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ABSTRACT

This is one in a series of SMSG supplementary and enrichment pamphlets for high school students. This series makes available expository articles which appeared in a variety of mathematical periodicals. Topics covered include: (1) the two most original creations of the human spirit; (2) mathematics of music; (3) numbers and the music of the east and west; and (4) Sebastian and the Wolf. (MP)

Financial support for the School Mathematics Study Group has been provided by the National Science Foundation.

Mathematics is such a vast and rapidly expanding field of study that there are inevitably many important and fascinating aspects of the subject which do not find a place in the curriculum simply because of lack of time, even though they are well within the grasp of secondary school students.

Some classes and many individual students, however, may find time to pursue mathematical topics of special interest to them. The School Mathematics Study Group is preparing pamphlets designed to make material for such study readily accessible. Some of the pamphlets deal with material found in the regular curriculum but in a more extended manner or from a novel point of view. Others deal with topics not usually found at all in the standard curriculum.

This particular series of pamphlets, the Reprint Series, makes available expository articles which appeared in a variety of mathematical periodicals. Even if the periodicals were available to all schools, there is convenience in having articles on one topic collected and reprinted as is done here.

This series was prepared for the Panel on Supplementary Publications by Professor William L. Schaaf. His judgment, background, bibliographic skills, and editorial efficiency were major factors in the design and successful completion of the pamphlets.

Panel on Supplementary Publications

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PREFACE

Music means different things to different people and exhibits many faces: musical sounds and tones; scales and modes; musical notation; harmony and dissonance; rhythm, melody and counterpoint; musical composition and forms; the human voice and choral music; orchestral and symphonic music; acoustics and the reproduction of music by phonograph, radio, T-V, sound motion pictures. In what ways, if any, are these various facets of music related to mathematics? What has mathematics contributed to musical notation? to the theory of composition? to the design of musical instruments? to the high-fidelity reproduction of music? Is the composer aware of mathematical relations involved in music and musical composition? Can the mathematician, as mathematician, enrich the domain of the musician? These are questions more easily asked than answered. Moreover, such answers as have been given are, for the most part, scattered through various periodicals, often inaccessible. That is why we have brought these essays together for your enjoyment. It is hoped that they will at least open new horizons for you, even if they do not answer your questions completely. You may then agree with Morris Kline when he says "the most abstract of the arts can be transcribed into the most abstract of the sciences, and the most reasoned of the arts is clearly recognized to be akin to the music of reason."

—William L. Schaaf

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THE MATHEMATICS TEACHER

THEODORE C. RIDOUT, "Sebastian and the 'Wolf' ", vol. 48 (1955), pp. 84-86.

FOREWORD

Nearly three hundred years ago, Leibniz, the philosopher and a coinventor of the calculus, had this to say: "Music is the pleasure that the human soul experiences from counting without being aware that it is counting". In more recent times, the renowned architect, theosophist and philosopher Claude Bragdon once observed that "music is number made audible, architecture is number made visible".

These two observations would seem to justify the conviction that music, at least in some of its aspects, is somehow intimately and inextricably associated with numbers and their properties. The early Greek mathematicians of the Pythagorean School were firmly convinced of this. Since ancient times, men have known that the pitch of a sound from a plucked string depends upon its length, and that if the ratios of the lengths of the strings are simple whole numbers, the resulting sounds will be harmonious. Specifically, Pythagoras was aware that lengths which sounded a note, its fifth and its octave were in the ratio 2:3:4. In fact, Pythagoras and his disciples believed that the distances of the astronomical planets from the earth were also in a musical progression, and that therefore the heavenly bodies, as they moved through space, gave forth harmonious sounds: whence arose the phrase "the harmony of the spheres". The Pythagoreans fully believed that the only explanation of the order and harmony and perfection in the Universe was to be found in the science of numbers, or *arithmetiké*.

Indeed, this conviction was so deep-rooted that for 1500 years, from the time of Pythagoras to the Middle Ages, men classified knowledge as the Seven Liberal Arts: the *trivium* (grammar, rhetoric, and logic) and the *quadrivium* (arithmetic, astronomy, geometry, and music). Furthermore, the mathematical sciences were thought of as follows: *numbers absolute*, or arithmetic; *numbers applied*, or music; *magnitudes at rest*, or geometry; *magnitudes in motion*, or astronomy.

What have other observers said about music and mathematics? Listen to J. J. Sylvester, the brilliant, poetic, temperamental British mathematician of the mid-nineteenth century who contributed so much to the theory of invariants and matrices: "Mathematics is the music of Reason. The musician *feels* Mathematics, the mathematician *thinks* Music". Or again, the opinion of Helmholz, more the physicist than the mathematician: "Mathematics and Music, the most sharply contrasted fields of scientific activity, are yet so related as to reveal the secret connection binding together all the activities of our mind". Finally, from the pen of Havelock Ellis, the celebrated author of the "Dance of Life" and perceptive interpreter of civilization and culture: "It is not surprising that the greatest mathematicians have again and again appealed to the arts in order to find some analogy to their own work. They have indeed found it in the most varied arts, in poetry, in painting, and in sculpture, although it would certainly seem that it is in music, the most abstract of all the arts, the art of number and of time, that we find the closest analogy."

The Two Most Original Creations of the Human Spirit

ELMER B. MODE

"The science of Pure Mathematics, in its modern developments, may claim to be the most original creation of the human spirit. Another claimant for this position is music."

A. N. WHITEHEAD, *Science and the Modern World.*

1. **Introduction.** In the quotation given above a great Anglo-American philosopher [1] characterized two distinct fields of human interest, one a science, the other an art. The arts and the sciences, however, are not mutually exclusive. Art has often borrowed from science in its attempts to solve its problems and to perfect its achievements. Science in its higher forms has many of the attributes of an art. Vivid aesthetic feelings are not at all foreign in the work of the scientist. The late Professor George Birkhoff, in fact, wrote as follows:

"A system of laws may be beautiful, or a mathematical proof may be elegant, although no auditory or visual experience is directly involved in either case. It would seem indeed that all feeling of desirability which is more than mere appetite has some claim to be regarded as aesthetic feeling." [2]

Serge Koussevitzky, noted conductor, has stated also that "there exists a profound unity between science and art." [3]

It is not, however, the purpose of this paper, to discuss the relationships between the sciences and the arts, but rather to enumerate some of the lesser known attributes which music and mathematics have in common. There is no attempt to establish a thesis.

