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ABSTRACT

The application of latent trait theory to classroom tests necessitates the use of small sample sizes for parameter estimation. Computer generated data were used to assess the accuracy of estimation of the slope and location parameters in the two parameter logistic model with fixed abilities and varying small sample sizes. The maximum likelihood procedure for estimating the parameters was compared to a method in which the observed relative frequencies were smoothed using an isotonic regression method prior to applying the maximum likelihood procedure. The isotonic method was considered promising because the smoothed relative frequencies yield more accurate estimates of the probability of correctly answering a test item given a particular level of the ability than do the observed relative frequencies. The results were presented in terms of variance and mean squared error of estimating the parameters. The results indicated that the isotonic procedure provided more accurate estimates of the location parameter whereas the maximum likelihood procedure provided more accurate estimates of the slope parameter. Since the isotonic method did provide for more accurate estimation of the location parameter, it was concluded that the isotonic method warrants further attention. The implications of the results for use with classroom tests were also discussed. (Author/CTM)

ACCURACY OF ESTIMATING TWO PARAMETER LOGISTIC LATENT TRAIT PARAMETERS AND IMPLICATIONS FOR CLASSROOM TESTS

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Abstract

The application of latent trait theory to classroom tests necessitates the use of small sample sizes for parameter estimation. Computer generated data were used to assess the accuracy of estimation of the slope and location parameters in the two parameter logistic model with fixed abilities and varying small sample sizes. The maximum likelihood (ML) procedure for estimating the parameters was compared to a method in which the observed relative frequencies were smoothed using an isotonic regression method prior to applying the ML procedure. The isotonic method was considered promising because the smoothed relative frequencies yield more accurate estimates of the probability of correctly answering a test item given a particular level of ability then do the observed relative frequencies.

The results were presented in terms of variance and mean squared error of estimating the parameters. The results indicated that the isotonic procedure provided more accurate estimates of the location parameter whereas the ML procedure provided more accurate estimates of the slope parameter. Since the isotonic method did provide for more accurate estimation of the location parameter, it was concluded that the isotonic method warrants further attention. The implications of the results for use with classroom tests were also discussed.

ACCURACY OF ESTIMATING TWO PARAMETER LOGISTIC LATENT TRAIT PARAMETERS AND IMPLICATIONS FOR CLASSROOM TESTS

Latent trait theory is an alternative to classical theory for evaluating classroom tests. Recent applications and suggested applications of latent trait theory (e.g. Baker, 1977; Hambleton and Cook, 1977; Lord, 1977; Marco, 1977; and Rentz and Bashaw, 1977) have most often been in the context of standardized tests or in other situations in which a large number of individuals are able to take the test items. Since classroom tests are quite often administered to a small number of individuals, the problem of possible low accuracy in estimation of the latent trait parameters due to small sample sizes must be considered before latent trait theory can be useful for classroom tests.

This study was an attempt to assess, using computer generated data, the accuracy of estimation in the two parameter logistic model with varying small sample sizes and to examine one method offering promise of improving the accuracy. Specifically, the frequently used maximum likelihood (ML) procedure for estimating the latent trait parameters was compared to a method in which the observed relative frequencies were smoothed using an isotonic regression method prior to applying the ML procedure. In the isotonic method, the smoothed relative frequencies (instead of the observed relative frequencies) were used as the data in the usual maximum likelihood procedure. The isotonic method was considered promising because it results in the smoothed relative frequencies which are entered into the maximum likelihood procedure to be monotonic non-decreasing along the ability scale.

It was believed that the isotonic method might lead to more accurate estimation of the latent trait parameters because smoothed relative frequencies exhibit a monotonic nondecreasing relationship that is consistent with the assumptions of the ML procedure. They might, therefore, yield more accurate estimates of the actual probabilities than do the observed relative frequencies (Ayer, Brunk, Ewing, Reed, and Silverman, 1955). The use of the smoothed relative frequencies also ensures that a non-negative slope estimate will result when the logistic parameters are estimated.

Comparisons of accuracy in estimation were based on mean squared error, variance, and bias in estimating the parameters. In addition, the obtained average variances were compared to an approximation to the Rao-Cramér lower bound for the variance in estimating each of the two parameters.

