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ABSTRACT

This is part one of a two-part SMSG Programed Algebra Text for high school students. The general plan of the course is to build upon the student's experience with arithmetic. The student is initially led to extract from his or her experience the fundamental properties of addition and multiplication. The text then introduces negative real numbers and extends the fundamental operations to develop the real number system. Chapter topics include: sets and the number line; numerals and variables; sentences; properties of operations; open sentences and English sentences; the real numbers; properties of addition; properties of multiplication; multiplicative inverse; properties of order; and subtraction and division. (MP)

PROGRAMED FIRST COURSE IN ALGEBRA

(REVISED FORM H)
PART I

Programed First Course in Algebra

Revised Form H

Student's Text, Part I

Prepared under the supervision of the
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PREFACE

TO THE STUDENT:

This text is written for you. It is not just a collection of problems, and it is not quite the same as other texts that you use. It is written so that you may study and learn algebra without too much outside help; it is called a programed text. It presents mathematics in an orderly way, asking questions along the way to be sure that you are following.

Mathematics is a pencil and paper subject, and as you read mathematics you should be prepared to work out details and examples to aid you in understanding. With this text, response sheets are provided for each section except for some of the review sections. In addition to the appropriate response sheet for the section on which you are working, provide yourself with paper for the purpose of doing necessary figuring as you go along.

For each item, you will record your response in the space provided on the response sheet, and then verify it. Most of the correct responses are in the shaded zones on the pages of your text. At some places, however, you will be told to refer to the Answer Key in the back of your book for the correct answer.

You will probably find it helpful to cover the shaded zone with a piece of paper or card while you make your response. Then you can check to see whether you were correct or not. In any case, you should write your response before you compare it with that provided. If you merely copy the given response, you will not be making the best use of the text.

Items which call for responses are of three general types. Those which occur most frequently are constructed response items like Items 1, 2, and 3 below.

1	The sum of 5 and 6 is ______.	11

In some cases you will be given a clue to record the entire answer indicated by the underline, as shown below:

2	The product of 6 and 5 can be written $6 \times$ ____.	6×5
3	$\frac{1}{3} + \frac{1}{2} = \frac{\square}{6}$.	$\frac{5}{6}$

Notice in Item 3 that a box has been used to show that the numerator of the fraction is missing, but that the entire fraction should be recorded on your response sheet. In Item 2 the underline is used to indicate that your response should include everything above the line.

A second type of item is a multiple choice question like Items 4 and 5. For each of these you will circle on the response sheet the letter of the response that you choose, and then read the appropriate part in the shaded zone which follows the item. In some cases, the shaded zone includes only a single discussion for all choices, as in 4 below.

4 In which of the following is there an error in addition?

[A] $5 + 7 + 13 = 25$

[B] $\frac{2}{3} + \frac{5}{3} + \frac{8}{3} = 5$

[C] $3.2 + 1 + .7 = 4.0$

In [C], $3.2 + 1 + .7$ may be written $3.2 + 1.0 + 0.7$, which equals 4.9, so this contains an error. [A] and [B] contain no errors, so [C] is the correct choice.

In other cases, there is a separate discussion for each choice as in 5 below.

5 In which of the following is the multiplication done correctly?

[A] $.3 \times .2 = .6$

[B] $63 \times 15 = 945$

[C] $\frac{2}{3} \times \frac{1}{7} = \frac{3}{21}$

[A] $.3 \times .2$ could be written $\frac{3}{10} \times \frac{2}{10}$, which equals $\frac{6}{100}$, or $.06$. This is not $.6$.

[B] Yes, this is done correctly.

[C] $\frac{2}{3} \times \frac{1}{7} = \frac{2 \times 1}{3 \times 7} = \frac{2}{21}$, so the multiplication was not done correctly.

If you are not really sure why a certain choice is correct, you will find it worthwhile to read the discussions for the other responses.

Other items are found in the form of practice sets of exercises. For these, spaces are provided on the response sheet, while the correct answers are given in the Answer Key at the back of each volume.

In most chapters, the final section includes a set of review problems. Some of these problems will serve to provide you with more practice on the topics of the chapter. Others have been written to challenge you with the opportunity of applying what you have learned to new situations. In some cases review problems have been designed to help you begin thinking about what is to come in succeeding chapters. Response sheets have been provided for those review sets which are in programed form. For others, answers will be found in the Answer Key.

Items marked with an asterisk(*) are considered optional--either they are more difficult, or they consist of material at a somewhat more advanced level. Omitting these items will not interrupt the continuity of what is to follow. However, if you are having no difficulty with the regular work of the text, you will find that the starred items provide better understanding and a sound background for further work in mathematics.

Your mathematical growth and the satisfaction and enjoyment you derive from your study of algebra will depend largely on reading and following directions carefully. You have a new and enriching experience ahead of you. We hope that you will make the most of it.

SUGGESTED REFERENCES

The following list of references is offered if you wish additional reading material on some of the topics mentioned in the text. The list draws on the volumes of the NEW MATHEMATICAL LIBRARY (NML), which is a series of monographs written for secondary school students and published by the School Mathematics Study Group. The references are by no means complete, but we hope that they will be helpful in directing your attention to available literature at the appropriate level. In the teachers' commentary is an additional list of references. These cover topics which are not necessarily included in this course but which may be of interest to you.

VOLUMES OF THE NEW MATHEMATICAL LIBRARY TO DATE:

Volume	Author and Title
1	Niven, Ivan NUMBERS: RATIONAL AND IRRATIONAL
2	Sawyer, W. W. WHAT IS CALCULUS ABOUT?
3	Beckenbach, E. and R. Bellman AN INTRODUCTION TO INEQUALITIES
4	Kazarinoff, N. D. GEOMETRIC INEQUALITIES
5	Salkind, C. T. THE CONTEST PROBLEM BOOK
6	Davis, P. J. THE LORE OF LARGE NUMBERS
7	Zippin, Leo USES OF INFINITY
8	Yaglom, I. M. GEOMETRIC TRANSFORMATIONS
9	Olds, C. D. CONTINUED FRACTIONS
10	Ore, Oystein GRAPHS AND THEIR USES
11	HUNGARIAN PROBLEM BOOK I
12	HUNGARIAN PROBLEM BOOK II
13	Aaboe, Asger EPISODES FROM THE EARLY HISTORY OF MATHEMATICS
14	Grossman, I. and W. Magnus GROUPS AND THEIR GRAPHS

Following each topic listed below is a set of number pairs such as (1 , 3). The first numeral refers to the volume in the series, and the second, in most cases, to the chapter in that volume. Thus, (1 , 3) refers to Volume 1, Chapter 3. In the case of Volume 6, which is divided by section instead of by chapter, the second numeral refers to the section specified. Volumes 11 and 12 are collections of problems of the Eötvös Competitions for the years 1894 through 1928. These are printed in chronological order. For these, the reference (11, 1899/3), for example, indicates Volume 11, Problem 3 of the 1899 competition. Those references which we consider to be challenging have an asterisk to the left of the reference designation.

Topic	References
Approximations to $\sqrt{2}$	(7 , 2)
Composite numbers	(6 , 20)
Decimal representation	(1 , 2), (1 , 3), (12 , 1907/3), (12 , 1917/2), (12 , 1925/2), (12 , 1927/2)
Divisibility	*(11 , 1899/3), (12 , 1925/1), (6 , 20)
Equivalence relation	*(10 , 7)
Identity element	(1 , 1)
Inequalities	(1 , 3), (3 , 1), (3 , 2); (3 , 3)
Graphs of inequalities	(3 , 3)
Linear equations	(6 , 22)
Natural numbers and integers	(1 , 1), (12 , 1906/3), *(12 , 1913/1), (12 , 1926/2)
Periodic decimals	(1 , 2)
π	(6 , 19)
Primes	(1 , 1), (1 , Append A), (6 , 20), (13 , 2)
Proofs	(1 , 2)

Properties of numbers	
Associative property	(1 , 1)
Closure	(1 , 1), (1 , 2), (1 , 4)
Commutative property	(1 , 1)
Distributive property	(1 , 1)
Quadratic equations	*(11 , 1896/2), *(11 , 1899/2),
Real numbers	
Irrational numbers	(1 , 3), *(1 , 4),
Rational numbers	(1 , 3), *(1 , 4)
Real numbers	(1 , 3)
Unique factorization	(1 , Append B)

CONTENTS

Chapter

Chapter 1

SETS AND THE NUMBER LINE

1-1. Sets and Subsets

We often wish to refer to collections of things, such as a herd of cows, a flock of sheep, a band of thieves, a set of dishes, the collection of numbers 1, 2, 3, 4, 5. Each of these collections is an example of a set. The things in a set we will call members or elements of the set. We can say that the elements of a set belong to the set.

1	We shall use the word ____ whenever we wish to talk about a collection of things.	set
2	The ____ of states of the United States has 50 members.	set
3	The set {1,3,5,7} has four ____, or members.	elements
4	{2,4,6} is the ____ of even counting numbers less than 7.	set

Above, we have introduced the use of braces, { }, to indicate that we are forming or displaying a set, such as {1, 3, 5, 7}. In this method of indicating a set, the order in which the elements are listed is not important, and no element is listed more than once. Notice that we use commas to separate the elements listed within the braces.

5	Use braces to write the set consisting of the last three letters of the alphabet. ____	{x,y,z}
6	1,2,3 is a list of numbers, but {1,2,3} is a ____ which has three members.	set

Often the elements of a given set share some characteristic. If they do, we may be able to use this characteristic to describe the set in words. On the other hand, we are free to form sets whose members have no simple relation to one another. An example might be the set, {a star, a pair of eyeglasses, a penny, the number 6}.

7 In {California, Colorado, Connecticut} the elements are states whose names begin with the letter _____. — **C**

8 In {Monday, Tuesday, Wednesday} the _____ are days of the week. — **elements, or members**

9 In {a,b,c,d,e} the elements are the first five _____ of the alphabet. — **letters**

Although a set may be described in words, it is sometimes easier to use braces and to list all the elements of a set:

> "{1,2,3,4}" is simpler than "the set of counting numbers less than five".

On the other hand, "the set of names of all citizens of the United States" is simpler than an actual listing of all names.

A capital letter is usually used to name a set. For example, we can write "F = {1,2,3,4}" or "F is the set of all counting numbers less than five".

We have referred several times to counting numbers. The set of counting numbers is the set of all the numbers you would use in counting: 1, 2, 3, and so forth. Thus, 1,037,518 is an example of a counting number, but $15\frac{7}{8}$ is not. Since the set of counting numbers cannot be listed completely, we might write {1,2,3,...}. Here we have written the set with enough elements to show the pattern and then have used the three dots to mean "and so forth". As we see here, sometimes we do list elements of a set in some chosen order to show a pattern that we have in mind. Thus, {1,3,5,7,...} indicates the set of odd counting numbers.

10 Which of the following is "the set of all counting numbers greater than six"?

[A] {6,7,8, ...}

[B] {8,10,12, ...}

[C] {7,8,9, ...}

[A] Is 6 greater than six? No.

[B] {8,10,12, ...} is the set of all even counting numbers greater than 6, but not of all counting numbers greater than six.

[C] Yes, {7,8,9, ...} is another way of writing "the set of all counting numbers greater than six". It might be read, "The set whose elements are 7, 8, 9, and so forth."

Another name that is used in referring to the counting numbers is the natural numbers. If we use the letter N to name this set, then

$$N = \{1,2,3, \ldots\}.$$

If we include the number 0 together with the counting (or natural) numbers, we obtain a new set, the set of whole numbers. If we use the letter W to name this set, then

$$W = \{0,1,2,3, \ldots\}.$$

11	What number is an element of W, the set of whole numbers, but not of N, the set of natural numbers? ______	0
12	If A = {4,5, , }, then A is the set of counting numbers which are greater than 3 and less than 8.	A = {4,5,6,7}
13	Is every element of N also an element of W? ______ (yes, no)	yes
14	Write the set B consisting of all two-digit whole numbers each of whose units digit is twice its tens digit. B = {12, , , } .	{12,24,36,48}
15	If C is the set of numbers less than 10 which are the squares of whole numbers, then C = ______.	C = {0,1,4,9}

An interesting question now arises. How shall we describe the set of whole numbers which are greater than 9 but less than 10? You may not wish to call this a set, since it has no elements. However, in mathematics we say that a set which contains no elements is the empty set or null set. We use the special symbol $\emptyset$ to denote the empty set. (Note: no braces are used when writing the empty set, $\emptyset$.)

How many elements does each of the following sets contain?

16	{0,1,2}	______	3
17	{0,1}	______	2
18	{0}	______	1
19	$\emptyset$	______	0

20 A symbol for the set of whole numbers between $\frac{4}{3}$ and $\frac{5}{4}$ is ______. $\emptyset$

21	The set of whole numbers less than 1 is ______.	{0}
22	The set of counting numbers less than 1 is ______.	∅
23	Are {0} and ∅ names for the same set? (If you are in doubt, look back at Items 16-19.) ______ (yes,no)	no

When the elements of a set can be counted with the counting coming to an end, the set is called a finite set. It will not always be desirable to list the elements of a finite set. The citizens of the United States form a finite set, but it would be quite a job to list them. The numbers from 1 to 1,000,000 form a finite set, too; again, it would be quite a job to list these.

When describing finite sets, it is also convenient to use three dots. For example, the set of counting numbers less than 10 could be written {1, 2, 3, ..., 9}. This is read, "the set consisting of one, two, three, and so forth, up to and including nine". Note the use of the comma after the three dots.

24 How would we write: E is the set of even numbers between 5 and 20 ?

[A] E = {6,8,10,12,14,16,18}

[B] E = {6,8,10, ..., 18}

[C] E = {6,8,10, ..., 20}

Either [A] or [B] is a correct selection; the three dots in [B] enable us to omit some members of the set, [C] is incorrect because 20 is not <u>between</u> 5 and 20.

25 Which of the following represent the same set of numbers?

M = {1, 3, 5, 7, 9, 11, 13, 15, 17, 19, 21, 23}

N = {1, 11, 21, 3, 13, 23, 5, 15, 7, 17, 9, 19}

P = {1, 3, 5, ..., 23}

[A] M and P only

[B] M and N only

[C] all three

The correct selection is [C]. The three dots in set (P) indicate that all the odd numbers from 7 through 21 are to be included. Set (N) is the same set with the elements listed in different order.

If it is not possible to count, (even if we had unlimited time), all the elements of a set with the counting coming to an end, the set is called an <u>infinite</u> set. As we mentioned earlier, a complete listing of the counting numbers is not possible. Thus, we say that the set of counting numbers is an infinite set.

26	If $R = \{0,1,2, \ldots,100\}$, $S = \{0,3,6, \ldots\}$, and T is the set of even numbers, which of these are infinite sets? [A] R and S [B] S only [C] S and T

The correct choice is [C]; both S and T are infinite sets. $\{0,1,2, \ldots,100\}$ is a finite set having 101 members.

Given the set $M = \{0,2,4,6,8\}$, and the set $L = \{0,3,6,9\}$, we can now form a set Q, of numbers which are elements of either M or L or both. Then $Q = \{0,2,3,4,6,8,9\}$. Note that although 0 and 6 are elements of both M and L, each is listed only once in Q. Given two sets A and B, the set whose members are elements of either A or B, or both A and B, is called the <u>union</u> of A and B. Thus, in our example above, Q is the union of M and L. The symbol for the union of sets is "$\cup$". For example, we write "$Q = M \cup L$".

27	Give the set of numbers which are elements of either $S = \{0,5,10,15,20,25\}$ or of T, the set of multiples of 10 which are less than 50, or of both S and T. [A] $\{0,5,10,15,20,25,30,40\}$ [B] $\{0,10,20\}$ [C] $\{0,5,10,15,20,25,10,20,30,40\}$

[A] is the correct selection. [B] listed only the numbers which are elements both of $S = \{0,5,10,15,20,25\}$ and of T, the set of multiples of 10 which are less than 50. [C] is not a proper choice because no element should be repeated in the listing.

28 If $M = \{1,2, \ldots,19\}$ and $N = \{17,18,19, \ldots,30\}$, then which of the following is the union of M and N ?

[A] $\{17,18,19\}$
[B] $\{1,2,3, \ldots,30\}$
[C] $\{30,29,28, \ldots,2,1\}$

Either [B] or [C] is a correct choice.

29 If $M = \{1,2,3, \ldots,19\}$, $N = \{17,18, \ldots,30\}$, and $R = \{1,2,3, \ldots,30\}$, then which of the following is true?

[A] R is the set whose members are elements of either M or N or both M and N.

[B] R is the union of M and N.

[C] $R = M \cup N$.

All three choices are correct.

If $R = \{0,1,2,3,4\}$, and $T = \{0,2,4\}$, then every element of T is also an element of R. In this case we say that T is a <u>subset</u> of R.

<u>Definition</u>. If every element of a set A belongs to a set B, then A is a <u>subset</u> of B.

There is one thing about the set $\emptyset$ which is important to remember: $\emptyset$ is a <u>subset</u> of <u>every</u> set. For example, suppose $A = \{1,2,3,4\}$. Now consider the set $\emptyset$. Is there an element of $\emptyset$ which does <u>not</u> belong to A ? Since $\emptyset$ does not have any elements, the answer is "no". Thus, $\emptyset$ is a subset of A.

If $P = \{0,1,2,5,7,8\}$, determine whether each of the following sets is a subset of P.

	Set	Is it a subset of P?	
30	$Q = \{0\}$	______	yes
31	$R = \{0,1,2,3,4\}$	______	no
32	$T = \{0,2,4\}$	______	no
33	$\emptyset$	______	yes
34	P	______	yes

Notice that P is a subset of P, since every element of P surely belongs to P. In fact, every set is a subset of itself.

35. Given the following sets:

X: the set of all whole numbers greater than and less than 7.

Y: the set of whole numbers less than 10.

Z: the set $\{1,2,3,4,5,6\}$.

Which of the following statements is not true?

[A] X and Z are the same set.
[B] X is a subset of both Y and Z.
[C] Y is a subset of X.
[D] Z is a subset of X.

The correct selection is [C]; you were asked for the statement which is not true. Notice that the same set may be described in many ways.

Item 35 above summarizes four points that had been mentioned previously. These are as follows:

(1) A set may be described in many ways.

(2) X is a subset of Y if every element of X is an element of Y.

(3) If X and Z are the same set, then X is a subset of Z and Z is a subset of X.

(4) A set is always a subset of itself.

36. Now let's see how many different subsets a certain finite set has. If $S = \{0,1\}$, the different subsets of S listed are:

[A] $\{0\}$, $\{0,1\}$, $\{1\}$, $\emptyset$
[B] $\{0\}$, $\{1\}$, $\{0,1\}$, $\{1,0\}$, $\emptyset$

The correct selection is [A]. Since $\{0,1\}$ and $\{1,0\}$ are two ways of listing the same set, these are not different subsets of S.

The following starred items are optional, as explained in the Preface.

*37	ϕ has only one subset. What is this subset? ______	ϕ
*38	{0} has two subsets; ϕ and ______.	{0}
*39	{0,1} has four subsets; {0,1}, {0}, {1} and ______.	
*40	{0,1,2} has ______ (how many) subsets.	
	These eight subsets are:	
*41	{0,1,2}, {0,1}, {0,2}, { , }, {0}, { }, {2}, ϕ.	{1,2}, {1}
*42	{0,1,2,3} has ______ (how many) subsets?	16

Here we have the start of a pattern for discovering the number of subsets in a finite set with any given number of elements.

*43. See if you can complete the following table and discover a rule for finding the number of subsets in any finite set:

number of elements	1	2	3	4	5	6	...	n, a counting number
number of subsets	2	4	8				...	
*in exponent form	2^1	2^2	2^3				...	

*also written--$2^1 = 2$, $2^2 = 2 \cdot 2$, $2^3 = 2 \cdot 2 \cdot 2$, etc.

When you finish, compare your table with the one on page i.

1-2. The Number Line

In this section we shall make a connection between numbers and points on a line. You have already had some experience with this idea when you used rulers, thermometers, etc. The scales on these instruments associate numbers with certain points on a line.

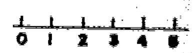
Let us begin by drawing a line.

On this line choose a point and label this point 0.

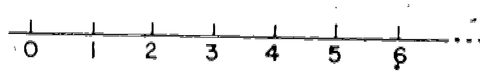
Next choose a second point to the right of the first and label it 1.

Using the distance between these two points as a unit of measure, mark off another point which is one unit to the right of the point labeled 1. Label this point 2.

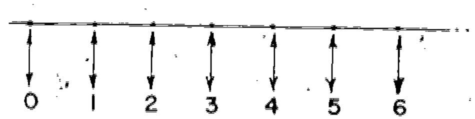
If we were to continue this process by marking off three more points, our line would look like ______.



We can think of this process as continuing without end, even though we cannot show the process beyond the margin of the page.

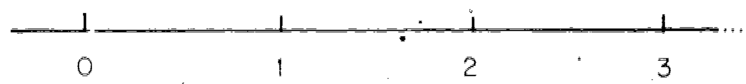


The use of three dots at the right of the line indicates that the process can be continued without end. This endless process associates each whole number with a point. A matching of points and numbers like this

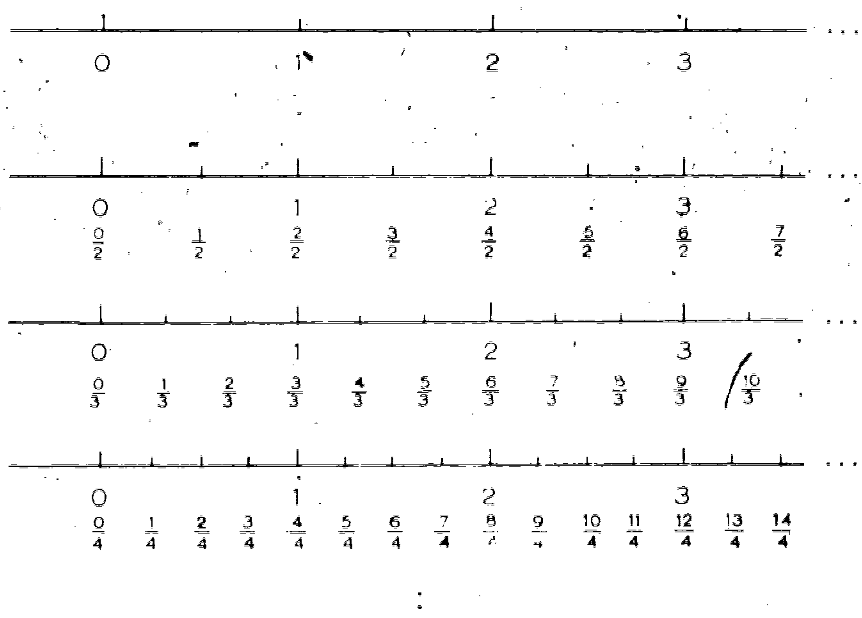


is an example of a correspondence. Since the set of whole numbers is an infinite set, this correspondence makes it clear that there are infinitely many points on the line.

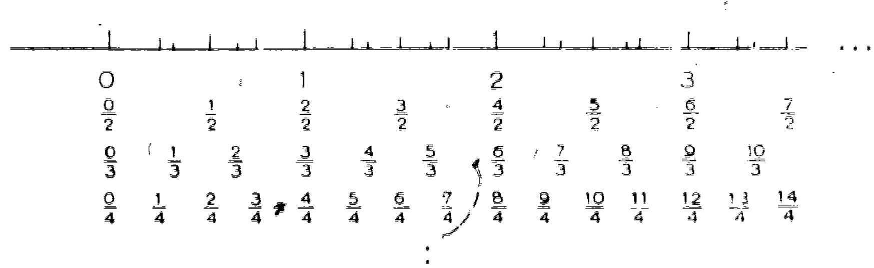
Return now to the line, some of whose points we have labeled with the whole numbers. The labeled points divide the part of the line to the right of the point labeled by 0 into intervals of unit length:



By dividing these intervals into halves, thirds, fourths, etc., we can label other points, making use of names other than 0, 1, 2, 3, ...



Here again we are to imagine that this process of division and labeling is continued without end. Now, if we take all the lines so obtained and put them together on one line in such a way that points labeled by the same number coincide, we have a labeling that looks like this:



The use of three dots below the diagram indicates an endless process.

In the preceding figure you will notice that many different names have been used to label the same point.

2	Notice that the point which corresponds to 2 has many labels: $\frac{4}{2}$, $\frac{6}{\square}$, $\frac{\square}{4}$, etc.	$\frac{6}{3}$, $\frac{8}{4}$
3	Similarly, the point associated with $\frac{3}{2}$ has the labels $\frac{3}{2}$, $\frac{6}{4}$, $\frac{9}{6}$, $\frac{\square}{8}$, etc.	$\frac{12}{8}$

All the names which label a given point are different names for the same number.

Names such as $\frac{5}{2}$, $\frac{4}{3}$, $\frac{9}{4}$ are called fractions.

<u>Definition</u>: We shall mean by a <u>fraction</u> a name or a symbol which indicates the division of one number by a second number (different from zero) where the name of the first number is written above a horizontal bar, and the name of the second is written below. The upper name is called the numerator and the lower name, the denominator.

The insistence in the definition that the second number must be different from zero comes from the fact that division by zero is not defined.

4 Which of the following are fractions?

$$\frac{25}{3} \qquad 8\frac{1}{3} \qquad \frac{2.79}{2} \qquad \frac{\frac{3}{4}}{5}$$

[A] $\frac{25}{3}$ and $8\frac{1}{3}$ [C] $\frac{25}{3}$, $8\frac{1}{3}$, and $\frac{2.79}{2}$

[B] $\frac{25}{3}$ and $\frac{\frac{3}{4}}{5}$ [D] $\frac{25}{3}$, $\frac{2.79}{2}$, and $\frac{\frac{3}{4}}{5}$

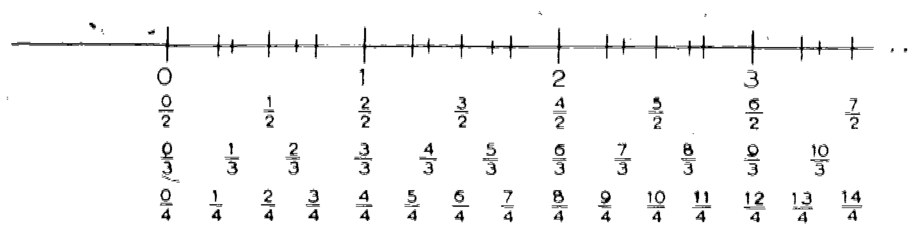
[D] is the correct choice. In [A], $8\frac{1}{3}$ is an abbreviation of $8 + \frac{1}{3}$ and is not written in the form of an indicated division of one number by another. The symbol $\frac{2.79}{2}$, which is in the proper form, has been omitted in [B]. [C] not only included $8\frac{1}{3}$, which should not have been included, it has omitted $\frac{\frac{3}{4}}{5}$ which meets the definition.

A number which can be represented by a fraction indicating the quotient of two whole numbers, excluding division by zero, is called a <u>rational</u> <u>number</u>.

	The number 15 is a whole number.	
5.	It may be written $\frac{\square}{1}$.	$\frac{15}{1}$
6	$\frac{15}{1}$ is a fraction indicating the quotient of two ______ numbers.	whole
7	Therefore, the whole number 15 is a ______ number.	rational
8	In a similar way, each whole number can be written as an indicated quotient of itself and ______.	1
9	Thus, every whole number is a ______ number.	rational

10	A number which can be represented by a ______ indicating the quotient of two whole numbers,	fraction
11	excluding division by zero, is called a ______ ______.	rational number
12	15 ______ (is, is not) a rational number.	is
13	$8\frac{1}{3}$ is a rational number since it can be written as the fraction $\frac{25}{3}$. ______ (true, false).	true
14	.61 may be written $\frac{\square}{100}$; hence,	$\frac{61}{100}$
15	.61 ______ (is, is not) a rational number.	is
	$\frac{3}{\frac{4}{5}}$ may be written as $\frac{15}{4}$; hence,	
16	$\frac{3}{\frac{4}{5}}$ ______ (is, is not) a rational number.	is

Look again at the line:



17 The "number names" that label a given point on the line are different names for the same ______ number. — rational

18 Every point which we have labeled corresponds to ______ (one, many) rational number(s). — one

19 Every number which can be represented as the indicated quotient of two whole numbers corresponds to one ______ on the line. — point

20 Thus, we have a correspondence between ______ on a — points

21 line and rational ______. — numbers

Notice that, as expected, $\frac{1}{2}$ and $\frac{2}{4}$ are labels for the same point. Similarly, if we had made more divisions on the line, $\frac{3}{4}$, $\frac{6}{8}$, ..., also would be labels for the same point. If the numerator and denominator of $\frac{1}{2}$ is multiplied by the same number (other than 0), then the resulting fraction is a name for $\frac{1}{2}$.

22 In which of the following sets may all of the numbers be used as labels for the same point?

[A] $(\frac{2}{3}, \frac{4}{6}, \frac{6}{9}, \frac{9}{6})$ [C] $(\frac{2 \times 1}{3 \times 1}, \frac{2 \times 2}{3 \times 2}, \frac{2 \times 3}{3 \times 3}, \frac{2 \times 5}{3 \times 5})$

[B] $(\frac{2}{3}, \frac{4}{6}, \frac{6}{9}, \frac{8}{12})$ [D] $(\frac{2 \times 0}{3 \times 0}, \frac{2 \times 1}{3 \times 1}, \frac{2 \times 2}{3 \times 2}, \frac{2 \times 3}{3 \times 3})$

[B] and [C] are correct selections. In [A], $\frac{9}{6}$ does not belong in the list. In [D], $\frac{2 \times 0}{3 \times 0}$ does not belong.

We shall call a line on which we associate points with numbers as we did above, a <u>number line</u>.

The number corresponding to a given point is called the <u>coordinate</u> of the point.

23 At the beginning of this section when we constructed the ______ line, the first point which we chose was assigned the coordinate 0. — number

24 The second point which we chose was assigned the ______ 1. — coordinate

25	The point halfway between the point whose coordinate is 2 and the point whose coordinate is 3 has the coordinate ____.	$\frac{5}{2}$
26	The number 85 ____ (is greater, is less) than the number 52.	is greater
27	The point with coordinate 85 is to the ____ (left, right) of the point with coordinate 52.	right
28	Since $\frac{1}{3}$ is ____ than $\frac{1}{2}$, the point whose	less
29	coordinate is $\frac{1}{3}$ lies to the ____ of the point whose coordinate is $\frac{1}{2}$.	left

	The number 63 is greater than the number 52 and less than the number 85; therefore, the point with	
30	coordinate ____ lies <u>between</u> the points with	63
31	coordinates ____ and 85.	52

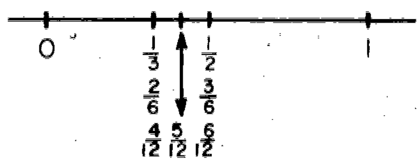
The number line strongly influences our way of thinking about numbers. It would not be unusual for a mathematician to use such language as, "63 lies between 52 and 85". When he talks like this he associates points on the number line so closely with the coordinates of the points that he does not call attention to this distinction every time. In much the same way, for example, when we say, "the point 4", we mean the point with coordinate 4.