2. **Number and Pitch.** The study of mathematics usually begins with the natural numbers or positive integers. Their symbolic representation has been effectively accomplished by means of a radix or scale of ten, the principle of place-value where the position of a digit indicates the power of ten to be multiplied by it, and a zero. The concept of number is most basic in mathematics. We cannot directly sense number. A cardinal number, such as five, is an abstraction which comes to us from many concrete instances each of which possesses other attributes not even remotely connected with the one upon which our interest is fixed. Such widely differing groups as the fingers of the hand, the sides of the pentagon, the arms of a starfish, and the Dionne quintuplets, are all instances of "fiveness," the property which enables each group to be matched or placed into one-to-one correspondence with the other. The establishment of such equivalence requires no knowledge of mathematics, only good eyesight. With these facts in mind we may state a definition familiar to mathematicians. *The (cardinal) number of a group of objects is the invariant property of the group and all other groups which can be matched with it.*

The positive integers constitute, however, but a small portion of the numbers of mathematics. The former mark off natural intervals in the continuum of real numbers. The difference between two small groups of objects is readily sensed; man finds no difficulty in distinguishing *visually*, at once, between *three* and *four* objects, but the distinction between, say, thirty-two and thirty-three objects calls for something more than good vision.

In music, study begins with notes or tones. In western music their symbolic representation is accomplished by means of a scale of seven, a principle of position, and the rest, which denotes cessation of tone. There is something permanent and unchangeable about a given note. You may sing it, the violin string may emit it, the clarinet may sound it, and the trumpet may fill the room with it. The quality or timbre, the loudness or intensity, and duration of one sound may be markedly different from another; yet among these differences of sound there remains one unchanging attribute, its pitch. This is the same for a single such note or any combination of them. The pitch of a note may then be defined as *the invariant property of the note and all other notes which may be matched with it.* Notes which can be matched are said to be in unison. Pitch, also, as an abstraction, derived from many auditory experiences. The establishment of pitch equivalence does not require a knowledge of music, only a keen ear.

The notes of the diatonic scale mark off convenient intervals in a continuum of pitches. Within a given range, the interval between two tones of the scale is, in general, readily sensed, but outside of such a range the human ear may fail to distinguish between or even to hear two differing tones. As a matter of fact, "tones" removed from the range of audibility cease to be such. As psychological entities they disappear and may be identified only as vibrations in a physical medium.

Invariance of pitch is an important musical property and the ability of a musician not playing a keyed instrument to maintain this property for a given note is a necessary, but not a sufficient condition for his artistry. This recalls the story of the distracted singing teacher who, after accompanying his none-too-apt pupil, sprang suddenly from the piano, thrust his fingers wildly through his hair, and shouted: "I play the white notes, and I play the black notes, but you sing in the cracks."

3. **Symbols.** Mathematics is characterized by an extensive use of symbols. They are indispensable tools in the work; they constitute the principal vehicle for the precise expression of ideas; without them modern mathematics would be non-existent. The most important mathematical symbols are, with few exceptions, in universal use among the civilized countries of the world.

Music also is distinguished by a universal symbolism. The creation of anything but the simplest musical composition or the transmission of significant musical ideas is difficult if not impossible without the symbols of music.

Incidentally it may be remarked that the page of a musical score and the page of a book in calculus are equally unintelligible to the uninitiated. There are very few fields of activity outside of mathematics (including logic) and music which have developed so extensively their own symbolic language. Chemistry and phonetics are nearest in this respect.

In both music and mathematics preliminary training involves the acquiring of technique. Mathematics demands such facile manipulation of symbols that the detailed operations become mechanical. We are encouraged to eliminate the necessity for elementary thinking as much as possible, once the fundamental logic is made plain. This clears the way for more complicated processes of reasoning.

"It is a profoundly erroneous truism, repeated by all copy-books and by eminent people when they are making speeches, that we should cultivate the habit of thinking of what we are doing. The precise opposite is the case. Civilization advances by extending the number of important operations which we can perform without thinking about them." [4]

In music also, the preliminary training involves a learning of technique. The aim here is to be able to read, or to write, or to translate into the appropriate physical actions, notes and combinations of them with such mechanical perfection that the mind is free for the creation and the interpretation of more profound musical ideas.

4. **Logical Structure.** The framework of a mathematical science is well known. We select a class of objects and a set of relations concerning them. Some of these relations are assumed and others are deduced. In other words, from our axioms and postulates we deduce theorems embracing important properties of the objects involved.

Music likewise has its logical structure. The class of objects consists of such musical elements as tones, intervals, progressions, and rests, and various relations among these elements. In fact, the structure of music has been formally described as a set of postulates according to the customary procedure of mathematical logic. [5]

In mathematics a development is carried forward according to the axioms or postulates. If these are obeyed the results are correct, in the mathematical sense, although they may not be interesting or useful. Mere obedience to law does not create an original piece of mathematical work. This requires technical skill, imagination, and usually a definite objective.

Music also has its axioms or laws. These may be as simple as the most obvious things in elementary mathematics — the whole equals the sum of all its parts — if we are counting beats in a measure; they may be less obvious to the layman, such as the canons of harmony or the structural laws of a classical symphony. Here again we may follow the laws of music scrupulously without ever creating a worth-while bit of original music. Technical skill, imagination, the fortunate mood, and usually a definite objective are requisites for the creation of a composition which not only exhibits obedience to musical laws but expresses significant ideas also. Occasionally the musician becomes bold and violates the traditional musical axioms so that the resulting effects may at first sound strange or unpleasant. These may become as useful, provoking, and enjoyable, as a non-Euclidean geometry or a non-Aristotelian logic. In such manner did Wagner, Debussy, Stravinsky, and others extend the bounds of musical thought. In mathematics as well as in music one may have to become accustomed to novel developments before one learns to like them.