METHOD AND DATA GENERATION

This study used the two parameter logistic model (Birnbaum, 1968):

$$P_g(\Theta) = \frac{\exp\,[1.7\,a_g\,(\Theta - b_g)]}{1 + \exp\,[1.7a_g\,(\Theta - b_g)]} \qquad (g = 1, 2, \ldots, k) \qquad (1)$$

In this model, $P_g(\Theta)$ is the probability that an examinee with ability Θ answers item g correctly, a_g and b_g are parameters associated with item g ($g = 1, 2, \ldots, k$), and k is the number of items on the test. The parameter a_g (often referred to as an index of item discrimination) is a slope parameter and b_g (often referred to as an index of item difficulty representing the ability level at which 50% of the individuals would be expected to correctly answer the item -- Baker, 1977) is a location

parameter. The constant of 1.7 was chosen so that the probabilities and parameters would correspond closely to those of a normal ogive.

Marginal distribution of ability. The marginal distribution of ability was assumed to be a logistic distribution with the distribution function:

$$F(\Theta) = \frac{\exp(1.7\Theta)}{1 + \exp(1.7\Theta)}$$

The marginal distribution was divided into 15 equal probability intervals that were fixed for all replications. The median of the interval was used as the ability level of all scores in that interval in subsequent calculations.

Item characteristic curves: The actual item characteristic curves were constructed using the two parameter logistic model, with a_g taking the values 0.5, 1.0, and 1.5 and b_g taking the values 0.0, 0.5, and 1.5. Sample size (N) was allowed to take on the values 45, 60, 90, 150, and 240. These values were chosen to cover the range of possible item parameters and sample sizes that are likely to occur with classroom tests.

Random generation of the relative frequencies. After the logistic parameters and sample size were fixed, the relative frequency of individuals correctly answering the item within each of the fifteen intervals on the ability scale was randomly generated using the IMSL (1977) Fortran IV subroutine, GGBIN. The generated binomial variates were based on the actual $P_g(\Theta)$ value from the two parameter logistic model (i.e., the probability of correctly answering the item for individuals at the median of the interval) and the sample size in the interval. The randomly generated value corresponded to the relative frequency of individuals with ability Θ that correctly answered the hypothetical test item.

Maximum likelihood estimation method. Maximum likelihood procedures were used to estimate the a_g and b_g parameters. The likelihood function for the two parameter logistic model for an individual item is proportional to e^L (Finney, 1971) where,

$$L = \sum_{i=1}^{m} r_i \cdot \ln [P_g (\Theta_i)] + \sum_{i=1}^{m} (n_i - r_i) \cdot \ln [1 - P_g (\Theta_i)],$$

i represents one of the m = 15 intervals along the ability scale, Θ_i is the median ability for the ith interval, n_i is the number of individuals in interval i, r_i is the number answering the item correctly, and $P_g (\Theta_i)$ is as given in equation (1). The maximum likelihood solution involves finding the a_g and b_g values which maximize L. The maximum likelihood equations are obtained by finding the partial derivatives of L with respect to a_g and b_g and setting them equal to zero. Direct solutions are not possible, so the values of a_g and b_g which maximize L are found by iteratively solving these two maximum likelihood equations:

$$\begin{aligned} &\sum_{i=1}^{m} \Theta_i [P_g(\Theta_i) - \frac{r_i}{n_i}] = 0 \\ &\sum_{i=1}^{m} [P_g(\Theta_i) - \frac{r_i}{n_i}] = 0 \end{aligned} \qquad (2)$$

The maximum likelihood solution was achieved by substituting starting values for a_g and b_g into equation (1), the equation for $P_g (\Theta_i)$, and using an iterative scheme to find the $\hat{a}_g$ and $\hat{b}_g$ (the estimated parameter) values which maximize L.

Parameters a_g and b_g were estimated using the IMSL (1977) Fortran IV subroutine ZSYSTM which uses Brown's algorithm. The convergence criterion was satisfied if, on two successive iterations, both estimates agreed to 9 significant digits. In order to minimize the number of iterations and lack of convergence problems the actual values of a_g and b_g were used as the starting values for the iterative process. Fortran IV double precision arithmetic was used for all of the computations.

