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ABSTRACT

This is part three of a three-part manual for teachers using SMSG junior high school text materials. Each chapter contains an introduction and a collection of sample test questions. Each section contains a discussion related to the topic at hand and answers to all the exercises. Chapter topics include: (1) non-metric geometry; (2) volumes and surface areas; (3) the sphere; and (4) relative error. (MP)

MATHEMATICS FOR JUNIOR HIGH SCHOOL

Commentary for Teachers

Volume II (Part 3)

(preliminary edition)

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NOTE TO TEACHERS

Based on the teaching experience of over 100 junior high school teachers in all parts of the country and the estimates of authors of the revisions (including junior high school teachers), it is recommended that teaching time for Part 3 be as follows:

Chapter	Approximate number of days
10	10
11	13
12	11
13	7
Total	41

Throughout the text, problems, topics and sections which were designed for the better students are indicated by an asterisk (*). Items starred in this manner should be used or omitted as a means of adjusting the approximate time schedule.

Chapter 10

NON-METRIC GEOMETRY

The material of this chapter will be new to almost all pupils. Most of it has been tried in a number of eighth grade classes with rather surprising success. There are a number of reasons for including it in the eighth grade curriculum. Among these are:

1. It helps develop spatial intuition and understanding.
2. It emphasizes in another context the role of mathematics in reducing things to their simplest elements.
3. It affords other ways of looking at objects in the world about us and raises fundamental questions about them.
4. It illustrates types of mathematical (geometric) reasoning and approaches to problems.
5. It gives an interesting insight into the meaning of dimension.

The general purposes of this Chapter are similar to those for Chapter 4 of Volume I. It is suggested that teachers read the beginning of the Commentary for Teachers of that chapter.

In this Chapter there is more emphasis on the use of models in understanding spatial geometry. There is correspondingly less emphasis on "sets" as such. This Chapter is more geometry than "sets." However, the terminology of sets, intersections, and unions is used throughout this Chapter. The students should already have some familiarty with these ideas. A quick run-over of the ideas of Chapter 4 of Volume I would be useful as background for most students. However, if sets, intersections, unions, and some geometrical terms can be explained as you go along, this material should be teachable without much specific background on the part of students.

The use of terms like "simplex" and the distinctions between 2- and 3-dimensional polyhedrons should contribute to precision of thought and language.

Reading. The text is written with the intention that the pupils read (and study) it. There may be occasional passages which some pupils will not follow easily. Pupils should be encouraged to read beyond these passages when they occur, and then go back and study them. Teachers will find it profitable to read some of the material with the pupils.

Materials. In teaching subject matter like this to junior high school students, models are of considerable use. Familiarity and facility with models should increase teaching effectiveness as well as improve basic understandings. The students should be encouraged to use cardboard or oak tag for their models of tetrahedrons and cubes. They will be asked to draw on the surfaces of these later. It is suggested that each student contribute one model of a regular tetrahedron and one of a cube for later class use. They will be needed later in the study of polyhedrons. Each student should keep a model of a cube and a model of a tetrahedron at home for use in homework. In this Commentary for Section 10-4 and 10-7 there are several suggestions for making models from blocks of wood. The boys in the class should be encouraged to make these.

Time. The material of this unit is recommended for study for a period of about two weeks. If class interest is high, some extra time could profitably be used.

A Word of Warning. In preparing themselves on this material, teachers should not expect to master it at one reading. Terminology and points of view can be assimilated gradually. Why not take the attitude of exploring the material together with the students?

10-1 Tetrahedrons

It is probably desirable for the teacher to follow the instructions in the text to make in class either a model of a regular tetrahedron or a model of a non-regular tetrahedron or both. Have each student leave one model of a tetrahedron with you. Have them keep one model of a tetrahedron to use at home.

Some boys might like to make models of tetrahedrons by sawing wood blocks.

Answers to Questions in Text

The edges of (PQRS) are (PQ), (PR), (PS), (QR), (QS), and (RS). The faces of (PQRS) are (PQR), (PQS), (PRS), and (QRS).

Answers to Exercises 10-1

1. (Construction)
2. (Construction)
3. The measure of the new angle DAC must not be more than the sum of the measures of angles BAC and BAD nor less than their differences. Otherwise, it won't fit. (This could be tried in class.)

10-2 Simplexes

In Chapter 4 we regarded points, lines, and planes as our "building blocks". Here (with some prior notion of lines and planes) we will use points, segments, triangular regions, and solid tetrahedrons as our "building blocks".

The discussion about "taking points between" and "dimension" probably will need to be read more than once by pupils. After one gets the general idea, the material is quite readable. A class discussion of the idea prior to the reading of the text by the pupils may be a good idea.

The discussion of "betweenness" illustrates an important fact about reading some mathematics. One sometimes has an easier time reading for general ideas and then rereading to fill in details.

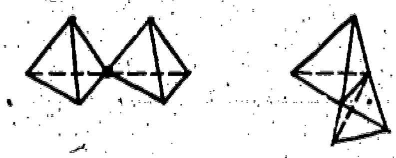
Answers to Exercises 10-2

1. (a) 3

   (b) 3

2. 2

3,4. The better students might be encouraged to draw figures for each of these problems, as:



5. The vertices of a 2-simplexes are blue.
   The edges will be red.
   The interior will be green.
   No. Just two.

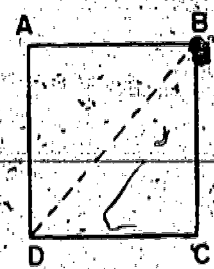
6. The vertices of the model are blue.
   The edges will be red.
   The faces will be green.
   The solid tetrahedron.

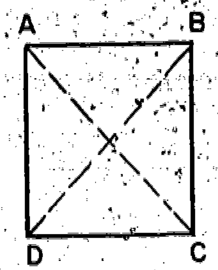
Note: When we say "color points" what we mean, of course, is "color the representations of the points on the paper or model".

Answers to Exercises 10-3

1. Constructions

2. There are a number of possible answers ... (PABD) and (PBCD) could be two of the 12 if one face is labeled as on the right.





3. 24 2-simplexes.

4. 24

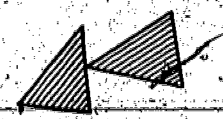
Yes, you can make the comparison. The top vertices of the six pyramids would correspond to the six points in the middle of the faces of the cube.

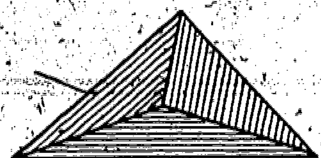
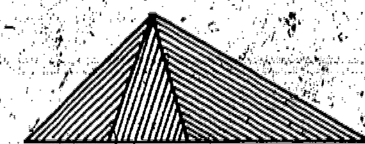
10-4 Polyhedrons

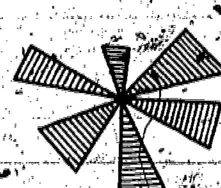
In class it would be a good idea to hold or fasten models of tetrahedrons together to indicate various 3-dimensional polyhedrons. Also, blocks of wood with some corners and edges sawed off (and maybe with a wedge-shaped piece removed) are good models of 3-dimensional polyhedrons.

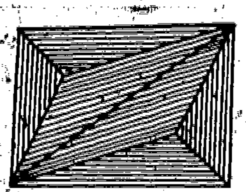
Models of tetrahedrons should be held together to indicate how the intersection of two could be exactly an edge of one but not an edge of the other. (They would not intersect "nicely.") Other examples can also be shown here.

Answers to Exercises 10-4

1. (a)  (b) 

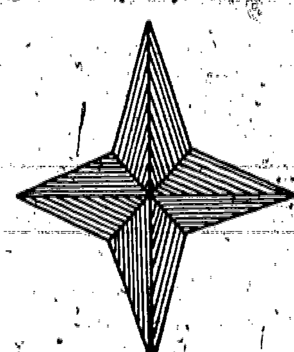
2.  or 

3. (a)  (b) 

   (c)  (one possible answer)

4. Draw the segments (BD), (GE), and (HE).

5. (a) Draw segments (BD), (BH), (DF), (FH), and either (BF) or (HD).

   (b) 

6. (a) (PQR), (XYZ), (XPQ), (XYQ), (YZQ), (ZQR), (XZP), and (PRZ).

   (b) (FPQR), (FXYZ), (FXPQ), (FXYQ), (FYZQ), (FZQR), (FXZP) and (FPRZ).

   (c) (QXYZ), (PXZQ) and (RPQZ)

10-5 One-Dimensional Polyhedrons

The pupils may enjoy drawing all sorts of simple closed polygons. Some should be in a plane. But they also should be encouraged to do some on surfaces of cubes, tetrahedrons, wooden blocks, etc.

Answers to Questions in Text in 10-5

Other polygonal paths from P to S are PRS, PRQS, PQS, PQRS. Five in all. Other simple closed polygons containing BE and GA are:

BEDFGAB
BEDCHFGAB
BEGAHCB
BEGAHFDCB
BEDFGAHCB

Answers to Exercises 10-5

1. 2 paths from A to B.
   3 simple closed polygons.

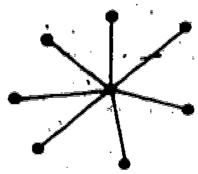
2. (a) 7

   (b) PQSP, PRSP, PQRP, QRSQ, PQRSP, PRQSP, PRSQP.

   (c) PQRSP (We are naming the vertices in order returning to a starting point.)

3. PABCDEFQ, PCBAFEDQ, PEDCBARQ (there are others)

4.

 or  or such.

5. There are many ways of doing it.

10-6 Two-Dimensional Polyhedrons

This section gives a good opportunity to draw a lot of 2-dimensional simplexes and polyhedrons and to illustrate again sets of 2-simplexes which intersect nicely.

Some drawing and illustrations should be in the plane and some should be on surfaces or 3-dimensional polyhedrons.

Answers to Questions in Text, 10-6

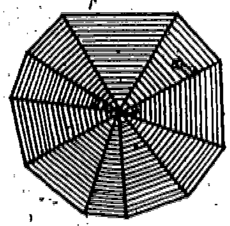
Students are asked if they see a relationship among the faces, edges, and vertices of certain polyhedrons. It would be difficult to discover the Euler relationship from just three cases. The relationship will be developed in detail in Section 10-8.

Answers to Table and to Exercises 10-6

1.