Benjamin Peirce defined mathematics as "the science which draws necessary conclusions." The operations from hypothesis to theorem proceed in logical order without logical hesitation or error. When the series of deductive operations flows swiftly and naturally to its inevitable conclusion, the mathematical structure gives a sense of satisfaction, beauty, and completeness. Sullivan characterizes the opening theme of Beethoven's Fifth Symphony as one which "immediately, in its ominous and arresting quality, throws the mind into a certain state of expectance, a state where a large number of happenings belonging to a certain class, can logically follow." [6] The same is true of the opening phrase of the prelude to *Tristan and Isolde*, or any really great enduring masterpiece.

An interesting departure from the usual logical structure of a musical composition occurs in the Symphonic Variations, "Istar" by Vincent d'Indy. Instead of the initial announcement of the musical theme with its subsequent variations, "the seven variations proceed from the point of complex ornamentation to the final stage of bare thematic simplicity." Philip Hale, the eminent Boston musical critic related the following anecdote in the Boston Symphony Bulletin of April 23, 1937.

"M. Lambinet, a professor at a Bordeaux public school, chose in 1905 the text 'Pro Musica' for his prize-day speech. He told the boys that the first thing the study of music would teach them would be logic. In symphonic development logic plays as great a part as sentiment. The theme is a species of axiom, full of musical truth, whence proceed deductions. The musician deals with sounds as the geometrician with lines and the dialectitian with arguments. The master went on to remark: 'A great modern composer, M. Vincent d'Indy, has reversed the customary process in his symphonic poem "Istar." He by degrees unfolds from initial complexity the simple idea which was wrapped up therein and appears only at the close, like Isis unveiled, like a scientific law discovered and formulated.' The speaker found this happy definition for such a musical work — 'an inductive symphony.' "

5. **Meaning.** A mathematical formula represents a peculiarly succinct and accurate representation of meaning which cannot be duplicated by any other means. It is concerned with the phenomenon of variability; it involves the function concept. "A mathematical formula can never tell us what a thing is, but only how it behaves." [7]

How true this is of music! A theme of great music compresses into a small interval of space or time, inimitably and accurately, a remarkable wealth of meaning. Music is not fundamentally concerned with the description of static physical objects, but with the impressions they leave under varying aspects. Debussy's "La Mer" is a fine example of this type of description. Music's interest is often not in the physical man but in his changing moods, in his emotions. One of the sources of the greatness of "Die Walküre" is Wagner's genius for portraying vividly the conflicting aspects of Wotan's nature — as god and as man.

The meaning of musical motive grows with study. It is usually exploited or developed and from it are derived new figures of musical expression. A good theme demands more than the casual hearing before its deep significance is completely appreciated. It is often worked up from an entirely insignificant motive as in Beethoven's Fifth Symphony or in Mozart's G minor symphony. In mathematics a basic formula or equation may have implications which can be understood only after much study. It may appear to be almost trivial as in the case of $a + b = b + a$ or it may be less obvious and more elegant as in the case of Laplace's equation,

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 .$$

Music consists of abstractions, and at its best gives expression to concepts which represent the most universal features of life. Beethoven's music expresses powerfully the great aspirations, struggles, joys, and tragedies of human existence. The Eroica symphony may have been composed with Napoleon in mind but it portrays far more than the career of a single man. It is a portrayal of the heroic in man and as such is universal in its application. It is well known that a musical passage or composition may produce different responses among people. The possibility of varying interpretation constitues one of the sources of music's uniqueness and a reason for its power. It is an evidence of its universality. Herein lies a fundamental difference between music and painting or sculpture. The effect of a musical episode is due to its wide potential emotional applicability; the effect of a painting or piece of sculpture is due to its concreteness. Attempts at abstract representations by painters have not been generally successful; attempts at stark realism in music have likewise failed. Music in its most abstract form, as for example, Bach's or Mozart's, often defies application to the concrete. It seems to be above mundane things, in the realm of pure spirit.

So it is with mathematics. Our conclusions are always abstract, and universal in their application, although they may have originated from a special problem. The possibilities of interpretation and application of a given theorem or formula are unlimited. Poincaré is reported to have said that even the same mathematical theorem has not the same meaning for two different mathematicians. What differing reactions may ensue when Laplace's equation is set up before an audience of mathematicians! What differing degrees of abstractness are suggested by the two equations previously written!

6. **The Creative Process.** "It is worth noting . . . that it is only in mathematics and music that we have the creative infant prodigy; . . . the boy mathematician or musician, unlike other artists, is not utilizing a store of impressions, emotional or other, drawn from experience or learning; he is utilizing inner resources. . . ." [8]

Statements of this type have led many to believe that mathematical talent and musical talent have more than an accidental relation. Some feel that mathematicians are more naturally drawn to music than musicians are to mathematics. As far as the writer has been able to ascertain, no serious investigations on the relation between the two talents have been published. A brief study of exceptionally gifted children yields no testimony that the child prodigy in music has more than the average mathematical sense, or that the child prodigy in mathematics has exceptional ability in music.

In a recent article, *Mind and Music,* [9] the inimitable English music critic, Ernest Newman, discusses the role that the subconscious mind might play in the creative processes of music. Hampered by a dearth of reliable testimony on this subject, he attempts, nevertheless, to estimate this role. Berlioz and Wagner had written of their creative experiences without attempting any self-analysis. So also had Mozart although Newman does not refer to him. Newman feels that the Memoirs of Hector Berlioz are not too reliable in this respect. Wagner's letters, however, seem to indicate that many of his musical ideas were the result of an upsurge from the unconscious depths of his mind of ideas long hidden but suddenly crystallizing. The activity of his conscious mind was often displaced by the upward thrust of these latent creative forces.

