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ABSTRACT

This is one in a series of SMSG supplementary and enrichment pamphlets for high school students. This series makes available expository articles which appeared in a variety of mathematical periodicals. Topics covered include: (1) a forerunner of Mascheroni; (2) Mascheroni constructions; and (3) can we outdo Mascheroni. (MP)

Financial support for the School Mathematics Study Group has been provided by the National Science Foundation.

Mathematics is such a vast and rapidly expanding field of study that there are inevitably many important and fascinating aspects of the subject which do not find a place in the curriculum simply because of lack of time, even though they are well within the grasp of secondary school students.

Some classes and many individual students, however, may find time to pursue mathematical topics of special interest to them. The School Mathematics Study Group is preparing pamphlets designed to make material for such study readily accessible. Some of the pamphlets deal with material found in the regular curriculum but in a more extended manner or from a novel point of view. Others deal with topics not usually found at all in the standard curriculum.

This particular series of pamphlets, the Reprint Series, makes available expository articles which appeared in a variety of mathematical periodicals. Even if the periodicals were available to all schools, there is convenience in having articles on one topic collected and reprinted as is done here.

This series was prepared for the Panel on Supplementary Publications by Professor William L. Schaaf. His judgment, background, bibliographic skills, and editorial efficiency were major factors in the design and successful completion of the pamphlets.

PREFACE

The early Greek geometers, particularly those of the Athenian school under the stimulus of Plato and Eudoxus, were deeply interested in the three classical problems of antiquity: (1) the duplication of a cube, or the determination of the edge of a cube whose volume is twice that of a given cube; (2) the trisection of an angle; and (3) the quadrature of a circle, or finding a square whose area is equal to that of a given circle. Solutions to these problems were readily found if the use of parabolas. hyperbolas and curves other than the circle were permitted. Ingenious solutions were found by Hippias, Archytas, Eudoxus and Menaechmus. But Plato objected to these solutions because they were "mechanical and not geometrical"; that is, it was necessary to use instruments other than the straightedge (unmarked ruler) and compasses. Thus, according to Plato: "The good of geometry is set aside and destroyed, for we again reduce it to the world of sense, instead of elevating it and imbuing it with the eternal and incorporeal images of thought, even as it is employed by God, for which reason He always is God."

It should be noted that no Greek geometer ever succeeded in solving these problems using straightedge and compasses only, that is, with Euclidean tools, and that these problems plagued mathematicians for upwards of two thousand years. Not until the latter part of the 19th century were mathematicians able to prove rigorously that these constructions were impossible with Euclidean tools. As it turns out, the test of constructibility under these restrictions is an algebraic test, involving a number of theorems in the algebra of the real number system.

The postulates laid down by Euclid were as follows: (1) A straight line may be drawn from any one point to any other point; (2) a finite straight line may be extended indefinitely to any length in a straight line; (3) a circle may be described from any center at any distance from that center. Postulates (1) and (2) define what we may do with a straightedge: it is permissible to draw any portion of a straight line determined by any two given points. Postulate (3) tells us what we are allowed to do with the compasses: it is permissible to draw a circle with a given center and passing through a given point.

It is important to note that neither instrument may be used to transfer a distance. In other words, the straightedge is not marked; and the com-

passes are such that if either leg is lifted from the plane, the instrument will automatically collapse. Hence Euclidean compasses are often called *collapsing compasses*. Modern compasses, as you know, stay open, and can be used as dividers for transferring distances as the draftsman does. You might think that modern compasses are more powerful than collapsing compasses, but it can be proved that any construction that can be effected with the modern compasses can also be performed with the collapsing compasses; the two instruments are mathematically equivalent.

In short, these three postulates are tantamount to allowing the ruler and compasses to be used:

(1) to draw a straight line through two given points;

(2) to describe a circle with a given center such that it passes through a given point.

These two operations are sufficient to enable us to carry out all the plane constructions of elementary Euclidean geometry. Indeed, the term *Euclidean construction* designates any construction which can be effected by employing these two operations repeated any finite number of times.

—William L. Schaaf

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ACKNOWLEDGMENTS

The SCHOOL MATHEMATICS STUDY GROUP takes this opportunity to express its gratitude to the authors of these articles for their generosity in allowing their material to be reproduced in this manner: the late Florian Cajori, who, at the time that his article was first published, was associated with the University of California; William F. Cheney, Jr., who, when his paper was first published, was associated with the University of Connecticut, at Storrs, Conn.; Julius H. Hlavaty, who was then associated with the Dewitt Clinton High School in New York City; and Nathan Altshiller Court, of the University of Oklahoma.

The SCHOOL MATHEMATICS STUDY GROUP is also pleased to express its sincere appreciation to the several editors and publishers who have been kind enough to allow these articles to be reprinted, namely:

AMERICAN MATHEMATICAL MONTHLY

(1) Florian Cajori, "*A Forerunner of Mascheroni*," vol. 36 (1929), p. 364-365.

THE MATHEMATICS TEACHER

(1) William F. Cheney, Jr., "*Can We Outdo Mascheroni?*," vol. 46 (March 1953), p. 152-156.

(2) Nathan Altshiller Court, "*Mascheroni Constructions*," vol. 51 (May 1958), p. 370-372.

(3) Julius H. Hlavaty, "*Mascheroni Constructions*," vol. 50 (Nov. 1957), p. 482-487.

FOREWORD

It has been known for nearly three hundred years that all Euclidean constructions can be effected with the compasses alone. This brief introductory article sketches the history and anticipates the development of the techniques involved in such constructions.

In these discussions of geometric constructions, it is assumed that all given geometric data are points. Thus if a line is "given," it may be replaced by two definite points lying on that line. Similarly, a given circle may be replaced (1) by its center and one point of the circle, or (2) by the center and two other points whose distance apart equals the radius, or (3) by three definite points of the circle.

If we wish to effect a construction by use of the straightedge alone, the given data must include at least four points, no three of which lie on a straight line. If we use the compasses in conjunction with the straightedge, a new point may be determined (1) as the intersection of a pair of straight lines, or (2) one of the intersections of a circle and a straight line, or (3) one of the intersections of two circles. (In the latter case, the second circle may be replaced by the common chord or the radical axis.)

A Forerunner of Mascheroni

Florian Cajori

The Italian Lorenzo Mascheroni who published in 1797 a well known work on the *Geometry of the Compasses,*[1] in which all constructions are effected without a ruler and by the use only of compasses, was anticipated by 125 years, as is now first shown by a Danish writer, Georg Mohr[2] whose book, *Euclides Danicus* of 1672, the Royal Danish Scientific Society at Kopenhagen has just published in *facsimile* and also in translation into German. The book was overlooked by mathematicians, notwithstanding the fact that there appeared two editions in 1672, one in Danish, the other in Dutch. There is nothing to indicate that Mascheroni had any knowledge of Mohr's book. The two worked independently. The *Euclides Danicus* in Dutch covers 36 pages and is a much smaller book than that of Mascheroni. It consists of two parts, the first part containing 54 constructions in Euclid's *Elements*, effected by the use of only the compasses. The last few problems call for the construction of a figure similar to a given figure and equal in area to another. An easy problem is the following: Given the line BA, to find the end point of a line twice as long. Draw an arc with A as center and AB as radius. Starting at B, apply to this arc, using the compasses, BA three times successively as a chord; the final intersection on the arc is the required point E of the straight line BAE. The second part of Mohr's book gives 24 constructions of various selected problems, ending with a rather involved problem on the erection of a sun dial.

Mohr's book is mentioned by some bibliographers but without a hint as to the nature of its content. From its title one might surmise that it was an edition of Euclid's *Elements*. Leibniz refers to him in a letter to Oldenburg (May 12, 1676) as "Georgius Mohr Danus, in geometria et analysi versatissimus." More is known of him than is indicated by the editors of the 1928 edition of his book. Cantor[3] refers to Mohr's trip to England and thence to France where, about 1676, he met Leibniz and informed him that Collins was in possession of infinite series for arc sin x and sin x. Before this, Oldenburg had mentioned Mohr, in a letter to Leibniz of September 30, 1675, as one well versed in algebra, and mechanics, who left with John Collins a manuscript on roots of $A + \sqrt{B}$ written in Dutch. This tract was forwarded to Leibniz. Born in 1640 in

[1] Lorenzo Mascheroni, *Geometria del compasso* (Pavia 1797, Palermo, 1901).

[2] Georg Mohr, *Euclides Danicus*, Amsterdam, 1672. Mit einem Vorwort von Johannes Hjelmslev und einer deutschen Uebersetzung von Julius Pál. Udgivet af det Danske Videnskabernes Selskab Kobenhavn. Hovedkommissionaer: Andr. Fr. Host & Son, Kgl. Hof-Boghandel 1928.