Parameter estimation methods. Two procedures were used to estimate the parameters. In the first (regular), the maximum likelihood solution was applied directly to the observed relative frequencies. In the second procedure (isotonic) the observed relative frequencies were initially smoothed using an isotonic regression method. In this procedure, the randomly generated relative frequencies of individual's correct answers were checked to determine if they were monotonic nondecreasing along the ability scale. Whenever a pair of relative frequencies did not meet this condition (a reversal), the pair was averaged. The procedure continued until no reversals remained. The maximum likelihood solution was then applied to the smoothed relative frequencies.

Data generation. The relative frequencies were randomly generated 100 times for each of the 45 possible combinations of a_g, b_g, and N. The parameters (a_g and b_g) were estimated for each of the replications using both of the procedures. The individual ability parameters were fixed and assumed known for all replications.

Assessing the accuracy of estimation. The accuracy of estimation was judged by comparing the mean squared errors (mean squared deviations about the actual parameter values) and variances (mean squared deviations about the average estimated parameter values) in estimating a_g and b_g for each

combination of the logistic parameters, sample size, and estimation method. Mean values of the parameter estimates were also examined in order to assess the amount of bias resulting from each of the estimation methods. The logistic parameters on some of the replications were not estimable by the maximum likelihood method. For comparative purposes, only those replications for which both the regular and isotonic methods proved estimable were included in the computations of the summary statistics.

Approximating variances, the formulas for which are derived in the Appendix, were also calculated. The approximating variance for the slope parameter is given by

$$\text{Approx. Var. } (a_g) = [1.7^2 \sum_{i=1}^{m} w_i \, (\Theta_i - b_g)^2]^{-1}$$

and the approximating variance for the location parameter is given by

$$\text{Approx. Var. } (b_g) = [\ (1.7\ a_g)^2 \sum_{i=1}^{m} w_i]^{-1},$$

where

$$w_i = n_i P_g(\Theta_i)\ [1 - P_g(\Theta_i)].$$

These approximating variances hold when the $P_g\ (\Theta_i)$ values follow the two parameter logistic distribution, the samples are random at each ability level with ability levels being fixed, and the number of individuals at each ability level is large (Berkson, 1953). Because the first two of these conditions were met in the design of the present study, the results may be used to assess how large a sample size is needed for the approximating variances to provide close approximations to the observed variances.

Results

A solution was attained for 99.36% of the solutions attempted using the regular method and for 99.93% of the solutions attempted using the isotonic method. The replications in which solutions were not found tended to be those with smaller sample sizes and larger parameter values. The only combination of parameters and sample size in which fewer than 90% of the solutions attempted were attained was with N=45, a_g=1.5, and b_g=1.5. For the regular method, 81% were solved and for the isotonic method, 83% were solved in this case. Only those items for which <u>both</u> the regular and isotonic methods produced solutions were included in subsequent analyses. For these items, the average number of iterations was 5.82 for the regular method and 5.92 for the isotonic method.

<u>Mean estimates of the parameters</u>. The mean values of the slope parameter (a_g) estimates are presented in Table 1 and the mean values of the location parameter (b_g) estimates are presented in Table 2. For all combinations of method, sample size, and logistic parameters except one (regular method, N=150, a_g=0.5, and b_g=1.5), the mean estimate of the slope parameter was greater than the respective actual slope parameter. In addition, the isotonic method produced larger estimates of the slope parameter than did the regular method in all cases. The bias evident by examination of Table 1 is in the same direction as the bias reported by Berkson (1955). There was a weak tendency for the bias to lessen as sample size increased.

The means of the estimated location parameter values do not appear to be consistently biased in either direction (Table 2). However, the means for the regular method were consistently greater in absolute value than were the means for the isotonic method.

Insert Tables 1 and 2 about here.

Accuracy of Estimation

The variance and approximating variance for the slope and location parameter estimates are presented in Tables 3 and 4. The mean squared errors in the slope and location parameter estimates are presented in Tables 5 and 6.