Suppose we are given two points whose coordinates are rational numbers. (Picture them as close together as you please.) Can we find a third point between the two given points? Using the idea of associating numbers with points, we shall try to answer this question by working with numbers. Let us begin by considering the numbers $\frac{1}{3}$ and $\frac{1}{2}$.

32	The number $\frac{1}{3}$ has many names such as $\frac{1}{3}, \frac{2}{6}, \frac{3}{9}, \frac{\square}{12}$, etc.	$\frac{4}{12}$
	The number $\frac{1}{2}$ also has many names such as	
33	$\frac{1}{2}, \frac{2}{4}, \frac{3}{6}, \frac{4}{8}, \frac{\square}{12}$, etc.	$\frac{6}{12}$
	Using the name $\frac{4}{12}$ for the number $\frac{1}{3}$ and $\frac{6}{12}$ for the number $\frac{1}{2}$, it is now easy to see that there is a number between $\frac{4}{12}$ and $\frac{6}{12}$.	

34	For example: $\frac{\square}{12}$ is one such number.	$\frac{5}{12}$
35	Since the number $\frac{5}{12}$ is greater than $\frac{1}{3}$ and less than $\frac{1}{2}$, the point $\frac{5}{12}$ must lie ______ the points with coordinates $\frac{1}{3}$ and $\frac{1}{2}$.	between
36	The answer to our question, "Can we find a third point between the two given points?" is ______ (yes, no) for the points whose coordinates are $\frac{1}{3}$ and $\frac{1}{2}$.	yes

The process for finding a number between $\frac{1}{3}$ and $\frac{1}{2}$ and locating the point on a number line which has this coordinate is illustrated below.



Although we have used a special case by choosing $\frac{1}{3}$ and $\frac{1}{2}$, a similar process can be used with any two rational numbers. Using this process we can show that between any two points with rational coordinates there is a third point.

37	Two rational numbers between $\frac{2}{5,000}$ and $\frac{3}{5,000}$ are $\frac{\square}{15,000}$ and $\frac{\square}{15,000}$.	$\frac{7}{15,000}$, $\frac{8}{15,000}$
	Return to our example using $\frac{1}{3}$ and $\frac{1}{2}$.	
	Now that we have found the number $\frac{5}{12}$ between $\frac{1}{3}$ and $\frac{1}{2}$, let us find a number between $\frac{1}{3}$ and $\frac{5}{12}$.	
38	Another name for $\frac{1}{3}$ is $\frac{\square}{24}$:	$\frac{8}{24}$
39	another name for $\frac{5}{12}$ is $\frac{\square}{24}$.	$\frac{10}{24}$
40	Certainly, ______ is between $\frac{1}{3}$ and $\frac{10}{24}$.	$\frac{9}{24}$, or $\frac{3}{8}$
41	It ______ (is, is not) possible to continue this process by finding a number between $\frac{1}{3}$ and $\frac{9}{24}$.	is

42	Since the above process can be continued indefinitely, there are ______ points (a finite number of, infinitely many) between the points with coordinates $\frac{1}{3}$ and $\frac{1}{2}$.	infinitely many

From this example we should see that, in general, between any two points having rational numbers as coordinates there are infinitely many points.

We have a process for associating rational numbers with points on the number line. Does our process assign a rational number to every point to the right of the point with coordinate zero? Surprisingly, the answer is no. Later in this course we shall prove this to you. Meanwhile, we assume that every point to the right of the point with coordinate 0 has a number coordinate, although some of these numbers are not rational.

In Chapters 1 through 5 we shall be concerned with the set of numbers consisting of the number 0 and the numbers corresponding to all points to the right of the point 0. When we speak of "numbers of arithmetic" we shall mean numbers of this set.

	The following statement, The set of numbers of arithmetic contains elements which are rational numbers and also elements which are not rational numbers,	
43	is a ______ statement. (true, false)	true
44	Every number of ______ may be associated either with the point on the number line whose coordinate is 0	arithmetic
45	or with a point to the ______ of the point with coordinate 0.	right
	Every point of a number line to the right of the point whose coordinate is zero may be associated with	
46	a ______ of ______.	number of arithmetic

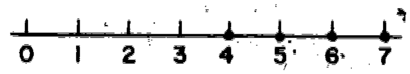
Now that we have the number line, let us show a use of it which will form the basis for many important applications later on.

Consider the set of whole numbers between 3 and 8.

47 Let us call the set F. Then, F = { , , , }. — F = {4,5,6,7}

48 We can associate every element of F with a ______ on a number line. — point

49 Each element of F is the ______ of exactly one point. — coordinate

Examine the number line: 

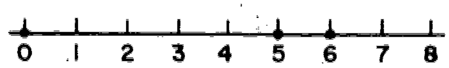
We have marked the points whose coordinates are

50 ______, ______, ______, and ______, with heavy dots. — 4,5,6,7

The set of points with coordinates 4, 5, 6, and 7 is called the graph of the set of numbers, {4, 5, 6, 7}. Indicating the points of the graph of a set by means of heavy dots as we have done above will be called "drawing the graph of the set" or "graphing the set".

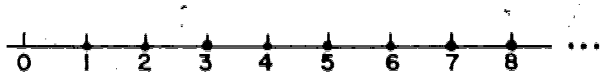
51 Draw the graph of the set $\{0,3,\frac{9}{2},2\}$. ______ — 

The graph of a set B has been drawn as follows:



52 B = { }. — B = {0,5,6}

The following figure appears on your response sheet.

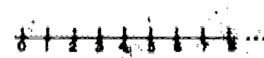
N 

W 

On the upper line we have drawn the graph of the set,

53 N, of ______ numbers (natural numbers). — counting

In your response sheet on the lower line, draw the graph of the set,

54 graph of the set, W, of whole numbers. ______ — 

Compare the graphs of set N and set W. For every

55 heavy dot on the upper line there ______ (is, is not) a heavy dot directly below on the lower line. — is

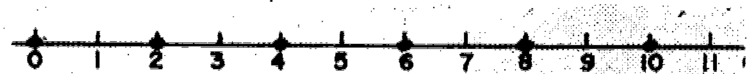
56	Thus, every point in the graph of the set ______	N
57	is also in the graph of the set ______.	W
58	Of the two sets, N, W, we can see that ______	N
59	is a subset of ______.	W

Given S = {0,3,4,7} and T = {0,2,4,6,8,10}.

The response for each of the following appears directly below the item.

60 Draw the graph of set S. ______

61 Draw the graph of set T. ______



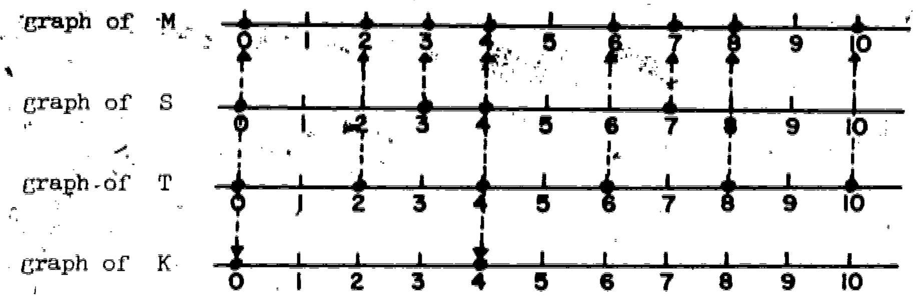
62 Draw the graph of the set, $M = S \cup T$. ______

63 Draw the graph of the set, K, of all numbers belonging to both S and T. ______

As you did this problem did you notice a scheme for drawing the graphs of set M and set K? If you did, you observed that for every dot on the graph of M (see p. 19) there is a dot directly below on one or both graphs of S and T.

You also observed that for every dot on the graph of K there is a dot directly above on both graphs of S and T.

The figure below summarizes your scheme.



Given two sets A and B, we can form a set consisting of all those elements which are members of both A and B. This set which we have formed from A and B is called the <u>intersection</u> of A and B. Thus in our example above, K is the intersection of S and T. The symbol for the intersection of sets is "$\cap$". For example, we write $K = S \cap T$.

64	If M = {1,2, ..., 19} and N = {17,18,19, ..., 30}, then which of the following is the intersection of M and N? [A] {17,18,19} [B] {1,2,3, ..., 30} [C] {30,29,28, ..., 2,1}

[A] is the correct choice; [B] and [C] are both $M \cup N$.

65	If M = {1,2,3, ..., 21} and N = {17,18,19, ..., 30} then which of the following is $M \cap N$? [A] {17,18,19,20,21} [B] {17,18,19} [C] {1,2,3, ..., 30}

[A] is correct. Although each element in [B] is a member of both M and N, these are not <u>all</u> the elements of both M and N; the set [C] is the union, not the intersection of M and N.

66	If M is the set of all elements each of which belongs to both set S and set T, then M is the ______ of S and T.	intersection

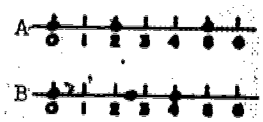
If $S = \{1,2,4,8,9\}$, $T = \{0,1,2,5,7,8\}$, and
67 $M = \{1,2,8\}$, then $M = S \_\_\_ T$. $S \cap T$

If $S = \{1,2,4\}$ and $T = \{0,1,3\}$, then the
68 union of S and T is $\{\_\_\_\_\_\_\_\_\}$. $\{0,1,2,3,4\}$

If $S = \{1,2,4\}$ and $T = \{0,1,3\}$, then
69 $S \cap T = \{\_\_\_\_\}$. $\{1\}$

If $S = \{1,2,4,8,9\}$ and $T = \{0,1,2,5,7,8\}$,
70 then $S \cap T = \{\_\_\_\_\_\_\}$. $\{1,2,8\}$

Consider the sets $A = \{0,2,5\}$ and $B = \{1,\frac{5}{2},4\}$.
Draw two numbers lines, one below the other, and
71 on these lines draw the graphs of A and B.



If set C is the set of numbers which are
elements of both A and B, what can you say
about set C from looking at the graphs of A
72 and B? ______ It has no elements, or it is the empty set.

73 What is the usual symbol for the set C? ______ $\emptyset$

Given fi te set $A = \{a,b,c,d,e\}$ and the finite
74 set B $,3,4,5\}$, each having ______ (how many) elements. 5

We can pair off, one-to-one, the elements of A
and B in many ways.

For example: 1, 2, 3, 4, 5
$\updownarrow$ $\updownarrow$ $\updownarrow$ $\updownarrow$ $\updownarrow$
75 a, b, c, d, ___ e

and 1, 2, 3, 4, 5
$\updownarrow$ $\updownarrow$ $\updownarrow$ $\updownarrow$ $\updownarrow$
76 d, ___, b, e, c a

The letters of the alphabet, in their usual order, can
77 be paired with the first ______ (how many) counting numbers 26
in their usual order as follows:

78 1, 2, ___, 4, ___, ___, ..., 25, 26 3,5,6
$\updownarrow$ $\updownarrow$ $\updownarrow$ $\updownarrow$ $\updownarrow$ $\updownarrow$ $\updownarrow$ $\updownarrow$
79 a, ___, c, ___, ___, ___, ..., ___, z b,d,e,f,y

	Consider $B = \{r,s,t,u\}$ and $C = \{r,s\}$. B and C	
80	are both ________ of B.	subsets
	However, C does not contain <u>all</u> the elements of B. C is said to be a <u>proper subset</u> of B.	
81	$G = \{1,2,3\}$ is a ________ subset of $H = \{0,1,2,3\}$.	proper
82	$E = \{2,4,6,8, \ldots\}$ is a ________ ________ of $N = \{1,2,3,4, \ldots\}$.	proper subset

	If we start with a finite set S, then a proper	
83	subset must have fewer ________ than the original set.	elements, or members
	We cannot pair off, one-to-one, the elements of a	
84	finite set S with the elements of a ________ ________ of S.	proper subset

The following starred items are optional.

	A startling development occurs if we consider an original set which is an <u>infinite set</u>.	
*85	$N = \{1,2,3,4, \ldots\}$ is an infinite ________.	set
*86	$E = \{2,4,6,8, \ldots\}$ is also an ________ ________, and	infinite set
*87	it is a proper ________ of N.	subset
	By means of the diagrams	
	$\begin{array}{cccccccc} 1, & 2, & 3, & 4, & 5, & \ldots, & 20, & \ldots \\ \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & & \updownarrow & \\ 2, & 4, & 6, & 8, & 10, & \ldots, & 40, & \ldots \end{array}$	
*88	we have paired off, <u>one-to-</u>________, the members of N with the members of a proper subset of N.	one-to-one

In fact, our rather vague description of an infinite set can now be made quite formal.

<u>Definition</u>: An infinite set, S, is a set whose elements may be paired off, one-to-one, with the elements of a proper subset of S.

*89 The set of multiples of 5 is an infinite ______, | set
*90 since {0,10,20,30, ...} is a proper ______ of | subset
{0,5,10,15, ...} and we can pair the elements of
these two sets one-to-one as below.

$$\begin{array}{ccccccc} 0 & 5 & 10 & 15 & 20 & 25 & \dots \\ \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \\ 0 & 10 & 20 & 30 & 40 & 50 & \dots \end{array}$$

Classify the following sets as finite or infinite.

All rational numbers between 1 and 2 whose numerators are whole numbers and whose denominators
*91 are whole numbers between 1 and 10. ______ | finite

*92 All rational numbers between 1 and 4. ______ | infinite

All rational numbers between 1 and 2 whose
*93 numerators are between 1 and 10. ______ | infinite

For the whole numbers, 0, 1, 2, ..., we notice that after each whole number there is a next whole number. For example, the next whole number after 583 is 584, and 584 is called the <u>successor</u> of 583.

94 Each whole number is followed by its ______. | successor

95 The successor of 567 is ______. | 568

96 The ______ of the number six hundred seventeen is the number six hundred eighteen. | successor

To find the successor of any whole number, the
97 number ______ is added to the whole number. The successor of [illegible] is [illegible]. | 1

The numbers, 5, 6, 7, ..., all follow the number 4.
98 Of these, the closest one is ______. | 5

The closest whole number following another is its
99 ______. | successor

Let's look at the rational numbers with the idea of successor in mind. Would adding 1 to say, [illegible], give us the next rational number after [illegible]? Is there a rational number closer to [illegible]?

100 How should you complete the following statement?

There ______ a next rational number after the number 2.

[A] is

[B] is not

Suppose there were a next rational number after 2. Between that number and 2 we know we can find a third number. (This process is shown in Items 32-36.) This third number is closer to 2 than the supposed "next" number. Thus no rational number can be next to 2, and the correct answer is [B].

101 Since there is no "next" rational number following a given rational number, the given rational does not have a ______.

[A] numerator

[B] denominator

[C] successor

[C] is the correct selection.

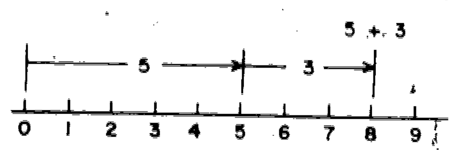
1-3. Addition and Multiplication on the Number Line

We have seen how to graph sets of numbers on the number line. Now let us use the number line to visualize addition and multiplication of numbers.

At the beginning we recall that

$$5 + 3$$

is a symbol for the number obtained by adding 3 to 5, namely, 8. This can be interpreted as moving from the point 0 to the point 5 and then moving from this point three units to the right, thereby locating a point with coordinate $5 + 3$, or 8.



As a different example, let us find

$$3 + 5$$

on the number line.

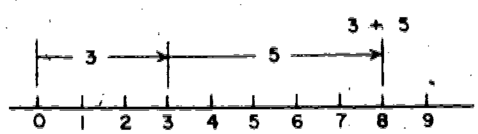
1 We move from the _______ with coordinate 0 to the point

2 point with coordinate _______, and then move 5 units 3

3. to the _______ (right, left), thereby locating the point with right

coordinate 3 + 5.

Thus:



Although the indicated additions "5 + 3" and "3 + 5" on the number line are different, we see by the diagrams that they terminate at the same point.

This emphasizes the fact that 5 + 3 and 3 + 5 are

4 different symbols for the same number, namely, _______. 8

Notice that although the number line continues indefinitely here, for convenience, we have omitted the three dots to the right of the line in the graph with an understanding that the line continues.

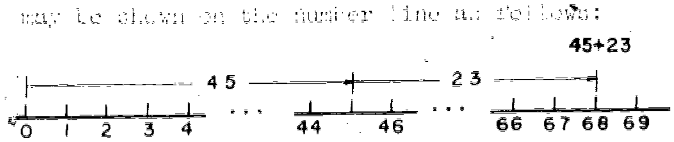
Indicate each of the following sums on the number lines provided on the response sheet.

5. 4 + 2

6. [illegible] + 4

Answers to 5 and 6 are on page i.

45 + 23 may be shown on the number line as follows:



7. Indicate the addition [illegible] on the number line provided on the response sheet. Answer on page i.

We may wonder whether addition on the number line is always possible. In other words, is the sum of any two rational numbers always a rational number? This question needs careful thought. We shall return to this question (and to similar ones) in later chapters.

Look at the example $5 + 3 = 8$. Notice that we can find a rational number so that when we add this number to 5, the result will be 8; we can also find a rational number so that when we add this to 3, the result will be 8. Recall that, in looking for the answer to the question,

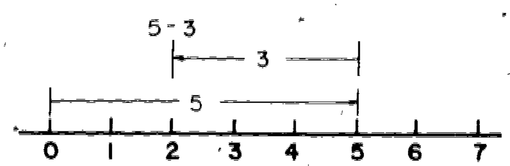
What number added to 5 is 8 ?,

we find this number by subtraction. That is, the same number, 3, is the answer to each of the questions

$$5 + \text{what number} = 8 \ ?$$
$$\text{and} \quad \text{What number} = 8 - 5 \ ?$$

Closely tied to this, is the procedure for indicating the subtraction of numbers on the number line. This procedure is not difficult.

$5 - 3$ may be interpreted as a symbol for the number obtained by subtracting 3 from 5. This may be shown on the number line as



Notice that we moved from the point with coordinate 0 to the point with coordinate 5 and then moved 3 units to the <u>left</u>, thereby locating the point with coordinate $5 - 3$; that is, 2. Notice that the coordinate $5 - 3$ is <u>2</u> since $3 + \underline{2}$ is 5.

Use the number line to indicate the following:

8. [illegible] - 4

9. 2 - [illegible]

10. [illegible]

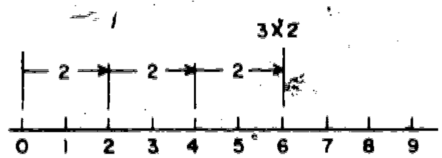
Answers to 8 and 9 are on page i; answer to 10 is on page ii.

The procedure for visualizing the multiplication of two counting numbers is similar to that of addition if we recall that, for example,

$$3 \times 2$$

is a symbol for the number obtained by adding three 2's.

Thus

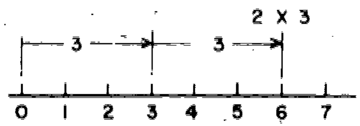


As a different example consider 2×3.

2×3 is a symbol for the number obtained by adding

11 ________ 3's. 2
(how many)

Thus



Although the indicated multiplications "3×2" and "2×3" on the number line are <u>different</u>, we see by the diagrams that they terminate at the same point.

12 This emphasizes the fact that 3×2 and 2×3 are different symbols for the same number, namely, ________. 6

Show the process of multiplication on the number line for each of the following.

13 4×2

14 2×4

15 3×5

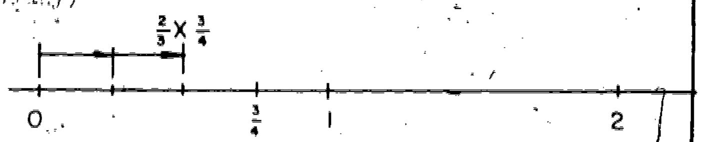
Answers to 13, 14, and 15 are on page ii.

We may visualize the multiplication of rational numbers as follows: Consider "$\frac{2}{3} \times \frac{3}{4}$", which is a

16 symbol representing two-thirds of ________. three-fourths

Since we are concerned with thirds, on the number line we divide the interval from 0 to $\frac{3}{4}$ into three equal intervals.

17 We can move from the point 0 to the right ______ (how many) of the smaller intervals: two



Indicate each of the following on the number line:

18 $\frac{1}{2} \times 4$

19 $\frac{2}{4} \times 2$

20 $\frac{3}{4} \times \frac{2}{3}$

Answers to 18, 19, 20, are on page ii.

You will have noticed that when we add or multiply two whole numbers we always obtain a whole number. Further, in all our examples using the number line, we end at a point which is either at the point 0 or is to the right of that point. If we tried to apply our method of indicating "subtraction" to 5 - 7, we would find ourselves moving to the left of the point with coordinate 0, where (as yet) we have not labeled any points. We shall learn about the possibility of labeling such points in Chapter 6.

Suppose we start with the set T = {1,2}. If we decide to add together two elements of T, we find that there are three possible sums, namely,

$$1 + 1 = 2,$$
$$1 + 2 = 3,$$
$$\text{and} \quad 2 + 2 = 4.$$

Thus the set S, whose elements are all possible

21 sums of two elements of T, is S = { , , }. {2,3,4}

22 S ______ (is, is not) a subset of T. is not

Let U be the set of all possible sums of two elements of V = {1,2,3,4}.

23 Then U = {______}. {2,3,4,5,6,7,8}

24 U contains ______ (how many) elements. 7

25 U ______ (is, is not) a subset of V. is not

Let us consider $Q = \{0,1\}$.

If we form a new set P, whose elements consist of all possible products of two elements of Q,

26 then $P = \{$ ______ $\}$. — $\{0,1\}$

27 Notice that P ______ (is, is not) a subset of Q. — is

(Actually, in this case, the set P is the same set as Q. Our attention, however, is directed to the fact that P is a subset of Q: every element of P is an element of Q.)

Here is another example:

Given the set of whole numbers, $W = \{0,1,2,3,\ldots\}$.

28 The sum of two whole numbers is itself a ______ number. — whole

The set of possible sums of two whole number

29 ______ (is, is not) a subset of W. — is

We shall say: If the sum of any two elements of a certain set is itself an element of the same set, then the set is closed under the operation of addition.

30 The set of whole numbers ______ (is, is not) closed under the operation of addition. — is

The sum of two even whole numbers is an even whole number.

Therefore, the set of all even whole numbers is

31 closed under the operation of ______. — addition

On the other hand, the sum of two odd whole numbers

32 is ______ (even, odd), so the set of all odd whole numbers — even

33 is not ______ under the operation of addition. — closed

34 Is $B = \{0,1\}$ closed under addition? ______ — no $[1 + 1 = 2]$

35	Is F = {0,4,8,12,...} closed under addition? ______	yes
36	If Z is a finite set of whole numbers that is closed under addition, then Z = {______}.	{0} [This is the only possible answer.]

The last response was a hard one. If you answered it correctly, or if you now see that {0} is the only answer, we hope you have gained understanding of what is meant by a set being "closed under the operation of addition."

	Are [illegible] generalize a bit?	
37	If [illegible] = {0,1}, then ______ (is, is not) closed under the operation of addition, and ______ (is, is not) closed under the operation of multiplication.	is not is

	O.K.? (If you missed the last response, remember that 0 × 0 = 0, 0 × 1 = 0 and 1 × 1 = 1 are the only possible products.)	
38	The set of all even whole numbers is ______ under the operation of multiplication.	closed
40	The set of all odd whole numbers ______ (is, is not) closed under multiplication.	is [The <u>product</u> of two odd numbers is odd.]

In particular, we have seen that <u>the set of whole numbers is closed under addition and under multiplication</u>.

W = {0,1,2,3,...}
E = {0,2,4,6,...}
Q = {0,1}
S = {0,6,12,18,...}
T = {1,3,5,7,...}
I = {0}

Which of the above are closed under the operation

*41 of addition? _______ W,E,S,I

*42 under the operation of multiplication? _______ W,E,Q,S,T,I

*43 under the operation of subtraction? _______ I

Chapter 2

NUMERALS AND VARIABLES

2-1. Numerals and Numerical Phrases

The symbols 6, VI, 5 + 1, and $\frac{12}{2}$ are all names for the same number. All of them name the coordinate of the same point on the number line.

Usually it is more convenient to use the name 6 than VI, 5 + 1, or $\frac{12}{2}$, because 6 is simpler. The simplest, or generally preferred, name of a number is called its <u>common</u> name. The common name for the numeral 2×7 is 14.

1	8×1, $6 + 2$, $\frac{32}{4}$, $10 - 2$ are all names for the same number. The common name of this number is ______.	8
2	The ______ name for $\frac{15}{3}$ is 5.	common
3	The common name for 32×15 is ______.	480

You are a person. You have a name. When people want to talk about you, they have to use your name, but you are not the same as your name. In the same way, the name of a <u>number</u> is not the same as the number. "2 is an even number" is a shorthand way of saying, "The number named by 2 is even."

A symbol which names a number is called a <u>numeral</u>. Thus, 8, 7 + 4, $\frac{17}{5}$, and 653 - 47 are all numerals. We use the word "numeral" when we want to emphasize the fact that a certain symbol <u>names</u> a number. In general, we shall continue to use phrases like "the number 2" instead of "the number named by 2" when there is no good reason for doing otherwise.

Consider the statement $5 + 7 = 2 \times 6$. This use of the equal sign illustrates its general use with numerals: An equal sign standing between two numerals asserts that the numerals name the same number. "$5 + 7 = 2 \times 6$" is an assertion which is true, since $5 + 7$ and 2×6 <u>are</u> numerals which represent (or name) the same number. On the other hand, "$5 + 7 = 5 + 5$" is an assertion which is false, since $5 + 7$ and $5 + 5$ do <u>not</u> name the same number.

4	$\frac{10}{2}$ is a numeral for the number we usually call ______.	5

number

numerals

number

product

sum

indicated

2×15, or 15×2

12	$(4 + [illegible]) \times 2 =$ ____ $\times 2$, since $4 + [illegible] = [illegible]$	13×2
13	$4 + ([illegible] \times 2) =$ $4 +$ ____, since $[illegible] \times 2 = [illegible]$	$4 + 18$
	Give the common name for each of the following.	
14	$([illegible] \times 6) + ([illegible] \times 7)$ [illegible] ____	28
15	$([illegible] \times 4) - ([illegible] \times [illegible])$ [illegible] ____	16
16	$[illegible] + ([illegible] \times 7)$ ____	23
17	$([illegible] \times [illegible]) - (7 \times 5)$ ____	5
18	$[illegible] \times ([illegible] + [illegible])$ ____	54

The symbol "×" for multiplication is often replaced by a dot "·". Hence "2 × 3" can be written as "2 · 3". We agree also that if two numerals are placed side by side we have an indicated product. For example "2(7 - 4)" means "2 × (7 - 4)". Notice, however, that "23" is already established as the common name for the number twenty-three and so cannot be interpreted as the indicated product "2 × 3". A similar exception is "$2\frac{1}{2}$" which means "$2 + \frac{1}{2}$" rather than "$2 \times \frac{1}{2}$". We may, however, write "2(3)" or "(2)(3)" in place of "2 × 3". Similarly, "$2 \times \frac{1}{2}$" might be replaced by "$2 \cdot \frac{1}{2}$" or "$2(\frac{1}{2})$".

	Give the common name for each of the following.	
19	$[illegible]([illegible])$ ____	48
20	$[illegible]([illegible])$ ____	30
21	$[illegible]([illegible] + [illegible])$ [illegible] ____	24
22	$[illegible]([illegible] + [illegible])$ [illegible] $\times$ ____ [illegible] ____	7, 21
23	$[illegible]([illegible] + [illegible])$ ____	50
24	$[illegible]([illegible] - [illegible])$ ____	10
25	$(4 + [illegible])([illegible] + [illegible])$ [illegible] $\times$ [illegible] ____	153
26	$([illegible] + 4)([illegible] - 4)$ ____	20
27	$([illegible] - [illegible])([illegible] + [illegible])$ ____	45
28	$([illegible] - [illegible])([illegible] + [illegible])$ ____	56

[illegible]

We like to be as concise as possible in writing. Very often we have to write something like "the sum of [illegible] and the product of 4 and 5." We have seen that the expression can be written conveniently as:

$$[illegible] + ([illegible] \times [illegible]).$$

We [illegible] the confusion about the meaning of [illegible] $\times$ 5 + 7 by the use of parentheses. We also avoid confusion of this sort by making the agreement that when no parentheses are used in such an expression, the multiplication is done first, before the addition. Using this agreement

$$[illegible] \times [illegible] \text{ means } [illegible]$$

while

$$([illegible] + [illegible]) \times [illegible] \text{ means } [illegible] \times [illegible].$$

[illegible] when there are no parentheses [illegible] multiplication precedes subtraction.

$$[illegible] - [illegible] \times [illegible] \text{ means } [illegible] - ([illegible] \times [illegible]), \text{ or } [illegible] - [illegible] = [illegible], \text{ or } [illegible].$$

[illegible] use the agreement that multiplication precedes addition or subtraction.

[illegible]	[illegible] $\times$ [illegible] ______	39
[illegible]	[illegible] $\times$ [illegible] ______	31
[illegible]	[illegible] $\times$ [illegible] ______	70
[illegible]	[illegible] $\times$ [illegible] ______	43
[illegible]	[illegible] $\times$ [illegible] ______	35
[illegible]	[illegible] $\times$ [illegible] ______	29
[illegible]	[illegible] $\times$ [illegible] + 7 ______	17
[illegible]	[illegible]([illegible] + [illegible]) ______	27
[illegible]	[illegible] $\times$ 7 ______	37
[illegible]	[illegible] $\times$ [illegible] = [illegible] ______	20

You will not need [illegible] the agreement stated above. However, when in doubt you should always insert parentheses to make sure that the meaning is clear.

An expression such as $\frac{5(6 - 2)}{13 - 3}$ is understood to be an indicated quotient of two numbers. Thus

$$\frac{5(6 - 2)}{13 - 3} = \frac{20}{10} = 2.$$

39. $\frac{2(1 + 3)}{4 - 1} = \frac{2 \times 4}{3} =$ ______ 8/3

40. $\frac{5(7 - 3)}{7} =$ ______ 20/7

41. $\frac{(7 - 3)(2 + 1)}{3(6 - 3)} =$ ______ 4/3

42. One pair of parentheses may lie within another. For example, what do you think is the correct common name for

$$\left(\left((3 + 4) \times 5\right) - 1\right) \times 2\ ?$$

[A] [illegible] [C] [illegible]

[B] [illegible] [D] none of these

The correct choice is [C]. If you chose [C] and are certain of your method, you may skip Items 43-48.

In $\left(\left((3 + 4) \times 5\right) - 1\right) \times 2$,

43. $3 + 4$ can be replaced by ______. 7

44. Now we have: $\left(\left((7) \times 5\right) - 1\right) \times$ ______ 2

45. $(7) \times 5 =$ ______ 35

46. We have: $\left((35) - \text{\_\_\_}\right) \times 2$ 1

47. $(35) - 1 =$ ______ 34

48. So we have: $(34) \times 2$. The result is ______. 68

A *numerical sentence* is a statement about numbers. For example, [illegible]

$$[illegible]$$

[illegible] "The product of [illegible] and the sum of [illegible] and 7 is the same as the difference of [illegible] and [illegible]."