The interpretation thus suggested is strikingly similar in many respects to that described by Jacques Hadamard in his *Essay on the Psychology of Invention in the Mathematical Field.* [10] This noted mathematician draws on the related experiences of Poincaré, Helmholtz, Gauss, and others to discuss the origin of the inspiration or sudden insight that contributes to, completes, or initiates an original work. The role of thoughts that lie vague and undiscernible in the subconscious, only to become, of a sudden, clear and discernible after a period of unsuspected incubation, is described in undogmatic terms. One cannot affirm, of course, that these opinions concerning the creative process are confined solely to music and mathematics, but it is interesting that they are voiced by two eminent scholars, one from each field.

The greatest works of music are distinguished by their intellectual content as well as by their emotional appeal. The sacred music of Bach, the symphonies of Beethoven, or the operas of Wagner, offer subjects for analysis and discussion, as well as opportunities for emotional experience. Each composer had ideas to "work out," ideas to be developed and clarified by the forms and artifices of music, the object being to make their full significance felt by the appreciative listener.

Mathematical creativity involves very much the same general development. Concepts must be clarified, operations carried out, latent meanings revealed. If these are significant and logically developed the result has a unity and a sense of completeness which brings intellectual and aesthetic satisfaction to both author and reader.

7. **Aesthetic Considerations.** To many, mathematics seems to be a forbidding subject. Its form seems to be more like that of a skeleton than that of a living, breathing, human body. This idea, is, of course, derived from its abstract character and from the demands which it makes for sharply defined concepts, terse methods of expression, and precise rules of operation. In a sense, mathematics lacks richness, if by richness we mean the presence of those impurities which impart savor and color. These impurities may be in the nature of concrete examples, illustrations from, or applications to fields other than mathematics. They may represent departures from the normal abstract logical development, and may make no contributions whatsoever to the formal struc-

ture which constitutes mathematics. But if we subtract from the richness of mathematics, we add also to its purity, for in mathematics the structure or form is more important than its applications. We may apply the mathematics to many problems associated with human existence, but these applications are not essential parts of pure mathematics, they lie apart from it.

"In music the flavor of beauty is purest, but because it is purest it is also least rich. . . . A melody is a pure form. Its content is its form and its form is its content. A change in one means a change in other. We can, of course, force an external content upon it, read into it stories or pictures. But when we do so we know that they are extraneous and not inherent in the music." [11]

In a different sense mathematics is over-rich for its fields are unlimited in extent and fertility.

"But no one can traverse the realm of the multiple fields of modern Mathematics and not realize that it deals with a world of its own creation, in which there are strangely beautiful flowers, unlike anything to be found in the world of external entities, intricate structures with a life of their own, different from anything in the realm of natural science, even new and fascinating laws of logic, methods of drawing conclusions more powerful than those we depend upon, and ideal categories very widely different from those we cherish most." [12]

One needs here but to change a few words in order to describe the unique and lovely creations of music. The melodies and harmonies of music are its own inventions. They are often mysteriously beautiful, incapable of description by other means and without counterpart elsewhere in the world about us. A musical composition may be of the utmost simplicity or of the most intricate character, yet it may "well-nigh express the inexpressible." It is exactly this ability to convey the "inexpressible" ideas that give mathematics and music much in common. The mathematics student who seeks always a meaning or picture of each new proposition often fails to appreciate the power of that which defies representation.

8. Conclusion. There is much of interest to those who love both music and mathematics, and much has been written by mathematicians on the bearings of one field on the other. Archibald has written delightfully of some of their human aspects as well as the scientific. Birkhoff has attempted the evaluation of musical aesthetics by quantitative methods. Miller and others have brought the instruments of physics to bear upon the problems of musical tone and acoustics.

Success in music and in mathematics also depend upon very much the same things — fine technical equipment, unerring precision, and abundant imagination, a keen sense of values, and, above all, a love for truth and beauty.

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Mathematics of Music

ALI R. AMIR-MOEZ

Though music may seem far removed from what many think are the cold logical aspects of mathematics, nevertheless, music, with its emotional appeal, has a mathematical foundation. The following article will show how highly mathematical are the sounds, the scales and the keys (the parts, so to speak) of music.

1. **Harmonics of a Sound:** When a sound is made, for example, by striking a string of a musical instrument, each particle of air next to the source of the sound vibrates. We shall call the number of vibrations of that particle of air in one second the number of vibrations of the sound. The larger this number is, the higher the pitch of the sound becomes.

Suppose a sound is called C, and its number of vibrations is c. That is, if, for example, the sound C, and its number of vibrations is c. That is, for example, the sound C makes a particle of air vibrate five hundred times in one second, we say $c=500$. It was discovered by Greek mathematicians that if after the sound C is heard we make another sound S whose number of vibrations is twice the number of vibrations of C, i.e., 2c, then S will be pleasant to hear. As far as the history of mathematics shows, this idea is due to Pythagoras. The sound T with three times as many vibrations, i.e., with 3c vibrations, is also pleasant to hear after C. This fact is true for sounds with vibrations c, 2c, 3c, 4c, 5c, etc. Usually, if we play these sounds successively in some order with a certain rhythm, we call it a melody. If we play a few of these sounds together, we call it harmony.

We shall call the sounds with vibrations 2c, 3c, 4c, etc. harmonics of C.

2. **A Primitive Scale:** In the work of Omar Khayyam*, it is mentioned that the study of the ratios of integers is essential for the science of music. That was the only mathematics used in the Greek theory of music. To explain the idea, we start with the sound C, and we suppose that C_1 is the name of the sound with 2c vibrations. Let us call G_1 the sound whose number of vibrations is 3c. (We shall explain why we have chosen these names.) If a sound with twice as many vibrations is a harmonic of a given sound, it is reasonable to believe that the sound G with one-half as many vibrations as G_1 is a harmonic of C. Thus we can say that the sounds C, G, and C_1 are harmonic of one another, and their vibrations are, respectively, c, $^{3c}/_{2}$, and 2c. We can compare these sounds and their vibrations by constructing the following table.