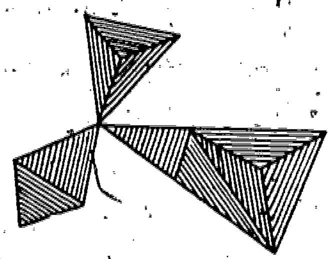
	F	E	V
Tetrahedron	4	6	4
Cube	6	12	8
Cube (with simplexes)	12	18	8

2. (a)



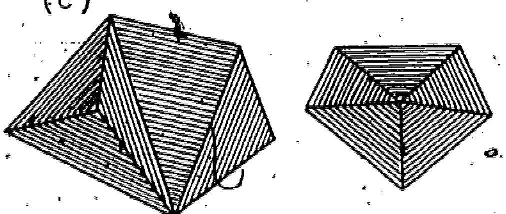
and many other possibilities.

(b)



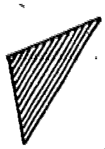
and many other possibilities.

(c)

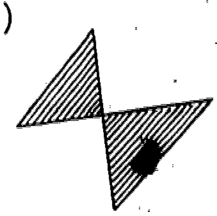


and many other possibilities.

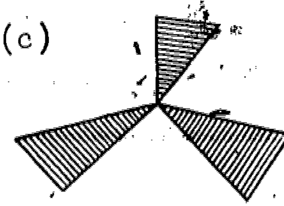
3. (a)



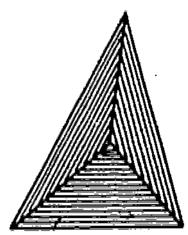
(b)



(c)



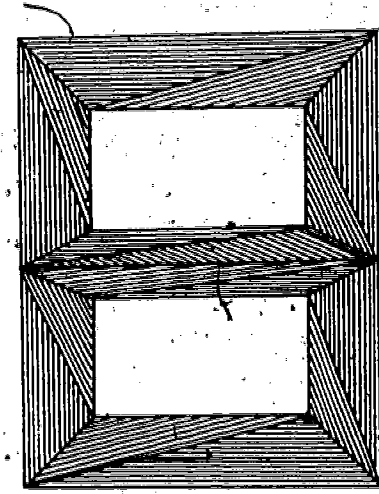
4. No, there won't be four. The union of the first three must itself be a 2-simplex. Yes, a tetrahedron.



5. F = 24 E = 36 V = 14.

6. 6 sets of 4-simplexes each.
12 original edges, and 6 sets of 4 new edges each.

7. This can be done in many ways. We have first subdivided it along AB into two figures like the second in the text of this section.



10-7 Three Dimensional Polyhedrons

In this section and the next section there are many fine opportunities to use the models which the pupils have prepared. It is suggested that the class be divided into three or four parts and that each part take several models of regular 3" tetrahedra and fasten them together (with cellulose tape or paste) to produce some rather peculiar looking 3-dimensional polyhedrons. They can also fasten models of cubes together to produce odd looking polyhedrons.

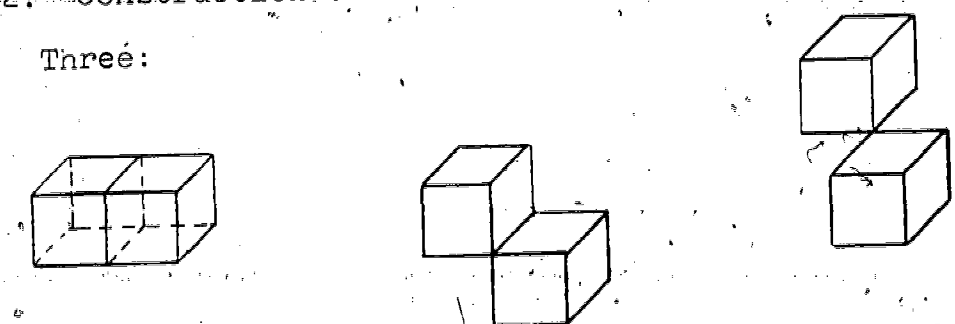
There is a good opportunity here to encourage some of the boys in the class to make models of 3-dimensional polyhedrons out of blocks of wood. Start with a block of wood and saw corners or edges off of it and possibly notches out of it. The surface you get will be a simple surface as long as no holes are punched through the solid. You could not punch holes through with an ordinary saw. Be sure that the surface is made up only of flat portions. A surface like this could be covered with paper and then colored or drawn on. It could then be re-covered with paper for further use. It is interesting that no matter how the block is cut up (without holes) every simple closed polygon on it will separate the surface into just two pieces. Also the models can be used for examples of the Euler Formula in the next section. The faces, edges, and vertices can be counted rather easily on objects like this.

Make a model of a "square doughnut" out of 8 cubes as suggested in the text. The surface will not be simple.

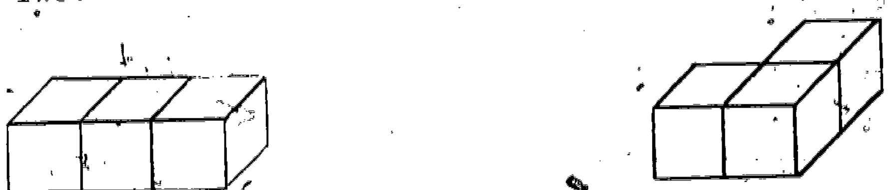
Answers to Exercises 10-7

1, 2. Constructions.

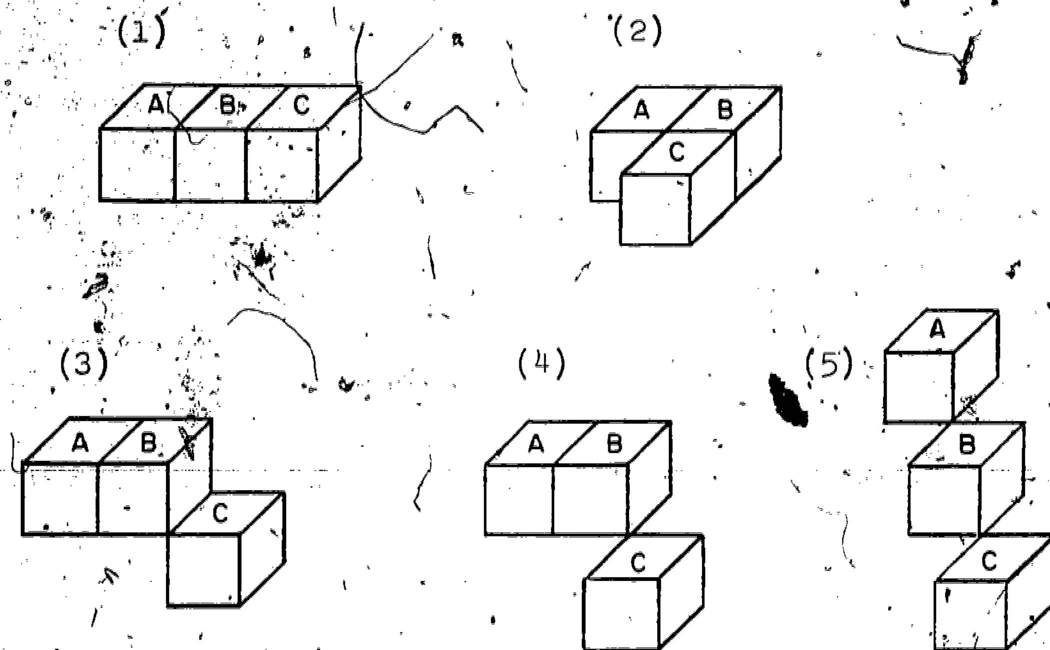
3. Three:



4. Two:



5. (a) Five of the many possibilities are as follows:



Intersections:

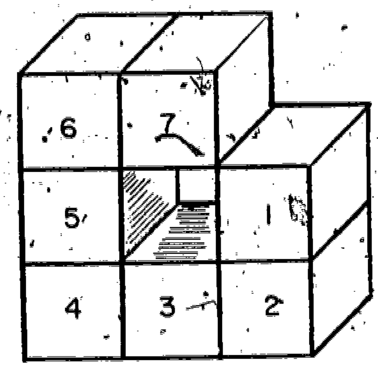
	A and B	A and C	B and C	A and B and C
1)	face	∅	face	∅
2)	face	edge	face	edge
3)	face	∅	edge	∅
4)	face	∅	vertex	∅
5)	vertex	∅	vertex	∅

∅ represents the empty set.

(b) No

(c) Yes.

*6.



The intersection of cube #7 with cube #6 is a face and the intersection of cube #7 with cube #1 is an edge. Thus the figure is not a "simple surface."

10-8 Counting Vertices, Edges, and Faces--The Euler Formula

Use models of tetrahedrons and cubes in explaining the material of this section to the class.

The counting of faces, edges, and vertices on any simple surface should be interesting and informative for the pupils. Use models made of wood for some of the examples.

Use the model of the "square doughnut" of the last section to help pupils count faces on this type of surface.

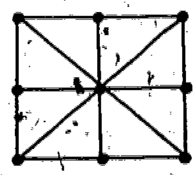
The teacher should assign particular problems on counting faces, edges, and vertices as appropriate for the particular class.

The teacher should subdivide the surface of a model of a cube in an irregular way and count faces, vertices, and edges as indicated in the text.

Answers to Exercises 10-8

1. The count is 12 faces, 18 edges, and 8 vertices. The 2-simplexes of the subdivision of the tetrahedron can be corresponded to the faces of the solid which is the union of the five tetrahedron.

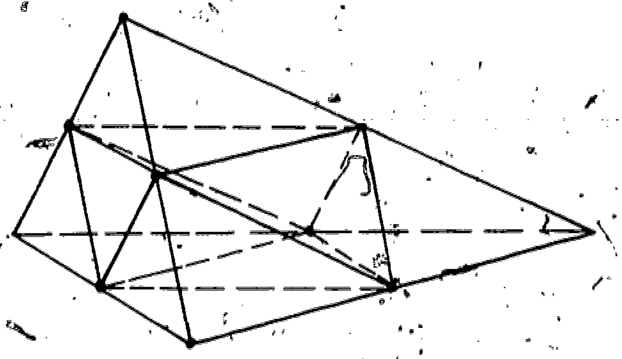
2. A face looks like →



There are:
[illegible] vertices,
48 faces,
72 edges (48 on interiors of faces of cube and 24 on edges of cube).

3. V + F - E should be 2.

4.



V = 10 V + F - E = 10 + 16 - 24 = 2.
F = 16
E = 24

5. (a) V = 16
F = 14
E = 28

V + F - E =
16 + 14 - 28 = 2

(b) V = 16
F = 14
E = 28

V + F - E =
16 + 14 - 28 = 2.

6. (a) V = 20
F = 18
E = 36

V + F - E = 2

(b) V = 18
F = 16
E = 32

V + F - E = 2.