Insert Tables 3, 4, 5, and 6 about here

Slope Parameter. The regular method provided more accurate estimates of the slope parameter than did the isotonic method (Tables 3 and 5) for most combinations of the parameters and sample sizes. The relative differences in accuracy for the two methods were greater in mean squared error than in variance primarily because the isotonic estimates of the slope parameter were more biased than were the regular estimates. The relative difference in accuracy for the two methods decreased as sample size increased. (This is as expected, since, as the sample size increases, the observed relative frequencies less frequently require smoothing.)

Location Parameter. The isotonic method provided more accurate estimates of the location parameter than did the regular method (Tables 4 and 6) for all combinations of the parameters and sample sizes. The relative differences between the mean squared errors and variances were in general, very small.

Approximating Variances. The variances for slope parameter estimates were in all cases larger than the asymptotic variances (Table 3). The approximating variances substantially underestimated the obtained variances for smaller samples (as expected) and larger parameter values. The variances (and mean squared errors) for the isotonic method were very similar to approximating variances for estimating the location parameter.

The isotonic method tended to result in variances that were actually lower than the approximating variances for small samples and lower values of the location parameter. The variances (and mean squared errors) for the regular method were similar to the approximating variances for the larger sample sizes.

Discussion

The average slope parameter estimates were larger for the isotonic method than for the regular method. A possible explanation for this result may be attempted using an analogy to linear least squares regression analysis. The latent trait maximum likelihood estimation problem may be solved using weighted linear regression analysis with the predictor being ability and the criterion being $1.7\hat{a}_g\ (\Theta_i-\hat{\delta}_g)$. In this solution, it is necessary to iteratively solve for the parameters and for the weights which are a function of the parameters (for example, see Berkson, 1955). The isotonic method initially results in the predictor and criterion being in the same nondecreasing monotonic ordering due to averaging the criterion values involved in reversals. If the isotonic method was used prior to linear least squares regression, the covariance between predictor and criterion (and, therefore, the slope) would increase in comparison to that which would be obtained if the isotonic method had not been used. (The variance of the predictor remains unchanged.) If this analogy and reasoning holds, it would explain why the slope estimate is greater under the isotonic method.

The smaller location parameter values found with the isotonic method are more difficult to explain. It may be that the slope and location parameters are related in the iterative scheme so that an increase in the slope results in a decrease in the location parameter.

The regular method led to more accurate estimation of the slope parameter whereas the isotonic method yielded more accurate estimation of the location parameter. The isotonic method was not more accurate for both parameters and does not provide a reduction in computational labor (no reduction was noted in the number of iterations required). It probably, therefore, should not be used exclusively in practice. A scheme that would be beneficial would be to fit the data with both methods and to use the regular method to estimate the slope parameter and the isotonic method to estimate the location parameter. However, the gain in accuracy over using just the regular method might not justify the additional computational labor. Another scheme would be to estimate the parameters using the isotonic method. Ignoring the isotonic estimate of the slope, the isotonic estimate of the location parameter would then be used as a fixed value for estimating the slope using the observed relative frequencies as data.

The results also indicated that the approximating variances approach closely the obtained variances for the larger sample sizes and at some combinations of the parameters for the smaller sample sizes. Thus, the approximating variances can be used to indicate the degree of accuracy to be expected when using two parameter latent trait theory with fairly large sample sizes. The use of the approximating variances assumes knowledge of student ability, that the logistic model actually holds, and that there is no bias in estimation. These are strong assumptions to make in applied settings (and in fact, the estimates of the slope will be biased) so that the approximating variances should be seen as a lower bound which would probably not be achieved--even by the regular method.

In relation to classroom testing, unless moderate sample sizes are used (say 100 or more students) the latent trait parameter estimates will be fairly inaccurate (with the degree of accuracy also depending on the actual parameter values). However, the problem of inaccuracy in estimating the classical indexes also exists (Baker, 1965). In comparing the results of the present study to those of Baker (1965), it appears that the estimates of the latent trait parameters are probably no less accurate than those for the classical indexes for small samples. Thus, if it is desired to present item analyses to instructors, no clear preference emerges for another comparison of the relative accuracy of estimation in the classical and latent trait theories. The choice should be made on the amount and quality of information each theory provides to the instructors.