[3] Moritz Cantor, *Vorlesungen über Geschichte der Mathematik*, vol. III, 2nd ed., p. 179.

Copenhagen, Mohr went to Holland as a young man and did not return to Denmark except for a short time in 1682.[4]

It may be proper to introduce here some historical remarks. Ever since the dawn of abstract geometry, mathematicians have taken delight in limiting the kind and number of instruments to be used in effecting geometric constructions. Apparently urged by the ideal of simplicity and economy in instrumental equipment, the Greeks ordained that in the science of geometry only an ungraduated straightedge and compasses shall be used. This limitation has given rise to some of the most interesting and famous discussions in geometry and analysis — on the squaring of the circle, trisection of an angle, duplication of a cube, and the inscription of regular polygons in a given circle. From time to time, further instrumental restrictions have been made in the interest of speculative geometry and mathematical recreation. Inspired perhaps by a remark of Pappus, the Arabic scholar Abû'l Wefâ[5] of the tenth century, in constructing the corners of the regular polyhedrons on a circumscribed sphere, set himself the condition that all construction be effected with a ruler and a *single* opening of the compasses. The German painter Albrecht Dürer and the Italian mathematicians of the sixteenth century, including Benedetti and Tartaglia,[6] effected many constructions under these limitations. J. V. Poncelet[7] in 1822 and Jakob Steiner[8] in 1833 went a step further and showed that all constructions possible with a straightedge and compasses can be effected also by the use of a straightedge, and a circle fixed in position and drawn once for all. In 1890 A. Adler[9], of Vienna, went still further and demonstrated that all these constructions can be made by the use of only an ordinary ruler with two parallel straight edges, or only a ruler in the form of a right angle, or only a ruler in the form of a fixed acute angle. If we take cognizance also of the fact that all constructions possible by the straightedge and compasses can be effected by the compasses alone, as was shown by Mohr, Mascheroni, and later writers, then the remarkable result stares us in the face that all Euclidean constructions can be made with *any one* of the four ordinary instruments of geometric construction taken singly; viz., the compasses, or the ruler with parallel straight edges, or the ruler with a right angle, or the ruler with an acute angle.

[4] See G. Eneström, Bibliotheca Mathemateca, (3) vol. 10 (1909-1910), pp. 71, 72; vol. 12 (1911-1912), p. 77.

[5] See Woepcke, Journal Asiatique, (5) vol. 5 (1855), pp. 241, 352-358.

[6] See Moritz Cantor, *op. cit.*, vol. II, 2nd ed., 1913, p. 566.

[7] J. V. Poncelet, *Traité des propriétés projectives des figures*, Paris, 1822, pp. 187-190.

[8] Jakob Steiner, *Ueber die geometrischen Constructionen ausgeführt mittels der geraden Linie und eines festn Kreises*, Berlin, 1833.

[9] A. Adler, *Ueber die zur Ausführung geometrischer Constructionsaufgaben zweiten Grades notwendigen Hülfsmittel*, Wiener Sitzungsberichte d. Akademie d. Wiss., Math.-Naturw. Classe, vol. 99 (1890), Abth. IIa, pp. 846-859.

FOREWORD

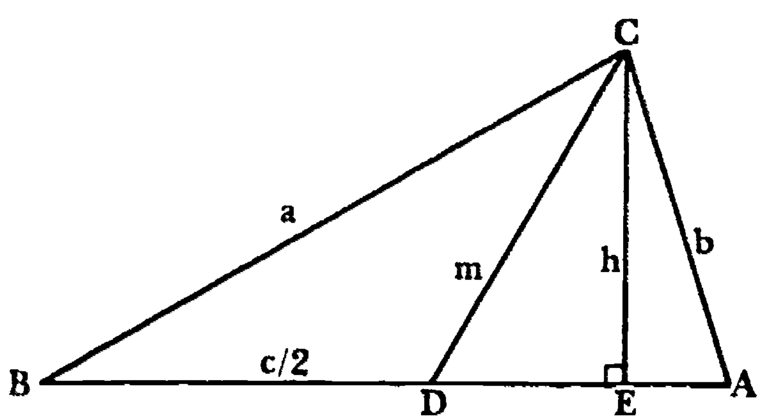
By using the methods of coordinate geometry, it can be proved that the analytical processes which are equivalent to the various steps of straightedge and compass construction are rational operations and the operation of taking a square root and no others. These operations may be combined and repeated in any way a finite number of times and appiied to the coordinates of the given points.

Actually, the only construction problems that can be solved by the use of the straightedge alone are those which depend, analytically, upon the solution of a linear equation whose solution involves only rational operations. Those problems and only those problems which can be solved by using straightedge and compasses are problems which depend analytically upon the solution of a second-degree algebraic equation whose solution involves rational operations together with the extraction of square roots only.

In the course of his discussion in the present essay, the author refers to a theorem of plane geometry which may not be familiar to you.

Theorem: In any triangle ABC, the median to side c ($= m$) is given by
$4m^2 = 2a^2 + 2b^2 - c^2$.

Proof:



In $\triangle BEC$: $a^2 = BE^2 + h^2$. (1)
In $\triangle DEC$: $m^2 = DE^2 + h^2$. (2)

Subtracting (1) from (2):

$$m^2 = a^2 + DE^2 - BE^2.$$

But $BD + DE = BE$.

Therefore $m^2 = a^2 - BD^2 - 2(BD)(DE)$ (3)

Using triangles AEC and DEC, together with the fact that $AE = DA - DE$, we have:

$$m^2 = b^2 - DA^2 + 2(DA)(DE). \qquad (4)$$

Now $BD = DA = \frac{c}{2}$.

Adding (3) and (4):

$$2m^2 = a^2 + b^2 - BD^2 - DA^2,$$

or $2m^2 = a^2 + b^2 - 2(BD)^2$.

But $BD = \frac{c}{2}$; hence

$$2m^2 = a^2 + b^2 - 2\left(\frac{c^2}{4}\right),$$

or $4m^2 = 2a^2 + 2b^2 - c^2$.

Mascheroni Constructions

Julius H. Hlav· ˈy

Introduction

Plato is credited with (or blamed for) restricting the geometer to the use of the compasses and straightedge alone. As Hogben points out, it was perfectly consistent for Plato to lay down this limitation: "Geometry was an aid to spiritual perfection. We are not expected to attain spiritual perfection and enjoy ourselves at the same time. So it was natural for those who held this belief to make geometry as difficult and unpalatable as generations of school children have found it." However that may be, we must not overlook that it was this very restriction which opened the way to a great deal of investigation in mathematics arising from attempts to solve certain problems by means of the straightedge and compasses alone. The long history of the three notorious problems of antiquity—the trisection of an angle, the doubling of the cube, and the squaring of the circle—owes its start to the Platonic restriction. The high point of the history of construction problems involving the straightedge and compasses was reached when, in the last century, it was demonstrated that all constructible numbers are algebraic, and when the impossibility of the solution of the three famous problems was proved.

It is natural to expect—and the expectation has been realized by experience—that if we permit the use of instruments other than the straightedge and the compasses, we can solve a greater variety of problems. For example, it is well known that the trisection of an angle is possible if we permit just a slight modification of the straightedge, e.g., putting one mark on the straightedge.

The Mascheroni Problem

It was left for an Italian mathematician (and it turned out later that he had been anticipated by a Danish mathematician) to prove that Plato missed a trick. In 1797, Lorenzo Mascheroni published his book *Geometria del Compasso* in which he showed that *all* the constructions of geometry that are performable by the use of the straightedge and com-

passes can also be performed by means of the compasses *alone*. The Danish mathematician Hjelmslev discovered in 1928 that one of his countrymen, Georg Mohr, had anticipated Mascheroni in his discovery. In *Euclides Danicus*, published in 1672, Mohr had solved the so-called Mascheroni problem.

The question naturally arises: "How can it be shown that all the constructions of Euclidean geometry can be done by means of the compasses alone?"

The number of constructions in geometry is infinite, and therefore we certainly cannot prove our problem by solving every construction problem with compasses alone. However, all Euclidean constructions are merely finite successions of the following four fundamental constructions:

I Drawing a circle with a given center and a given radius

II Finding the points of intersection of two circles

III Finding the points of intersection of a line and a circle

IV Finding the point of intersection of two lines.

It is obvious that the first two problems offer no difficulty since our instruments are the compasses. It is necessary, however, to make some convention about the meaning of a straight line, since a straight line cannot be drawn with compasses alone. We will say that a straight line is given if two of its points are given. It must be possible, however, to find as many points on the line so given as we desire. The solution of this problem is easy and will be made clear in the rest of the article.

We must therefore show that it is possible to solve problems III and IV with the compasses alone. The solution of these two problems, as well as many other Mascheronian constructions, can be simplified considerably by introducing the theory of inversions. However, since that theory is not a usual part of elementary geometry, we will give the constructions and solutions that do not explicitly involve this theory. The reader may consult Courant or Yates (see the bibliography) for the constructions by use of inversions. We will conclude with additional problems—some worked out and others merely proposed.