In this sentence we have two numerals, $2(3 + 7)$ and $23 - 3$, connected by "$=$". Each numeral plays the role of a phrase in English. When we want to emphasize this role of a numeral in a sentence we speak of a <u>numerical</u> <u>phrase</u>. $2(3 + 7)$ and $23 - 3$ are examples of numerical phrases.

The sentence

$$2(3 + 7) = 23 - 3$$

is a true sentence, as you can readily check.

Now consider the numerals, or numerical phrases, $(3 + 1)(5 - 2)$ and 10. We can make of them the sentence

$$(3 + 1)(5 - 2) = 10.$$

<u>This</u> happens to be a <u>false</u> sentence. (Check to be sure.)

False sentences, as well as true ones, will turn out to be important in your later work. A numerical sentence can be true or false, but not both.

Determine which of the following sentences are true and which are false.

		(true, false)	
49	$(5 + 3)2 = 5 + 3(2)$		false
50	$3(4 + \frac{1}{3}) = 3(4) + 3(\frac{1}{3})$		true
51	$20 - 5(2) = 30$		false
52	$\frac{4 + 7}{2} = 4 + \frac{7}{2}$		false
53	$(\frac{12}{3} + 6) - 2 = \frac{12}{3} + (6 - 2)$		true

54 Which of the following represent(s) 72 as an indicated product?

X. $12 \times 5 + 12$

Y. 36×2

Z. $30 + 42$

[A] X and Y

[B] Y only

[C] Y and Z

$12 \times 5 + 12$ <u>contains</u> an indicated product, but the numeral is not an indicated product. $30 + 42$ is an indicated sum. Your choice should have been [B].

55	Which of the following correctly translates the numerical sentence $3(5 + 7) = 36$ into an English sentence? [A] Three times five plus seven is thirty-six. [B] Three times the sum of five and seven is thirty-six.	

The correct choice is [B]. Choice [A] fails to show that we first form the sum of 5 and 7. It could mean $(3 \times 5) + 7 = 36$, which is a false sentence.

Which of the following numerical sentences are true and which are false?

56	$4(5) + 4(8) = 4(13)$	(true, false)	true
57	$(3 + 7)4 = 3 + 7(4)$	______	false
58	$3(8 + 2) = 6 \times 5$	______	true
59	$2(5 + \frac{1}{2}) = 2(5) + 2(\frac{1}{2})$	______	true
60	$\frac{16}{2} + 4 - 3 = \frac{16}{2} + (4 - 3)$	______	true
61	$23 - 5(2) = 36$	______	false
62	$\frac{7 + 2}{2} = 7 + \frac{2}{2}$	______	false

Some of the important ideas of this section are reviewed in Items 63-73 below.

63	The names of numbers, as distinguished from numbers themselves, are called ______.	numerals
64	In $(8 + 5) \times 7$, the numeral $8 + 5$ is enclosed in ______.	parentheses
65	The numeral $8 + 5$ is an indicated ______,	sum
66	while 13×7 is an ______ ______.	indicated product
67	A numerical sentence may be true or false but not ______.	both
68	$2(3 + 7) = (2 + 3)(4 + 0)$ is ______ (true, false).	true

Insert parentheses in each of the following expressions so that the resulting sentence is true. For example:

Given	We Write	
$2 + 3 \cdot 4 + 5 = 45$	$(2 + 3)\cdot(4 + 5) = 45$	
$2 + 3 \cdot 4 + 5 = 29$	$2 + 3 \cdot (4 + 5) = 29$	
69 $2 + 3 \cdot 4 + 5 = 25$	______	$(2+3)\cdot 4+5 = 25$
70 $2 + 3 \cdot 4 + 5 = 19$	______	$(2+3\cdot 4)+5 = 19$, or $2+(3\cdot 4)+5=19$
71 $3 + 4 \cdot 6 + 1 = 43$	______	$(3+4)\cdot 6+1 = 43$
72 $3 + 4 \cdot 6 + 1 = 28$	______	$3+(4\cdot 6)+1 = 28$, or $(3+4\cdot 6)+1=28$
73 $3 \cdot 5 + 7 = 36$	______	$3\cdot(5+7) = 36$

2-2. Some Properties of Addition and Multiplication

Here is a problem: Find the common name for $0.8 + 7 + 3.2$. How would you proceed?

Perhaps you would think: $0.8 + 7 = 7.8$, and $7.8 + 3.2 = 11$. Your work might be written this way:

$$(0.8 + 7) + 3.2 = 7.8 + 3.2 = 11,$$

or you might notice first that $7 + 3.2 = 10.2$. In this case you would think:

$$0.8 + (7 + 3.2) = 0.8 + 10.2 = 11.$$

Both methods lead to the same result.

Let us compare carefully the two numerals

$$(0.8 + 7) + 3.2$$

$$0.8 + (7 + 3.2).$$

1 Both use only the numerals 0.8, 7, and ______. — 3.2

In both, the order of the numerals is the same.

2 Each involves only one kind of operation, namely ______. — addition

However, the two numerals differ as to the way in which 0.8, 7, and 3.2 are grouped.

3 The grouping is shown by ______. — parentheses

But in spite of the different grouping, the two

4 numerals name the same ______. — number

We can use these two numerals to make a true sentence:

$$(0.8 + 7) + 3.2 = 0.8 + (7 + 3.2)$$

Let's try some more examples of this sort:

5 $(5 + \frac{3}{2}) + \frac{1}{2} = \_\_\_ + \frac{1}{2}$ — $6\frac{1}{2}$

$= 7$

6 $5 + (\frac{3}{2} + \frac{1}{2}) = 5 + \_\_\_$ — 2

7 $= \_\_\_\_\_\_$ — 7

Thus the sentence $(5 + \frac{3}{2}) + \frac{1}{2} = 5 + (\frac{3}{2} + \frac{1}{2})$

8 is a ______ sentence. — true

The sentence $(1.2 + 1.8) + 2.6 = 1.2 + (1.8 + 2.6)$

9 is a ______ sentence. We can check this by noting that — true

$$(1.2 + 1.8) + 2.6 = 3 + 2.6$$

10 $= \_\_\_\_\_\_$ — 5.6

11 while $1.2 + (1.8 + 2.6) = 1.2 + \_\_\_\_$ — 4.4

12 $= \_\_\_\_\_\_.$ — 5.6

The sentence $(\frac{1}{3} + \frac{1}{2}) + \frac{2}{3} = \frac{1}{3} + (\frac{1}{2} + \frac{2}{3})$ is a

13 ______ sentence. We can verify this by showing that — true

14 $(\frac{1}{3} + \frac{1}{2}) + \frac{2}{3} = \frac{\square}{6}$ — $\frac{9}{6}$

15 and that $\frac{1}{3} + (\frac{1}{2} + \frac{2}{3}) = \frac{\square}{6}$. — $\frac{9}{6}$

From the previous true sentences, it appears that we have discovered: If we have any three numbers, the sums

(first number + second number) + third number

and

first number + (second number + third number)

are equal.

	We can say this as follows: "If you add a second number to a first number, and	
16	then add a third number to their ______, the result	sum
17	is the same as if you add the second and ______	third
18	numbers and then add their sum to the ______ number."	first

You have been computing sums like

$$0.8 + 7 + 3.2$$

for many years. Usually you compute such a sum in two steps. You think

$$0.8 + 7 + 3.2 = 7.8 + 3.2 \quad \text{or} \quad 0.8 + 7 + 3.2 = 0.8 + 10.2$$
$$= 11 \qquad\qquad = 11$$

In other words, you usually add several numbers in steps. At each step you add two numbers.

Addition is an example of what mathematicians call a binary operation. When we have a way of assigning a single number to each pair of numbers we are dealing with a binary operation.

19	Addition is a ______ operation, because we can find exactly one number which is the sum of two given numbers.	binary
20	Another binary ______ is multiplication, because for two numbers we can find a number which is their product.	operation

Now let us go back. We concluded: If we have any three numbers, the sum indicated by

(first number + second number) + third number

is the same as

first number + (second number + third number).

We have discovered a relationsip involving addition which holds for all the numbers of arithmetic. This kind of general statement about addition is called a property of addition. (The use of the word "property" here is similar to its use in the sentence: "Sweetness is a property of sugar.")

The particular property of addition we have discovered is called the associative property of addition.

You will discover that in all the algebra which follows, in both theory and applications, our main concern will be with those properties which hold for all numbers with which we work. In these first few chapters, we will be considering the set of numbers of arithmetic. Therefore, we may omit the phrase "for numbers of arithmetic" when writing the properties.

	The word association suggests some sort of group. The associative property of addition has to do with the grouping of numbers in an indicated sum.	
21	$8 + (9 + 6) = (8 + 9) +$ _____ illustrates the	$(8 + 9) + 6$
22	_____ property of addition.	associative
	Illustrate the associative property of addition:	
23	$(15 + 18) + 23 =$ ___ + (___ + ___)	$15 + (18 + 23)$
24	$9 + (25 + 2) =$ _____	$(9 + 25) + 2$
25	The preceding items provided practice in applying the associative property of _____.	addition
26	"$4 + (3 + 5) = (4 + 3) + 5$" is an example of the associative _____ of _____.	property, addition
27	"$(7 + 6) + 3 = 7 + (6 + 3)$", is another example of the _____ _____ of _____.	associative property, addition

The associative property eliminates the need for any parentheses in a phrase like [illegible], since the grouping does not affect the sum.

28 Which of the following sentences do you know to be true by using only the associative property of addition?

W. $6 + 4 + 8 = (6 + 4) + 8 = 6 + (4 + 8)$

X. $(6 + 4) + 8 = 6 + (8 + 4)$

Y. $(6 + 8) + 4 = 6 + (8 + 4)$

Z. $(4 + 8) + 6 = 6 + (8 + 4)$

[A] W and Y only [C] X and Y

[B] all of them [D] Y and Z

The only correct selection is [A]. The associative property of addition does not allow us to change the order of the numbers, as is done in (X) and (Z).

Suppose you wished to find the sum $(5 + \frac{3}{2}) + \frac{1}{2}$. Can the associative property of addition be used to make the addition simpler? Yes, you would certainly suggest writing $5 + (\frac{3}{2} + \frac{1}{2})$ to represent the same sum.

So there are cases, such as this one, where regrouping by moving the parentheses in accordance with the associative property of addition will lead to a simpler way to find the sum.

29 Which of the following additions cannot be done in a simpler way by using only the associative property of addition?

E. $(\frac{39}{7} + 6) + \frac{101}{7}$ G. $\frac{3}{4} + \frac{1}{3}$

F. $\frac{3}{5} + (\frac{7}{5} + 8)$ H. [illegible] $+ (14 + 20)$

[A] E only [C] E and G

[B] G only [D] E, F and H

The correct choice is [C]. In (E), $(\frac{39}{7} + 6) + \frac{101}{7} = \frac{39}{7} + (6 + \frac{101}{7})$ by the associative property of addition, and the new expression is not simpler than the old. In (G), $\frac{3}{4} + \frac{1}{3}$ is not a situation in which the associative property of addition applies, since only two numbers are added.

30 If you earn $3 today and $5 tomorrow, will you earn the same amount as you would if you earn $5 today and $3 tomorrow? ______ yes

31	Does walking 3 miles before lunch and 5 miles after lunch cover the same distance as walking 5 miles before lunch and 3 miles after lunch? _____.	yes
32	If two numbers are added in different orders, the sum is the same. $4 + 9 = 13$ and $9 + 4 =$ ___.	13
33	$6 + 0 =$ ___ and $0 + 6 =$ ___. Thus we see:	6, 6
34	$6 + 0 =$ ___ $+ 6$	0 + 6

The exercises which you have just completed suggest another property of addition: For any two numbers,

first number + second number = second number + first number.

This property of addition is called the <u>commutative property of addition</u>.

	We can state the commutative property of addition as follows:	
35	If two numbers are added in different orders, the results are the _____.	same
36	The true sentence $9 + 8 = 8 +$ ___ illustrates the commutative property of addition.	8 + 9

37	$5 + 6 = 6 + 5$ by the _____ property of addition.	commutative
	The <u>associative</u> property of addition is concerned with the <u>grouping</u> of numbers.	
38	$(6 + 7) + 4 = 6 + ($ ___ $+$ ___ $)$ illustrates the associative property.	6 + (7 + 4)
	On the other hand, the <u>commutative</u> property of addition is concerned with the <u>order</u> in which we add two numbers.	
39	$(6 + 7) + 4 = 4 + (6 + 7)$ illustrates the _____ property of addition.	commutative
	It might help you to remember the word commutative and its meaning if you recall that a commuter goes back and forth.	

Let us review what we have learned.

40 We say that addition is a binary ______. operation

Two properties of addition are the associative
41 and ______ properties. commutative

$(4 + 9) + 3 = 3 + (4 + 9)$ is a true sentence which
42 illustrates the ______ property. (Did you notice how we used $(4 + 9)$?) commutative

43 $4 + (9 + 3) = (4 + 9) + 3$ illustrates the ______ property of addition. associative

44 $(4 + 9) + 3 = (9 + 4) + 3$ illustrates the ______ property of addition. commutative

Multiplication is also a binary operation. It might occur to you at this point to ask whether multiplication also has the associative and commutative properties. Make a guess. Then go on.

Let us consider the indicated product

$$7 \times (8 \times 3).$$

45 $7 \times (8 \times 3)$ is the product of 7 and ______. 24, or 8×3

46 The common name for $7 \times (8 \times 3)$ is ______. 168

If we are interested in seeing whether or not multiplication has the associative property, we can compare the indicated products $7 \times (8 \times 3)$
47 and $\underline{(\ \ \times\ \ ) \times 3}$. $(7 \times 8) \times 3$

48 $(7 \times 8) \times 3 = \underline{\ \ \times 3}$ 56×3

49 $= \underline{\qquad}$ 168

We have found that

$$(7 \times 8) \times 3 = 7 \times (8 \times 3)$$

50 is a ______ sentence. true

Another true sentence illustrating the same pattern is

51 $(5 \times 3) \times 7 = \underline{5 \times (\ \ \times\ \ )}$. $5 \times (3 \times 7)$

These and similar examples suggest to us that
52 multiplication has the ______ property, as does addition. associative

The associative property of multiplication may be stated as follows:

Given any three numbers, the products

(first number × second number) × third number

and

first number × (second number × third number)

are equal.

	Thus multiplication, like addition, is an operation	
53	that has the ______ property.	associative
	We could describe it by saying:	
	If you multiply a second number by a first number,	
54	and then multiply a third number by their ______,	product
	the result is the same as if you multiply the	
55	third number by the ______ number and then	second
56	multiply their product by the ______ number.	first
	The meaning of $7 \times 8 \times 3$ is clear, since the	
57	way the three numbers are ______ does not matter.	grouped

Now let us ask whether or not multiplication, like addition, also has the <u>commutative</u> property.

58	$5 \times 4 = 4 \times 5$, since $5 \times 4 = 20$ and $4 \times 5 =$ ___	20
59	$3 \times 4 = \underline{4 \times \quad}$	4×3
60	$241 \times 3 = 3 \times 241$, since $241 \times 3 =$ ___	723
61	and $3 \times 241 =$ ___	723
62	The indicated products $\frac{3}{4} \times \frac{1}{3}$ and $\frac{1}{3} \times \frac{3}{4}$ are both names for ______.	$\frac{1}{4}$
63	If two numbers are multiplied in different orders, the results are the same. For example, $4 \times 9 = 36$ and $9 \times 4 =$ ___.	36

The pattern illustrates this general property of multiplication:

Given two numbers,

first number × second number = second number × first number.

This property of multiplication is called, of course, the <u>commutative property of multiplication</u>.

We can summarize the results of this section: Addition and multiplication are two different binary operations both of which have the associative and commutative properties. Later, we shall see that not all binary operations have these properties.

For each of the following true sentences, tell what property of numbers is illustrated.

64	$(\frac{3}{5} + \frac{3}{2}) + \frac{1}{2} = \frac{3}{5} + (\frac{3}{2} + \frac{1}{2})$	______ property of ______	associative, addition
65	$\frac{3}{5} \times (\frac{3}{2} \times \frac{1}{2}) = (\frac{3}{5} \times \frac{3}{2}) \times \frac{1}{2}$	______ property of ______	associative, multiplication
66	$221 + (17+19) = (221+17) + 19$	______ property of ______	associative, addition
67	$6 + (8 + 4) = 6 + (4 + 8)$	______ property of ______	commutative, addition.
68	$(31 \times 13) \times 17 = 31 \times (13 \times 17)$	______ property of ______	associative, multiplication
69	$(31 \times 13) \times 17 = 17 \times (31 \times 13)$	______ property of ______	commutative, multiplication

The following example shows how properties of multiplication can be used to simplify a computation. Tell what property of multiplication is illustrated by each step.

70	$\frac{1}{5} \times (26 \times 5) = \frac{1}{5} \times (5 \times 26)$	______	commutative
71	$= (\frac{1}{5} \times 5) \times 26$	______	associative
72	$= 1 \times 26$ [since $\frac{1}{5} \times 5 =$ ___]		1
	$= 26$		

In the example, the work was made easier by rearranging and regrouping terms. We could do this because of the commutative and associative properties of multiplication.

In each exercise, you are to find the common name. Look for the easiest method; rearrange and regroup the numbers if it is helpful.

73 $4 \times 7 \times 25 =$ ______ 700

74 $\frac{1}{5} \times \frac{23}{7} \times 5 =$ ______ $\frac{23}{7}$

75 $73 + 62 + 27 =$ ______ 162

76 $(3 \times 4) \times (7 \times 25) =$ ______ 2100

Here is a true sentence.

$$15(7) + 15(3) = 15(7 + 3).$$

Does it illustrate any property we have studied? Think a moment; then read on.

The sentence does not illustrate a property of this section. An easy way to see this is to recognize that it involves both addition and multiplication, which leads us to suspect that there may be number properties other than those in this section.

2-3. Distributive Property

Problem: Suppose there were 15 boys in a stamp club and each brought 7 British stamps and 3 French stamps to an exhibit. How many stamps were brought in all?

In order to find how many stamps were brought, you

1 might think: 15(7), or ______, British stamps were brought. 105

2 Also, 15(__), or ______, French stamps were brought. 3, 45

For the total number of stamps, we find:

3 $15(\_\_) + 15(\_\_) = 105 + 45$ 7,3

4 $= \_\_\_\_$ 150

On the other hand, you might solve this problem in a different way. You might think:

5 The total number of boys was ____; 15

6 each brought 7 + ____ stamps. 7 + 3

Thus the total number of stamps which were brought

7 can be written: $15(\_\_ + \_\_) = 15(\_\_)$ (7 + 3), 10

8 $= \_\_\_\_$ 150

9 Since $15(7) + 15(3) = \_\_\_\_$ 150

10 and $15(7 + 3) = \_\_\_\_$ 150

we see that we obtain the same result, whether we use the first method or the second.

11 $15(7) + 15(3) = 15(\_\_ + \_\_)$ 15(7 + 3)

The problem we have just discussed can be solved in two different ways. Let us look at another example, which will illustrate the same pattern.

Problem: A candy store sells 150 boxes of candy, each containing $\frac{1}{2}$ pound of chocolates and $\frac{1}{3}$ pound of caramels. How many pounds of candy are sold?

Try to find two ways to do the problem above, following the pattern used in Items 1-11. Test to see whether or not the two methods lead to the same result. Then check your work by doing Items 12-22.

One method: The number of pounds in each box is

12 $\frac{1}{2} + \_\_\_\_$, or ____. $\frac{1}{2} + \frac{1}{3}$, $\frac{5}{6}$

13 Hence, there are ____$(\frac{5}{6})$ pounds in all. $150(\frac{5}{6})$

Another method:

14 In all, there are $150(\_\_)$ pounds of chocolates $150(\frac{1}{2})$

15 and ____$(__)$ pounds of caramels. $150(\frac{1}{3})$

16 Hence, there are $150(\_\_) + \_\_\_\_(\_\_)$ pounds in all. $150(\frac{1}{2}) + 150(\frac{1}{3})$

17	Since $150(\frac{1}{2}) + 150(\frac{1}{3}) = \_\_\_ + 50$	75
18	$= \_\_\_\_$	125
19	and $150(\frac{1}{2} + \frac{1}{3}) = 150(\quad)$	$150(\frac{5}{6})$
20	$= \_\_\_\_$	125
	we observe that the two methods give the same result. Thus, we see that the sentence	
	$$150(\frac{1}{2}) + 150(\frac{1}{3}) = 150(\frac{1}{2} + \frac{1}{3})$$	
21	is a ____ sentence.	true
	The sentence	
22	$$150(\_\_ + \frac{1}{3}) = 150(\frac{1}{2}) + 150(\frac{1}{3})$$	$\frac{1}{2}$
	is also a true sentence.	

23 We have found that

$$150\ (\frac{1}{2}) + 150(\frac{1}{3}) \quad \text{and} \quad 150(\frac{1}{2} + \frac{1}{3})$$

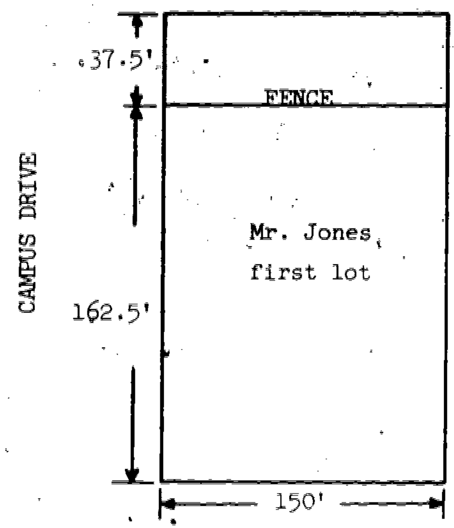
are two names for the same number. How would we write the number $12(\frac{2}{3} + \frac{3}{4})$ in a second way following the pattern of this discussion?

[A] $12(\frac{2}{3}) + 12(\frac{3}{4})$

[B] $(\frac{2}{3} + \frac{3}{4})12$

In both cases, the numeral names the same number as $12(\frac{2}{3} + \frac{3}{4})$. However, [A] is the correct choice, since this follows the pattern of the discussion.

Problem: Mr. Jones owns a city lot of rectangular shape, 150 feet deep, with a front of 162.5 feet. Adjacent to his lot, and separated from it by a fence, is another lot with the same depth, but with a front of only 37.5 feet.



What are the areas, in square feet, of each of these two lots and what is the sum of their areas?

24 The area of Mr. Jones' lot in square feet is 150(162.5), or ______. The total area of the two separate lots, in square feet, can be written as 24,375

25 $150(162.5) + 150(\_\_\_\_\_\_)$. 37.5

26 $150(162.5) + 150(37.5) = 24,375 + \_\_\_\_\_\_$ 5,625

27 $= \_\_\_\_\_\_$. 30,000

28 The total area of the two separate lots is ______ square feet. 30,000

Suppose that Mr. Jones buys the second lot and removes the fence. What is the area of the entire lot Mr. Jones now owns?

After Mr. Jones buys the neighboring lot, the area of his property, in square feet, can be written as

29 $150(162.5 + \_\_\_\_\_\_)$. — 37.5

30 $\therefore 150(162.5 + 37.5) = 150(\_\_\_\_\_\_)$ — 200

31 $150(200) = \_\_\_\_\_\_$ — 30,000

Therefore the area of the entire lot Mr. Jones now

32 owns is ______ square feet. — 30,000

Our results lead us to conclude that

33 $150(162.5 + 37.5) = 150(\_\_\_\_\_\_) + 150(\_\_\_\_\_\_)$ — 162.5, 37.5

34 is a ______ sentence. — true

If we examine the diagram showing the lots, we can see that

$150(162.5 + 37.5)$ is the number of square feet in the area of the largest rectangle.

$150(162.5)$ and $150(37.5)$ are, respectively, the number of square feet in the areas of the two smaller rectangles.

Since the largest rectangle is composed of the two smaller ones, the number of square feet in the area of the largest also corresponds to $150(162.5) + 150(37.5)$. Thus, each of the numerals $150(162.5 + 37.5)$ and $150(162.5) + 150(37.5)$ represents the number of square feet in the same area, and without computing we may assert the truth of the sentence

$$150(162.5 + 37.5) = 150(162.5) + 150(37.5).$$

[illegible] [illegible] is a ______ (true, false) sentence. — true

[illegible] We can [illegible] this by noting that [illegible] = ______ — 1075

while [illegible] = ______ + 75, and — 1000

1000 + 75 = ______. — 1075

	Decide which of the following sentences are true and which are false.		
39	$13(19 + 1) = 13(19) + 13(1)$	(true, false)	true
40	$2(\frac{1}{3} + \frac{1}{4}) = 2\frac{1}{3} + 2\frac{1}{4}$	______	false
41	$3(2.5) + 3(1.5) = 3(2.5 + 1.5)$	______	true
42	$4 \cdot 2 + 4 \cdot 5 = 4 \cdot 7$	______	true
43	$3(2) + 6(3) = 9(6)$	______	false
44	$15 \cdot 2 = 7 \cdot 2 + 8 \cdot 2$	______	true

The items above reveal a number pattern which can be stated as follows:

Given any three numbers of arithmetic,

first number × (second number + third number)

and

(first number × second number) + (first number × third number)

are equal.

This property of numbers is called the <u>distributive property</u> of multiplication over addition, or, as we shall frequently say, the <u>distributive property</u>.

	The distributive property can be described by saying:	
	If the sum of the second and third of three numbers is	
45	______ by the first number, the result is the same as	multiplied
46	if the ______ of the first and third is added to the	product
47	product of the first and ______.	second

	Complete each of the following illustrations of the distributive property:	
48	$20(\frac{5}{4} + \frac{1}{5}) = \underline{20(\quad) + 20(\quad)}$	$20(\frac{5}{4}) + 20(\frac{1}{5})$
49	$\underline{\quad(2 + 3)} = 20 + 30$	$10(2 + 3)$
50	$\underline{18(\quad + \quad)} = 18(\frac{2}{3}) + 18(\frac{1}{2})$	$18(\frac{2}{3} + \frac{1}{2})$

51 $30(\underline{\quad + \quad}) = 30(\frac{4}{7}) + 30(\frac{3}{7})$ — $30(\frac{4}{7} + \frac{3}{7})$

52 $19(\frac{4}{5} + \frac{1}{5}) = \underline{\quad(\frac{4}{5}) + \quad(\frac{1}{5})}$ — $19(\frac{4}{5}) + 19(\frac{1}{5})$

The true sentence

53 $13(19 + 1) = 13(19) + 13(\_\_\_)$ — 1

illustrates the distributive property.

Of course

$13(19) + 13(1) = 13(19 + 1)$

54 also illustrates the _______ property. — distributive

$13(19 + 1)$ is an indicated product of two numbers,

55 13 and ___ + ___. — 19 + 1

56 $13(19) + 13(1)$ is an indicated _______ of two — sum

57 numbers, $13(19)$ and ___( ). — 13(1)

In the sentence

$13(19) + 13(1) = 13(19 + 1)$

58 the same number, _______, has been expressed in two — 260

ways: as an indicated sum and as an indicated

59 _______. — product

If you are asked to compute $13(19) + 13(1)$ you may,

60 if you wish, compute $13(\_\_\_\_\_\_)$ instead. Since — 20

$13(20)$ is easier to compute than $13(19) + 13(1)$,

61 the _______ property helps us here to simplify a — distributive

computation.

Which is easier to compute, $150(162.5 + 37.5)$ or

62 $150(162.5) + 150(37.5)$? _______ ~~is~~ easier. — 150(162.5+37.5)

63 $150(162.5 + 37.5)$ is an indicated _______. — product

If we compare $12(\frac{2}{3} + \frac{3}{4})$ and $12(\frac{2}{3}) + 12(\frac{3}{4})$, we

note that both name the same number. The common

64 name for this number is _______. — 17

It is easier to compute this common name by using

65 the indicated _______, $12(\frac{2}{3}) + 12(\frac{3}{4})$. — sum

60

66 Remember, the ______ property tells us something about a certain indicated sum and a certain indicated product.

distributive

67 The distributive property is the basic number property that involves ______ (how many) operations, addition and multiplication.

two

68 Which of the following statements is correct by use of the distributive property of multiplication over addition? (Try to answer <u>without any computation</u>, so that you can see whether or not you recognize the pattern.)

[A] $15 + (12 \times 7) = (15 + 12) \times (15 + 7)$

[B] $4(2.5 + 4.5) = 4(2.5) + 4.5$

[C] $24(\frac{1}{8} + \frac{1}{6}) = 24(\frac{1}{8}) + 24(\frac{1}{6})$

The sentence $15 + (12 \times 7) = (15 + 12) \times (15 + 7)$ is false, which proves that addition is <u>not</u> distributive over multiplication. The sentence $4(2.5 + 4.5) = 4(2.5) + 4.5$ is also false; application of the distributive property would give the true sentence $4(2.5 + 4.5) = 4(2.5) + 4(4.5)$. The correct choice is [C].

Look at the sentence

$$(4 + 3)8 = 4(8) + 3(8) .$$

It is clear that this sentence is true, since $(4 + 3)8 = 56$ and $4(8) + 3(8) = 56$.

Moreover, this sentence resembles in some respects those which we have used to illustrate the distributive property. It does, in fact, state that a certain indicated product, $(4 + 3)8$, and a certain indicated sum, $4(8) + 3(8)$, name the same number.

Let us compare our sentence with one which illustrates the distributive property as we have stated it.

Our sentence:	$(4 + 3)8 = 4(8) + 3(8)$
Distributive property is illustrated by:	$8(4 + 3) = 8(4) + 8(3)$

Look at the pair of sentences above. $(4 + 3)8$ and $8(4 + 3)$ are equal by the commutative property of multiplication. In the same way, $4(8) = 8(4)$ and $3(8) = 8(3)$ illustrate the commutative property of multiplication. Thus without any computation at all we could be sure that

$$(4 + 3)8 = 4(8) + 3(8)$$

is a true sentence. Our conclusion would be based on the fact that this sentence can be obtained from an illustration of the distributive property by applying the commutative property of multiplication three times.

Here is another example.

We are sure, without computing, that

$$6(8 + 2) = 6(8) + 6(2)$$

69 by the ______ property. — distributive

We are also sure, without computing, that

70 $6(8 + 2) = (8 + 2)\_\_\_$ — 6

71 $6(8) = 8(\_\_\_)$ — 6

72 $6(\_\_\_) = 2(6)$ — 2

73 All these equalities follow from the ______ — commutative

74 property of ______. — multiplication

Hence, we can be sure that

75 $(8 + 2)6 = \underline{8(6) + 2(\quad)}$ — $8(6) + 2(6)$

The two examples just completed suggest an alternate pattern for the distributive property. It can be stated as follows: For any three numbers of arithmetic, the indicated product

(first number + second number) × third number

names the same number as the indicated sum

(first number × third number) + (second number × third number).

We now have several patterns. Complete the following, to illustrate the various patterns.

