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ABSTRACT

Discussed are some issues and queries about research in language and language acquisition. In particular, the area of inquiry is the logic subjacent to communication. In question here are the foundations of communication. What, if anything, underlies language? The unorthodox position developed in this paper has ambitious assumptions and slim empirical foundation. Nevertheless, it has strong links with some classical intellectual theories. The general argument to be made runs roughly as follows: The human animal is, to a great extent, characterized by the mathetic capacity. This innate capacity, which is related to the primal abilities of classifying and rule-generating and following, is essentially mathematical in nature and underlies all cognitive activity including language. Relative to their own sets of assumptions, human beings tend to be logical or consistent beings. A rich and not well-developed area for understanding the nature of language and its acquisition, and intellectual growth in general, is the study of children in situations where their implicit sets of rules or assumptions become clear. The clarity, precision, and power of certain elementary mathematical structures make them ideal vehicles for the study of the mathetic capacity and its manifestations. (Author/MK)

LANGUAGE, LOGIC AND MATHEMATICS: REFLECTIONS ON ASPECTS OF EDUCATION AND RESEARCH

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De omnibus rebus et quibusdam aliis.

- Latin phrase

La langue est une raison humaine qui a ses raisons, et que l'homme ne connaît pas.

- Claude Lévi-Strauss (1966, p.250)

These headings are large subjects in themselves, and many large books have been written on one or another of their multitudinous aspects. I can only acknowledge a congenital weakness for biting off more than I can chew; I have always had a fellow feeling for a graduate student I once heard of, who proposed doing a thesis on the influence of the eighteenth century on the nineteenth.

- Douglas Bush (Bonner, 1963, p.v)

Philosophy is written in that great book which ever lies before our gaze--I mean the universe--but we cannot understand if we do not first learn the language and grasp the symbols in which it is written. The book is written in the mathematical language, and the symbols are triangles, circles and other geometrical figures, without the help of which it is impossible to conceive a single word of it, and without which one wanders in vain through a dark labyrinth.

- Galileo Galilei

In science we have a highly developed special use of symbols, built on convention, and resulting in the boldest abstractions that have ever been made. Scientific symbolization is, I think, always genuine language, in the strictest sense, and the symbolism of mathematics the greatest possible refinement of language.

- Susanne Langer, 1956 (1964, p.61)

Mathematics has come to be identical with philosophy for modern thinkers, though they say it should be studied for the sake of other things.

- Aristotle (*Metaphysics* I, 992a 32)

The slenderest knowledge that may be obtained of the highest things is more desirable than the most certain knowledge obtained of lesser things.

- St. Thomas Aquinas

I Introduction

Language, for all its kingly role, is in some sense a superficial emboidery upon deeper processes of consciousness, which are necessary before any communication, signaling, or symbolism whatsoever can occur.

- Benjamin Lee Whorf, 1941 (1966, p.279)

The general purpose of this paper is to raise for discussion some issues and queries about research in language and language acquisition. In particular the area of inquiry will be the logic subjacent to communication. In question here are the foundations of communication. What, if anything, underlies language?

The general argument to be made runs roughly as follows:

> The human animal is, to a great extent, characterized by the mathetic capacity. This innate capacity, which is related to the primal abilities of classifying and rule-generating and following, is essentially mathematical in nature and underlies all cognitive activity including language. Relative to their own sets of assumptions, human beings tend to be logical or consistent beings. A rich, and not well-explored, area for understanding the nature of language and its acquisition, and intellectual growth in general, is the study of children in situations where their implicit sets of rules or assumptions become clear. The clarity, precision, and power of certain elementary mathematical structures make them ideal vehicles for the study of the mathetic capacity and its manifestations.

The foregoing position is not an orthodox one. Its assumptions are ambitious and its empirical foundations slim. Nevertheless, it has strong links with some classical intellectual theories. In the following sections of the paper an attempt will be made to develop this position.

II Assumptions

> *The reader has a right to know how the author's opinions were formed. Not, of course, that he is expected to accept any conclusions which are not drawn out by argument. But in discussions of extreme difficulty, like these, when good judgement is a factor, and pure ratiocination is not everything, it is prudent to take every element into consideration.*
>
> — C.S. Peirce, 1897 (1955, p.1)

> *It often happens, therefore, that in criticising a learned book of applied mathematics, or a memoir, one's whole trouble is with the first chapter, or even with the first page. For it is there, at the very outset, where the author will probably be found to slip in his assumptions. Farther, the trouble is not with what the author does say, but with what he does not say. Also it is not with what he knows he has assumed, but with what he has unconsciously assumed. We do not doubt the author's honesty. It is his perspicacity which we are criticising.*
>
> — A.N. Whitehead, 1925 (1964, p.29)

Bearing the opinions of Peirce and Whitehead in mind, it is perhaps worth briefly describing some of the conscious assumptions which underlie this paper. We will first mention a few intellectual characteristics of the author and then adumbrate some value positions.

The writer of this work could be described as a North American mathematician and educator with strong interests in research, psychology, and the history of ideas. As a mathematician in the Pythagorean tradition he is specially conscious of the use, and misuse, of his discipline in other academic areas and in western culture in general. As an educator he particularly values research which can be applied to help children cope more effectively with their individual environments. As a student of the history of ideas he is skeptical of the emphasis placed on instrumentalism and the quantifiable aspects of life in North America.