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Chapter 11

VOLUMES AND SURFACE AREAS

General Remarks

The development of the main portions of this chapter is along empirical and intuitive lines. We have made generous provisions for discovery activities. The entire chapter is built around the construction and frequent use of the various models the pupils are to study.

At the end of this chapter you will find diagrams for constructing models of the geometric solids we will consider. We suggest you use the following steps to be taken in the construction of the models:

(a) Place a sheet of stiff paper such as oak tag behind the diagram in the text.

(b) With a compass point or pin point perforate the text page at all the vertices.

(c) Remove the stiff paper and join the holes by using a sharp pencil and a straight edge so that your diagram resembles that of the text.

(d) With a pair of scissors cut along the boundary lines of the figure or the stiff paper.

(e) Fold your cut-out along the inner edges of your solid.

(f) Use glue to paste the tabs on the inner edges of your solid.

(g) Seal each such edge by using tape, since some models are to be used as containers.

(h) Take good care of your models since they are to be used frequently.

We suggest that you take some time in class helping the pupils construct the first models. Be sure that they construct their models carefully. You will notice that in some cases a base is not sealed since such models will be used as containers. It is convenient at times to attach this base temporarily also.

We have assumed that your pupils are somewhat familiar with the following geometric terms: point, line, line segment, ray, parallel lines, intersecting lines, perpendicular lines, angles, measure of an angle, triangle, altitude and base of a triangle, base and altitude of a parallelogram, square, regular polygon, hexagon, perimeter of a polygon, circumference of a circle, Pythagorean property, congruent triangles, surface areas, and volumes.

You might perhaps take an inventory to see whether or not this is true. If preliminary work needs to be done we recommend Chapters 7, 8, 10, and 11 of SMSG '60 - '61 text for Junior High School Mathematics, Volume I. The subject of measurement is extensively developed in Chapters 7 and 8. The subject of parallels, parallelograms, triangles, and right prisms is covered in Chapter 10, and circles and cylinders in Chapter 11.

If the pupils already know a good deal about areas of plane figures and volumes of right prisms respectively, Sections one and three could serve as a brief review of the main ideas.

If on the other hand your pupils have had little or no preparation for area and volume, Sections one and three should definitely be supplemented. In these sections we have done more reviewing than developing since we were not introducing the ideas of area and volume for the first time. Chapters 7, 8, 10 and 11 of SMSG '60 - '61 Junior High School Mathematics, Volume I, text should most certainly be consulted in expanding upon the brief treatment in Sections 1 and 3 of this unit.

Encourage your pupils to estimate areas and volumes prior to calculations and experiments. We feel that these activities will help them to appreciate more the ideas of area and volume than the results obtained by calculations alone. It will be necessary for you to supply such problems, as they have not been included in the text.

The activities of finding volumes by fitting inch cubes which pupils have constructed, and of pouring sand and salt into the interiors of the various prisms, should help them to understand the concept of volume.

You will notice that we have not done much articulating with the ideas developed in Chapter 10. We suggest you take advantage of these ideas whenever they will help you to clarify and unify the basic ideas presented in this Chapter.

The dimensions were given for the various patterns for your convenience.

It is suggested that 13 days be allowed for this Chapter.

Answers to Review Exercises 11-1a

1. 48 sq. in.
2. $3\frac{3}{4}$ sq. ft.
3. 169 sq. in.
4. $12\frac{1}{4}$ sq. ft.
5. 240 sq. in.
6. 32.5 sq. cm.
7. 23.92 sq. ft.
8. 65.1 sq. cm.
9. 703 sq. ft.
10. Approx. 64 sq. in.
11. (a) 10 sq. in. (b) 70 sq. in.
12. 20 sq. in.

Answers to Exercises 11-1b

1. 76 sq. in.
2. 576 sq. in.
3. 247 sq. cm.
4. Approx. 61 sq. ft.
5. Approx. 12 sq. ft.

*6. 24 in.

*7. (a) 51,480 sq. ft.

(b) $ 15,444.00

Answers to Exercises 11-1c

1. 1,038 sq. in.
2. 2,750 sq. in.
3. 17,400 sq. ft.
4. 3,080 sq. in.

Answers to Exercises 11-1d

1. (a) 25π sq. in. (d) 20.25 sq. in.

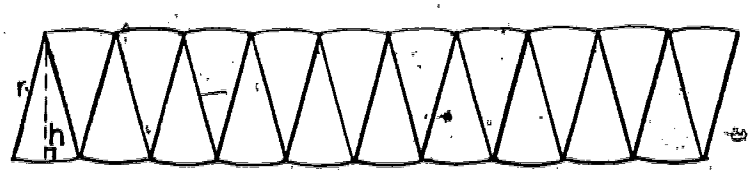
(b) 100π sq. in. (e) 225π sq. in.

(c) 400π sq. in. (f) 196π sq. in.

2. 25π, 100π, 400π.

Doubling the radius quadruples the area.

3. (a) BRAINBUSTER. Arrange the 20 congruent triangles as in the following figure



$$A_{\square} = bh$$

$$h \approx r;\ b \approx \frac{1}{2}C = \frac{1}{2}(2\pi r) = \pi r$$

$$A_{\odot} \approx bh = (\pi r)(r) = \pi r^2$$

(b) We would say that the area of the circle is approximately equal to the area of the circumscribed polygon. It will always be less than the area of the circumscribed polygon since there will always be vertices of the polygon which are not points of the circle. Therefore there is always some portion of the interior of the circumscribed polygon which is not contained in the interior of the circle. However for larger values of n the areas are very close and we can think of the area of the interior of the circle as the lower limit of the area of interior of the circumscribed polygon.

Thus:

(Area of inscribed polygon)<(Area of circle)<(Area of circumscribed polygon). For very large values of n we can replace $(<)$ by $(\approx)$.

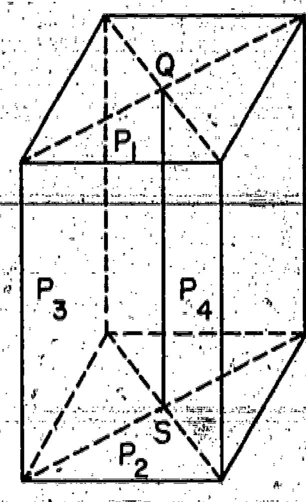
<u>Exercises 11-2</u>

1. Listing examples.
2. Examination of models.
3. Construction and examination of models.
4. ℓ_1 and ℓ_2 cannot be skew lines since they lie in the same plane r. ℓ_1 and ℓ_2 cannot intersect since they lie in parallel planes p_1 and p_2. Hence ℓ_1 and ℓ_2 can only be parallel.
5. (a) A line perpendicular to a plane is perpendicular to all lines in that plane.

   (b) The side opposite the right angle of a right triangle is the hypotenuse of the right triangle.

   (c) The hypotenuse of a right triangle is the longest side of a right triangle.

6. (a) Draw P_3 and P_4 containing $\overline{QS}$.



(b) P_3 and P_4 are perpendicular to P_2 since they contain $\overline{QS}$ a line perpendicular to P_2.

(c) P_3 and P_4 are perpendicular to P_1 since P_1 is parallel to P_2.

(d) $\overline{QS}$ is perpendicular to P_1 since P_3 and P_4 contain $\overline{QS}$.

Exercises 11-3

1. (a) 4 cubic inches
   (b) 10 cubic inches
   (c) $4\frac{1}{2}$ cubic inches
2. (a) 16 square inches
   (b) 28 square inches
   (c) $16\frac{1}{2}$ square inches
3. Surface area is $33\frac{1}{2}$ square inches.
   Volume is 9 cubic inches.
4. Construction. Surface area is 29.7 square inches.
   Volume is 11.5 cubic inches.

5. (a) Perimeter of base of Model 4 is 6 inches.
Perimeter of base of Model 5 is 6 inches.
Perimeter of base of Model 7 is 6 inches.
Perimeter of base of Model 8 is 6 inches.

(b) Yes.

(c) For rhombus based right prism, 7.9 cubic inches
For right rectangular prism, 9 cubic inches
For hexagon based right prism, 10.5 cubic inches
For right circular cylinder, 11.5 cubic inches

(d) No.

(e) See (c)

(f) The interior of a circle has the largest area.

6. Volume for Model 4, 9 cubic inches
Volume for Model 6, 9 cubic inches
Yes. They are equal.
Perimeter of base of Model 4, 6 inches
Perimeter of base of Model 6, $7\frac{1}{4}$ inches
No. The perimeters are not equal.

Exercises 11-4

1. Class activity
2. Yes. It is the perpendicular distance between the bases.
3. No. It is not the perpendicular distance between the bases.
4. Too large, since the lateral edge of an oblique prism is greater than the height.

Exercises 11-5

1. (a-c) Draw line segments $\overline{QS}$ and $\overline{QT}$. Consider triangles AQS and AQT to be right triangles since AQ is perpendicular to the base.
   Now, $\overline{AS}$ and $\overline{AT}$ are hypotenuses of their respective triangles. Therefore $\overline{AS}$ and $\overline{AT}$ are 5 inches in length (3, 4, 5 right triangles)

   (d) Yes.

   (e) Yes.

2. (a) It would work for any regular polygonal region.

   (b) It would apply to any lengths.

   (c) Lateral edges

3. (a) Use the Pythagorean property.

   (b) $(AS)^2 = (AO)^2 + (SQ)^2$

   $(13)^2 = 12^2 + (SQ)^2$

   $169 - 144 = (SQ)^2$

   $25 = (SQ)^2$ same development for QT

   $5 = (SQ)$ $5 = QT$

   (c) Yes. (e) Yes.

   (d) Yes. (f) Yes.

4. (a) No.

   (b) Lateral edges, regular

5. Construction

6. (a) Four. (b) Equal.

7. 600 square inches.

8. (a) 340 square inches.
   (b) 13 inches.
9. (a) 360 square feet.
   (b) $\sqrt{194}$

Exercises 11-6

1. (a) 28 cu. in.
   (b) 800 cu. cm.
   (c) 288,000 cu. ft.
2. 3 cu. in.
3. 12 meters
4. 82,944,000 cu. ft.
   3,072,000 cu. yd.
5. $144\sqrt{3}$ sq. in.
6. The volume is doubled. If the side is doubled and height halved, then the formula

$$V = \frac{1}{3} \times s^2 \times h \text{ becomes } V = \frac{1}{3} \times (2s^2) \times \frac{h}{2}$$
$$= \frac{1}{3} \times 4s^2 \times \frac{h}{2}$$
$$= \frac{2}{3} \times s^2 \times h$$
$$= 2 \times (\frac{1}{3} \times s^2 \times h)$$

Exercises 11-7

1. $T = \pi r^2 + \pi rs$ or $T = \pi r(r + s)$
2. Lateral area is 36π sq. ft., total area is 45π sq. ft.

