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ABSTRACT

This is one in a series of manuals for teachers using SMSG high school supplementary materials. The pamphlet includes commentaries on the sections of the student's booklet, answers to the exercises, and sample test questions. Topics covered include order on the number line, properties of order, solution of inequalities, and graphs of open sentences in two variables. (MP)

SCHOOL
MATHEMATICS
STUDY GROUP

SP-13

SUPPLEMENTARY and ENRICHMENT SERIES

INEQUALITIES

Teachers' Commentary

Edited by Philbrick Bridgess

SM56

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PREFACE

Mathematics is such a vast and rapidly expanding field of study that there are inevitably many important and fascinating aspects of the subject which, though within the grasp of secondary school students, do not find a place in the curriculum simply because of a lack of time.

Many classes and individual students, however, may find time to pursue mathematical topics of special interest to them. This series of pamphlets, whose production is sponsored by the School Mathematics Study Group, is designed to make material for such study readily accessible in classroom quantity.

Some of the pamphlets deal with material found in the regular curriculum but in a more extensive or intensive manner or from a novel point of view. Others deal with topics not usually found at all in the standard curriculum. It is hoped that these pamphlets will find use in classrooms in at least two ways. Some of the pamphlets produced could be used to extend the work done by a class with a regular textbook but others could be used profitably when teachers want to experiment with a treatment of a topic different from the treatment in the regular text of the class. In all cases, the pamphlets are designed to promote the enjoyment of studying mathematics.

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TEACHERS COMMENTARY

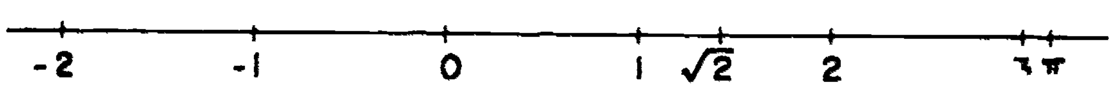
INEQUALITIES

Foreword

The principal objective of this pamphlet on inequalities is to help the pupil develop an understanding and appreciation of some of the algebraic structure exhibited by the real number system, and the use of this structure as a basis for the techniques of algebra. More specifically, we are interested in an exploration of the order properties of addition and multiplication of real numbers.

The principal objective of this COMMENTARY FOR TEACHERS is to give all possible aid to the teacher as he leads the pupil toward the above objective. Just as we urge the pupil to read his text carefully, we also urge the teacher to make full use of this commentary.

The pupil is more or less familiar with the <u>class</u> of objects which we call the real numbers. This includes the negative rationals (via experiences such as those involving gains and losses, degrees below zero on a thermometer, etc.) as well as certain irrationals such as $\sqrt{2}$ or π. The "picture" which goes with the class of real numbers is the full number line,



in which it is taken for granted that there is a number for each point and conversely. Notice that the order relation is implicit in this picture, although the operations of addition and multiplication are virtually absent.

This Commentary for Teachers is intended to be more than an answer book. You will find in it discussions of what we are doing, why we choose to present a particular topic in a certain manner, and what comes later to which this topic is leading.

CONTENTS

Chapter 1

ORDER ON THE NUMBER LINE

One of the great unifying and simplifying concepts of all mathematics, the idea of set, is of importance throughout mathematics in many ways--in classifying the numbers with which we work, in examining the properties of the operations upon these numbers, in solving equations and inequalities, in factoring polynomials, in the study of functions, etc. Formal work in the use of the symbolsim of sets is not attempted. The language of sets is introduced as needed.

We do not introduce much of the standard set notation, such as set building notation, Σ, $\supset$, $\subset$, $\cup$, $\cap$, because we found that there was not enough use for them to make it worthwhile. There is, however, no objection to the teacher using any of these if he so desires.

The teacher should be aware of three common errors made by pupils in working with the empty set. The most important error is the confusion of $\{0\}$ and $\emptyset$. $\{0\}$ is a set containing 0 as an element. This point, however, may need further emphasis by the teacher. A less significant mistake is to use the words "<u>an</u> empty set" or "<u>a</u> null set" instead of "<u>the</u> empty set". There is but one empty set, though it has many descriptions. A third error is the use of the symbol, $\{\emptyset\}$, instead of just $\emptyset$.

Incidentally, the symbol used for the empty set is the same as that used as a vowel in the Danish alphabet.

In case the teacher has had no experience in working with sets, a few of the basic definitions are given below:

1. A <u>set</u> is a collection of elements or members with some common characteristic. The common characteristic may simply be membership in the set.

2. The set which contains no elements is called the <u>empty set</u>, or <u>null set</u>, and is denoted by the symbol $\emptyset$.

3. If every element of set A belongs to set B, then A is a <u>subset</u> of B.

4. A set can be indicated by the use of "curly brackets" and a listing of the elements as: $\{2,3,4\}$ (Note: an element is never repeated), or by a description as: the set of all real numbers greater than 5.

5. If the elements of a set can be counted, with the counting coming to an end, the set is a <u>finite</u> set. The only exception to this rule is the set $\emptyset$, which cannot be counted, but it is nevertheless finite.

6. The graph of a set is the set of points whose coordinates are elements of the set.

Some of the characteristics of the number line are:

1. The coordinate of a point is the number associated with the point on the number line.

2. Every number may be associated with a point on the number line, and every point on the number line corresponds to a number--though not necessarily a rational number.

3. There are infinitely many points on the number line; there are infinitely many points between each pair of points on the number line.

The emphasis here is on the fact that <u>coordinate</u> is the <u>number</u> which is <u>associated</u> with a point on the line, "coordinate" and "associated" and "corresponding to" must become part of the pupil's vocabulary. He must not confuse <u>coordinate</u> with <u>point</u>, nor <u>coordinate</u> with the name of the number.

In graphing real numbers, the pupil should be aware that the number line pictured which he draws is only an <u>approximation</u> to the true number line situation. Consequently, any information which he deduced from his number line picture is only as accurate as his drawing.

A common misunderstanding is that some numbers on the line are real and others are irrational. The pupil should be encouraged to say, at least for the time being, that "-2 is a real number which is a rational number and a negative integer; $\frac{3}{2}$ is a real number which is a rational number; $-\sqrt{2}$ is a real number which is a negative irrational number". The point is that every integer is also a rational number and a real number; every rational number is a real number; and every irrational number is a real number.

Answers to Problem Set 1-1

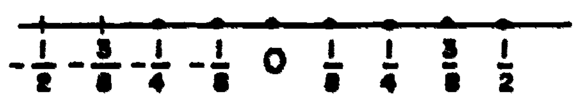
1. (a) right (d) left
   (b) left (e) right
   (c) left (f) right

2. (a)

(Note: the use of arrows on the basic number line is a matter of personal preference.)

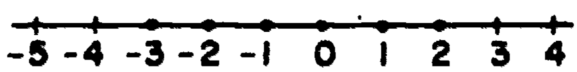
(b)

(c)



(d)

(Note: Points for -4 and 3 are not included.)



1-2. Sentences Involving Inequalities.

The words "true" and "false" for sentences seem preferable to "right" and "wrong" or "correct" and "incorrect" because the latter all imply moral judgments to many people. There is nothing illegal, immoral, or "wrong" in the usual sense of the word about a false sentence. The pupil should be encouraged to use only "true" and "false" in this context.

We have been doing two kinds of things with our sentences: We talk about sentences, and we use sentences. When we write

"$3 + 5 = 8$" is a true sentence,

we are talking about our language; when, in the course of a series of steps, we write

$$3 + 5 = 8,$$

we are using the language. When we talk about the language, we can perfectly well talk about a false sentence, if we find this useful. Thus, it is quite all right to say

"3 + 5 = 10" is a false sentence;

but it is far from all right to use the sentence

$$3 + 5 = 10$$

in the course of a proof. When we are actually _using_ the language, false sentences have no place; when we are talking about our language, they are often very useful.

The concept of a sentence is to be enlarged in three ways: (1) We increase the variety of relations which our sentences can express, so that inequalities are included along with equations, (2) We write _open_ sentences which involve variables, and for which the notion of a _truth set_ becomes important. It is essential that the pupil consider both equations and inequalities as sentences, as objects of algebra with equal right to our attention, and as equally interesting and useful types of sentences, (3) We consider compound sentences as well as simple sentences.

We believe that the pupil will expect the relation "is less than" for the real numbers to have the same meaning as it did when he was doing arithmetic.

It is possible that some pupils may wonder if there is some other definition of "is less than" which might have been chosen. If they raise the question, the following discussion of an interpretation which at first glance seems plausible may be helpful. Although "is less than" was defined to mean "is to the left of" on the number line for the numbers of arithmetic, it could clearly be there interpreted as "is closer to 0 than". It is then possible that "<" for the real numbers might well have this latter meaning! On the other hand, the example of the thermometer does not agree with this interpretation, nor would such familiar things as the variation in the height of tides or elevations above and below sea level.

There is also a good mathematical reason for rejecting this interpretation. The mathematician is never really interested in a relation as such, but rather in the properties it enjoys. _Whatever_ meaning is attached to "is less than" we want to be able to say, for example, that precisely one of the sentences "$3 < -3$" and "$-3 < 3$" is true. The plausible interpretation does not permit this comparison, since neither of the points -3 and 3 is closer to 0 than the other.

We expect the pupil to say that

$<$ means "is to the left of",

$\leq$ means "is to the left of or is the same number as",

$\geq$ means "is to the right of or is the same number as",

$\nless$ means "is not to the left of",

$\ngtr$ means "is not to the right of",

on the real number line. He should definitely have a feeling for the meaning of "<" since much of the further discussion of order is framed for "is less than".

In attempting to compare negative rational numbers, the pupils should be aware that the multiplication property of 1 can be applied to convert these into rational numbers represented by fractions with the same denominator. Thus, to compare $-(\frac{7}{13})$ and $-(\frac{9}{17})$, we have

$$-(\frac{7}{13}) = -(\frac{7}{13} \times \frac{17}{17}) = -(\frac{119}{13 \times 17}),$$
$$-(\frac{9}{17}) = -(\frac{9}{17} \times \frac{13}{13}) = -(\frac{117}{13 \times 17}).$$

Since $13 \times 17 = 17 \times 13$ by the commutative property of multiplication, it is easy to compare the numb s

$$-(\frac{119}{13 \times 17}) \text{ and } -(\frac{117}{17 \times 13}).$$

Answers to Problem Set 1-2

1. (a) $3 < -1$ is false, since 3 is to the right of -1 on the number line and is, therefore, greater than -1. Note the easy comparison of a positive and a negative number.

   (b) $2 < (-\frac{7}{2})$ is false, since 2 is to the right of $(-\frac{7}{2})$ and is, therefore, greater than $(-\frac{7}{2})$.

   (c) $-6 < 0$ is true, since -6 is to the left of 0 and is, therefore, less than 0.

The truth status of (d) through (t) can be determined in the same manner as (a), (b) and (c) above; consequently, we simply list the results.

(d) $0 > -4$ is true

(e) $6 > -7$ is true

(f) $1 \not< 1$ is true

(g) $-1 \not> -2$ is false

(h) $-3 \not> 5$ is true

(i) $4 + 3 < 3 + 4$ is false

(j) $5(2 + 3) > 5(2) + 3$ is true

(k) $\frac{1}{2} + \frac{1}{3} \not< 1$ is false

(l) $5 + 0 \not< 5$ is true

(m) $2 > 2 \times 0$ is true

(n) $0.5 + 1.1 = 0.7 + 0.9$ is true

(o) $5.2 - 3.9 < 4.6$ is true

(p) $2 + 1.3 > 3.3$ is false

(q) $2 + 1.3 \not> 3.3$ is true

(r) $4 + (3 + 2) < (4 + 3) + 2$ is false

(s) $\frac{2}{3}(8 + 4) < \frac{2}{3} \times 8 + \frac{2}{3} \times 4$ is false

(t) $5 + (\frac{2}{5} + \frac{3}{5}) \neq (4 - 1)2$ is false

2. The purpose of this exercise is to lead the pupil to compare two numbers. By the time the pupil completes the exercise, he should see that there is only one true sentence in each group of three sentences below.

(a) $-3.14 < -3$ is true.
$-3.14 = -3$ is false.
$-3.14 > -3$ is false.
-3.14 is to the left of -3 on the number line and is therefore less than -3.

(b) $2 < -2$ is false.
$2 = -2$ is false.
$2 > -2$ is true.

(c) Simplifying the names of the numbers $\frac{5 + 3}{2}$ and 2×2, we have the pair 4 and 4.
$4 < 4$ is false, or $\frac{5 + 3}{2} < 2 \times 2$ is false.
$4 = 4$ is true, or $\frac{5 + 3}{2} = 2 \times 2$ is true.
$4 > 4$ is false, or $\frac{5 + 3}{2} > 2 \times 2$ is false.

(d) If we write -0.001 and $-(\frac{1}{1000})$ as common fractions, we have the same number twice in the pair, instead of two distinct numbers. Thus,

$$-(\tfrac{1}{1000}) < -(\tfrac{1}{1000}) \text{ is false.}$$

$$-(\tfrac{1}{1000}) = -(\tfrac{1}{1000}) \text{ is true.}$$

$$-(\tfrac{1}{1000}) > -(\tfrac{1}{1000}) \text{ is false.}$$

(e) I discovered that there is only one true relationship if you compare two numbers on the basis of less than, equal to or greater than.

3. (a) $\frac{3}{5} > (-\frac{6}{10})$. Notice that in this and in most of the following exercises the multiplication property of 1 may be used to facilitate the comparison.

(b) $\frac{3}{5} > \frac{3}{6}$ (e) $-\frac{3}{5} < \frac{3}{6}$

(c) $\frac{9}{12} > \frac{8}{12}$ (f) $-\frac{3}{5} < (-\frac{3}{6})$

(d) $-\frac{173}{29} > (-\frac{183}{29})$

1-3. <u>Open Sentences</u>

An open sentence involving one variable has a "truth set" defined as the set of numbers for which it is true. We have no need at this time to introduce a name for the set which makes a sentence <u>false</u>. The word "solution set" is also used for "truth set", particularly for sentences which are in the form of equations.

<u>Answers to Problem Set 1-3a</u>

1.	$0 + 1 \neq 5$	True	
	$2 + 1 \neq 5$	True	
	$4 + 1 \neq 5$	False	
	$6 + 1 \neq 5$	True	Truth set is $\{0,2,6\}$
2.	$1 < 3$	True	
	$2 < 3$	True	
	$3 < 3$	False	
	$4 < 3$	False	Truth set is $\{1,2\}$

3. $1 \nless 3$ False
   $2 \nless 3$ False
   $3 \nless 3$ True
   $4 \nless 3$ True Truth set is $\{3,4\}$

4. $2(-2) + 3 = 1$ False
   $2(-1) + 3 = 1$ True
   $2(0) + 3 = 1$ False
   $2(1) + 3 = 1$ False Truth set is $\{-1\}$

5. In a similar manner the truth set is found to $\{-10, -5, 0\}$.

6. $\{-2,-1,0,1\}$

7. If the pupil has not met the symbol for the empty set before, he may say, "There is no member of the domain which is in the truth set", or similar expressions. In symbols we write $\emptyset$, or $\{\ \}$.

In the next set of exercises, formal methods of solving sentences are not to be encouraged. These methods will be introduced after the properties of order have been considered. For now, the pupil should try a few numbers to determine the truth set.

Answers to Problem Set 1-3b

1. The set of all real numbers greater than 4.
2. The set of all real numbers except 6.
3. The set of all real numbers greater than or equal to 100.
4. The set of all real numbers.
5. $\{-1\}$
6. $\emptyset$
7. The set of all real numbers less than $\frac{8}{9}$.
8. The set of all real numbers less than or equal to $\frac{21}{2}$. (SMSG prefers $\frac{21}{2}$ to $10\frac{1}{2}$ to lessen the subsequent confusion over the operations involved in $10\frac{1}{2}$ versus $5\frac{x}{y}$.)
9. The set of all real numbers.
10. The set of all real numbers except 3, -3 and 0. Here is an opportunity to remind the students that $\frac{3}{0}$ is not a number.
11. (5) and (6) are finite sets. The others are infinite sets.

1-4. Graphs of Open Sentences

We shall soon start saying "graph of the open sentence" instead of the more clumsy but more nearly precise "graph of the truth set of the open sentence".

In Example 3(e) do not fuss over the "plotting" of the empty set. Either no graph at all or a number line with no points marked is all right. It has been suggested that a $\emptyset$ be placed at the end of the unmarked number line.

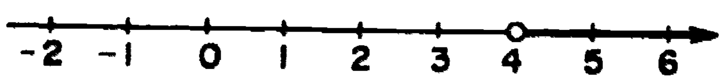
For convenience in doing problems involving the number line, you might find it helpful to duplicate sheets of number lines for the pupils' use.

In any of the Problem Sets in which the pupil is asked to find the truth set, see that he checks his results to be sure he has found all numbers which make the sentence true.

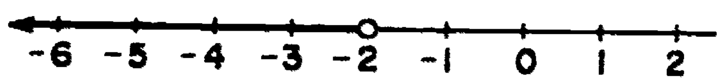
The pupil should realize the importance of the domain in finding truth sets and in graphing the truth sets.

Answers to Problem Set 1-4

1. (a)



(b)



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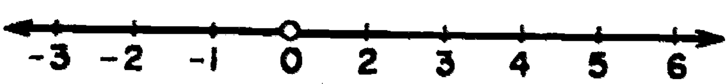

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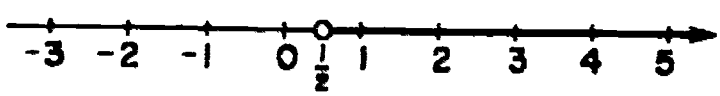


(e)

2. (a) $x = -1$ Note: The example asked for an open sentence, since there are more than one possibilities. 2(a) might be $2x = -2$, $x + 2 = 1$, $2x + 4 = 2$, etc.

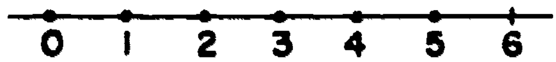
(b) $x > 2$ (c) $x > -3$ (d) $x \neq -2$

3. (a)

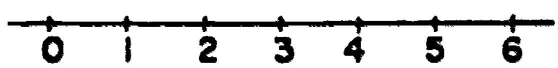
(b)

(c)

(d)



(e)



ϕ or null set

1-5. Compound Inequalities: Or

We use the word "clauses" to denote a sentence which is part of a compound sentence, just as in the corresponding situation in English. The word is convenient but not very important. Thus, $x \geq 2$ is a compound sentence with connective <u>or</u>: x equals 2 or x is greater than 2.

"x equals 2" is one clause;
"x is greater than 2" is the other.

There are two interpretations of the word <u>or</u>. We are using it in the <u>inclusive</u> sense. A pupil might say to another pupil, "I will get an A in mathematics or an A in French." This does not preclude the possibility that he would get an A in both subjects. Then there is the <u>exclusive</u> meaning which is implied when we say, "The Yankees or the Indians will win the pennant." We mean exactly one of the two will win.

Answers to Problem Set 1-5a

1. $3 \geq -5$; $3 > -5$. True. $3 = -5$. False. Since one clause is true, the compound sentence is true.

2.	True	8.	False	14.	False
3.	False	9.	True	15.	True
4.	True	10.	True	16.	True
5.	False	11.	True	17.	True
6.	True	12.	True	18.	False
7.	True	13.	True		

Answers to Problem Set 1-5b

1. (a) True for any element of set of integers 7 or larger.
   False for any element of set of integers 6 or smaller.

   (b) True for any integer not greater than 8.
   False for any integer greater than 8.

   (c) True for no integer. $\emptyset$
   False for all integers.

   (d) True for any integer less than 4.
   False for any integer equal to 4 or greater than 4.

   (e) True for no integers. $\emptyset$
   False for all integers.