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Table 1. Average values for the estimated slope (a_g) parameter

Sample Size	Method	Slope (a) 0.5, Location (b) 0.0	0.5: 0.5	0.5: 1.5	1.0: 0.0	1.0: 0.5	1.0: 1.5	1.5: 0.0	1.5: 0.5	1.5: 1.5
45	Regular	0.551	0.504	0.613	1.103	1.144	1.200	1.680	1.710	1.993
	Isotonic	0.696	0.650	0.760	1.303	1.335	1.388	1.945	1.937	2.248
60	Regular	0.553	0.582	0.578	1.077	1.121	1.139	1.687	1.618	1.663
	Isotonic	0.658	0.695	0.704	1.219	1.270	1.286	1.880	1.799	1.806
90	Regular	0.534	0.551	0.534	1.036	1.029	1.144	1.543	1.595	1.684
	Isotonic	0.610	0.629	0.620	1.130	1.126	1.242	1.655	1.709	1.791
150	Regular	0.500	0.521	0.491	1.036	1.064	1.034	1.572	1.568	1.619
	Isotonic	0.547	0.569	0.546	1.092	1.125	1.096	1.637	1.633	1.684
240	Regular	0.526	0.514	0.515	1.008	1.030	1.061	[illegible]	1.570	1.532
	Isotonic	0.556	0.545	0.552	1.043	1.065	1.099	1.575	1.611	1.569

Table 2. Average values for the estimated location (b_g) parameter

Sample Size	Method	Location (b)								
		0.0			0.5			1.5		
		Slope (a)								
		0.5	1.0	1.5	0.5	1.0	1.5	0.5	1.0	1.5
45	Regular	0.061	0.030	-0.015	0.583	0.522	0.513	1.402	1.561	1.391
	Isotonic	0.029	0.027	-0.014	0.448	0.486	0.492	1.181	1.432	1.340
60	Regular	0.012	0.042	0.002	0.538	0.500	0.518	1.494	1.610	1.559
	Isotonic	0.014	0.038	0.002	0.453	0.470	0.501	1.253	1.485	1.507
90	Regular	0.029	-0.007	0.001	0.520	0.505	0.507	1.535	1.489	1.496
	Isotonic	0.021	-0.006	0.001	0.460	0.483	0.496	1.347	1.425	1.466
150	Regular	0.003	-0.009	-0.021	0.502	0.498	0.493	1.577	1.565	1.495
	Isotonic	0.002	-0.009	-0.021	0.465	0.484	0.486	1.440	1.515	1.479
240	Regular	-0.009	0.001	-0.001	0.528	0.505	0.490	1.549	1.496	1.498
	Isotonic	-0.008	0.001	-0.001	0.503	0.496	0.486	1.461	1.470	1.487

Table 3. Variance and approximating variance for the estimated slope (a_g) parameter.

Sample Size	Method	Slope (a)								
		0.5			1.0			1.5		
		Location (b)								
		0.0	0.5	1.5	0.0	0.5	1.5	0.0	0.5	1.5
45	Regular	.0697	.0529	.0524	.1061	.1593	.3422	.3492	.3349	.7532
	Isotonic	.0749	.0606	.0670	.1265	.1937	.3831	.4827	.3822	.9016
	Approx.	.0470	.0405	.0220	.1010	.0932	.0763	.2055	.1992	.2187
60	Regular	.0425	.0547	.0570	.0955	.1518	.3209	.2500	.1661	.5360
	Isotonic	.0442	.0633	.0595	.1062	.1759	.3835	.3200	.2033	.5322
	Approx.	.0353	.0304	.0165	.0757	.0699	.0572	.1542	.1494	.1640
90	Regular	.0267	.0311	.0317	.0781	.0735	.1070	.1147	.1415	.2451
	Isotonic	.0266	.0349	.0334	.0846	.0898	.1094	.1262	.1558	.2509
	Approx.	.0235	.0203	.0110	.0505	.0466	.0381	.1028	.0996	.1094
150	Regular	.0146	.0126	.0126	.0454	.0460	.0507	.0760	.0923	.1188
	Isotonic	.0145	.0126	.0122	.0491	.0505	.0508	.0819	.0959	.1347
	Approx.	.0141	.0122	.0066	.0303	.0279	.0229	.0617	.0598	.0656
240	Regular	.0126	.0112	.0128	.0249	.0195	.0393	.0436	.0543	.0645
	Isotonic	.0124	.0116	.0122	.0260	.0204	.0400	.0435	.0552	.0644
	Approx.	.0088	.0076	.0041	.0189	.0175	.0143	.0385	.0374	.0410