There are three significant value judgements implicit in the remainder of this paper which deserve identification. These interrelated concerns are about formalism, holism, and interdisciplinarity. All three can be related to an analogy between architecture and the potency of a mathematical discipline related by Von Neumann (1961):

> As a mathematical discipline travels far from its empirical source, or still more, if it is a second and third generation only indirectly inspired by ideas coming from "reality", it is beset with very grave dangers. It becomes more and more purely aestheticizing, more and more purely *l'art pour l'art*. This need not be bad, if the field is surrounded by correlated subjects, which still have closer empirical connections, or if the discipline is under the influence of men with an exceptionally well-developed taste. But there is a grave danger that the subject will develop along the line of least resistance, that the stream, so far from its source, will separate into a multitude of insignificant branches, and that the discipline will become a disorganized mass of details and complexities.
> In other words, at a great distance from its empirical source, or after much "abstract" inbreeding, a mathematical subject is in danger of degeneration. At the inception the style is usually classical; when it shows signs of becoming baroque, then the danger signal is up.
>
> (p.2063)

Despite the increasing use of mathematics in many academic disciplines and some strong unifying influences (Bourbaki, 1966), mathematics would not seem to have completely avoided the dangers of degeneration Von Neumann foresaw (Coleman, 1976). However, if some branches of mathematics can be said to be baroque in the Von Neumann sense, there are many other disciplines which have long since entered the rococo stage. In these areas one characteristically meets pages of highly formal, quasi-mathematical material which bears no visible link to any recognizable version of "reality". Exemplars of wretched excesses of this type would seem to appear in much of the literature in contemporary economic and political theory. Perhaps the discipline where this excessive 'formalism' has been most rampant, however, has been twentieth-century philosophy in the English-speaking world with the extensive influence of the schools of logical positivism and linguistic analysis (Mundle, 1970).

The range, scale, and organization of contemporary intellectual activity is such that it is most difficult for scholars to break out of the narrow, highly-specialized niches in which they find themselves. The growing awareness of the severity of the global environmental situation (Waddington, 1977) has illustrated perhaps better than anything else, however, the dangers implicit in overlooking the holistic aspects of general contexts of academic questions. In particular, scientists can no longer afford to ignore questions of value.

One corollary of this is that there is a real need for research endeavours which are at least interdisciplinary (Toulmin, 1977) and which have the potential to become transdisciplinary (Margenau, 1971).

III Issues

(a) Introduction

> *Mathematics, science, and language constitute the principal functions in the activity of man, by means of which he rules over nature and maintains order in its midst. The origin of these functions is to be found in the three modes in which the will to live of the individual man shows itself: 1. the mathematical attitude of mind, 2. mathematical abstraction, and 3. the imposition of the will by means of sounds.*
>
> - L.E.J. Brouwer, 1928 (Kneebone, 1963, p.321)

In this part of the paper an attempt will be made to identify some of the general issues which arise in any effort to understand human cognitive behaviour and specially those which relate mathematics to language and its acquisition. Because of the magnitude of the area and the strict limitation of space, the treatment will be impressionistic rather than comprehensive. The first two sections deal with the relation of mathematics to two other contenders for primal intellectual activity, language and logic. In the third section some connections between mathematics and linguistics are mentioned. The final four sections deal, respectively, with innateness, cognitive development, rules, and mathesis.

(b) On Mathematics and Language

> *Sellers: How's your mathematics?*
> *Secombe: I speak it like a native.*
>
> - The Goon Show

> *Mathematics, far from being a mere language, is the very instrument of structuration.*
>
> - Jean Piaget (1971b, p.338)

> *Spoken language is merely a series of squeaks.*
>
> - A.N. Whitehead, 1929 (1969, p.308)

Every attempt to construct a comprehensive framework of general ideas in which human experience can be interpreted must grapple with the phenomena of language and mathematics and their interrelation. One intellectual tradition which can be traced back at least to the time of Pythagoras has always given mathematics the fundamental role in its general philosophic framework. Scholars who could be called members of this Pythagorean tradition would

include Plato, Kepler, Leibniz, Whitehead, and Heisenberg. Characteristic features of thought in the Pythagorean tradition are an emphasis on the qualitative, aesthetic dimension of mathematics; a strong belief in harmony and the existence of central order; and a sense that knowledge, as well as power, is wonder (Thompson, 1971; Higginson, 1977a).

That thinkers of the mathematical profundity of Leibniz and Whitehead should create general philosophies which are highly mathematical in orientation is not surprising. Perhaps less expected is the emphasis given to the role of mathematics in their general surveys of thought and culture by thinkers who come from less specialized backgrounds (Korzybski, 1958; Spengler, 1939). The case of Jung who found mathematics as a schoolboy to be "sheer terror" (1965, p.29) and who late in life saw "that it might be possible to take a further step into the unity of psyche and matter through research into the archetypes of the natural numbers" (Von Franz, 1974, p.ix), is another interesting case.

There are two versions of the 'mathematics is a language' thesis which one can observe. The less sophisticated of the two is the 'mathematics is *only* a language' which would seem to be postulated mainly by anxious humanists who have little insight into the range and depth of contemporary mathematics. The other version can be seen in the attempts of Leibniz to construct a 'universal characteristic' (Wiener, 1951):

> a language which would embrace both the technique of discovering new propositions and their critical examination--a language whose signs or characters would play the same rôle as the signs of arithmetic for numbers and those of algebra for quantities in general.
>
> (p.18)

There are also signs of this idea in Whitehead's work (1956):

> I am impressed by the inadequacy of language to express our conscious thought, and by the inadequacy of our conscious thought to express our subconscious. The curse of philosophy has been the supposition that language is an exact medium. Philosophers verbalize and then suppose the idea is stated for all time. Even if it were stated, it would need to be restated for every century, perhaps every generation. Plato is the only one who knew better and did not fall into this trap. When ordinary methods failed him, he gave us a myth, which does not challenge exactitude but excites revery. Mathematics is more nearly precise, and comes nearer to the truth. In a thousand years it may be as commonly used a language as is speech today.
>
> (p.295)

Among present-day humanists George Steiner (1969, 1971, 1975) is unique in his sensitivity to this possible direction in the evolution of language:

> It is a commonplace of current sociology and 'media-study' that this primacy of the 'logic'--of that which organizes the articulations of time and of meaning around the *logos*--is now drawing to a close. Increasingly, the word is caption to the picture. Expanding areas of fact and of sensibility, notably in the exact sciences and the non-representational arts, are out of reach of verbal account of paraphrase. The notations of symbolic logic, the languages of mathematics, the idiom of the computer, are no longer meta-dialects, responsible and reducible to the grammars of verbal cognition. They are autonomous communicatory modes, claiming and expressing for themselves an increasing range of contemplative and active pursuit. . . . If music is one of the principal 'languages outside the word', mathematics is another.
>
> (1971, pp.86,95)

