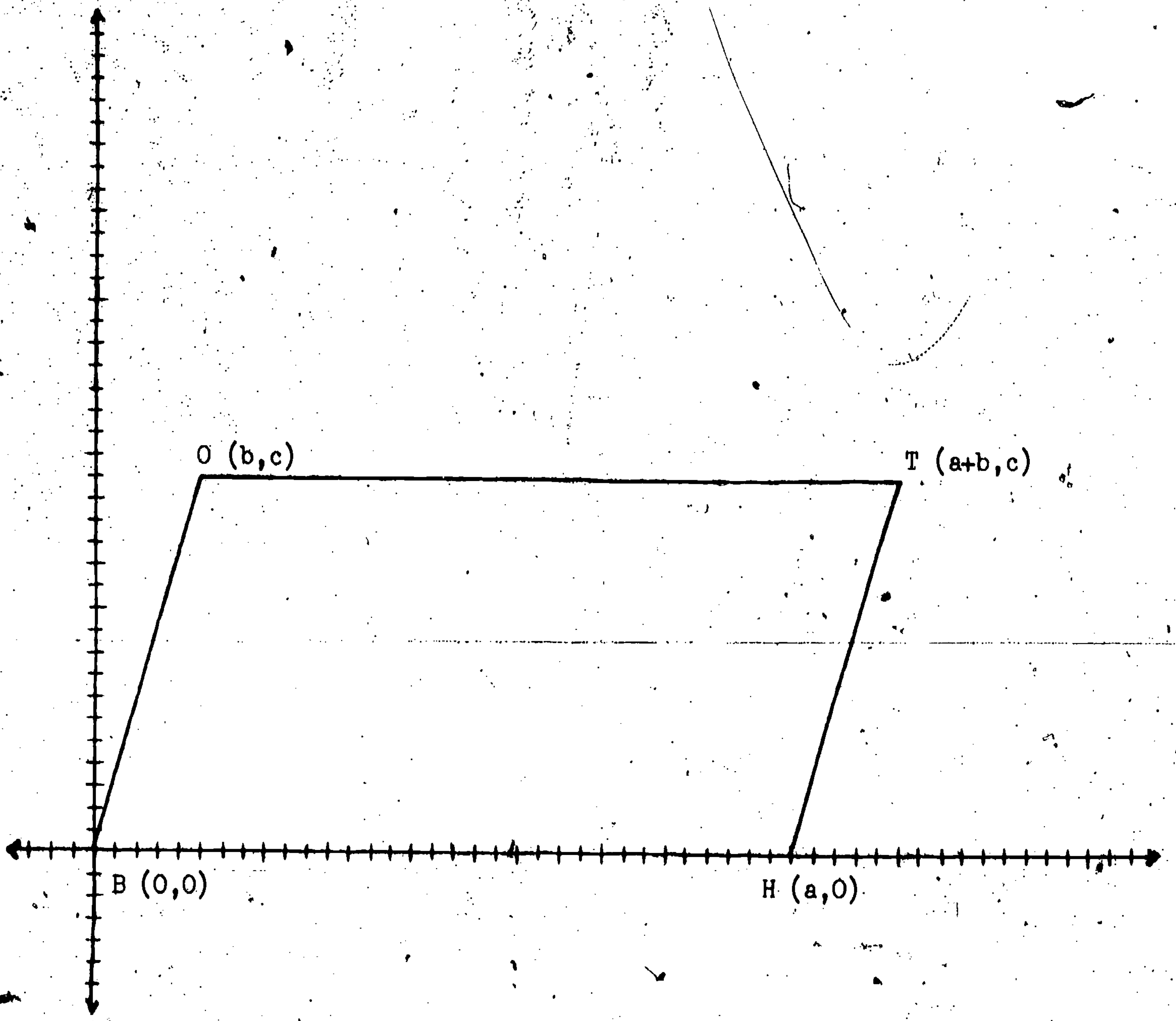


Figure 5

Check list:

Are there congruent sides?

Are there parallel sides?

Are there right angles?

BOTH is a ____________________.

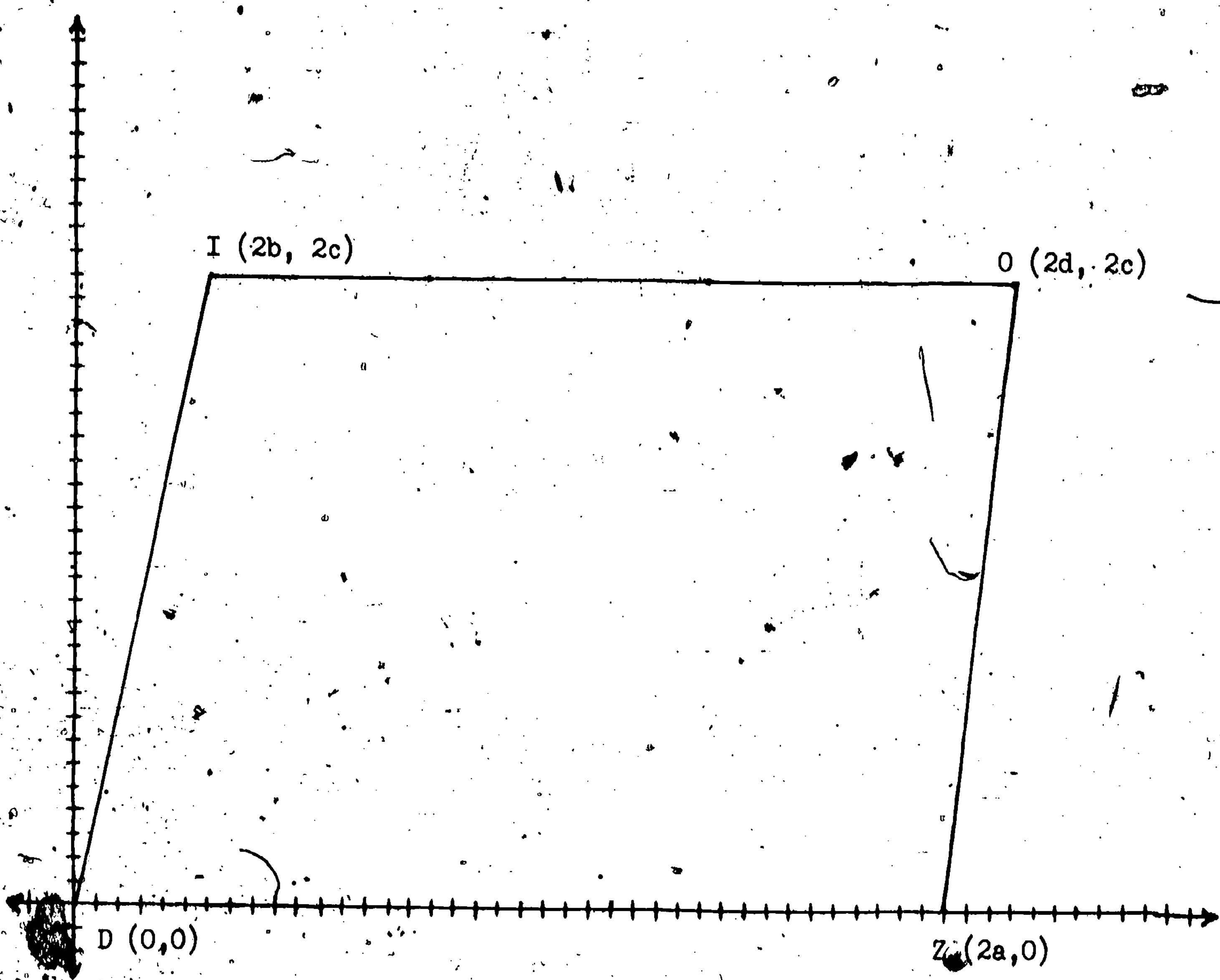


Figure 6

Check list:

Are there congruent sides?