   (f) True for all integers.
   False for no integer. $\emptyset$

   (g) True for all integers.
   False for no integers. $\emptyset$

   (h) True for all integers less than 2.
   False for all integers greater than or equal to 2.

   (i) $h + 2 < 9$ is true for all integers less than 7.
   $5h > 9$ is true for all integers greater than or equal to 2. The compound sentence is true if either of these conditions is met. Thus, the compound sentence is true for all integers. The compound sentence is false for no integer. $\emptyset$

   (j) $x > 0$ or $x < 0$ is true for all integers except 0.
   $x > 0$ or $x < 0$ is false for 0.

2\. (a) A sentence whose truth set is the set of all real numbers not equal to 3 is $y \neq 3$. Another is $y > 3$ or $y < 3$. Note that although the sentences above are mathematically equivalent, their English translations differ. and it is the former sentence which describes the required set directly.

(b) A sentence whose truth set is the set of all real numbers less than or equal to -2 is $v \leq -2$. Another is $v \ngtr -2$.

(c) A sentence whose truth set is the set of all real numbers not less than $-(\frac{5}{2})$ is $x \nless -(\frac{5}{2})$. Another is $x \geq -(\frac{5}{2})$.

Notice here that the alternate form is easier to comprehend. This may suggest to the pupil a clearer description in English for the required set.

3\. If p is any positive real number, and n is any negative real number, then n is to the left of zero, and p is to the right of zero; thus, n is to the left of p. (Recall that this principle was used to speed the comparison of numbers in many of the preceding exercises.) It follows that "$n < p$" and "$n \neq p$" are true statements and "$p < n$" is false. The statement "$n \leq p$" is true since the statement means $n < p$ or $n = p$, and, though the second statement is false, the first is true.

1-6. Compound Inequalities: And

If we were making more use of set language we would say that the truth set of a compound open set with connective and is the intersection of the truth sets of each of the two clauses. The truth set of the compound open sentence with connective and consists of the elements which belong to both truth sets of the two clauses. On the other hand, the truth set of a compound open sentence with connective or is the union of the truth sets of each of the two clauses. It is a set consisting of the elements which belong to either one or the other or both of the truth sets of the clauses.

Although we do write "$1 < x < 3$" as a shorthand for "$1 < x$" and "$x < 3$", we **NEVER** write, for example, "$-2 < x \geq 1$" as a shorthand for "$-2 < x$ or $x \geq 1$". The pupil cannot help but read "$-2 < x \geq 1$" as "x is greater than -2 and greater than or equal to 1"; in other words, he would read "$-2 < x \geq 1$" as a conjunction, when what is wanted is a disjunction. The same is true of "$2 < x < -1$". It would appear that x is between 2 and -1. What is intended is that "$x < -1$ or $x > 2$" and it must be so written.

Answers to Problem Set 1-6a

1. (a) False
   (b) True
   (c) True
   (d) False
   (e) True
   (f) True

2. (a) $\{0,1\}$
   (b) $\{-4,-3,-2,-1,0,1,2\}$
   (c) $\{-7\}$
   (d) $\emptyset$
   (e) $\{-4,-3,-2,-1,0,1,2,3\}$
   (f) $\emptyset$

Answers to Problem Set 1-6b

1. (a) True
   (b) False
   (c) False
   (d) True
   (e) False
   (f) True

2. (a) False
   (b) True
   (c) True
   (d) True
   (e) False
   (f) False
   (g) False
   (h) True
   (i) False
   (j) True

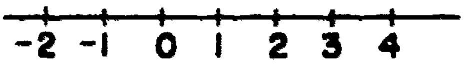
3. (a) True
   (b) True
   (c) False
   (d) False
   (e) True
   (f) False

4. (a) $\{1,2,3\}$ (0 is not in domain)
   (b) $\{0,2,4\}$
   (c) $\emptyset$
   (d) $\{0,-1,-2,-3,-4\}$
   (e) $\emptyset$
   (f) $\{-5,-1,1\}$

5. (c)

Answers to Problem Set 1-7

1. 

2.  ϕ

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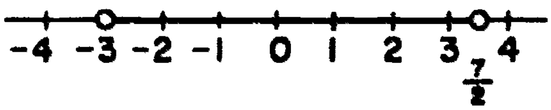
4. 

5.  ϕ

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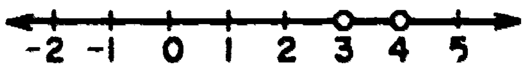
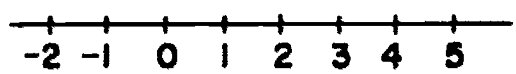
10. 

11. 

12.  ϕ

Chapter 1

Suggested Test Items

1. Which of the following sentences are true and which are false?
   (a) $5 \leq 6 + 1$
   (b) $-9 + 11 \ngtr -5$
   (c) $-8 - 2 > -3$
   (d) $5 + 3 \neq 7$

2. Which of the following sentences are true and which are false?
   (a) $\frac{6 - 3}{3} > \frac{3}{2}$ and $6 \ngtr 2 - 1$
   (b) $6 \neq 5 + 1$ or $4 \leq 3$
   (c) $6 + 1 = 4 + 3$ or $-6 \nless -7$
   (d) $\frac{4}{7} + \frac{3}{4} > \frac{3}{5} + \frac{5}{7}$ and $4 \nless 5.3$

3. Which of the following sentences are true and which are false?
   (a) $(18 - 10) - 4 = 18 - (10 - 4)$
   (b) $(18 - 10) - 4 \neq 18 - (10 + 4)$
   (c) $-3 - 4 > -8$ or $6 + 5 > 5 + 6$
   (d) $7 + 0 = 7$ and $-7(0) = -7$
   (e) $4 > 6$ or $5 + 2 = 10$
   (f) $7 \nless 3$ or $17.813 + .529 = 8.777 + 18.442$

4. Determine whether each sentence is true for the given value of the variable.

(a) $3t + 4 = 15$; 2 (c) $20 + 2x \nless 10$; -5

(b) $4x + 33 < 7$; -7 (d) $\frac{x}{2} + \frac{1}{3} \neq \frac{6x}{12} + \frac{3}{9}$; 1

5. If the variables have the values assigned below, determine whether the sentence is true.

(a) $3x = 4 + y$, x is 2 and y is 2.

(b) $5x < 2 + y$, x is -3 and y is -18.

6. Determine the truth sets of the following open sentences:

(a) $2x + 1 < -5$ (b) $4 + x \ngtr 2x + 1$

7. Draw the graphs of the truth sets of the open sentences:

(a) $x + 5 = -6$ (c) $2x \leq -7$

(b) $x + 1 > 3$ (d) $x \ngtr -2$

8. Draw the graphs of the truth sets of the compound open sentences:

(a) $x > 3$ and $x < 4$ (c) $x = 3$ or $x < -2$

(b) $x \leq 5$ and $x > -4$ (d) $x < -3$ and $x > 4$

9. Which of the open sentences A, B, C, D, and E below has the same truth set as the open sentence "$p \geq 7$"?

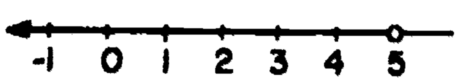
A. $p > 7$ or $p = 7$ D. $p \nless 7$

B. $p > 7$ and $p = 7$ E. $p \neq 7$

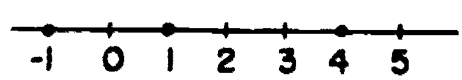
C. $p \ngtr 7$

10. Write open sentences whose truth sets are the sets graphed below:

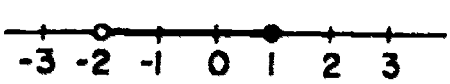
(a)



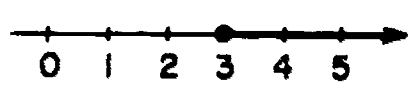
(c)



(b)



(d)

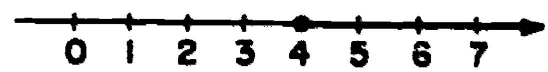


11. For each of the sentences in column I, select the appropriate graph of its truth set in column II.

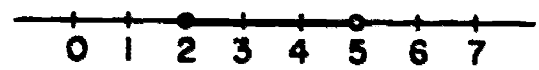
I

II

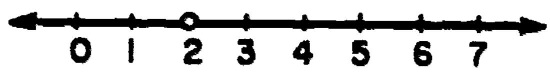
(a) $6x > 18$ and $2x < -5$



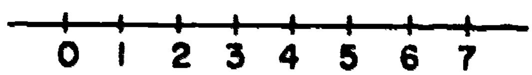
(b) $y < -3$



(c) $b \neq 2$



(d) $t \ngtr 4$



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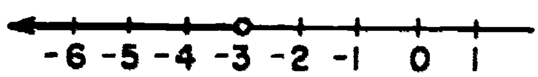
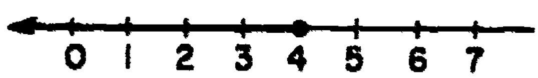
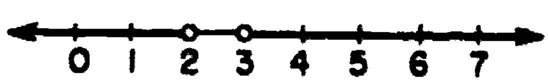
(e) $d \geq 2$ and $d < 5$



(f) $x \geq 4$



(g) $w < 2$ and $w > 3$



Chapter 2

PROPERTIES OF ORDER

2-1. Comparison Property

In stating the comparison property and, later on, the transitive property, we are following the convention established in Chapter 1 to the effect that whenever we write a sentence about numbers, we are tacitly asserting the truth of that sentence.

The comparison property here given is also called the trichotomy property of order. Notice that it is a property of $<$; that is, given any two different numbers, they can be ordered so that one is less than the other. When the property is stated, we must include the third possibility that the numerals name the same number. Hence, the name "trichotomy".

Although "$a < b$" and "$b > a$" involve different orders, these sentences are exactly the same thing about the numbers a and b. Thus, we can state a trichotomy property of order involving "$>$" as:

> For any number a and any number b,
> exactly one of these is true:
>
> $$a > b, \quad a = b, \quad b > a.$$

If, instead of concentrating attention on the order relation, we concentrate on the two numbers, then if $a \neq b$, either "$a < b$" or "$a > b$" is true, but not both. Here we fix the numbers a and b and then make a decision as to which order relation applies. It is purely a matter of which we are interested in: the numbers or the order. The comparison property is concerned with an order.

Answers to Problem Set 2-1

1. (a) $-2 < -1.6$

   (b) $-2 < 0$

   At this point pupils should recall that if a number is negative, it will be located to the left of 0 on the real number line; if positive, the right of 0. Consequently, a negative number is always less than a positive number or zero. Using this fact makes easier Problem 1(c) and others in this problem set.

(c) $-(\frac{2 \times 3 \times 4}{5}) < (\frac{2 \times 3 \times 4}{5})$

(d) $-16 = -(\frac{32}{2})$

(e) $12 = (5 + 2)(\frac{1}{7} \times \frac{36}{3})$

Note the use of the associative property in simplifying the second number:

$(5 + 2)(\frac{1}{7} \times \frac{36}{3}) =$

$7(\frac{1}{7} \times 12) =$

$7(\frac{1}{7})(12) =$

12

(f) $-2 < 2.$

2. Use only "<" to make true sentences.

(a) $-3 < 2$

(b) $-(\frac{5}{2}) < \frac{4}{2}$

(c) $-(\frac{6}{5}) < -(\frac{4}{5})$

(d) $\frac{4}{5} < \frac{11}{10}$

(e) $-(\frac{4}{50}) < \frac{11}{100}$

(f) $\frac{205}{26} < \frac{103}{13}$. Note that $\frac{103}{13} = \frac{206}{26}$.

(g) $\frac{2}{3} < \frac{13}{15}$. $\frac{2}{3} = \frac{10}{15}$ and $\frac{10}{15} < \frac{13}{15}$.

(h) $-(\frac{25}{238}) < \frac{12}{119}$

(i) $-1.5 < -\sqrt{2}$. Recall that $1.41 < \sqrt{2} < 1.42$.

(j) $1.5 + 3 < \sqrt{2} + \pi$.
Here the pupil may write $1.5 + 3 = 1.4 + 3.1$ and use the inequalities $1.4 < \sqrt{2}$ and $3.1 < \pi$ to show that $1.4 + 3.1 < \sqrt{2} + \pi$. Finally, by the transitive property $1.5 + 3 < \sqrt{2} + \pi$. Here he has anticipated the property "If a, b, c and d are real numbers for which $a < b$ and $c < d$, then $a + c < b + d$", from Section 2-5.

2.

3. If a is a real number and b is a real number, then exactly one of the following is true:

$$a = b, \quad a > b, \quad b > a.$$

Restating Exercise 1 using pairs of numbers given:

(a) In 1(a) either $-2 > -1.6$ or $-1.6 > -2$. Since -1.6 is to the right of -2, $-1.6 > -2$ is the true sentence.

(b) $0 > -2$.

(c) $\left(\frac{2 \times 3 \times 5}{5}\right) > -\left(\frac{2 \times 3 \times 4}{5}\right)$.

(d) $-16 = -\left(\frac{32}{2}\right)$

(e) $12 = (5 + 2)\left(\frac{1}{7} \times \frac{36}{3}\right)$

(f) $2 > -2$

4. The best statement would be: "If a is a real number and b is a real number, then exactly one of the following is true: $a \geq b$ or $a < b$.

Some may say "For any real numbers a and b, $a \geq b$ or $a \leq b$. If $a \leq b$ <u>and</u> $b \leq a$, then $a = b$". The last sentence of this particular statement is reasonable and innocent in appearance. Surprisingly enough, it turns out to be one of the most useful criteria for determining that two variables have the same value! In many instances in the calculus, for example, one is able to show by one argument that $a \leq b$ and by another that $b \leq a$. He is then able to conclude that $a = b$. Given two <u>numerals</u>, it is usually trivial to check whether or not they name the same number. In the case of two numbers, of course, we have <u>complete</u> information. It is only when our information about two "numerals" is <u>incomplete</u> that a statement like, "If $a \leq b$ and $b \leq a$, then $a = b$", can possibly be useful as a tool.

2-2. Transitive Property

Any attempt to illustrate this transitive property of $<$ with triples of <u>integers</u> is likely to be met with a vociferous "So what!" by your pupils. On the other hand, not only can this property be illustrated with fractions as in the text, but the pupil can also

begin to appreciate its usefulness and perhaps be inclined to say his "So what!" in a quieter voice.

The pupil will, in the course of his mathematical training, see many other relations which have a transitive property: "is equal to" for numbers, "is a factor of" for positive integers, "is congruent to" for various geometric figures, etc.

What is an easy way to tell that $\frac{337}{113} < 3$? By the multiplication property of 1, $3 = 3 \times \frac{113}{113} = \frac{339}{113}$, so that $\frac{337}{113} < 3$. Similarly, $3 = 3 \times \frac{55}{55} = \frac{165}{55}$, so that $3 < \frac{167}{55}$.

Answers to Problem Set 2-2

1. (a) $-(\frac{1}{5}) < \frac{3}{2}$, $\frac{3}{2} < 12$, $-(\frac{1}{5}) < 12$.

(b) $-\pi < \sqrt{2}$, $\sqrt{2} < \pi$, $-\pi < \pi$.

(c) $-1.7 < 0$, $0 < 1.7$, $-1.7 < 1.7$

(d) $-(\frac{27}{15}) < -(\frac{3}{15})$, $-(\frac{3}{15}) < -(\frac{2}{15})$, $-(\frac{27}{15}) < -(\frac{2}{15})$

(e) $$\frac{12(\frac{1}{2} + \frac{1}{3})}{3} = \frac{12 \cdot \frac{1}{2} + 12 \cdot \frac{1}{3}}{3} = \frac{6 + 4}{3} = \frac{10}{3}$$

Then the order is

$$-(\frac{5}{3}) < \frac{6}{3}, \quad \frac{6}{3} < \frac{12(\frac{1}{2} + \frac{1}{3})}{3}, \quad -(\frac{5}{3}) < \frac{12(\frac{1}{2} + \frac{1}{3})}{3}$$

(f) $$\frac{3 \times (27 + 6)}{9} = \frac{3 \times 33}{9} = 11 \text{ or } \frac{22}{2}.$$

$$\frac{(2 \times 3) + (7 \times 9)}{6} = \frac{6 + 63}{6} = \frac{69}{6} = \frac{23}{2}.$$

$$\frac{(99 \times 3)\frac{1}{3}}{2} = \frac{99 \times (3 \times \frac{1}{3})}{2} = \frac{99}{2}.$$

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Since $\frac{22}{2} < \frac{23}{2}$, $\frac{23}{2} < \frac{99}{2}$, and $\frac{22}{2} < \frac{99}{2}$, we find that

$$\frac{3 \times 27 + 6}{9} < \frac{(2 \times 3) + (7 \times 9)}{6}$$

$$\frac{(2 \times 3) + (7 \times 9)}{6} < \frac{(99 \times 3)\frac{1}{3}}{2}$$

$$\frac{3 \times (27 + 6)}{9} < \frac{(99 \times 3)\frac{1}{3}}{2}$$

(g) $3^2 = 9$, $4^2 = 16$, and $(3 + 4)^2 = 49$. Therefore, $3^2 < 4^2$, $4^2 < (3 + 4)^2$, $3^2 < (3 + 4)^2$.

(h) $-(\frac{1}{2}) = -(\frac{6}{12})$, $-(\frac{1}{3}) = -(\frac{4}{12})$, $-(\frac{1}{4}) = -(\frac{3}{12})$. Thus,

$-(\frac{6}{12}) < -(\frac{4}{12})$, $-(\frac{4}{12}) < -(\frac{3}{12})$, $-(\frac{6}{12}) < -(\frac{3}{12})$, and

$-(\frac{1}{2}) < -(\frac{1}{3})$, $-(\frac{1}{3}) < -(\frac{1}{4})$, $-(\frac{1}{2}) < -(\frac{1}{4})$.

(i) $1 + \frac{1}{2} = \frac{3}{2}$ or $\frac{6}{4}$.

$1 + (\frac{1}{2})^2 = 1 + \frac{1}{4} = \frac{5}{4}$.

$(1 + \frac{1}{2})^2 = (\frac{3}{2})^2 = \frac{9}{4}$, or

$\frac{5}{4} < \frac{6}{4}$, $\frac{6}{4} < \frac{9}{4}$, $\frac{5}{4} < \frac{9}{4}$, or

$1 + (\frac{1}{2})^2 < 1 + \frac{1}{2}$, $1 + \frac{1}{2} < (1 + \frac{1}{2})^2$, $1 + (\frac{1}{2})^2 < (1 + \frac{1}{2})^2$.

2. Of three real numbers a, b and c, if $a > b$ and $b > c$ then $a > c$.
Illustrations from Problem 1:
1(a) $-(\frac{1}{5})$, $\frac{3}{2}$, 12: $12 > \frac{3}{2}$, $\frac{3}{2} > -(\frac{1}{5})$, $12 > -(\frac{1}{5})$.
1(g) 3^2, 4^2, $(3 + 4)^2$: $(3 + 4)^2 > 4^2$, $4^2 > 3^2$, $(3 + 4)^2 > 3^2$.

3. Art is heavier than Bob.
Bob is heavier than Cal.
Conclusion: Art is heavier than Cal. Let Art's, Bob's and Cal's weights be represented respectively by the numbers a, b and c.
From the first sentence: $a > b$. From the second sentence: $b > c$. From the transitive property as stated in Problem 2, $a > c$; that is, Art is heavier than Cal.

4. The transitive property for "=" is: For all real numbers a, b, and c, if $a = b$ and $b = c$, then $a = c$. If Art weighs the same as Bob and Bob and Cal are equal in weight, we know Art and Cal must weigh the same. If $3 + 4 = 7$ and $7 = 5 + 2$, then $3 + 4 = 5 + 2$.