Two other regions where language and mathematics overlap are in the areas of the study of symbolic forms and in etymology. The work of Ernst Cassirer (1955, 1970) is in a class of its own in the first of these areas. His thesis that *animal symbolicum* is a more accurate 'definition' of man than *animal rationale* (1970, p.28) is a very potent one for the study of language and mathematics. The examination of mathematical terminology in different languages can yield great insights into the relations between different languages as Menninger (1969) has shown so clearly (Beilin, 1975). In addition to this we have the phenomenon noted by Brown (1973):

> One of the most beautiful facts emerging from mathematical studies is [illegible] very potent relationship between the mathematical process and ordinary language. There seems to be no mathematical idea of any importance or profundity that is not mirrored, with an almost uncanny accuracy, in the common use of words, and this appears especially true when we consider words in their original, and sometimes long forgotten, senses.
>
> (pp.90-91)

(c) On Mathematics and Logic

> *LOGIC, n. The art of thinking and reasoning in strict accordance with the limitations and incapacities of the human misunderstanding. The basic of logic is the syllogism, consisting of a major and a minor premise and a conclusion--thus:*
>
> Major Premise: *Sixty men can do a piece of work sixty times as quickly as one man.*
>
> Minor Premise: *One man can dig a post-hole in sixty seconds; therefore--*
>
> Conclusion: *Sixty men can dig a post-hole in one second.*
>
> *This may be called the syllogism arithmetical, in which, by combining logic and mathematics, we obtain a double certainty and are twice blessed.*
>
> \- Ambrose Bierce, *The Devil's Dictionary*, 1906 (1958)

> *In the following we shall use the notions "formal," "logical," and "mathematical" as synonyms.*
>
> \- W. Pankow, 1976 (Jantsch and Waddington, 1976, p.22)

Logic is a discipline with a long history (Kneale and Kneale, 1962). The last century has seen, in particular, the extensive development (Van Heijenoort, 1967) of the very abstract branch of logic, that of formal, symbolic, or mathematical logic (Kleene, 1952; Curry, 1977). The development of symbolic logic provides a good illustration of the Von Neumann thesis described earlier. Although pages of symbolic logic may appear highly 'degenerate', the advent of computers has meant that many parts of the discipline have been provided with substantial links with 'reality' and have continued to thrive.

As in the case of language, the question of the relationship between mathematics and logic has been to a great extent a dispute about primacy and 'territoriality'. One of the three commonly recognized (Eves and Newsom, 1965) schools of mathematical philosophy, that of Logicism, is founded on the premise that logic is the parent of mathematics. Although there are clear indications in their work that Dedekind and Frege held this position, the best known exponent of this school was Russell (1937) who argued in several of his works that "all mathematics follows from symbolic logic" (p.9).

The extent to which this view has been accepted by mathematicians (who are notorious for their lack of interest in foundations) varies considerably

from individual to individual (Putnam, 1975). The 'formalist' school under Hilbert could not accept the whole logistic argument and the 'intuitionist' school led by Brouwer found it antithetic to their position. A common contemporary view of the question is stated by Kneebone (1963):

> The earlier critical movement led eventually to the attempts of Frege and Russell to reduce mathematics without residue to pure logic; but in the more recent history of the study of foundations of mathematics logic has been found insufficient by itself, and appeal has had to be made once again to some kind of intuition.
>
> (p.357)

There has been, in a sense, therefore, a return to the position outlined by C.S. Peirce prior to the publication of much of Russell's work. Writing of conversations with his father who was a distinguished Harvard professor, Peirce stated:

> He, a mathematician, and I, a logician, held daily discussions about a large subject which interested us both; and he was struck, as I was, with the contrary nature of his interest and mine in the same propositions.
>
> (1955, p.141)

There seems to be no doubt that symbolic logic is appropriate for the 'thinking' of computers. When it comes to the case of human thought the appropriate logic is, however, much less obvious (Henle, 1962). It is clear, though, that human thought cannot be modelled in anything other than a highly idealized form on the fully-developed version of mathematical logic. In considering this question it appears that the only definition of logic which can be universally applied to acts of human behavior and cognition is the very basic one of logic as "the study of consistent sets of beliefs" (Hodges, 1977, p.13). This opens the door to the consideration of several different types of logic which may be in conflict for a given individual at a given moment, as for instance when a child's social logic may not coincide with his moral logic.

From this focus on consistency one can come full circle to see logic as a derivative of mathematics. In the words of Peirce

> Our native capacity for thinking rigourously (susceptible, of course, to development through training and practice) is the only thing from which mathematics can be 'derived'. . . . That is why it may be truly said that 'mathematics lays the foundation on which logic builds'.
>
> (Goudge, 1969, p.58)

(d) On Mathematics and Linguistics

> *To be human requires the study of structure.*
> *To be animal merely requires its enjoyment.*
>
> \- A.N. Whitehead, 1938 (1958, p.105)

> *The last point is a request to the English-speaking reader. In France, certain half-witted 'commentators' persist in labelling me a 'structuralist'. I have been unable to get it into their tiny minds that I have used none of the methods, concepts, or key terms that characterize structural analysis.*
>
> \- Michel Foucault (1970, p.xiv)

In any history of twentieth-century mathematics which may come to be written, primacy of place will have to be given to the work and influence of the polycephalic French mathematician Nicolas Bourbaki (1966, 1971, 1974; Halmos, 1968). Some forty years ago Bourbaki began a most ambitious project; the creation of a unified treatise on mathematics which would embrace all branches of the discipline. In this project it soon became clear that the concept of 'structure' was to play a crucial role. Bourbaki (1971) saw structures as the "tools of the mathematician" (1971, p.31) and the basic building blocks of "The Architecture of Mathematics" (the title of an important paper published in 1949).

In his attempt to "present the entirety of the mathematical universe" (1971, p.32), Bourbaki saw "at the center" three "mother-structures". These were algebraic structures, order structures, and topological structures, having, respectively, to do with composition, rank, and proximity. Classic, representative examples of each of these structures were groups, lattices, and topologies.