We first observed this pattern:

76 $3(9 + 11) = 3(9) + \underline{\quad(\quad)}$ — 3(11)

	The fact that $3(9 + 11)$ and $3(9) + 3(11)$ name the same number can also be written:	
77	$3(9) + 3(11) = \underline{3(\quad\quad)}$	$3(9 + 11)$
	In the sentence $3(9 + 11) = 3(9) + 3(11)$	
78	we apply the ______ property of multiplication	commutative
	to each of the indicated products	
79	to obtain $(9 + 11)3 = 9(3) + \underline{\quad(\quad)}$.	$11(3)$
	And finally, we also see that	
80	$9(3) + 11(3) = \underline{(\quad\quad\quad)3}$.	$(9 + 11)3$

The four patterns are summarized below. You will need to become familiar with all of them.

$$3(9 + 11) = 3(9) + 3(11)$$
$$3(9) + 3(11) = 3(9 + 11)$$
$$(9 + 11)3 = 9(3) + 11(3)$$
$$9(3) + 11(3) = (9 + 11)3$$

	Follow the pattern of any convenient form of the distributive property to complete each example.	
81	$12(3 + 4) = 12(\underline{\quad}) + 12(\underline{\quad})$	$3, 4$
82	$\underline{3(\quad) + \quad(7)} = 3(5 + 7)$	$3(5) + 3(7)$
83	$(3 + 11)2 = \underline{(\quad) + (\quad)\quad}$	$(3)2 + (11)2$
84	$7 \cdot 8 + 6 \cdot 8 = \underline{(\quad + \quad) \cdot 8}$	$(7 + 6) \cdot 8$
85	$27(\frac{7}{8}) + 27(\frac{1}{8}) = \underline{27(\quad\quad\quad)}$	$27(\frac{7}{8} + \frac{1}{8})$

Just as the associative and commutative properties of addition and multiplication often make computations simpler, so also the distributive property sometimes helps us to compute more easily.

60

Write the common name for each numeral. Use the distributive property to help you work quickly.

[illegible] $(.[illegible])(.[illegible]) + (.[illegible])(.[illegible])$ ______ .3

(Did you notice that [illegible] + [illegible] = [illegible]?)

[illegible] [illegible] ______ ______ 4, 13

[illegible] $8(4\frac{1}{2}) = 8(4 + \frac{1}{2})$ = ___( ) + ___( ) $8(4) + 8(\frac{1}{2})$

[illegible] = ______ 36

[illegible] [illegible] = ______ 57

In order to compute quickly [illegible] we may first note that

[illegible] [illegible] = [illegible](___ + ___) (11 + 7)

= [illegible]

[illegible] = [illegible] $\frac{1}{2}(18) + \frac{2}{3}(18)$

[illegible] Since it is easy to see that $\frac{1}{2}(18)$ = ______ 9

[illegible] and $\frac{2}{3}(18)$ = ______, we find that 12

[illegible] [illegible] = ______ 21

More important than these aids to mental manipulation is the role of the distributive property in much of our arithmetic technique. For example, consider the computation of the product.

[illegible]

Here we are applying the distributive property as follows:

[illegible] 23 = 20 + ______ 3

[illegible] Hence (23)62 = (______ + ___)62 20 + 3

[illegible] = (20)62 + ( )62 (20)62 + (3)62

99	(20)62 = ____	1240
100	(3)62 = ____	186
101	(23)62 is thus the sum of 186 and ____.	1240

As you understand better the structure of the number system, as shown by the properties of numbers, you can see the ideas behind the computational processes you learned in the elementary grades.

	Here are some numerical phrases. Write the common name for each. (Try to use the number properties discussed in this chapter to make your computations easy.)	
102	$\frac{1}{[illegible]}$([illegible] + 5)10 = ____	22
103	[illegible]([illegible]) + [illegible]([illegible]) ____	54
104	[illegible]([illegible]) + [illegible] ____	27
105	7([illegible]) + [illegible] ____	70
106	[illegible]([illegible]) + [illegible] ____ (Did you notice that [illegible] = [illegible]([illegible]), so that [illegible]([illegible]) = [illegible]([illegible]) + [illegible](1)?)	120
107	([illegible] + [illegible]) × 4 × $\frac{1}{[illegible]}$ = ____	24

2-4. Variables

The pattern or form of a problem is often of primary importance.

	Consider this simple exercise in mental arithmetic: "Take [illegible], add [illegible], multiply by [illegible], and divide by [illegible]."	
1	The numbers which you think of in succession as you do the mental arithmetic are [illegible], [illegible], [illegible], ____.	8
	Consider further this same exercise. The numerical phrases which indicate the pattern for the thinking are:	

Instructions		Numeral
Take 9	$\longrightarrow$	9
Add 3	$\longrightarrow$	9 + 3
Multiply by 2	$\longrightarrow$	2(9 + 3)
Divide by 3	$\longrightarrow$	______

$\frac{2(9 + 3)}{3}$

Let us try another exercise.

"Take 6, add 2, multiply by 7, and divide by 4."

Instructions		Numeral
Take 6	$\longrightarrow$	6
Add 2	$\longrightarrow$	______
______ 7 (multiply by, add)	$\longrightarrow$	___(___)___
______ 4	$\longrightarrow$	______

6 + 2

multiply by, 7(6 + 2)

divide by, $\frac{7(6 + 2)}{4}$

The numerical phrase, $\frac{7(6 + 2)}{4}$, although a numeral for the number 14, has an advantage over 14 in that it preserves the pattern or form of our exercise.

Complete the numerical phrase which gives the pattern for the mental exercise, "Take 7, multiply by 3, add 3, multiply by 2, and divide by 12." ______

$\frac{2(3(7) + 3)}{12}$

Let us now try an exercise of this sort with the added feature that we choose in the beginning any member of the set S, where

$$S = \{1, 2, 3, \ldots, 1000\}.$$

The instructions this time are:

"Take any number from S, multiply by [illegible], add [illegible], divide by [illegible], and subtract [illegible]."

Notice that there are 1000 possible exercises, one for each choice of an element from the set S. If we start with the number 17, which is in S, we have:

	Instructions		Numeral	
9	Multiply by 3	$\longrightarrow$	3(___)	$3(17)$
10	Add 12	$\longrightarrow$	3(17) + ___	$3(17) + 12$
11	Divide by 3	$\longrightarrow$	______	$\frac{3(17) + 12}{3}$
12	Subtract 4	$\longrightarrow$	______	$\frac{3(17) + 12}{3} - 4$

Here is how we might find the common name for $\frac{3(17) + 12}{3} - 4$:

Since $12 = 3(4)$,

$$\frac{3(17) + 12}{3} - 4 = \frac{3(17) + 3(4)}{3} - 4.$$

By use of the distributive property,

13 $$\frac{3(17) + 3(4)}{3} - 4 = \frac{3(\_\_ + \_\_)}{\_\_\_\_} - 4$$ $\frac{3(17 + 4)}{3} - 4$

$$= (17 + 4) - 4$$

14 $$= \_\_\_\_\_\_$$ 17

For this exercise we selected the number 17 from S, and after following the instructions, we obtained the same number, 17. What would have happened if we had selected some other number from S? Would our result have been the same number as that with which we started? One way to find out would be to select each member of S in turn, thus working 1000 problems.

Let us look for an easier way. Instead of writing down the particular number selected, let us just represent it by the letter n, which is the first letter of the word "number".

	Instructions		Numeral	
	Take a number from S	→	n	
15	Multiply by 3	→	___(n)	3(n)
16	Add 12	→	______	3(n) + 12
	Divide by 3	→	$\frac{3(n) + 12}{3}$	
			$\frac{3(n) + 3(4)}{3}$	
17	Using the distributive property		$\frac{3(\_\_\_\_\_\_)}{\_\_\_}$	$\frac{3(n + 4)}{3}$
			n + 4	
18	Subtract 4	→	______ - 4	n + 4 - 4
19			______	n

Remember that n can represent any element of the set S. In this exercise, if we select any number from S and follow the instructions, we shall always obtain the same number as that chosen at the start. If a letter is used, as in this example, to denote one of a given set of numbers, it is called a variable. In a given computation involving a variable, the variable is a numeral which represents a definite though unspecified number from a given set of numbers.

20	n was used as a ______ to denote a number from set S.	variable
21	If a number, denoted by n, is selected from S and the instructions are "multiply by 2 and add 3", the number which results can be denoted by ______.	2(n) + 3, or 2n + 3

In Items 1-14, we considered the case for which the variable had the value 17. Notice the use of the word "value". In general, when we wish to make a statement such as "let n be 17", we often use the words "let n have the value 17" or "let the value of n be 17".

22	The set from which we chose [illegible] was $S = \{1,2,3,\ldots,1000\}$. We could have chosen ______ (how many) different values for the variable.	1000

The set of numbers from which the values of the variable may be chosen is called the domain of the variable.

[illegible]	Thus, in the problem "Select a number from S, multiply by 3, add 12, divide by 3, and subtract 4", we were [illegible] that n was to be selected from set S, which is $\{1,2,3,\ldots,1000\}$. The set S is the ______ of the variable.	domain
[illegible]	If the domain of a variable is the set of numbers of arithmetic, then the possible values for the variable are the number [illegible] and the numbers corresponding to points to the ______ of [illegible] on the number line.	right

Unless the domain of a variable is specifically stated, for the present we shall assume it to be the set of all numbers of arithmetic.

When a phrase contains at least one variable, it is called an open phrase.

[illegible]	______ is an open phrase which could mean "a number increased by 5". The value of the open phrase depends on the ______ of n.	n + 5 value
[illegible]	Suppose that the value of n must be chosen from the set $\{7, \ldots\}$. Then the ______ of n for this example is $\{7, \ldots\}$.	domain 7
[illegible]	If n is [illegible], then the value of [illegible] is ______.	12
[illegible]	[illegible] we may [illegible] for a value of n [illegible] in the domain of n. The domain of n was given as ______.	{7,12}
	If n is [illegible], the value of [illegible] is ______.	17

Given the variable n whose domain is $\{8, 12, 18\}$, the only possible values for the given open phrases are:

	Open Phrase	Values	
31	$n - 8$	$0,\ \_\_\_\_,\ 10$	0, 4, 10
32	$6n$	$\_\_\_\_,\ 72,\ \_\_\_\_$	48, 72, 108
33	$\frac{n}{12}$	$\frac{2}{3},\ \_\_\_\_,\ \_\_\_\_$	$\frac{2}{3}$, 1, $\frac{3}{2}$

Find the possible values of these phrases if the domain of n is $\{2, 5, 10\}$.

34	$\frac{2(n + 1)}{3}$	____, ____, ____	2, 4, $\frac{22}{3}$
35	$\frac{3(n) - 2}{4}$	____, ____, ____	1, $\frac{13}{4}$, 7
36	$\frac{(n - 1)(n + 3)}{n + 1}$	____, ____, ____	$\frac{5}{3}$, $\frac{16}{3}$, $\frac{117}{11}$
37	$\frac{3n(n + 2)}{2n + 4}$	____, ____, ____	3, $\frac{15}{2}$, 15

"Twice the sum of t and 3" is indicated by which of these phrases:

38 $2t + 3$ or $2(t + 3)$? ____ — $2(t + 3)$

39 In the phrase $2(t + 3)$, we call "t" a ____. — variable

40 Its domain is understood to be the ____ ____ ____, since a domain was not otherwise stated. — numbers of arithmetic

If 5 is added to twice n, and the sum is divided by 3, the result is represented by which of the following two phrases:

41 $\frac{2n + 5}{3}$ or $\frac{2n}{3} + 5$? ____ — $\frac{2n + 5}{3}$

42 The domain of the variable in the phrase $\frac{2n + 5}{3}$ is understood to be the numbers of ____. — arithmetic

A phrase may contain more than one variable. In this case, the number represented by the phrase depends upon the value assigned to each variable.

If a is [illegible], [illegible] is [illegible], c is $\frac{[illegible]}{4}$, m is 1, and n is [illegible], find the value of (that is, the number represented by) each of the following:

[illegible]

43. [illegible] ______ $\frac{3}{2}$, or $1\frac{1}{2}$

44. ______ $\frac{39}{2}$, or $19\frac{1}{2}$

45. [illegible] ______ 15

46. [illegible] ______ 27

47. [illegible] ______ $\frac{9}{4}$

48. [illegible] ______ $\frac{3}{2}$

If a is [illegible], [illegible], find the value of each of the following:

[illegible] ______ 0

[illegible] ______ 13

[illegible] ______ 0

[illegible] ______ 9

[illegible] ______ 10

[illegible]

[illegible]

If a is [illegible], b is 2, and c is 4, the value of:

59. $\dfrac{(3a + 4b) - 2c}{[illegible]}$ = ______

60. $\dfrac{([illegible]a - 4b) + 5c}{[illegible]}$ ______

61. $\dfrac{\left(\frac{7a}{[illegible]} + \frac{[illegible]b}{[illegible]}\right) - \frac{2c}{[illegible]}}{[illegible]}$ ______

62. $\dfrac{(1.5a + [illegible].7b) - 2.[illegible]c}{7}$ ______

Chapter [illegible]

SENTENCES

[illegible]-1. Open Sentences

When we make assertions, or statements, about numbers we write sentences, such as

$$([illegible] + 1)([illegible]) \quad [illegible]0 \quad \text{(which is a false sentence)}$$

and

$$4 + [illegible] = [illegible] + 4 \quad \text{(which is a true sentence).}$$

Sentences involve the use of verbs. In [illegible] examples above we have used the verb symbol "=" which means "equals" or "is equal to." Other verb forms may also be used. In particular, the symbol "≠" means "does not equal" or "is not equal to."

Thus, $[illegible] + 4 \neq 2[illegible] \cdot 2$ is a true sentence, and $[illegible] + 4 \neq \frac{2[illegible]}{2}$ is a false sentence. Do you see why?

Judge each of the following sentences as true or false.

1	$4 + [illegible] \quad 10 + [illegible]$	______	false
2	$4 [illegible] \neq 4 + [illegible]$	______	true
[illegible]	$[illegible] + [illegible] \neq [illegible] + [illegible]$	______	false
4	[illegible]	______	true
[illegible]	$[illegible]([illegible] \times [illegible]) \quad ([illegible] \times [illegible]) [illegible]$	______	true

Insert the symbol "=" or "≠" to complete each of the following sentences so as to make it true.

[illegible]	[illegible]	______	[illegible]	=
[illegible]	[illegible]	______	$[illegible] \cdot \frac{1}{[illegible]}$	=
[illegible]	[illegible]	______	[illegible]	≠
[illegible]	$1[illegible] + [illegible]$	______	[illegible]	≠
10	$[illegible](\frac{[illegible]}{[illegible]})$	______	$(\frac{1}{[illegible]})$	[illegible]
11	$(\frac{[illegible]}{[illegible]})$	______	$\frac{[illegible]}{[illegible]0}$	≠

We can verify that [illegible] + [illegible] = [illegible] + [illegible] by recognizing that 5 + [illegible] and 8 + 4 are both names for the same number. The common name of this number is 12. Surely [illegible] is a true sentence.

We can also recognize that [illegible] is a true sentence by noticing that it is an instance of the commutative property of addition.

12	12([illegible]) = ([illegible])12 is a true sentence, since this is an instance of the ______ property of multiplication.	commutative
13	[illegible] + [illegible] = [illegible] + [illegible] is a ______ (true, false) sentence.	true
14	7 × ([illegible] × [illegible]) = (7 × [illegible]) × [illegible] is a true sentence, since this is an instance of the ______ property of multiplication.	associative
15	The true sentence [illegible] represents an instance of the ______ property.	distributive

16 Which of the following phrases correctly completes the statement, "A numerical sentence is ______."

[A] An assertion about numbers

[B] A statement that two numerals are equal

[C] An application of the commutative, associative or distributive properties of numbers

[D] A sentence containing numerals

[E] A true statement about numbers

If you chose [A] you are correct; go to Item 23. If you made any other choice, continue with Item 17.

17	[illegible] is a numerical ______ but it is not a statement that two numerals name the same number. Hence, [B] is incorrect.	sentence
18	[illegible] is a ______ (true, false) numerical sentence.	true
19	[illegible] is a true numerical ______, but it is not an instance of any of the properties of numbers. Hence, [C] is incorrect.	sentence

[illegible] + [illegible] = [illegible] + [illegible] is an instance of the commutative

20 property of ______. | addition

Although the instruction, "Write [illegible] times 2 on the blackboard" is a sentence containing numbers, it is not a numerical sentence since it does not make an

21 assertion about ______. Thus, [D] is incorrect. | numbers

22 The statement [illegible] ______ (is, is not) true, but it is a numerical sentence. Thus, [E] is incorrect. | is not

23 Show how best to complete the statement, "A numerical sentence is true or false depending upon whether or not ______."

[A] the two numbers named by the numerical phrases are the same

[B] the relationship of the two numerical phrases are the same

[C] the commutative, associative and distributive properties can be used

[D] two numerical phrases are separated by a verb symbol

You should review Items 1-22 unless you chose [B], which is the correct response.

We have no trouble deciding whether or not a numerical sentence such as

[illegible] + [illegible] = 7

is true. (Is it?)

However, consider the sentence

[illegible] + x = [illegible].

Is this sentence true? You may say that you don't know what number "x" represents and without this information you cannot decide.

24 Is the sentence, "He is a doctor", true or false?

[A] True

[B] False

[C] I don't know

If you chose [illegible] or [B] you are not thinking carefully. You cannot decide the truth of the sentence until "he" is identified. The variable "x" may be used in much the same way as a pronoun in English. [C] is correct.

We say that sentences such as

$$[illegible] + x [illegible] 7$$

and

$$x + 5 \neq 9$$

which contain variables, are <u>open</u> sentences. The word "open" is suggested by the fact that we cannot decide whether they are true since we do not know what number x represents.

In the same way, a phrase like "$5x + [illegible]$" or "$7p + [illegible]$" involving one or more variables, is called an <u>open</u> <u>phrase</u> since we do not know what number is represented by such a phrase.

The following is a list of sentences and phrases. Identify which are open sentences, (OS), open phrases, (OP), numerical sentences, (NS), or numerical phrases, (NP), by writing OS, OP, NS or NP in the spaces provided.

25 $7x = 12$ ______ OS

26 $5 + 7 \neq 15$ ______ NS

27 $4x + [illegible] = 2y$ ______ OS

28 $4y + q$ ______ OP

29 $37 + 12$ ______ NP

30 $5y + 6[illegible]$ [illegible] $2x$ [illegible] ______ OS

31 $\frac{37}{12} + \frac{57}{[illegible]} \neq 520$ ______ NS

32 $\frac{a + b - 7x}{46}$ ______ OP

33 $a + (b + c) = (a + b) + c$ ______ OS

In each of the following open sentences, the variables have given values. Tell whether each sentence is true or false for the given values of the variables.

34 $7 + x = 12$; let x be [illegible] ______ (true, false) true

35 $7 + x \neq 12$; let x be [illegible] ______ false

36 $y + [illegible] \neq 11$; let y be [illegible] ______ true

37 $t + [illegible] = 11$; let t be [illegible] ______ false

38	$\frac{2x+1}{7} \neq$ [illegible]; let x be [illegible]	______	true
39	$\frac{2x+1}{7} \neq$ [illegible]; let x be [illegible]	______	false
40	$2y + 3x =$ 2[illegible]; let x be 4 and y be [illegible]	______	false
41	$2y +$ [illegible] $=$ 2[illegible]; let x be [illegible] and y be 4	______	true
42	$2a -$ [illegible] $\neq (2a + 4) - b$; let a be [illegible] and b be [illegible]	______	fal[illegible]
43	$2a -$ [illegible] $\neq (2a + 4) - b$; let a be [illegible] and b be [illegible]	______	false

In the above problems it was easy to decide whether or not the sentence was true for the given values of the variables.

Suppose we are given an open sentence and the domain of the variable. How shall we determine the values, if any, of the variable that will make the sentence true? One way would be to guess various numbers until we hit on a "truth" number. However, after a first guess, a bit of thinking could guide us.

Consider the open sentence "$2x + 11 = 28$." We will try to "guess" a value of x large enough so that $2x + 11$ is greater than 28.

If we let x be 9, then $2x + 11$ is the same as $2(9) + 11$, or 29.

44 But 29 is ______ (greater, smaller) than 28. If we try 8 for x, $2x + 11$ is the same as $2(8) + 11$, or 27. — greater

It is clear that 8 is too small.

The next step would be to try a number between 8 and 9. Suppose we try $8\frac{1}{2}$. Then

45 $2x + 11$ is $2(\_\_) + 11$, — $8\frac{1}{2}$

46 or ______. — 28

47 Now we have the ______ (true, false) sentence — true

$$2(8\tfrac{1}{2}) + 11 = 28$$

or $28 = 28$.

48	We can say that the open sentence $2x + 11 = 28$ is true if x is __________ (what number).	$8\frac{1}{2}$

49	Do you think there are other values of x which would make the open sentence $2x + 11 = 28$ true? [A] Yes [B] No

If you try numbers greater than $8\frac{1}{2}$, you will always get a number greater than $2(8\frac{1}{2}) + 11$. But $2(8\frac{1}{2}) + 11$ is another name for 28. If you try numbers less than $8\frac{1}{2}$, you will always get a number less than $2(8\frac{1}{2}) + 11$ or less than 28. There can be only one "truth" number for this open sentence. [B] is correct.

For some special open sentences we may be able to avoid the guessing process. Recall that we know that

$$(234)(478) = (478)(234)$$

is a true sentence without multiplying, since this is an instance of the of the commutative property of multiplication.

In the same way,

$$3(m + 4) = 3m + 12$$

is a true sentence, no matter what number m represents, since this is an instance of the distributive property.

In the following set of exercises, determine what numbers, if any, will make each of the following open sentences true. You may work these on practice paper by the method used above, but record only the "truth numbers" on your response sheet.

	Open Sentence	Value(s) of the variable for which the sentence is true	
50	$12 -$ [illegible]	________	4
51	[illegible]	________	10
52	$3x - 2 = 10$	________	4
53	$3x - 2 = 15$	________	$\frac{17}{3}$
54	$4x + 3x = 14$	________	2
55	$4x - 3x = 14$	________	14

56	$s + 3 = s + 2$ (careful!)	______	no numbers
57	$t + 3 = 3 + t$	______	all numbers
58	$t + 2t \neq 2t + 3t$	______	all numbers
59	$(x + 1)2 \neq 2x + 2$	______	no numbers

60 Which of the following is a true statement?

[A] Every open sentence has only one value of the variable which makes it true.

[B] Every open sentence has many values of the variable which make it true.

[C] An open sentence may have one value of the variable, no values of the variable, or many values of the variable which make it true.

Examine Items 55, 56, and 57. You will see that [C] is the correct response.

Recall that x^2 means "x multiplied by x" or "$x \cdot x$" and is read as "x squared". x^2 may also occur in an open sentence.

From the set of numbers of arithmetic find values of the variable which make the following open sentences true. Again record only the truth numbers on your answer sheet.

	Open Sentence	Value(s) of the variable for which the sentence is true	
61	$x^2 = 9$	______	3
62	$4 - x^2 = 0$	______	2
63	$x^2 = x$	______	1; 0
64	$x^2 - 5x = 0$	______	5, 0
65	$x + 2 = 9$	______	7
66	$(x - 1)^2 = 4$	______	3
67	$4 + x^2 = 0$	______	no numbers of arithmetic
68	$x^2 + 7 = 7$	______	0
69	$x^2 = \frac{9}{16}$	______	$\frac{3}{4}$
70	$7 + x = 5$	______	no numbers of arithmetic

8

You may have been tempted to respond "-4" to Item 70. Remember that we are considering only numbers of arithmetic. In Chapter [illegible] we shall see how we may extend our notion of "numbers" to include such numbers as "-4".

* A number of interest to us later is a value of x for which "$x^2 = 2$" is a true sentence. We call this number the square root of 2, and write it $\sqrt{2}$. Later you will find that $\sqrt{2}$ is the coordinate of a point on the number line. Approximately where on the number line would it lie?

* Let us try to guess a "truth number" for the open sentence $x^2 = 2$. Since $(1)^2$ is less than 2, and $(2)^2$ is greater than 2, we might try 1.5.

71 $(1.5)^2 =$ ______ 2.25

Since $(1.5)^2$ is greater than 2, suppose we next try 1.4 for x.

72 $(1.4)^2 =$ ______ 1.96

which is less than 2.

$(1.4)^2$ is very close to 2, however, so the point corresponding to $\sqrt{2}$ would be very near the point

73 corresponding to ______ on the number line. 1.4

3-2. Truth Sets of Open Sentences.

Let the domain of the variable in the open sentence

$$3 + x = 7$$

be the set of all numbers of arithmetic. If we specify that x has a particular value, then the resulting sentence will be a numerical sentence which is either true or false.

Complete the following table based on the open sentence $3 + x = 7$. (This is similar to the exercises in 3-1.)

	If x is	the sentence	is	
	0	$3 + 0 = 7$	false	
1	1	$3 + \_\_ = 7$	false	1
2	$\frac{1}{2}$	$3 + \frac{1}{2} = 7$	____	false
3	____	$3 + 2 = 7$	false	2
4	4	$3 + 4 = 7$	____	true
5	6	$3 + \_\_ = 7$	false	6

We can see that the open sentence "$3 + x = 7$" can be thought of as a "sorter": the elements of the domain are sorted into two subsets. One of these subsets is the set of all numbers of the domain which make the resulting numerical sentence true. The other subset is the set of all those numbers of the domain which make the sentence false. In the example we are considering, 4 is a member of the first subset and $0, 1, \frac{1}{2}, 2, 6$ all belong to the second subset.

If we start with any open sentence and any given domain, the open sentence will always act as a sorter, sorting the domain into two subsets.

Definition. The truth set of an open sentence in one variable is the set of all numbers from the domain of the variable which make the sentence true.

Test whether or not the given number is a member of the truth set of the given open sentence.

	For the sentence	the number	belongs to the truth set	
	$10 + x$ [illegible] 12	[illegible]	yes	
6	$y +$ [illegible] $\neq 11$	[illegible]	____	yes
7	$\frac{[illegible]x + 2}{7} \neq 4$	[illegible]	____	yes
8	$2x + 1$ [illegible] $2(x + 1)$	[illegible]	____	no
9	$x + \frac{1}{x}$ [illegible] 2	[illegible]	____	no
10	[illegible]m [illegible] $m + 2m$	[illegible]	____	yes
11	$n^{[illegible]} + 2n \neq n(n + 2)$	[illegible]	____	no

We see that 4 is the only number which makes $3 + x = 7$ true. Hence, the truth set of $3 + x = 7$ is $\{4\}$.

It may happen that the truth set may contain more than one member; in fact, every element of the domain may be in the truth set. For example, we have seen that $3(n + 4) = 3n + 12$ is true for all numbers of arithmetic.

	Open sentence	domain	truth set	
12	$3(x + 5) = 17$	$\{0,\frac{1}{3},\frac{2}{3},1\}$	______	$\{\frac{2}{3}\}$
13	$x^2 - (4x - 3) = 0$	$\{1,2,3,4\}$	______	$\{1,3\}$
14	$4y = y + 3y$	$\{1,2,3,4\}$	______	$\{1,2,3,4\}$
15	$x(x + 1) = 3x$	$\{0,1,2\}$	______	$\{0,2\}$
16	$3 + x = 7$	$\{0,1,2\}$	______	$\emptyset$

In Item 16, the truth set was $\emptyset$ because the domain of the variable did not include 4. It is not difficult to write an open sentence having $\emptyset$ as its truth set even though the domain of the variable is the set of all numbers of arithmetic.

17 Fill in the blanks so that each of the following open sentences has $\emptyset$ for its truth set.

(P) $x + 5 = x +$ ______

(Q) $x^2 +$ ______ $= 0$

Which of the alternatives below is correct? (Remember the domain is the set of numbers of arithmetic.)

[A] In (P) use any number except 5, and in (Q) any number except 0.
[B] In (P) use any number except 5, and in (Q) use 0.
[C] In (P) use any number and in (Q) use any number.

In (P), if we use any number except 5, the resulting sentence will have truth set $\emptyset$. In (Q), if we use 0, we have $x^2 + 0 = 0$. The truth set of this sentence is $\{0\}$. The only correct choice is [A].

	Let the domain of the variable in each of the following be the set of all numbers of arithmetic. Find the truth set for each open sentence:		
18	$n - 5 = 7$	______	{12}
19	$2n - 5 = 7$	______	{6}
20	$3n - 5 = 7$	______	{4}
21	$5n - 5 = 7$	______	$\{\frac{12}{5}\}$
22	$2n + 14 = 2(n + 7)$	______	set of all numbers of arithmetic
23	$2n + 14 = n + 7$	______	$\emptyset$

Many formulas used in science and business are in the form of open sentences in several variables. For example, the formula $V = \frac{1}{3}Bh$ is used to find the volume of a cone. The variable h represents the number of units in the height of a cone; B represents the number of square units in the base of this cone; V represents the number of cubic units in the volume. When values are specified for all but one of the variables in such a formula, the resulting open sentence contains one remaining variable. Then the truth set of this sentence gives information about the value of this variable.

24 In the formula $V = \frac{1}{3}Bh$, if we want an open sentence containing h as the only variable, for what variables must we specify values?

[A] V and B [B] V, B, and h

If V, B and h were all specified, the resulting sentence would not be an open sentence. You should have selected [A].

Let us consider a particular cone whose volume is 66 cubic feet and the area of whose base is 33 square feet. We use this information to form the open sentence

$$66 = \frac{1}{3}(33)h.$$

It is not difficult to determine that the truth set of this sentence is {6}. Thus we are able to conclude that the height of this particular cone is 6 feet.

Here are some more problems involving the use of formulas. For each problem write the appropriate open sentence, find the corresponding truth set, and then determine the answer to the question asked. The correct answers are on page iii.

25. The formula used to change a temperature F measured in degrees Fahrenheit to the corresponding temperature C in degrees Centigrade is

$$C = \frac{5}{9}(F - 32).$$

Find the value of C when F is 86.

26. The formula used to compute simple interest is

$$i = prt,$$

where i is the number of dollars of interest, p is the number of dollars of principal, r is the interest rate, and t is the number of years. Find the value of t when i is 120, r is 0.04, and p is 1000.

27. A formula used in physics to relate pressure and volume of a given amount of a gas at constant temperature is

$$pv = PV,$$

where V is the number of cubic units of volume at P units of pressure and v is the number of cubic units of volume at p units of pressure. Find the value of V when v is 600, P is 75, and p is 15.

28. The formula for the area of a trapezoid is

$$A = \frac{1}{2}(B + b)h,$$

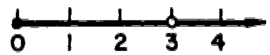
where A is the number of square units in the area, B is the number of units in the one base, b is the number of units in the other base, and h is the number of units in the height. Find the value of B when A is 20, b is 4, and h is 4.

The graph of a set S of numbers, it will be recalled, is the set of all points on the number line whose coordinates are the members of S, and only those points. Let us review some examples.

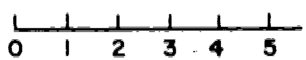
(1) $A = \{0, 2, \frac{7}{2}\}$.

Its graph is indicated by heavy dots at the points whose coordinates are 0, 2, and $\frac{7}{2}$.