Solution of Problem III

Problem: To find the points of intersection of a line and a circle

Case 1: Given circle O with radius r and a line defined by points A and B, where A and B are not collinear with O.

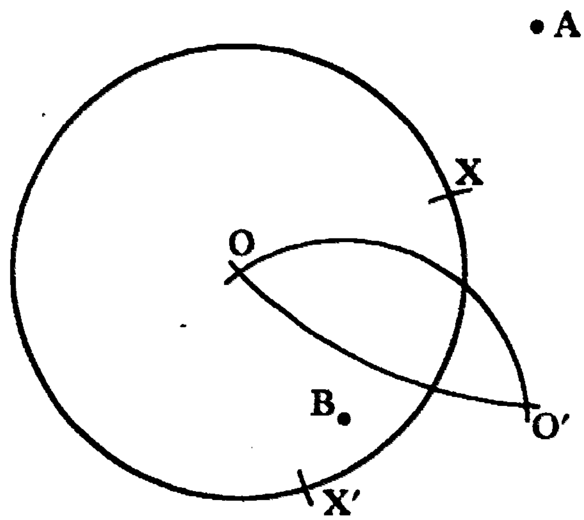


FIGURE 1

Construction:

1. With A as center and radius AO draw an arc through O (problem I).
2. With B as center and radius BO draw an arc through O (problem I).
3. Determine O' point of intersection of these two circles (problem II).
4. With O' as center and radius r intersect circle O in X and X′ (problems I and II).

Points X and X′ are the required points of intersection.

Proof:

$$AO = AO', BO = BO'$$

by construction.

AB is the perpendicular bisector of OO'. (Two points equally distant from the ends of a line segment determine . . .)

$$OX = O'X, OX' = O'X'$$

by construction.

X and X′ lie on AB. (Points equally distant from the ends of a line lie on the perpendicular bisector.)

Case 2: Given circle O with radius r, and a line through the center of the circle and determined by A and O.

We can draw the arc of an arbitrary circle with center A and intersecting circle O in points B and C (Problems I and II).

It is clear that the midpoints of the arcs BC are the required points.

Therefore, if we succeed in bisecting an arc we have solved the problem.

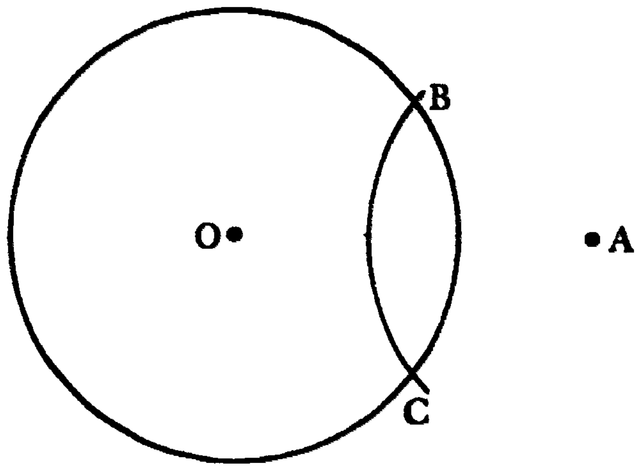


FIGURE 2

Auxiliary problem IIIa: To bisect an arc of a circle
Given: Arc BC with center O and radius r
To find: X, the midpoint of arc BC

Construction:

With B and C as centers and radius r draw arcs DO and OE.
Construct $OD = OE = BC$.
Determine point F such that

$$DF = EF = CD\,(= BE).$$

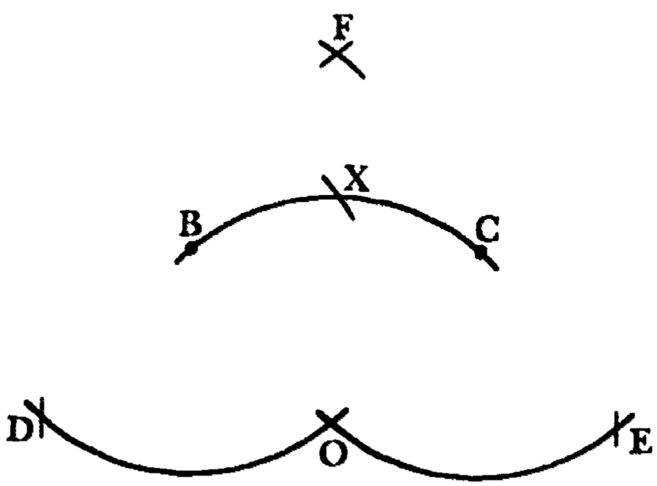


FIGURE 3

With OF as radius and D (or E) as center cut arc BC in X.
Point X is the required midpoint of arc BC.

Proof:

Since $BC = DO$ and $BD = OC$, BC is parallel to DO.

Since $BC = OE$ and $BO = CE$, BC is parallel to OE.

Therefore, D, O, E are collinear.

Since $DO = OE$ and $DF = EF$, OF is the perpendicular bisector of DE.

Therefore, OF is perpendicular to DC and OF bisects BC and arc DC.

It remains, therefore, to show that X lies on OF.

To prove that X is on OF, we will show that $\angle XOE$ is a right angle.

We can do this by proving

$$\overline{EX}^2 = \overline{OX}^2 + \overline{OE}^2.$$

Since CO is the median of triangle CDE,

$$4\overline{OC}^2 = 2\overline{DC}^2 + 2\overline{CE}^2 - \overline{DE}^2.$$

But by construction $DE = 2OE, DC = EF, OC = CE$.

$$\begin{aligned} 4\overline{OC}^2 &= 2\overline{EF}^2 + 2\overline{OC}^2 - 4\overline{OE}^2 \\ 4\overline{OE}^2 + 2\overline{OC}^2 &= 2\overline{EF}^2 \\ 2\overline{OE}^2 + \overline{OC}^2 &= \overline{EF}^2 \end{aligned}$$

In right triangle OEF, $\overline{EF}^2 = \overline{OE}^2 + \overline{OF}^2$

$$\begin{aligned} 2\overline{OE}^2 + \overline{OC}^2 &= \overline{OE}^2 + \overline{OF}^2 \\ \overline{OE}^2 + \overline{OC}^2 &= \overline{OF}^2. \end{aligned}$$

Since X is on arc BC, $OC = OX$; and $OF = EX$ by construction,

$$\overline{OE}^2 + \overline{OX}^2 = \overline{EX}^2.$$

We have now demonstrated that it is possible to bisect an arc, and, thus, we can complete the solution of Problem III.

The complete construction may then be carried out as follows:

With center A and any radius (we have used the radius of the given circle, for convenience) intersect circle O in points B and C.

With B and C as centers and radius BO draw arcs OD and OE.

With BC as radius and O as center intersect arcs OD and OE in D and E.

With centers D and E and radius DC construct arcs intersecting in F

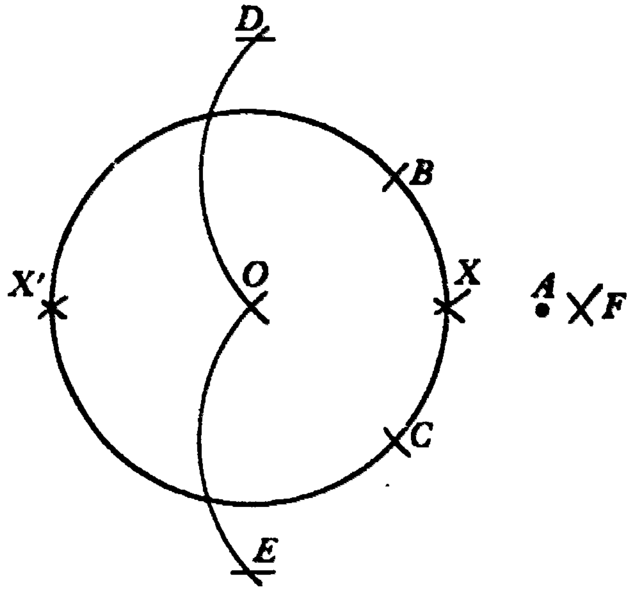


FIGURE 4

With center E and radius OF intersect circle O in points X and X'.

We can check our construction by cutting the circle O with a circle with center D and radius OF.