Are there parallel sides?

Are there right angles?

DIOZ is a ____________

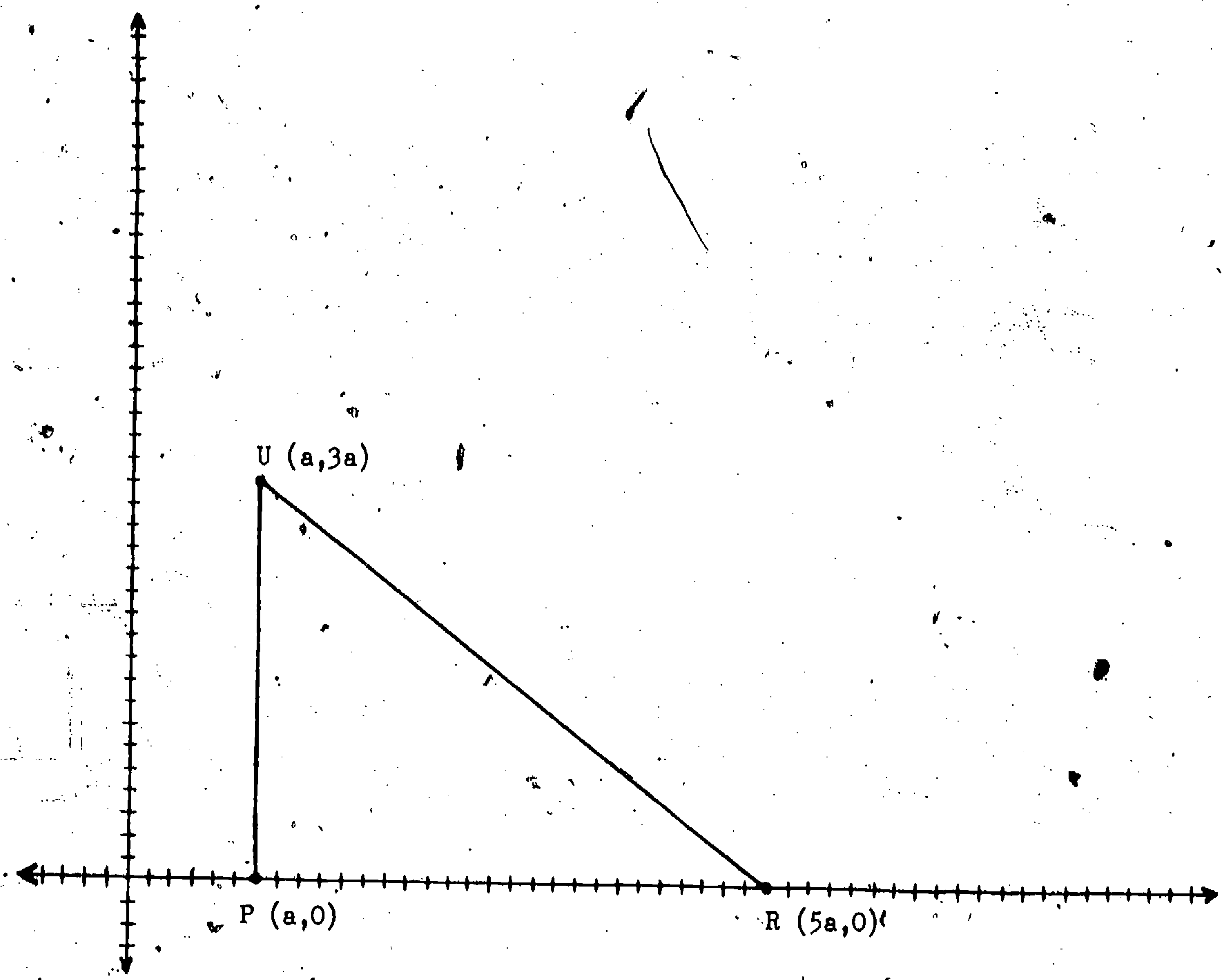


Figure 7

Check list:

Are there congruent sides?

Is there a right angle?