(2) B is the set of all numbers of arithmetic except 3. Its graph is shown by a heavy line which begins at the point whose coordinate is 0 with a heavy dot, omits the point with coordinate 3, and continues to the right with an arrow. The arrow indicates that all the points to the right are on the graph. We use an open circle at the point whose coordinate is 3 to indicate that 3 is not included in the set.



(3) The set C is the null set, $\emptyset$. Its graph has no points marked on the number line as there are no elements in the null set.



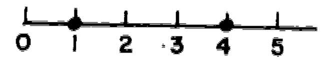
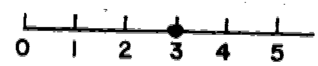
Since we can graph any set of numbers, in particular we can graph the truth sets of open sentences.

Complete the following table.

	Sentence	Truth Set	Graph	
29	$x = 2$	______		$\{2\}$
30	$x \neq 1$	all numbers of arithmetic except 1	______	
31	$3+x=(7+x)-4$	______		all numbers of arithmetic
32	$y(y+1) = 3y$	______	______	$\{0,2\}$ 
33	$3x = 7$	______	______	$\{\frac{7}{3}\}$ 
34	$2x+1 = 2(x+1)$	______		$\emptyset$

90

35 Which graph is <u>incorrect</u>?

Sentence	Graph of Truth Set
[A] $(x - 1)(x - 4) = 0$	
[B] $\frac{x}{4} = \frac{6}{8}$	
[C] $3x + 4 \neq 13$	
[D] $(4 + x) + (x + 3) = 2x + 7$	

The truth set of $(x - 1)(x - 4) = 0$ is $\{1,4\}$. The truth set of $\frac{x}{4} = \frac{6}{8}$ is $\{3\}$. The truth set of $3x + 4 \neq 13$ is the set of all numbers of arithmetic except 3. The truth set of $(4 + x) + (x + 3) = 2x + 7$ is the set of all numbers of arithmetic. Therefore, the only incorrect graph is that of [C]. If you selected [C], you should begin Section 3-3 unless you desire more practice.

Here are some more practice problems.

State the truth set and draw its graph for each of the following open sentences. (Answers are given on page iii.)

	Sentence	Truth Set	Graph
36.	$x + 3 = 7$		
37.	$2x = x + 3$		
38.	$x + x \neq 2x$		
39.	$x + 3 = 3 + x$		
40.	$(x)(0) = x$		
41.	$3 + x \neq 6$		
42.	2x + [illegible] 8		
43.	$5 \neq 3n + 1$		
44.	$y(1) \neq y$		
45.	$x^2 = 2x$		

3-3. Sentences Involving Inequalities

If we think of two different numbers of arithmetic, we are always able to say that one of the numbers is less than the other. Thus, of 4 and 9, 4 is less than 9; of $\frac{3}{4}$ and $\frac{7}{8}$, $\frac{3}{4}$ is less than $\frac{7}{8}$, etc.

	For each of the following pairs of numbers, determine which is less than the other.	
1	1.2, 2.1	1.2
2	0.4, .04	.04
3	.001, .002	.001
4	0.9, 1	0.9
5	$\frac{1}{11}$, $\frac{1}{12}$	$\frac{1}{12}$

We have a mathematical symbol to express this relationship between different numbers: we write "$1.1 < 1.2$" and read this as "1.1 is less than 1.2". We may also write "$1.2 > 1.1$" and read this as "1.2 is greater than 1.1".

Note that in a true sentence the point of the symbol ">" (the small end) is directed toward the smaller of the two numbers.

6	[illegible] + 7 $\underset{(<,>)}{\underline{\qquad}}$ 4 + 7 is a true sentence.	<
	Since [illegible] × 0 is another name for 0, the sentence	
7	([illegible] × 0) [illegible] is $\underset{(\text{true, false})}{\underline{\qquad\qquad}}$.	true
8	2 > (2 × 0) is $\underset{(\text{true, false})}{\underline{\qquad\qquad}}$	true
9	$\frac{1}{4}$ [illegible] $\frac{1}{9}$ is $\underset{(\text{true, false})}{\underline{\qquad\qquad}}$.	false
	Just as "≠" is read "is not equal to",	
10	"≯" is read "is ______ greater than".	not
11	"≮" is read "is ______ ______ ______".	not less than

State whether the following sentences are true or false.

12	$4 + 3 < 3 + 4$	______	false
13	$5(2 + 3) > 5(2) + 3$	______	true
14	$5 + 0 \not> 5$	______	true
15	$0.5 + 1.1 \not> 0.7 + 0.9$	______	true
16	$2 + 1.3 >$ [illegible]	______	false
17	$4 + (3 + 2)$ [illegible] $(4 + 3) + 2$	______	false
18	$\frac{2}{3}(8 + 4) < (\frac{2}{3} \times 8) + (\frac{2}{3} \times 4)$.	______	false

We know that $8 < 12$. If we locate on a number line two points having coordinates 8 and 12, we notice that the point with coordinate 8 is to the <u>left</u> of the point with coordinate 12.

19 The point with coordinate 5 is to the ______ of the point with coordinate 9. — left

Locate points with coordinates $\frac{5}{2}$ and 2.2. Which number corresponds to the point farther to the

20 left? ______ — 2.2

In general, the lesser of two numbers will correspond

21 to the point to the ______ of the other. — left

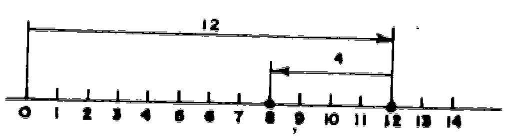
22 $2.2 \underset{(<,>)}{\_\_\_\_} \frac{5}{2}$ is a true sentence. — $<$

23 The point 2.2 is to the ______ of the point $\frac{5}{2}$. — left

We may interpret $8 < 12$ as meaning that the point with coordinate 8 is to the left of the point with coordinate 12.

Recalling Section 1-3 we may indicate addition and subtraction on a number line.

or $8 + 4 = 12$



or $12 - 4 = 8$

24	$8 < 12$ and $8 + \_\_\_\_ = 12$.	4
25	$12 > 8$ and $12 - \_\_\_\_ = 8$.	4
26	$2.2 < \frac{5}{2}$ and $2.2 + 0.3 = \_\_\_\_$.	$\frac{5}{2}$ or 2.5
27	$\frac{3}{4} > \frac{1}{4}$ and $\_\_\_\_ - \frac{1}{2} = \frac{1}{4}$.	$\frac{3}{4}$

The ideas suggested by Items 24-27 will be developed at greater length in later chapters.

28	2.39 is ____ (greater, less) than 1.08.	greater
29	2.39 is to the ____ (left, right) of 1.08 on the number line.	right
30	$2.39 - \_\_\_\_ = 1.08$.	1.31

Indicate which of the following sentences are true and which are false.

31	$5.2 - 3.9 < 4.6$	____	true
32	$\frac{1}{2} + \frac{1}{[illegible]} \not< 1$	____	false
33	$7.6 + (1.4 + 1.5) \not< \frac{22}{2}$	____	false
34	$1.1 \not> \frac{6}{[illegible]}$	____	true
35	$2 < \frac{[illegible]}{[illegible]}$	____	false

Just as there are open sentences containing the verb symbols "=" and "≠", there are open sentences containing the verb symbols "<", "≮", and ">", "≯".

$x + 2 > 4$ is an open sentence because we cannot decide whether it is true or false.

36	The open sentence $x + 2 > 4$ is read, "the sum of the number x and the number 2 is ____ than 4".	greater
37	The open sentence $x + 2 < [illegible]$ is read, "the sum of the number x and the number 2 is ____ than the number [illegible]".	less

38 Consider the open sentence $x + 2 > 6$. Since $5 + 2 = 7$, and 7 is greater than 6, 5 is a member of the ______ set of the open sentence, $x + 2 > 6$. — truth

39 3 ______ (is, is not) a member of the truth set of $x + 2 > 6$, — is not

40 because $3 + 2 = 5$ and 5 is ______ than 6. — less

41 Which elements of the set $\{0,1,3,4,6,7\}$ are members of the truth set of $x + 1 < 5$? ______ — 0, 1, 3

Just as in English a sentence like "Bill caught the ball" is called a simple sentence, in mathematics a sentence like $x + 2 > 4$ is called a simple sentence. Simple sentences are sentences containing only one verb form.

A simple sentence involving the symbol "$=$" is often called an <u>equation</u>. A simple sentence in which the verb form involves the idea of "is less than" or "is greater than" is called an <u>inequality</u>.

State which of the following are inequalities and which are equations.

42 $2x + 5 = 1$ ______ — equation

43 $x + 1 < 3$ ______ — inequality

44 $x + 5 > 0$ ______ — inequality

The remainder of this section will be devoted chiefly to finding and graphing the truth sets of open sentences involving inequalities.

How do we find the truth set of an open sentence such as $x + 2 > 4$?

Suppose x is <u>greater</u> than 2. For example, if x is 3, then $x + 2$ becomes $3 + 2$, or 5.

45 5 ______ (is greater than, is less than) 4. — is greater than

If x is 8, then $x + 2$ is $8 + 2$, or 10.

46 If x is 8, then $x + 2$ ______ $(<, =, >)$ 4. — $>$

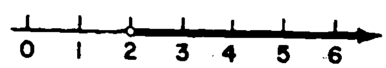
47 If x is $\frac{5}{2}$, then $x + 2$ is ______, which is greater than 4. — $\frac{9}{2}$

In general, if x is any number greater than 2,

48 then $x + 2 > 4$ is a ______ (true, false) sentence. — true

49	Suppose x is <u>less</u> than 2. If x is 1, then $x + 2$ becomes 3, and 3 ______ 4. (<, =, >)	$<$
50	If x is $\frac{3}{2}$, $x + 2 > 4$ is a ______ sentence. (true, false)	false
51	If x is 0, then $x + 2$ is $0 + 2$ or 2. It is certainly true that 2 ______ 4. (<, =, >)	$<$
52	In general, if x is a number less than 2, then $x + 2 > 4$ is a ______ sentence. (true, false)	false
53	If x is 2, the sentence $x + 2 > 4$ is ______. (true, false) Thus, 2 is <u>not</u> in the truth set of $x + 2 > 4$.	false

Every number greater than 2 makes the sentence "$x + 2 > 4$" true, while 2 or any number less than 2 makes it false. That is, the truth set of the sentence "$x + 2 > 4$" is the set of all numbers greater than 2. The graph of this truth set is the set of all points on the number line whose coordinates are greater than 2. The graph would look like this:

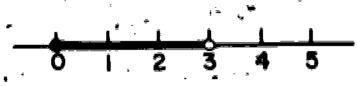


Recall that the open dot at the point with coordinate 2 indicates that 2 is <u>not</u> in the truth set, while the heavy line and the arrowhead indicate that the coordinates of all points to the right of this point <u>are</u> elements of the truth set.

As another example, consider the inequality $1 + x < 4$. When x is 3, or $3\frac{3}{4}$, or 4, or 5.7, or 10, or any number equal to or greater than 3, then $1 + x$ is <u>not</u> less than 4. When x is any number less than 3, then $1 + x < 4$ is true. Do you agree? Test this for some numbers less than 3.

54	The truth set of $1 + x < 4$ is the set of all numbers of arithmetic from 0 to 3, including 0, but ______ 3. (including, not including)	not including
55	The graph of the truth set of $1 + x < 4$ is drawn with ______ at 3 since 3 is not in the truth set. (an open dot, a heavy dot)	an open dot

The graph of the truth set of $1 + x < 4$ is



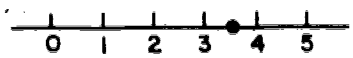
In which of the following does the graph correctly represent the truth set of the open sentence?

56 $2 + x < 4$ 

(yes, no) yes

57 $3x = 5$ 

_____ no

58 $2x = 7$ 

_____ yes

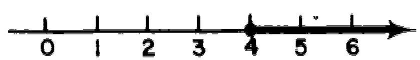
59 $x > 1$ 

_____ no

60 $2x > 5$ 

_____ no

Which of the following open sentences correctly describes this graph?



61 $x \not> 4$, $x \not< 4$, $x \neq 4$ _____ $x \not< 4$

Here are some practice problems in graphing truth sets of equations and inequalities. The correct answers are on page iii.

62. x [illegible]
63. $x \neq$ [illegible]
64. $x > 5$
65. $x + y = 4$
66. $x + y \neq 4$
67. $2x$ [illegible]
68. $2x \neq$ [illegible]
69. $x + y > 4$
70. $x + y < 4$
71. $m < 2$
72. $m > 2$
73. $\frac{x}{2}$ [illegible] 2
74. $x(x + 2)$ [illegible] $4x$
75. $3a + 2 = 3(a + 2)$

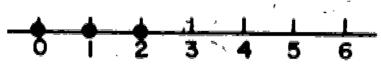
97

Thus far in this section the domain for x has been the set of all the numbers of arithmetic. Let us consider a problem where the domain is a finite set.

76 If the domain of x is $\{0,1,2,3,4,5\}$ which of the following is the correct graph of the truth set of $x + 3 < 6$? _____ B

A.

B.



77 Graph B is the graph of a/an ________ set. (finite, infinite) finite

3-4. <u>Compound Sentences</u>

All the sentences discussed so far have been simple sentences. That is, they have contained only one verb form. Let us consider a sentence such as

$$4 + 1 = 5 \text{ and } 6 + 2 = 8.$$

Your first impression may be that we have written <u>two</u> sentences. But actually the sentence is one <u>compound sentence</u>, with the connective <u>and</u> between two clauses. In mathematics, as well as in English, we encounter sentences which are compounded out of simple sentences.

1 Select the compound sentences from the following:

M. $3(5 + 2)$ and $4(6 + 3)$

N. $4 > 3$ and $5 > 4$

P. $2 > 3$ and $4 \neq 4$

[A] M [B] N [C] N and P [D] all of them

The correct choice is [C]. (M), "$3(5 + 2)$ and $4(6 + 3)$" is not a sentence, since there is no verb form. (N) and (P) <u>are</u> compound sentences. (M) might be called a compound phrase.

	Indicate which of the following are compound sentences.		
2	$2 + 3 = 5$	______ (yes, no)	no
3	$5 + 2 = 7$ and $2 + 4 \neq 8$	______	yes
4	$6 + 3 < 7$ and $2 \times 5 = 10$	______	yes
5	$4 + 3 = 7 + 2$	______	no
6	$5 + (3 + 7)$ and $3(2 + 1)$	______	no

The English sentence, "Mary carried an umbrella and John wore a raincoat," is true only if both clauses are true. That is, the compound sentence is true only if Mary actually carried an umbrella and John actually wore a raincoat. In other words, the sentence is false if either clause is false.

7 Which of the three sentences below is true?

(A) $6 + 1 \neq 5$ and $5 > 2 + 4$

(B) $3 + 1 \ngtr 6$ and $3 + 1 \neq 5$

(C) $6 - 5 \neq 1$ and $3 \nless 2 + 4$

In the compound sentence "$6 + 1 \neq 5$ and $5 > 2 + 4$" the second clause is false. In "$3 + 1 \ngtr 6$ and $3 + 1 \neq 5$" both clauses are true, while in "$6 - 5 \neq 1$ and $3 \nless 2 + 4$" both clauses are false. Hence, only the compound sentence in (B) is true.

8	[illegible] $\ngtr 1 + 2$ and $4 + 7 > 10$ is a ______ (true, false) sentence,	true
9	since both clauses are ______ (true, false).	true
	In order to decide whether the compound sentence [illegible] and [illegible]	
10	is true, we must test whether ______ (only one, both) clause(s) are true.	both
	In fact, the compound sentence [illegible] and [illegible]	
11	is ______ (true, false), even though [illegible] is true,	false
12	since [illegible] is ______.	false

Remember, a compound sentence with the connective "<u>and</u>" is true if <u>all</u> of its clauses are true; otherwise, it is false.

	Indicate which of the following compound sentences are true and which are false.		
13	$4 \cdot$ [illegible] $= 1$[illegible] and 11 [illegible] $10 + 1$, and [illegible]	(true, false)	true
14	$5 - \frac{11}{2} = \frac{1}{2}$ and $6 < \frac{2}{3} \times$ [illegible]	________	false
15	[illegible] $>$ [illegible] $+ 2$ and $4 +$ [illegible] 11	________	false
16	[illegible] $+ 3 >$ [illegible] $\times \frac{1}{3}$ and $4 \neq \frac{1}{2}$ [illegible]	________	true
17	$\frac{7}{8}$ [illegible] $\frac{11}{12}$ and [illegible]$.2 +$ [illegible]$.4 \neq 12.$[illegible]	________	false
18	[illegible]$.2$[illegible] $+ 0.$[illegible] $\neq 6.2$[illegible] and $6.2 + 0.$[illegible] 6.9[illegible]	________	true

Now consider this sentence: $4 + 1$ [illegible] or [illegible] $+ 2$ [illegible] 7. This is another type of compound sentence, one with the connective <u>or</u>. Here we must be very careful. While the meaning of "and" in English is clear, the word "or" in English may be used to show two different meanings. If we say, "The Yankees will win the pennant or the Indians will win the pennant", we mean that exactly <u>one</u> will win. If we say, "The class president must be a good student or he must play football", it is possible that he might be both a good student and a football player.

We eliminate this double meaning for the "or" in mathematics by agreeing that, unless otherwise stated, a compound sentence with the connective "<u>or</u>" is true if <u>one or more</u> of its clauses is true.

	A compound sentence with the connective "or" is <u>true</u>	
19	if one or more of its clauses is ______; otherwise	true
20	it is ______.	false
	If a false clause is connected to a true clause by the connective "or", then the resulting compound	
21	sentence is ______.	true
	Indicate which of the following compound sentences are true and which are false.	
22	[illegible] and [illegible] (true, false)	true
23	[illegible] or [illegible] ________	true

24	$3 < 2 + 2$ and $7 > 6 + 2$	______ (true, false)	false
25	$3 < 2 + 2$ or $7 > 6 + 2$	______	true
26	$4 = \frac{1}{2} + \frac{3}{2}$ and $7 < 2 + 3$	______	false
27	$4 = \frac{1}{2} + \frac{3}{2}$ or $7 < 2 + 3$	______	false

28 A compound sentence with the connective "or" is false only if

[A] at least one clause is false.

[B] all the clauses are false.

Since the compound sentence with the connective or is true if one or more of its clauses is true, we cannot be sure that it is false if we know only that at least one of its clauses is false. [B] is correct.

4. In the table that follows you are given (a) whether Clause 1 is true or false, (b) the connective to be used, and (c) whether Clause 2 is true or false. Determine whether each resulting compound sentence is true or false.

	(a) Clause 1	(b) Connective	(c) Clause 2	(d) Sentence	
29	TRUE	and	TRUE	______ (true, false)	true
30	TRUE	and	FALSE	______	false
31	FALSE	and	TRUE	______	false
32	FALSE	and	FALSE	______	false
33	TRUE	or	TRUE	______	true
34	TRUE	or	FALSE	______	true
35	FALSE	or	TRUE	______	true
36	FALSE	or	FALSE	______	false

Let us summarize:

A compound sentence with the connective AND is true only if all of its clauses are true. It is false if any clause is false.

A compound sentence with the connective OR is false only if

all of its clauses are false. It is true if any clause is true.

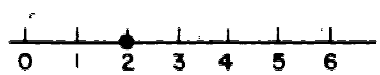
Items 29 and 30, taken together, lead to the following observation, which is of importance later: if we know that the compound statement,

"Clause 1 or Clause 2"

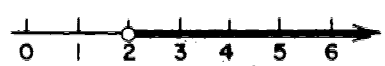
is <u>true</u>, and if we also know that Clause 1 is false, we may conclude that Clause 2 <u>must</u> <u>be</u> <u>true</u>.

	Consider the compound open sentence	
	$x = 2$ or $x > 2$.	
37	This sentence will be true if x has a value which makes either clause _______.	true
38	Thus, the truth set of "$x = 2$ or $x > 2$" is the _______ of the truth set of "$x = 2$" and the truth set of "$x > 2$".	union
39	The truth set of $x = 2$ is _______.	{2}
40	The truth set of $x > 2$ is the set of numbers which are _______.	greater than 2
41	Therefore, the truth set of "$x = 2$ and $x > 2$" is the set consisting of the number _______ and all	2
42	numbers _______ than 2.	greater

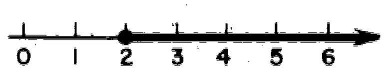
The graph of the truth set of $x = 2$ is



The graph of the truth set of $x > 2$ is

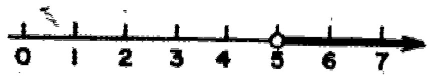


Since a graph of the truth set of an open sentence is a set of points, we may state that the graph of the truth set of "$x = 2$ or $x > 2$" is the union of the graphs of the truth sets of the separate clauses. Thus, the graph of the truth set of "$x = 2$ or $x > 2$" is

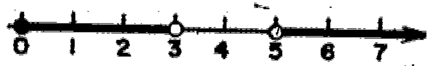


43 The graph of the truth set of $x < 3$ is ______. — see answer below

44 The graph of the truth set of $x > 5$ is ______. — see answer below

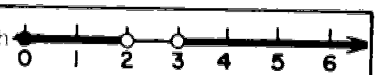


45 Therefore, the graph of the truth set of $x < 3$ or $x > 5$ is ______. — see answer below



It has required many words, carefully chosen, to describe the various connections among open sentences, truth sets, and graphs. We have consistently referred to the "graph of the truth set of an open sentence". In the future, let us agree to shorten this phrase to the "graph of a sentence".

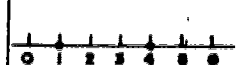
46 An open sentence for the graph



is ______ or $x >$ ______. — $x < 2$ or $x > 3$

47 The graph of the open sentence x [illegible] 1 or x [illegible] 4 is ______.

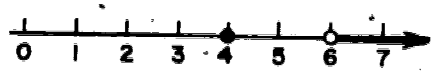
48 The graph of the open sentence x [illegible] 1 or x [illegible] 4 is ______.



49 The graph of the open sentence x [illegible] or x [illegible] is ______.

50 The graph of the open sentence x [illegible] or x [illegible] is ______.

Write an open sentence for each of the following graphs:

51  $x = 4$ or ______

$x = 4$ or $x > 6$

52  ______ or $x = 6$ or ______

$x < 4$ or $x = 6$ or $x > 6$

53  ______

$x < 4$ or $x > 6$

Instead of writing the compound open sentence, "$x > 2$ or $x = 2$", we shall use an abbreviation: "$x \geq 2$", which is read, "x is greater than or equal to 2". Similarly, the symbol $\leq$ is read, "is less than or equal to".

54	We may write $x < 5$ or $x = 5$ as ______.	$x \leq 5$
55	$x \geq 3$ is read, "x is ______."	x is less than or equal to 3
56	We may write $x = \frac{3}{2}$ or $x > \frac{3}{2}$ as ______.	$x \geq \frac{3}{2}$

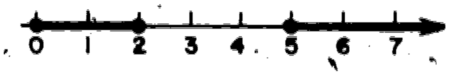
One further remark may be made. We have introduced the symbols $\nless$, $\neq$, $\ngtr$. Do you see how these symbols are related to compound sentences involving inequality?

57	$x \nless 7$ means the same as $x \underline{\quad} 7$. $(\leq, \geq)$	$x \leq 7$
58	"$x < 7$ or $x > 7$" may be written as ______.	$x \neq 7$
59	$x \geq 7$ means the same as ______.	$x \nless 7$

Remember: $x + 3 \geq 4$ is a compound open sentence having two clauses.

$x + 3 \geq 4$ and $x = 5$ is a compound open sentence with three clauses.

Write an open sentence for each of the following graphs:

60  $x \leq 2$ or x ______ — $x \leq 2$ or $x \geq 5$

61  $x \leq 2$ or x ______ — $x \leq 2$ or $x > 5$

62  __________ — $x < 2$ or $x \geq 5$

So far we have considered only the graphs of truth sets of compound open sentences with the connective "or". Now we shall study the graphs of compound open sentences with the connective "and". Sentences such as $x = 2$ and $x = 5$ offer no difficulty.

We recall that if two phrases are joined by "and",

63 the resulting compound sentence is ______ (true, false) if — **true**

both clauses are true, but false if either clause

64 is ______ (true, false). Thus, a compound sentence with the — **false**

connective "and" is true if and only if all of its clauses are true.

Since the two clauses of "$x = 2$ and $x = 5$" cannot

65 both be ______ (true, false) at the same time, the truth set — **true**

66 of "$x = 2$ and $x = 5$" is the ______ set. — **empty**

How many points are on the graph of the truth set of

67 "$x = 2$ and $x = 5$"? ______ (none, one, two) — **none**

How many points are on the graph of the truth set of

68 "$x = 2$ and $x = 7$"? ______ — **none**

Which of the following is the truth set of the sentence,

69 "$x = 9$ or $x = 11$"? $\{9\}$, $\{11\}$, $\{9, 11\}$, $\emptyset$. ______ — $\{9, 11\}$

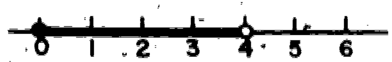
Which of the following is the truth set of the sentence,

70 "$x = 9$ and $x = 11$"? $\{9\}$, $\{11\}$, $\{9, 11\}$, $\emptyset$. ______ — $\emptyset$

We shall now turn our attention to compound sentences such as "$x > 2$ and $x < 4$" in which the connective "and" joins two inequalities.

The graph of the truth set of $x > 2$ is

The graph of the truth set of $x < 4$ is



The compound sentence, "$x > 2$ and $x < 4$" is true only if both clauses are true. Therefore, the graph of the truth set consists of those points which are included in both the graphs of the truth sets of the separate clauses. Thus, the graph of the truth set of "$x > 2$ and $x < 4$" is

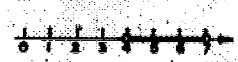
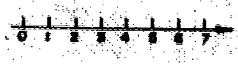
In short, the truth set of

$$x > 2 \text{ and } x < 4$$

is the intersection of the truth set of $x > 2$, and the truth set of $x < 4$. In words, we may say that the truth set of $x > 2$ and $x < 4$ is the set of numbers of arithmetic between 2 and 4.

71	The truth set of "$x > 1$ and $x < 6$" is the set of numbers of arithmetic between ______ and ______.	1 and 6
72	An open sentence whose truth set is the set of numbers between 5 and 9 is ______ and ______.	$x > 5$ and $x < 9$
73	The graph of "$x > 2$ and $x < 4$" is the ______ of the graph of $x > 2$ and the graph of $x < 4$.	intersection

74	The truth set of "$x < 2$ and $x > 4$" is the intersection of the truth set of ______ and the truth	$x < 2$
75	set of ______.	$x > 4$
76	The truth set of $x < 2$ and $x > 4$ is ______.	$\emptyset$

	Graph each of the following:	
77	$x > 4$ and $x < 7$	
78	$x < 4$ and $x > 7$	 ϕ
79	$x > 4$ or $x < 7$	
80	$x < 4$ or $x > 7$	

81	Sometimes we write "$x > 2$ and $x < 4$" as "$2 < x < 4$". This can also be read "x is between 2 and 4". Thus, we shall write "$x > 1$ and $x < 6$" as ___ $< x <$ ___.	$1 < x < 6$
82	The compound sentence "$x > 5$ and $x < 7$" written in this abbreviated form is ______. This sentence is often read, "5 is less than x and x is less than 7".	$5 < x < 7$
83	x is between 4 and 9 is written ______.	$4 < x < 9$
84	$3 < x < 7$ means that x is ______ 3 and 7.	between

Using the abbreviated notation we would write "x is either 2, 4, or any number of arithmetic between 2 and 4" as

$$2 \leq x \leq 4.$$

Do you see what would be meant by

$$2 \leq x < 4 \text{ ?}$$

3-5. Summary and Review

We have examined some numerical sentences and have seen that each one can be classified either as true or false, but not both.

We have also established a set of symbols to indicate relations between numbers:

"$=$" means "is" or "is equal to"

"$\neq$" means "is not" or "is not equal to"

"$<$" means "is less than"

"$>$" means "is greater than"

"$\leq$" means "is less than or is equal to"

"$\geq$" means "is greater than or is equal to"

We have discussed compound sentences which have two clauses. If the clauses are connected by the word or, the sentence is true if at least one clause is true. Otherwise it is false. If the clauses are connected by and, the sentence is true if both clauses are true. Otherwise it is false.

An open sentence is a sentence containing one or more variables.

The truth set of an open sentence containing one variable is the set of all those numbers which make the sentence true. The open sentence acts as a sorter. It sorts the domain of the variable into two subsets: a subset of numbers which make the sentence true, and a subset of numbers which make the sentence false.

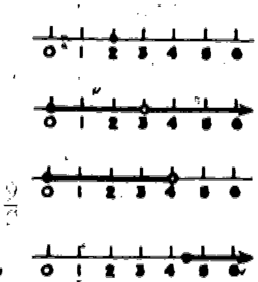
The graph of a sentence is the graph of the truth set of the sentence.

The following items provide further practice in working with open sentences, both simple and compound. For each open sentence that follows, state the truth set and construct its graph. Remember that the domain of the variable is the set of all numbers of arithmetic. Answers are on page iv.

Some examples of the form of your answers are:

Open Sentence	Truth Set
$x + 3 = 5$	$\{2\}$
$2x \neq x + 3$	The set of all numbers of arithmetic except 3.
$x + 1 < 5$	The set of all numbers of arithmetic less than 4.
$2x \geq 9$	The set of all numbers consisting of $\frac{9}{2}$ and all numbers greater than $\frac{9}{2}$.

Graph



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1. $a + 3$ [illegible] 14

2. $5 + v$ [illegible] 15

3. $2x$ [illegible]

4. [illegible] 1 [illegible] and [illegible] + t [illegible] 4

5. [illegible] or 2 + t [illegible] 1

6. x [illegible] x

7. $x + 2$ [illegible] or $x + 4$ [illegible] 6

8. $\frac{x}{[illegible]}$ [illegible]

9. $t + 4$ [illegible] or $t +$ [illegible] $\neq 5$

10. [illegible] $\neq a +$ [illegible]

11. [illegible]x^2 [illegible] 12

12. $(-x)^2$ [illegible]

13. [illegible] + t [illegible] 12 or [illegible] + 1 $\neq 6$

14. $5x + 3$ [illegible] 19

15. $(x - 1)^2$ [illegible] 4

16. [illegible]x [illegible] $(3 + x)$ [illegible] 2

17. 2 [illegible] x

18. $t + 6 \leq 7$ and $t + 6 \geq 7$

19. $3(x + 2)$ [illegible] $3x + 6$

20. $t + 2 \neq 5$ and $3 + 2$ [illegible] 5

21. x [illegible] 2 or x [illegible] 3

22. x [illegible] $\frac{4}{2}$ or $x + 2$ [illegible] 5

23. x [illegible] 3 and $x + 1$ [illegible] 4

24. $x + 1$ [illegible] 4 and $x + 2$ [illegible] 5

25. $x > 2$ or $x < 4$

26. $x > 3$ or x [illegible] 3

27. $x > 3$ or $x = 1$

28. $3 < x < 6$

29. $x < 2$ or $x \geq 5$

30. $x < 3$ or x [illegible]

31. $x \neq 3$ and $x \neq 4$

32. x [illegible] 3 or $x < 2$ or $x > 4$

33. $2 \leq x \leq 5$

34. $x < 3$ and $x > 2$ and $x < 4$

Chapter 4

PROPERTIES OF OPERATIONS

4-1. Identity Elements

	Consider the truth sets of the open sentences,	
	[illegible]	
1	The truth set of each of these sentences is ______.	{0}
2	What values of n make the sentence "$n + 0 = n$" true? ______	all values
3	The sum of any number and ______ is equal to the number.	0

Here we have an interesting property which we can state in words: For every number of arithmetic a, $a + 0 = a$. By the commutative property of addition we can see that $0 + a = a$ is also true. Either of these forms will be referred to as the <u>addition property of 0</u>, and we state this property as follows:

For every number a, $a + 0 = a$ and $0 + a = a$.