Although the Bourbakist (1971) approach with its emphasis on the axiomatic method is in some ways a highly formal one, the Bourbakists see themselves as having a dynamic version of formalism:

> From the axiomatic point of view mathematics appears on the whole as a reservoir of abstract *forms*--the mathematical structures; and it sometimes happens, without anyone really knowing why, that certain aspects of experimental reality model themselves after certain of these forms, as if by a sort of preadaptation. . . . It is only in this sense of the word "form" that the axiomatic method can be said to be a "formalism"; the unity that it confers on mathematics is not the supporting framework of formal logic, the unity of a lifeless skeleton, but the nourishing sap of an organism in full development.
>
> (p.36)

Within mathematics the Bourbakist position is quite a controversial one. There is no doubt, however, that the 'mother-structures' conception, which differs so significantly from the more traditional classifications of mathematics (as being the study of space and quantity or having the 'branches' of arithmetic, geometry, and analysis) has been a potent stimulus to mathematical research. It is perhaps somewhat more surprising to see the considerable impact this methodology has had on other academic disciplines. In the past decade 'structuralist' techniques have been consciously applied in fields as diverse as literature and anthropology (Lane, 1970; Levi-Strauss, 1969). At least a few structuralists (Piaget, 1971a) are aware of the extent to which the model for their endeavours is the Bourbakist treatise. In the area of linguistics both Piaget and Chomsky have been considerably influenced by the structuralist approach.

The scientific study of language, or linguistics, in common with many other contemporary disciplines, has become highly mathematized. What is somewhat unusual is the extent to which it is the contemporary developments in mathematics rather than the classical techniques which are being utilized (Harris, 1969; Lyons, 1970). While in many areas it is the quantitative, statistical side of mathematics which is used most, in linguistics, with the exception of areas like computational stylistics, these techniques are not particularly important. In contrast, the structural side of mathematics, particularly with regard to algebraic structures, is proving to be highly productive when applied to linguistic problems (American Mathematical Society, 1961; Luce et al, 1963).

As examples, it is perhaps worth briefly mentioning two areas of contemporary mathematics which have implications for the study of language. The first of these is the theory of finite-state automata, and the second is coding theory.

A finite-state automaton, or finite-state machine can be defined, in mathematical terms, as (Birkhoff and Bartee, 1970):

> A 5-tuple $[A, S, Z, \S, \Omega]$, where A is a finite list of input symbols: $A = \{A_0, A_1, \ldots, A_n\}$, Z is a list of output symbols: $Z = \{Z_0, Z_1, \ldots, Z_m\}$, S is a set of internal states: $S = \{S_0, S_1, \ldots, S_r\}$, § is a next-state function from $S \times A$ into S and Ω is an output function from $S \times A$ into Z. (p.66)

Finite-state automata and their Turing-machine cousins have, over the last thirty years, been objects of intense interest to scholars in computer science (Preparata and Yeh, 1973) and artificial intelligence (Burks, 1970). They have, as well, been applied extensively in the area of linquistics (Gross, 1972).

Coding theory, on the other hand, seems to have been virtually ignored by linguistics. This is somewhat unusual since coding theory can be seen as an attempt to construct a mathematical language (Van Lint, 1973; Berlekamp, 1968). The fundamental problem from which coding theory evolves is that of creating a means of communication between computers which is optimally efficient. That is, a code which produces the smallest number of communication errors most economically.

Even a brief study of automata and coding theory reveals some interesting features. From the study of automata one is impressed by the complexities which can be generated from very simple components (this is, of course, the 'secret' of the modern-day computers) following a small number of explicit rules. A striking characteristic of coding theory is the extent to which an attempt to construct an efficient language leads directly to a consideration of algebraic questions, in particular the theory of groups, and to mathematical concepts, not only of considerable utility but also of great elegance and power. The extensive redundancy of natural language in contrast with algebraic codes is specially striking.

In short, one has here, in one area of concern, two examples of an intriguing phenomenon which may well be quite significant; the "algebraization" of mathematics (Eilenberg, 1969). At the root, this phenomenon is the technique of isolating rules and observing the properties which emerge in the structures generated by the interaction of these rules.

(e) On Innate Capacities

> *Everything primary, and consequently everything genuine, in man works as the forces of nature do,* unconsciously. *What has passed through the consciousness thereby becomes an idea: consequently the expression of it is to a certain extent the communication of an idea. It follows that all the genuine and proved qualities of the character and of the mind are primarily unconscious and only as such do they make a deep impression. What man performs unconsciously costs him no effort, and no effort can provide a substitute for it: it is in this fashion that all original conceptions such as lie at the bottom of every genuine achievement and constitute its kernel come into being. Thus only what is inborn is genuine and sound: if you want to achieve something in business, in writing, in painting, in anything, you must* follow the rules without knowing them.
>
> - Schopenhauer (1970, pp.175-176)

> *The natural operations of the human mind are algebraic in character. That is, we are all born potential algebraists!*
>
> - A.J. Coleman (1977)

In the vast literature on language and language acquisition (Abrahamsen, 1977) no question has been more actively contested than the innateness hypothesis. The most recent chapter of this long-standing dispute between nativists and environmentalists (Osser, 1971, 1975) began with Chomsky's (1959) classic review of Skinner's *Verbal Behavior.* The essence of Chomsky's argument was that the behavioristic model of language acquisition was inadequate since it could not explain children's ability to generate novel sentences. To explain this phenomenon Chomsky postulated the existence of an innate language capacity in the human animal, The counterattacks on this position were numerous and strong and though the debate was a heated one for many years a sort of stalemate now seems to exist.

This question is, of course, only the contemporary manifestation of a classic philosophical debate between rationalists and empiricists (Stich, 1975). Chomsky, in particular, is well aware of the intellectual background of this issue and frequently stresses his debt to Descartes.