SOLUTION OF PROBLEM IV

Problem: To find the point of intersection of two lines AB and CD

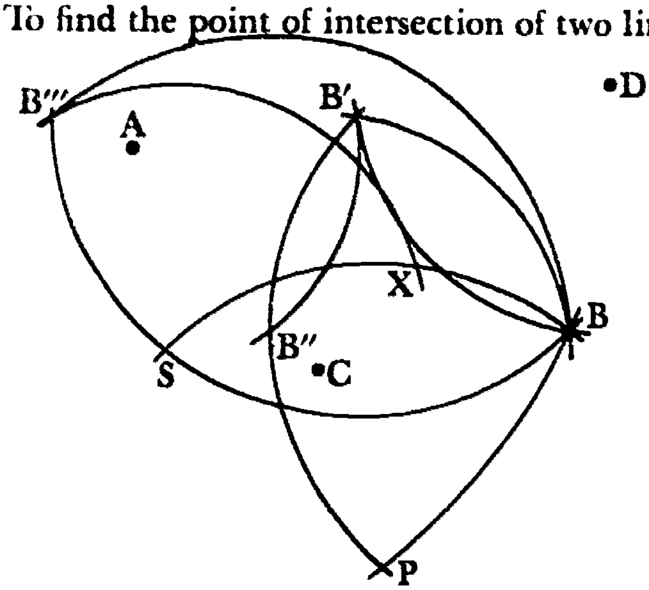


FIGURE 5

Construction:

Let us adopt the following notation: $K(Y)$ is to mean: "A circle with K as center and passing through Y and having therefore the radius KY."

1. Find the symmetric B' of B with respect to CD. (Draw circles $C(B)$ and $D(B)$ from which it follows:

a. $DB' = DB$ and b. $CB' = CB$)

2. Find the symmetric B'' of B' with respect to AB. (Draw circles $A(B')$ and $B(B')$ from which it follows:

a. $AB'' = AB'$ and b. $BB'' = BB'$)

3. Find the symmetric B''' of B with respect to $B'B''$. Draw circles $B'(B)$ and $B''(B)$ from which it follows:

a. $B'B''' = B'B$ and b. $B''B''' = B''B$)

4. Cut $B(B')$ in P by $B'''(B)$ and it follows:

a. $B'''B = B'''P$ and b. $BP = BB'$

5. Cut $B'(B)$ in S by $P(B)$ and it follows:

a. $B'B = B'S$ and b. $PS = PB$

6. Cut $P(B)$ in X by $S(B''')$ and it follows:

a. $PB = PX$ and b. $SB''' = SX$

X is the required point of intersection of AB and CD.

Proof:

It has to be proved that X lies on AB and on CD.

1. $B'(S) = P(S)$

(Since $B'S = BB'$ [5a], $BB' = PB$ [4b], $PB = PS$ [5b])

2. $SX = SB'''$ by 6b

3. arc SX = arc SB'''

4. $\angle SBX = \frac{1}{2}$[arc SX]

$\angle SBB''' = \frac{1}{2}$[arc SB''']

5. and $\angle SBX = \angle SBB'''$ and X lies on BB'''

6. AB is the perpendicular bisector of $B'B''$ by 2 above.

7. $B''B''' = B'''B'$

(Since $B''B''' = B''B$ [3b], $B'''B' = B'B$ [3a], $B''B = B'B$ [2b])

8. Therefore, X lies on AB.

9. Triangles $B'''BP$ and XBP are isosceles and have $\angle XBP$ in common. ($B'''B = B'''P$ [4a], $PX = PB$ [6a]).

10. Triangles $B'''BP$ and XBP are similar.

11. $BB''':PB = PB:BX$

12. $BB''':BB' = BB':BX$, by substitution in 4b.
13. $\angle B'''BB' = \angle XBB'$, by identity.
14. Triangles $B'''BB'$ and PXB' are similar.
15. But triangle $B'''BB'$ is isosceles, by 3a.
16. Therefore, triangle BXB' is isosceles and $BX = B'X$.
17. And since CD is the perpendicular bisector of BB' (by 1),
18. X must lie on CD, and X is the point of intersection of AB and CD.

Miscellaneous Problems

1. To bisect a line segment.

Given: Line segment AB.

Construction:

Draw circles $A(B)$ and $B(A)$.

On circle $B(A)$ find point C diametrically opposite to A. (Mark off AB on $B(A)$ three times.)

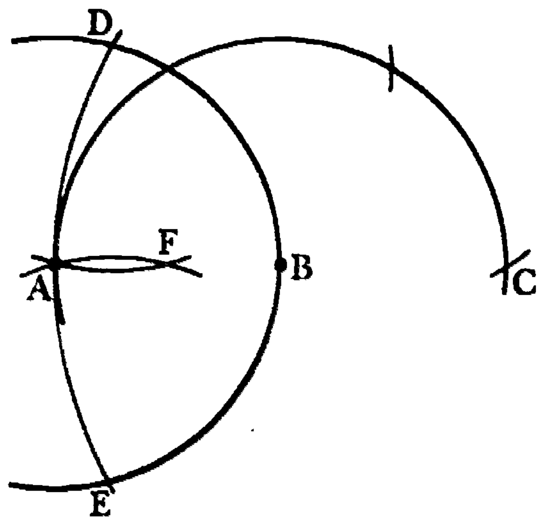


Figure 6

Draw arc with $C(A)$ intersecting $A(B)$ in D and E.

Draw $D(A)$ and $E(A)$ intersecting in F. F is the required point.

Proof:

Since $AD = AE$, $CD = CE$ and $DF = EF$, AC is the perpendicular bisector of DE and F lies on CA.

Isosceles triangles ADF and CAD are similar.

Therefore

$$CA:AD = AD:AF$$

or

$$2AB:AB = AB:AF$$

and

$$AB = 2AF.$$

2. To construct a regular pentagon

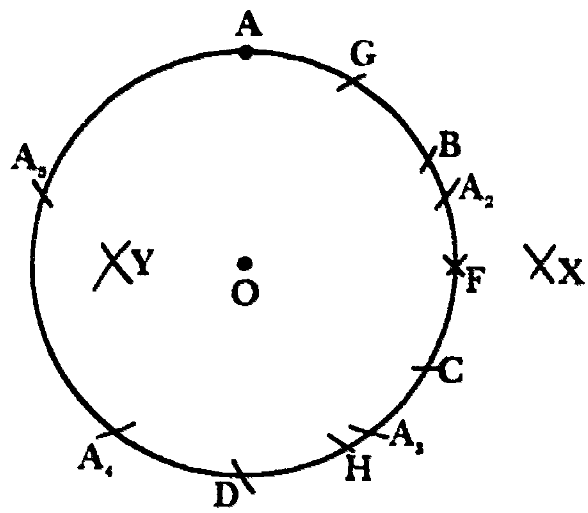


FIGURE 7

Construction:

Take any point A on circle O.

Find point D diametrically opposite to A.

Find F, the midpoint of arc BC. (See auxiliary construction in the solution of Problem III above.)

From F mark off points G, H so that $FG = FH = AO$.

Find point Y so that $GY = HY = OX = (AF)$.

Then AY is the side of the required pentagon.

The proof is left for the reader.

3. To construct a line through a given point parallel to a given line.

4. To erect a perpendicular to a given line at one extremity of the given line.

5. To drop a perpendicular to a given line from a given point outside the given line.

6. To construct the fourth proportional to three given line segments.

7. To construct the mean proportional between two given line segments.

8. To construct the tangents to a given circle from a given external point.

9. To find the center of a given circle. (This is called Napoleon's problem. Napoleon is reported to have challenged Mascheroni to solve this problem, and Mascheroni succeeded with a very elegant solution.)

Bibliography

COURANT, RICHARD and ROBBINS, HERBERT. *What Is Mathematics?* New York: Oxford University Press, 1941.

Among the many excellent things in this book is a good and complete treatment of geometrical constructions. The theory of inversions is introduced in the treatment of the Mascheroni problem.

There is also a treatment of the other, analogous problem of construction — restriction to the use of the straightedge alone. Steiner's solution of these constructions is presented, using the straightedge and a given circle with a given center.

Constructions with other tools, mechanical instruments, and linkages are also dealt with.

LANASCOL, A. QUEMPER DE. *Géométrie du Compas.* Paris, 1925.

The fifteen chapters of this book are a treasure-trove of constructions. All the constructions of elementary geometry are carefully carried out and demonstrated. For many of the problems there are many alternate solutions given, and there is a wealth of historical information on various solutions. For example, there are four distinct solutions of the construction of the fourth proportional. Mascheroni's own constructions are given for many of the problems.

STEINER, JACOB. "Geometrical constructions with a ruler, given a fixed circle with its center." Translated by M. E. Stark. *Scripta Mathematica.* Vol. XIV, pp. 187-264. (September-December 1948).

The first English translation of Steiner's original booklet with an introduction and notes by R. C. Archibald.

YATES, ROBERT C. *Geometrical Tools, A Mathematical Sketch and Model Book.* St. Louis: Educational Publishers, revised 1949.