5. The transitive property for $\geq$ would be: For all real numbers a, b and c, if $a \geq b$ and $b \geq c$, then $a \geq c$. If $\pi \geq 3.14$ and $3.14 \geq 2$, then $\pi \geq 2$.

6. (a) The non-positive real numbers are the set of numbers less than or equal to 0; in other words, the set comprised of 0 and all negative numbers.

   (b) The non-negative real numbers are the set of numbers greater than or equal to 0; in other words, the set consisting of zero and all the positive numbers.

7. (a) $-(\frac{15}{8})$ and $-(\frac{25}{12})$. Some students may observe that $-(\frac{15}{8}) = -(1\frac{7}{8})$ and $-(\frac{25}{12}) = -(2\frac{1}{12})$. Referring to the number line, they will see that $-(\frac{25}{12}) < -(\frac{15}{8})$.
   Some may reason as follows:

   $-(\frac{15}{8}) > -(\frac{16}{8})$ or -2.

   -2 or $-(\frac{24}{12}) > -(\frac{25}{12})$.

   If $-(\frac{15}{8}) > -2$ and $-2 > -(\frac{25}{12})$, then $-(\frac{15}{8}) > -(\frac{25}{12})$.

   (b) $-(\frac{17}{35})$ and $-(\frac{7}{13})$.

   Note that $-(\frac{17}{35}) > -(\frac{1}{2})$ or $-(\frac{17}{34})$, and $-(\frac{1}{2})$ or $-(\frac{7}{14}) > -(\frac{7}{13})$.

   If $-(\frac{17}{35}) > -(\frac{1}{2})$ and $-(\frac{1}{2}) > -(\frac{7}{13})$, then $-(\frac{17}{35}) > -(\frac{7}{13})$.

   (c) $-(\frac{145}{28})$ and $-(\frac{104}{21})$.

   Using mixed numbers, $-(\frac{145}{28}) = -(5\frac{5}{28})$ and $-(\frac{104}{21}) = -(4\frac{20}{21})$.

   If $-(5\frac{5}{28}) < -5$ and $-5 < -(4\frac{20}{21})$, then $-(5\frac{5}{28}) < -(4\frac{20}{21})$ and $-(\frac{145}{28}) < -(\frac{104}{21})$.

7. (c) continued

Alternatively,

If $-(\frac{145}{28}) < -(\frac{140}{28})$ or -5, and $-(\frac{105}{21})$ or $-5 < -(\frac{104}{21})$,

then $-(\frac{145}{28}) < -(\frac{104}{21})$.

(d) $-(\frac{192}{46})$ and $-(\frac{173}{44})$.

If $-(\frac{192}{46}) < -(\frac{184}{46})$ or -4, and $-(\frac{176}{44})$ or $-4 < -(\frac{173}{44})$,

then $-(\frac{192}{46}) < -(\frac{173}{44})$.

2-3. Order of Opposites

The pupil must learn to designate the opposite of a given number by means of the definition. Do not say and do not let the pupil say, "To find the opposite of a number, change its sign." This is very imprecise (in fact, SMSG consistently avoids attaching a "sign" to the positive numbers) and will lead to a purely manipulative algebra which we want to avoid at all costs.

The opposite of the opposite of the opposite of a number is the opposite of that number. What is the opposite of the opposite of a negative number? The number, of course!

The pupil is well aware that the lower dash "-" is read "minus" in the case of subtraction. We prefer to retain the word "minus" for the operation of subtraction and not use it as an alternative word for "opposite of". Thus, the dash attached to a variable, such as "$-x$" will be read "opposite of".

If x is a positive number, then $-x$ is a negative number. The opposite of any negative number x is the positive number $-x$, and $-0 = 0$. Thus, the pupil should not jump to the conclusion that when n is a real number, then $-n$ is a negative number; this is true only when n is a positive number.

We do not like to read "$-x$" as "negative x". A negative number is the opposite of a positive number only. To read "$-x$" as "negative x" implies that x is positive, but we want pupils to think of x as any real number. Some teachers read "$-x$" as the negative of x. In this usage the "negative of x" is synonymous with "the opposite of x". We prefer the latter.

Answers to Problem Set 2-3a

1. (a) If x is positive, then the opposite of x is negative.
   (b) If x is negative, then the opposite of x is positive.
   (c) If x is zero, then the opposite of x is zero.

2. (a) If the opposite of x is a positive number, then the number x itself must be negative.

   (b) If the opposite of x is a negative number, then the number itself must be positive.

   (c) If the opposite of x is 0, then the number x itself must be 0, for 0 is its own opposite.

3. (a) Since every real number has an opposite, it follows that every real number is the opposite of some real number.

   (b) Yes. See 3(a).

   (c) Every negative number is the opposite of some (positive) real number; hence, the set of negative numbers is a subset of the set of all opposites.

   (d) Some opposites, namely opposites of negative numbers, are not negative numbers. Hence, the set of opposites is not a subset of the set of negative numbers.

   (e) No. See 3(d).

In order to motivate the "property for opposites", here it would be well to consider several other pairs of numbers; for example, a pair of distinct positive numbers, a pair of distinct negative numbers, 0 and a positive number, and 0 and a negative number.

Answers to Problem Set 2-3b

1. (a) $-\frac{1}{6} < \frac{2}{7}$ and $-\frac{2}{7} < \frac{1}{6}$.

   (b) $-\pi < \sqrt{2}$ and $-\sqrt{2} < \pi$.

   (c) $\pi < \frac{22}{7}$ and $-\frac{22}{7} < -\pi$.

   The pupils may need to be told that $\pi = 3.1416$ to 4 decimal places and they may determine $\frac{22}{7} = 3.1428$ to four decimal places.

(d) $3(\frac{4}{3} + 2) = 3 \times \frac{4}{3} + 3 \times 2$, by the distributive property
$= 4 + 6$
$= 10$

$\frac{5}{4}\,(20+8) = (\frac{5}{4})(20)+(\frac{5}{4})(8)$, by the distributive property
$= 25 + 10$
$= 35.$

Since $10 < 35$ and $-35 < -10$, we have
$3(\frac{4}{3} + 2) < \frac{5}{4}(20 + 8)$ and $-\frac{5}{4}(20 + 8) < -3(\frac{3}{4} + 2)$.

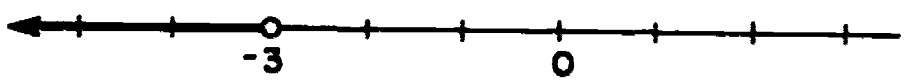
(e) $-(\frac{8 + 6}{7}) = -\frac{14}{7} = -2.$

Then $-(\frac{8 + 6}{7})$ and -2 are names for the same number.

2. (a) $x > 3$.

(b) $x > -3$.

(c) $-x > 3$.



The pupil may use a simple trial-and-error process in the exercise above, or he may reason along some line such as this: The sentence says "opposite of x is greater than 3". Hence, "the opposite of x" would describe numbers such as $\frac{13}{4}$, 4, 6.7, 10, 1000, etc. If such numbers are opposites of members of the set we are seeking, the set itself includes $-\frac{13}{4}$, -4, -6.7, -10, -1000, etc.

It would be gratifying if the pupils observe that the open sentences "$-x > 3$" and "$x < -3$" have the same truth set before they graph the sentences.

(d) $-x > -3$.

3. (a) The sentence states "the opposite of x is not equal to 3". There are many numbers whose opposites are not equal to 3; in fact, there is only one number whose opposite does equal 3, and this is, of course, -3. Hence, the required set is the set of all real numbers except -3.

(b) $-x \neq -3$.
By the reasoning of part (a) the required set here is found to be the set of all real numbers except 3.

(c) $x < 0$.
The truth set for this sentence is the set of all real numbers less than 0; that is, the set of negative numbers.

(d) $-x < 0$.
Here the set required is the set of all real numbers whose opposites are less than zero. Now if the opposites of all members of this set are less than zero, the members of the set must be greater than zero; in other words, "$-x < 0$" and "$x > 0$" have the same truth set. Thus, the truth set is the set of all positive real numbers.

(e) $-x \geq 0$.
The sentence states that the opposite of x is greater than or equal to zero; that is, that the opposite of x must be either zero or a positive number. Hence, x itself must be either zero or a negative number, and the truth set is comprised of such members. Recall that a brief description of this set is the set of <u>non-positive numbers</u>.

(f) $-x \leq 0$.
Here the reasoning would parallel (e) above: Each member of the truth set is either zero or a positive number. This set is described briefly as the <u>non-negative numbers</u>.

4. A variety of answers is possible here.
   (a) A is the set of all non-negative real numbers $x \geq 0$, $-x \leq 0$, $x \nless 0$, $-x \ngtr 0$.

   (b) B is the set of all real numbers not equal to -2.
   $x \neq -2$, $-x \neq 2$.
   Also, $x > -2$ or $x < -2$.

   (c) C is the set of all real numbers not greater than -3.
   $x \ngtr -3$, $-x \nless 3$.
   Also, $x \leq -3$, $-x \geq 3$.

(d) $\emptyset$.
$x \geq 0$ and $x < 0$.
Also, $-x \leq 0$, and $-x > 0$.

(e) E is the set of all real numbers.
$x \geq 0$ or $x < 0$.
Also, $-x \leq 0$ or $-x > 0$.

At this point the pupil may not have performed operations of addition and multiplication with the real numbers, so these sentences should be free of such operations; or at least, he should be warned against the danger of using operations with which he is not familiar.

5. (a) $x < 1$.

(b) $-2 < x$ and $x \leq 1$.
This open sentence can be written much more suggestively as: $-2 < x \leq 1$.
We would read this "x is greater than -2 <u>and</u> less than or equal to 1". This terminology emphasizes the number line picture and suggests strongly that x is 1 or is between -2 and 1. Remember, we <u>never</u> write "$-2 < x \leq 1$" as a shorthand for "$-2 < x$ or $x \leq 1$".

6. (a) 7.2 is the opposite of -7.2. The greater is 7.2.

(b) -3 is the opposite of 3. The greater is 3.

(c) -0 is the opposite of 0. They are the same number.

(d) $\sqrt{2}$ is the opposite of $-\sqrt{2}$. The greater is $\sqrt{2}$.

(e) -17 is the opposite of 17. The greater is 17.

(f) 0.01 is the opposite of -0.01. The greater is 0.01.

(g) -2 is the opposite of $-(-2)$. The greater is $-(-2)$.

(h) $-(1 - \frac{1}{\pi})^2$ is the opposite of $(1 - \frac{1}{\pi})^2$. The greater is $(1 - \frac{1}{\pi})^2$.

(i) $-\left(1 - (\frac{1}{\pi})^2\right)$ is the opposite of $\left(1 - (\frac{1}{\pi})^2\right)$. The greater is $\left(1 - (\frac{1}{\pi})^2\right)$.

(j) $(\frac{1}{2} - \frac{1}{3})$ is the opposite of $-(\frac{1}{2} - \frac{1}{3})$. The greater is $(\frac{1}{2} - \frac{1}{3})$.

The opposites of (a), (d), (f), (g) and (h) may be given in terms of the opposites of the opposites, e.g.,

(a) The opposite of -7.2 is $-(-7.2)$.

(d) The opposite of $\sqrt{2}$ is $-(\sqrt{2})$, etc.

7. The relation "$\succ$" does <u>not</u> have the comparison property. For example, 2 and -2 are different real numbers but neither is further from 0 than the other; in other words, none of the statements "$-2 = 2$", "$-2 \succ 2$" and "$2 \succ -2$" is true.

The transitive property for "$\succ$" would read: If a, b and c are real numbers and if $a \succ b$ and $b \succ c$, then $a \succ c$. This is certainly a true statement, as can be seen by substituting the phrase "is further from 0 than" for "$\succ$" wherever it occurs.

The relations "$\succ$" and "$>$" have the same meaning for the numbers of arithmetic: "is further from 0 than" and "is to the right of" mean the same thing on the arithmetic number line.

8. (a) $s > -100$; s is the number representing John's score.

(b) $n \leq 0$ and $n \geq -200$; n is the number representing my financial condition in dollars.

(c) $d - 10 > 25$; d is the number of dollars in Paul's original debt. (Some pupils may observe that the variables s, n and d would ordinarily be further restricted to be rational numbers represented by fractions whose denominators are 100.)

9. Following the hint:

$-\frac{13}{42}$ and $-\frac{15}{49}$ are to be compared.

$\frac{13}{42}(\frac{7}{7}) = \frac{91}{294}$ (Multiplication property of 1.)

$\frac{15}{49}(\frac{6}{6}) = \frac{90}{294}$

$\frac{91}{294} > \frac{90}{294}$ and $-\frac{91}{294} < -\frac{90}{294}$. Thus, $-\frac{13}{42} < -\frac{15}{49}$.

In order to compare two negative rational numbers, we use the multiplication property of 1 to compare their opposites and then use the property of opposites: For real numbers a and b, if $a < b$, then $-b < -a$.

We can describe this briefly as: In order to compare two negative (or two positive) rational numbers, represent them by fractions with the same denominator and compare the numbers represented by their numerators.

2-4. Order Relation for Real Numbers

One of our main objectives in this pamphlet is to study one phase of structure of the real number system. A system of numbers is a set of numbers and the operations on these numbers. Hence, we do not really have the real number system until we define the operations of addition and multiplication for negative numbers as they pertain to the order relation as well as to the relation of equality.

Order in the real numbers was introduced in this chapter. In this section we continue with order and obtain its properties with respect to addition and multiplication. There is an important shift in our point of view on order in this chapter. Previously we have tended to use order as a convenient way to discuss certain properties of real numbers. In this sense "<" or ">" were hardly more than fragments of our language. Now we treat "<" as an order relation. A similar shift in point of view has to be made earlier in the case of addition, for example. In arithmetic, the sign "+" in the expression "25 + 38" is nothing more than a reminder or command to carry out a previously learned process to obtain "63". The idea of "+" as an operation to be studied for its own sake is quite a different notion of addition from that in arithmetic. Thus, in this chapter, the order relation becomes a mathematical object in its own right.

In talking about a given pair of numbers we may, and frequently do, shift from "less than" to "greater than" and back again without trouble. However, this tends to obscure the idea of order relation and is not permissible when we are studying the order relation "<" itself. We are making a big issue of this matter because it is mathematically important for the student to begin thinking of order relation and not just order. On the other hand, do not belabor the point with the pupils. It is not essential that they be able to explain it, etc. If you think about the

order relation and are careful to discuss it correctly, then the pupil will automatically learn to think about an order relation as a mathematical object rather than as a convenient way of discussing a pair of real numbers.

Answers to Problem Set 2-4

1. (a) $-\frac{3}{2} < -\frac{4}{3}$

   (b) $-(7) < -(-7)$

   (c) Cannot tell; if m is a real number, then exactly one of the following is true: $m < 1$, $m = 1$, $1 < m$.

2. If $m > 4$ and $4 > 1$, then $m > 1$; transitive property.

3. (a) true
   (b) true
   (c) true
   (d) true for any real number a

4. (a) true
   (b) false
   (c) false
   (d) truth set consists of all positive numbers.

5. (a) $3 < 3 + x$

   $0 < x$

   Three possible descriptions of the truth set are:

   The set of numbers x such that $0 < x$.

   All numbers greater than 0.

   The set of all positive numbers.

   (b) $3 + x < 3$

   $x < 0$

   The set of negative numbers.

6. (a) the set of all numbers less than 3
   (b) the set of all numbers greater than -3
   (c) the set of all numbers greater than 3
   (d) the set of all numbers less than -3

2-5. Addition Property of Order

Addition is naturally tied in with order by the fact that $x + y$ is obtained on the number line by going to the right from x if y is positive and going to the left is y is negative. However, this is not the property which we want to emphasize. The property which is regarded as basic is the addition property of order. It is easy to prove that the first property mentioned above follows from the basic one. For example, if $a = 0$, then the condition $a < b$ says that b is positive. The addition property gives $c < b + c$, which says that if b is positive, then $b + c$ is to the right of c, etc.

$(-3) + (-3) < (-\frac{1}{2}) + (-3)$ is a true sentence. If c has successively the values $\frac{1}{2}$, 0, -7, the sentence is true in each case. A statement of the addition property of order in words is: "If one number is less than another, and to each of these is added the same number, the order remains unchanged". The corresponding property of equality is the addition property of equality.

Answers to Problem Set 2-5a

1. (a) true (c) true
   (b) true (d) true

2. a, b, c are real numbers and if $a \leq b$, then

$$a + c \leq b + c.$$

If a, b, c are real numbers and if $a > b$, then

$$a + c > b + c.$$

If a, b, c are real numbers and if $a \geq b$, then

$$a + c \geq b + c.$$

3. If $a < b$, then $a + c < b + c$. addition property of order
   If $c < d$, then $b + c < b + d$. addition property of order
   Hence, $a + c < b + d$. transitive property

4. In the following example, call the pupils' attention to the fact that we could not test the truth set by substituting numbers into the original sentence, but had, instead, to verify the truth set by reversing the steps which we took in obtaining it. We have learned to recognize that certain equations are equivalent because we know that the steps we

took to obtain one from the other __are__ reversible. Soon we shall apply to inequalities this same procedure of recognizing equivalent sentences by the fact that the intervening steps are reversible.

(a) If $3 + x < (-4)$ is true for some x,
then $x < (-4) + (-3)$ is true for the same x,
and $x < -7$ is true for the same x.
Thus, if x is a number which makes the original sentence true, then $x < -7$.

If $x < -7$ is true for some x,
then $3 + x < (-7) + 3$ is true for the same x,
and $3 + x < (-4)$ is true for the same x.
Hence, the truth set is the set of all real numbers less than -7.

The pupils should follow the above form in finding truth sets of this problem. The truth sets are as follows:

(b) The set of all numbers greater than -1.

(c) The set of all x such that $x < -5$. (Either form of the answer is correct)

(d) The set of all numbers greater than $\frac{4}{3}$. (Either form of the answer is correct)

(e) The set of all x such that $x \geq \frac{7}{3}$.

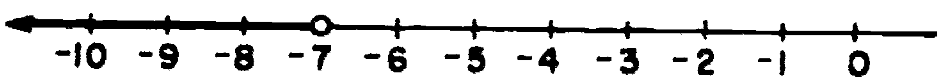
(f) The set of all numbers greater than 4.

(g) The set of all x such that $-7 < x$.

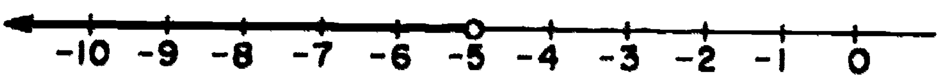
(h) The set of all numbers greater than -4.

(i) The set of all x such that $x \leq -1$.

5. (a)



(c)



(h)

6. Theorem: If $a < b$, then $-b < -a$.

Proof:

If $a < b$,	assumption
then $a + \big((-a)+(-b)\big) < b + \big((-a)+(-b)\big)$.	addition property of order
$\big(a + (-a)\big) + (-b) < \big(b + (-b)\big) + (-a)$	commutative and associative properties of addition
$0 + (-b) < 0 + (-a)$	addition property of opposites
$-b < -a$	addition property of zero

7. Theorem: If $0 < y$, then $x < x + y$.

Proof:

For every a, b, c, if $a < b$, then $a + c < b + c$.	addition property of order
Let $a = 0$, $b = y$, $c = x$:	
If $0 < y$, then $0 + x < y + x$.	
If $0 < y$, then $x < y + x$.	addition property of zero
If $0 < y$, then $x < x + y$.	commutative property of addition

---

Proof of Theorem 2-5a.

We may write the addition property of order thus:

"If $a < b$, then $c + a < c + b$"

because of the commutative property of addition.

The sentence $4 = (-2) + 6$ can be changed to this sentence involving order: $(-2) < 4$.

The number y such that $7 = 5 + y$ is 2, found by the addition property of equality. In the same manner, the truth set of $-3 = -6 + y$ is found to be 3, again a positive number.