$\triangle$ PUR is a __________ triangle

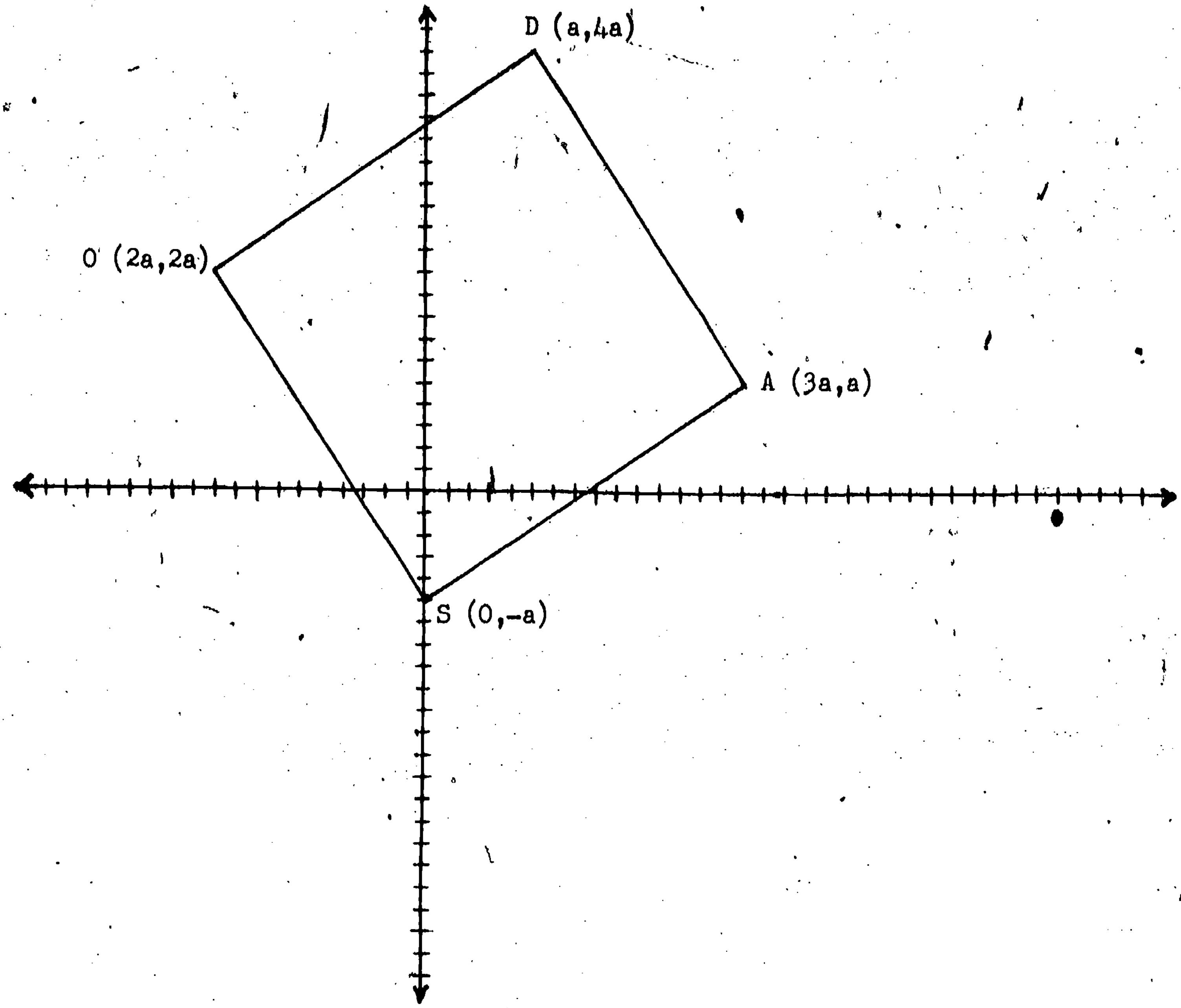


Figure 8

Check list:

Are there congruent sides?

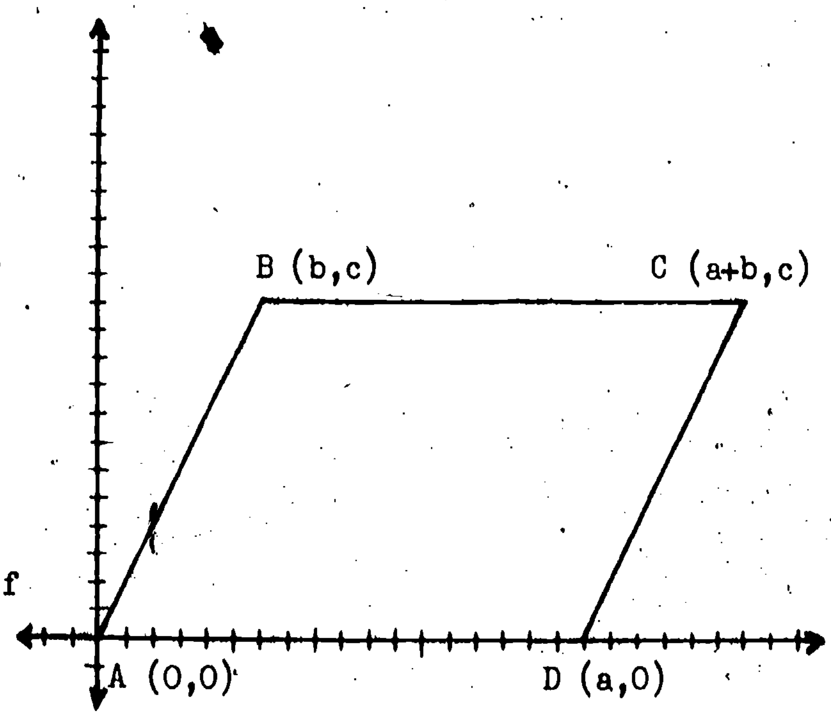
Are there parallel sides?

Are there right angles?

SODA is a ________________

Exercises:

Perhaps less easy to do is giving coordinates to a figure where only its type is known. For example, what coordinates would correctly describe the vertices of a figure if it was known only to be a parallelogram? Here is a proposed solution and some questions about that solution.



1. Must $\overline{AD}$ be placed on the x axis?
2. Could a point such as (5,0) be used for point D?
3. Would coordinates (a,c) work for point B instead of (b,c)?
4. Why was C selected as the y coordinate of point C?
5. Could a new constant such as d be used for the x coordinate of point C?
6. Why was a+b selected as the x coordinate of point ?.
7. Choose coordinates for the vertices of a figure where all that is known is that the figure is a trapezoid.

In exercises 8 through 15, establish general coordinates for the vertices of a figure if it is known only that the figure is a:

8. scalene triangle
9. isosceles triangle
10. right triangle
11. equilateral triangle
12. quadrilateral
13. rectangle
14. rhombus
15. square

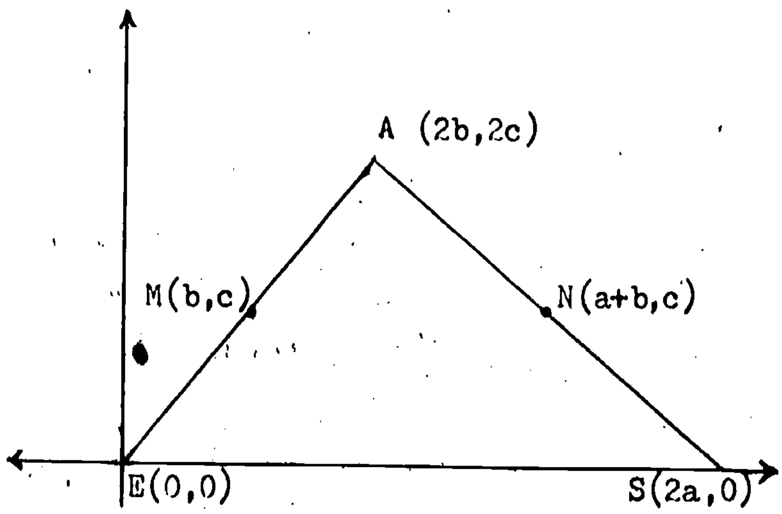
COORDINATE PROOFS

Activity VII

If one is going to do proofs, they may as well be done as easily as possible. Here is a method of proof which sometimes makes the act of proving a statement true unbearably easy.

Prove: The segment joining the midpoints of two sides of a triangle is parallel to and one-half the length of the third side.

$\triangle EAS$ is any triangle. By the midpoint formula M and N are the midpoints of AE and AS respectively.