Sound	C	G	C_1
c	1	$\frac{3}{2}$	2

*Omar Khayyam, "Discussion of Difficulties in Euclid," *Scripta Mathematica* V. 24, pp. 275 - 303 (1959).

The first line of the table shows the name of each sound, and the second line shows the corresponding number of vibrations. For example, under G we see $^{3}/_{2}$, which means that G has $^{3c}/_{2}$ vibrations in a second.

The names chosen here are actually those chosen in the scale. If C is the natural C of the scale, then G has $^{3}/_{2}$ as many vibrations as C. C_1 is the next so-called C, which is usually called the octave of C.

In this scale we have only three sounds. If we play C, G, and octave of C on the piano, we can almost see how they sound. Of course, we cannot make much music with three sounds.

3. **Oriental Scale:** Let us extend the idea of section 2 further. We take the fifth and seventh harmonics of C, *i.e.*, the sounds whose numbers of vibrations are 5c and 7c. We call these sounds, respectively, E_2 and K_2. We shall explain the choice of the subscripts shortly. Let us compare these sounds with C_1, the octave of C, and with C_2, the octave of C_1. Note that 2c is the number of vibrations of C_1, and 4c is the number of vibrations of C_2. Thus, if E_1 is a sound with half as many vibrations as E_2, then we see that $^{5c}/_{2}$ is the number of vibrations of E_1. Similarly, we can choose a sound K_1 whose number of vibrations is $^{7c}/_{2}$. If we compare these sounds according to their pitch, we get them in the order C_1, E_1, K_1, C_2. This is clear because

$$2 < \frac{5}{2} < \frac{7}{2} < 4.$$

Since these sounds are all harmonics of C, the sounds E and K, which have half as many vibrations as E_1 and K_1, respectively, *i.e.*, $^{5c}/_{4}$ and $^{7c}/_{4}$, are also harmonics of C. As in section 2, we can make a table as follows:

Sound	C	E	G	K	C_1
c	1	$\frac{5}{4}$	$\frac{3}{2}$	$\frac{7}{4}$	2

These five sounds together approximately constitute the oriental scale.

4. **Middle-East Scale:** If we proceed with what was done in section 3, we get more sounds in the scale. Since the sounds with vibrations 2c, 4c, 8c, 16c, *etc.* do not contribute to the scale, we choose the sounds between them. In particular, let us call D_3 the sound with 9c vibrations. We also choose P_3, H_3, and B_3 with vibrations, respectively, 11c, 13c, and 15c. As before, we may choose D_2, P_2, H_2, and B_2 with vibrations $^{9c}/_{2}$, $^{11c}/_{2}$, $^{13c}/_{2}$, and $^{15c}/_{2}$, respectively. Then we choose D, P, H, and B with vibrations $^{9c}/_{8}$, $^{11c}/_{8}$, $^{13c}/_{8}$, and $^{15c}/_{8}$, respectively. We shall construct a table as before.

Sound	C	D	E	P	G	H	K	B	C_1
c	1	$\frac{9}{8}$	$\frac{5}{4}$	$\frac{11}{8}$	$\frac{3}{2}$	$\frac{13}{8}$	$\frac{7}{4}$	$\frac{15}{8}$	2

A scale may be made out of these sounds with eight names in the scale instead of seven. Before we discuss this set of sounds, we make a table using the theoretical (physical) sounds of the scale.

Sound	C	D	E	F	G	A	B	C_1
c	1	$\frac{9}{8}$	$\frac{5}{4}$	$\frac{4}{3}$	$\frac{3}{2}$	$\frac{5}{3}$	$\frac{15}{8}$	2

If we compare P and F, we see that that the ratio of the number of vibrations of P to the number of vibrations of F denoted by

$$\frac{P}{F} = \frac{11}{8} : \frac{4}{3} = \frac{33}{32}$$

This shows that P is sharper than F. This is where the middle-east music is different from the physical scale. The sound H with $^{13c}/_{8}$ vibrations is not used in the middle-east music. Thus, C, D, E, P, G, K, B, C_1 approximately constitute the sounds of the middle-east scale. We see that

$$\frac{K}{A} = \frac{7}{4} : \frac{5}{3} = \frac{21}{20}$$

Therefore, K is also sharper than A.

5. **Tones and half-tones:** If we study the physical scale, we observe that

$$\frac{D}{C} = \frac{9}{8}, \frac{E}{D} = \frac{10}{9}, \frac{F}{E} = \frac{16}{15}, \frac{G}{F} = \frac{9}{8}, \frac{A}{G} = \frac{10}{9}, \frac{B}{A} = \frac{9}{8}, \frac{C_1}{B} = \frac{16}{15}$$

This suggests the idea of small and large intervals or tones and half-tones. We shall write this as follows:

Sound	C		D		E		F		G		A		B		C
Tone		1		1		$\frac{1}{2}$		1		1		1		$\frac{1}{2}$	

The above table indicates which interval is a tone and which is a half-tone. For example, between E and F is a half-tone. But, we really should say large and small intervals.

6. **Geometric Progression:** An ordered set of numbers is called a geometric progression when the ratio of each one to its predecessor is always the same. For example, the set

$$5,\ 10,\ 20,\ 40,\ 80, \ldots.$$

is a geometric progression. The ratio is 2, that is, the ratio of each number to the one before it is two. Indeed, we can produce as many members of this set as we desire.