(For the time being, by "number" we mean "number of arithmetic", unless stated otherwise.)

Since adding 0 to any number gives identically the same number, the number 0 is called the <u>identity element</u> for addition.

4	So far we have been concerned with the addition property of ______.	0
5	We shall find it useful at times to remember that ______ is the identity element for addition.	0

Is there an identity element for multiplication? Consider the truth sets of the following open sentences:

[illegible]

6	The truth set of each of these sentences is ______.	{1}
7	The product of any number and 1 is equal to the ______.	number
8	The identity element for multiplication is ______.	1
9	In general, for any number n, $n(1) =$ ______.	n
10, 11	1 is the ______ element for the operation of ______.	identity, multiplication

Since $a(1) = 1(a)$, by the commutative property of multiplication, we state the multiplication property of 1 as:

For every number a, $a(1) = a$ and $(1)a = a$.

12 We can often use the multiplication property of 1 or the addition property of 0 to simplify the work of finding a common name of a number. In which of the following could neither of these properties be used?

[A] $27 + (\frac{47}{3} + \frac{7}{3}) - 18 = 27$

[B] $163(\frac{7}{12} + \frac{1}{6} + \frac{1}{4}) = 163$

[C] $(6 - 6)(46) = 0$

[A] $27 + (\frac{47}{3} + \frac{7}{3}) - 18 = 27 + 18 - 18$
$= 27 + (18 - 18)$
$= 27 + 0$
$= 27$

The last step is true by the addition property of 0.

[B] $163(\frac{7}{12} + \frac{1}{6} + \frac{1}{4}) = 163\left(\frac{7}{12} + (\frac{1}{6} + \frac{1}{4})\right)$ Associative property
$= 163\left(\frac{7}{12} + (\frac{2}{12} + \frac{3}{12})\right)$
$= 163(\frac{12}{12})$
$= 163(1)$
$= 163$

The last step is true by the multiplication property of 1.

[C] $(6 - 6)(46) = 0(46)$
$= 0$

Here we did not use either the multiplication property of 1 or the addition property of 0. The correct choice is [C].

	Look at the following sentences: $5(0) = 0$ $3(0) = 0$ $0(0) = 0$ $\frac{1}{2}(0) = 0$ $m(0) = 0$ Each time that we find the product of a number and 0 our result is 0.	
13	Consider the open sentence, $m(0) = 0$. Can a number m be found that will make this sentence false? ______ Try different values for m.	no
14	This sentence, $m(0) = 0$, is true for every number, ______.	m
15	This property of numbers is called the ______ ______ of 0.	multiplication property
16	Simply stated: Any number times ______ equals ______.	0,0

Since $a(0) = 0(a)$, by the commutative property of multiplication, we state the multiplication property of 0:

For every number a, $a(0) = 0$ and $0(a) = 0$.

The multiplication property of 0 is very useful. No matter how many numbers are multiplied, as long as one of them is zero, the product is still zero. Don't ever be fooled by a problem such as

$$10 \times 9 \times 8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1 \times 0.$$

The answer is simply 0.

	Find the common name for each of the following, and tell which of the three preceding properties was used:	
17	$\frac{7}{8}\left((3.7 + 0.3) - 4\right) =$ ______	0
18	The property used was the ______ property of ______.	multiplication,0
19	$\left(\frac{2}{3} - \frac{1}{3}(2)\right) + 17 =$ ______	17
20	The property used was the ______ property of ______.	addition, 0
21	$\left(\left(\frac{16}{3} - 5\right)(3)\right)(5280) =$ ______	5280
22	The property used was the ______ property of ______.	multiplication,1

If you missed any of Items 17, 19, or 21, turn to page vi where you will see how they can be done.

Remember that the number 1 has many names. For example, all of the following are names for 1:

$$\frac{4}{4}, \frac{3}{3}, \frac{6}{6}, \frac{11}{11}, \frac{105}{105}.$$

23	For every value of n, <u>except</u> <u>0</u>, $\frac{n}{n}$ = ______.	1

We shall often use names like these for 1. Note that no matter which name we use, the number 1 is the identity element for multiplication. For instance,

$$"n(\tfrac{4}{4}) = n"$$

is just another way of saying

$$"n(1) = n".$$

This multiplication property of 1 is useful in arithmetic in working with rational numbers. Suppose we wish to find a numeral for $\frac{5}{6}$ in the form of a fraction with 18 as its denominator. Since $\frac{5}{6} = \frac{5}{6}(1)$, we may use some other name for 1. Of the many possibilities, such as $\frac{2}{2}, \frac{3}{3}, \frac{5}{5}, \ldots,$ we choose "$\frac{3}{3}$" because 3×6 is 18.

	We know that	$\frac{5}{6} = \frac{5}{6}(1)$	
24	which may be written	$\frac{5}{6} = \frac{5}{6}(\frac{\square}{3})$	$\frac{5}{6}(\frac{3}{3})$
25	or	$\frac{5}{6} = \frac{5(3)}{6(\square)}$	$\frac{5(3)}{6(3)}$
26	to show that	$\frac{5}{6} = \square$.	$\frac{15}{18}$

Suppose we now wish to add $\frac{7}{9}$ to $\frac{5}{6}$. To do this we desire other names for $\frac{7}{9}$ and $\frac{5}{6}$, names which are fractions with the <u>same</u> denominator. What denominator should we choose?

	The denominator we choose must be a multiple of	
27	both ______ and ______.	9, 6
28	But it cannot be 0 since division by ______ is not possible.	0
	Non-zero multiples of 9 are members of the set,	
29	{9,18, , , ...}.	{9,18,27,36,...}

30 Non-zero multiples of 6 are members of the set, $\{\_,\_,\_,\ldots\}$. $\{6,12,18,\ldots\}$

31 Possible choices for our denominator are the common non-zero multiples of both 9 and 6; namely, the elements of the set, $\{\_,36,\_,\ldots\}$. $\{18,36,54,\ldots\}$

Although we could choose as our denominator any number in the set $\{18,36,54,\ldots\}$, if we choose the smallest of these numbers, we will find the computation to be easier.

32 _____ is the smallest. 18

33 This is called the least common multiple of _____ and _____. 9, 6

Return again to the problem of adding $\frac{7}{9}$ and $\frac{5}{6}$.

To change $\frac{7}{9}$ to a fraction with denominator 18, we proceed as follows:

$$\frac{7}{9} = \frac{7}{9}(1) = \frac{7}{9}\left(\frac{2}{2}\right)$$
$$= \frac{7(2)}{9(2)}$$

34 $\frac{7}{9} = \frac{\square}{18}$. $\frac{14}{18}$

35 From Item 26 we know $\frac{5}{6} = \square$. $\frac{15}{18}$

36 Then $\frac{5}{6} + \frac{7}{9} = \frac{\square}{18} + \frac{\square}{18}$ $\frac{15}{18} + \frac{14}{18}$

37 or $\frac{5}{6} + \frac{7}{9} = \square$. $\frac{29}{18}$

Let us use the multiplication property of 1 to find a common name for:

$$\frac{\frac{2}{3} + 5}{\frac{4}{7}}$$

	Using the multiplication property of 1: $\dfrac{\frac{2}{3}+5}{\frac{4}{7}} = \dfrac{\frac{2}{3}+5}{\frac{4}{7}}\left(\frac{21}{21}\right)$	
38	We chose $\frac{21}{21}$ because 21 is the least common multiple of ______ and ______.	3, 7
39	We can write: $\dfrac{\frac{2}{3}+5}{\frac{4}{7}} \quad \dfrac{(\frac{2}{3}+5)(21)}{\square(21)}$	$\frac{4}{7}$
40	Using the distributive property: $\dfrac{\frac{2}{3}(\square) + 5(\square)}{\frac{4}{7}(21)}$	$\dfrac{\frac{2}{3}(21) + 5(21)}{\frac{4}{7}(21)}$
41	Multiplying: $= \dfrac{14 + \square}{12}$	105
42	and adding: $= \square$.	$\frac{119}{12}$

43 In which of the following is the multiplication property of 1 used correctly to find a common name for the number? If you use paper and pencil to follow these, you will find them quite simple.

E. $\frac{1}{2} + \frac{1}{3} = \frac{1+1}{2+3} = \frac{2}{5}$

F. $\frac{7}{12} + \frac{5}{18} = \frac{7}{12}\left(\frac{3}{3}\right) + \frac{5}{18}\left(\frac{2}{2}\right) = \frac{21}{36} + \frac{10}{36} = \frac{31}{36}$

G. $\dfrac{7 + \frac{2}{3}}{\frac{5}{6}} = \dfrac{(7 + \frac{2}{3})(6)}{(\frac{5}{6})(6)} = \dfrac{42 + 4}{5} = \dfrac{46}{5}$

H. $\dfrac{\frac{1}{2} + \frac{3}{5}}{\frac{3}{20}} = \dfrac{(\frac{1}{2} + \frac{3}{5})(10)}{\frac{3}{20}(20)} = \dfrac{5 + 6}{3} = \dfrac{11}{3}$

[A] E and F [C] F and G

[B] F, G, and H [D] all four

In E no use is made of the multiplication property of 1, and the result is incorrect. Done correctly, it would be:

$$\frac{1}{2} + \frac{1}{3} = \frac{1}{2}\left(\frac{3}{3}\right) + \frac{1}{3}\left(\frac{2}{2}\right) = \frac{3}{6} + \frac{2}{6} = \frac{5}{6}$$

The multiplication property of 1 is used correctly in (F) and (G).

The multiplication property of 1 is used incorrectly in (H), since the number is multiplied by $\frac{10}{20}$, which is not a name for 1. Done correctly, this would be:

$$\frac{\frac{1}{2} + \frac{3}{5}}{\frac{3}{20}} = \frac{\frac{1}{2} + \frac{3}{5}}{\frac{3}{20}} \cdot \left(\frac{20}{20}\right) = \frac{\left(\frac{1}{2} + \frac{3}{5}\right)20}{\frac{3}{20}(20)} = \frac{10 + 12}{3} = \frac{22}{3}.$$

The only correct choice is [C].

4-2. Closure

Let us develop further what we already know from Chapter 1 about a set of numbers being closed under the operation of addition.

	If a and b are whole numbers (a and b may both be the same number), then we know that	
1	a + b is also a _______ number.	whole

We also know that whenever a and b are whole numbers, there is <u>exactly</u> <u>one</u> number, a + b, which is their sum.

2	That is, 4 + 3 = _______ and there is <u>exactly</u> <u>one</u> whole number, namely, 7, which is the sum of 4 and 3.	7
3	Another way of saying that _______ is the only whole number which is the sum of 4 and 3, is to say, "The sum of 4 and 3 is a <u>unique</u> whole number".	7
	The only whole number which is the sum of 5 and 6	
4	is _______; thus, we may say that the sum of 5 and	11
5	6 is a u - - - - - (six-letter word) whole number.	unique

Frame		Answer
6	There is ________ (exactly, more than) one whole number which is	exactly
7	the sum of 9 and 8; namely, the number ______.	17
8	The sum of 9 and 8 is a ______ whole number.	unique

We recognize that when we add whole numbers a and b, the unique (exactly one) number we obtain is an element of the set of whole numbers.

Frame		Answer
	We may express the preceding statement by saying that the set of whole numbers is <u>closed</u> under the operation	
9	of ______.	addition
	Think of any two even numbers. Find their sum. Is it an even number?	
	If a and b are two <u>even</u> numbers, their sum,	
10	a + b, ________ (is, is not necessarily) an even number.	is
	Just as the set of whole numbers is closed under the operation of addition, the set of even numbers is	
11	likewise ______ under addition.	closed
	Is the set of all odd numbers closed under addition? Let's check this.	
12	1 + 1 = 2, 5 + 3 = ______, 17 + 15 = ______.	8, 32
13	2, 8, and 32 ________ (are, are not) in the set of odd	are not
14	numbers; they are ______ numbers.	even
	The set of odd numbers is not closed under addition	
15	because we do not obtain an ______ number when we add two odd numbers.	odd
	Consider the set, S = {1,4,7,9}. Since 9 is in S and 4 is in S, but 9 + 4 is <u>not</u> in S,	
16	S ________ (is, is not) closed under addition.	is not

11?

17 Which of the following sets are closed under the operation of addition?

$X = \{1,2,3,4,5,6,7,8, \ldots\}$

$Y = \{1,3,5,7,9,11, \ldots\}$

$Z = \{0,2,4,6,8,10,12, \ldots\}$

[A] X and Y are closed under addition.
[B] X and Z are closed under addition.
[C] Y and Z are closed under addition.
[D] All three are closed under addition.

The only correct choice is [B]. $X = \{1,2,3,4,5,6,7,8, \ldots\}$ and $Z = \{0,2,4,6,8,10,12, \ldots\}$ are closed under addition since the sum of any two counting numbers is a unique counting number and the sum of any two even numbers is always a unique even number.

The set $Y = \{1,3,5,7,9,11, \ldots\}$ is not closed under addition. Y is the set of odd counting numbers, and the sum of any two odd numbers is always an even number.

The set of all numbers of arithmetic is closed under addition because the sum of any two numbers of arithmetic is always a unique number of arithmetic. Remember - we use the word "unique" to mean <u>exactly one</u>. We state this property as follows:

<u>Closure Property of the Set of Numbers of Arithmetic Under Addition</u>: For all numbers of arithmetic a and b, the number $a + b$ is a unique number of arithmetic.

All of the following numbers, $\frac{2}{3}$, $\frac{1}{5}$, 3, $\frac{1}{10}$, $\frac{3}{100}$, $\frac{3}{5}$,

18 __________ numbers of arithmetic. **are**
(are, are not)

Find the following sums:

19 $\frac{2}{3} + \frac{1}{5} =$ ______ **$\frac{13}{15}$**

20 $3 + \frac{1}{10} =$ ______ **$\frac{31}{10}$**

21 $\frac{3}{100} + \frac{3}{5} =$ ______ **$\frac{63}{100}$**

22 Each of the above sums is a ______ of ______. **number of arithmetic**

Just as the set of the numbers of arithmetic is closed under addition, from our experience we are aware that the set of the numbers of arithmetic is closed under multiplication. The product of any two numbers of arithmetic is always exactly one number of arithmetic. We state this property as follows:

<u>Closure Property of the Set of Numbers of Arithmetic under Multiplication</u>: For all numbers of arithmetic a and b, the number ab is a unique number of arithmetic.

Note that we use the symbol ab to indicate a times b. $a \times b$, $a(b)$, $(a)b$, $(a)(b)$, $a \cdot b$, and ab name the same number.

Find the following products:

23 $\frac{2}{3} \times \frac{1}{5} =$ ______ $\frac{2}{15}$

24 $3 \times \frac{1}{10} =$ ______ $\frac{3}{10}$

25 $\frac{3}{100} \times \frac{3}{5} =$ ______ $\frac{9}{500}$

26 The numbers $\frac{2}{3}$ and $\frac{1}{5}$, 3 and $\frac{1}{10}$, $\frac{3}{100}$ and $\frac{3}{5}$, are ______ of ______ as are their products. numbers of arithmetic

27 From experience we are aware that the set of all numbers of arithmetic is ______ under multiplication. closed

28 Is the set of all whole numbers closed under multiplication? ______ yes

The following two examples further illustrate the idea of closure.

29 Which of the following sets are closed under the operation of multiplication?

$K = \{1,3,5,7,9,11,13\}$

$L = \{1,3,5,7,9,11,13, \ldots\}$

$M = \{0,3,6,9,12,15,18, \ldots\}$

[A] K and L [C] all are closed

[B] K and M [D] L and M

The only correct choice is [D]. $L = \{1,3,5,7,9,11,13, \ldots\}$ and $M = \{0,3,6,9,12,15,18, \ldots\}$ are closed under the operation of multiplication. L is the set of odd counting numbers, and any element of L multiplied by any element of L is always an element of L. For example, $3 \times 5 = 15$, and 15 is an odd counting number. Also, $9 \times 11 = 99$, and 99 is an odd counting

(continued)

(29 Continued)

number. Set M is the set of multiples of three, which is closed under multiplication because the product of any two multiples of three is itself a multiple of three.

The set K = {1,3,5,7,9,11,13} is not closed under multiplication. For example, if we multiply 9 × 11 we obtain the unique number 99, but it is not an element of set K.

0 and 1 are two special numbers. They are the identity elements for addition and multiplication. Let us look at addition and multiplication in I = {0,1}. In order to do this easily, we shall make "addition" and "multiplication tables" as shown below.

+	0	1
0	0	1
1	1	2

×	0	1
0	0	0
1	0	1

30 Which of the following statements is true for the set I = {0,1}?

[A] I is closed under addition but not under multiplication.

[B] I is closed under multiplication but not under addition.

[C] I is closed under addition and multiplication.

The correct selection is [B]. I = {0,1} is closed under multiplication but not under addition. All of the numbers in the multiplication table are elements of I, but the number 2, which is in the addition table, is not in I.

4-3. Associative and Commutative Properties of Addition and Multiplication.

We have discussed a number of patterns for forming true sentences about numbers and have seen that these patterns are closely related to many of the techniques of arithmetic. For instance, we considered (in Section 2-2) sentences such as:

$$(7 + 8) + 5 = 7 + (8 + 5)$$

and

$$(1.2 + 1.8) + 2.6 = 1.2 + (1.8 + 2.6).$$

These sentences are true. Verify this fact!

These true sentences display the pattern known as the associative property of addition.

1	$(3 + 5) + 2 = 3 + (5 + 2)$ is a ________ sentence. (true, false)	true
2	It is an illustration of the ________ property of addition.	associative
3	The numerical phrase, $(3 + 5) + 2$, is a number made up of three numbers: 3, 5, and 2. In the phrase the second number, 5, is grouped together with the ________ number, 3.	first
4	In the phrase, $3 + (5 + 2)$, the second and third numbers are ________ together.	grouped
5	Our sentence, $(3 + 5) + 2 = 3 + (5 + 2)$ is a statement involving the same set of numbers, (, ,), grouped differently under addition.	(3,5,2)

We can describe this pattern about three numbers and the addition operation as follows:

6 7 8	"If you add a second number to a first number, and then add a ________ number to their sum, the result is the same as if you add the third number to the ________ and then add their ________ to the ________ number."	third second sum, first

Suppose we try to find a concise mathematical statement which describes the associative property of addition:

10	Since the first number can be any ______ of arithmetic, we can represent it by the variable a.	number
11	We will not represent the second number by a since the second number may be different from the ______ number.	first
12	The second number will be represented by ______ (the same, a different) variable.	a different
13	Let's represent the ______ number by the variable b.	second
14	Likewise, the third number will be represented by a variable other than ______ or ______, since the third	a,b or b,a
15	number may be different from the ______ or ______ number.	first,second
16	Represent the ______ number by the variable c.	third

Let's continue with our effort to state the associative property as a more brief mathematical statement.

	Given any numbers of arithmetic a, b, and c, the number obtained by adding the second to the first	
17	is ___ + b.	a + b
18	Now consider adding the third number, ______ (a, b, or c), to the number a + b.	c
19	An expression for this sum is (a + b) + ______.	(a + b) + c
20	However, the number obtained by adding the sum of the second and third numbers, (b + ___), to the first	(b + c)
21	number, a, is a + (______).	a + (b + c)
22	The associative property of addition tells us that (a + b) + c ______ (=, ≠) a + (b + c).	=

The symbols a, b, and c are variables. Their domain is the set of numbers of arithmetic. Using these variables we may state the associative proerty of addition: For all values of the variables a, b, and c it is true that

$$(a + b) + c = a + (b + c).$$

This kind of statement is usually shortened by saying:

For any numbers of arithmetic, $(a + b) + c = a + (b + c)$.

Compare the brevity of this statement with the same statement written out in Items 6-9.

23	A precise mathematical statement of the associative property of addition is: For any numbers of arithmetic a, b, and c, $(a + b) + c =$ ______.	$(a+b)+c=a+(b+c)$
24	As a consequence of the associative property of addition, every sentence of the <u>form</u> or <u>pattern</u> $(a + b) + c = a + (b + c)$ will be a true sentence if a, b, and c are ______ of arithmetic.	numbers
25	Consider the addition example, $1.2 + 8.8 + 8.8$. By the associative property, this example can be written two ways, either $(1.2 + 8.8) + 8.8$ or ______.	$1.2 + (8.8+8.8)$
26	The sentence, $(1.2 + 8.8) + 8.8 = 1.2 + (8.8 + 8.8)$ is true due to the ______ property of addition.	associative
27	Finding the common name from the phrase, $(1.2 + 8.8) + 8.8$ is ______ (easier, more difficult) than finding it from $1.2 + (8.8 + 8.8)$ since it is obvious that $1.2 + 8.8 = 10$.	easier

Another property developed in Chapter 2 was the commutative property of addition. In the language of algebra we state the commutative property of addition:

For any numbers of arithmetic a and b

$$a + b = b + a.$$

Only two variables are needed to state this property.

To simplify the computation in the example $\frac{1}{3}+\frac{1}{2}+\frac{2}{3}$, the commutative and associative properties of addition are both useful.

28 The ______ property of addition gives us the choice of how we group our numbers. — associative

29 Grouping $\frac{1}{3}$ and $\frac{1}{2}$, we may write $\frac{1}{3}+\frac{1}{2}+\frac{2}{3}$ as _____. — $(\frac{1}{3}+\frac{1}{2})+\frac{2}{3}$

$(\frac{1}{3}+\frac{1}{2})+\frac{2}{3}$ may be written as $(\frac{1}{2}+\frac{1}{3})+\frac{2}{3}$ by making

30 use of the ______ ______ of addition to change the order in $(\frac{1}{3}+\frac{1}{2})$ to $(\frac{1}{2}+\frac{1}{3})$. — commutative property

31 But $(\frac{1}{2}+\frac{1}{3})+\frac{2}{3}=$ ______ because of the associative property of addition. — $\frac{1}{2}+(\frac{1}{3}+\frac{2}{3})$

Now we can find a common name for $\frac{1}{2}+(\frac{1}{3}+\frac{2}{3})$.

32 $$\frac{1}{2}+(\frac{1}{3}+\frac{2}{3})=\frac{1}{2}+\underline{\quad}$$ — $\frac{1}{2}+1$

$$=\frac{3}{2} \text{ or } 1\frac{1}{2}$$

The example above illustrates that using these properties intelligently will simplify your work at times. Later on you will find these properties applied to more fundamental mathematical investigations.

You have learned before that there are associative and commutative properties for <u>multiplication</u>, also. How would you write these in the language of algebra?

Since the associative property of multiplication involves three numbers, a statement of this property

33 will contain ______ variables. — three

A statement of the associative property of multiplication is:

For any numbers of arithmetic, a, b, and c,

34 $(ab)c = a(\quad)$. — $(ab)c = a(bc)$

A statement of the commutative property of multiplication

35 involves ______ variables. — two

A statement of the commutative property of multiplication is:

36 For any numbers of arithmetic a and b, ab ______. — $ab = ba$

Associative and Commutative Properties for Multiplication:

For any numbers of arithmetic a, b, and c

$$(ab)c = a(bc)$$

and

$$ab = ba.$$

37 If m, n, s, u, v, and y are any numbers of arithmetic, which of the following true sentences is not an illustration of either the associative property of addition or the associative property of multiplication?

[A] $2u(vz) = (2uv)z$

[B] $(\frac{m}{2} + \frac{n}{2}) + \frac{s}{2} = \frac{m}{2} + (\frac{n}{2} + \frac{s}{2})$

[C] $5z(uv) = (uv) \cdot 5z$

If you will recall that $2u$ can be thought of as the name for a single number by the closure property of multiplication for the numbers of arithmetic, then you will see that $2u(vz) = (2uv)z$ is true by the associative property of multiplication.

"$(\frac{m}{2} + \frac{n}{2}) + \frac{s}{2} = \frac{m}{2} + (\frac{n}{2} + \frac{s}{2})$" is an illustration of the associative property of addition for the numbers of arithmetic, since if m, n, and s are any numbers of arithmetic, then $\frac{m}{2}$, $\frac{n}{2}$, and $\frac{s}{2}$ are also numbers of arithmetic.

$5z(uv) = (uv) \cdot 5z$ is a true sentence, but it illustrates the commutative property of multiplication, not the associative property for addition or the associative property for multiplication. You are correct. Remember: commutativity has to do with order; associativity has to do with grouping.

38 If x, y, m, and n are numbers of arithmetic, which of the following sentences are true?

R. $(3x)(9y) = (9y)(3x)$

S. $1.7 + 2.9 = 2.9 + 1.7$

T. $3 + (\frac{5}{2}m + \frac{1}{2}n) = (3 + \frac{1}{2}n) + \frac{5}{2}m$

[A] S and T only

[B] R and T only

[C] R and S only

[D] all of them

(R) is an instance of the commutative property of multiplication.
(S) is an instance of the commutative property of addition.
Hence, both are true sentences. (T) is also a true sentence since

$$3 + (\frac{5}{2}m + \frac{1}{2}n) = 3 + (\frac{1}{2}n + \frac{5}{2}m) \quad \text{commutative property of addition}$$

$$= (3 + \frac{1}{2}n) + \frac{5}{2}m \quad \text{associative property of addition.}$$

Hence, the correct choice is [D].

Because of the associative and commutative properties of addition and multiplication for the numbers of arithmetic, you can reorder and regroup the terms in an indicated sum or the factors of an indicated product of several numbers.

Consider, for example, the product

$$4 \cdot a \cdot 3 \cdot b \ .$$

We can write the numerals 4, a, 3, b in many orders. (There are 24 orders to be exact, as you can verify for yourself.) We may write, for example,

$$4 \cdot a \cdot b \cdot 3, \text{ or}$$
$$a \cdot b \cdot 4 \cdot 3, \text{ or}$$
$$b \cdot 4 \cdot 3 \cdot a, \text{ etc.}$$

Also, we may group the factors $3, 4, a, b$ in many ways.

So, we can form many true sentences, of which these are samples:

$$(4a)(3b) = (3a)(b)(4)$$
$$(4)(3a)(b) = (4a)(3b)$$

If we want a short name for $(4a)(3b)$, we can think:

$$(4a)(3b) = (4 \cdot 3)(ab)$$
$$= 12ab$$

In a similar way, we can write the product $x \cdot y \cdot x$ in many ways. xyx and xxy are numerals for the same number. You will recall that x^2y is another way to write xxy. Thus, another name for $(xy)x$ is x^2y.

	To write another name for $(2r)(5s)$, we might think:		
39	$(2r)(5s) = (2 \cdot 5)(r \_\_)$		s
40	$= \_\_\_ rs$		10rs
	To write another name for $(17a)(3ab)$, we might think:		
41	$(17a)(3ab) = (17 \cdot 3)(\_ \cdot \_)(b) = 51a^2b$		$(a \cdot a)$
42	You remembered, of course, that $a \cdot a =$ ____.		a^2
43	A short name for $(3x)(4)$ is ____ x.		12x

	Read the following carefully and complete each statement.	
44	$(3d)d = 3(d \cdot \_\_) = 3d^2$	d
45	$(5p)(3q) = (5 \times 3)(\_\_\_) = \_\_\_\_$	(pq), 15pq
46	$(8m)(2n) = \_\_\_\_$	16mn
47	$(9y)(7x) = \_\_\_\_$	63xy
48	$(4c)(3abc) = \_\_\_\_$	$12abc^2$

Were you bothered by the arrangement in the previous responses? What do you note about 10rs, 12x, 15pq, 16mn, 63xy, and $12abc^2$?

	In each case the variables in the above responses	
49	were written _______ (first, last)	last
	In the response $12abc^2$, 12 is written	
50	before ____ c^2.	abc^2

In products we usually write variables last. You may also have observed that when several variables are involved in a product, we usually write them in alphabetical order. This is not a hard and fast convention, but "3ab" is usually written rather than "3ba" unless there is a special reason for not doing so. And, of course, you already know it is mathematically correct to do this since multiplication is commutative and associative.

51	$(\frac{1}{2}x)(6y) = \underline{\quad xy}$	$3xy$
52	$(.2q)(4p) = \underline{\qquad}$	$.8pq$
53	$100a^2b = (10a)(\underline{\qquad})$	$(10ab)$
54	$100a^2b = (25ab)(\underline{\qquad})$	$(4a)$
55	$(\frac{1}{2}cd)(6am) = \underline{\qquad}$	$3acdm$
56	$3xy = (\frac{6}{5}x)(\underline{\qquad})$	$(\frac{5}{2}y)$
	In 51 through 56 above we made use of the associative and commutative properties of multiplication.	
57	$7xy^2 \underset{(=,\neq)}{\underline{\qquad}} (7yx)(y)$	$=$
58	$64a^2bc^2 \underset{(=,\neq)}{\underline{\qquad}} (16abc)(4abc)$	$\neq$
59	$x^2y^2 \underset{(=,\neq)}{\underline{\qquad}} (xy)(yx)$	$=$
60	$(8a)(b^2)$, $(8)(ab)(b)$, $(4b)(2ab)$, $(24ab)(\frac{1}{3}ab)$ $\underset{\text{(are, are not)}}{\underline{\qquad\qquad}}$ all indicated products equal to $8ab^2$.	are not
61	In the preceding item, the indicated product $(\underline{\quad})(\underline{\quad})$, is not equal to $8ab^2$.	$(24ab)(\frac{1}{3}ab)$

All the properties with which we have been concerned are the commutative properties of addition and multiplication for the numbers of arithmetic. Why are we so concerned whether or not binary operations like addition and multiplication are commutative? Let us try <u>division</u>, for example. Recall that

$$6 \div 3$$

means "6 divided by 3". How would you write this example to test commutativity? We want to ask:

	Is "$6 \div 3 = 3 \div 6$" a true sentence?	
62	Since $6 \div 3 = \underline{\qquad}$ and $3 \div 6 = \underline{\qquad}$, the	2, $\frac{1}{2}$
63	above sentence is $\underline{\qquad}$.	false

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Since $6 \div 3 = 3 \div 6$ is false, we can conclude that $a \div b = b \div a$ is not true for all numbers of arithmetic, and hence that division is not commutative.

Let us test the operation of division for associativity by using the following sentence:

$$(12 \div 6) \div 2 = 12 \div (6 \div 2).$$

If this sentence is true, then we can consider the possibility that division is associative. If this sentence is false, we can state that division is not associative.