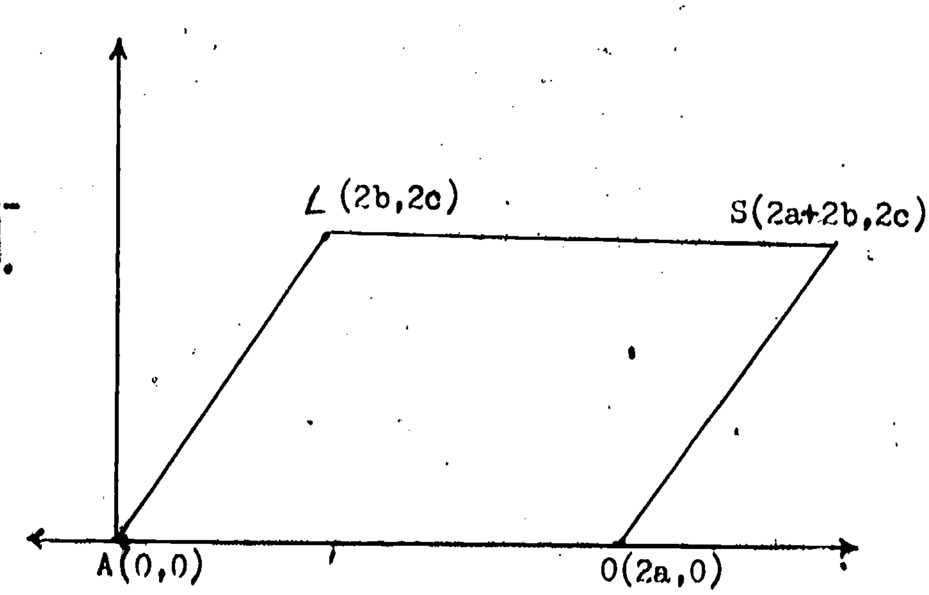
Slope $\overline{ES}$ = slope $\overline{MN}$ = 0
Therefore, $\overline{ES} \parallel \overline{MN}$.

$MN = \sqrt{a^2} = a$
$ES = 2a$
Therefore, $MN = 1/2\ ES$, and the proof is complete. (Q.E.D.)

Notice the selection of the coordinates of A as (2b, 2c) rather than (b, c) and similarly (2a, 0) for S. This was done in anticipation of working with midpoints.

Prove the diagonals of a parallelogram bisect each other.

▱ ALSO is any parallelogram. The midpoint of diagonal $\overline{LO}$ is (b + a, c). The midpoint of diagonal $\overline{SA}$ is (b + a, c). Therefore $\overline{LO}$ and $\overline{SA}$ bisect each other.



Complete the following proofs, using coordinates as an aid.

1. Prove the opposite sides of a parallelogram are congruent.
2. Prove the diagonals of a rectangle are congruent.
3. Prove that a rhombus is equilateral.
4. Prove the diagonals of a rhombus are perpendicular bisectors of each other.
5. Prove the midpoint of the hypotenuse of a right triangle is equidistant from the three vertices of the triangle.
6. Prove the segment whose endpoints are the midpoints of the diagonals of a trapezoid is parallel to the bases and has length equal to average of the base lengths.
7. Prove the segments joining the midpoints of the opposite sides of a quadrilateral bisect each other.

NECESSARY CONDITIONS

Exercise 1

In the last activity you were dealing with necessary conditions on parallelograms, rectangles, rhombuses and squares. For example, if a quadrilateral is a parallelogram, it is necessary that its diagonals bisect each other.

Observe that any condition necessary for parallelograms is also a necessary condition for rectangles, rhombuses and squares. Hence, the diagonals of a rectangle bisect each other; the diagonals of a rhombus bisect each other; and the diagonals of a square bisect each other.

The following is a table to be completed by you. Its purpose is to catalogue the necessary conditions for parallelograms, rectangles, rhombuses and squares.

Necessary conditions for parallelograms, rectangles, rhombus and squares. Put a check in the blocks that indicate such conditions.

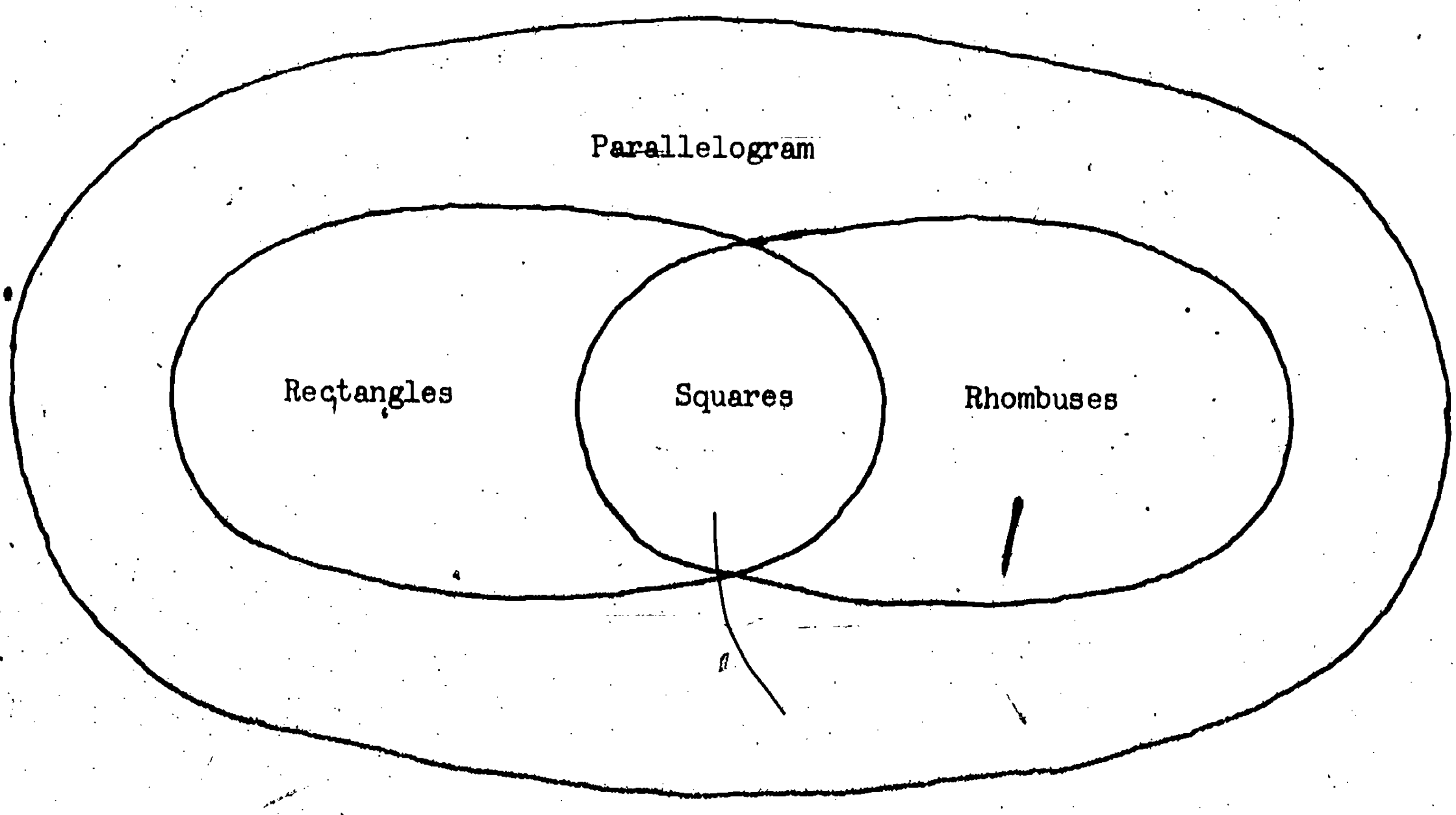
If a quadrilateral is a / then it is necessary that	Parallelogram	Rectangle	Rhombus	Square
opposite sides are parallel				
opposite sides are congruent				
opposite angles are congruent				
the diagonals bisect each other	✓	✓	✓	✓
the diagonals are congruent				
the diagonals are perpendicular				
the diagonals are ⊥ bisectors of each other				
the diagonals bisect the angles				
is equiangular				
is equilateral				
is equiangular and equilateral				