If one member of a set and the ratio are given, we can always produce as many members as needed. For example, if ½ is a member of the progression and the ratio is $\sqrt{2}$, then we can write some of the members of this progression, such as $\frac{1}{2}$ $\frac{1}{2}\sqrt{2}$, $\frac{1}{2}(\sqrt{2})(\sqrt{2}) = \frac{1}{2}(\sqrt{2}^2), \frac{1}{2}(\sqrt{2}^3) \ldots.$

7. **Geometric Means:** For two numbers, the geometric mean of them is the square root of the product of them. This is a sort of average, similar to one-half of the sum, which is called the arithmetic mean. As for the arithmetic average of a few numbers, we add them and divide the sum by the number of

them; for the geometric mean of a few numbers we multiply them and take the root of order equal to the number of them. For example, the geometric mean of

$$5, \; 7, \; 2, \; 6$$

is

$$\sqrt[4]{(5)(7)(2)(6)} = \sqrt[4]{420} \; .$$

8. **Modern Scale:** Since two sounds are compared in terms of the ratio of their number of vibrations rather than the difference of the number of vibrations, in order to make all intervals equal and call each one a "half tone," we need to take the geometric mean of twelve half-tones of the scale. Thus the number of vibrations of the sounds in the modern scale form a geometric progression which has 1 as a member and $\sqrt[12]{2}$ as its ratio. Thus the modern scale can be shown in the following table:

Sound	C	D	E	F	G	A	B	C_1
c	1	$(\sqrt[12]{2})^2$	$(\sqrt[12]{2})^4$	$(\sqrt[12]{2})^5$	$(\sqrt[12]{2})^7$	$(\sqrt[12]{2})^9$	$(\sqrt[12]{2})^{11}$	2

As we observe, the power of $\sqrt[12]{2}$ increases by 2 whenever we have a tone: and it increases by one whenever we have a half-tone.

The modern scale is not really as natural to the ear as the old Greek scale; but with slight training, the ear gets used to it. The important fact is that modulation from one key to another becomes extremely easy.

There is one disadvantage in the modern scale, namely, the third harmonic of C, *i.e.*, G, becomes slightly flat. The sound G is called the dominant of the scale and, being flat, makes the music dull. We shall show this fact mathematically. In the modern scale

$$\frac{G}{C} = (\sqrt[12]{2})^7 = 1.498$$

But, in the natural scale

$$\frac{G}{C} = \frac{3}{2} = 1.5$$

This mistake is always corrected in the violin. This is one of the reasons that an orchestra with string instruments sounds much better than a piano solo.

9. **Major Keys:** A sample of the scale of a major key is the one in section 8. This is called "C major" since it starts with C. C is also called the tonic of the scale. In any major key, the sound (notes) of the scale have the same relation to one another as the ones in C major. That is, the interval between the third and fourth elements is one half-tone; also the interval between the seventh and eighth elements is a half-tone, and the other intervals are all one tone.

The next major key is G major. This has been chosen for two reasons. One is that the note G is the third harmonic of C; the other is that this key has a higher pitch. Note that going from C to its second harmonic does not change the scale. The table of the scale of this key is as follows:

Sound	G	A	B	C_1	D_1	E_1	$F_1\#$	G_1
c	$(\sqrt[12]{2})^7$	$(\sqrt[12]{2})^9$	$(\sqrt[12]{2})^{11}$	2	$(\sqrt[12]{2})^{14}$	$(\sqrt[12]{2})^{16}$	$(\sqrt[12]{2})^{18}$	$(\sqrt[12]{2})^{19}$

We observe that in order to have the interval between the seventh and eighth, *i.e.*, subtonic and tonic, a half tone, we have to use $F_1\sharp$ (F_1 sharp) with vibrations $(\sqrt[12]{2})^{18}$ instead of F_1 with $(\sqrt[12]{2})^{17}$.

If we choose the fifth note of this scale as the first of a new scale, we get the key of D major. This key needs two sharps.

The reader may try this idea and work out tables for several major keys which come after G major.

As it was possible to get major keys with higher pitch, it is also possible to get major keys with lower pitch.

Suppose we look at the table in section 8 and consider a scale for which the fifth note is C. This will have the following table:

Sound	F_0	G_0	A_0	$B_0\flat$	C	D	E	F
c	$(\sqrt[12]{2})^{-7}$	$(\sqrt[12]{2})^{-5}$	$(\sqrt[12]{2})^{-3}$	$(\sqrt[12]{2})^{-2}$	1	$(\sqrt[12]{2})^{2}$	$(\sqrt[12]{2})^{4}$	$(\sqrt[12]{2})^{5}$

Here we have to use $B_0\flat$, *i.e.*, B flat, in order to have the interval between the third and fourth notes be a half-tone.

If we proceed in this way, each lower key has an extra flat. We leave it to the reader to produce many major keys and write tables for the corresponding scales.

10. **Minor Keys:** To imitate the crying sound of middle-east music, minor keys seem to be proper. Most older pieces written in minor keys avoid the very large interval followed by a half-tone, but we find this combination of sounds in recent pieces.

Many forms of minor keys have been considered. We shall describe only one of the most recent pieces.

To obtain a new scale, instead of going to the third harmonic of C, we may go to the fifth harmonic of C. But, this key is not the simplest minor key. Thus we move down to A, whose fifth harmonic is approximately C. The table of the scale for A minor is the following:

Sound	A_0	B_0	C	D	E	F	G#	A
c	$(\sqrt[12]{2})^{-3}$	$(\sqrt[12]{2})^{-1}$	1	$(\sqrt[12]{2})^{2}$	$(\sqrt[12]{2})^{4}$	$(\sqrt[12]{2})^{5}$	$(\sqrt[12]{2})^{8}$	$(\sqrt[12]{2})^{9}$

As the physical scale shows, it is desirable to have a half-tone interval between the subtonic and the tonic of a scale. This brings $G\sharp$ into the scale. As we see, the interval between F and $G\sharp$ is one and a half tones.

Other minor keys are obtained from this in a manner similar to that by which the major keys are obtained from C major. We leave it to the reader to obtain them.

It would be very interesting for one to compare his knowledge of music with what has been said here.

Numbers and the Music of the East and West

ALI R. AMIR-MOEZ

Let C be a sound whose number of vibrations is c. Having heard C, any sound with number of vibrations equal to kc, $k = 1, 2, \ldots$, is pleasant to hear. These sounds are called the harmonics of C. If we play a few of these sounds successively with some rhythm, a pleasant melody is made. If we play a few of these sounds together, a rich sound comes out and it is called harmony. A sound C with vibrations $2c$ is called the octave of C.

Let us consider two octaves of C, say C^{I} and C^{II}, and construct the following table.