Three aspects of this issue are worth noting here. First, there is the question of motivation. Chomsky (1975) does not see the study of language as an end in itself, but rather as a means of understanding some more general human characteristics:

> One reason for studying language--and for me personally the most compelling reason--is that it is tempting to regard language, in the traditional phrase, as a 'mirror of the mind'. . . . By studying the properties of natural languages, their structure, organization, and use, we may hope to gain some understanding of the specific characteristics of human intelligence. We may hope to learn something about human nature; something significant, if it is true that human cognitive capacity is the truly distinctive and most remarkable characteristic of the species . . . the study of this particular human achievement . . . may serve as a suggestive model for inquiry into other domains of human competence and action that are not quite so amenable to direct observation.
>
> (pp.3-4)

It is Chomsky's (1975) opinion that the question of innateness is not so much a question of 'if' as it is a question of 'what'.

> Every 'theory of learning' that is even worth considering incorporates an innateness hypothesis. . . . The question is not whether learning presupposes innate structure--of course it does; that has never been in doubt--but rather what these innate structures are in particular domains.
>
> (p.13)

One of the criticisms of the existence of 'LADS' (language acquisition devices) which Chomsky seems to have most difficulty in rebutting, is one put forward by Putnam (1971). Putnam questions the existence of a *specific* language capability and suggests that "what must be 'innate' are heuristics . . . general multipurpose learning strategies" (p.138). Chomsky (1972) has responded to this suggestion by essentially saying 'let's try it and find out'.

> . . . a nondogmatic approach to this problem can be pursued . . . through the investigation of specific areas of human competence, such as language, followed by the attempt to devise a hypothesis that will account for the development of this competence. If we discover through such investigation that the same 'learning strategies' are sufficient to account for the development of competence in various domains, we will have reason to believe that Putnam's assumption is correct. If we discover that the postulated innate structures differ from case to case, the only rational conclusion would be that a model of mind must involve separate 'faculties' with unique or partially unique properties. I cannot see how anyone can resolutely insist on one or the other conclusion in the light of the evidence now available to us.
>
> (pp.86-87)

Despite the importance of the thesis raised by Putnam, there seems to have been little work done directly on this issue. The growth of computer technology has encouraged the development of information-processing models of human cognition which indirectly address this question. These range from the specific and practical (Farnham-Diggory, 1972) through the mechanistic (one well-known computer scientist is reputed to have stated, "After all, what is the human brain except a computer made of meat?") to the esoteric (Lilly, 1974).

(f) On Cognitive Development

We would suggest that learning mathematics may be viewed as a microcosm of intellectual development.

- Bruner and Kenney, 1965 (1968, p.420)

Every newborn child provides in embryonic form the sum total of possibilities, but each culture and period of history will retain and develop only a chosen few of them. Every newborn child comes equipped, in the form of adumbrated mental structures, with all the means ever available to mankind to define its relations to the world and its relations to others. But these structures are exclusive. Each of them can integrate only certain elements out of all those that are offered. Consequently, each type of social organization represents a choice, which the group imposes and perpetuates. In comparison with adult thought, child thought is a sort of universal substratum the crystallizations of which have not yet occurred, and in which communication is still possible between incompletely solidified forms.

- Claude Lévi-Strauss (1969, p.93)

Psychological history may eventually show that Piaget was really talking about mathematical abilities all along.

- Sylvia Farnham-Diggory (1972, p.485)

Any attempt to understand the development of a cognitive functioning in humans must take into consideration the massive research programme of the Genevan school under the leadership of Jean Piaget. In North America in the past decade Piaget has become something of an educational cult figure. There are, however, reasons to feel that the 'interpretation' of Piaget's work which has been made by many psychologists and educators is a distorted one (Higginson, 1976). There would seem to be at least four reasons for this: the scale of the research, its epistemic context and the roles attributed to language, and to mathematics and logic.

It has been estimated that in over half a century of work in the field Piaget has written the equivalent of fifty five-hundred-page books on child development. All too often this edifice is reduced to an oversimplified description of 'four stages' and educators scurry away to devise diagnostic tests and means for accelerating children through the stages. Hence one gets the ironic situation of having the holistic, constructive Piagetian position being interpreted in reductionistic terms with the major focus in application being on those activities children are *unable* to do.

This emphasis has been so constant and powerful that even knowledgeable observers such as Smedslund (1977) now begin to see any reaction against it as a rejection of the Piagetian thesis:

> When I meet a small child I always take for granted that, within his limited sphere of activity and given his own premises, he is logical, and my problem is to understand what his expressions mean, and hence to grasp his existential situation. In so far as Piagetian psychologists focus on logicality as a variable (e.g. conserver or non-conserver) and give only peripheral attention to the problem of determining children's understanding of instructions and situations, I think they are making an epistemological error and are out of step with everyday human life as well as with all useful psychological practice. It may be objected that Piaget is not really denying the logicality of children at any stage, but is merely studying the various *forms* of logic they have attained. However, it is a matter of historical record that children who failed on tasks were often simply described as non-logical, and that the problem of criteria of understanding has received relatively scant attention in Piagetian literature.
>
> (p.4)

It seems likely that behind this tendency is a failure to fully comprehend the intellectual context of Piaget's work. Genetic epistemology is a long way removed from stimulus-response learning theories but if Piaget is, intellectually speaking, closer to Kant than he is to Skinner, the implications of this fact have not filtered through to many educators or researchers. There does not seem, for instance, to be any widespread awareness of the fundamental paradoxes inherent in the very common activity of designing behavioral objectives for Piagetian stages.

The secondary role of language is another aspect of the Genevan theory which causes difficulty. It is traditional to see language and thought as more closely identified than in the Piagetian (1968) view.