SUFFICIENT CONDITIONS

Exercise 2

The converse of a necessary condition is a sufficient condition. The converse of the statement "If a quadrilateral is a parallelogram, then it is necessary that its diagonals bisect each other" could be stated as "The fact that the diagonals of a quadrilateral bisect each other is sufficient to guarantee that the quadrilateral is a parallelogram.

Observe that any condition sufficient to guarantee that a figure is a square or rectangle or rhombus, is sufficient to guarantee that the figure is a parallelogram. For example, if a quadrilateral is equiangular and equilateral, it is certainly a square. But it is also a rectangle, a rhombus, and a parallelogram. The Venn diagram below indicating how the sets of squares, rectangles, rhombuses and parallelograms fit together may be helpful in visualizing this result.



Sufficient conditions for parallelograms, rectangles, rhombuses and squares.

Put a check in the blocks that indicate sufficient conditions.

Square	Rhombus	Rectangle	Parallelogram	If a quadrilateral it is then sufficient to guarantee that it is a
				has opposite sides parallel
				has opposite sides congruent
				has opposite angles congruent
				has diagonals which bisect each other
				has congruent diagonals
				has perpendicular diagonals
				has diagonals which are $\perp$ bisector of each other
				has diagonals which bisect the angles
				is equiangular
				is equilateral
✓	✓	✓	✓	is equiangular and equilateral

LINEAR INEQUALITIES

Activity VIII

A linear equation has a straight line as its geometric representation. A single inequality is represented by all points on one side of a line. If we think of a linear equation as dividing a plane in half, the solution of an inequality in two dimensions consists of all points in a half-plane.

Example: Graph the solution of: $2x + 36 \leq 6$.

Solving for y in the usual manner: $y \leq -2/3\, x + 2$.

This says that the points satisfying the inequality lie on or below the line $y = -2/3\, x + 2$.

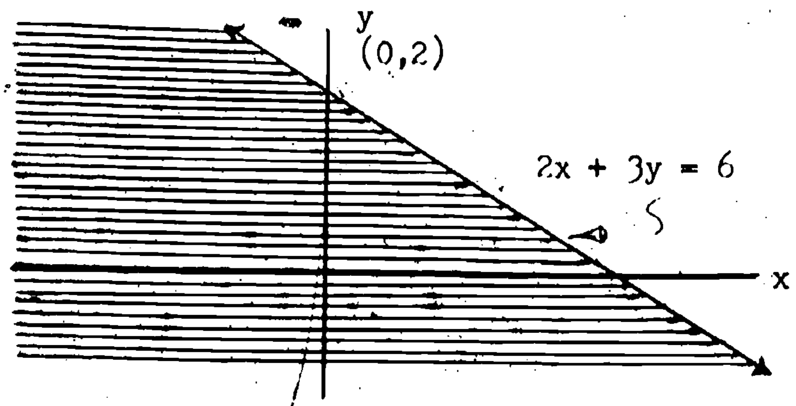


Fig. 1

Now consider a system of two linear inequalities, such as:

1) $x \leq 3$ and 2) $3x + 2y \leq 6$

The solution of a pair of inequalities consists of a section of a plane which we shall call the solution space. The solution space is the set of all possible solutions.

The first inequality states that x must lie less than or equal to 3. Solving for y on the second inequality yields:

$$y \leq -3/2\, x + 3.$$

The solution space for this system consists of points which are both to the left of $x = 3$ and below $3x + 2y = 6$. This may be symbolized as $\{(x,y):\ x \leq 3 \cap 3x + 2y \leq 6\}$. The solution space is shown in Figure 2.

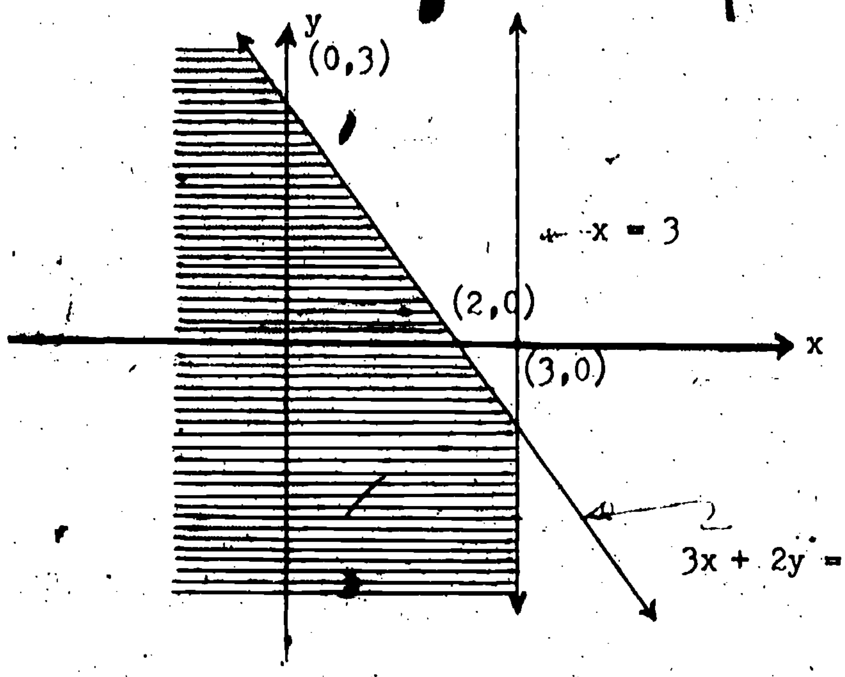


Fig. 2

Consider another system involving two inequalities.