Sound	C	G	C'	G'	C''
c	1	$\frac{3}{2}$	2	3	4

As shown in the table let G^{I} be the third harmonic of C. Since the sound G^{I} with $3c$ vibrations is pleasant to be heard with C, it is reasonable to think that G with half as many vibrations, i. e., —c, would also make harmony with C. In fact this sound is called the dominant of the scale. This way we obtain only three sounds in the scale. Now in order to get more sounds in the scale let us consider three octaves of C, say C^{I}, C^{II}, and C^{III}. Let us construct the following table.

Sound	C	E	G	K	C'	E'	G'	K'	C''	E''	G''	K''	C'''
c	1	$\frac{5}{4}$	$\frac{3}{2}$	$\frac{7}{4}$	2	$\frac{5}{2}$	3	$\frac{7}{2}$	4	5	6	7	8

As shown in the table the 5th harmonic of C is called E^{II}. Again it is conceivable that since E^{II} is a harmonic of C, also E^{I} with vibrations —c, and E with vibrations —c will be harmonic with C. A similar process can be used for the 7th harmonic of C, called K^{II}, and K^{I}, K could be found accordingly. This sound K is missing in the physical scale. If we consider only these five sounds, the scale will be as follows:

Sound	C	E	G	K	C'
c	1	$\frac{5}{4}$	$\frac{3}{2}$	$\frac{7}{4}$	2

An approximation of these sounds appears frequently in the music of the Far East.

Now let us consider four octaves of C, say C^{I}, C^{II}, C^{III}, and C^{IV}. In order to get more sounds in the scale, we construct the following table by considering the harmonics of orders 9, 10, 11, 12, 13, 14, and 15 of C, and putting tones with — of these frequencies in the scale.

Sound	C	D	E	P	G	H	K	B	C^{I}	C^{II}	C^{III}	D^{III}	E^{III}	P^{III}	G^{III}	H^{III}	K^{III}	B^{III}	C^{IV}
c	1	$\frac{9}{8}$	$\frac{5}{4}$	$\frac{11}{8}$	$\frac{3}{2}$	$\frac{13}{8}$	$\frac{7}{4}$	$\frac{15}{8}$	2	4	8	9	10	11	12	13	14	15	16

Now theoretically speaking the sounds of the scale stand as follows:

Sound	C	D	E	F	G	A	B	C
c	1	$\frac{9}{8}$	$\frac{5}{4}$	$\frac{4}{3}$	$\frac{3}{2}$	$\frac{5}{3}$	$\frac{15}{8}$	2

Comparing this table with the previous one we see that

$$P:F=\frac{11}{8}:\frac{4}{3}=\frac{33}{32};$$

i. e., P is sharper than F. Some people are of the opinion that half of a half tone exists in the music of the middle east, but this is not true. The difference is actually the use of P instead of F. As we shall show in the modern or practical scale, F is slightly sharper in theoretical scale, but it is still too flat to be equal to P.

We see also that H and K are replaced by A. An experiment at the School of Science, University of Teheran, made it certain that H, i. e., the 13th harmonic of C does not exist in the music of the middle east. But K is also used instead of A. The ratio

$$K:A=\frac{7}{4}:\frac{5}{3}=\frac{21}{20}$$

shows that K is sharper than A.

Now if we look at the intervals in the theoretical scale we see that

$$D:C=\frac{9}{8},\ E:D=\frac{10}{9},\ F:E=\frac{16}{15},\ G:F=\frac{9}{8},\ A:G=\frac{10}{9},\ B:A=\frac{9}{8},\ \text{and}\ C:B=\frac{16}{15}.$$

This has suggested the idea of tones and half tones for today's music. Looking at the preceding intervals we have approximately 12 half tones between C and C^{I}. In practical scale all half tones are at equal intervals, i.e., we have to have the geometric mean of these half tone intervals. Clearly the product of intervals,

$$i.\ e., \frac{9}{8}\ \frac{10}{9}\ \frac{16}{15}\ \frac{9}{8}\ \frac{10}{9}\ \frac{9}{8}\ \frac{16}{15}=2.$$

Therefore the geometric mean, considering 12 half tones, will be $\sqrt[12]{2}$. We can really describe each half tone as a term in a geometric progression whose first term is 1 and its ratio is $\sqrt[12]{2}$.

Let us compute the ratio of the dominant and the tonique, i. e., G:C. Clearly G is the 8th term of the progression and its frequency will be

$(\sqrt[12]{2})^7 \approx 1.498$ which is a little more flat than $\frac{3}{2} = 1.5$. This is the place that in violin is always corrected. The advantage of the modern scale is that modulation from one key to another is very convenient. In fact musicians say:

> "If the sensitive ear of a musician does not distinguish these slight differences of sounds, the mathematicians ear of course wouldn't distinguish them either."

But the musician's ear has been trained to appreciate the sounds of practical scale. However any simple melody of this sort sounds harsh to a Persian tribesman.

Sebastian and the Wolf

THEODORE C. RIDOUT

This is not another parody on Little Red Ridinghood. It is rather a tale of how a musician battled a "wolf," and how a schoolteacher fought the same fight in the classroom.

The "wolf" we are interested in is a beast that plagued the makers of musical instruments for many centuries. Although many attacked him with vigor, and Johann Sebastian Bach laid him fairly low, the ghost of the critter still haunts our concert halls and keeps turning up in the physics laboratory. Since he has a mathematical origin, it seems fitting that we should discuss him in these pages.

The problem is something like this. You tune your violin by fifths, adjusting string tension until you hear a perfect fifth when two strings are sounded together. The fifth is a basic unit in musical tuning.

But so, also, is the octave. Take a very simple one-stringed instrument, Pythagoras' monochord. Every time we quadruple the tension on the string we double the rate of vibration, and the tone goes up an octave. Starting at a single vibration per second, the process of doubling the frequency of vibration takes us up in geometric progression to frequencies of 2, 4, 8, 16, . . . until we reach 256, where we pause for breath and call the tone "middle C." If we double again we get "upper C," at a frequency of 512 per second, and so on, until the pitch is too high for even your dog to hear.