> Language is not enough to explain thought because the structures that characterize thought have their roots in action and in sensorimotor mechanisms that are deeper than linguistics.
>
> (p.98)

The fundamental role played in Piagetian theory by logico-mathematical constructs is hard to fully appreciate. While some mathematicians (notably Freudenthal, 1973) are not fully convinced of the mathematical soundness of Piaget's thinking, there is no doubt that the Genevan literature is built around some quite sophisticated pieces of mathematics. It seems quite likely

that the average social scientist (who may well *be* a social scientist largely because of an inability to handle mathematics) will not make much of material such as the following:

> . . . the gravitational field can be represented by a tensor whose components change with the system of co-ordinates. But the tensor itself is an invariant. This tensor coincides with the tensor that defines the metric of space.
>
> (Piaget, 1977, p.174)

In his conception of the relation between thought and mathematical processes Piaget comes close to a classic position outlined by Boole (1958) in 1854:

> The laws of thought, in all its processes of conception and of reasoning, in all those operations of which language is the expression or the instrument, are of the same kind as are the laws of the acknowledged processes of Mathematics . . . upon the very ground that human thought, traced to its ultimate elements, reveals itself in mathematical forms, we have a presumption that the mathematical sciences occupy, by the constitution of our nature, a fundamental place in human knowledge, and that no system of mental culture can be complete or fundamental, which altogether neglects them.
>
> (p.423)

The theory of Piaget (1971a) has many features in common with that of Chomsky:

> While the logical positivists, enthusiastically followed by Bloomfield, wanted to reduce mathematics and logic to linguistics and the entire life of the mind to speech, Chomsky and his followers base grammar on logic and language on the life of reason. . . . Chomsky actually arrived at this conception of linguistics structure by combining mathematico-logical concepts and techniques of formalization (algorithms, recursive devices, abstract calculi, and especially the algebraic concept of the monoid or semigroup) with ideas taken from general linguistics on the one hand (especially the conception of syntax as 'creative') and from psycholinguistics on the other (for example, the idea of the speaker-hearer's 'competence' in his own language).
>
> (pp.83-84)

Despite these considerable similarities the advocates of the two schools have tended to accentuate the differences in position. At times these considerations have become moderately heated. Chomsky (1976) has in one instance characterized the Genevan representation of his position as constituting, once again, "a chapter in the history of dogmatism" (p.19).

Piaget's (1971b) speculations on the nature of innate capabilities are of interest:

> To suppose that the ultimate origin of the coordinations underlying logico-mathematical structures is to be found at the very center of the most highly generalized functioning of the living organization is itself a solution of a kind, insofar as it concerns the harmony between these coordinations or structures and the outer environment.
>
> (p.345)

Another intriguing observation made by Piaget (Beth and Piaget, 1966) concerns the relation of natural mental structures to the Bourbakist mother-structures:

> In 1952 a small colloquium was held at Melun near Paris on 'mathematical and mental structures'. This colloquium opened with two papers, the first by J. Dieudonné on Bourbakist structures and the other by myself on mental structures. Now, without knowing Bourbaki's work at the time, we found, merely by attempting to classify the different operational structures observed empirically in the development of the child's intelligence, three types of structures. These were to begin with irreducible, combining with each other later in different ways: structures of which the reversible form is inversion or annulment (A-A=0), and which we may describe by referring to algebraic or group models; structures whose form of reversibility is reciprocity, and which must be described in terms of relation and order; and structures basic to the continuum, especially spatial structures whose elementary forms, surprisingly enough, are of a topological character, and appear before metric and projective constructions! This convergence between these two entirely independent accounts impressed the members of this colloquium, especially the two authors themselves (of whom, if we may say so, the first is known for his wilful ignorance of psychology and the second for his unwilful ignorance of mathematics. . .).
>
> (p.168)

(g) On Rules

> *Any system of thought, whether within the sciences or the arts, aims to produce expressible* forms *, and the distinction between form and formlessness can be expressed by one word:* rules. *Sometimes these rules are deliberately set up and known; sometimes we merely conform to them without knowing them in any articulate sense (as we obey the rules of the land without knowing much about them). Sometimes the rules are dictated by Nature (as an artist or sculptor is restricted to some extent by the properties of paint or stone). Form requires rules, perhaps better described as 'constraints'. rules which are mostly man-made (self- or socially-imposed).*
>
> \- Colin Cherry, 1971 (1973, p.271)

> *The notion of the importance of pattern is as old as civilization. Every art is founded on the study of pattern. . . . Mathematics is the most powerful technique for the understanding of pattern, and for the analysis of the relationships of patterns.*
>
> \-- A.N. Whitehead, 1941 (Schilpp, 1951, pp.677-678)

It would be possible to make a strong argument for the case that much of the scientific progress which has been made in the last three hundred years is due to the increasing sensitivity in different fields of the implications of the concept of rule. Two specific areas where this can be easily illustrated are mathematics with the concept of function (Bochner, 1966; Kline, 1972), and computer science with the concept of algorithm (Knuth, 1968). One reason why this does not seem immediately obvious is the obfuscation introduced by the existence of a large number of cognate terms such as law, regularity, pattern, relation, and form.

The structuralist position, which has been noted in a previous section, is one which is particularly rule-related, given that structures are essentially nothing more than sets of rules. In light of this, it is most surprising to see how little work has been done by structuralist researchers on the role (itself a rule-oriented word) of rules in language and cognitive development. In the places, for example, where Piaget's experiments deal with the concept of rule (1962, 1965) he focuses on the moral aspects of child development.

Toulmin (1974) has commented on this situation:

> Questions about 'rules' and their role in human conduct, arise repeatedly nowadays in methodological discussions of mental philosophy, cognitive psychology and linguistics alike. Yet the term 'rules' itself--like the associated term, 'concept'--remains one of the great *unanalyzed analysanda* of cognitive theory. Thus, the well-known disagreement between Skinner (1957) and Chomsky (1959) over linguistic behavior turned on the question, whether it was possible to account for the development of our capacity to follow linguistic 'rules' by appeal to the 'laws' of operant conditioning alone. So one might have expected psycholinguists to attack, quite directly, the consequent analytical problems about the nature and function of rules. Yet by now, a dozen years later, they have done scarcely anything to demonstrate the special features of 'rule-governed' and 'rule-following' behavior, and so to establish the significance of rules for theoretical psychology.
>
> (p.186)

Language acquisition is a process which is specially rich in rule applications and the fact that the area has not been more intensively studied is quite surprising for the insights to be gained from the observation of children's language rules would seem to be considerable. (A favourite example is that of the child who on first seeing a fish fork, which has three tines, called it a "threek"). The work which has been done in this area has dealt with children's rules for creating plurals and verb tenses. Commenting on this research Slobin (1971) has written:

> . . . evidently children are especially sensitive to patterned regularities. As soon as the pattern is noticed, the child will try to apply it as broadly as possible, thus producing words which are regular, even if they have never been heard before. One cannot help but be impressed with the child's great propensity to generalize, to analogize, to look for regularities--in short, to seek and create order in his language.
>
> (p.50)

Perhaps an even more fundamental intellectual ability than rule-forming or rule-following is the related intellectual ability of discriminating or classifying. The importance of the classifying ability to language was recognized by Jespersen (1922) over half a century ago:

> *The Classifying Instinct.* Man is a classifying animal: in one sense it may be said that the whole process of speaking is nothing but distributing phenomena of which no two are alike in every respect, into different classes on the strength of perceived similarities and dissimilarities. In the name-giving process we witness

> the same ineradicable and very useful tendency to see likenesses and to express similarity in the phenomena through similarity in name. Professor Hempl told me that one of his little daughters, when they had a black kitten which was called *Nig* (short for Nigger), immediately christened a gray kitten *Grig* and a brown one *Brownig*.
>
> (pp.888-889)

The recent work of Brown (1973) can be taken as an example of the power and depth of the mathematical and logical edifice which can be constructed from the starting point of classification:

> The theme of this book is that a universe comes into being when a space is severed or taken apart. The skin of a living organism cuts off an outside from an inside. So does the circumference of a circle in a plane. By tracing the way we represent such a severance, we can begin to reconstruct, with an accuracy and coverage that appear almost uncanny, the basic forms underlying linguistic, mathematical, physical, and biological science and can begin to see how the familiar laws of our own experience follow inexorably from the original act of severance.
>
> (p.v)

(h) On Mathesis

All things began in Order, so shall they end, and so shall they begin again; according to the Ordainer of Order, and the mysticall Mathematicks of the City of Heaven.

- Sir Thomas Browne, 1658 (Needham, 1968, p.iii)

'Mathematizing' may well be a creative activity of man.

- Hermann Weyl (1963, p.219)

There is present on many levels in nature a tendency toward order, form, and symmetry; hence in living systems toward organic coordination; this tendency being realized when circumstances are favorable.

- Lancelot Law Whyte (1974, p.20)

And clothest Mathesis in rich ornaments,
That admirable mathematique skill.

- George Peele, 1593 (*O.E.D.*, p.1743)

We have a direct awareness of mathematical form as an archetypical structure.

- G. Spencer Brown, 1969 (1973, p.xviii)

In the previous sections reference has been made to developments in several fields which share an emphasis on order, structure, or form. There are as well concomitant publications in other arts and sciences which share this emphasis (Capra, 1975; Whyte et al, 1969; Bronowski, 1973; Thom, 1975; Alexander, 1964; Waddington, 1970; Polanyi, 1964; Pirsig, 1975; Radnitzky, 1973; Von Bertalanffy, 1967; Jolley, 1973; Thompson, 1969; Weyl, 1952; UNESCO, 1972; Whyte, 1968, 1974).

It is possible to see this movement as a renascent form of Pythagoreanism with its fundamental characteristic of harmony or symmetry. Heisenberg (1972) has written:

> But what was there in the beginning? A physical law, mathematics, symmetry? In the beginning was symmetry!
>
> (p. 133)

In a similar vein Abdus Salam (1972) has stated:

> We have always found that whenever a postulated symmetry principle was appearing to fail in natural phenomena, this must be due to some still greater symmetry, with which it must be in conflict. We may, at a given time, fail to comprehend the aesthetics of nature. When, however, the full and final picture emerges, one has invariably found that the symmetries this exhibits are profounder still.
>
> (p.78)

The 'algebraization of mathematics' which has been mentioned previously is also a reflection of this neo-Pythagorean trend since the theory of groups is nothing more than the mathematization of the concept of symmetry. Inherent in this theory are the ideas of balance, inverse, unity, duality, and equilibrium (Weyl, 1952). Piaget's work, with its concepts of equilibration and structures, can be seen to be highly consistent with this trend as well. In addition, one can find explicit statements such as the following in his work (1971):

> Mathematics today is taking a decidedly qualitative trend, and its involvement with isomorphisms and morphisms of all kinds has opened up such broad structuralist perspectives that there is apparently no field--human, biological, or physical--that cannot now be reduced to fairly elaborate mathematization.
>
> (p.340)

In the English language there is no term in common use which refers to the dynamic human ability to impose intellectual order or form on a situation. This was not always the case. In earlier times the word mathesis (from the Greek root mathein, 'to learn', as in 'polymath') was used to refer to mental discipline, learning, or science. Implicit in the term was the idea that mathesis had to do with the process of mathematizing rather than with the end product of this process which was mathematics.

An attempt must now be made to synthesize the foregoing with the original question of the foundations of communication in mind. Taking some of the adumbrated issues into account the rather bold mathetic conjecture emerges.

> Underlying human cognition in general and human communication in particular is the mathetic or mathematizing capacity of the human animal. This inborn capacity is built on the abilities to classify and to create and follow rules. It manifests itself in the production and utilization of symbol systems, and depends to a great extent on an inherent tendency to equilibrium, order, and symmetry in many spheres.

IV Research

The process of the acquisition of language always involves an active and productive attitude. Even the child's mistakes are very characteristic in this respect. Far from being mere failures that arise from an insufficient power of memory or reproduction, they are the best proofs of activity and spontaneity on the part of the child. In a comparatively early stage of its development the child seems to have gained a certain feeling of the general structure of its mother tongue without, of course, possessing any abstract con-sciousness of linguistic rules. It uses words or sentences that it has never heard and that are infractions of the morphologic or syntactic rules. But it is in these very attempts that the child's keen sense for analogies appears.

- Ernst Cassirer, 1944 (1970, p.249)

Three, three, twenty-three; two threes, twenty-three.