1) $2x + y \geq 4$ 2) $x + 3y \leq 6$

Solving both inequalities for y:

1) $y \geq -2x + 4$ 2) $y \geq -1/3\ x + 2$

The solution space for this system consists of those points which are both above the line $2x + y = 4$ and above $x + 3y = 6$.

$$\{(x,y):\ 2x + y \geq 4 \cap x + 3y \geq 6.\}$$

Figure 3 illustrates the solution space.

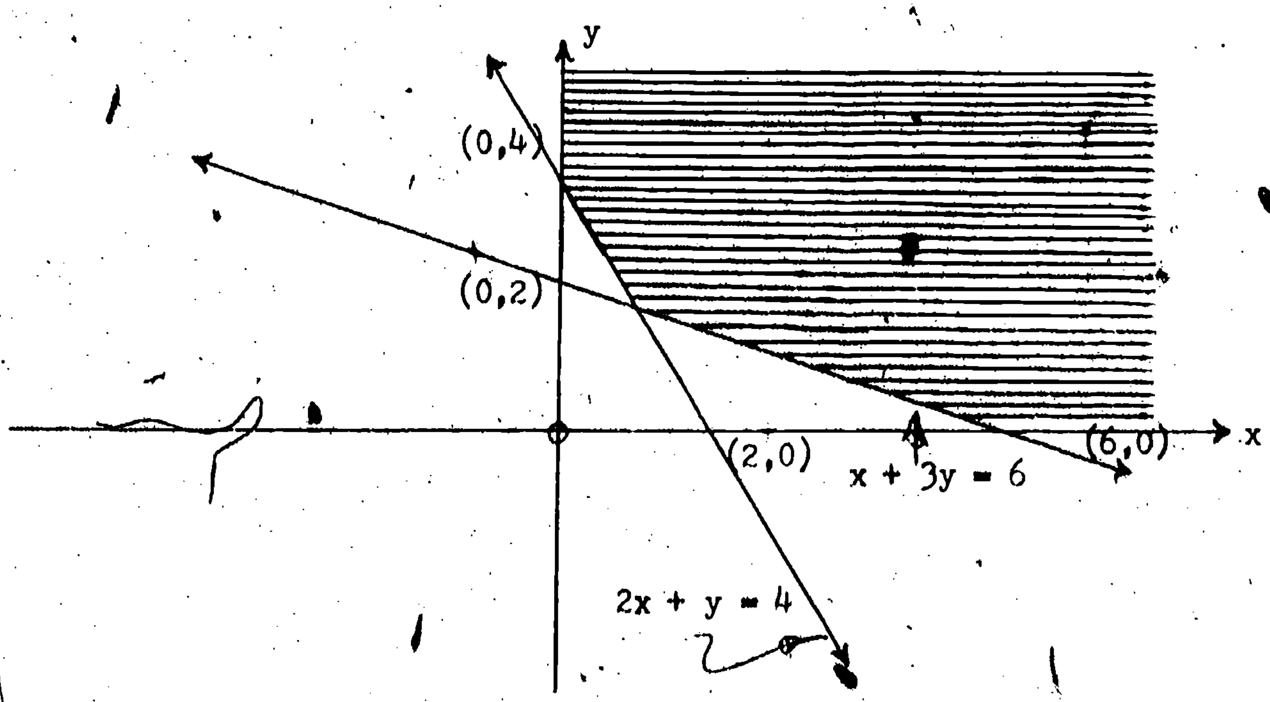


Fig. 3

Exercises

Find the general solutions and graph the solution space, if they exist, for each problem.

1. $3x + 3y \le 5$
   $x + y \ge 0$

2. $2x + 3y \ge 6$
   $3x + y \ge 2$

3. $2x + y \ge 4$
   $2x + y \le -3$

4. $2x + 3y \ge 5$
   $x + y \le 2$

5. $7x + 4y \le 28$
   $x - 2y \le 8$

6. $2x - 3y \ge 2$
   $x + y \ge 5$

7. $2x - y \ge 5$
   $x + y \ge 2$

8. $3x + y \ge 6$
   $x + 2y \ge 4$

9. $x - y \le -3$
   $2x - 2y \ge 4$

10. $y \ge 2x + 2$
    $y \le -x - 2$

11. $y \le x - 1$
    $y \ge 2x - 2$

12. $x + y \ge 4$
    $x + y \le 6$
    $2x - y \le 7$

13. $x + y \le 4$
    $x - y \le 1$

NON-NEGATIVE CONSTRAINTS

Activity IX

In the work to come, we shall deal with quantities, prices of goods, and other variables which cannot have negative values. In the case of two variables, x and y, this restriction is expressed by writing $x \geq 0$, $y \geq 0$ with the other inequalities of the problem. This implies then that only the points in the first quadrant, including the x and y axes will be considered. For instance, the system:

$$x \geq 0$$
$$y \geq 0$$
$$x + 2y \leq 8$$
$$2x + y \leq 6$$

has as its solution space the shaded area I shown in Figure 4.

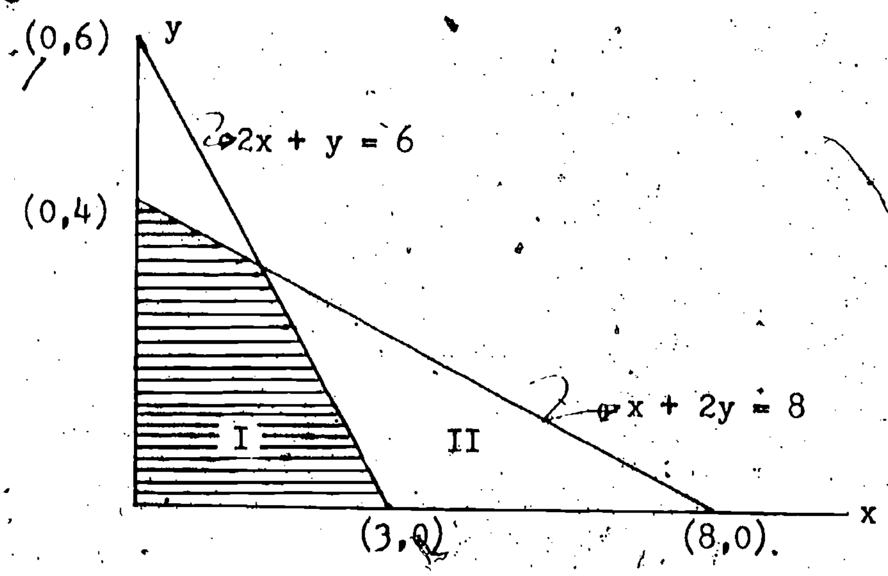


Fig. 4

Consider next the system:

$$x \geq 0$$
$$y \geq 0$$
$$x + 2y \leq 8$$
$$2x + y \geq 6$$

The solution space for this system is area II on Figure 4.

Another system can be illustrated with the following problem.