This is an excellent little book, published in workbook form, with constructions to be done in the book. Our special problem is treated on pages 42-53, and the approach is through inversions. The topics covered in this book are: the straightedge and the modern compasses; dissection of plane figures; the compasses; folds and creases; the straightedge; line motion linkages; the straightedge with immovable figure; the assisted straightedge; parallel and angle rulers; higher tools and quartic systems; general plane linkages. Complete bibliographies are given on each of these topics.

FOREWORD

One does not ordinarily associate the name of Napoleon with the development of mathematics, yet he is supposed to have said that the advance and perfecting of mathematics are closely related to the prosperity of a nation. Suffice it to say that Napoleon did have contacts with a number of mathematicians, including Monge, Fourier, Poncelet, and Mascheroni. It is the latter in whom we are here interested. When Lorenzo Mascheroni in 1797 published his celebrated *Geometria del Compasso*, he proved that any construction that can be performed with the straightedge and compasses could also be executed with the compasses alone. To be sure, no straight lines appear in such constructions; points are not determined by the intersection of the two straight lines; a straight line is regarded as having been "constructed" or obtained when the locations of the two points lying on the line are known. All constructions are made with the compasses, but without restriction to a fixed radius.

Mascheroni asserted that such constructions with compasses alone were more accurate than those involving the use of a straightedge as well. It was Napoleon who proposed to French mathematicians the problem of dividing a circle into four equal parts by using only the compasses; Mascheroni solved the problem by applying the radius three times to the circle.

About a century later, the famous Viennese geometer August Adler, in his *Theorie der Geometrischen Konstruktionen*, proved that Mascheroni's claims were correct, namely, that all Euclidean constructions can be carried out by the use of the Euclidean compasses alone. However, in so doing, Adler used the idea of inversion of a circle, a concept unknown to Mascheroni, having been put forth by Jacob Steiner in 1824. Other mathematicians, including E. W. Hobson and H. P. Hudson, have also corroborated and contributed to Mascheroni geometry. It is also of interest to note that in 1822 Poncelet proved that all constructions possible with Euclidean tools can be performed with the straightedge alone, if we are given a fixed circle with its center in the plane of construction.

Mascheroni Constructions

By N. A. Court

In a recent issue of THE MATHEMATICS TEACHER,[1] Dr. J. H. Hlavaty called attention to the curious so-called Mascheroni constructions, or geometric constructions with compasses alone. The readers of Dr. Hlavaty's attractive article may be interested in some supplementary notes of a historical and bibliographical nature.

A BYPRODUCT OF THE MASCHERONI CONSTRUCTIONS

Plato enjoined seekers after geometrical knowledge to carry out their geometric constructions with only an unmarked ruler and compasses. What prompted such a restriction is an open question which has led to considerable speculation. It seems certain, however, that it never occurred to the great philosopher that such a seemingly innocent limitation imposed upon the permissible construction tools could possibly have a bearing upon the nature of the problems which can then be solved. For example, the Greek geometers soon discovered that they could devise a number of methods for the trisection of an angle if they disregarded Plato's restriction, but they were never able to solve that problem while complying with the restriction. They came to the conclusion that Plato's restriction was a severe handicap at times, though this, of course, did not show that a "legitimate" solution of the trisection problem could not be found. This view was shared by succeeding generations of mathematicians, and for about two thousand years geometers awaited the genius who would finally conquer the trisection problem, only finally to discover that this "messiah" will never come.

By outbidding Plato in "puritanism," Mascheroni brought the question of the role of construction tools in geometry to the fore at a time that was ripe and ready to deal with it. The matter was taken up by Poncelet (1788–1867), Steiner (1796–1867), and others. These preliminary studies paved the way for Gauss (1777–1855), who finally provided the definitive answer to the question concerning which problems can and which cannot be solved with ruler and compasses.

[1] L, No. 7 (November 1957).

GEORG MOHR

Strictly speaking, the term "Mascheroni construction" is a misnomer, for the Italian mathematician was not the first one to discard the ruler and to carry out geometrical constructions with the compasses alone; he was anticipated 125 years earlier by a Danish mathematician named Georg Mohr. In 1672, Mohr simultaneously published, in Amsterdam, a Dutch edition and a Danish edition of a book bearing the title *Euclides Danicus*. This book contains Mascheroni's basic result and a goodly number of his problems.

Very little is known about Georg Mohr. He was born in 1640 in Copenhagen, and it is surmised that like many of his Scandinavian contemporaries he left Denmark to study at the then flourishing Dutch universities. No other writings of Mohr are known. Leibniz, in a letter daced May 12, 1676, and addressed to H. Oldenburg, then secretary of the Royal Society of London, refers to "Georgius Mohr Danus in Geometria et Analysi versatissimus" ("the Dane Georg Mohr very well versed in geometry and analysis").[2]

The contemporaries of Mohr may have known a good deal more about him than we do now, but they appear to have paid little attention to his *Euclides Danicus*. In fact, the book seems to have passed entirely unnoticed. Bibliographical references to this work are very scant, and those extant seem to take the book for some kind of compilation of Euclid's *Elements*. A copy of the Danish edition of the *Euclides Danicus* came to light only very recently, and by sheer accident, when a Danish student happened upon the book in a secondhand bookshop. He showed the book, for appraisal, to his teacher, Professor Johannes Hjelmslev of the University of Copenhagen. The latter, realizing the book's historical importance, published a facsimile copy of it together with a German translation, *Georg Mohr, Euclides Danicus, Amsterdam, 1627*, in Copenhagen in 1928.

Strange as may be the reappearance of a book which was ignored for more than two and a half centuries, this find does not in any way detract from the merits of Mascheroni's work. At the time the Italian mathematician wrote his *Geometria del compasso*, nobody knew anything about either Mohr or his book. Mascheroni concludes the preface to his renowned book with the explicit statement that he knows of no work of the same kind as his, and there is not the slightest ground for doubting his word.

[2] C. I. Gerhardt, *Leibnizens Mathematisch. Schriften* (Berlin), I (1849), 88.

Mathematician and Poet

Mascheroni's fame as a mathematician is largely based on his *Geometria del compasso*, but not exclusively so; he is also the author of several other books. The author of the article on Mascheroni in the *Great Soviet Encyclopedia*,[3] now approaching completion, credits the Italian mathematician with having been the first to introduce into mathematical analysis the *sine integral* and the *cosine integral*, that is, the functions

$$\int_0^x (\sin t\,dt)/t \text{ and } \int_x^\infty (\cos t\,dt)/t.$$

In addition to being a gifted mathematician, Mascheroni was also a talented poet — a rather rare combination. To consider him as "the greatest poet among mathematicians" is to belittle him, for literary men are just as eager to claim him as one of their own as mathematicians are to consider him as belonging to their clan. The articles devoted to Mascheroni in the French *Grande Encyclopédie*[4] and the *Enciclopedia Italiana*[5] characterize him as both "mathematician and poet." Moreover, both articles were written, not by mathematicians, but by professors of literature. There is more than one edition of Mascheroni's collected poetical works.

Napoleon

Although a member of a monastic order, Mascheroni had sympathy for the French Revolution and was a great admirer of Napoleon. The book *Problemi per gli agrimensori* or *Problems for Surveyors*, which Mascheroni published in 1793, included a dedication, in verse, to Napoleon. During Napoleon's campaign in northern Italy, the two men became acquainted with one another, and the successful general learned directly from the Italian scholar about the latter's geometrical discoveries. When, shortly after that, Mascheroni published his *Geometria del compasso*, he made use of his poetic talent to place at the head of his work a dedicatory ode to Napoleon, a poem of considerable literary merit. Napoleon, on his part, repaid his friend by being instrumental in bringing the author's work to the attention of the learned circles of France.

There is, in this connection, a historically authenticated anecdote that is worth relating. In December of 1797 there took place in Paris a brilliant gathering of prominent writers and scholars, with the immortal

[3] XXVI, 423.
[4] XXIII, 360.
[5] XXII, 496.

Lagrange and Laplace among them. A most conspicuous member of the company was the young and victorious General Napoleon Bonaparte, who happened to be a former pupil of Laplace in a military school. In the course of the evening, the victor at Arcole and Rivoli had occasion to entertain Lagrange and Laplace with a kind of solution of some problems of elementary geometry that was completely unfamiliar to either of the two world-famous mathematicians. Legend has it that after having listened to the young man for a considerable while, Laplace, somewhat peeved, remarked, "General, we expected everything of you, except lessons in geometry."

Echoes of the above conversation prompted the young A. M. Carette, who had just graduated from the famous Ecole Polytechnique founded by Gaspard Monge (1746–1818), to translate Mascheroni's book into French. The translation was published in 1798, one year after the original had come off the press. Thirty years later, in 1828, Carette published a second edition in which he included a biography of Mascheroni, but the dedicatory poem to Napoleon was left out—the political complexion of France had changed.