Table 4. Variance and approximating variance for the estimated location (b_g) parameter

Sample Size	Method	Location (b) 0.0, Slope (a) 0.5	1.0	1.5	0.5, 0.5	1.0	1.5	1.5, 0.5	1.0	1.5
45	Regular	.3288	.0516	.0304	.3212	.0497	.0303	.2733	.2128	.0765
	Isotonic	.1215	.0436	.0279	.1595	.0405	.0264	.1442	.1296	.0564
	Approx.	.1423	.0460	.0259	.1467	.0494	.0287	.1860	.0835	.0591
60	Regular	.1571	.0318	.0200	.1596	.0431	.0227	.2744	.2753	.0809
	Isotonic	.0861	.0282	.0185	.0933	.0365	.0210	.1400	.1566	.0638
	Approx.	.1068	.0345	.0195	.1101	.0371	.0215	.1395	.0627	.0443
90	Regular	.1055	.0256	.0117	.1217	.0290	.0154	.1981	.0869	.0417
	Isotonic	.0751	.0237	.0111	.0752	.0256	.0146	.1122	.0658	.0363
	Approx.	.0712	.0230	.0130	.0734	.0247	.0144	.0930	.0418	.0295
150	Regular	.0385	.0138	.0069	.0653	.0116	.0090	.1488	.0879	.0258
	Isotonic	.0325	.0131	.0067	.0510	.0107	.0088	.1043	.0710	.0243
	Approx.	.0427	.0138	.0078	.0440	.0148	.0086	.0558	.0251	.0177
240	Regular	.0270	.0085	.0061	.0456	.0090	.0075	.0985	.0352	.0156
	Isotonic	.0243	.0082	.0060	.0384	.0088	.0074	.0719	.0320	.0153
	Approx.	.0267	.0086	.0049	.0275	.0093	.0054	.0349	.0157	.0111

Table 5. Mean squared error for the estimated slope (a_g) parameter

Sample Size	Method	Slope (a)								
		0.5			1.0			1.5		
		Location (b)								
		0.0	0.5	1.5	0.0	0.5	1.5	0.0	0.5	1.5
45	Regular	.0722	.0529	.0651	.1167	.1799	.3822	.3817	.3788	.9967
	Isotonic	.1131	.0830	.1344	.2181	.3060	.5336	.6812	.5733	1.461
60	Regular	.0453	.0614	.0631	.1014	.1664	.3401	.2849	.1799	.5624
	Isotonic	.0690	.1014	.1010	.1539	.2487	.4655	.4642	.2927	.6252
90	Regular	.0279	.0337	.0329	.0794	.0743	.1277	.1165	.1505	.2790
	Isotonic	.0386	.0516	.0477	.1014	.1057	.1679	.1502	.1995	.3354
150	Regular	.0146	.0130	.0127	.0467	.0501	.0518	.0811	.0969	.1330
	Isotonic	.0167	.0174	.0143	.0575	.0661	.0600	.1006	.1135	.1684
240	Regular	.0133	.0114	.0130	.0250	.0204	.0431	.0448	.0591	.0656
	Isotonic	.0156	.0137	.0149	.0279	.0247	.0498	.0490	.0676	.0693

Table 6. Mean squared error for the estimated location (b_g) parameter

Sample Size	Method	Location (b) 0.0			0.5			1.5		
		Slope (a)								
		0.5	1.0	1.5	0.5	1.0	1.5	0.5	1.0	1.5
45	Regular	.3326	.0525	.0306	.3281	.0502	.0305	.2829	.2165	.0883
	Isotonic	.1223	.0443	.0281	.1621	.0407	.0264	.2457	.1342	.0820
60	Regular	.1572	.0336	.0200	.1611	.0431	.0230	.2744	.2873	.0844
	Isotonic	.0863	.0297	.0185	.0956	.0374	.0210	.2009	.1569	.0638
90	Regular	.1064	.0256	.0117	.1221	.0290	.0154	.1993	.0870	.0417
	Isotonic	.0755	.0237	.0111	.0768	.0258	.0146	.1356	.0713	.0375
150	Regular	.0385	.0139	.0073	.0653	.0116	.0091	.1547	.0921	.0258
	Isotonic	.0325	.0131	.0071	.0522	.0109	.0089	.1079	.0712	.0247
240	Regular	.0270	.0084	.0061	.0464	.0090	.0076	.1009	.0352	.0156
	Isotonic	.0244	.0082	.0060	.0384	.0088	.0076	.0735	.0329	.0155