3. Radius is 9 feet, lateral area is 135π sq. ft., total area is 216π sq. ft., volume is 324π cu. ft.

4. Height is 36 inches, slant height is 39 inches, lateral area is 585π sq. inches.

Sample Questions - Chapter 11

True - False

T 1. A square is a parallelogram.

F 2. A trapezoid may have two pairs of parallel sides.

F 3. If the base and height of a triangle are both doubled, the area is multiplied by 2.

F 4. If the radius of a circle is doubled, the area is also doubled.

T 5. A regular decagon can be separated into ten congruent triangles.

F 6. The bases of a triangular right prism can be only right triangles.

T 7. Any two parallel faces of a rectangular right prism may be considered as the bases of the prism.

F 8. The length of the lateral edges of an oblique prism is the distance between its bases.

T 9. The volume of a pyramid varies directly as its height.

F 10. The volume of a cone is one third that of a cylinder whose bases are congruent to the base of the cone.

T 11. A tetrahedron is always a pyramid.

F 12. A pyramid is always a tetrahedron.

F 13. A regular pyramid must be a tetrahedron.

T 14. Some pyramids have altitudes which do not intersect the closed region of the base.

T 15. The slant height of a right circular cone is never equal to the height of the cone.

Completion Questions

1. If adjacent sides of a rectangle are congruent, the figure is a square
2. If two, and only two, opposite sides of a quadrilateral are parallel, the figure is a trapezoid.
3. If adjacent sides of a parallelogram are congruent, the figure is a rhombus or square.
4. If each side of a rectangle is doubled, its area is multiplied by 4.
5. If the height of a triangle is doubled, its area is multiplied by 2.
6. If an octagon is separated into congruent triangles by connecting the center with each vertex, the area is 8 times the area of each triangle.
7. The area of a polygon inscribed in a circle $<$ ($<$, $>$, or $=$) area of the circle.
8. If the radius of a circle is doubled, the area is multiplied by 4.
9. If the radius of a circle is multiplied by 3, the area is multiplied by 9.
10. If all sides of a polygon have sides of equal measure, it is called a regular polygon.
11. In the blank following the name of each figure write the formula used to find its area.
    (a) Parallelogram $A = bh$
    (b) Rhombus $A = bh$
    (c) Triangle $A = \frac{1}{2}bh$
    (d) Circle $A = \pi r^2$
    (e) Trapezoid $A = \frac{1}{2}h(b_1 + b_2)$
    (f) Regular polygon $A = \frac{1}{2}hp$

12. The length of the base of a rhombus is 12 inches and the area is 6 square inches. The altitude is $\frac{1}{2}$ in.

13. A trapezoid has an altitude of 7 inches, and the average of the two bases is 14 inches. The area is 98 square inches.

14. One of the five congruent triangles into which a regular pentagon is divided has an area of 13.5 square inches. The area of the pentagon is 67.5 square inches.

15. If two planes are parallel the distances from different points of one plane to the other plane are equal .

16. If a line is perpendicular to one of two parallel planes, it is also perpendicular to the second plane

17. The measure of the edge of a cube is 4 inches. The surface area is 96 square inches.

18. The formula for the volume of a cube is $V = s^3$.

19. A triangular right prism has for its bases right triangles the lengths of whose sides are 6 inches, 8 inches and 10 inches. The lateral edges measure 15 inches. The surface area is 408 square inches.

20. The volume of the prism in Problem 9 is 360 cubic inches.

21. An oblique prism has square regions for bases whose edges are 3 inches in length. The distance between the bases is 7 inches. The volume is 63 cubic inches.

22. The radius of the base of a right circular cylinder is 8 feet and its altitude is 35 feet. The surface area is 688π square feet.

23. In Problem 22, the volume is 2240π cubic feet.

24. In Problem 22, if the cylinder is open at one end the surface area is 624π square feet.

25. The following solids have bases that are equal in measure, and volumes that are equal in measure. How do their altitudes compare? (Use <, >, and =).

(a) A cylinder and a cone.
The altitude of the cylinder < the altitude of the cone.

(b) A prism and a cone.
The altitude of the prism < the altitude of the cone.

(c) A prism and a cylinder.
The altitude of the prism = the altitude of the cylinder.

(d) A pyramid and a prism.
The altitude of the pyramid > the altitude of the prism.

(e) A cone and a pyramid.
The altitude of the cone = the altitude of the pyramid.

(f) A cylinder and a pyramid.
The altitude of the cylinder < the altitude of the pyramid.

26. A cone whose base radius measures 60 cm. has a slant height which measures 1 meter. The lateral area is 6000π sq. cm.

27. A cone whose base radius is 2 inches has an altitude of 4 inches. Its volume is $\frac{16\pi}{3}$ cubic inches.

28. A pyramid has an altitude which measures 4 inches, and a rhombus for a base whose side measures 4 inches and whose altitude measures $3\frac{1}{8}$ inches. Its volume is $\frac{50}{3}$ or $16\frac{2}{3}$ cubic inches.

Chapter 12

THE SPHERE

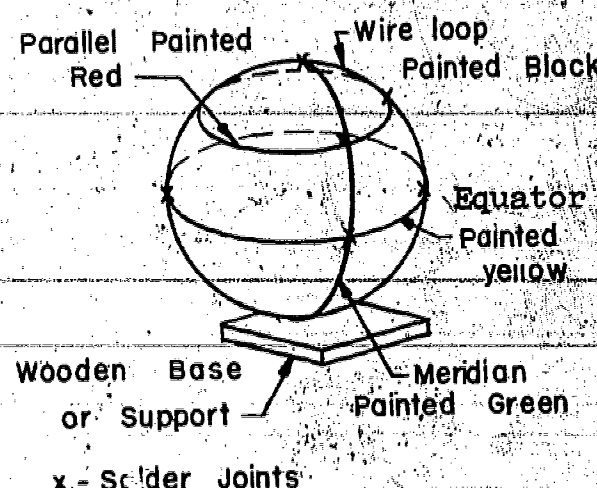
The main purposes of this chapter are:

1. To develop enough of the properties of a sphere so that pupils will appreciate the significance of longitude and latitude as a means of location of points on the surface of the earth.
2. To extend the use of proofs as a form of deductive reasoning. While many of the properties introduced in this chapter are not proved in a formal manner, two fundamental properties are proved in Section 3.
3. To introduce and use the formulas for the volume and area of a sphere. In this connection the teacher should find especially useful the chapter on the sphere in "Concepts of Informal Geometry" by R. D. Anderson (Studies in Mathematics, Volume 5 of the School Mathematics Study Group.)

Although many objects commonly used in everday living are spherical we usually find that very little attention is given to the definition of a shpere or to the study of its properties. Indeed, the shape of the earth should be sufficient reason for devoting considerable attention to studying the sphere.

Many pupils display strong interest in the study of locating points on the surface of the earth. This interest can be motivated without introducing extremely difficult ideas about navigation. The teacher should capitalize on this interest as much as possible.

Teachers may find it helpful to encourage some pupils to make models of spheres. In Sections 3 and 4, it would be especially helpful to have a class size, wire model of the earth, using lengths of wire to represent the equator, parallels of latitude, meridians, etc. Perhaps a pupil who has



great difficulty understanding properties of the sphere when these are presented in rather abstract fashion would find it especially helpful to construct such a wire model.

The number of days recommended for this chapter is 10 or 11 days. ~~The teacher should perhaps be warned that the class should~~ not be carried away by the geography to too great an extent. Of course if this could be carried over into geography classes cooperativly, time could be saved and interest heightened in both courses.

12-1. Introduction

The purpose of this section is to start the pupils thinking in terms of a sphere. Each pupil should have as part of his equipment, a ball, or other spherical object, on which he can draw circles and over which he can place a string to trace paths of circles. Also, the classroom should be equipped with a large globe and a spherical solid. The spherical solid should be large enough for general classroom use and should be painted or covered in such a way that the teacher can draw lines on the surface. Such a sphere will be extremely useful in all sections of this chapter.

Answers to Exercises 12-1

1. Baseball, basketball, softball, billiards, bowling, croquet, golf, lacrosse, jai-lai, soccer, tennis, marbles, polo, volleyball, water polo, ping pong, ball and jacks, handball, tetherball, medicine ball, beachball, cageball.
2. Storage tanks for petroleum products, balloons, as in toys, weather balloons, other planets, satellites.
3. A bobber, or float, may be spherical in shape when using a rod and line.

   Floats for nets are sometimes made of hollow, spherical metal containers. The U. S. Navy used huge spherical steel floats in hanging anti-submarine nets during war time.

4. Some light fixtures consist of a suspended sphere. Some valves used in pressure controls, etc., use ball bearings or spherical parts.
5. (a) circle
   (b) sphere
   (c) no thickness
6. For instance, a solid cube.
7. A sphere is a set of points in space such that all points in the set are the same distance from a particular point, C called the center of the sphere.
8. (a) The set of points is the interior of the sphere.
   (b) The set of points is the exterior of the sphere.

12-2. Great and Small Circles.

This section is chiefly concerned with the intersections of lines and planes with a sphere. It will be necessary to recall some of the material presented in chapters covered previously which deal with geometric ideas. It is expected that most pupils will accept without any question that the intersection of a plane and a sphere is a circle. However, a proof for this property is outlined in Problem 8 of the exercises for this section.

The exercises here are exploratory. Not only should they be worked on outside of class, but they should serve as a basis for much of the class discussion at this time. It would be appropriate to spend two, or perhaps three class days on this section since thinking about the surface of a sphere will be new to most of the pupils. The time spent here in informal consideration should save time later on.

The ideas of meridian and parallel of latitude are touched on lightly here because of their connection with great and small circles. Their use as a coordinate system of reference in locating points is discussed in Section 4.

Answers to Exercises 12-2

1. (a) Yes. Every point on a sphere has an antipode.

   (b) No. Any point on a sphere has only one antipode.

2. (a) AB.

   (b) It is farther from A to B.

   (c) No. There is no point on a straight line through the interior more distant than B because the circle with center at A and radius AB touches the great circle through A and B only at the point B. There is no point on the great circle more distant than B measured on the great circle since B is half way around.

   (d) Yes. Since A and B are antipodal points they are points on a line segment containing C. Any great circle must be on a plane containing C. Such a plane contains $\overline{AC}$ and thus contains $\overline{AB}$.