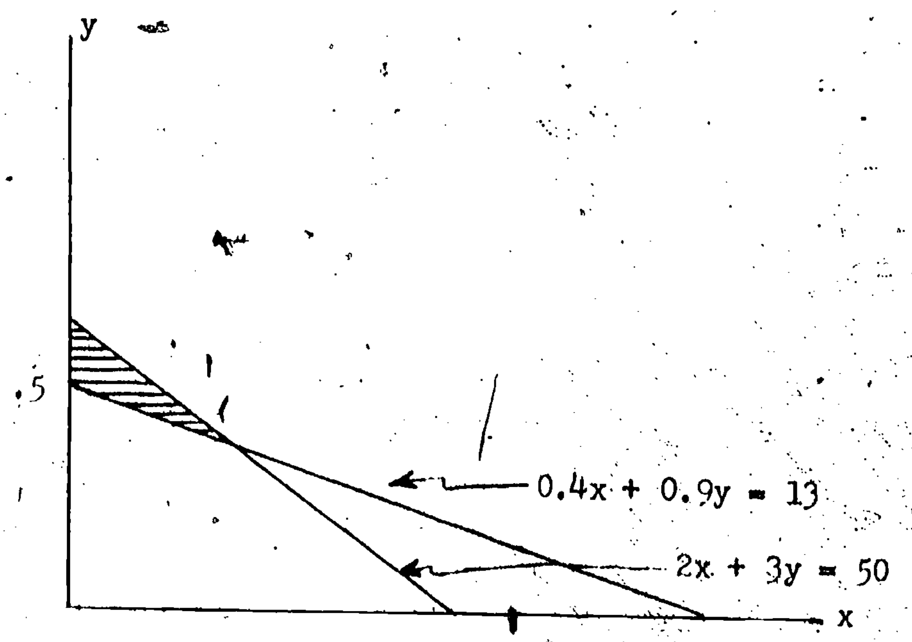
Food type	Units in mixture	ounces of nutrient/unit	ounces of wt./unit
X	x	0.4	2
Y	y	0.9	3

A mixture is to be made of food types X and Y. If the mixture is to contain not less than 13 ounces of nutrient, and is not to weigh more than 50 ounces, what combination of the foods are permissible?

The nutrient constraint is $0.4x + 0.9y \geq 13$. The weight constraint is $2x + 3y \leq 50$. Take these two constraints along with $x \geq 0$ and $y \geq 0$. The solution space for the system:

$$x \geq 0$$
$$y \geq 0$$
$$0.4x + 0.9y \geq 13$$
$$2x + 3y \leq 50$$

is shown below.



Exercises

Construct a graph showing the solution space.

1. $x \geq 0$
   $y \geq 0$
   $y \geq 3$
   $x \leq 4$

2. $x \geq 0$
   $y \geq 0$
   $x \leq 3$
   $x \leq y$

3. $x \geq 0$
   $y \geq 0$
   $2x + y \geq 6$
   $x + 4y \leq 8$

4. $x \leq 3$
   $y \leq 2$
   $x + 3y \leq 9$

5. $x \geq 0$
   $y \geq 0$
   $7x + 4y \leq 28$
   $x + 2y \leq 8$

6. $x \geq 0$
   $y \geq 0$
   $2x + 3y \leq 12$
   $x - 2y \geq 2$

7. $x \geq 0$
   $y \geq 0$
   $4x + y \leq 12$
   $x + 4y \geq 8$

8. $x \geq 0$
   $y \geq 0$
   $4x + y \geq 12$
   $x + 4y \leq 8$

9.

Storage Unit	Cost/Unit	Storage Space
A	\$3.00	x
B	\$8.00	y

Sufficient storage space is available for 500 units (total) and \$2400 is available to spend on the items. Graph the solution space, showing permissible combinations of items which may be purchased and stored without exceeding total space and money restrictions.

10.

Food Type	Units in Mixture	Ounces of wt / unit	Ounces of Nutrients/unit
X	x	3	0.5
Y	y	5	1.0

A mixture is to have at least 8 ounces of nutrients, and the total weight is not to exceed 45 ounces. Graph the solution space, showing all possible combinations of the two food types.

11.

Department	Hours required to produce one unit A	B	Units of A	Units of B	Hours available
I	1	1	x	y	5
II	2	1	x	y	7

The table shows that it takes one hour for department A to produce product A, while it takes department II two hours to make one unit of A. The table also shows the number of hours available for making products A and B. Graph the solution space, showing all permissible combinations of items which may be produced.

12.

Department	Hours available	Hours required to make one unit A	B
I	120	4	3
II	60	1	3
III	175	7	5

Let x be the number of units of products A and y be the number of units of products B. Graph the solution space showing all permissible combinations of items which may be produced.

13.

Department	Hours required to make one unit A	B
I	2	1
II	1	1
III	1	3

If the hours available in I, II, and III are 12, 7, and 15 respectively, graph the solution space showing all permissible combinations of items which may be produced.

14.

Department	Hours available	Hours required to produce one unit A	B
I	120	4	3
II	40	1	2

Each unit of A contributes $3 to overhead and profit while each unit of B contributes $4. Graph the solution space showing permissible combinations that can be made in the available time if total contribution to overhead and profit is to be at least $100.

LINEAR PROGRAMMING

Activity X

In applied mathematics, we are often interested in a number of ways of accomplishing a certain objective. For example, some combinations of foods will provide a satisfactory diet, but some combinations are more costly than others, and we are interested in finding the minimum cost of providing dietary requirements. Again, there are many combinations of products that a plant can manufacture, and we are interested in the combination which yields a maximum profit.

The variables in the real world are called constraints; that is, ~~situations are subject to restrictions in which, in most cases, the~~ variables do not take on negative values.

When the problem is one of finding the maximum (or minimum) value of a system of inequalities, when the constraints and the objectives function are linear, we have a problem in linear programming.

Example 1

Suppose that an airline agrees to provide space on a special tour to Lower Slobobia, for at least 180 first-class and 320 tourist passengers. It must use two or more of its type-X planes. Each type X plane has 30 first-class and 50 tourist seats. The type Y plane has 30 first-class seats and 70 tourist seats. The flight cost is \$1200 for type X plane and \$1800 for each type Y plane. Total cost is to be a minimum. How many of each kind of plane should be used?