Appendix

A method which provides an approximation to the Rao-Cramér lower bound for the variances of $\hat{a}_g$ and $\hat{b}_g$ (the item subscript, g, will be omitted) were used.

Berkson's (1953) description of a proof for a slightly different parameterization of the logistic model was followed. First, some results will be stated regarding the logistic distribution which will be used in the derivation. As stated in the text, the distribution function for the logistic distribution used here is:

$$P(\theta_i) = \exp\,[1.7a\ (\theta_i - b)]\ /\ \{1 + \exp\,[1.7a\ (\theta_i - b)]\}. \tag{1}$$

If both numerator and denominator are divided by $\exp\,[1.7a\ (\theta_i - b)]$ then,

$$P(\theta_i) = \{1 + \exp\,[-1.7a\ (\theta_i - b)]\}^{-1}. \tag{2}$$

Also,

$$\begin{aligned} 1 - P(\theta_i) &= 1 - \exp\,[1.7a\ (\theta_i - b)]\ /\ \{1 + \exp\,[1.7a\ (\theta_i - b)]\} \\ &= \frac{1 + \exp\,[1.7a\ (\theta_i - b)] - \exp\,[1.7a\ (\theta_i - b)]}{\{1 + \exp\,[1.7a\ (\theta_i - b)]\}} \\ &= \{1 + \exp\,[1.7a\ (\theta - b)]\}^{-1}. \end{aligned} \tag{3}$$

Finally, dividing (1) by (3) we have,

$$P(\theta_i)\ /\ [1 - P(\theta_i)] = \exp\,[1.7a\ (\theta_i - b)]. \tag{4}$$

To find the approximating lower bound to the variance, it is necessary to examine the likelihood function. First, the likelihood function will be placed in a more convenient form.

From the text,

$$L = \sum_{i=1}^{m} r_i \cdot \ln P(\Theta_i) + \sum_{i=1}^{m} (n_i - r_i) \cdot \ln [1 - P(\Theta_i)]$$

$$= \sum_{i=1}^{m} r_i \cdot \ln P(\Theta_i) + \sum_{i=1}^{m} n_i \ln [1 - P(\Theta_i)] - \sum_{i=1}^{m} r_i \cdot \ln [1 - P(\Theta_i)].$$

Combining the first and third terms,

$$L = \sum_{i=1}^{m} r_i \cdot \ln \{P(\Theta_i) / [1 - P(\Theta_i)]\} + \sum_{i=1}^{m} n_i \ln [1 - P(\Theta_i)]$$

By substituting (3) and (4) into the equation,

$$L = \sum_{i=1}^{m} r_i \cdot \ln \{\exp [1.7a (\Theta_i - b)]\} + \sum_{i=1}^{m} n_i \ln \{1 + \exp [1.7a(\Theta_i - b)]\}^{-1}$$

$$= 1.7 \sum_{i=1}^{m} r_i \, a(\Theta_i - b) - \sum_{i=1}^{m} n_i \ln \{1 + \exp [1.7a (\Theta_i - b)]\}. \quad (5)$$

To obtain the lower bound for the variance of $\hat{a}$, it is necessary to find the first and second partial derivatives with respect to a. Taking the first derivative we get,

$$\frac{\partial L}{\partial a} = 1.7 \sum_{i=1}^{m} r_i (\Theta_i - b) - 1.7 \sum_{i=1}^{m} n_i (\Theta_i - b) \{1 + \exp [1.7a (\Theta_i - b)]\}^{-1}$$

$$\cdot \exp [1.7a (\Theta_i - b)].$$

By using result (2) and rearranging terms,

$$\frac{\partial L}{\partial a} = 1.7 \sum_{i=1}^{m} r_i (\Theta_i - b) - 1.7 \sum_{i=1}^{m} n_i (\Theta_i - b) \{1 + \exp [-1.7a(\Theta_i - b)]\}^{-1}.$$