   (e) No. A, C and D are three points in a plane but are not in a line. Only one plane can contain three such points, and that plane in the drawing contains the great circle shown. No great circle can pass through A and D without containing C. Such a plane passing through C must contain B.

3. (a) More than any number you can name.

   (b) More than any number you can name.

   (c) No. Antipodal points are endpoints of the diameters of a circle. Hence, the plane of any circle on the sphere through these points contains the center of the sphere and hence, is the plane of a great circle.

4. (a) No. One example would be small circles on parallel planes intersecting a sphere. There are others.

(b) Yes. Consider two particular great circles. Each of these lies on a plane containing the center of the sphere. The intersection of these planes is a line since the intersection is not empty (it contains C). This line contains a diameter which is also a diameter of each great circle. This diameter contains a pair of antipodal points. These points lie on each great circle. Hence, the two great circles intersect.

5. (a) One. A meridian is a semi-circle (half a great circle), from pole to pole and thus is a curve on only a half of the surface of a sphere.

(b) One.

(c) No. Parallel planes do not intersect. Hence, circles on two parallel planes cannot intersect.

6. (a) D is in the interior of the sphere.

(b) No. E might be in the interior of the sphere, that is, just beyond $\overline{AC}$ on the line, but it might be in the exterior of the sphere. E might also be on the sphere, antipodal to A.

(c) A line passing through the center of the sphere intersects the sphere in two distinct points which are antipodal. One of these is A. If B is on $\overline{AC}$ and on the sphere it must be point A or the antipode of A.

(d) Yes, by definition of antipodal points.

(e) $\overline{AC}$ is called the radius. $\overline{BC}$ is also a radius.

7. (a) The interior of a sphere is the set of all points in space including the center itself, such that the distance of each point from the center of the sphere is less than the radius.

(b) The exterior of a sphere is the set of all points in space such that the distance of each point from the center of the sphere is greater than the radius.

(c) A sphere is the set of all points in space such that the distance of each point from the center of the sphere is the radius (or, equal to the radius).

8. (a) 90°, $\overline{AC}$ is perpendicular to m. Thus $\overline{AC}$ is perpendicular to any line on m.

(b) A right angle.

(c) A right triangle. The union of three segments determined by three points not in a straight line defines a triangle. Since one of the angles involved is 90°, ARC is a right triangle.

(d) They are the same. Both are right angles.

(e) A right triangle.

(f) AP = AR.

In Δ ACP, $(\overline{AP})^2 = (\overline{CP})^2 - (\overline{CA})^2$

In Δ ACR, $(\overline{AR})^2 = (\overline{CR})^2 - (\overline{CA})^2$

But $\overline{CR} = \overline{CP}$ (Radii of the sphere)

$\therefore \quad (\overline{AP})^2 = (\overline{AR})^2$

(g) The results will be the same for any point of the intersection set.

(h) All such segments are congruent.

(i) Yes. Because each point of the intersection set is the same distance from A, thus defining a circle.

12-3. <u>Properties of Great Circles.</u>

Mathematically, this section is the most important in the chapter. Up to this point, the background was in the process of being developed. After this section, the chief concern is in the application of these results.

The shortest path in a plane between two points is along a straight line, and the shortest path on a sphere between two points is along a great circle. Hence, the importance of great circles.

No attempt is made to prove either of these basic facts. Although this property of a great circle may not be as obvious as that for a straight line, some experimentation should make it seem reasonable. Pupils should be encouraged to experiment with lengths of string and a globe until they become convinced of this.

In many other respects, great circles on a sphere behave quite differently from straight lines on a plane. Any two non-antipodal points determine a great circle, but each great circle intersects every other great circle in <u>two</u> points. These are the fundamental facts proved in this section.

With these results, the pupils should be able to go back over the previous set of exercises with greater understanding. They should, then, be able to pin down the reasons for some of the results which may have seemed doubtful. This should be done before going on to the applications. It might be advisable for the teacher to pause briefly at this point and review the previous exercises.

Answers to Exercises 12-3.

1. (a) Using a globe with diameter 12 inches, the distance is about $6\frac{1}{2}$ inches.

   (b) About $9\frac{1}{2}$ inches on the above globe.

   (c) The great circle path is shorter.

2. (a) About $8\frac{1}{2}$ inches.

   (b) About 12 inches.

   (c) (b) is longer than (a)

3. The "best route" asked for here would be a matter of opinion. There is no shortest route in the sense that these two points are antipodes. A safest route might be to travel directly north from Singapore. This route offers more possible "crash-landing" places as much of this route is overland.

4. The shortest path between two points where one is due north of the other is a great circle path through the North Pole.

5. (a) The results in Problem 1 may be used to verify the fact that a great circle route is shorter than a route following a small circle.

   (b) When traveling between any two points on the equator.

6. No. Three points in space determine a plane. Any plane containing three points on a sphere intersects the sphere. Since the intersection set of the plane is not the empty set or a set of one point it must be a circle. If the plane passes through the center the points are on a great circle of the sphere. Otherwise they are on a small circle.

*7. White - a polar bear because the camp would have to be close to the North Pole. If the camp were at the North Pole, he would go north from any point to get back to camp. A second possibility might be that he had his camp two hours walk north of a small circle whose length was 12 miles, in which case he would shoot the bear at the point where he started his eastward trip. A third possibility would be that the small circle had length 6 miles so he would go around twice. In fact, the small circle could be of length 12/n miles for any positive integer n. Except for the bear, the camp might have been in the vicinity of the South Pole but there are no bears in the Antarctic.

12-4. Locating Points on the Surface of the Earth

This section is concerned with setting up a coordinate system on a sphere. In order to have a coordinate system on a surface, it is necessary that each pair of coordinates locate a unique point and desirable that each point have a unique pair of coordinates. The results of the previous sections show that the usual system of longitude and latitude has these two properties except for the North and South Poles and points on the 180th meridian.

Because the system seems so obvious, the use of meridians and parallels of latitude in locating points on the earth is generally treated very casually. It is all too frequently assumed that because a pupil can locate points by this system he understands the fundamental properties involved. Or, there is no concern about such understanding. This section attempts to bring out several important ideas:

1. That such a system uses a reference line for longitude and a reference line for latitude. Any meridian may serve as a zero meridian, or reference line, but the Greenwich meridian has been designated for this purpose. Similarly, any parallel of latitude may be used as a reference line, but the equator has been designated, and

with obvious advantages, as the line of 0° latitude.

2. Directed movement on the surface of the earth is done with understanding only when locations are known. That is, to guide something or someone on earth, one must know where the starting point is.

3. Longitude and latitude help us locate points on various hemispheres, but there is a subtle difference in the notation used for longitude and latitude. For longitude we measure east and west from the zero meridian (Greenwich Meridian) to the 180° meridian. For latitude, we measure north or south of the equator, but stop at the poles (which may be thought of as the 90° parallel-except that they are points, not circles).

Students interested in astronomy may be encouraged to learn more about the Tropic of Cancer, Arctic Circle, etc. It might be interesting for some students to prepare reports describing how these lines came to be designated, how they received their names, and so on. Also, students interested in geography can extend their study of antipodal points. There might be some cooperation, at this point, with the teachers of science or social studies.

Answers to Exercises 12-4.

1. (a) About 74° W. 41° N. (e) About 2° E. 49° N.

   (b) About 88° W, 42° N. (f) About 38° E, 56° N.

   (c) About 122° W, 38° N. (g) About 42° W, 23° S.

   (d) About 0°, 51° N. (h) About 144° E, 37° S.

2. 0°, 51° N. Greenwich is on the zero meridian.

3. 42° E, 0°. Any point on the equator has a latitude of zero.

4\. (a) The dividing line between U.S. and Soviet Troops in Korea was fixed at the 38th parallel by the Yalta and Potsdam conferences. After the Korean War the the dividing line was set at approximately this line.

(b) Most of the states have at least partial boundaries along parallels of latitude. Among these are :

(1) Northern boundary of Pennsylvania at 42nd parallel;

(2) Northern boundary of South Dakota close to 46th parallel;

(3) Nebraska - Kansas boundary at the 40th parallel;

(4) California - Oregon at 42nd parallel.

(c) In 1844 the United States claimed from Great Britain the whole of the Pacific Coast as far as Alaska, that is, to the 54° 40' parallel of latitude and the slogan of those in Oregon territory was "fifty-four forty or fight" but in the Oregon treaty of 1846 the boundary was fixed at the 49th parallel.

(d) The Missouri Compromise provided that, except in Missouri, there should be in the Louisiana Purchase no slavery north of the $36\frac{1}{2}^{\circ}$ parallel.

(e) The Mason and Dixon Line was originally the boundary between Pennsylvania and Maryland which was by charter supposed to have been the 40th parallel but was in fact a little below that. The Line was eventually considered to be the boundary which separated "slave states" from "free states".

(f) The complete name for Ecuador is "La Republica del Ecuador" which is Spanish for "The Republic of the Equator".

5. (a) Buenos Aires, Argentina
   (b) Wellington, New Zealand

6. (a) Pusan, Korea
   (b) Madrid, Spain

7. (a) Buenos Aires is located on approximately the antipode of Pusan, Korea.
   (b) Wellington and Madrid are located on points which may be considered antipodes.
   (c) These points suggest antipodes.

8. Answers will vary. For most of the cities located in the 48 states between Canada and Mexico, antipodal points will be located in the Indian Ocean between Africa and Australia.

9. The northernmost tip of Alaska.

10. The North Pole.

11. The State of Hawaii, 155° W, 20° N.

12. Guesses.

13. (a) Reno is at about the 120th meridian and Los Angeles at the 118th. Hence, the meridian of Reno is west of that of Los Angeles.
    (b) A little west of Quito, Ecuador.
    (c) The latitude of Portland, Oregon is 46°; that of Seattle is 47°; that of San Francisco is 38°. Hence, Portland is the closest.
    (d) The latitude of London is 51°, of Madrid is 41°, and Casablanca is 33°. Hence, Madrid is closest.

14. Various answers.

*15. No. A point on the sphere determines the intersection of a parallel of latitude and a meridian. Only one point is determined by each such intersection.

*16. Yes. The North Pole and the South Pole. Both the poles have a latitude of 90°, one being south the other north. However, both poles may be thought of as having an infinite number of locations with respect to the zero meridian. Also any point on the 180° meridian would be both 180° W and 180° E.

*17. To find the antipodal location of a point without using a map or globe, observe the following:

(1) The latitude will be the same except for direction. That is, for 45° N, the antipode will have latitude 45° S.