To test associativity for division we want to ask:

64	Is $(12 \div 6) \div 2 = 12 \div$ ______ a true sentence?	$12 \div (6 \div 2)$
65	$(12 \div 6) \div 2$ is a name for ______.	1
66	$12 \div (6 \div 2)$ is a name for ______.	4
67	$(12 \div 6) \div 2 = 12 \div (6 \div 2)$ is a ______ (true, false) sentence.	true

We can therefore conclude that division is not associative. It is neither commutative nor associative.

If you are interested in seeing some other binary operations, complete Items 68-76; otherwise, begin Section 4-4.

*68 Let us define another binary operation as follows:

Let $2 ** 3$ be defined to mean $(2)(2)(2)$; and $3 ** 2$ to mean $(3)(3)$. In general, $a ** b$ will be defined to mean a has been used as a factor b times, where a and b are counting numbers. Is the following sentence true?

$$5 ** 2 = 2 ** 5$$

[A] yes [B] no

Since $5 ** 2$ means $(5)(5)$ or 25 and $2 ** 5$ means $(2)(2)(2)(2)(2)$ or 32, $5 ** 2 = 2 ** 5$ is a false sentence, and [B] was the correct choice.

	Let us see if the operation "**" as defined in Item 68 is associative: Is (2 ** 3) ** 2 = 2 **(3 **2) a true sentence?	
*69	(2 ** 3) = (2)(2)(2) = ______	8
*70	(2 ** 3)** 2 = (2)(2)(2) × ( )( )( ) = 8 × 8	(2)(2)(2)
*71	(2 ** 3)** 2 = ______	64
*72	2 **(3 ** 2) = 2 ** ((3)( )) = 2 ** 9	(3)
*73	2 ** 9 = (2)(2)(2)(2)(2)(2)(2)(2)(2) = ______	512
	We can conclude that the sentence	
*74	(2 ** 3)** 2 = 2 **(3 ** 2) is ______ (true, false) and that	false
*75	the operation "**" ______ (is, is not) associative.	is not

We have found that there exist reasonable binary operations which are neither commutative nor associative. You may complain that such operations are artificial. On the contrary, the "**" operation is actually used in the language of certain digital computers. A machine is much happier if you give it all its instructions on a line, and so a horizontal notation which writes 2^3 as 2 ** 3 was devised for this operation and here the order of the numbers makes a great difference.

You may think that a binary operation must be both commutative and associative, or else that neither of these properties holds. This is not the case. Consider the following binary operation:

*76 We define a o b to be $\frac{a + b}{2}$ for all numbers of arithmetic, a and b. We can easily check to see that this is a commutative operation since a o b = $\frac{a + b}{2}$ and b o a = $\frac{b + a}{2}$.

$$\frac{a + b}{2} = \frac{b + a}{2}$$

is true since a and b are numbers of arithmetic and a + b = b + a is true by the commutative property of addition for the numbers of arithmetic.

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Now let us consider the associative property for this operation. Let $a = 3$, $b = 2$, and $c = 4$ and test to see if

$$(3 \circ 2) \circ 4 = 3 \circ (2 \circ 4).$$

Is this operation associative?

[A] yes [B] no

$$(3 \circ 2) \circ 4 = \frac{\frac{3+2}{2} + 4}{2} = \frac{3 + 2 + 8}{4} = \frac{13}{4}$$

and

$$3 \circ (2 \circ 4) = \frac{3 + \frac{2+4}{2}}{2} = \frac{6 + 2 + 4}{4} = \frac{12}{4}.$$

This is not an associative operation and [B] was the correct choice. As you go on with your study of mathematics, you will see other examples of binary operations that may be associative or commutative, or neither, or both.

4-4. The Distributive Property

Our previous work with numbers has shown us many illustrations of the distributive property of the numbers of arithmetic.

$$15(7 + 3) = 15(7) + 15(3)$$

and

$$\frac{1}{3}(12) + \frac{1}{4}(12) = (\frac{1}{3} + \frac{1}{4})12$$

are both true sentences which illustrate the distributive property.

1	$(\frac{2}{5} + \frac{3}{4})20 = \frac{2}{5}(20) + \frac{3}{4}(20)$ also illustrates the ______ property.	distributive
2	The phrase $(\frac{2}{5} + \frac{3}{4})20$ is an indicated ______ of $(\frac{2}{5} + \frac{3}{4})$ and 20.	product
3	The phrase $\frac{2}{5}(20) + \frac{3}{4}(20)$ is an indicated ______ of $\frac{2}{5}(20)$ and $\frac{3}{4}(20)$.	sum

4 5	The true sentence $(\frac{2}{5} + \frac{3}{4})20 = \frac{2}{5}(20) + \frac{3}{4}(20)$ states a relation between an indicated ______ and an indicated ______.	product, or sum; sum, or product

The example above points out that the distributive property relates indicated sums and indicated products. As you continue your work in mathematics you will find what an important role this property plays.

You may find it difficult to decide whether a number is an indicated sum or an indicated product. You must look carefully at the numerals and the indicated operations. For example, $3 + x$ is an indicated sum since the number x is added to 3. On the other hand, $3x$ may be thought of as an indicated product since x is multiplied by 3.

6	$x + 12$ is an indicated ______.	sum
7	$ax(3 + x)$ is an indicated ______.	product
8	$(a + b)n$ is an indicated ______.	product
9	$an + bn$ is an indicated ______.	sum

If you were correct on Items 6-9, you may skip to Item 13. If not, continue with Item 10.

10 The phrase $y(x + b)$ is an

[A] indicated sum

[B] indicated product

Although $(x + b)$ does indicate a sum, the entire open phrase, $y(x + b)$ is an indicated product. It is the product of y and $(x + b)$. [B] is correct.

11 The phrase $yx + yb$ is an

[A] indicated sum

[B] indicated product

Although yx and yb are indicated products, the entire open phrase $yx + yb$ is an indicated sum. It is the sum of yx and yb. [A] is correct.

12	The phrase $(4a + b)p$ is an [A] indicated sum [B] indicated product

$(4a + b)p$ is an indicated product. It is the product of $(4a + b)$ and p. By now you surely observed further that $(4a + b)$ is the indicated sum of b and the indicated product, $4a$. However, the entire open phrase $(4a + b)p$ is an indicated product. [B] is correct.

Let's look at some of our examples for indicated sums and indicated products again.

Indicated Sums	Indicated Products
$xy + zw$	$(x + y)(z + w)$
$3x^2 + 4xy$	$x(3x + 4y)$

Notice that $xy + zw$ is the indicated sum of two products; namely, xy and zw. On the other hand, $(x + y)(z + w)$ is the indicated product of two sums; namely, $(x + y)$ and $(z + w)$. Let us note also that the indicated sum $3x^2 + 4xy$ and the indicated product $x(3x + 4y)$ illustrate the distributive property since $3x^2 + 4xy$ is the same number as $x(3x + 4y)$.

13	The distributive property relates indicated sums and indicated ______. Hence, it is used in two situations:	products
14	The indicated product $3(x + y)$ may be written as the indicated ______ $3x + 3y$.	sum
	On the other hand, we may write the indicated sum $3x + 3y$ as an indicated product:	
15	$3x + 3y = 3(\quad)$.	$3(x + y)$
16	$6 + x$, $4a + 2$, $\frac{5}{3}m + \frac{3}{5}n$ are all indicated ______.	sums
17	$(3 + x)y$, $3(x + y)$, $4a(7x + 5y)$ are all indicated ______.	products

Complete each of the following:

	Indicated Product	=	Indicated Sum	
18	3(____)	=	3m + 3n	3(m + n)
19	a(x + y)	=	______	ax + ay
20	x(____)	=	x^2 + xz	x(x + z)
21	b(d + 1)	=	______	bd + b [Remember: b·1 = b]
22	x(y + 3)	=	______	xy + 3x

Notice in Item 22, that by using the distributive property, we get x(y + 3) = xy + x3. However, we have agreed to write x3 as 3x. The commutative property of multiplication makes this permissible.

The indicated sum 15x + 25y can be written as an indicated product. To do so we proceed as follows:

23 15x + 25y = 5(__) + 5(__) 5(3x) + 5(5y)

= 5(3x + 5y)

Just as we have stated the associative and commutative properties in mathematical language, we can state the <u>distributive property</u>: For all numbers of arithmetic a, b, and c, a(b + c) = ab + ac, and (b + c)a = ba + ca.

Although there is only one distributive property, we shall learn to recognize it in any of the following patterns.

24	a(b + c) = ab + ____	ab + ac
25	ab + ac = a(____)	a(b + c)
26	(b + c)a = ba + ____	ba + ca
27	ba + ca = (b + c)____	(b + c)a

In order to understand the patterns of the distributive property better, examine the following. If all of the variables used represent numbers of arithmetic, write indicated products as indicated sums and indicated sums as indicated products.

28	m(p + q) = ______	mp + mq
29	rs + rt = ______	r(s + t)

30	$(h + k)x =$ ______	$hx + kx$
31	$mn + qn =$ ______	$(m + q)n$

Use the distributive property to write the following indicated products as indicated sums.

32	$3(10 + 5) = 3(10) + \ \ (\ \ )$	$3(10) + 3(5)$
33	$3(m + 2) = 3(m) +$ ______ $= 3m + 6$	$3(m) + 3(2)$
34	$(m + 3)2 =$ ______ $+ 3(2)$	$m(2) + 3(2)$
35	$=$ ______ $+ 6$	$2m + 6$
36	$\frac{1}{2}(6 + c) =$ ______	$\frac{1}{2}(6) + \frac{1}{2}(c)$
37	$=$ ______ $+ \frac{1}{2}c$	$3 + \frac{1}{2}c$
38	$(a + b)4 =$ ______	$a(4) + b(4)$
39	$=$ ______	$4a + 4b$
40	$(x + y)b =$ ______ $+ y(b)$	$x(b) + y(b)$
41	$=$ ______	$xb + yb$ or $bx + by$
42	$c(d + e) =$ ______	$c(d) + c(e)$
43	$=$ ______	$cd + ce$
44	$a(a + 2) = a(a) +$ ______	$a(a) + a(2)$
45	$=$ ______	$a^2 + 2a$

Omit the middle step if you wish in the following.

46	$0.7(3 + n) =$ ______	$2.1 + 0.7n$
47	$(c + d) =$ ______	$4c + 4d$
48	$b(a + 1) =$ ______ (Remember: $b \cdot 1 = b$)	$ab + b$
49	$2(\ell + w) =$ ______	$2\ell + 2w$

Use the distributive property to write these indicated sums as indicated products.

50	$3(4) + 3(7) = \underline{3(\quad)} = 3(11)$	$3(4 + 7)$
51	$3a + 3b = \underline{\qquad}$	$3(a + b)$
52	$3c + 3d = \underline{\qquad}$	$3(c + d)$
53	$3(5) + 7(5) = \underline{(\quad)5} = 10(5)$	$(3 + 7)5$
54	$ma + na = \underline{\qquad}$	$(m + n)a$
55	$ab + a(4) = \underline{\qquad}$	$a(b + 4)$
56	$ab + 4a = \underline{\qquad}$	$a(b + 4)$
57	$2a + a^2 = \underline{\qquad}$ Recall: $a^2 = a(a)$ or $a \cdot a$	$(2 + a)a$
58	$x^2 + xy = \underline{\qquad}$	$x(x + y)$
59	$4c + 3c = \underline{\qquad}$	$(4 + 3)c$

Complete each of the following:

60	$ay + y = \underline{a \cdot y + 1 \cdot \quad}$	$a \cdot y + 1 \cdot y$
61	$= \underline{(a + \quad)y}$	$(a + 1)y$
62	$y + y^2 = \underline{(\quad)y + (\quad)y}$	$(1)y + (y)y$
63	$= \underline{(\quad)y}$	$(1 + y)y$
64	$y + y^2 = \underline{y(\quad) + y(\quad)}$	$y(1) + y(y)$
65	$= \underline{y(\quad)}$	$y(1 + y)$

Write the indicated sum, $2rs + 2rt$, as an indicated product.

We can write the indicated sum, $2rs + 2rt$, as an indicated product in three ways:

66	1) $2rs + 2rt = \underline{2(\quad) + 2(\quad)}$	$2(rs) + 2(rt)$
67	$= \underline{2(\quad)}$	$2(rs + rt)$
68	2) $2rs + 2rt = \underline{r(\quad) + r(\quad)}$	$r(2s) + r(2t)$
69	$= \underline{\qquad}$	$r(2s + 2t)$
70	3) $2rs + 2rt = \underline{2r(\quad) + 2r(\quad)}$	$2r(s) + 2r(t)$
71	$= \underline{2r(\quad)}$	$2r(s + t)$

Although all three ways are correct, the third is usually preferred.

$54 + 36x^2$ could be written as

$$6(9) + 6(6x^2) \text{ or}$$
$$9(6) + 9(4x^2)$$

or several other ways; however, the following is usually preferred since 18 is the largest number which fits the pattern:

72	$54 + 36x^2 = \underline{18(3) + 18(\quad)}$	$18(3) + 18(2x$
73	$= \underline{18(\qquad)}$	$18(3 + 2x^2)$
74	$54x^2y + 36xy^2 = \underline{18xy(\quad) + 18xy(\quad)}$	$18xy(3x) + 18xy($
75	$\underline{18xy(\qquad)}$	$18xy(3x + 2y)$

Write each open phrase as an indicated product, if possible:

76	$3a + 5a = (3 + 5)a = \underline{\qquad}a$	8a
77	$17b + 3b = \underline{(17 + \quad)b} = 20b$	$(17 + 3)b$
78	$6x + x = \underline{(6 + \quad)x} = 7x$	$(6 + 1)x$
79	$5y + \frac{7}{2}y = (\frac{10}{2} + \frac{7}{2})y = \underline{\qquad}$	$\frac{17}{2}y$
80	$8c + 2c = \underline{\qquad}\ \underline{\qquad}$	$(8 + 2)c$, 10c
81	$1.2m + 3.8m = \underline{\qquad}$	5.0m or 5m
82	$\frac{1}{4}m + \frac{4}{2}m = (\frac{1}{4} + \frac{4}{2})m = \underline{\qquad}$	$\frac{9}{4}m$
83	$5a + 3b = \underline{\qquad}$	$5a + 3b$

The commutative and associative properties of addition, together with the distributive property, make it possible to write some open phrases in simpler form. For example:

(i) $3x + 2y + 4x + 6y = (3x + 4x) + (2y + 6y)$

(ii) $= (3 + 4)x + (2 + 6)y$

$= 7x + 8y$

The properties used in the work shown above were:

84	In step (i), the ______ and ______ properties of addition.	associative, commutative
85	In step (ii), the ______ property.	distributive

Where possible, use the associative, commutative, and distributive properties to write the following in simpler form:

86	$\frac{2}{3}a + 3b + \frac{1}{3}a = \underline{(\frac{2}{3}a + \quad) + 3b}$	$(\frac{2}{3}a + \frac{1}{3}a) +$
87	$= \underline{(\quad\quad)a + 3b}$	$(\frac{2}{3} + \frac{1}{3})a + 3b$
88	$= \underline{(1)a + \quad\quad}$	$(1)a + 3b$
89	$= \underline{\quad\quad\quad}$	$a + 3b$
90	$5m + 2n + 5m = \underline{\quad\quad}$	$10m + 2n$
91	$4a + 3b + 2b + 8a = \underline{(4a + 8a) + (\quad\quad)}$	$(4a+8a)+(3b+2$
92	$= \underline{(\quad\quad)a + (\quad\quad)}$	$(4+8)a + (3+2$
93	$= \underline{\quad\quad}$	$12a + 5b$
94	$7x + 13y + 2x + 3y = \underline{\quad\quad}$	$9x + 16y$
95	$4x + 2y + 3 + 3x = \underline{\quad\quad}$	$7x + 2y + 3$
96	$1.3x + 3.7y + 6.2 + 7.7x = \underline{\quad\quad}$	$9x + 3.7y + 6$
97	$2a + \frac{1}{3}b + 5 = \underline{\quad\quad}$	$2a + \frac{1}{3}b + 5$

How might we write $3(x + y + z)$ as an indicated sum? By the associative property we might group the x and the y, to write

98	$3(x + y + z)$ as $\underline{3((x + y) + \quad)}$	$3((x + y) + z)$
99	Then, by the ______ property,	distributive
100	$3((x + y) + z) = \underline{3(x + y) + \quad\quad}$.	$3(x + y) + 3z$
	Once more, using the distributive property,	
101	$3(x + y) + 3z = \underline{(3x + \quad) + 3z}$	$(3x + 3y) + 3z$
	$= 3x + 3y + 3z.$	

From Items 98-101 we arrive at the statement

$$3(x + y + z) = 3x + 3y + 3z$$

which is true for all values of the variables. By a similar process we could extend the pattern of the distributive property to the sum of three or more numbers.

	Write each of the following as an indicated sum:	
102	$(c + f + g)h = \underline{ch + fh + \quad\quad}$	$ch + fh + gh$
103	$3a(a + b + c) = \underline{\quad + \quad + 3ac}$	$3a^2 + 3ab + 3ac$
104	$6(2s + 3r + 7q + a) = \underline{\quad\quad}$	$12s+18r+42q+6a$
105	$(2x + 3y + 4z)7z = \underline{14xz + \quad + \quad}$	$14xz+21yz+28z^2$
106	$(\frac{1}{2}a + \frac{3}{4}b + \frac{1}{3}c)8 = \underline{\quad\quad}$	$4a + 6b + \frac{8}{3}c$

Another important application of the distributive property is illustrated by the following example:

Write $(x + 2)(x + 3)$ as an indicated sum without parentheses. Before looking into this matter more closely, let's look at another similar example,

$$(x + 2)(b + c).$$

	The closure property of addition assures us that if	
107	x is a number, $(x + 2)$ is a ______.	number
	Then $(x + 2)(b + c)$ may be thought of as $a(b + c)$, thus	
108	$\begin{array}{ccc} a & (b\ + & \underline{\quad}) \\ \uparrow & \uparrow & \uparrow \\ \overbrace{(x + 2)} & (b\ + & c) \end{array}$	c
	In the same way, $(x + 2)(x + 3)$ may also be thought of as	
109	$\begin{array}{ccc} a & (b\ + & c) \\ \uparrow & \uparrow & \uparrow \\ \overbrace{(x + 2)} & (x\ + & \underline{\quad}) \end{array}$	3
	We can now apply the distributive property, $a(b + c) = ab + ac$, as follows:	
110	$\begin{array}{ccccc} a\ (b + c) & = & a \cdot b & + & a \cdot c \\ \overbrace{(x + 2)}(x + 3) & = & \overbrace{(x + 2)}(x) & + & \underline{\overbrace{(\quad\quad)}(3)} \end{array}$	$(x + 2)(3)$

The remaining steps are displayed below.

$$\underbrace{(x+2)}_{a}\underbrace{(x+3)}_{(b+c)} = \underbrace{(x+2)}_{a}\cdot\underbrace{(x)}_{b} + \underbrace{(x+2)}_{a}\cdot\underbrace{(3)}_{c} \qquad \text{(i)}$$

$$= \Big(x(x) + 2(x)\Big) + \Big(x(3) + 2(3)\Big) \qquad \text{(ii)}$$

$$= x^2 + 2x + 3x + 6 \qquad \text{(iii)}$$

$$= x^2 + (2x + 3x) + 6 \qquad \text{(iv)}$$

111 $= x^2 + (\quad\quad)x + 6$ (v) — $x^2 + (2+3)x + 6$

$$= x^2 + 5x + 6 \qquad \text{(vi)}$$

In going from step (i) to (ii), the distributive

112 property was used ________ time(s). — two
(how many)

113 In going from step (iv) to (v), the ________ property was used again. — distributive

114 The property used from (iii) to (iv) was the ________ property of addition. — associative

From the items above we have arrived at the sentence $(x + 2)(x + 3) = x^2 + 5x + 6$, which is true for all values of the variable x.

115 Write the phrase $(a + 5)(a + 8)$ as an indicated sum.

[A] $40 + a^2$

[B] $a^2 + 5a + 8a + 40$

[C] $a^2 + 13a + 40$

Your work might look somewhat like this:

$$(a + 5)(a + 8) = (a + 5)a + (a + 5)8$$
$$= a^2 + 5a + a(8) + 40$$
$$= a^2 + 5a + 8a + 40$$
$$= a^2 + (5 + 8)a + 40$$
$$= a^2 + 13a + 40$$

Although on the third line of the work shown we have the same indicated sum as in [B], a better choice is [C].

Use the distributive property to write each open phrase as an indicated sum following the pattern used in Items 111-114.

116. $(y + 2)(y + 9) =$ ______

117. $(x + 1)(x + 5) =$ ______

118. $(2x + a)(x + a) =$ ______

119. $(3x + 4)(4x + 3) =$ ______

120. $(x + y)(x + y) =$ ______

After you have completed all of Items 116-120, turn to page vi and compare your work with that given there.

4-5. The Numbers of Arithmetic

Let us summarize our conclusions about the operations of addition and multiplication in the set of numbers of arithmetic. In the statements that follow, we are concerned only with numbers of arithmetic.

1. Closure Property under Addition: For all numbers a and b, there is exactly one number $a + b$.
2. Closure Property under Multiplication: For all numbers a and b, there is exactly one number ab.
3. Commutative Property of Addition: For all numbers a and b, $a + b = b + a$.
4. Commutative Property of Multiplication: For all numbers a and b, $ab = ba$.
5. Associative Property of Addition: For all numbers a, b, and c, $(a + b) + c = a + (b + c)$.
6. Associative Property of Multiplication: For all numbers a, b, and c, $(ab)c = a(bc)$.
7. Distributive Property: For all numbers a, b, and c, $a(b + c) = ab + ac$ and $(b + c)a = ba + ca$.
8. Addition Property of 0: For every number a, $a + 0 = a$ and $0 + a = a$.
9. Multiplication Property of 1: For every number a, $a(1) = a$ and $(1)a = a$.
10. Multiplication Property of 0: For every number a, $a(0) = 0$ and $0(a) = 0$.

Although this chapter has dealt largely with the operations of multiplication and addition, you also are familiar with division and subtraction. We will discuss division and subtraction briefly here, but a more complete treatment will be encountered in a later chapter.

Given two numbers of arithmetic (different from 0) we may divide each by the other. If a and b are non-zero numbers of arithmetic, then

$$\frac{a}{b} \text{ and } \frac{b}{a} \text{ are both numbers of arithmetic.}$$

For example, $\frac{12}{4}$ and $\frac{4}{12}$ both name numbers of arithmetic. Notice that if $\frac{a}{b} = c$ is a true sentence, then $a = bc$ is also a true sentence. For example, $\frac{12}{4} = 3$ and $12 = 4 \times 3$ are both true sentences. We notice that division is closely related to multiplication.

In particular, we may "divide" the identity element for multiplication by any non-zero number of arithmetic. If a is any number of arithmetic, except zero, then $\frac{1}{a}$ is a number of arithmetic and

$$a \cdot \frac{1}{a} = 1.$$

Finally, you remember that if a is any number of arithmetic, except 0, then $\frac{0}{a} = 0$. Of course, we never divide by 0.

In the set of numbers of arithmetic, subtraction presents a somewhat different situation. If $a \neq b$, then one of the symbols "$b - a$" and "$a - b$" names exactly one number of arithmetic; the other symbol does not name a number of arithmetic. For example, "$12 - 4$" names a number of arithmetic, but "$4 - 12$" does not.

If $a < b$, then $b - a$ is a number of arithmetic. If this number is c, then $a + c = b$ is a true sentence. For example, $12 - 4 = 8$ and $4 + 8 = 12$ are both true sentences. We notice that subtraction is closely related to addition.

In particular, if a is any number of arithmetic other than 0, then "$0 - a$" does not name a number of arithmetic.

If the domain of x is the set of numbers of arithmetic, then the truth set of $4 + x = 0$ is $\emptyset$.

Eventually we shall extend our idea of "number" in such a way that "$4 - 12$" and "$0 - 4$" will both name "numbers" and, further, the truth set of $4 + x = 0$ will not be the empty set.

4-6. Review

This set of review problems is included to provide an opportunity for reviewing the mathematical ideas learned to date. For this section, and for review sections at the close of other chapters, no response sheets are provided. If more practice is desired with the distributive property, Problems 17-37 may be done. The answers to the problems in this section will be found on page vii.

1. (a) Write a description of the set H if

$$H = \{21, 23, 25, \ldots, 49\}.$$

   (b) Consider the set A of all whole numbers greater than 20. Is H a subset of A?

   (c) Classify sets H and A as finite or infinite.

2. (a) Find the coordinate of a point which lies on the number line between the two points with coordinates $\frac{3}{4}$ and $\frac{5}{6}$.

   (b) How many points are between these two?

3. Consider the set

$$T = \{0, 3, 6, 9, 12, \ldots\}.$$

   Is T closed

   (a) under the operation of addition?

   (b) under the operation of "averaging"?

4. Let the domain of the variable t be the set R of all numbers between 3 and 5, inclusive.

   (a) Draw the graph of the set R.

   (b) Decide whether each of the following numbers is in the domain of the variable t.

$$\frac{3}{2},\ \pi,\ \frac{17}{4},\ 4\frac{1}{3},\ \frac{11}{2},\ \sqrt{2}$$

5. In which of the following does the sentence have the same truth set as the sentence "$x \leq 5$"?

   (a) $x > 5$ or $x = 5$

   (b) $x < 5$ and $x = 5$

   (c) $x \not> 5$

   (d) $x \not< 5$

   (e) $x \neq 5$

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6. Find the truth set for each of the following sentences.

(a) $n - 5 = 7$
(b) $2n - 5 = 7$
(c) $3n - 5 = 7$
(d) $4n - 5 = 7$
(e) $6n - 5 = 7$
(f) $12n - 5 = 7$

7. If m is a number of arithmetic, find the truth set of

(a) $m + m = 2$
(b) $m + m = 2m$
(c) $m \geq 2m$
(d) $m + 3 < m$

8. If the domain of m is the set of counting numbers, find the truth set of the open sentences in (a) through (d) in Problem 7.

9. Let T be the truth set of

$$"x + 3 = 5 \quad \text{or} \quad x + 1 = 4".$$

(a) Is 3 an element of T?
(b) Is 2 an element of T?
(c) Is $\emptyset$ a subset of T?

10. If S is the truth set of

$$"x + 1 < 5 \quad \text{and} \quad x - 1 \geq 2"$$

draw the graph of S.

11. Consider the open sentence

$$2x \leq 1.$$

What is the truth set if the domain is the set of

(a) all counting numbers?
(b) all whole numbers?
(c) all numbers of arithmetic?

12. (a) Is the following sentence true?

$$\frac{17}{2}(8 + 1) = \frac{17}{2}(4 + 5)$$

(b) Do you have to perform any multiplication to answer part (a)? Explain.

13. Which of the following sentences are true?

 (a) $5(4 + 2) = (4 + 2)5$

 (b) $10(\frac{1}{2} + \frac{1}{3}) = \frac{21}{2} + \frac{21}{3}$

 (c) $10(\frac{1}{2} + \frac{1}{3}) = 5 + \frac{10}{3}$

 (d) $4 + 7 + 1.817 = 5 + 1.817 + 6$

 (e) $12 \times 8 + 12 \times 92 = 1200$

 (f) $(5 \times \frac{25}{4})16 = 500$

14. Explain how the property of 1 is used in performing the calculation

$$\frac{\frac{3}{5} + \frac{2}{3}}{\frac{3}{4}}$$

15. Explain why

$$3x + y + 2x + 3y = 5x + 4y$$

is true for all values of x and y.

16. Write these indicated products as indicated sums without parentheses.

 (a) $(x + 1)(x + 1)$

 (b) $(x + 2)(x + 2)$

In the remainder of the non-starred problems use the distributive property to state indicated products as indicated sums, and indicated sums as indicated products.

17. $(1 + x)5 =$ ______

18. $\frac{3}{4}a + \frac{2}{4}a =$ ______

19. $33 + 15(11) =$ ______

20. $c + 2c =$ ______

21. $(\frac{1}{2} + \frac{1}{3})y =$ ______

22. $x(y + 1) =$ ______

23. $xy + x =$ ______

24. $3ax + 2ay =$ ______

25. $(5 + y)y + 2y =$ ______

26. $b + 3ab =$ ______

27. $2a(a + b + c) =$ ______

28. $(u + 2v)(u + v) =$ ______

29. $(a + 1)^2 =$ ______

 Remember: $(a+1)^2 = (a+1)(a+1)$

Use the properties to write the following in simpler form:

30. $17x + x =$ ______

31. $2x + y + 3x + y =$ ______

32. $3(x + 1) + 2x + 7 =$ ______

33. $1.6a + .7 + .4a + .3b =$ ______

34. $by + 2by =$ ______

35. $9x + 3 + x + 2 + 11x =$ ______

*36. Here you are going to see how to test whether a whole number is exactly divisible by 9. Keep a record, as you go, of the properties of addition and multiplication which are used. Try the following:

$$\begin{aligned}
2357 &= 2(1000) + 3(100) + 5(10) + 7(1) \\
&= 2(999 + 1) + 3(99 + 1) + 5(9 + 1) + 7(1) \\
&= 2(999) + 2(1) + 3(99) + 3(1) + 5(9) + 5(1) + 7(1) \\
&= \big(2(999) + 3(99) + 5(9)\big) + \big(2(1) + 3(1) + 5(1) + 7(1)\big) \\
&= \big(2(111) + 3(11) + 5(1)\big)9 + (2 + 3 + 5 + 7) \\
&= (222 + 33 + 5)9 + (2 + 3 + 5 + 7)
\end{aligned}$$

Is 2357 divisible by 9? Try the same procedure with 35874. Can you formulate a general rule to tell when a whole number is divisible by 9?

*37. Look for the pattern in the following calculation:

$$\begin{aligned}
19 \times 13 &= 19(10 + 3) \\
&= 19(10) + 19(3) && \text{(what property?)} \\
&= 19(10) + (10 + 9)3 \\
&= 19(10) + \big(10(3) + 9(3)\big) && \text{(what property?)} \\
&= \big(19(10) + 10(3)\big) + 9(3) && \text{(what property?)} \\
&= (19 + 3)10 + 9(3) && \text{(what properties?)}
\end{aligned}$$

The final result indicates a method for "multiplying teens" (whole numbers from 11 through 19): Add to the first number the units digit of the second, and multiply by 10; then add to this the product of the units digits of the two numbers.

Use the method to find:

$15 \times 14 =$ ______

$13 \times 17 =$ ______

$11 \times 12 =$ ______

Chapter 5

OPEN SENTENCES AND ENGLISH SENTENCES

5-1. English and Mathematical Language

Problem: Henry and Charles were opposing candidates in a class election. Henry received 30 more votes than Charles, and 516 members of the class voted. Find the number of votes that Charles received.

	In this problem you know that Charles and Henry together	
1	received ______ votes.	516
2	You know that Henry ________ (won, lost), because he received	won
3	________ (how many) more votes than Charles.	30
4	You want to find the number of ________ that Charles received.	number of votes

One way to solve this problem is to use trial and error. We can simply guess until we obtain the correct result.