According to the laws of physics, if C has a frequency of 256, the perfect fifth above it, G, will have a frequency of 256×1.5, which is 384. Here C:G $=$ 1:1.15, or 4:6. Moreover the triad (or chord) C-E-G sounds best if the frequencies of these notes have the exact ratio 4:5:6. These combinations are pleasing to the ear because the harmonics, or overtones, as well as the fundamental tones, combine with a minimum of conflicts or undesirable beats.

Filling in, we have the following notes and their relative frequencies:

C (frequency k) – D$\left(\frac{9}{8}k\right)$ – E$\left(\frac{5}{4}k\right)$ – F$\left(\frac{4}{3}k\right)$ – G$\left(\frac{3}{2}k\right)$ – A$\left(\frac{5}{3}k\right)$ – B$\left(\frac{15}{8}k\right)$ – C$(2k)$

This is known as a pure scale, and its tones have exact harmonic relations to the keynote. Music played on an instrument so tuned sounds rich and ethereally beautiful.

So much for theory. I will now retire to my workshop and construct a piano. Starting at a very low C (about 32 vibrations per second) I go up by perfect fifths along the musical alphabet. The notes will be C, G, D, A, E, B, F♯, C♯, G♯, E♭, B♭, F, and C. From bottom C to top C is just seven octaves. This looks like the beginning and the end of a complete and perfect keyboard, and I flatter myself I can fill in the other notes in proper ratios to make a shining row of ivories. But first I had better check my fifths.

Going up from C to G, I increased my frequency by the correct factor, 1.5. From G up to D, I again multiplied by 1.5. In all, I multiplied twelve times by this factor, so that my top C vibrates at a frequency that is $(1.5)^{12}$ times that of the bottom C, or 129.75 times the original.

But suppose I go up by octaves? Starting at the same C as before and doubling frequency until I have gone up seven octaves, I arrive at a frequency that is 2^7, or 128, times the original. I now have *two* top C's, and there is a difference of about a quarter of a semitone between them. What to do?

Various solutions have been offered. For many centuries a compromise known as the mean-tone scale was used. This can best be illustrated by arranging the twelve fifths around the dial of a clock, with C at twelve o'clock, G at one o'clock, and so on. Instead of the fifths being tuned in the correct ratio 1.5, they are tuned in the compromise ratio $\sqrt[4]{5}$, which equals 1.49535, introducing a very slight error. This error multiplied twelve times leaves a gap somewhere. Suppose I have the gap come after 11 on the clock, between F and top C on my keyboard. Instead of a normal interval of 7 semitones here, I have an interval of 7.4 semitones.

A musical composition involving this overgrown interval would be anything but harmonious, and might even howl like a wolf in the forest; hence the interval came to be known as the *quinte-de-loup*, or wolf fifth. One could play only in certain keys that did not run afoul of this beast.

Mathematicians of course came to the rescue, but their aid was not always appreciated. As far back as 1482 a Spaniard named Bartolo Rames proposed an equalized tuning in which all semitones should go up in the ratio of $\sqrt[12]{2}$, or 1.05946 times the note below. This divides the error exactly between successive intervals, slightly reducing each fifth. so that a complete set of twelve fifths can begin and end on exactly the sam tones as a set of seven octaves. With the intervening notes filled in, we have a slightly imperfect but very useful tuning known as equal temperament.

This solution made little headway, in spite of various advocates, until the appearance of a mighty figure on the scene. This was none other than Johann Sebastian Bach, who proposed equal temperament for all keyboard instruments, and proceeded to tune his clavichord and harpsichord accordingly. He was thus able to play in any one of the tweleve possible keys, and could modulate from one key to another without encountering any wolves.

To demonstrate the system he composed a series of twenty-four preludes and fugues, making use of all twelve keys. This was in 1722. He later repeated the process, giving us in all forty-eight pieces under the title "The Well-Tempered Clavichord." The term "well-tempered" of course refers to equal temperament. It is generally agreed that this work was chiefly responsible for the present universal use of equal temperament.

It was many generations before the mean-tone scale was abandoned in the tuning of pipe organs; and Bach was forced to compromise with the wolf at the co: ole. For this reason his organ compositions are written only in the simpler ke .

My a. bra class was naturally interested in all this. Plainly $\sqrt[12]{2}$ raised to the twelfth power is 2, so that twelve semi-tones will fit perfectly into an octave, forming a continuous chromatic scale. We checked the value of the radical on a slide rule. The physics department demonstrated a sound disk for us, proving to all the neighborhood that the frequencies under discussion produced a musical scale. The disk contained 8 rows of holes, corresponding to the notes of the scale, as follows: C (24 holes), D (27), E (30), F (32), G (36), A (40), B (45), C (48). When spun by an electric motor and played upon with a jet of compressed air, such a disk gives off "musical" tones approaching the power of a steam calliope, and in the exact harmonic ratios of the pure scale.

My students wrote papers on such topics as the clavichord, pipe organs, orchestration, electronic instruments, acoustics, and so on. One or two whose musical background was stronger than the mathematical were somewhat shocked to find figures encroaching on the province of the Muse. These dissenters were cheered, however, to know that Robert Smith in 1759 characterized equal temperament as "extremely coarse and disagreeable," and that Helmholtz in 1852 considered that it made every note on the piano sound "false and disagreeable," and that on the organ it produced a "hellish row." Helmholtz had used just intonation, as he called it, or pure tuning, for his experimental harmonium, and like many a musical expert, became so conditioned to perfect harmonies that he found those of equal temperament very distasteful.

Thus, thanks to the Queen of Sciences, the "wolf" has now become a thing of the past, though my young daughter tells me that one or two of his cubs show up occasionally at the high-school band rehearsals.

FOR FURTHER READING AND STUDY

The following references are not all equally significant. Some of them are quite general, others are either more scholarly or more technical. All of them, however, are relevant. Those that are probably most illuminating and readily accessible have been indicated by an asterisk. Not to have been so labeled does not imply that the reference is any way unscholarly or without merit.

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