(2) For longitude, compute the distance half way around the earth from that point, that is, the antipode must be on a meridian 180° from the point. Subtract the longitude from 180. The difference is labeled with the direction opposite to the original longitude. For example: to find the antipode meridian for 40° W. $180 - 40 = 140$. The antipodal meridian is 140° E.

(a) 100° E, 25° N.

(b) 80° W, 65° S

(c) 0°, 49° N.

Note: The answer to 17 (c) is the location of Greenwich, England.

*18. The Tropic of Cancer is approximately at $23\frac{1}{2}^\circ$ North. This angle is the angle of maximum tilt of the earth's axis and the Tropic of Cancer marks the maximum distance north where the sun can appear directly overhead. The Artic Circle is approximately the southern border of the "land of the midnight sun", that is, where on June 21st the sun does not set. Actually due to various inaccuracies and discrepancies this phenomenon exists somewhat below the Arctic Circle.

*19. The International Date Line coincides with the 180th meridian for about half its length. Elsewhere it only approximates this.

*20. The location of the Antipodes is about 180° W and 51° S. When it is midnight at Greenwich, it is noon at the Antipodes because the islands are "twelve hours around the earth." When it is the middle of summer in Greenwich, it is midwinter at the Antipodes because of the tilt of the earth. No.

12-5. Volume and Area of a Spherical Solid

In the beginning, some attention is given to a very rough estimate of the volume of a sphere by comparing it with that of the inscribed and circumscribed cubes. The latter is quite simple but the teacher may not want to lay much stress on the former. The proof for the formula for the volume of a sphere is given in the reference mentioned at the beginning of the commentary for this chapter. The teacher may want to show this to some bright student.

No attempt is made to justify the formula for the surface area. This could be indicated as follows. The volume of a pyramid is one third of the altitude times the area of the base. Hence, if the sphere were thought of as composed of pyramids with vertices at the center and with bases little curved regions on the sphere, it would seem reasonable that the following would hold:

$$V = \frac{1}{3} r \cdot A$$

where r is the radius of the sphere (altitude of the pyramids) and A is the surface area. Then since $V = \frac{4}{3} \pi r^3$, the formula for the area follows.

Answers to Exercises 12-5.

1. (a) $113\frac{1}{7}$ cu. in. (e) 735.91 cu. in.

   (b) $4190\frac{10}{21}$ cu. ft. (f) 1204.75 cu. in.

   (c) $268\frac{4}{21}$ cu. yd. (g) 2483.71 cu. mm.

   (d) $905\frac{1}{7}$ cu. cm. (h) 310.46 cu. ft.

   Answers to (e) - (h) are given correct to two decimal places.

2. Answers to the following are given correct to no more than two decimal places.

   (a) 113.04 sq. in. (e) 393.88 sq. in.

   (b) 1256 sq. ft. (f) 547.11 sq. in.

   (c) 200.96 sq. yd. (g) 886.23 sq. mm.

   (d) 452.16 sq. cm. (h) 221.56 sq. ft.

3. (a) r = 25, surface area = 7850 sq. ft., number of gallons of paint is 19.625, cost $117.75.

   (b) Volume is approximately 65,400 cu. ft., number of gallons of oil is approximately 491,000 and the value of the oil is approximately $63,800.

4. Volume of the bowl is 1024 cu. in., the weight of the sugar is 32 pounds and mother pays $3.20.

5. r is 9 in., the surface area of one globe is 1017.36 sq. in., and the total cost of the plastic is $353.25.

6. r = 20 ft., the volume of the gas is 33493.33 cu. ft.

7. (a) The volume is multiplied by 8 since

$$\frac{4}{3}\pi(2r)^3 = \frac{4}{3}\pi(8r^3) = 8\left(\frac{4}{3}\pi r^3\right).$$

The surface area is multiplied by 4, since

$4\pi(2r)^2 = 4\pi(4r^2) = 4\cdot(4\pi r^2)$.

(b) The volume is multiplied by 27 and the surface area by 9, for similar reasons.

8. Call the radii R, and r. Then $\frac{R}{r} = \frac{3}{2}$.

(a) The volumes are $\frac{4}{3}\pi R^3$ and $\frac{4}{3}\pi r^3$ and so

$$\frac{\text{Volume of one sphere}}{\text{Volume of other sphere}} = \frac{\frac{4}{3}\pi R^3}{\frac{4}{3}\pi r^3} = \left(\frac{R}{r}\right)^3 = \left(\frac{3}{2}\right)^3 = \frac{27}{8}$$

(b) The surface areas are $4\pi R^2$ and $4\pi r^2$, so

$$\frac{\text{Surface area of one}}{\text{Surface area of the other}} = \frac{4\pi R^2}{4\pi r^2} = \left(\frac{R}{r}\right)^2 = \left(\frac{3}{2}\right)^2 = \frac{9}{4}$$

9. The following results were obtained by using a regulation softball and standard half-gallon, paper milk carton.

Carton:	Sphere:
$\ell \approx 3\frac{3}{4}''$	$d \approx 3\frac{10}{16}''$
$w \approx 3\frac{3}{4}''$	$r \approx 1\frac{13}{16}$ or $\frac{29}{16}$

Water level raised $\approx 1\frac{3}{4}''$

Volume $\approx$ 24.6 cu. in. $\qquad r^3 \approx \frac{24389}{4096} \approx 5.9$

$$\pi r^3 \approx \frac{536558}{28672} \approx 18.0$$

(using $\frac{22}{7}$)

$18.0 < 24.6$

$24.6 \div 18.0 \approx 1.36$, showing that the constant is about $\frac{4}{3}$.

12-6. Finding Lengths of Small Circles.

This section is not an essential one, but it gives a pupil the means of computing the lengths of circles of latitude given in Section 12-4. It also gives him a chance to use again the trigonometry which was developed in the previous chapter.

Answers to Exercises 12-6.

1. (a) 24150 miles
   (b) 6470 miles
   (c) 17680 miles
2. (a) 21650 miles
   (b) 10825 miles
   (c) 10 degrees
   (d) $\frac{10}{180}$
   (e) $\frac{10}{180}$ (10825 miles) $\approx$ 600 miles
   (f) $\frac{600}{400}$ hrs. = 1.5 hrs. = 1 hr. 30 min.
3. (a) Distance $\approx$ 3190 miles.
   (b) Distance $\approx$ 4260 miles.

*4. The earth makes one-half of a revolution in 12 hours. We might say that the longitude of the point on the earth "directly under" the sun changes by 180° in 12 hours. Hence, the longitude change is 15° per hour, and the answers are:

(a) Since the difference in longitude is 60° the sun-time difference is 4 hours. Thus the sun-time at 70° W is 3:00 a.m.

(b) Since the difference in longitude is 80° the sun-time difference is $\frac{80}{15}$ hours, or 5 hours and 20 minutes. Thus the sun-time at 10° E is 12:20 p.m.

5. The longitude change is 80°, which is equivalent to 5 hours and 20 minutes.

(a) The time when the sun is "directly over" city E is 5 hours and 20 minutes before 7:00 a.m., that is 1:40 a.m.

(b) No. All points having the same longitude will have the same sun-time.

(c) See (b)

6. The length of the circle of latitude at 40° is approximately $25000(\cos 40^\circ) \approx 19150$ miles. This will be the approximate distance the sun moves relative to the earth at this latitude in 24 hours. Hence, the distance for one hour's difference in sun-time will be

$$\frac{19150}{24} \approx 800 \text{ miles.}$$

A is approximately 800 miles east of B.

Sample Questions for Chapter 12

Part I. True - False.

T 1. All points on a parallel of latitude are the same distance from the North Pole.

F 2. The North Temperate Zone on the earth's surface is bounded by two great circles.

F 3. All points on the zero meridian are the same distance from the equator.

T 4. All diameters for a given sphere have the same length.

T 5. For a given sphere, the center of a great circle of the sphere is the center of the sphere.

F 6. All small circles of a given sphere have equal lengths.

F 7. For a given sphere, two great circles cannot have the same center.

T 8. The diameter of any meridian is an axis of the sphere.

T 9. All great circles passing through the North Pole must pass through the South Pole.

F 10. The intersection set of a line and a sphere must contain two points, assuming the set is not the empty set.

T 11. A plane which intersects a sphere in only one point is said to be tangent to the sphere.

T 12. The radius of a given sphere is R. If a point X is located a distance G from the center of the sphere, and $G > R$, then X is in the exterior of the sphere.

T 13. If the radii of two circles of a given sphere are not equal, then not both the circles are great circles of the sphere.

F 14. Parallels of latitude lie on planes with are parallel to the axis of the earth.

F 15. The longitude of the North and South Poles is usually written as 90°.

T 16. The International Date Line follows part of the 180° meridian.

T 17. The cities of New York and London are located as points on a globe. Only one great circle may pass through these two points.

F 18. If a point is located at X 100° E, 0°, its antipode is located at 100° W, 0°.

T 19. Every great circle of a sphere has its center at the center of the sphere.

F 20. The line through two centers of small circles of a sphere must pass through the center of the sphere.

F 21. Through any two points of a sphere there is just one small circle.

F 22. Through any two points of a sphere there is at least one small circle.

F 23. If you double the radius of a sphere, you double its surface area.

T 24. If you double the radius of a sphere you multiply its volume by 8.

T 25. Suppose the center of one small circle is just as far from the center of the sphere as the center of another small circle. The lengths of the circles must be equal.

Part II. Multiple Choice.

b 1. Which of the following statements is true?

I. The area of a sphere is equal to four times the area of a great circle of the sphere.

II. The area of a sphere is equal to eight times the area of any small circle of the sphere.

III. The formula for the area of a sphere is $A = 4\pi r^2$.

IV. The formula for the area of a sphere is $A = \pi r^2 h$.

(a) Only I is true.

(b) Only I and III are true.

(c) Only II and III are true.

(d) Only II and IV are true.

(e) All of the statements are true.

d 2. The formula for the volume of a sphere is:

(a) $V = 4\pi r^2$

(b) $V = \pi r^2 h$

(c) $V = \pi r^3$

(d) $V = \frac{4}{3}\pi r^3$

(e) $V = 2\pi r^2 h$

e 3. Which of the following are true for one specific sphere?

I. All radii are congruent.

II. All diameters are congruent.

III. The length of a diameter is equal to twice the length of a radius.

IV. All great circles have the same length.

(a) Only I and II are true.

(b) Only III and IV are true.

(c) Only I and III are true.

(d) Only I, II, and III are true.

(e) All of the statements are true.

c 4. A sphere has a radius r. Point A is located a distance x from the center of the sphere. If A is in the exterior of the sphere, which one of the following must be true?