EDITORIAL NOTE. There is reason to believe that the idea of undertaking geometric constructions with compasses alone was suggested to Mascheroni by the earlier work of Giambattista Benedetti (1530–1590), a Venetian by birth, who wrote on the geometry of so-called *rusty compasses*, or compasses of fixed opening. Such investigations seem first to have been considered with some success by the Arabian mathematician Abû'l-Wefâ (940–998).

A German edition of Mascheroni's work, entitled *L. Mascheronis Gebrauch des Zirkels*, was prepared by J. P. Grüson and brought out in Berlin in 1825. It is based on Carette's 1798 French edition of the work. The subject is treated in English in various places, for example: A. Cayley, *Messenger of Mathematics*, Vol. 14 (1885), pp. 179–181; A. Cayley, *Collected Papers*, Vol. 12, pp. 314–317; E. W. Hobson, *Mathematical Gazette*, Vol. 7 (1913), pp. 49–54; H. P. Hudson, *Ruler and Compasses*, Longmans Green and Company, Inc. (1916), reprinted in *Squaring the Circle and Other Monographs*, Chelsea Publishing Company (1953), pp. 131–143; J. L. Coolidge, *Treatise on the Circle and Sphere*, Clarendon Press (1916), pp. 186–188. The work by Hudson gives not only Mascheroni's exposition, but also a more modern treatment, due to A. Adler of Vienna, which employs geometric inversion.

A bibliography of Mascheroni's mathematical writings is given in *L'Intermédiare des mathématiciens*, Vol. 19 (1912), p. 92. For a picture of Mascheroni, after an engraving by F. Redenti, from a drawing made from life, see D. E. Smith, *History of Mathematics*, Vol. 1, Ginn and Company (1923), p. 516. Here one can also find reference to three Italian biographies of Mascheroni.

The particular problem, "to divide the circumference of a circle into four equal parts by compasses only," has become known as *Napoleon's problem*. Although the solution to this problem appears in Mascheroni's work, it is narrated that Napoleon proposed the problem to the French mathematicians.

FOREWORD

We have already suggested the distinction between the Euclidean compasses and modern compasses; the former is a collapsible instrument, whereas with the modern compasses (with a set radius) not only can we describe a circle (as with the Euclidean compasses), but we can also carry a distance from one place to another, an operation which is properly executed by an instrument known as the *dividers.*

Every operation that can be performed with straightedge and dividers can be performed with straightedge and Euclidean compasses. The converse is not true, however; the straightedge and dividers can do more than the straightedge alone, but not as much as the straightedge and Euclidean compasses.

Both Mascheroni and Adler showed how it is possible to avoid determining a point as the intersection of two straight lines. Naturally, if we use the compasses only, a straight line cannot be "drawn" but it can be determined: we consider it as having been "constructed" when two points lying on the line are known or have been found. Once a line has been so "constructed," it is also possible, still with the compasses alone, to determine any other desired points on the line.

Mascheroni's methods are older than those of Adler. Since Mascheroni freely used his compasses as dividers, his methods are often shorter. Thus Mascheroni describes a given circle C(AB) in a single operation; Adler, using Euclidean compasses, must use a construction using five circles to accomplish the same result. However, Adler's methods are more elegant and more powerful.

Can We Outdo Mascheroni?

By WM. FITCH CHENEY, JR.

LORENZO MASCHERONI was for many years professor of mathematics at the University of Pavia (some twenty-two miles south of Milan), where Christopher Columbus had once been a student. Mascheroni was an Italian. He was born in 1750 and died in 1800. During his life, he published a considerable number of mathematical writings, the best known of which was his *Geometry of the Compass,* which first appeared in 1797. In it he showed how all standard constructions usually performed with straightedge and compass could be carried out with the compass alone. Mascheroni claimed that the compass was more accurate than the straightedge, since few, if any, "straightedges" are really straight, and they tend to skid more easily than a compass when in use.

Mascheroni used the "Modern Compass," which retained its setting when lifted from the paper, rather than the more elegant "Classical Euclidean Compass of the Greeks," which would close up if either point was raised from the drawing surface. In his constructions, Mascheroni frequently reflected points across lines, but did not know of inversion, which was discovered by Steiner in 1824, and was later named by Liouville "the Transformation by Reciprocal Radii."

As is well known, the mechanics of reflecting the point, P, across the line, AB, consists in swinging two classical arcs through P, centered at A and B respectively, until they meet again at P'. Inversion, on the other hand, replaces P by Q on the same produced radius of the inversion circle centered at O, such that the radius, r, of that circle is the geometric mean of OP and OQ. If $OP = r$, so does OQ. *If* $\frac{1}{2} r < OP$, the mechanics of inversion consists in drawing $P \circ O$, (that is, the circle centered at P and through O), to intersect the inversion circle at S and T, and then reflecting O across ST to Q. This construction is readily justified through the similar isosceles triangles, SOP and OQS. It requires three classical arcs. If P is less than $\frac{1}{2} r$ distant from O, its distance from O must be doubled, to P_1, by letting $O \circ P$ cut $P \circ O$ at U and V, and $U \circ V$ cut $P \circ O$ at P_1. If P_1 is more than $\frac{1}{2} r$ from O, it is inverted to Q_1 by the process outlined above, and Q_1 then replaced by Q, twice as far from O. If P_1 is not

more than ½ r from O, doubling must continue to P_n, which is. After inverting P_n to Q_n, the distance of the latter from O must then be doubled the same number of times to reach Q, the inverse of the original P. To double any distance with compass only takes three classical arcs.

In 1890, Professor August Adler of Vienna published a book on *The Theory of Geometric Constructions*, and devoted Chapter Three to the consideration of Mascheroni's constructions. Professor Adler used the classical compass throughout his work, and relied largely on inversion. He could do everything with the classical compass that Mascheroni could do with his modern one.

Still later, in 1916, Hilda P. Hudson published a book on *Ruler and Compasses* (Longmans, Green and Co.), in Chapter Eight of which she commented on the work of Mascheroni and Adler, and focused attention on the number of arcs necessary in their various constructions, and whether or not they were all classical. It is largely from the stimuli of these publications that the present paper has been produced.

It is said that Napoleon Bonaparte delighted in stumping his engineers with the problem of quadrisecting a given circumference with a compass as the only tool. This problem is now generally known as "Napoleon's Problem," although it had been known before Napoleon's time. Mascheroni solved this problem by first stepping off the radius of the given circle around its circumference to locate in succession the four points, A, B, C and D, at 60° intervals. (See Fig. 1. In the figures of this paper, all construction arcs are numbered in the order in which they are

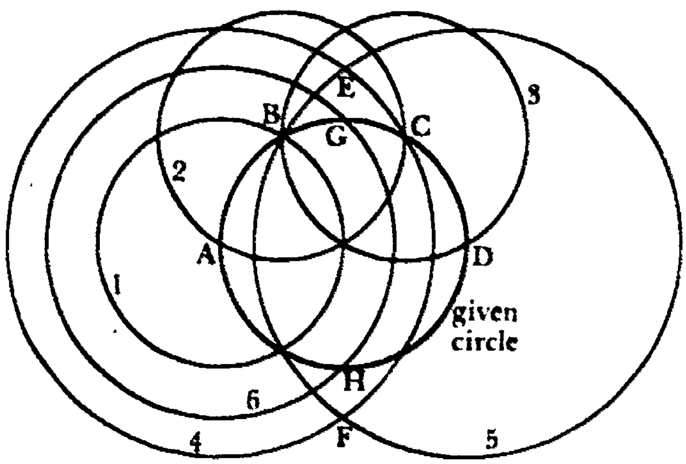


FIGURE 1

drawn.) Thus AD was a diameter. He then let $A \circ C$ cut $D \circ B$ at E and F. If the center of the given circle was called O, he then swung an arc centered at A and with radius equal to OE, to cut $O \circ A$ at G and H. This resulted in the arcs, $AH = HD = DG = GA = 90°$. To justify this

construction, note that in triangle AOC the cosine law tells us that $AC = r\sqrt{3}$. The law of Pythagoras applied to right triangle AOE makes $OE = r\sqrt{2}$ and in any circle, $r\sqrt{2}$ is the length of the chord to subtend a $90°$ arc. This construction of Mascheroni to solve Napoleon's Problem requires six arcs, the last of which is necessarily modern.