The reasons for Theorem 2-5b are as follows:

If $y = z + (-x)$,	
then $x + y = x + \big(z + (-x)\big)$	addition property of equality
$= \big(x + (-x)\big) + z$	associative and commutative properties of addition

In the next paragraph, y is negative, y is 0, or y is positive. Exactly one of these is true, by the comparison property.

Answers to Problem Set 2-5b

1. (a) $-24 < -15$, $-24 + 9 = -15$, $b = 9$

   (b) $-\frac{5}{4} < \frac{63}{4}$, $-\frac{5}{4} + \frac{68}{4} = \frac{63}{4}$, $b = \frac{68}{4} = 17$

   (c) $\frac{7}{10} < \frac{6}{5}$, $\frac{7}{10} + \frac{5}{10} = \frac{6}{5}$, $b = \frac{5}{10} = \frac{1}{2}$

   (d) $-\frac{1}{2} < \frac{1}{3}$, $-\frac{3}{6} + \frac{5}{6} = \frac{2}{6}$, $b = \frac{5}{6}$

   (e) $-345 < -254$, $-345 + 91 = -254$, $b = 91$

   (f) $-\frac{33}{13} < -\frac{98}{39}$, $-\frac{99}{39} + \frac{1}{39} = -\frac{98}{39}$, $b = \frac{1}{39}$

   (g) $-0.21 < 1.47$, $-0.21 + 1.68 = 1.47$, $b = 1.68$

   (h) $(\frac{3}{2})(-\frac{5}{4}) < (-\frac{2}{3})(\frac{4}{5})$, $(-\frac{225}{120}) + (\frac{161}{120}) = (-\frac{64}{120})$, $b = \frac{161}{120}$

2. We must show there __is__ a number b such that $c = a + b$, and then show that b is negative.

   If b is $c + (-a)$,
   then $b = c + (-a)$.

   $a + b = a + (c + (-a))$ addition property of equality
   $= (a + (-a)) + c$ commutative and associative properties of addition
   $= 0 + c$ addition property of opposites
   $a + b = c$ addition property of zero

   Hence, there is a number b such that $c = a + b$.

   Exactly one of these is true:

   $0 < b$, $b = 0$, or $b < 0$.

   If $0 < b$,
   then $a < a + b$. addition property of order
   $a < c$, since $c = a + b$.
   But $a < c$ and $c < a$ is a contradiction.
   If $b = 0$,
   then $a + b = a$. addition property of zero
   $c = a$, since $c = a + b$.
   But $c = a$ and $c < a$ is a contradiction.
   Therefore, $b < 0$.

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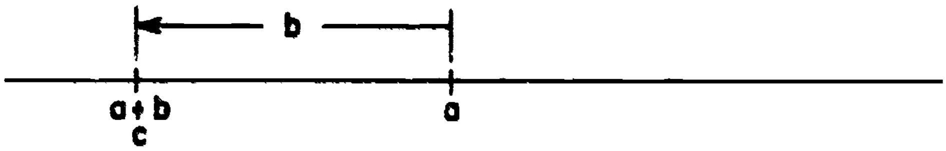
3. (a) true (d) true
   (b) false (e) true
   (c) true (f) true

4. (a) $5 < 13$, $8 < 13$, (also $5 < 8$)
   (b) $(-3) < (-1)$, $(-1) < 2$, $(-3) < 2$
   (c) $5 + 2 = 7$, $5 = 7 + (-2)$

5.



Locate a. Since $b < 0$, we move $|b|$ units to the left to locate $a + b$. c is another name for $a + b$. c is to the left of a, so $c < a$.

6. (a) false (b) true (c) true

7. (a) If x is the number,
   then $x + 5 < 2x$.
   If $x + 5 < 2x$ is true for some x,
   then $5 < x$ is true for the same x.
   Thus, if x is a number which makes the original sentence true, then $5 < x$.
   If $5 < x$,
   then $x + 5 < 2x$ is true for the same x.
   Hence, the truth set is the set of all numbers greater than 5.

   (b) If x is the number of dollars Moe would pay,
   then $x + 130$ is the number of dollars Joe would pay,
   and $x + x + 130 \leq 380$.
   The truth set is the set of all numbers less than or equal to 125.
   Thus, Moe would pay no more than \$125.

   (c) If n is the number,
   then $6n + 3 > 7 + 5n$.
   The truth set is the set of all numbers greater than 4.
   The number is any number greater than 4.

(d) If y is the number of students in the class, the $2y \geq y + 26$.
The truth set is the set of all numbers greater than or equal to 26.
The number of students in class is at least 26.

(e) If s is the score the student must make on the third test, then

$$\frac{82 + 91 + s}{3} \geq 90,$$

and

$$82 + 91 + s \geq 3(90).$$

The truth set is the set of all numbers greater than or equal to 97.
The student must score at least 97 on the third test.

There may arise in this problem a question of the domain of s. Some pupils will claim a domain of $0 \leq s \leq 100$ is implied, some may assert that s may be any number, and some may ask, "What about 30π?", etc. It is a good opportunity to note the importance of thinking about the domain of the variable.

(f) If n is the number of years in Norman's age, then $n + 5$ is the number of years in Bill's age, and

$$n + 5 + n < 23.$$

The truth set is the set of all numbers less than 9.
Norman is younger than 9 years old.

2-6. Multiplication Property of Order

The fundamental property is given in Theorem 2-6a. We "discovered" it by induction, but it is provable. This shows that the multiplication property of order is not independent of the other properties but can be deduced from them. It comes from the addition property of order by way of the distributive property which links addition and multiplication.

Since $\frac{1}{4} < \frac{2}{7}$,

$\frac{1}{4}(5) < \frac{2}{7}(5)$ if $a < b$ and $c > 0$, $ac < bc$

and $\frac{5}{4} < \frac{10}{7}$.

Similarly, since $-\frac{5}{6} < -\frac{14}{17}$,

$(-\frac{14}{17})(-\frac{1}{3}) < (-\frac{5}{6})(-\frac{1}{3})$ if $a < b$ and $c < 0$, $bc < ac$

and $\frac{14}{51} < \frac{5}{18}$.

Again, since $\frac{5}{3} < \frac{7}{4}$,

$\frac{7}{4}(-\frac{1}{4}) < \frac{5}{3}(-\frac{1}{4})$ if $a < b$ and $c < 0$, $bc < ac$

and $-\frac{7}{16} < -\frac{5}{12}$.

The reasons for the steps of the proof of the first case of Theorem 2-6a are as follows:

1. If x and z are two real numbers such that $x < z$, then there is a positive number y such that $z = x + y$.
2. Multiplication property of equality.
3. Distributive property.
4. The product of two positive numbers is positive.
5. If $z = x + y$ and y is a positive number, then $x < z$.

The pupils may notice that this is an example of a proof done by translation back and forth between statements about order and statements about equality.

The multiplication property of order, seen from the standpoint of the numbers rather than the order relation, could be stated: If one number is less than another, and both are multiplied by a positive number, the order remains unchanged; if one number is less than another number, and both are multiplied by a negative number, the order is reversed.

The reasons for the steps of the proof of Theorem 2-6b are as follows:

If $x > 0$,

$(x)(x) > 0(x)$. multiplication property of order (If $a < b$, then $ac < bc$ if c is positive.)

If $x < 0$,

$(x)(x) > (0)(x)$. multiplication property of order (If $a < b$, then $bc < ac$ if c is negative.)

If $x = 0$, $x^2 = 0$; and if $x \neq 0$, $x^2 > 0$.

Hence, for all real numbers x, $x^2 \geq 0$.

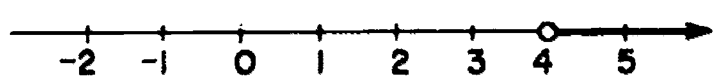
To the members of $-8x < -8$ we must add $5x + 2$ to obtain the original sentence, $(-3x) + 2 < 5x + (-6)$.

Multiplying the members of $1 < x$ by -8, we obtain the sentence, $-8x < -8$.

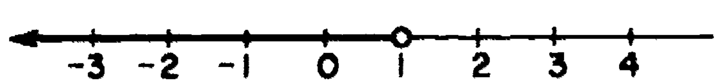
Answers to Problem Set 2-6

1. (a) the set of all numbers greater than 4.
   (b) the set of all numbers less than 1.
   (c) the set of all x such that $x < -4$.
   (d) the set of all numbers greater than $\frac{4}{3}$.
   (e) the set of all negative numbers.
   (f) the set of all x such that $x > \frac{3}{2}$.
   (g) the set of all numbers less than $\frac{17}{6}$.
   (h) the set of all numbers greater than $-\frac{4}{5}$.
   (i) the set of all numbers greater than 2.

2. (a)



   (b)



3. (a) If Sally has b books, then Sue has $b + 16$ books and

$$b + 16 + b > 28.$$

The truth set consists of all numbers greater than 6 so that Sally has at least 6 books.

(b) If x bulbs were planted, then

$$15 < \frac{5}{8}x \quad \text{and} \quad 15 > \frac{3}{8}x.$$

The truth set consists of all integers between 24 and 40.

4. If $a < b$, then $a = b + e$, where $e < 0$. (See Problem Set 2-5b, Problem 2).

Then, if $c < 0$, $ac = (b + e)c$. multiplication property of order

$ac = bc + ec$ distributive property

If $c < 0$, then ec is positive,

and $bc < ac$. If $z = x + y$ and y is positive, then $x < z$.

5. If $c < 0$, then $0 < (-c)$, and $(-c)$ is positive.
If $a < b$, then $a(-c) < b(-c)$,

$-(ac) < -(bc)$,

and $bc < ac$. If $a < b$, then $-b < -a$.

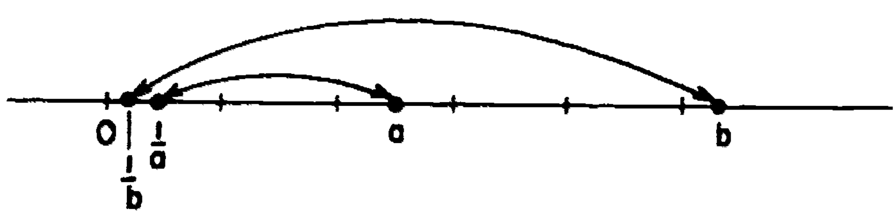
6. If $a < b$, $a > 0$ and $b > 0$,

then
$$a(\tfrac{1}{a} \cdot \tfrac{1}{b}) < b(\tfrac{1}{a} \cdot \tfrac{1}{b}).$$

$$(a \cdot \tfrac{1}{a})(\tfrac{1}{b}) < (b \cdot \tfrac{1}{b})(\tfrac{1}{a})$$

$$(1)(\tfrac{1}{b}) < (1)(\tfrac{1}{a})$$

$$\tfrac{1}{b} < \tfrac{1}{a}$$



7. If $a < b$, where $a < 0$ and $b < 0$,
then

$$(\tfrac{1}{a} \cdot \tfrac{1}{b}) \text{ is positive.}$$

$$a(\tfrac{1}{a} \cdot \tfrac{1}{b}) < b(\tfrac{1}{a} \cdot \tfrac{1}{b})$$

$$(a \cdot \tfrac{1}{a})(\tfrac{1}{b}) < (b \cdot \tfrac{1}{b})(\tfrac{1}{a})$$

$$\tfrac{1}{b} < \tfrac{1}{a}$$

The relationship holds when both a and b are negative.

8. If $a < b$, $a < 0$, $b > 0$, then $\frac{1}{a} < 0$ and $\frac{1}{b} > 0$; hence, $\frac{1}{a} < \frac{1}{b}$, by the transitive property.

   The relation $\frac{1}{b} < \frac{1}{a}$ does not hold if $a < 0$ and $b > 0$. This can also be disproved by one "counter example":

   If $a = -2$ and $b = 2$, then $\frac{1}{a} = -\frac{1}{2}$ and $\frac{1}{b} = \frac{1}{2}$.

   Since $-\frac{1}{2} < \frac{1}{2}$, the relation is false.

9. Addition property of order: if a, b, and c are real numbers, and if $a > b$, then $a + c > b + c$.

   Multiplication property of order: if a, b, and c are real numbers, and if $a > b$, then

   $$ac > bc, \text{ if } c \text{ is positive;}$$
   $$bc > ac, \text{ if } c \text{ is negative.}$$

10. If $a > 0$ and $b > 0$ and $a < b$, then by the multiplication property of order, we have

    $$a \cdot a < a \cdot b \text{ and } a \cdot b < b \cdot b.$$

    If $a^2 < ab$ and $ab < b^2$,

    then $a^2 < b^2$, by the transitive property of order.

Although we have given considerable preparation for the shift in point of view indicated here, most pupils obviously will not be able to appreciate fully its significance at this time. We have, however, no intention of proving everything from here on, but will continue as we have in the past, "discovering" properties and giving an occasional simple proof. The principal change is in our attitude toward the properties discovered; namely, that they could be proved if we had the time and experience to do so.

The best pupils will get the idea easily and we hope that most pupils will be able to think of the real number system deductively. The traditional geometry course, as well as the SMSG geometry course, requires a deductive approach to geometry. The SMSG geometry course also assumes the real numbers to be given axiomatically. Therefore, the deductive point of view is important not only for advanced mathematics but also for later high school courses.

The missing fundamental property, which would enable us to obtain everything about the real numbers, is called the <u>completeness axiom</u>. It can be stated in a number of forms, one of the most convenient being in terms of "least upper bounds". Before stating it, we first define an "upper bound" of a set as follows:

> Let S be a set of real numbers and b a real number such that $s \leq b$ for every s in S. Then b is called an <u>upper bound</u> for S. If there does not exist an upper bound for S which is <u>less</u> than b, then b is called a <u>least upper bound</u> for S.

We can now state the

> <u>Completeness Axiom</u>. If S is any set of real numbers for which there is an upper bound, then there exists a least upper bound for S. The completeness axiom is needed, for example, to prove the existence of $\sqrt{2}$. As a matter of fact, it is required to prove the existence of any irrational number.

<u>Answers to Review Problems</u>:

1. (a) $-100 < -99$ (d) $\frac{6}{7} > \frac{5}{8}$
   (b) $0.2 > 0.1$ (e) $3.4 - 4 > 3(4 - 4)$
   (c) $-(-3) < -(-7)$ (f) $x^2 + 1 > 0$

2. (a) true (d) false
   (b) false (e) false
   (c) true (f) true

3. (a) not equivalent (d) equivalent
   (b) equivalent (e) not equivalent
   (c) equivalent (f) not equivalent

4. (a) the set of all real numbers less than (-5).
   (b) the set of all real numbers greater than (-1).
   (c) the set of all real numbers greater than (-6).
   (d) the set of all real numbers less than (-3).
   (e) the set of all real numbers less than or equal to 91.
   (f) the empty set.

5. (a) the set of all real numbers greater than 2.
   (b) $\{2\}$.
   (c) the set of all negative real numbers.
   (d) the set of all real numbers except zero.
   (e) the set of all non-negative real numbers less than 90.
   (f) $\emptyset$

6. (a) $\{2,3\}$
   (b) $\{0,1,2,3\}$
   (c) $\{-2,-3\}$
   (d) $\emptyset$
   (e) $\{-3\}$
   (f) $\{0,1,2,3\}$

7. (a) $6 < 3x + 2 < 10$ if and only if $4 < 3x < 8$, by the addition property of order. Then $\frac{4}{3} < x < \frac{8}{3}$ by the multiplication property of order. Thus, the truth set is the set of real numbers x, such that $\frac{4}{5} < x < \frac{8}{3}$.
   (b) The set of all real numbers y such that $-\frac{11}{2} < y < -\frac{3}{2}$.
   (c) $-2 < \frac{2w + 3}{5} < 2$ if and only if $-10 < 2w + 3 < 10$

$$-13 < 2w < 7$$
$$-\frac{13}{2} < w < \frac{7}{2}.$$

Note: the answers will be left in this form, and not be stated here as truth sets, for the sake of brevity.

   (d) $-1 < 3 - x < 1$ if and only if $-4 < -x < -2$.
   Note carefully the change here because the multiplier is negative: $2 < x < 4$ or $4 > x > 2$.
   (e) $-\frac{25}{3} < y < \frac{1}{3}$
   (f) $0 \leq m \leq \frac{1}{2}$
   (g) $\frac{1}{2} < a \leq \frac{7}{2}$
   (h) $-\frac{1}{18} < p < \frac{5}{18}$
   (i) $-1 \leq \frac{4 - 3x}{2} \leq 1$ if and only if $-2 < 4 - 3x \leq 2$

$$-6 \leq -3x \leq -2$$
$$\therefore \frac{2}{3} \leq x \leq 2$$
$$(\text{also, } 2 \geq x \geq \tfrac{2}{3})$$

(j) $-1 \leq \frac{4 - 5x}{-2} \leq 1$ if and only if $-2(1) \leq 4 - 5x \leq (-2)(-1)$

Note application of theorem: If $a < b$ and $c < 0$, then $bc < ac$.

$$-2 \leq 4 - 5x \leq 2$$
$$-6 \leq -5x \leq -2$$

By the theorem: If $a < b$, $-b < -a$ $\quad 2 \leq 5x \leq 6$

Therefore, $\quad \frac{2}{5} \leq x \leq \frac{6}{5}$

8. If A is the number of square units in the area,

$$24 \leq A < 28.$$

9. If A is the number of square units in the area,

$$24 \leq A < 35.$$

10. If A is the number of square units in the area,

$$25.5225 < A < 26.5625.$$

11. (a) If p is the number of plants at the beginning of the second year,

$$p > \tfrac{3}{4}(240) \quad \text{and} \quad p < \tfrac{5}{6}(240);$$

that is, $180 < p < 200$.

If n is the number of seeds at the end of the second year,

$$n > (180)(240) \quad \text{and} \quad n < (200)(240);$$

that is, $43{,}200 < n < 48{,}000$.

(b) If s is the number of seeds at the end of the second year,

$$s > (180)(230) \quad \text{and} \quad s < (200)(250);$$

that is, $41{,}400 < s < 50{,}000$.

12. (a) If the side of a square is x inches long, then the side of the triangle is $x + 3.5$ inches long, and

$$4x = 3(x + 3.5).$$

The length of the side of the square is 10.5 inches.

(b) If the rate of the current is x miles per hour, then the rate of the boat downstream is $x + 10$ miles per hour, and

$$x + 10 \leq 25.$$

The rate of the current is equal to or less than 15 miles per hour.

(c) If x is the number of hours spent on the job, then

$$3 \leq x \leq 5.$$

Mary can expect to spend from 3 to 5 hours on the job.

(d) If x is the number of hours Jim must work,

$$1.5x \geq 75.$$

Jim must work at least 50 hours.

13. If x is the number of quarts of white paint, then $3x$ is the number of quarts of grey paint and

$$x + 3x = 7.4$$
$$4x = 4.7$$
$$x = 7, \quad 3x = 21$$

Thus, the man bought 1 gallon and 3 quarts of white and 5 gallons and 1 quart of grey. The information on the cost of the paint was unnecessary.

14. Proof: Either $\sqrt{a} < \sqrt{b}$, $\sqrt{a} = \sqrt{b}$, or $\sqrt{a} > \sqrt{b}$.

Assume $\sqrt{a} < \sqrt{b}$

then $a < b$ If $x < y$ then $x^2 < y^2$

$a < b$ and $a > b$ is a contradiction.

Assume $\sqrt{a} = \sqrt{b}$

then $a = b$

$a = b$ and $a > b$ is a contradiction.

Thus, $\sqrt{a} > \sqrt{b}$.

15. If $a > b$

$a + (-b) > b + (-b)$, addition property of order

$a - b > b + (-b)$, definition of subtraction

$a - b > 0$. addition property of opposites

16. If $(a - b)$ is a positive number, then $a > b$.
If $(a - b)$ is a negative number, then $a < b$.
If $(a - b)$ is zero, then $a = b$.