(a) $x = r$

(b) $x < r$

(c) $x > r$

(d) $x = 2r$

(e) None of the above answers is correct.

d 5. If the radius of a sphere is doubled, the volume of the original sphere is multiplied by:

(a) two

(b) four

(c) six

(d) eight

(e) ten

e 6. The volume of a sphere whose radius is 3 units is ...

(a) 3π

(b) $(3\pi)^2$

(c) 9π

(d) 12π

(e) 36π

c 7. What is the length of a great circle of a sphere whose diameter is 10 inches?

(a) About 20 inches.

(b) About 25 inches.

(c) About 30 inches.

(d) About 35 inches.

(e) About 40 inches.

b 8. On the earth, the distance from the North Pole to the point located at 76° W, 0° is about ...

(a) 4000 miles

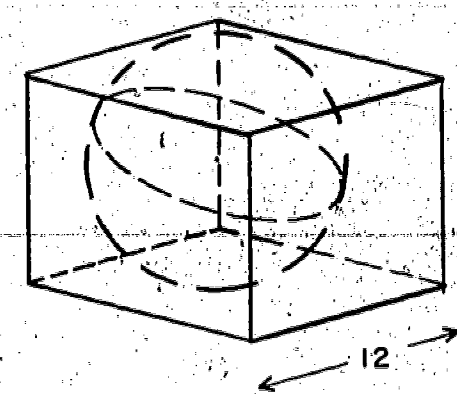
(b) 6000 miles

(c) 10,000 miles

(d) 12,000 miles

(e) 25,000 miles

c 9. In the figure at the right the sphere is inscribed in a cube whose edge has a length of 12 inches. The volume of the sphere is nearest:



(a) 300 cu. in.

(b) 500 cu. in.

(c) 1,000 cu. in.

(d) 1,500 cu. in.

(e) 2,000 cu. in.

a 10. The surface of the sphere shown in the drawing for Question 9 is about:

(a) 450 square inches

~~(b) 550 square inches~~

(c) 600 square inches

(d) 650 square inches

(e) 700 square inches

c 11. The radius of sphere A is twice as long as the radius of sphere B. The surface of sphere A is how many times that of sphere B?

(a) 2

(b) 3

(c) 4

(d) 8

(e) 10

d 12. A and B are on two points on earth such that B is directly north of A. If A and B move due west, what can be said of their paths?

(a) Their paths will cross.

(b) Their paths will be on the same line.

(c) Their paths will move in opposite directions.

(d) Their paths will not cross.

(e) Their paths would cross if continued around the sphere twice.

b 13. A is on a point located at 100° W, 25° N. B is on a point located at 120° W, 25° N. If A and B move due south, then:

(a) Their paths cannot intersect.

(b) Their paths will intersect.

(c) Their paths will move them farther apart.

(d) Their paths will intersect in a line.

(e) Their paths will intersect in many separate points.

b 14. Which one of the following cities has the greatest measure for latitude?

(a) New York City

(b) London

(c) Madrid, Spain

(d) Tokyo

(e) San Francisco

d 15. A formula for the total surface of a hemisphere (that is, the curved surface and the flat region) is . . .

(a) πr^2

(b) $2 \pi r^2$

(c) $2\frac{1}{2} \pi r^2$

(d) $3 \pi r^2$

(e) $\frac{4}{3} \pi r^2$

a 16. The radius of the earth is about 4,000 miles and the radius of the moon is about 1,000 miles. Which of the following ratios describes how the area of the earth compares with the area of the moon.

(a) 16:1 (c) 8:1 (e) 2:1

(b) 12:1 (d) 4:1

Chapter 13

RELATIVE ERROR

General Remarks

Suggested time allotment: about two weeks

The process of measurement plays such an important part in contemporary life that everyone should have a clear understanding of its nature. A substantial part of the arithmetic taught in the elementary school relates to measurement. Most of the early work in measurement is designed to familiarize children with common units of measurement and their use, and with the ratios between their measures.

The basic concept to be developed in this unit is that the process of measurement of a single thing yields a number which represents the approximate number of units. This is in contrast to the process of counting separate objects, which yields an exact number. When the number of separate objects is rounded or estimated, the resulting number is treated as an approximation in the same sense as a measurement. Since measurements are approximate, calculations made with their measures, such as sums or products, yield results which are also approximate.

It is a fundamental principle of scientific discipline that no direct measurement can be achieved with complete and total accuracy. (Accuracy here is being used in the sense of being correct, not in the technical sense that is developed later in the chapter. The teacher here might note that "accurate" as a word with a definite meaning and "correct" in the usual sense might be quite different.) Many errors may enter into the act of measurement. They are:

(a) Mistakes

(b) Instrumental errors

(c) External errors

(d) Personal errors.

Preventable mistakes come about through ignorance, carelessness, or improper use of the measuring instrument. These mistakes can be corrected by means of a system of checks, by proper education and by careful inspection during the measuring process.

Instrumental errors are inherent in the measuring device. These errors are the results of imperfections in materials and in the manufacturing process. They are also the end products of economics in the system of manufacturing. To make an instrument more precise requires better materials, more time and superior manufacturing techniques, all of which cause higher costs.

External errors are environmental errors introduced by causes known and unknown. Wind currents may affect the careful weighing of an object. Temperature changes may cause corresponding changes in the dimensions of both objects measured and the measuring instrument. Some of these errors can be controlled to a degree by measuring in a controlled environment. Some of the errors of known origin can be corrected by applying correction factors, which, in effect, nullify the external errors. Some external errors are so complicated in origin that they defy analysis and complete correction.

Personal errors are due to human imperfections. Most measurements have been made by human beings in one way or another. No two persons react to a situation in exactly the same manner. Nevertheless, personal errors can be reduced by proper selection and training of personnel in the measuring operation. (This latter factor may explain to the teachers why industries spend great sums of money simply to teach people how to read measuring instruments. Pupils may then see the consequent importance of this in school work.)

Errors are also characterized as determinate and indeterminate. We have already discussed some of the consequences of these errors under types of errors above. Determinate errors are ones that occur in a known pattern throughout a series of measurements. Such errors can be analyzed, accounted for and corrected. Instrumental errors and external errors caused by changes in temperature are of the determinate variety.

Indeterminate errors are the more insidious and more difficult to recognize and cope with. They have no modus operandi and their occurrence is haphazard and inconsistent. Human errors introduced by fatigue and anxiety are of the indeterminate variety. A sudden gust of wind during the measuring operation will introduce an indeterminate error. The only way to discover and manage this type of error is by repeated observations and measurements.

The unit selected for a given measurement should be suitable for the thing to be measured, and for the purpose for which the measurement is to be used. The unit is not necessarily a standard unit, but may be a multiple of a standard unit, or a sub-division of a standard unit. For example, the height of an airplane above the ground may be stated using 100 feet as a unit. The height of a person may be stated to the nearest half-inch, that is, using the half-inch as a unit. The result of measurement should be stated so as to indicate what unit was used, but this is not always done. In this Chapter, $2\frac{3}{4}$ in. implies that the unit used was $\frac{1}{4}$ inch, and that the result is stated to the nearest fourth-inch. The result, "$2\frac{3}{4}$ in." thus applies to any measurement which is within half of the unit on either side of the $2\frac{3}{4}$ inch mark; that is more than $2\frac{5}{8}$ and less than $2\frac{7}{8}$ inches. We have used the term "greatest possible error" to refer to the amount by which the actual measurement may vary from the stated result. The use of the word "error" may cause some difficulty. It should be emphasized that the word is used here to mean that a measurement such as $2\frac{3}{4}$ inches represents any of a series of measurements ranging from $(2\frac{3}{4} - \frac{1}{8})$ inches to $(2\frac{3}{4} + \frac{1}{8})$ inches. This meaning should be distinguished from the more familiar use of the term to mean mistakes in using the measuring instrument, mistakes in reading the scale, or mistakes resulting from use of a faulty instrument, such as a poorly marked ruler.

The two terms, "precision" and "accuracy" are sometimes confusing. In common usage, precision is the size of the unit of measurement used; the smaller the unit, the more precise the measurement. This means, of course, that the more precise the measurement, the smaller is the greatest possible error. Accuracy of a measurement is the percent the greatest possible error is of the measurement. That is, accuracy refers to the relative error.

In order to develop clearly the concepts of unit of measurement and greatest possible error, it is suggested that the students be given considerable practice in measuring lengths of line segments to the nearest inch, nearest half-inch, nearest tenth of an inch, and so on. They should be asked to state the greatest possible error for measurements, and to indicate within what range a stated measurement must fall.

Students sometimes have difficulty in seeing why measurements with the same significant digits in the same order have the same accuracy, regardless of the unit of measurement. A development along the following lines is sometimes helpful. Significant digits are defined as the digits in the numeral which show the number of units. The measurements below are analyzed to show the unit, the number of units, and the possible error.

Measurement	Unit	Number of Units	Greatest Possible Error
3,570 ft.	10 ft.	357	5 ft.
0.0357 ft.	0.0001 ft.	357	0.00005 ft.

Each of these measurements contains three significant figures. To determine the accuracy, we find the relative error, or percent of error. In the first case, we have the ration $\frac{5}{3750}$. In the second case, the ratio is $\frac{0.00005}{0.0357}$, which equals $\frac{5}{3750}$.

Students also have difficulty in understanding why measurements with a larger number of significant digits have greater accuracy. An example similar to that above can be used.

Measurement	Unit	Number of Units	Greatest Possible Error
3.57 ft.	0.01 ft.	357	0.005 ft.
35.70 ft.	0.01 ft.	3570	0.005 ft.

Relative error of 3.57 ft. is $\frac{0.005}{3.57}$, or $\frac{5}{35700}$

Relative error of 35.70 ft. is $\frac{0.005}{35.70}$, or $\frac{5}{3570}$.

The student should constantly be cautioned that the rules presented for computation with approximate data are "rough" and are not universally applicable.

Bakst in his Approximate Computation* makes this important point:

> "Generally speaking, the technique of Approximate Computation is not mechanical. The performance of numerical processes may be thought of as mechanical, but the arithmetic of Approximate Computation can be fully appreciated if and only if the interpretative processes are predominant. Only when a student is conscious of the nature of the data and can interpret the approximativeness and the meaning of the numerical results obtained by him, will he understand the importance of Approximate Computation as a fundamental part of applied mathematics."

For instance, when we carry a computed number like π or a trigonometric function to one more significant digit than the least precise of the approximate factors, we are attempting to minimize the error introduced by the numbers arising from calculation rather than measurement.