If we wished to solve Napoleon's Problem with the classical compass, we could replace Mascheroni's last step by reflecting E across BB' to E' and drawing $A \circ E'$ to cut $O \circ A$ at G and H. (Here B' is the other point of intersection of $O \circ A$ with $A \circ O$. See Fig. 2.) This quadrisects

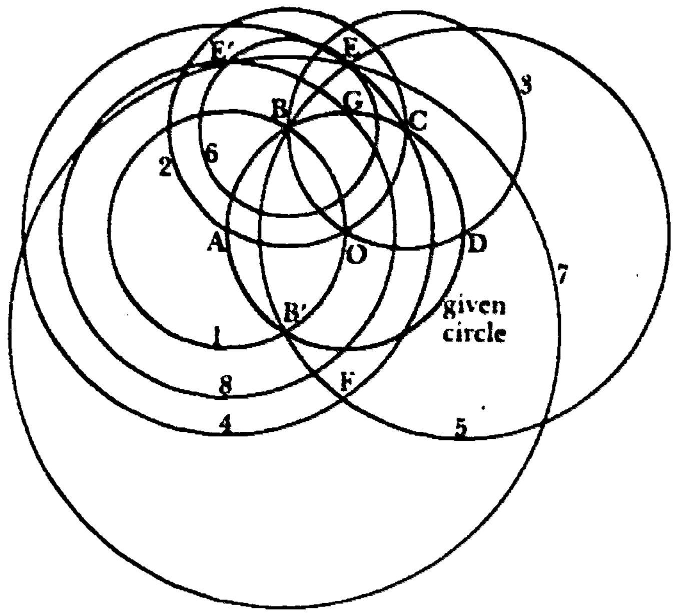


FIGURE 2

the circumference with eight classical arcs. However, we note that the circle $O \circ E$ completes the quadrisection of circle $B \circ O$, with the point diametrically opposite O lying on $A \circ C$. Hence we may assume that $B \circ O$ is given, (see Fig. 3), and construct in turn $O \circ B$, $C \circ B$, $A \circ C$, $D \circ B$ and $O \circ E$ to quadrisect $B \circ O$ by the points, O, P, Q and R. This requires only five arcs, and they are all five classical. (It is of interest in passing to note that this construction simultaneously quadrisects the circumference of $C \circ O$.)

A second and ancient problem solved by Mascheroni with compass only, was to construct the lengths of the sides, d and p, of the regular decagon and pentagon inscriptible in a given circle, whose radius may be taken as unit of measure. Mascheroni first quadrisected the given circle, $O \circ A$, as described above. (See Fig. 4.) He then let $G \circ O$ cut $O \circ A$ at J and K so that arcs $AJ = JB = BG = GC = CK = KD = 30°$.

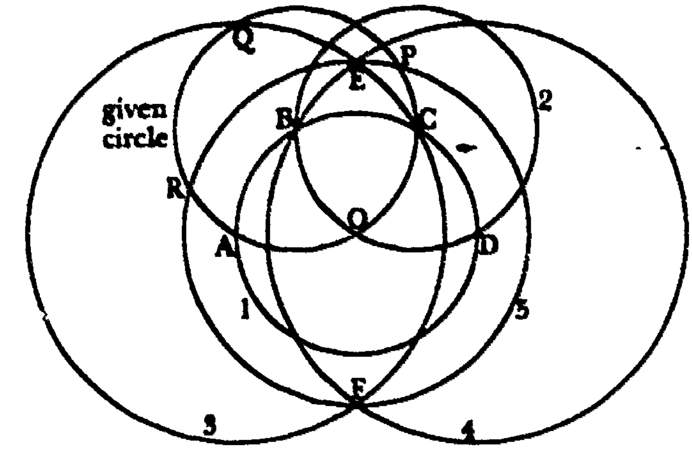


FIGURE 3

Finally he let $J \circ C$ cut $K \circ B$ inside $O \circ A$ at L. This construction makes $OL = d$ and $AL = p$, as is shown in the following paragraphs.

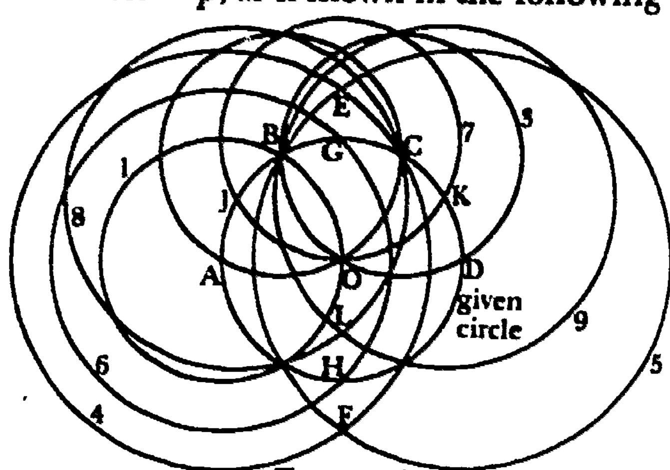


FIGURE 4

If, in a circle of unit radius, we inscribe a regular decagon of side d, then one side and two radii to its ends form an isosceles triangle, ABO, whose angles are 36°, 72° and 72°. (See Fig. 5.) If AC bisects angle A, then triangles ABC and ACO are isosceles, with triangle ABC similar to triangle ABO. From this similarity arises the quadratic equation, $d^2 + d = 1$, whose positive root is the famous Fibonacci ratio, $(\sqrt{5} - 1)/2$. Now in Figure 4, the sides of triangle JKL are by construction $\sqrt{3}$, $\sqrt{2}$ and $\sqrt{2}$. Hence its altitude from L is $\frac{1}{2}\sqrt{5}$, and $LO = \frac{1}{2}\sqrt{5} - \frac{1}{2} = d$.

Since connecting alternate vertices in a regular decagon produces a regular pentagon, we see that the perpendicular from A to BC, in Figure 5, would bisect it at M. Then $AM = \frac{1}{2}\,p$ and $MC = \frac{1}{2}(1 - d)$. In the right triangle, AMC, $AM^2 + MC^2 = AC^2$, or $p^2/4 + (1 - d)^2/4 = d^2$. Since $d = 1 - d^2$, this reduces to $p^2 = 1 + d^2$, so that p is the hypothenuse

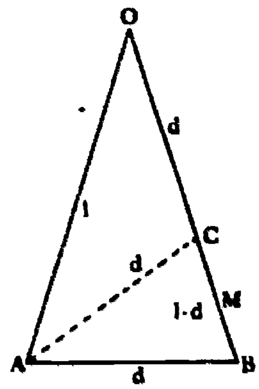


FIGURE 5

of a right triangle whose legs are 1 and d. But in Figure 4, ALO is a right triangle with legs 1 and d, so its hypothenuse, $AL = p$.

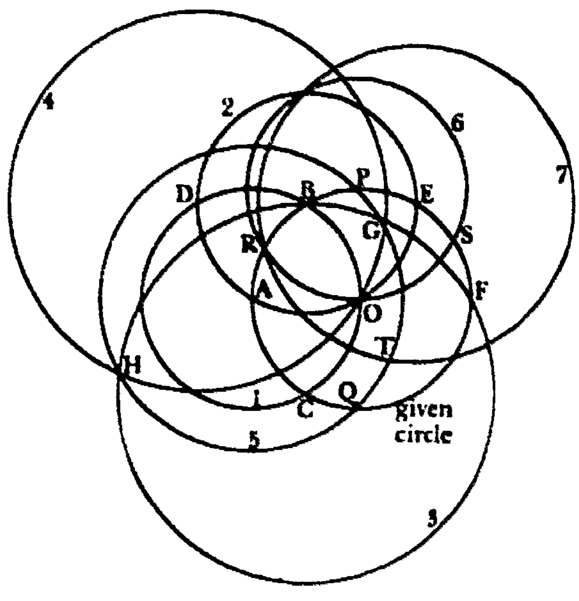


FIGURE 6

In this construction, Mascheroni used a total of nine arcs, one of which was modern. Figure 6 illustrates an alternative construction to improve on this plethora. With $O \circ A$ given, let $A \circ O$ cut $O \circ A$ at B and C. Let $B \circ O$ meet $A \circ O$ again at D, and meet $O \circ A$ again at E. Let $C \circ B$ meet $O \circ A$ again at F, and meet $D \circ O$ at G and H. Then $A \circ G$ meets $O \circ A$ at P and Q, such that arcs $AQ = QF = FP = PA = 90°$. Let $P \circ O$ meet $O \circ A$ at R and S, and let $E \circ R$ meet $A \circ G$ inside $O \circ A$ at T. Then $OT = d$ and $ST = p$. In this construction, all the arcs used are classical, and there are only seven of them.