17. If a, b, and c are real numbers, and $a > b$, then $a - c > b - c$.

If $a > b$,
$a + (-c) > b + (-c)$, addition property of order
$a - c > b - c$, definition of subtraction

Suggested Test Items

1. Find the truth sets of the following open sentences and draw their graphs.
   (a) $(-x) + 5 < (-8) + |-8|$
   (b) $\frac{3}{2}x + (-3) > x + (-4)$
   (c) $(-7) + (-y) < \frac{3}{7} + (-\frac{3}{7})$
   (d) $37 + (-6r) + 7 > 9r + (-7r) + 8 + (-2r)$
   (e) $5n - 3 > 2n + 9$
   (f) $4(3 - x) > 12$

2. If p, q, and t are real numbers and $p < q$, which of the following sentences are true?
   (a) $p + t < q + t$, if $t > 0$
   (b) $p + t > q + t$, if $t < 0$
   (c) $pt < qt$, if $t > 0$
   (d) $pt > qt$, if $t < 0$
   (e) $\frac{1}{p} < \frac{1}{q}$

3. If n is a non-negative number and x is a non-positive number, which of the following are true?
   (a) $x \leq 0$
   (b) $n \nless x$
   (c) $n \geq 0$
   (d) $x \leq n$
   (e) $x > 0$
   (f) $n > x$

4. We know that the sentence "$4 < 7$" is true. What true sentences result when both numbers are

(a) increased by 5
(b) decreased by 5
(c) multiplied by 5
(d) multiplied by (-5)
(e) multiplied by 0

5. Write an open sentence for each of the following problems. State the truth sets and answer the questions.

(a) Tom has \$15 more than Bill. After Tom spends \$3 for meals, the two boys together have at least \$60. How much money does Bill have?

(b) If 13 is added to a number and the sum is multiplied by 2, the product is more than 76. What is the number?

(c) Tom works at the rate of p dollars per day. After working 5 days he collects his pay and spends \$6 of it. If he then has more than \$20 left, what was his rate of pay?

(d) A farmer discovered that at least 70 per cent of a certain kind of need grew into plants. If he has 245 plants, how many seeds did he plant?

6. Which of the followi. sentences are true for every a and every b?

(a) If $a + 2 = b$, then $b < a$.
(b) If $a + (-3) = b$, then $b < a$.
(c) If $(a + 5) + (-2) = b + 5$, then $b < a$.
(d) If $a < 4$ and $4 > b$, then $a < b$.
(e) If $a + 2 < 7$ and $b + 2 > 7$, then $a < b$.

7. Given $\frac{7}{9}$, $\frac{2}{3}$, $\frac{3}{4}$ and n. In each part of this problem make as many statements involving "$<$" about n and the given numbers as you can, if you know:

(a) $n < \frac{7}{9}$ (b) $n < \frac{2}{3}$ (c) $n < \frac{3}{4}$

8. A man has three ore samples, each having the same volume. The sample of lead outweighs the sample of iron. The sample of gold outweighs the sample of lead. Which is a heavier ore sample, gold or iron? What property of real numbers is illustrated here?

Chapter 3

SOLUTION OF INEQUALITIES

3-1. Equivalent Inequalities.

Just as for equations, the thing we must look for in establishing the fact that two inequalities are equivalent is whether the operations we perform, can be reversed to carry us back from the simpler one to the given one. If they can be reversed, we know that the truth set of the original inequality is a subset of the truth set of the new inequality and the truth set of the new one is a subset of the original one. The two truth sets are therefore identical.

As with equations, you may want to point out to your students that inequalities may be simplified if we know how to "undo" some of the indicated operations. Indicated additions can be "undone" by adding the opposite, and indicated multiplications can be "undone" by multiplying by the reciprocal.

In Example 1 and Example 2 the truth set of the final inequality is the truth set of the original inequality because only operations yielding equivalent inequalities were used. No checking is necessary.

Answers to Problem Set 3-1

1. (a) $x + 12 < 39$

   $x < 27$

   The truth set is the set of all real numbers less than 27.

   (b) $\frac{5}{7}x < 36 - x$

   $\frac{12}{7}x < 36$

   $x < 36(\frac{7}{12})$

   $x < 21$

   The truth set is the set of all real numbers less than 21.

   (c) The set of all real numbers greater than $\sqrt{2}$.

   (d) The set of all real numbers less than $\sqrt{3}$.

   (e) The set of all real numbers greater than 2.

(f) The set of all real numbers less than 12.

(g) The set of all real numbers.

(h) $\emptyset$

(i) The set of all real numbers.

2. (a) $1 < 4x + 1$ and $4x + 1 < 2$

$0 < 4x$ and $4x < 1$

$0 < x$ and $x < \frac{1}{4}$

The truth set is the set of all real numbers between 0 and $\frac{1}{4}$.

(b) $4t - 4 < 0$ and $1 - 3t < 0$

$4t < 4$ and $1 < 3t$

$t < 1$ and $\frac{1}{3} < t$

The truth set is the set of all real numbers between $\frac{1}{3}$ and 1.

(c) The set of all real numbers between $-\frac{1}{2}$ and $\frac{1}{2}$.

(d) The set of all real numbers which are either less than $-\frac{1}{2}$ or greater than $\frac{1}{2}$.

*(e) $|x - 1| < 2$

On the number line the distance between x and 1 must be less than 2. Hence,

$$1 - 2 < x < 1 + 2$$
$$-1 < x < 3$$

The set of all real numbers between -1 and 3.

*(f) $|2t| < 1$

$2|t| < 1$

$|t| < \frac{1}{2}$

On the number line the distance between t and the origin must be less than $\frac{1}{2}$.

The set of all real numbers between $-\frac{1}{2}$ and $\frac{1}{2}$.

*(g) $|x + 2| < \frac{1}{2}$

$|x - (-2)| < \frac{1}{2}$

The set of all real numbers between $-\frac{5}{2}$ and $-\frac{3}{2}$.

*(h) $|y + 2| > 1$

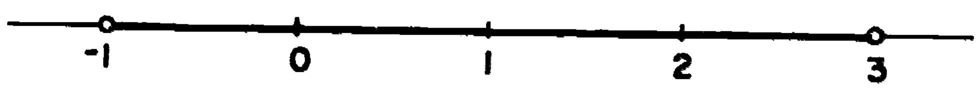
$|y - (-2)| > 1$

The set of all real numbers which are either less than -3 or greater than -1.

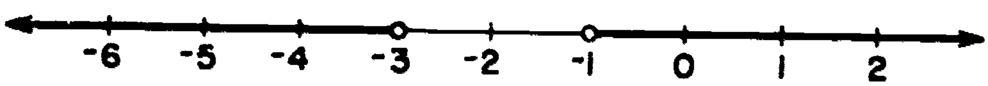
3. (a)

(c)

(e)



(h)



4. $3y - x + 7 < 0$

$3y < x - 7$

$y < \frac{1}{3}(x - 7)$

When $x = 1$, $y < \frac{1}{3}(1 - 7)$

$y < -2$

The truth set is the set of all real numbers less than -2.

$3y - x + 7 < 0$

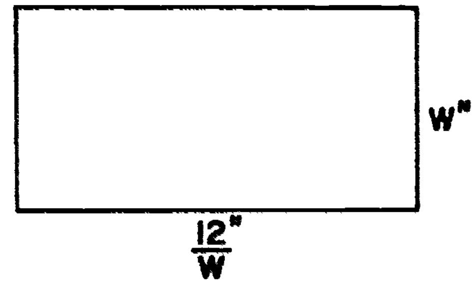
$-x < -3y - 7$

$x > 3y + 7$

When $y = -2$, $x > 3(-2) + 7$

$x > 1$ The truth set is the set of all real numbers greater than 1.

5. If the rectangle is w inches wide, it is $\frac{12}{w}$ inches long, since the area is 12 square inches. Then $\frac{12}{w} < 5$. Since, by the nature of the problem, $w > 0$,



$$12 < 5W$$

$$\frac{12}{5} < w$$

The width of the rectangle is greater than $2\frac{2}{5}$ inches.

6. If n is the negative number, then

$$n < \frac{1}{n} \quad \text{and} \quad n < 0$$

$$n^2 > 1 \quad \text{and} \quad n < 0$$

$$n < -1$$

The truth set is the set of all real numbers less than -1.

At this point the pupils may have no formal way of solving $n^2 > 1$. They can, however, go back to the method of making an intelligent guess and verifying it with the help of the number line. This is still a useful method when we do not have a better one. In Section 3-2 the student may look more closely at solving sentences such as $n^2 - 1 > 0$.

3-2. Polynomial Inequalities

The statement about when a product of several non-zero numbers is positive and when it is negative can, of course, be proved by using the commutative and associative properties to group the negative factors in pairs. The product of each pair of negative factors is positive, and the product of these pairs and all the positive factors will still be positive. Hence, the whole product will be negative only when there is an odd number of negative factors. The class should be encouraged by discussion to fill in these details.

5()

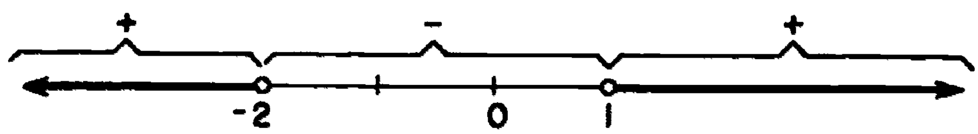
For instance, when $x = 2$, it is sufficient to recognize that $(2 + 3)$ is positive, $(2 + 2)$ is positive, $(2 - 1)$ is positive. Hence, the product is positive. Similarly for $x = -\frac{5}{2}$, $(\frac{5}{2} + 3)$ is positive, $(-\frac{5}{2} + 2)$ is negative, $(-\frac{5}{2} - 1)$ is negative. Since there are two negative factors, the product is positive.

The truth set of $(x + 3)(x + 2)(x - 1) > 0$ is the set of all x such that $-3 < x < -2$ or $x > 1$.

The truth set of $(x + 3)(x + 2)(x - 1) \geq 0$ is the set of all x such that $-3 \leq x \leq -2$ or $x \geq 1$.

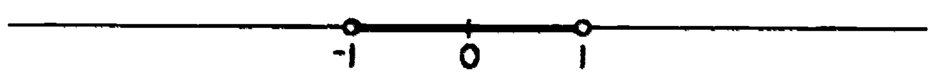
Answers to Problem Set 3-2a

1. (a) $(x - 1)(x + 2) > 0$



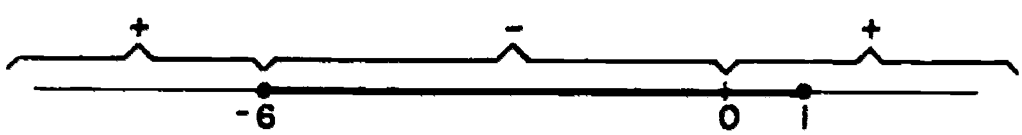
The set of numbers less than -2 or greater than 1.

(b) $y^2 < 1$



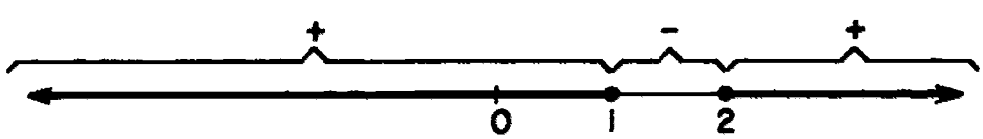
The set of numbers less than 1 and greater than -1.

(c) $t^2 + 5t \leq 6$



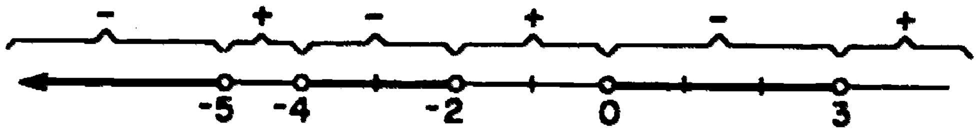
The set of numbers (greater than or equal to -6) and (less than or equal to 1).

(d) $x^2 + 2 \geq 3x$



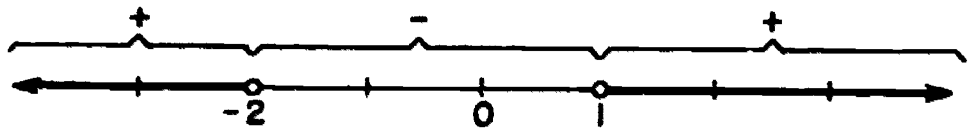
The set of numbers (less than or equal to 1) or (greater than or equal to 2).

(e) $(s + 5)(s + 4)(s + 2)(s)(s - 3) < 0$



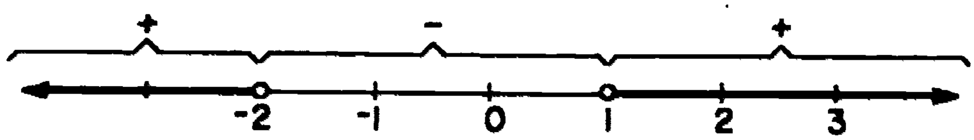
The set of numbers (less than -5) or (greater than -4 and less than -2) or (greater than 0 and less than 3).

(f) $2 - x^2 < x$

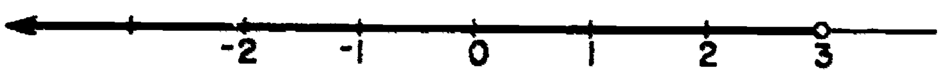


The set of numbers less than -2 or greater than 1.

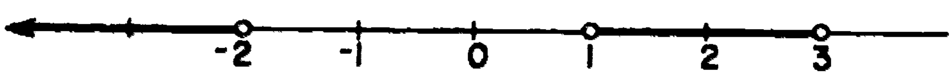
2. $(x + 1)(x - 1) > 0$ and $x < 3$.
The graph of $(x + 2)(x - 1) > 0$ is



The graph of $x < 3$ is



The truth set of "$(x + 2)(x - 1) > 0$ and $x > 3$" is the set of all numbers each of which is in <u>both</u> the above truth sets.



The set of numbers (less than -2) or (greater than 1 and less than 3).

3. If $(x + 2)(x - 1)$ is positive and $x - 3$ is negative, then $(x + 2)(x - 1)(x - 3)$ is negative; thus, every solution of the sentence in Problem 2 is a solution of $(x + 2)(x - 1)(x - 3) < 0$. This inequality is therefore a likely candidate, and when we graph its truth set we get the graph drawn in Problem 2 so that the two sentences are equivalent.

If x is a solution of $(x + 2)^2(x - 1) > 0$ then $(x + 2)^2$ must not be 0 and therefore must be positive, being a square. Multiplying by the positive number $\frac{1}{(x + 2)^2}$ we obtain

$x - 1 > 0$. Going backwards we see that, if $x - 1 > 0$, $x > 1$ and hence, $x \neq -2$, so that $x + 2 \neq 0$ and so $(x + 2)^2$ must be positive. Multiplying $x - 1 > 0$ by this positive number gives us $(x + 2)^2(x - 1) > 0$. Hence, $(x + 2)^2(x - 1) > 0$ and $x - 1 > 0$ are equivalent sentences.

The truth set of $(x + 2)^2(x - 1) \leq 0$ is the set of all numbers that are not in the truth set of $(x + 2)^2(x - 1) > 0$. The latter set we've just seen to be all x such that $x > 1$. Thus, the truth set of $(x + 2)^2(x - 1) \leq 0$ is the set of all x such that $x \leq 1$.

The product of x and $(x - 1)^3$ will be negative if and only if either $x < 0$ and $(x - 1)^3 > 0$ or $x > 0$ and $(x - 1)^3 < 0$.

The first clause is equivalent to "$x < 0$ and $x - 1 > 0$"; that is, to "$x < 0$ and $x > 1$". Since no number is both less than 0 and greater than 1, this sentence has no solution. The second clause is equivalent to "$x > 0$ and $x - 1 < 0$" and this has its truth set consisting of all x between 0 and 1. The truth set of $x(x - 1)^3 < 0$ is, thus, the set $0 < x < 1$.

The truth set of $x(x - 1)^3 \geq 0$ will be all the numbers not in the truth set of $x(x - 1)^3 < 0$. The latter set we have just seen to be the set $0 < x < 1$. The numbers not in this set consist of all $x \leq 0$ together with all $x \geq 1$. Therefore,

"$x(x - 1)^3 \geq 0$" is equivalent to "$x \leq 0$ or $x \geq 1$".

We have used above the fact that where the factor $x - 1$ occurs three times there are three factors changing together from negative to positive as x crosses 1, so their product changes from negative to positive. Some students may enjoy extending this idea to polynomials with the same factor four or five times, and then generalizing the situation.

A factor which is a positive real number such as $x^2 + 2$ will not change the product from a negative to a positive number or vice versa. For this reason the truth set of $(x^2+2)(x-3) < 0$ is the truth set of $x - 3 < 0$; that is, all x such that $x < 3$; and the truth set of $(x^2 + 2)(x - 3) \geq 0$ is the set of all x such that $x \geq 3$.

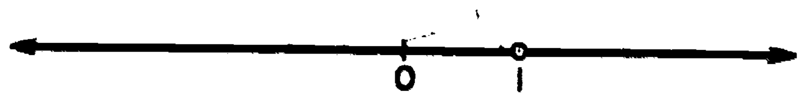
Answers to Problem Set 3-2b

1. $x^2 + 1 > 2x$

   $x^2 + 1 - 2x > 0$

   $(x - 1)^2 > 0$

   Since $a^2 > 0$ for all real numbers a except zero, the truth set is the set of all real numbers except 1.

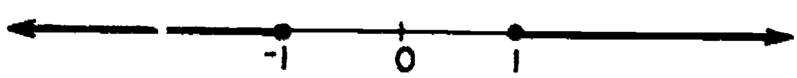


2. $x^2 + 1 < 0$

   The truth set is $\emptyset$.

3. $(t^2 + 1)(t^2 - 1) \geq 0$

   The set of numbers (less than or equal to -1) or (greater than or equal to 1).

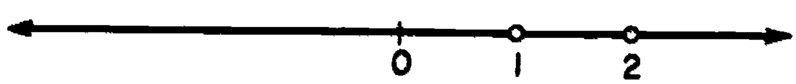


4. $4s - s^2 > 4$

   The truth set is $\emptyset$.

5. $(x - 1)^2(x - 2)^2 > 0.$

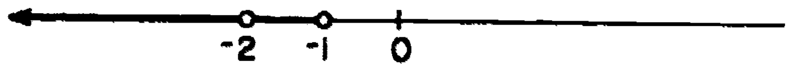
   The set of all real numbers except 1 and 2.



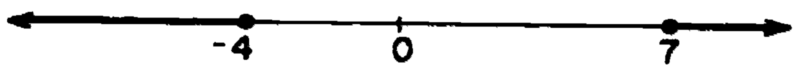
6. $(y^2 - 7y + 6) \leq 0$

   The set of numbers (greater than or equal to 1) and (less than or equal to 6).

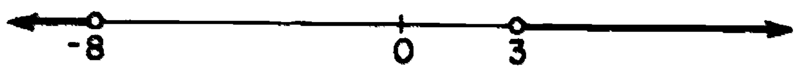
7. $(x + 2)(x^2 + 3x + 2) < 0$
The set of numbers less than -1, except -2.



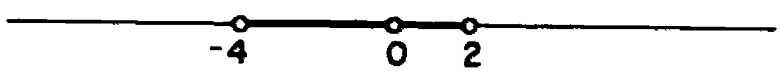
8. $3y + 12 \leq y^2 - 16$
The set of numbers (less than or equal to -4) or (greater than or equal to 7).



9. $x^2 + 5x > 24$
The set of numbers less than -8 or greater than 3.