*Twelfth Yearbook of the National Council of Teachers of Mathematics, Washington, D.C.: National Council of Teachers of Mathematics, 1937. p. 18.

If it seems desirable, this unit might be increased in scope by reports on such topics as the history of standard units of measurement, various measuring devices used in industry and science, and the work of the United States Bureau of Standards.

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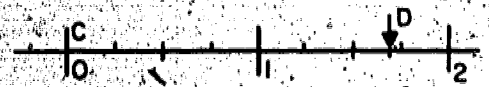
3. Swain, R. L. UNDERSTANDING ARITHMETIC. New York: Rinehart and Co., 1957. pp. 231-49.

4. The National Counicl of Teachers of Mathematics. THE GROWTH OF MATHEMATICAL IDEAS, GRADES K-12 (Twenty-Fourth Yearbook). Washington, D.C.: The National Council of Teachers of Mathematics, 1959. pp. 182-228.

Answers to Exercises 13-1.

1.  $CD = 1\frac{3}{4}$ inches.

2. Between $1\frac{5}{8}$ inches and $1\frac{7}{8}$ inches; $\frac{1}{8}$ inch from $1\frac{3}{4}$ inches.

3. The point D may be anywhere between $1\frac{5}{8}$ inches $(1\frac{3}{4} - \frac{1}{8})$ inches and $1\frac{7}{8}$ inches $(1\frac{3}{4} + \frac{1}{8})$ inches.

4. (a) $\frac{1}{2}$ inch

   (b) $1\frac{1}{4}$ inches and $1\frac{3}{4}$ inches.

   (c) $(1\frac{1}{2} \pm \frac{1}{4})$ inches

   (d) $\frac{1}{4}$ inch

5. (a) Between $2\frac{9}{32}$ and $2\frac{11}{32}$.

   (b) $\frac{1}{32}$

6. $\frac{1}{16}$; $\frac{1}{16}$

7. $\frac{1}{8}$ inch.

8. (a) $\frac{1}{10}$ cm.

   (b) $(3\frac{7}{10} \pm \frac{1}{20})$

   (c) $\frac{1}{20}$ cm.

9. $\frac{1}{200}$ cm.

10. One-half of the unit used.

Answers to Exercises 13-2.

1. 3.20 inches.

2. 4.0 inches.

3. (a) 5.2 feet.

   (b) 0.68 feet.

   (c) Both have the same precision.

4. $12\frac{1}{2}$ and $13\frac{1}{2}$ years old.

5. (a) a. 100 feet   d. 0.1 foot

   b. 1 foot   e. 0.0001 foot

   c. 10 feet   f. 0.1 foot

   (b) a. 50 feet   d. 0.05 foot

   b. 0.5 foot   e. 0.00005 foot

   c. 5 feet   f. 0.05 foot

6. (a) e is the most precise.

   (b) a is the least precise.

   (c) Yes, d and f have the same precision.

7. (a) 420<u>0</u>

   (b) 23,<u>0</u>00

   (c) 48,0<u>0</u>0,000

8. (a) 2   (e) 4

   (b) 4   (f) 2

   (c) 1   (g) 4

   (d) 4   (h) 3

9. (a) 4   (d) 3

   (b) 4   (e) 5

   (c) 2   (f) 2

Answers to Exercises 13-3

1. (a) 0.5 foot (e) 0.005 inches
   (b) 0.05 inch (f) 0.0005 foot
   (c) 5 miles (g) 500 miles
   (d) 0.5 foot (h) 5 miles
2. (a) 0.0096 (e) 0.00071
   (b) 0.012 (f) 0.083
   (c) 0.0019 (g) 0.0093
   (d) 0.0014 (h) 0.000093
3. (a) 0.05 foot;0.54% (c) 5 feet;0.54%
   (b) 0.0005 foot;0.54% (d) 500 feet;0.54%
4. The percents of error are the same for each measurement. In each case, the greatest possible error was the same fractional part of the measurement.
5. (a) 0.1 foot (d) 1,000 miles
   (b) 0.001 foot (e) 0.1 foot
   (c) 10 feet (f) 0.001 inch
6. (c) 3 (c) 3
   (b) 4 (d) 4
7. (a) 0.00096 (c) 0.0014
   (b) 0.000096 (d) 0.00014
8. Yes. As the number of significant digits increases the relative error decreases. The larger the number of significant digits the greater the accuracy.
9. 7,812' inches has the greatest accuracy.
   0.2 inch has the least accuracy.

10. (a) $36\frac{1}{2}$ inches, $22\frac{1}{4}$ inches, $46\frac{2}{7}$ inches, $32\frac{3}{8}$ inches, $27\frac{0}{16}$ inches.

(b) 3 inches, 82.4 inches, 4.62 inches, 3.041 inches, 0.3762 inches.

*11. 6 feet $\pm \frac{1}{2}$ foot.

$3\frac{1}{2}$ inches $\pm \frac{1}{8}$ inch.

7.2 miles $\pm$ 0.05 mile.

3 yards 4 inches $\pm \frac{1}{4}$ inch.

3.2 inches $\pm$ 0.005 inch.

12. (a) 4 (e) 4 (i) 5

(b) 4 (f) 2 (j) 2

(c) 5 (g) 5 (k) 2

(d) 2 (h) 4 (ℓ) 8

13. (a) 4.63×10^8 (f) 4.00×10^{-5}

(b) 3.27×10^5 (g) 3.68×10^6

(c) 4.62×10^{-4} (h) 8×10^{-8}

(d) 3.2004×10^1 (i) 7.2×10^{10}

(e) 2×10^0

14. e, j, i, c, g, a, f, b, d, h

15. BRAINBUSTER. $\frac{.0005}{3\frac{1}{2}} = \frac{5 \times 10^{-5}}{3.5} = \frac{1}{7 \times 10^4}$

$$\frac{5 \times 10^6}{(8.6)(6\times10^{12})} = \frac{10^7}{(2)(8.6)(6\times10^{12})} = \frac{1}{1.032\times10^7}$$

If two fractions have the same numerator, then the one with the larger denominator has the smaller value. Since, by definition, a measurement with a smaller relative error is the more accurate, the astronomer made the more accurate measurement.

Answers to Exercises 13-4

1. (a) $\frac{3}{4}$ inch.

   (b) $\frac{7}{8}$ inch.

   (c) 0.055 inch.

   (d) 0.15 inch.

   (e) 0.0510 inch.

   (f) $\frac{7}{32}$ inch.

2. (a) 731.8

   (b) 145.1

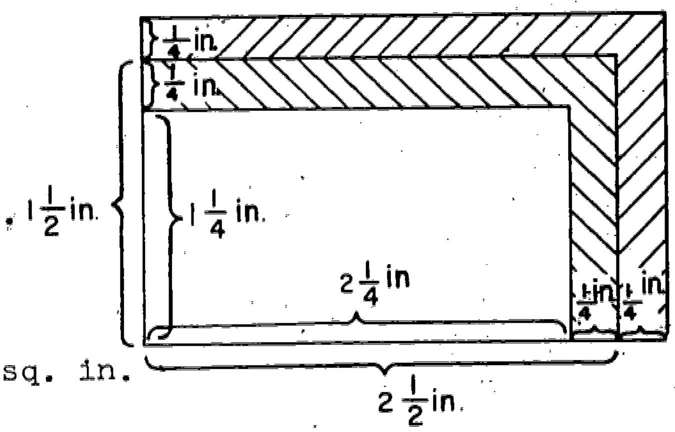
   (c) 1,758.03

3. (a) 1.0

   (b) 734

   (c) 4780

Answers to Exercises 13-5.



1. Largest area = $4\frac{13}{16}$ sq. in.

   Smallest area = $2\frac{13}{16}$ sq. in.

   Difference = 2 sq. in.

   Area is ($3\frac{3}{4}$ sq. in.) or 4 sq. in.

   (to the nearest $\frac{1}{2}$ sq. in.)

2. (a) 150

   (b) 17,000

   (c) 11×10^4

3. (a) 4.4

   (b) 1.14

   (c) 3.92×10^5

4. 2.55×10^5 (or 255,000) square rods.

5. (a) 11 inches

   (b) 145.6 feet

   (c) 20 miles

6. 46.9 pounds. Three significant digits are used here because 75 is an exact counting number and does not affect the number of significant digits.

7. 250

8. (a) 13 feet (0.375×34)

   (b) 220 feet (3.27×67)

Sample Questions for Chapter 13

Part I. True - False.

T 1. Counting separate objects is considered to be an exact measurement.

T 2. The smaller the unit, the more precise is the measurement.

F 3. The greatest possible error would be one-sixteenth inch if the length of a line is measured to the nearest $\frac{1}{4}$ inch.

T 4. The smaller the percent of error, the greater is the accuracy of the measurement.

F 5. If a measurement of a line is stated to be 10.0 inches, it implies that the line was measured to the nearest inch.

F 6. The more precise the measurement, the greater is the possible error.

T 7. A measurement of 300 miles has the same greatest possible error as a measurement of 700 miles.

T 8. There are four significant digits in the measurement 7,003 miles.

T 9. The term "greatest possible error" of a measurement does not refer to a mistake made in the measurement.

F 10. The greatest possible error of the sum of several approximate measurements is the same as the greatest possible error of the least precise measurement.

Part II. Multiple Choice.

b 1. The most precise measurement is:

(a) 26 $\frac{1}{2}$ inches

(b) 26.0 inches

(c) 260 inches

(d) 26 $\frac{1}{4}$ inches

c 2. The number with the greatest accuracy is the one with the least:

(a) precision

(b) possible error

(c) percent of error

(d) number of significant digits

a 3. The greatest possible error in the sum of 45.5 in., 36.05 in., 51.1 in. is:

(a) 0.105 in.

(b) 0.05 in.

(c) 0.055 in.

(d) 0.005 in.

Part III Completion.

1. The measurement of a line segment was stated to be $1\frac{1}{8}$ inches.

The measurement might be stated as $1\frac{1}{8} \pm$ ($\frac{1}{16}$) inches.

2. The greatest possible error in a measuremnt is always ($\frac{1}{2}$) of the unit used.

3. A measurement of $34\frac{1}{4}$ inches has the same precision as a measurement of $5\ (\frac{0}{4})$ inches.

Part IV. Miscellaneous.

1. Measure the length of the line segment $\overline{AD}$ to the nearest eighth of an inch. A____________D $(1\frac{4}{8})$ in.

2. Compute the percent of error in the measurement 25 inches. How would this compare with the percent of error in the measurement 25 miles?

   1. $(\frac{0.5}{25} = 0.02 = 2 \text{ percent})$
   2. (Same)