Plane type	No. of seats First class	Tourist
X	30	40
Y	30	80

Let x and y represent the number of planes of type X and of type Y. If C represents the total cost, then

$$C = 1200x + 1800y.$$

The airline wants to minimize C subject to the following constraints:

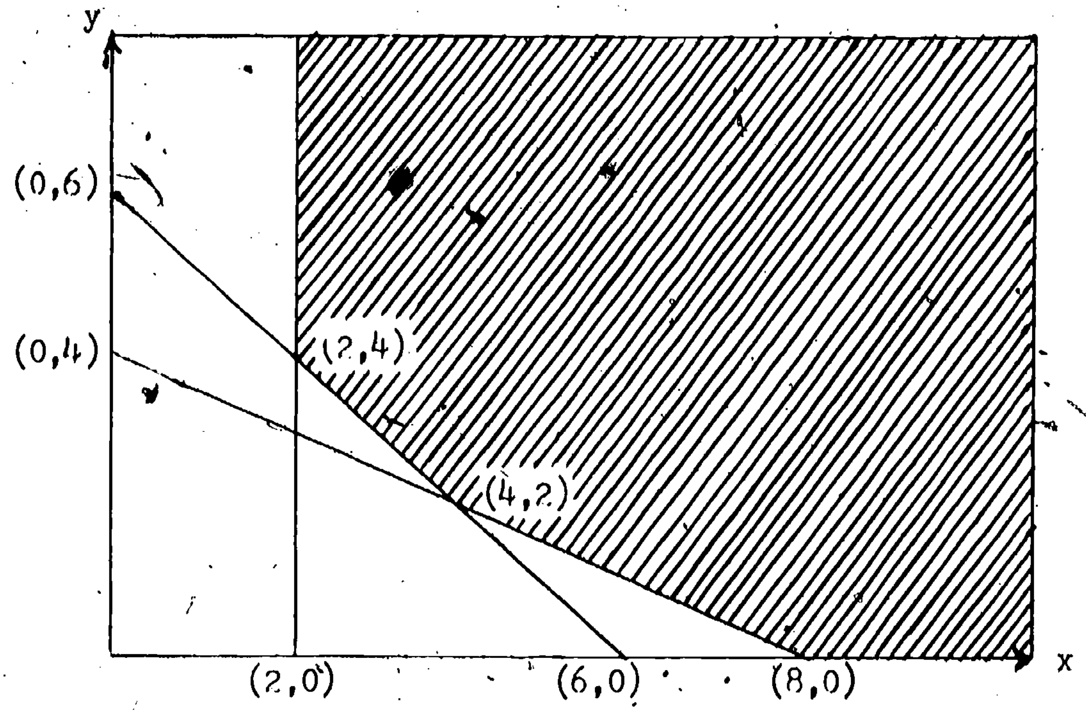
1\) $x \geq 2$ (must use 2 or more type X planes)

$y \geq 0$ (cannot use a negative number of type Y planes)

$30x + 30y \geq 180$ (# of first class seats)

$40x + 80y \geq 320$ (# of tourist seats)

The solution space is shown in the figure below.



The shaded region has the following characteristics:

1. Where it is bounded, the boundary is determined by straight lines. The points where boundary lines intersect are called <u>corner points</u>.

2. The region is <u>convex</u>.

It can be shown that 1200x + 1800y has a minimum value over the shaded region. To minimize this expression over the region, evaluate it at the three corner points.

At corner (2,4) $1200x + 1800y = 9,600$

At corner (4,2) $1200x + 1800y = 8,400$

At corner (8,0) $1200x + 1800y = 9,600$

The minimum cost in this problem is $8,400 and the airline should use 4 type X and 2 type Y planes.

<u>Example 2</u>

Product	Number of units made	Profit per unit	Hours required per unit Dept. I (4 hrs. avail.)	Dept. II (6 hrs. avail.)
A	x	$1.00	1	1
B	y	$ .50	1	2

Determine the maximum profit that can be achieved. Keep in mind the 4 and 6 hour time limitations in Department I and Department II.

The profit achieved may be expressed as: $P = x + .5y$

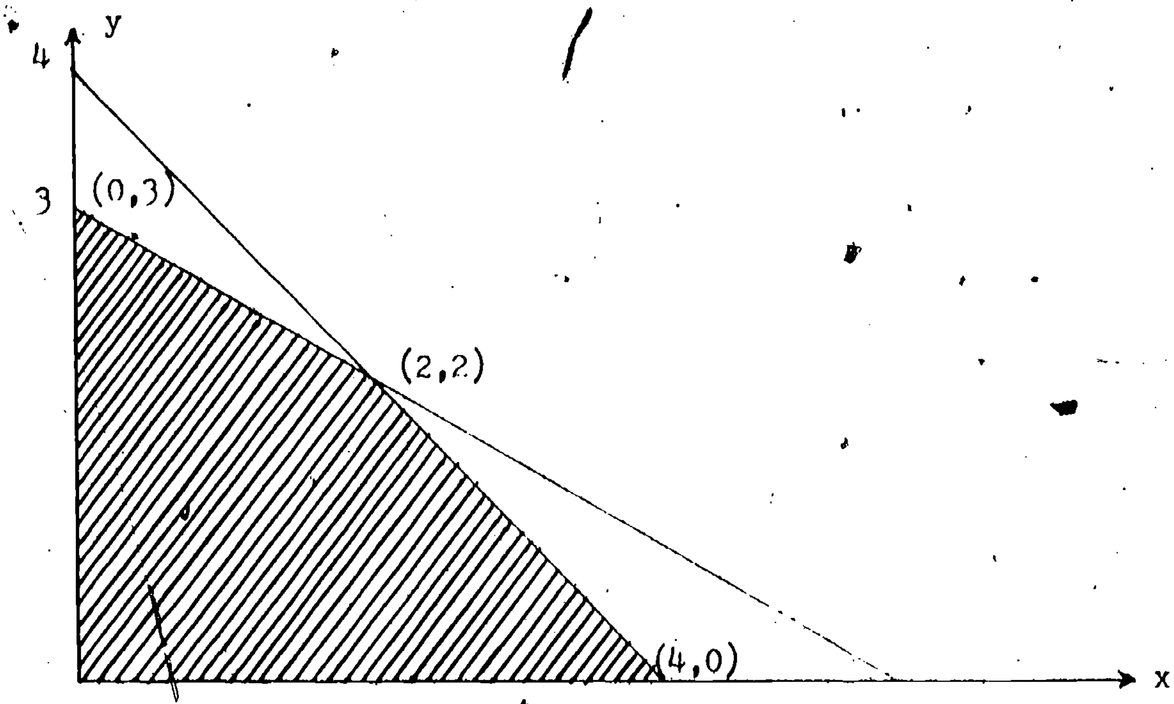
$x + y \leq 4$ (x units of A and y units of B in Department I.)

$x + 2y \leq 6$ (x units of A and y units of B in Department II.)

$x \geq 0$ (cannot produce a negative number of items.)

$y \geq 0$

The solution space is shown in the figure below.



We now check the corners to see which yields the maximum profit P.

At corner (0,3) $x + .5y = \$3.50$

At corner (2,2) $x + .5y = \$3.00$

At corner (4,0) $x + .5y = \$4.00$

The solution of the problem is to make 4 units of product A and 0 units of product B.

Exercises:

Pg. 119-125 Bowen, Earl K, Mathematics, with Applications in Management and Economics. Richard D. Irwin, Inc. Homewood, Ill. 1967.

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Chapter VI Kemeny, John G., et. al., Introduction to Finite Mathematics, Prentice-Hall, Inc., Englewood Cliffs, N.J. 1957.