10. $|x|(x - 2)(x + 4) < 0$
The set of numbers greater than -4 and less than 2, except 0.



3-3. Rational Inequalities

The method shown here is also applicable to finding the graph of the open sentence $(x - y - 2)(x + y - 2) < 0$ which we shall meet in Chapter 4. It is important to consider the punctuation in "$a > 0$ and $b > 0$, or $a < 0$ and $b < 0$". The comma after "$b > 0$" is significant.

Plotting the solution set of a quadratic inequality on the number line can yield one of four possible figures:

(a) the whole line

(b) the empty set

(c) all the points of a segment except its endpoints

(d) all the points not on a segment

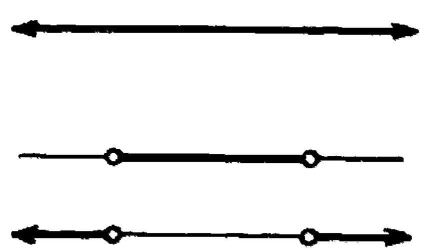


Figure 3-3a

These facts become a little less mysterious if the graph of the associated quadratic function is drawn. Cases (a) and (b) are illustrated in Figure 3-3b. The whole line is the solution set of $ax^2 + bx + c > 0$ and of $a_1x^2 + b_1x + c_1 < 0$. The empty set is the solution set of $ax^2 + bx + c < 0$ and of $a_1x^2 + b_1x + c_1 > 0$.

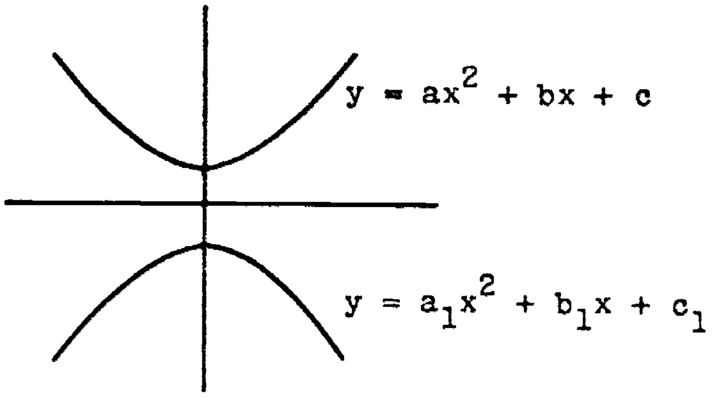


Figure 3-3b

Cases (c) and (d) are illustrated in Figure 3-3c. All the points of $\overline{AB}$ except A and B constitute the solution set of $ax^2 + bx + c < 0$ and of $a_1x^2 + b_1x + c_1 > 0$. All the points of the line not on $\overline{AB}$ constitute the solution set of $ax^2 + bx + c > 0$ and $a_1x^2 + b_1x + c_1 < 0$.

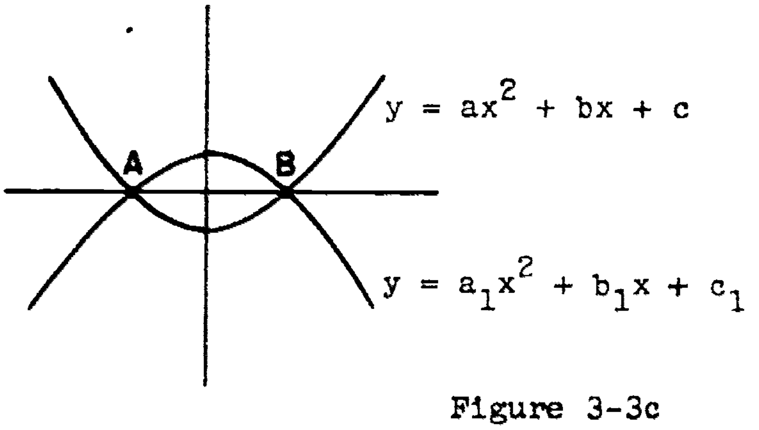


Figure 3-3c

If the pupils have had quadratic functions, this method should be mentioned. It can be extended to cover inequalities of the type: $(x - a)(x - b)(x - c) > 0$, etc., if the pupil is familiar with the graph of the cubic function.

Answers to Problem Set 3-3a

1. $x^2 - 3x + 2 > 0$ if $(x - 2)(x - 1) > 0$
   $x - 2 > 0$ and $x - 1 > 0$, or $x - 2 < 0$ and $x - 1 < 0$
   $x > 2$ and $x > 1$, or $x < 2$ and $x < 1$

   $\therefore$ Solution set is the set of all numbers less than 1 or greater than 2.

2. The set of all numbers x such that $2 < x < 3$.
3. The set of all numbers x such that $-4 < x < \frac{2}{3}$.
4. The set of all numbers x such that $\frac{1}{2} < x < 1$.
5. The set of all numbers x such that $x < -\frac{1}{4}$ or $x > 4$.
6. The set of all numbers except 1.
7. The set of all numbers x such that $x < -5$ or $x > -1$.
8. If $-x^2 - 4x + 5 < 0$, then $x^2 + 4x - 5 > 0$.
   Solution set is the set of all numbers x such that $x > 1$ or $x < -5$.
9. $2x^2 + 4x + 5 < 0$ is equivalent to $x^2 + 2x + \frac{5}{2} < 0$. This can be written $x^2 + 2x + 1 - 1 + \frac{5}{2} < 0$ or
   $(x + 1)^2 + \frac{3}{2} < 0$.
   Since this expression is never less than 0, the solution set is $\emptyset$.
10. $x^2 - 2\sqrt{5}x - 4 > 0$ is equivalent to $x^2 - 2\sqrt{5}x + 5 - 5 - 4 > 0$
    or $(x - \sqrt{5})^2 - 9 > 0$.
    Hence, $(x - \sqrt{5} - 3)(x - \sqrt{5} + 3) > 0$.
    Thus, $x - \sqrt{5} - 3 > 0$ and $x - \sqrt{5} + 3 > 0$, or
    $x - \sqrt{5} - 3 < 0$ and $x - \sqrt{5} + 3 < 0$
    Therefore, $x > \sqrt{5} + 3$ or $x < \sqrt{5} - 3$

    The solution set is the set of all numbers greater than $\sqrt{5} + 3$ or less than $\sqrt{5} - 3$.

If $a > b$ and $c < 0$, then $ac > bc$, but if $a > b$ and $c < 0$, then $ac < bc$. This is the difficulty involved in finding the solution set of $\frac{x + 2}{x - 3} > 0$.

If we multiply by $x - 3$, is $x - 3 > 0$ or is $x - 3 < 0$? If $x - 3 > 0$, then $(x - 3)(\frac{x + 2}{x - 3}) > 0(x - 3)$; or $x + 2 > 0$. We then must find the solution set of $x - 3 > 0$ <u>and</u> $x + 2 > 0$, which is the set of all numbers greater than 3.

But if $x - 3 < 0$, then $(x - 3)(\frac{x + 2}{x - 3}) < 0(x - 3)$; or $x + 2 < 0$.

Now we must find the solution set of $x - 3 < 0$ <u>and</u> $x + 2 < 0$, which is the set of all numbers less than -2.

Our complete solution set is, therefore, the set of all numbers greater than 3 <u>or</u> less than -2.

<u>Answers to Problem Set 3-3b</u>

1. (a) The set of all numbers x such that $-\frac{3}{2} < x < 4$.

   (b) The set of all numbers x such that $x < -3$ or $x > \frac{2}{5}$.

   (c) The set of all numbers x such that $x > -\frac{2}{3}$ or $x < -\frac{7}{4}$.

   (d) The set of all numbers x such that $x < \frac{2}{5}$ or $x > \frac{5}{2}$.

   (e) The set of all numbers x such that $-2 < x < -\frac{4}{3}$ or $x > 1$.

   (f) The set of all numbers x such that $\frac{5}{3} < x < 4$ or $x < \frac{3}{2}$.

   (g) The set of all numbers x such that $-\frac{5}{2} < x < -\frac{1}{5}$ or $x < -4$.

   (h) $\emptyset$ (The quotient of two positive numbers is positive.)

2. $\sqrt{1 + 2x} < x - 1$

   We observe that $\sqrt{1 + 2x}$ is defined for $x \geq -\frac{1}{2}$ and that if there is an x such that $x - 1$ is greater than a non-negative number, then $x > 1$.

   Thus: $\sqrt{1 + 2x} < x - 1$ and $x > 1$

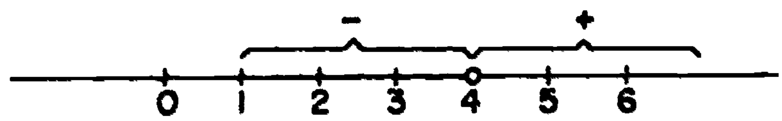
   is equivalent to

   $$1 + 2x < x^2 - 2x + 1 \text{ and } x > 1$$

   is equivalent to

   $$0 < x(x - 4) \text{ and } x > 1.$$

We need consider only values of $x > 1$.



Thus, the truth set consists of every number greater than 4.

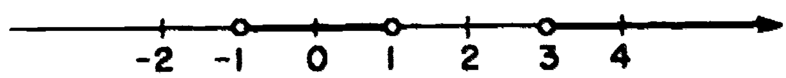
The graph:

3. The first two sentences are reversible since

(i) If $a < b$, then $a^2 < b^2$.

(ii) If $0 < a < b$, then $\sqrt{a} < \sqrt{b}$.

(a)



(b)

(c)

Chapter 4

GRAPHS OF OPEN SENTENCES IN TWO VARIABLES

4-1. Inequalities in Two Variables

It is very helpful in this chapter if the pupil has had some work with slope and y-intercept. If he puts the equation in the "y-form", as $y = mx + b$, then m is the slope and b is the y-intercept.

Upon attempting to locate points such as $(-2,5)$, $(-1,2)$, $(0,2)$, $(1,4)$, $(2,8)$, and $(3,10)$, the pupils will soon note that the points are not all on one line, but that they all lie above the line which is the graph of the open sentence "$y = 3x$", since along the y-axis "greater than" means "above". The open sentence whose graph is the set of points for which the ordinate is greater than 3 times the abscissa is "$y > 3x$".

Answers to Problem Set 4-1

1. The open sentence whose truth set is the set of ordered pairs for which the ordinate is two greater than the abscissa is "$y = x + 2$". The graph of the set of points associated with this set of ordered pairs is the line shown in the figure.

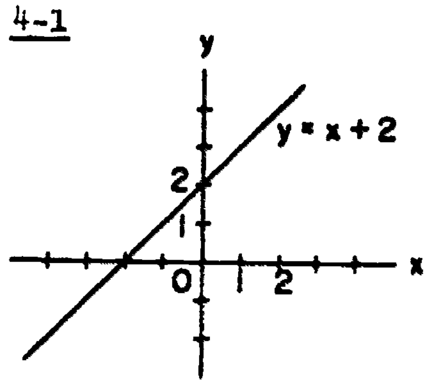


Figure for Problem 1

It is not possible to draw the graphs of both of the sentences "$y > x + 2$" and "$y \geq x + 2$", because in the first one the line whose equation is "$y = x + 2$" is dotted, and in the second one it is a solid line.

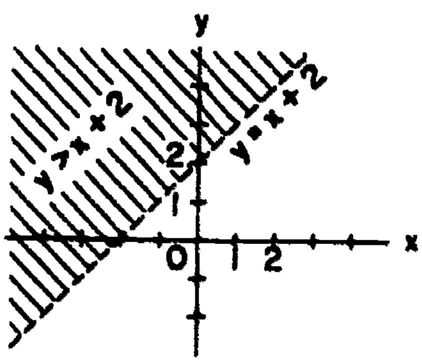


Figure for Problem 1(a)

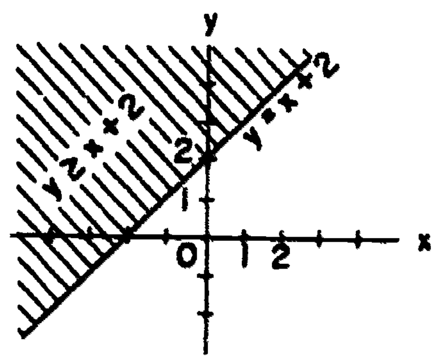


Figure for Problem 1(b)

2. (a)

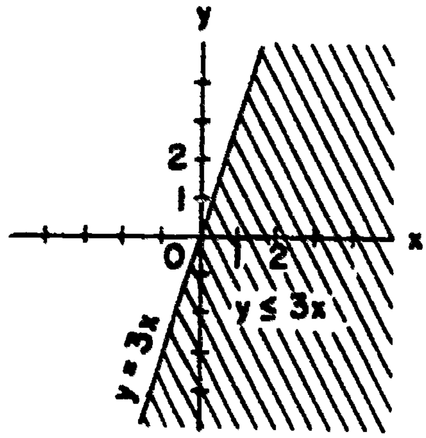


Figure for Problem 2(a)

(d)

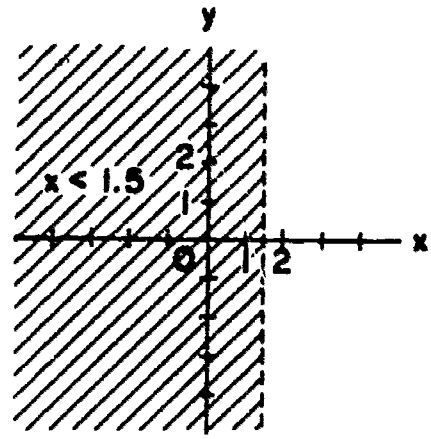


Figure for Problem 2(d)

(b)

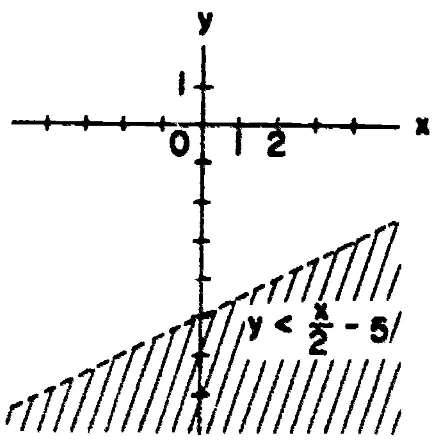


Figure for Problem 2(b)

(e)

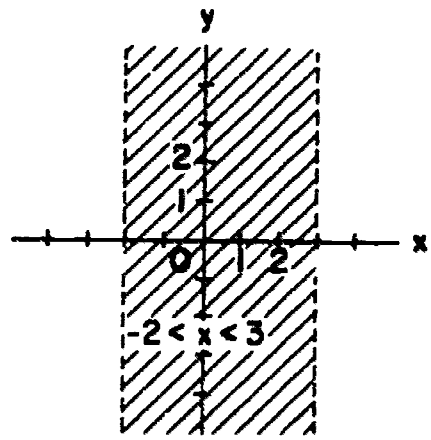


Figure for Problem 2(e)

(c)

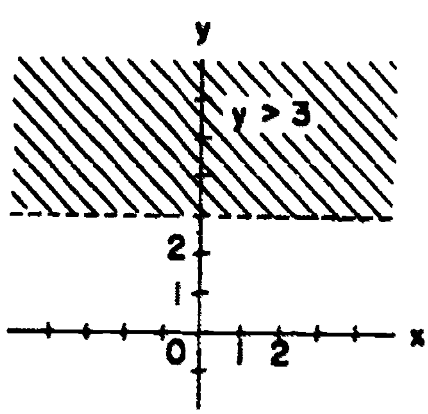


Figure for Problem 2(c)

(f)

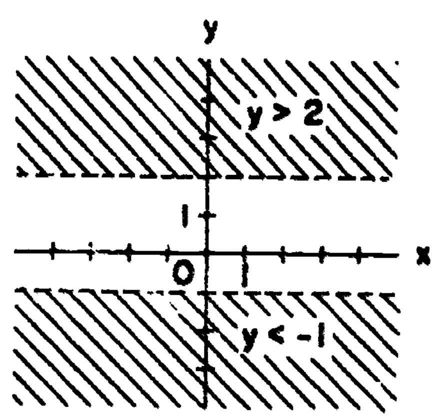


Figure for Problem 2(f)

3. (a)

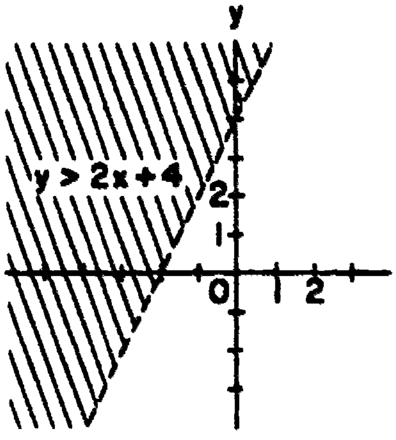


Figure for Problem 3(a)

(c)

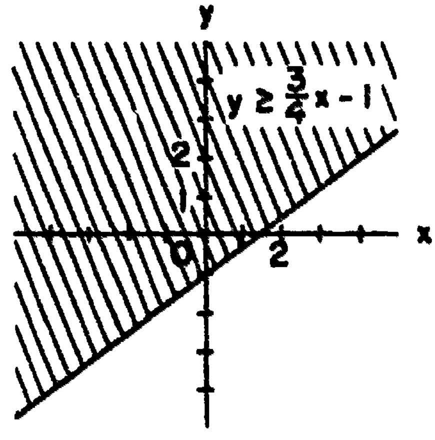


Figure for Problem 3(c)

(b)

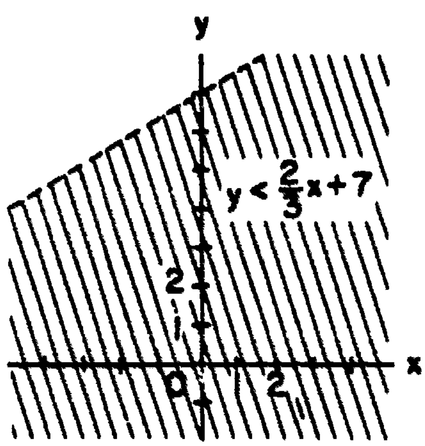


Figure for Problem 3(b)

(d)

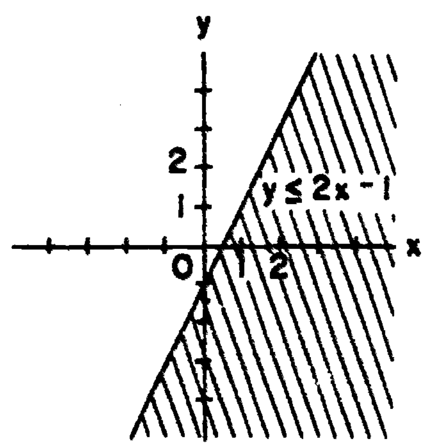


Figure for Problem 3(d)

69

4\. (a)

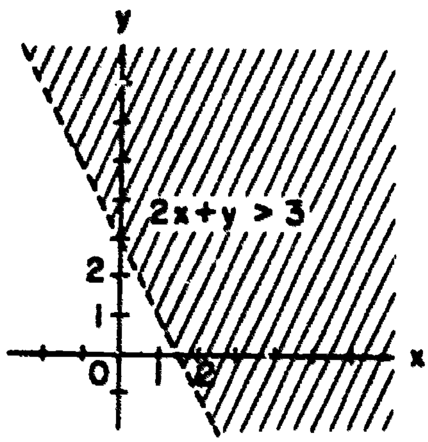


Figure for Problem 4(a)

(c)

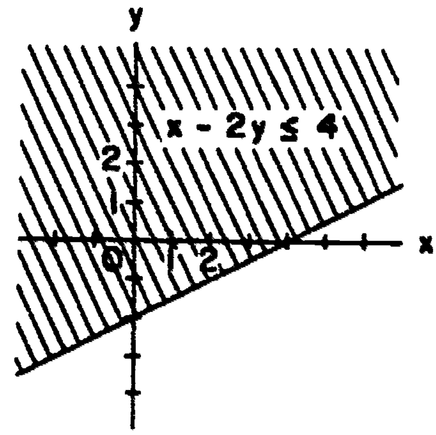


Figure for Problem 4(c)

(b)

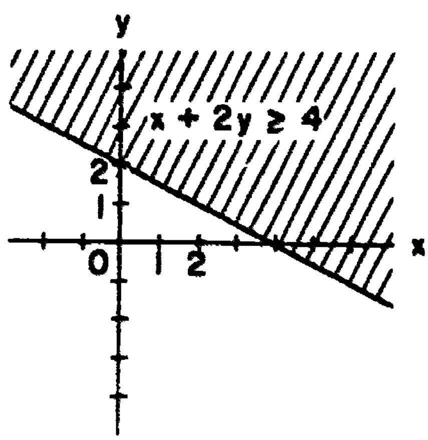


Figure for Problem 4(b)

(d)

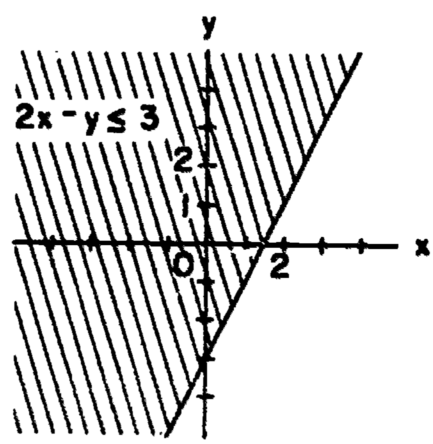


Figure for Problem 4(d)

5\. (a) $y \leq \frac{3}{2}x + 2$

(b) $y > -x + 4$

4-2. <u>Graphs of Open Sentences Involving Integers Only</u>.