Taking the second partial derivative with respect to a,

$$\frac{\partial^2 L}{\partial a^2} = 0 - 1.7 \sum_{i=1}^{m} n_i (\Theta_i - b) (-1) \{1 + \exp [-1.7a (\Theta_i - b)]\}^{-2}$$

$$\cdot \exp [-1.7a (\Theta_i - b)] \cdot (-1.7)(\Theta_i - b)$$

$$= -(1.7)^2 \sum_{i=1}^{m} n_i (\Theta_i - b)^2 \cdot \{1 + \exp [-1.7a (\Theta_i - b)]\}^{-2}$$

$$\cdot \exp [-1.7a (\Theta_i - b)].$$

Note that,

$$\{1 + \exp[-1.7a(\Theta_i - b)]\}^{-2} \cdot \exp[-1.7a(\Theta_i - b)]$$

$$= \{1 + \exp[-1.7a(\Theta_i - b)]\}^{-1} \cdot \frac{1}{\{1 + \exp[-1.7a(\Theta_i - b)] \cdot \exp[1.7a(\Theta_i - b)]\}}$$

$$= \{1 + \exp[-1.7a(\Theta_i - b)]\}^{-1} \cdot \{1 + \exp[1.7a(\Theta_i - b)]\}^{-1}.$$

Now, using (2) and (3) the above expression reduces to,

$$P(\Theta_i)\,[1 - P(\Theta_i)]. \tag{6}$$

Therefore, substituting back into the equation for $\partial^2 L/\partial a^2$ we have,

$$\frac{\partial^2 L}{\partial a^2} = -(1.7)^2 \sum_{i=1}^{m} n_i (\Theta_i - b_g)^2 P(\Theta_i)\,[1 - P(\Theta_i)]$$

Define, $w_i = n_i P(\Theta_i)\,[1 - P(\Theta_i)]$.

Then,

$$\frac{\partial^2 L}{\partial a^2} = -(1.7)^2 \sum_{i=1}^{m} w_i (\Theta_i - b_g)^2$$

and the approximate lower bound [which is $-1 / \left(\frac{\partial^2 L}{\partial a^2}\right)$] for the variance in estimating a is,

$$\frac{1}{(1.7)^2 \sum_{i=1}^{m} w_i (\Theta_i - b_g)^2} \tag{7}$$

To obtain the approximate lower bound for b, it is necessary to find the first and second partial derivatives of L with respect to b. Using the form of L in equation (5),

$$\frac{\partial L}{\partial b} = -1.7a \sum_{i=1}^{m} r_i - \sum_{i=1}^{m} n_i \{1 + \exp[1.7a(\Theta_i - b)]\}^{-1} \exp[1.7a(\Theta_i - b)] \cdot (-1.7)a.$$

By rearranging terms and using results (1) and (2),

$$\frac{\partial L}{\partial b} = -1.7a \sum_{i=1}^{m} r_i + 1.7a \sum_{i=1}^{m} n_i \{1 + \exp[-1.7a(\Theta_i - b)]\}^{-1}$$

Finding the second partial with respect to b,

$$\frac{\partial^2 L}{\partial b^2} = 0 + 1.7a \sum_{i=1}^{m} n_i \, (-1) \, \{1 + \exp [-1.7a \, (\Theta_i - b)]\}^{-2} \cdot \exp [-1.7a \, (\Theta_i - b)] \cdot (1.7a)$$

Rearranging terms,

$$\frac{\partial^2 L}{\partial b^2} = -(1.7a)^2 \sum_{i=1}^{m} n_i \, P(\Theta_i) \, [1 - P \, (\Theta_i)].$$

By taking, $w_i = n_i \, P(\Theta_i) \, [1 - P(\Theta_i)]$,

$$\frac{\partial^2 L}{\partial b^2} = -(1.7a)^2 \sum_{i=1}^{m} w_i$$

Then the approximate lower bound to the variance in estimating b is,

$$\frac{1}{(1.7a)^2 \sum_{i=1}^{m} w_i} \, .$$