- Adam (age 2½, looking at '33')

It is the intention of this section to mention some research work which is consistent with the mathetic conjecture. As has been previously noted, there has not been a great deal of activity in the linguistics area on children's rule-generating and rule-following behavior. In the case of educational research the traditional paradigm has not been a very productive one (Higginson, 1977b). It is perhaps not coincidental that the assumptions underlying the traditional paradigm are quite different from those underlying the mathetic conjecture. If one looks for research concerned with learning, hence eliminating, for instance, the majority of the work of the Genevan school, there are relatively few studies which fall into our category. Nevertheless, there have been some classic studies and there is some work in progress which would appear to be quite significant.

Perhaps the most fully-developed research programme of a mathetic type is the one under the direction of Herbert Ginsburg (1977). Working with colleagues at Cornell and Illinois (Easley and Zwoyer, 1975), Ginsburg, who has a strong background in cognitive psychology, is investigating "children's mathematical behavior". Making very good use of clinical interview techniques, Ginsburg is finding that children bring a wide and ingenious range of individual strategies and techniques to standard arithmetical tasks. In particular, young children feel secure and are quite effective when employing counting

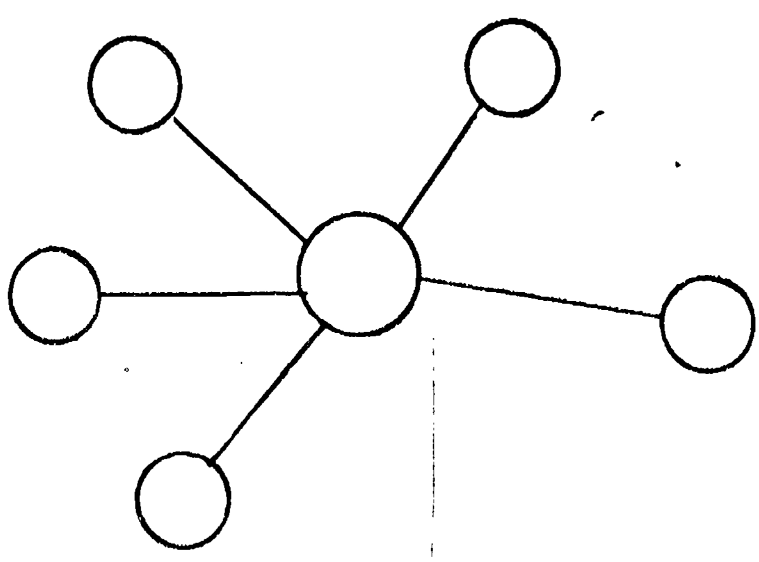
strategies for arithmetic questions.

As an example of the sort of rule uncovered by Ginsburg, consider the seven-year-old child who has been given the question

$$\begin{array}{r} 15 \\ +\ \underline{17} \end{array} \quad \text{and who has responded by writing} \quad \begin{array}{r} {}^{1}15 \\ +\ \underline{17} \\ 32 \end{array}$$

Under standardized evaluation devices this answer is "correct" and it would be most unlikely for the child in question to be given any further attention by the teacher. In the interview, however, it appears that the child's rule is to "always carry the smaller number". There may well, therefore, be considerable difficulty at a later period when the child meets questions such as 15 + 17 + 19. Of particular interest in the journal which Ginsburg edits is the paper by Erlwanger (1973) which raises many questions about the effects of individualized instruction in mathematics. In a recent issue of this same journal the investigation of children's, rule-following, mathematical behavior was extended to consider problems in the teaching of reading.

Highly compatible with the results of the Ginsburg group are some of the findings of Bates and Higginson (1975) who have developed a method of teaching basic arithmetic operations using a concept called "frames". This method was developed in an inner-city elementary school to try to minimize on problems raised by language complexity, the lack of a unifying gestalt for basic operations and the absence of any means of linking activity and symbolization. A 'frame' is a topological construction consisting of a centre and arms. The number of arms can vary.



A Five-Armed Frame.

To develop all basic operations only the 'gathering' action (for addition and its inverse 'dispersing' for subtraction) and the idea of duplication are needed. Beginning at the kindergarten level with physical objects and using a highly divergent pedagogy ("Make ten frames") children advance quite rapidly through the "three modes of representation - enactive, ikonic, and symbolic" (Bruner et al, 1966, p.1). As in Ginsburg's observations, counting was seen to be a primal ability.

Whether or not one wishes to grant mathematical concepts any special place in the human make-up, the fact that they manifest themselves with such great clarity makes them excellent vehicles for the observation of language development. Unfortunately, in many classrooms, this potential clarity is never realized as teachers themselves do not use mathematical terminology with any consistency. Faced with a barrage of horizontal and vertical versions of "plus, add, +, all together, sum" it is not surprising that children create "unorthodox" methods. One can sympathize with the youngster who, seeing '15 - 6' and hearing 'fifteen take-away six' from his teacher, feels that the answer should be fifteen. Some work in this general area of language in the mathematics curriculum is being done at present in the United Kingdom (Tahta and Love, 1976; Brown and Küchemann, 1976; Otterburn and Nicholson, 1976).

In most cases children learning mathematics have been faced by low-structure mathematics taught informally (e.g. Nuffield) or by high-structure mathematics taught formally (e.g. SMSG). An interesting variation on this basic theme comes when children are permitted to deal informally with formal mathematical structures. Often the results of this can be quite significant. Some of the work of Dienes (1960, 1965), in particular his development of 'logic' or 'attribute' materials, which would seem to be a natural extension of some of Vygotsky's (1962) experimental work in concept formation, would fall into this category. The work of Lowenthal (1976) and Cordier et al (1977) using games and graphs would seem to be very promising. Other work of this type has been carried out by Allen (1970), the CSMP project in the United States, and by Higginson (1973).

The potential for mathetic research with children using computers would seem to be very great. With few exceptions, however, this potential has not

been realized in contemporary research. The exceptions would have to include the exciting and profound developments initiated by Iverson (1972), Papert (1972), and Landa (1974).

Other related projects include the classic study of children's language by the Opies (1959), the structural learning approach of Scandura (1971) and his group, and the work on metacognition being developed by Burnett et al (1977).

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