This section is included because it is hoped the pupil will realize that open sentences do not necessarily include all real numbers as possible members of their truth sets, and will recognize the corresponding situation so far as the graphs are concerned.

The teacher will find additional material on this particular topic under the heading of "Lattice Points" in many algebra texts or texts on elementary functions.

Each ordinate is one-third the corresponding abscissa. Hence, we get ordered pairs of integers only for abscissas which are multiples of 3. 1 and 2 are not multiples of 3, so they cannot be abscissas.

Answers to Problem Set 4-2

1. (a)

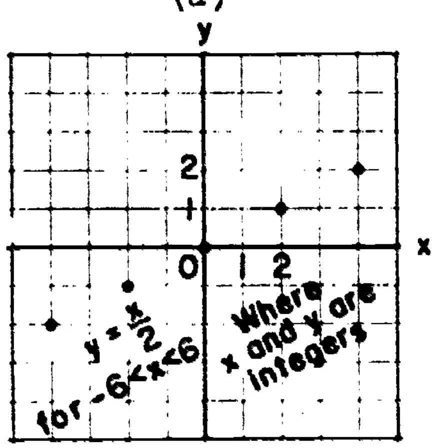


Figure for Problem 1(a)

(b)

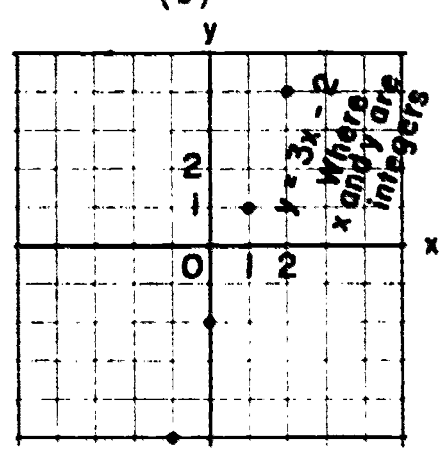


Figure for Problem 1(b)

(c)

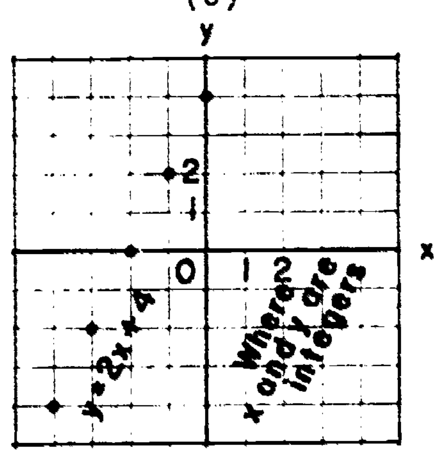


Figure for Problem 1(c).

2. (a)

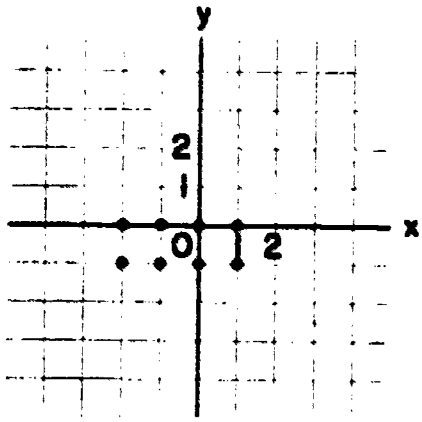


Figure for Problem 2(a)

(b)

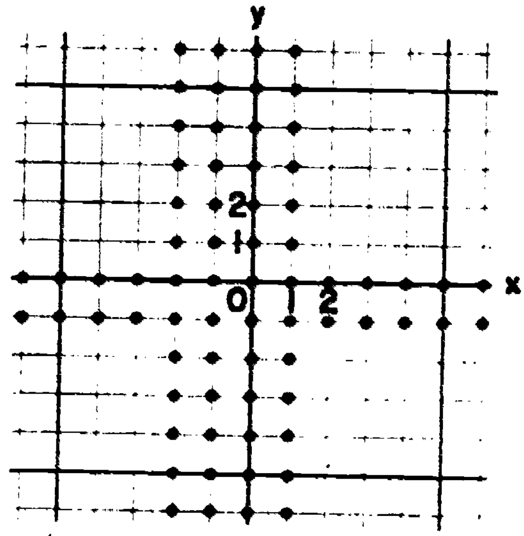


Figure for Problem 2(b)

(c)

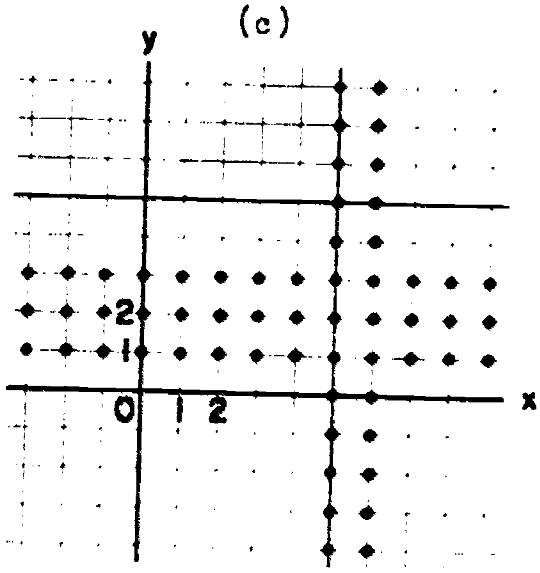


Figure for Problem 2(c)

(d)

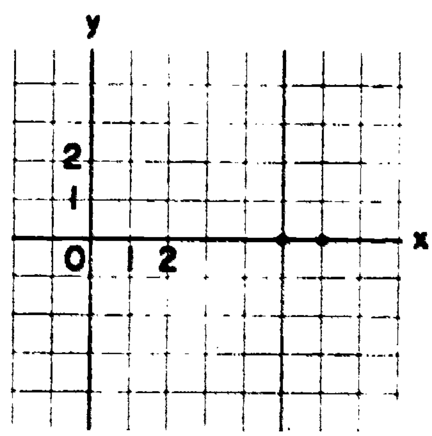


Figure for Problem 2(d)

3. $-2 < x < 0$ and $2 < y < 4$ where x and y are integers.

4. (a) $y = -3x + 1$, where x and y are integers.

   (b) $4 < x < 8$ and $-2 < y < 2$, where x and y are integers or $5 \leq x \leq 7$ and $-1 \leq y \leq 1$, where x and y are integers.

   (c) $y = -x - 3$ for $-8 < x < -4$, where x and y are integers.

   (d) $x = -2$ and $2 < y < 7$, where x and y are integers.

   (e) $x = -4$ or $-7 < y < -2$, where x and y are integers.

   (f) $-2 < x < 2$ and $-4 < y < 3$, where x and y are integers.

   (g) $x < 6$ and $y > 0$ and $y \leq x$, where x and y are integers; or $x \leq 5$ and $y \geq 1$ and $y \leq x$, where x and y are integers.

   (h) $x > -6$ and $y < 6$ and $y \geq x + 6$, where x and y are integers.

4-3. Systems of Inequalities.

The system of inequalities denoted by

$$\begin{cases} x + 2y - 4 > 0 \\ 2x - y - 3 > 0 \end{cases}$$

is a shorthand for the compound open sentence

$$x + 2y - 4 > 0 \ \underline{\text{and}} \ 2x - y - 3 > 0.$$

The graph of this system is, of course, the graph of the compound open sentence.

In drawing the graph of an inequality

$$Ax + By + C > 0, \quad (B \neq 0)$$

write the inequality in the form

$$(1) \quad y > -\frac{A}{B}x - \frac{C}{B} \quad (B > 0),$$

$$(2) \quad y < -\frac{A}{B}x - \frac{C}{B} \quad (B < 0),$$

whichever is appropriate.

On the line

$$(3) \quad Ax + By + C = 0,$$

$y = -\frac{A}{B}x - \frac{C}{B}$. Thus, for a given value of x, (x,y) satisfies equation (1) if the point (x,y) is above the line, since the ordinate of this point is greater than $-\frac{A}{B}x - \frac{C}{B}$. On the other hand, for a given value of x, (x,y) satisfies equation (2) if the point (x,y) is below the line. In general, the graph of (1) is the set of points above the line (3); the graph of (2) is the set of points below the line(3).

The graph of the system

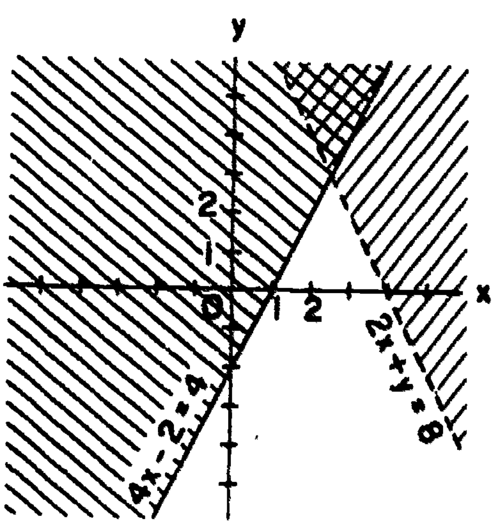
$$\begin{cases} 3x - 2y - 5 = 0 \\ x + 3y - 9 \leq 0 \end{cases}$$

is the portion of the line $3x - 2y - 5 = 0$ which is below or on the line $x + 3y - 9 = 0$.

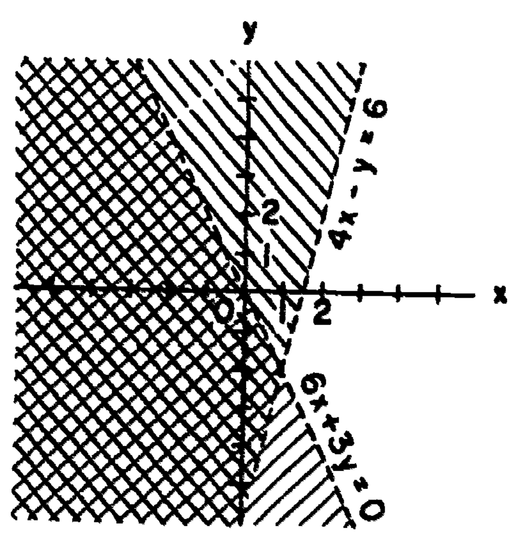
Answers to Problem Set 4-3a

The truth set of each open sentence in Problems 1, 2, 5, 6, 7 consists of all points in the doubly shaded regions of the graph, together with all points of the solid line boundaries of these regions. In Problems 3 and 4, the truth set is the portion of the solid line inside the shaded region.

1.

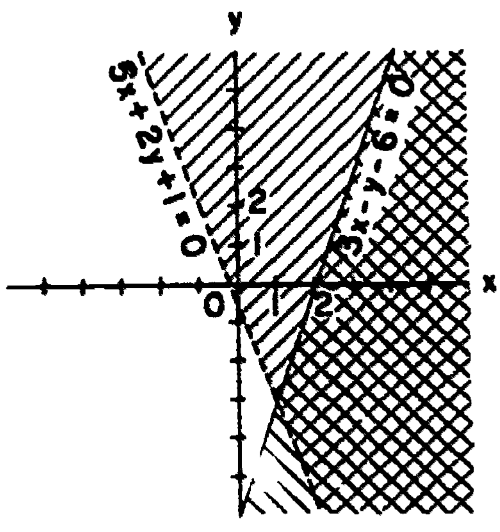


2.

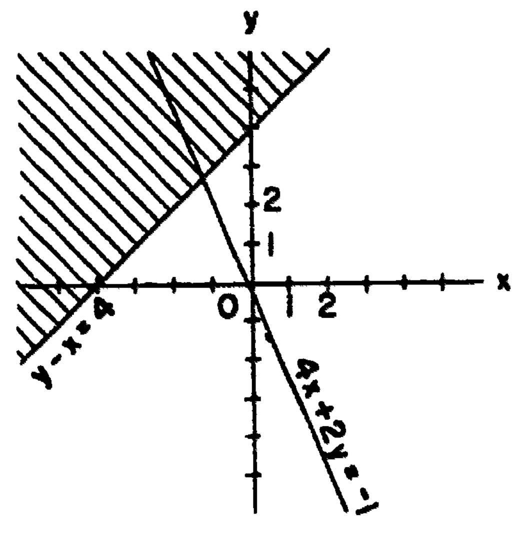


7?

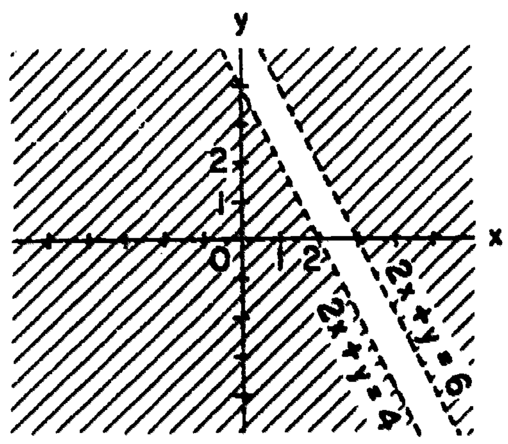
3.



4.

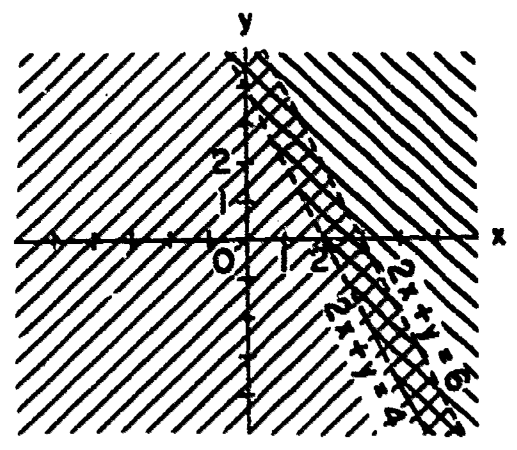


5.

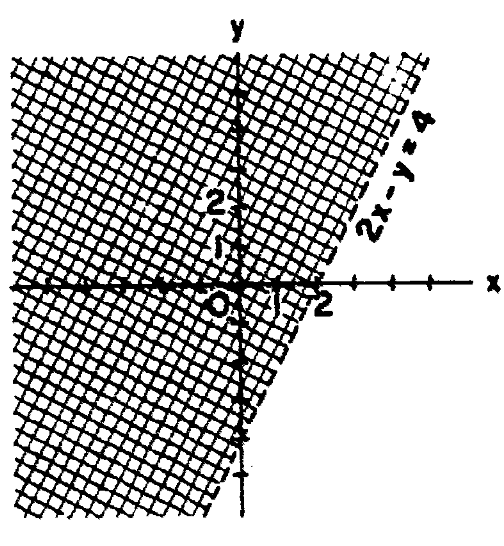


The empty set, $\emptyset$.

6.



7.



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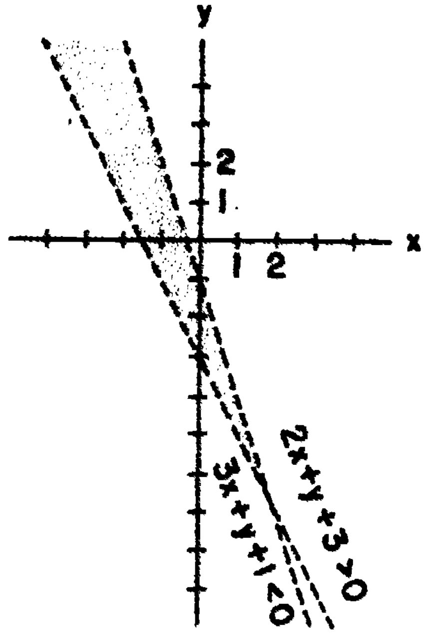
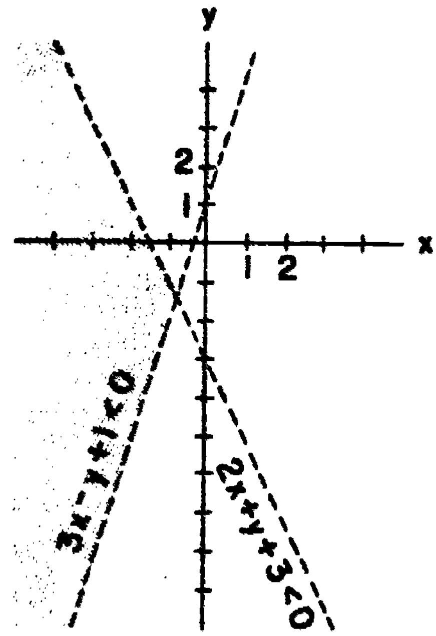
Answers to Problem Set 4-3b

The graphs of all the open sentences in Problems 1-3 of this set of problems consists of all points in all the shaded areas together with all points of the solid line boundaries of these regions. In Problem 4, the graph is the doubly shaded region.

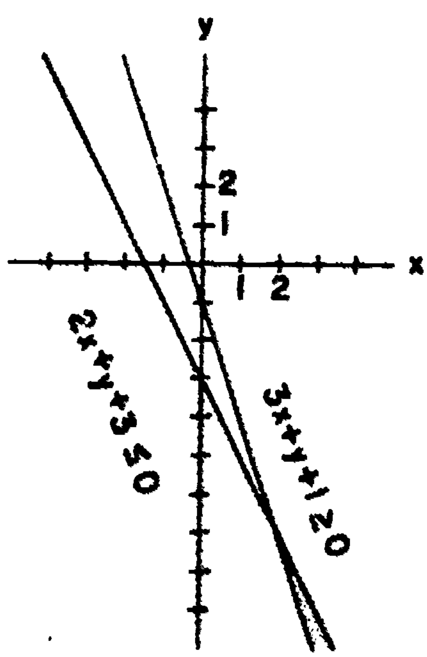
1.