A third fundamental compass construction contributed by Mascheroni was the location with compass only of the midpoint of an arbitrary arc of a given circle with known center. (See Fig. 7.) When given the arc

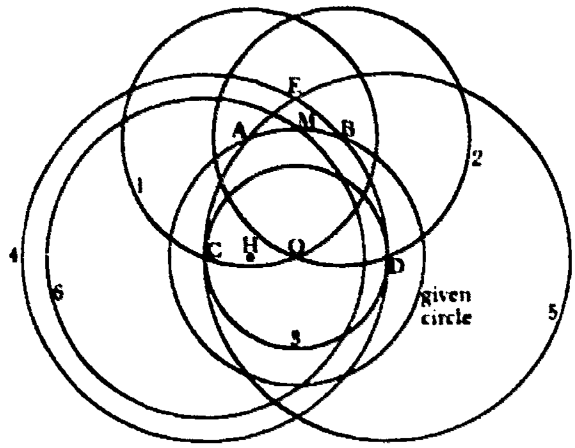


FIGURE 7

AB on $O \circ A$, he would first draw $A \circ O$ and $B \circ O$, and cut them at C and D respectively by a circle centered at O and with radius equal to chord AB, so that CD was a diameter of this last circle. He then let $C \circ B$ cut $D \circ A$ at E. Lastly he swung an arc centered at C, with radius equal to OE, to cut arc AB of $O \circ A$ at M. This required only six construction arcs, but the third and sixth of these necessitated the use of a modern compass. To justify this construction, let $OA = 1$ and $AB = CO = OD = s$. Now if M is the midpoint of arc AB, COM is a right triangle of legs s and 1, so that $CM = \sqrt{1 + s^2}$. Now if H is the midpoint of CO, the altitude HA of isosceles triangle COA is $\sqrt{1 - s^2/4}$, and it is also the altitude of right triangle HDA, whose base is $3s/2$. Thus $AD = \sqrt{1 - s^2/4 + 9s^2/4} = \sqrt{1 + 2s^2} = DE$. Finally, in right triangle ODE, $OE = \sqrt{1 + 2s^2 - s^2} = \sqrt{1 + s^2} = CM$, as desired.

Hilda Hudson solves this same problem entirely with classical arcs, for which she specifies the number necessary as fourteen. The following construction effects a definite reduction from that number. (See Fig. 8.) Assume that arc AB is given on $O \circ A$. Let $A \circ O$ cut $O \circ A$ at C. Let $C \circ B$ cut $A \circ O$ at D. Let $O \circ D$ cut $A \circ O$ at E. Let $B \circ O$ cut $O \circ D$ at F. Let $F \circ A$ cut $E \circ B$ at G, and cut $O \circ A$ at H. Reflect G across AH to K. Then $E \circ K$ bisects arc AB at M. All of the arcs used in this construction are classical, and there are only nine of them.

Mascheroni pointed out that all geometrical constructions ordinarily made with straightedge and compass were reducible to the location of a series of points found by the intersections of lines and, or, circles. Con-

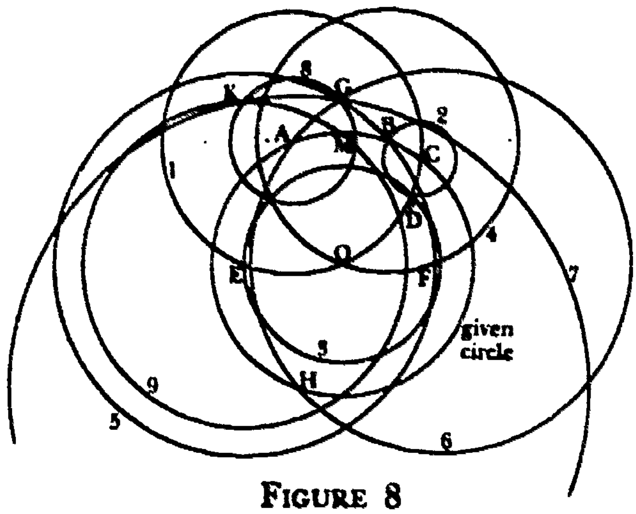


FIGURE 8

structing the intersection of two circles is obviously independent of the use of a straightedge. The intersections of the line AB (determined by the two points, A and B) with the circle $C \circ D$, he determined by reflecting C and D across AB to E and F respectively, and drawing $E \circ F$, which must cut $C \circ D$ in the desired points. This requires only five arcs, all of which are classical. However, one arc, (and hence 20% of the work) may be saved by the following trivially obvious procedure. First reflect C across AB to E, the intersection of $A \circ C$ with $B \circ C$. Let $A \circ C$ cut $C \circ D$ at G. Let $B \circ G$ cut $A \circ C$ at H. Then $E \circ H$ will cut $C \circ D$ in the desired points. It should be noted that if A, B and C are collinear, the reflection of $C \circ D$ across AB coincides with $C \circ D$, and hence fails to determine the desired points of intersection. In this case the solution depends on bisecting the two arcs into which $A \circ D$ divides $C \circ D$, using a construction already presented in this paper.

The remaining obstacle to eliminating the necessity of the straightedge in geometrical constructions is the location of the intersection of the lines joining two given pairs of points. Mascheroni solved this problem with eleven arcs, most of which were modern. Adler solved it by inverting the two given lines with respect to an arbitrary circle, and then reinverting the proper intersection point of the two corresponding circles.* This construction of Adler's used only classical circles, but

* For the convenience of readers unfamiliar with inversion theory, the following proof is included that all the points on a straight line go into all the points on a circle thru the center of invension. Assume that $O \circ A$ is the inversion circle, that M is the foot of the perpendicular from 0 onto an arbitrary line, and that P is an arbitrary point on that line. Let N be the inverse of M, and Q of P, so that $OM \cdot ON = OQ \cdot OP$, and triangles OQN and OMP are similar. Therefore OQ is perpendicular to NQ, and as P moves on the arbitrary given line, Q moves on the circle whose diameter is ON. Furthermore, if C is the center of this circle, $OC = \frac{1}{2} ON = 1/(2 \cdot OM)$, so that C may be found by reflecting 0 across the given line and inverting the resulting point.

according to Hilda Hudson, it took thirty-six of them in the general case. She reduced this number to sixteen by her construction. The present paper effects a further reduction. (See Fig. 9.)

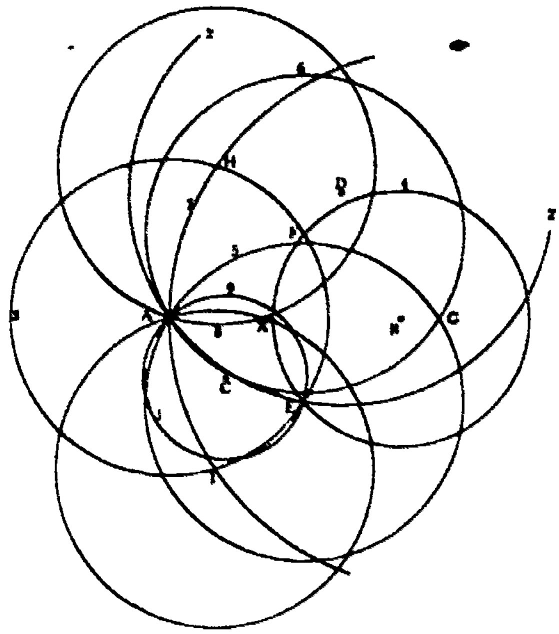


FIGURE 9

If the two given lines are AB and CD, their point of intersection, X, may be located as follows. Reflect A across CD to E. Reflect E across AB to F. Reflect A across EF to G. Let $G \circ A$ cut $A \circ E$ at H and I. Then reflecting A across HI will give X, the desired point of intersection of AB and CD. The justification of this construction lies in the facts that a line inverts into a circle through the center of inversion, and that its center is the inverse of the reflection of the center of inversion across that line. (See footnote.) In the construction of this paragraph, the inversion circle is $A \circ E$, which coincides with $A \circ F$. The inverse of the line CD is $E \circ A$. If C and D were both reflected across AB, the resulting line, $C'D'$, would cut CD at X, and its inverse circle would, by symmetry, be $F \circ A$. But $E \circ A$ cuts $F \circ A$ at G, whose inverse must thus be X.

If AB is perpendicular to CD, the above procedure breaks down because F coincides with E. But then $A \circ E$ cuts $E \circ A$ at K and L, $K \circ L$ cuts $E \circ A$ at G, and X is found from G as before. At the most then, by the constructions just described, the intersection of two general lines may be determined by only nine arcs, all of which are classical.

The table compares the numbers of arcs used for the constructions cited. There is a challenge in the fact that further simplifications have not in all cases been proved impossible.

The problems cited above are basic. Many similar problems admit of fascinating short cuts, usually within the comprehension of the best high school students. Their investigation, where time permits, provides both a valuable review of fundamental geometrical facts and a strong stimulus to further study.

The problem	Number of arcs used by Mascheroni, admitting modern ones	Least previously published number of arcs, using classical arcs exclusively	Minimum number of arcs in this paper, using classical arcs exclusively
Napoleon's	6	8	5
Constructing sides of inscribed decagon and pentagon	9	11	7
Bisecting an Arc	6	14	9
Intersection of general line and circle	5	5	4
Intersection of two general straight lines	11	16	9

For Further Reading and Study

Some of the following references pertain to geometric constructions in general as well as to Mascheroni geometry.

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