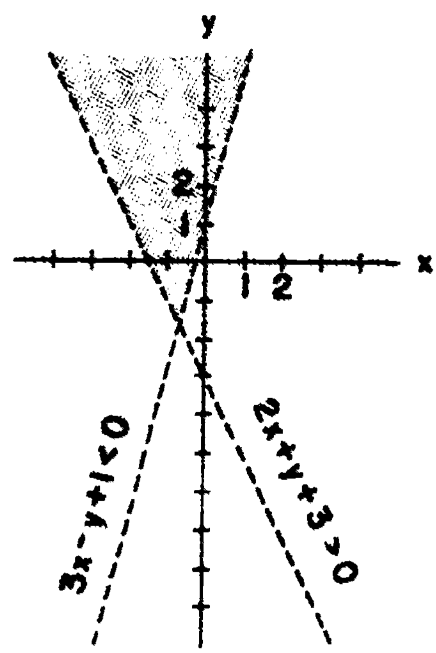
2.



3.



4.



The graph of

$$(x - y - 2)(x + y - 2) < 0$$

is the graph of

$$x - y - 2 < 0 \quad \text{and} \quad x + y - 2 > 0$$

or

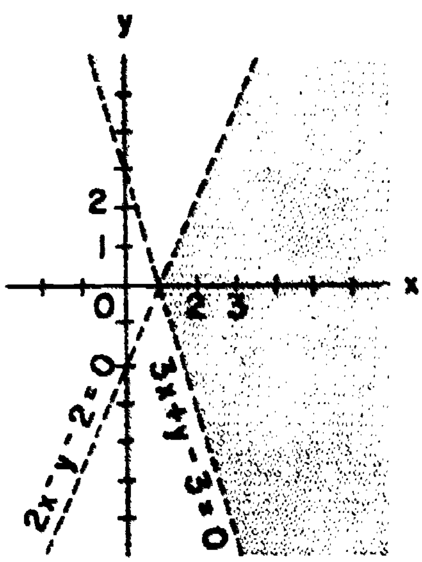
$$x - y - 2 > 0 \quad \text{and} \quad x + y - 2 < 0.$$

The graph consists of the two regions which are singly shaded.

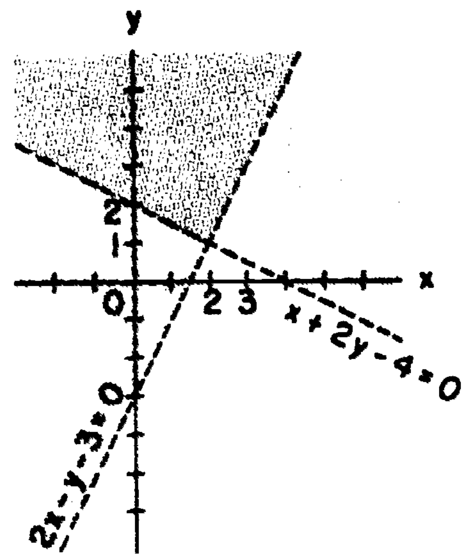
Answers to Problem Set 4-3c

The graphs of the truth sets here consist of all points in the unshaded and the doubly shaded regions of the figures.

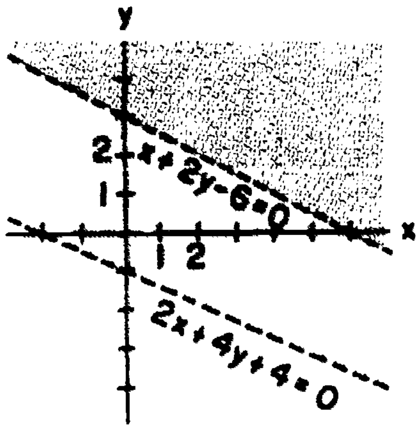
1. (a)



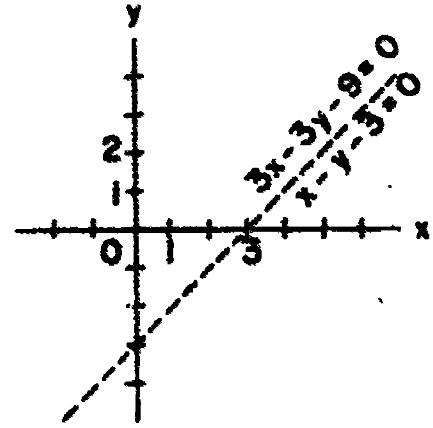
(b)



(c)

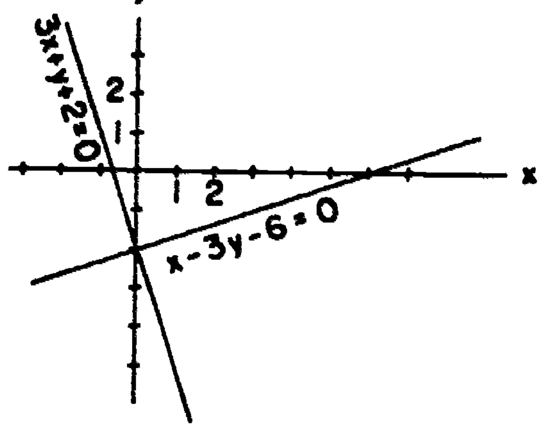


(d)



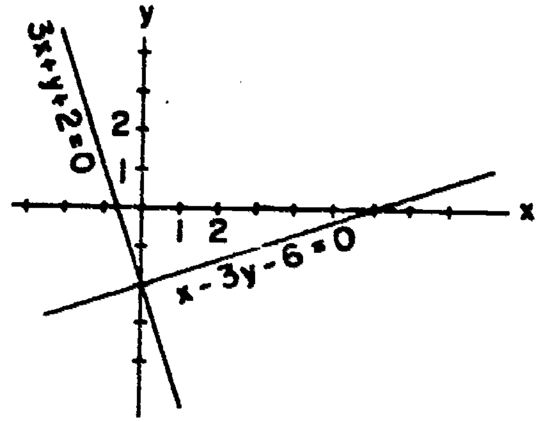
The null set, ∅.

2. (a)



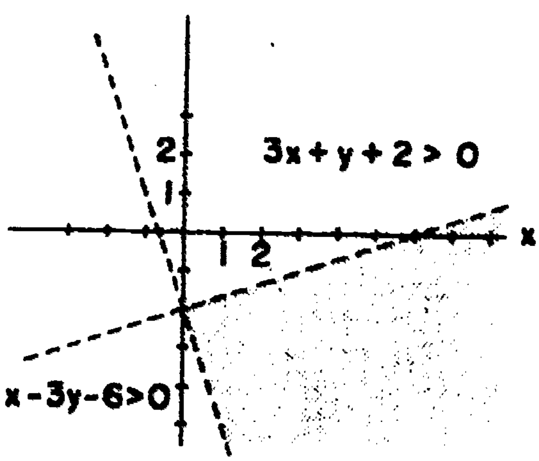
Truth set: {0,-2}

(b)



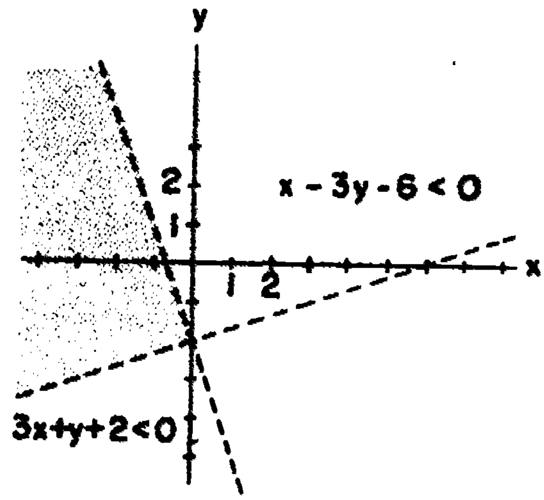
Truth set:
All points on both lines.

(c)



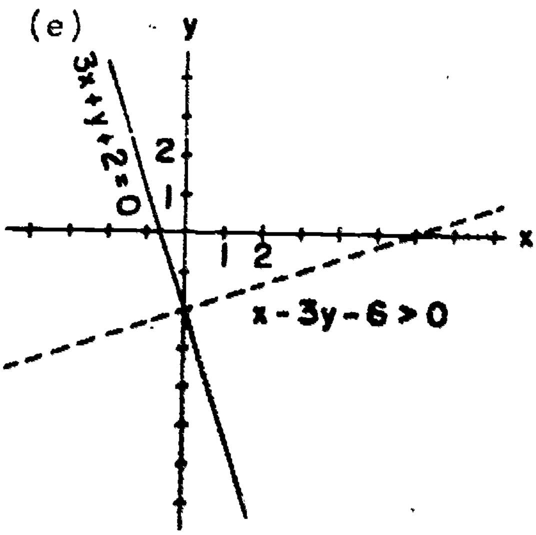
The truth set is doubly shaded region.

(d)



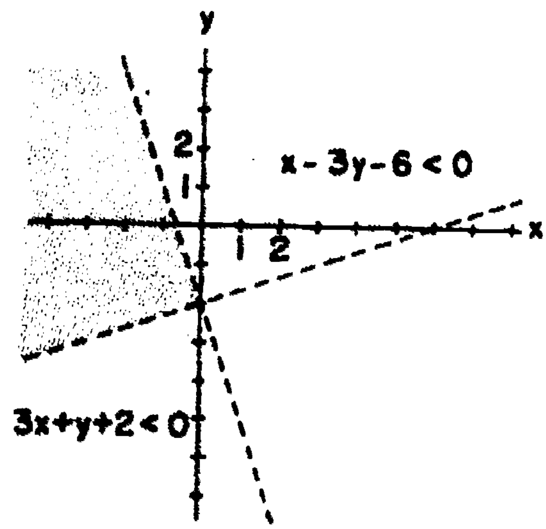
The truth set is doubly shaded region.

(e)



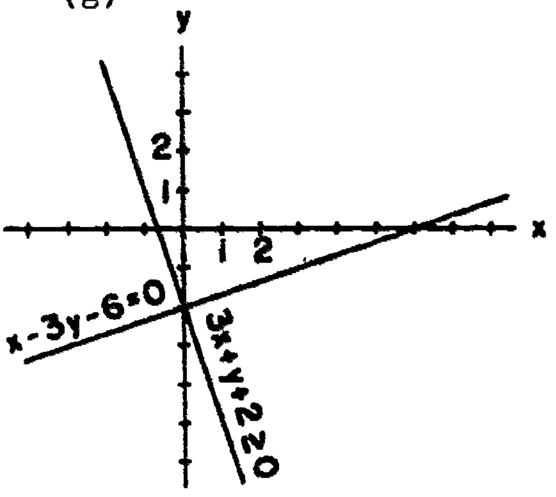
The truth set is set of points on solid line within shaded region.

(f)



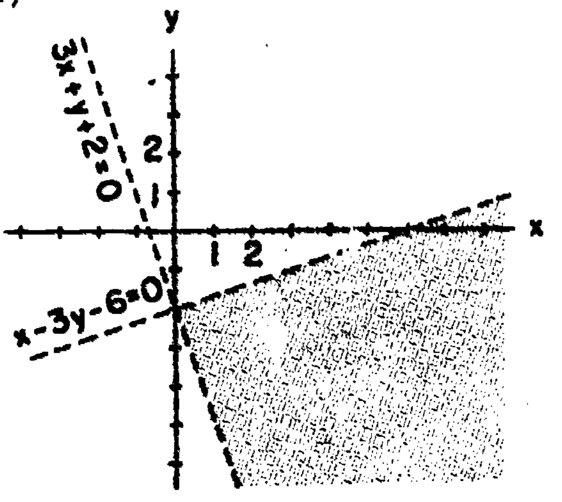
The truth set is whole shaded region.

2. (g)



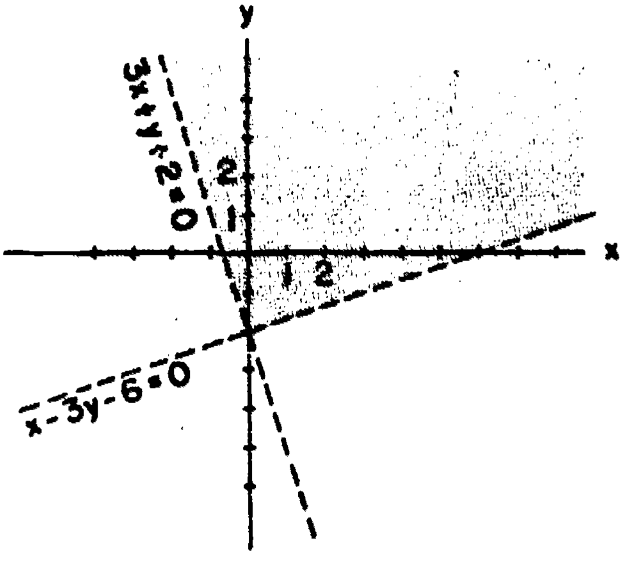
Truth set is whole shaded area and both lines.

(h)



Truth set is the doubly shaded and the unshaded region.

(i)



Truth set is the doubly shaded and the unshaded regions.

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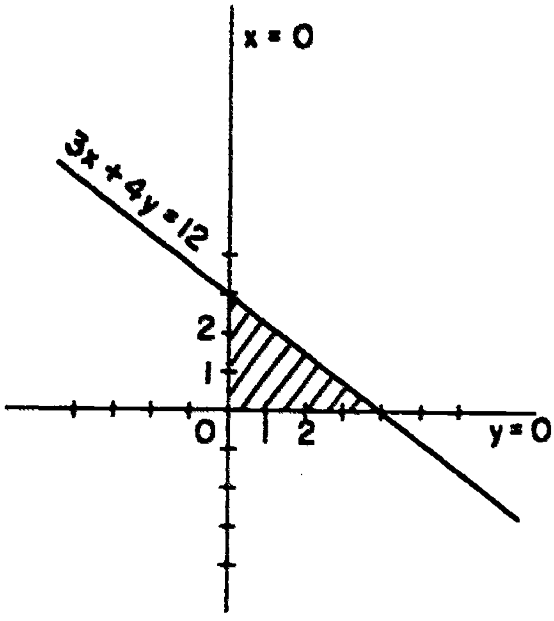
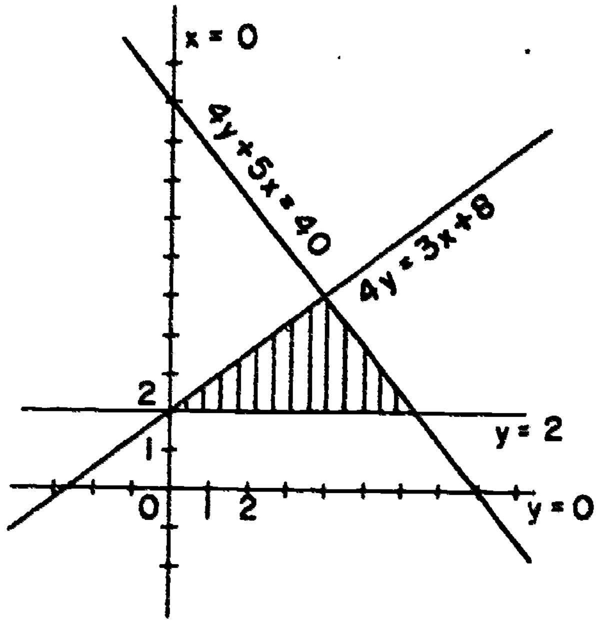
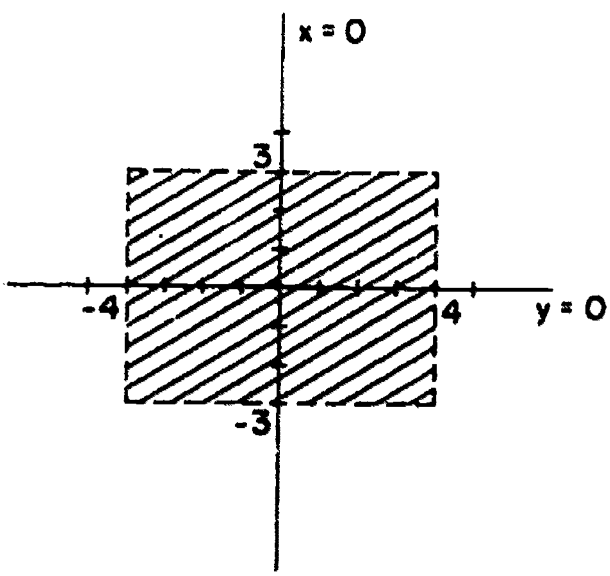
3. (a)



(b)



(c)



4. If r is the number of running plays and p is the number of pass plays, then $3r$ is the number of yards made on r running plays and $20(\frac{1}{3})$ p is the number of yards made on p passing plays. Since the team is 60 yards from the goal line,

$$3r + \frac{20}{3}p \geq 60$$

if they are to score.

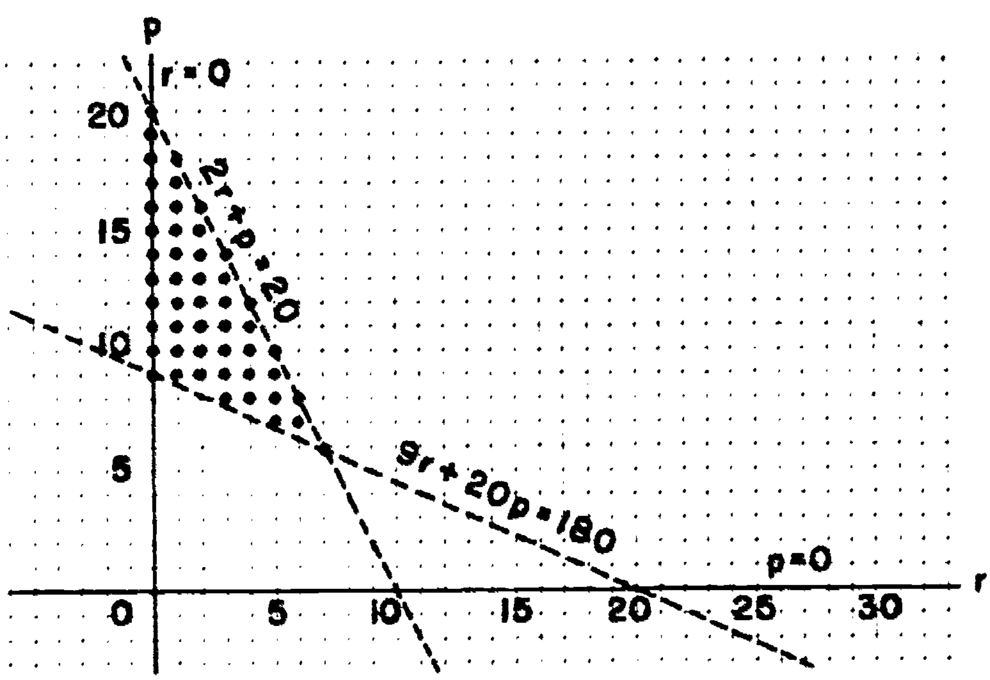
$30r$ seconds are required for r running plays, and $15p$ seconds are required for p passing plays; therefore,

$$30r + 15p \leq 5(60).$$

These two inequalities give the equivalent system

$$20p + 9r \geq 180 \quad (p \text{ and } r \text{ are non-negative integers.})$$
$$p + 2r \leq 20$$

The graph of this system is:



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It is evident there are 48 different combinations of r and p which will assure success; for example, 2 running and 10 passing, etc. However, there are some combinations which leave a smaller time remaining, thus giving the opponents less time to try to score. These are the points of the graph nearest the line $p + 2r = 20$.

This by no means exhausts the possibilities for graphing inequalities. If the pupil has had absolute value, there are many interesting graphs, such as: $y > |x|$; $|x| + |y| < 4$; $|x| - |y| < 4$; $\frac{x + y}{x + |y|} = 1$. If the pupil has had functions with one or two variables of second degree, there are many unusual graphs, such as $y > 3x - x^2$; $x^2 + y \leq 25$; $(x+4)^2 \geq x^2 + y^2$.