	For example, we might test whether 200 is the number of votes Charles received.	
	We would reason: If Charles had received 200 votes,	
5	then Henry would have received 200 + ______ votes.	200 + 30
6	But 200 + 230 = ______.	430
	This would mean only 430 votes were cast.	
	But we know that 516 votes were cast.	
7	We see that 200 ________ (is, is not) the number of votes Charles received.	is not

8 Did Charles receive exactly 300 votes?

[A] Yes

[B] No

If Charles had received 300 votes, then Henry would have received 330 votes. But this would mean that 630 votes were cast. 300 is not the number of votes Charles received. [B] is the correct choice.

9. Did Charles receive exactly 243 votes?

[A] Yes

[B] No

If Charles received 243 votes, Henry received 273 votes. Together, they received 516 votes, which is indeed the number of votes cast. [A] is correct.

Trial and error is a tedious way of solving problems like this. Do you notice that for each guess you are repeating the same pattern?

The pattern is: "Choose a whole number. Add to the number you chose the sum of the number and 30. Test whether the result is 516." Think back. Does this remind you of a game described in an earlier section?

In dealing with a somewhat similar pattern in Section 2-4 we found it helpful to represent the number chosen by a symbol n. We can now summarize the pattern we have followed in our problem about votes.

Choose a number $\longrightarrow$	n
Add to the number you chose the sum of the number and 30 $\longrightarrow$	$n + (n + 30)$
Compare your result with 516 $\longrightarrow$	Compare $n + (n + 30)$ with 516

Consider

$$n + (n + 30) = 516.$$

We have called such a mathematical sentence an

10 open ________. — open sentence

We have learned that an open sentence "sorts" the elements of the domain of the variable into two subsets--those elements which make the resulting

11 numerical sentence true, and those which make it _____. — false

12	The set of elements which make the resulting numerical sentence true is the ______ set.	truth

The pattern followed above leads us to conclude: The number of votes Charles received is an element of the truth set of the open sentence

$$n + (n + 30) = 516.$$

Suppose that you know that the truth set of this sentence is $\{243\}$. Then you can check that 243 is indeed the number of votes Charles received. You can be sure that 243 is the <u>only</u> possible answer to the given problem, because the truth set contains no other numbers.

13	This example illustrates a general procedure that can be used in solving "word problems" such as the one given. In such problems you are often looking for a certain <u>number</u> (here, the ______ of votes Charles received). There are three major steps:	number
14	(1) We find an open sentence such that the number we are looking for belongs to the ______ set of the sentence. (2) We find the truth set of the open sentence. (3) We test the elements of the truth set in the given problem.	truth

Step (1) is illustrated in the box before Item 10. Steps (1) and (3) really involve translating the problem into the instructions: Find the truth set of

$$n + (n + 30) = 516$$

and test each element of the truth set in the original problem.

It takes practice to become skillful at finding open sentences that fit word problems. In this chapter we are going to concentrate on this step--finding open sentences.

In later chapters we will give much attention to finding the truth sets of open sentences, that is, to Step (2).

Our aim in this chapter is to learn to translate word problems into instructions about finding truth sets of open sentences. It will help you to do this if you practice thinking about what problem situations might fit certain mathematical expressions.

In the example, we saw that if the variable n represents the number of votes Charles received, then
15 the phrase n ______ represents the
16 ______ of votes Henry received.

Answers: $n + 30$; number

How might a problem involve the phrase "$n + 5$"? In a problem, "n" might represent "the number of students in your class" or "the number of dollars in your allowance", or "the number of years in your age".

In the following table you are told what n represents in a problem. Give an appropriate meaning that $n + 5$ might have.

	n	$n + 5$	
17	a certain number	the ______ (sum, difference) of the number and 5.	sum
18 19	the number of students in your class	the ______ of students if 5 ______ (more, fewer) enrolled in the class.	number more
20 21 22	number of years in your age	the ______ of years in the age of your sister who is ______ years ______ (older, younger) than you.	number 5 older

You can see from these examples that the same open phrase may be related to many situations. The meaning you choose for a phrase depends on the interpretation you have chosen for the variable. However, remember that a value of a variable is <u>always</u> a number. Likewise, the corresponding value of an open phrase is a number.

23 Let us consider the open phrase $7w$. Which of the following meanings might w and $7w$ have in a problem?

[A] w is the water in the well and $7w$ is seven times the water in the well.

[B] w is the number of wolves in a zoo, and $7w$ is the number of wolves in seven zoos.

[C] w is the number of dollars I paid for one bushel of wheat and $7w$ is the number of dollars I paid for 7 bushels of wheat.

[A] Don't you mean the <u>number</u> of gallons of water or the number of pounds of water? Remember that the variable must always represent a number. Go back to Item 23 and read the choices again.

[B] This is a possible choice, but would be a correct one only if you knew that each zoo had the same number of wolves. Is there a better selection?

[C] You are correct. The interpretation about the wolves in a zoo would also be correct, provided you know that each of the seven zoos has the same number of wolves. It is more reasonable to expect each of several bushels of wheat to cost the same amount than to expect each of several zoos to contain the same number of wolves.

The interpretation you use for the variable, and hence for the open phrase, should be sensible. You are not likely to find a situation in which y represents the number of pencils on your desk and "$2,500,000 + y$" is a phrase related to a reasonable problem. On the other hand, the phrase "$2,500,000 + y$" might occur in connection with a problem about populations.

24 Suppose there were 2,500,000 persons in the city of Detroit in 1950. Then the open phrase "$2,500,000 + y$" could represent:

[A] the number of persons living in Detroit in 1960 if y is the number by which the population increased in 10 years.

[B] the population of Detroit y years after 1950.

[C] the increase in population if the population is y more than 2,500,000.

[A] You are correct.

[B] You have not been careful in translating the open phrase. If y is the number of years after 1950, what would we say about the population in $1950 + y$? We don't have enough information to write an open phrase for the population. [A] is the correct choice.

[C] The "increase in population" suggests that something is added to 2,500,000, but the open phrase would not indicate the increase. [A] is the correct choice.

25 Choose the sensible meaning for the variable in the open phrase, $.05 + k$.

R. k is the number of cows in the barn.
S. The rate of interest is increased by k per cent.
T. k is the number of dollars in my purse.

[A] R only [B] S only [C] S and T

S is sensible, and T is also good. I might have k dollars and 5¢ in my purse so that the open phrase $.05 + k$ represents the number of dollars in my purse. Thus, [C] is the correct choice. Don't you agree that we would not usually add the number of cows and the number $.05$?

Before we consider other open phrases we will review some of the ideas that we will need as we go along.

26 In the open phrase "$3x + 25$" the symbol "+" indicates the operation of _______. — addition

27 The symbol "+" in the open phrase $x + 5$ indicates that we are forming the _________ (sum, difference) of the two numbers, x and 5. — sum

28 The open phrase $2w + 3$ might be read: "3 _______ (more, less) than $2w$". — more

29 The phrase $2W + 3$ may also be read: "$2w$ increased by _______". — 3

30	The phrase $6t + 7$ may be read: "6t ______ by 7".	increased
31	The symbol " - " indicates that the open phrase $x - 5$ might be translated: "x ______ by 5". (increased, decreased)	decreased
32	Other translations of the phrase $x - 5$ might be: "5 ______ than x" or "5 is subtracted from (more, less)	less
33	______".	x

Note: We must be sure in each case that our variable represents a number. We are perfectly correct in saying: "John is n years old," or "I have t dollars." In each case, if we substitute a number for the variable, we have a sensible statement. On the other hand, we must avoid saying: "d is dimes" since that does not indicate whether d is the number of dimes or the number of cents in the value of the dimes.

In the following table you are given open phrases and you are to complete the table as suggested by the first example. There are many possibilities in each case. Choose a sensible one!

Open Phrase	Meaning of Variable	English Phrase
$3x + 25$	x is the number of cents Tom earns in one hour.	The number of cents Tom earned in three hours if he gets a bonus of 25 cents.
34. $n + 7$	n is John's age now, in years.	
35. $n - 7$		Joan's weight in pounds after she lost seven pounds.
36. $\frac{y}{2}$		Larry's wages in dollars for one week.
37. $2r + 5$	r is the cost in cents of one bunch of rhubarb.	
38. $a + b$	a is my age in years and b is my sister's age in years.	

After completing your table, turn to page xi and read the completions suggested there. Although the answers you gave may be different, they should represent numbers. The ideas expressed should be similar.

Of course, the indicated interpretations are just suggestions. It would be fun for you to write others for the given open phrases. Therefore, in the space below we have written only the open phrases--you are to find for each a good English interpretation. Be sure to specify the meaning of the variable first, so that it represents a number, and then to find a corresponding meaning for the open phrase. You must be very alert, since we shall not record these answers and you will have to find your own errors. Keep your results. We shall use them again in the next section.

Open Phrase	Interpretation of Variable	English Phrase
$3x + 25$		
$n + 7$		
$n - 7$		
$\frac{y}{2}$		
$2r + 5$		
$a + b$		

Go on to Section 5-2 when you have looked over your work and feel it is correctly done. If this program is being used under a teacher's supervision, have your teacher read your interpretations.

5-2. English Phrases and Open Phrases

In the section which you have just completed, you practiced finding interpretations for open mathematical phrases. In this section you will learn to translate English phrases into mathematical language. Once you have learned to do this with ease, you will be able to solve "word problems".

Suppose you were asked to write an open phrase indicating the number of feet in y yards. Recall that there are 3 feet in 1 yard. Hence

$3y$

would be an open phrase for the number of feet in y yards.

Write an open phrase for each of the following English phrases.

1 The number of pints in k quarts is _______. $2k$

2 The number of inches in f feet is _______. $12f$

3 The successor of the whole number n is _______. $n + 1$
(For example, 5 is the successor of 4.)

4 If n is an even number the next even number is _______. $n + 2$

5 In one pound there are _______ ounces. 16

6 An open phrase for the number of ounces in k pounds and t ounces is _______. $16k + t$

7 In one quarter there are _______ cents, and in one 25
8 dollar there are _______ cents. Then an open 100
phrase representing the total number of cents in
9 d dollars and k quarters is _______. $100d + 25k$

An open phrase for the number of cents in d dollars, k quarters, m dimes and n nickels
10 is _______. $100d+25k+10m+5n$

An open phrase for the number of cents in t
11 dimes and t nickels is _______. $10t + 5t$

An open phrase for the <u>number of coins</u> in a stack of coins containing d dimes and n nickels
12 is _______. $d + n$

An open phrase for the <u>number of cents</u> in d dimes
13 and n nickels is _______. $10d + 5n$

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14	Jack has only dimes and quarters. If m represents the number of quarters he has, and if n represents the number of dimes he has, then $m + n$ represents: [A] the number of cents Jack has. [B] the number of coins Jack has.

The number of coins Jack has is $m + n$. Hence [B] is correct. Notice that the value of Jack's coins, in cents, is $25m + 10n$.

In the above examples you were told the number represented by each variable. You probably noticed that, having chosen an interpretation for certain variables, you represented other numbers by phrases involving these variables. In order to do this it was necessary to look carefully at the relationships expressed in the English phrases.

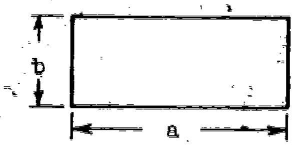
In many problems, you will need to use only a single variable. It will be necessary for you to choose the number you wish to represent by a variable. (There are often two or more possibilities.) You will usually want to express other numbers in the problem as phrases involving the single variable you have chosen.

	Write an open phrase for the number of years in John's age if John is seven years older than Dick.	
15	The number of years in John's age is expressed in terms of the number of years in ______ age.	Dick's
16	The relationsip "older than" can be expressed by ______________. (addition, multiplication)	addition
	We can choose the variable n to represent the number of years in Dick's age.	
17	Then the number of years in John's age is represented by the phrase ______.	$n + 7$

Many problems involve geometric figures. For such problems it helps to draw a diagram.

Suppose a represents the number of units in the length of a rectangle and b represents the number of units in the width.

18 The number of square units in the area of this rectangle is ______. — ab



19 The number of units in the perimeter of the rectangle is ______. — $2a + 2b$, or $2(a + b)$

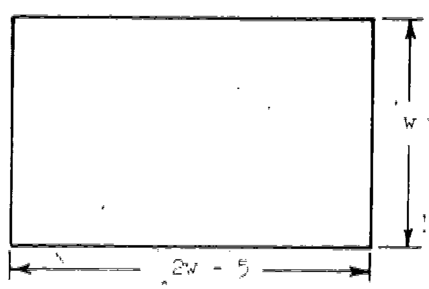
The length of a rectangle is 5 feet less than twice the width. Write open phrases for the number of feet in the perimeter and the number of square feet in the area.

20 Since the length is given in terms of the ______, — width

21 we might choose w as the number of feet in the ______. — width

22 Then an open phrase for the number of feet in the length, expressed in terms of the variable "w", would be ______. — $2w - 5$

The rectangle might be represented by the following figure on which the length and width are expressed in terms of the variable w.



23 An open phrase for the perimeter of the rectangle is ______. — $2(2w - 5) + 2w$, or $2((2w-5)+w)$

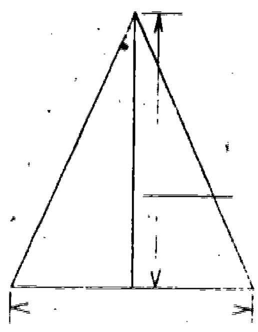
24 An open phrase for the area of the rectangle is ______. — $(w)(2w - 5)$

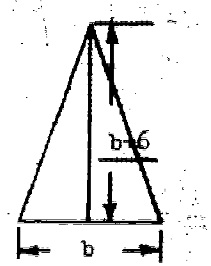
The number of feet in the height of a triangle is six more than the number of feet in the base. If b is the variable representing the number of feet in the base then _______ is an open phrase for the number of feet in the height.

25 $b + 6$

Indicate on the following figure how this might be shown.

26





How shall we write an open phrase for the part of the house a man can paint in one day, if he can paint the entire house in d days?

Suppose that he painted a house in 8 days. How much of the house did he paint in one day? _______

27 $\frac{1}{8}$

Similarly, if he can paint the house in d days, an open phrase for the part of the house he can paint in one day is _______.

28 $\frac{1}{d}$

29-34. In Section 5-1 you were asked to write English interpretations of open phrases and to keep these interpretations. Now take your English phrases, cover the mathematical phrases, and look for open phrases to fit. Write these open phrases on your response sheet for Items 29-34. Now uncover the open phrases and check to see if they are the same as the open phrases which you have just written.

If they are not, you have not translated correctly in both directions. Review from the beginning of this chapter, and then continue with Item 35. If you have written the open phrases correctly, go to Item 35.

If a pipe fills $\frac{1}{5}$ of a swimming pool in one hour,

35 the part filled in 3 hours would be ______. — $3(\frac{1}{5})$, or $\frac{3}{5}$

36 The part filled in x hours would be ______. — $\frac{x}{5}$, or $(\frac{1}{5})x$

The width of a rectangle is 4 inches less than the

37 length. Let the variable y represent the ______ of inches in the length of the rectangle. — number

An open phrase for the number of inches in the

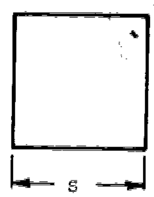
38 width is ______. — $y - 4$

In many problems you need facts you have learned in earlier mathematics courses. In working with such problems, it helps to think first about the relationships that you know apply. Remember, drawing a figure is often helpful.

For example, in a problem about a square you might need to remember:

The number of square units in the area of this square

39 is ______. (s is the number of units in the side.) — s^2



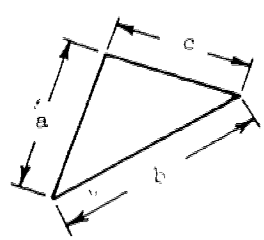
The number of units in the perimeter of the square

40 is ______. — $4s$

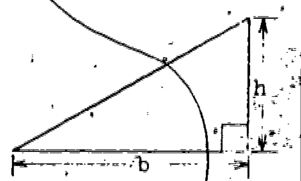
In a problem about a triangle you may need to remember:

The number of units in the perimeter of this triangle

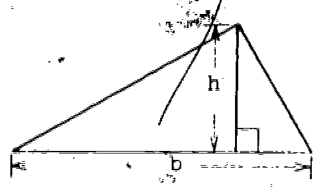
41 is ______. (a, b, c are the numbers of units in the respective sides. — $a + b + c$



42 The number of square units in the area of this triangle is ______. (b is the number of units in the base and h the number of units in the altitude.) $\frac{1}{2}bh$



43 Similarly, the number of square units in the area of this triangle is ______. $\frac{1}{2}bh$



We should also look at some basic formulas which are frequently used in everyday problems.

44 The distance d in feet traveled by an object which moves at the rate of r feet per hour for t hours is given by the sentence $d =$ ______. $d = rt$

If d is a distance in <u>miles</u>, r is the rate in <u>miles</u> per hour, and t is the number of hours traveled at this rate, the sentence is again

45 ______. $d = rt$

The simple interest i earned in t years by p

46 dollars at r per year can be stated $i =$ ______. $i = prt$

The total amount A after t years is written

47 ______. $A = p + prt$

For each of the following write an English interpretation of the given open phrase. In each case, identify the variable explicitly. Some possible interpretations for these are given on page xi.

48. $n + 7n$

49. $(2r - 5) + 7$

50. $3x + (2x + 1) + x$

51. $5000 + 4y$

52. $y + .04y$

*53. $x(x + 3)$ (Hint: This phrase might be interpreted as the expression for the number of square units in an area.)

*54. $\frac{1}{2}y(y + 5)$

In the following items, identify the variable explicitly and write the open phrase requested. Check your phrases with those on page xi

55. Write a phrase for the number of inches in the length of a second side of a triangle, if it is three inches longer than the first side.

56. The admission price to a performance of "The Mikado" is $2.00 per person. Write a phrase for the total number of dollars received in terms of the number of people who bought tickets.

57. When a tree grows it increases its radius each year by adding a ring of new wood. If a tree has r rings now, write an open phrase for the number of growth rings in the tree twelve years from now.

58. Three sons share in an inheritance. Let x represent the number of dollars in the inheritance.

(a) Write an open phrase for the number of dollars in one son's share which is one-half of the inheritance.

(b) Write an open phrase for the number of dollars of the second son's share, which is fifty dollars more than one-tenth of the inheritance.

(c) Write an open phrase for the third son's share.

(d) Write an open phrase for the sum of the three sons' shares.

Remember to draw a figure for this geometric problem:

59. One side of a triangle is x inches long and a second side is y inches long. The length of the third side is one-half the sum of the lengths of the first two sides.

(a) Write an open phrase for the number of inches in the length of the third side.

(b) Write an open phrase for the number of inches in the perimeter of the triangle.

60. A plant grows a certain number of inches per week. It is now 20 inches tall. Write an open phrase giving the number of inches in its height five weeks from now.

*61. Suppose that when a man immerses his arms in hot water, the temperature of his feet will rise one degree per minute, beginning at 10 minutes after his arms are put in the water. Write an open phrase for the rise in temperature of the man's feet at any time (more than ten minutes) after his arms are immersed.

5-3. Interpreting Open Sentences.

In many "word problems" you are concerned with finding a quantity by the use of certain given conditions. The conditions are stated in English sentences. As we have seen, one step in solving a word problem is to find an open sentence which has in its truth set the number (or numbers) we wish to find. We have already seen how an open phrase can have many interpretations. Now let us examine how we can relate open sentences to problems.

We first review the meanings of some of the verb forms used in mathematics.

1	"_____" means: "is" or "is equal to".	"="
2	"≠" means: "is _____" or "is _____ _____ _____".	is not, is not equal to
3	"_____" means: "is less than".	"<"
4	">" means: "_____ _____ _____".	is greater than
5	"≤" means: "is _____ than or is _____".	is less than or is equal to
6	"_____" means: "is greater than or is equal to".	"≥"

Consider the open sentence

$$2x = 500,000 .$$

We could translate this as "two times x is five hundred thousand". But in using mathematics we often arrive at open sentences like

$$2x = 500,000,$$

from more concrete situations. For example, we might know: Twice the number of soldiers in the army is 500,000. Then if x represents the number of soldiers in the army, we have:

$$2x = 500,000 .$$

One step in finding the number of soldiers in the army is to write the open sentence $2x = 500,000$. (The second step is to find the truth set, which in this case is easy. However, we are not interested here in actually finding truth sets.)

In the following table we have specified an interpretation either for the variable or for the open sentence. Complete the table. Then turn

to page xii to see some suggested completions. (Of course yours may be different.)

Open Sentence	Interpretation of Variable	Situation which would lead to open sentence
7. $3x - 2 = 784$	x is	The number of books in the library is tripled. Two books are lost. There are 784 books left.
8. $n + 7 = 82$	Jonathan weighs n pounds.	
9. $\frac{c}{2} = 17$		A class that has the same number of boys as girls has 17 boys.

Consider the open sentence $45 + 3x = 108$. If x is the number of sets of books, this sentence would be suggested by the following situation: "A book salesman is paid \$45 a week, plus \$3 for each set of books he sells. In one week he was paid \$108."

Another situation, if x is the number of small cartons, might be: "A freight shipment consisted of a box weighing 45 pounds and a number of small cartons weighing 3 pounds each. The whole shipment weighed 108 pounds."

10 Which of the following is not a situation that would lead to the open sentence $45 + 3x = 108$?

[A] Mary bought 3 bunches of carrots and a 45-cent box of berries. Her bill came to \$1.08.

[B] Jane had a piece of cloth 108 inches long. She used 45 inches for a tablecloth, and divided the rest into three equal lengths for place mats.

[C] John worked the same number of hours each day for three days. He had previously worked 45 hours on the same job. His total pay was \$108.

You should have selected [C], which does not lead to the open sentence. The open phrase "$45 + 3x$" concerns the number of hours John worked, but \$108 is the total pay.

Let us consider the open sentence

$$t(t + 2) = 425.$$

An appropriate situation is the following:

t	$t + 2$	$t(t + 2) = 425$
number of feet in the width of a rectangle.	the number of feet in the length of a rectangle with length 2 feet more than the width.	A rectangle whose length is 2 feet more than its width has an area of 425 square feet.

Below is a set of open sentences. Write your own English interpretations for each of these. Then turn to page xii for other suggestions.

11. $2n = 500$
12. $a + (2a + 3a) = (a + 2a) + 3a$
13. $5n + 8n = 65$
14. $4x = 100$
15. $y + 5 = 5 + y$
16. $\frac{1}{4}a + \frac{1}{2}b = 6$

We have also worked with sentences such as

$$x + 30 < 40.$$

We could read this as "x plus 30 is less than 40". Again let us consider where we might have obtained this sentence. Put in the format used before we have

Variable	Open Sentence
x	$x + 30 < 40$
number of dollars in my wallet	If you give me 30 dollars, I shall still have less than 40 dollars in my wallet.

Variable	Open Sentence
x	$x + 30 < 40$
number of points made by a football team during the first half.	If the team makes 30 points in the last half, they will still have less than 40 points.

No matter what interpretation [illegible] another number.

17	Below are two proposed interpretations of the open sentence p + 10,000 [illegible]. Which is a possible interpretation? [A] A star is [illegible] light years farther from the earth than a second star. The sum of their distances from the earth is greater than [illegible] light years. p is the distance of the second star from the earth. [B] 10,000 people moved into a town of population p, and the total population is greater than [illegible].

[A] is not correct. If the variable p represents the distance of the second star from the earth, then the first star is p + 10,000 light years from the earth. But then the total distance would be (p + 10,000) + p. The open phrase (p + 10,000) + p does not appear in the original open sentence. [B] is the correct choice.

	Variable	Open Sentence	
	Consider the open sentence [illegible]		
18	p is the ______ of	In my pocket, I have [illegible]	number
19	quarters in my pocket.	______ than $ [illegible]. (more, less)	more

5-4. Writing Open Sentences

By now you should be ready to write open sentences for some problems.

A board 44 inches long is to be cut into two pieces in such a way that one piece will be three inches longer than the other. How long should the shorter piece be?	
Since one piece is to be 3 inches longer than the other, we might let k represent the number of inches in the length of the shorter piece.	
The number of inches in the length of the longer piece is given by the phrase ______.	$k + 3$
A phrase for the sum of the two lengths is ______ + (___ + ___).	$k + (k + 3)$
Since the board is 44 inches long, the sum of the two lengths must be ______ inches.	44
We thus arrive at the open sentence ______ = 44.	$k + (k+3) =$ [illegible]

Here is what you should notice in the above example: We have gone from an English sentence about a board to an open sentence about numbers.

English sentence:

$$\underbrace{\text{length of shorter piece}}_{k}\ \text{plus}\ \underbrace{\text{length of longer piece}}_{(k + 3)}\ \text{is}\ 44\ \text{inches.}$$

Open sentence: $k + (k + 3) = 44$

Since at this time we are not concerned with finding the truth set of the open sentence, we shall not actually answer the question, "How long should the shorter piece be?" It should be clear to you, however, that there probably is a value of k for which the sentence is true.

A man left $[illegible] for his widow, a son and a daughter. The widow received $[illegible],000 and the daughter received twice as much as the son. How much did the son receive?
We know: "Son's share plus daughter's share plus $[illegible] is $[illegible]."

5 If we let the variable n represent the son's inheritance in dollars, then the daughter received ______ dollars. 2n

A phrase for the total amount of money is n + ______ + 5000. n + 2n + 5000

Recalling that the total amount of money was $10,500, we write the open sentence ______. n+2n+5000=10500

This is an appropriate result. However, we might have proceeded in a different way.

Let's suppose that we had chosen to let the variable x represent the daughter's inheritance.

Then the son received ______ dollars. $\frac{1}{2}x$

A phrase for the entire inheritance is ______. $\frac{1}{2}x + x + 5000$

10 An open sentence for the problem would then be ______. $\frac{1}{2}x+x+5000=10500$

Each of these open sentences is found to fit the word problem. Notice that the sentence we obtain depends upon the meaning of the variable. If we were to find the truth set of each sentence and complete the problem, we would arrive at the same answer for the son's inheritance. In other words, there would be a single answer to the question in the problem, regardless of the method used. In this case, since the question asked is, "How much did the son receive?", we would probably prefer the first open sentence, in which the variable represents the amount received by the son.

A bridge 2500 feet long has three spans, one of which is 100 feet longer than either of the other two, which are of equal length. How long is each shorter span?

11 The question asked concerns the ______ of each shorter span. length

A stated [illegible] is that the longest span is ______ feet longer than either of the other two. 100

An English sentence suggested by the problem is "The sum of the lengths of the spans is ______ feet." 2500

If the number of feet in the length of one of the shorter spans is f, then the number of feet in the length of the longest span is ______. f + 100

14

15	An open phrase for the total length of the bridge is (______) + 2f.	$(f + 100) + 2f$
16	An open sentence which results from the fact that the bridge is 2500 feet long is ______.	$(f+100)+2f=2500$

In some problems you will find that in addition to stated facts and relationships, some relationships are merely implied. The nature of the problem determines the implied relationship. For example, if you and I set out on a trip from the same place but at different times, and one eventually overtakes the other, the implied relationship is that we have traveled the same distance. After we have determined from a first reading what question the problem asks, we reread very carefully to discover all the relationships, stated and implied.

	Consider this problem: You start out on a trip at 40 miles per hour. One hour later I follow from the same place at 50 miles per hour. In how many hours will I overtake you?	
17	The question asks us to find the number of ______ (hours, miles).	hours
18	I started one hour later than you. Hence you traveled one hour ______ (more, less) than I traveled.	more
19	The word "distance" is not mentioned in the problem. However, if we both start at the same place and I overtake you, what is implied is that we both travel the same ______. The distance I traveled equals the distance you traveled.	distance
20	If x is the number of hours that I traveled, then the number of hours that you traveled is represented by the open phrase ______.	$x + 1$
21	My rate is ______ mph (miles per hour) and your rate is ______ mph.	50
22	We can use the formula d ______. Then the distance that I traveled can be represented by the open phrase ______.	$d = rt$ $50x$

	The distance that you traveled can be represented by	
23	the open phrase ______.	$40(x + 1)$
	Since the number of miles that each of us traveled	
24	is the same, we can write the open sentence ______.	$50x = 40(x + 1)$
	A set of 43 students was separated into two classes. One class was assigned to Mr. Smith and the other to Miss Jones. If Mr. Smith had 5 more students than Miss Jones, how many students did Miss Jones have?	
	A stated relationship is that the number of students	
25	in Mr. Smith's class was ______ than the (more, less)	more
	number in Miss Jones' class.	
	Using this relationship, if Miss Jones had y students, we can express the number Mr. Smith had	
26	as ______.	$y + 5$
	The English sentence implied is that the number of	
27	students in the two classes was ______.	43
28	From this, we get the open sentence ______.	$y + (y+5) = 43$
	Let us consider a different approach to the problem about the number of students assigned to Miss Jones and Mr. Smith. Suppose we decide to write an open phrase using the fact that there were in all 43 students.	
	If Miss Jones had y students, the number Mr. Smith	
29	had can be expressed by the open phrase ______.	$43 - y$
	Then the English sentence from which we get an open sentence is:	
30	"Mr. Smith had ______ more students than Miss Jones."	5
	An open sentence following from this relationship	
31	would be ______ $y + 5$.	$43 - y = y + 5$
	Notice that these are not the only possible open sentences for this problem. For example, we might have chosen to represent the number of students	

165

assigned to Mr. Smith by n. Then we could use the relationship expressed by the sentence, "Mr. Smith had 5 more students than Miss Jones", to represent the number in Miss Jones' class by the phrase ______. $n - 5$

In this case, the fact that there were 43 students in all would lead us to the open sentence

$n + ($ ______ $)$ ______. $n + (n-5) = 43$

"Jake has $1.[illegible] in his pocket, all in nickels, dimes, and quarters. He has one more quarter than he has dimes, and the number of nickels he has is one more than twice the number of dimes. How many dimes has he?"

Let d represent the number of dimes Jake has in his pocket. Then an appropriate open sentence is

______. $10d+25(d+1) + 5(2d+1) = 165$

Your result might have had the terms $10d$, $25(d + 1)$, and $5(2d + 1)$ in some other order. If your result was incorrect, or if you had trouble, complete Items [illegible] to [illegible]. If not, go on to Item [illegible].

If the variable d represents the number of dimes in his pocket, then he has ______ quarters. $d + 1$

We can represent the number of nickels by the phrase

______. $2d + 1$

[illegible] which are implied by the nature of this problem are the values in cents of dimes, nickels, and quarters. ?

The value in cents of the dimes in his pocket is given by the open phrase ______. $10d$

The value in cents of the quarters in his pocket is

[illegible] ______ $)$. $25(d + 1)$

The value in cents of the $(2d + 1)$ nickels is ______. $5(2d + 1)$

The sum of the values in cents of the dimes, quarters, and nickels is ______ cents. 165

Thus we obtain the open sentence ______. $10d+25(d+1) + 5(2d+1) = 165$

160

42 For the following problem, which is an appropriate open sentence?

> I bought 23 postage stamps, some of them 5-cent stamps and some of them 8-cent stamps. If the total cost was $1.18, how many of each kind did I buy?

[A] x is the number of 5-cent stamps. $5x + 8(23 - x) = 1.18$.

[B] x is the number of 5-cent stamps. $5x + 8(23 - x) = 118$.

You should note that if x is the number of 5-cent stamps, $5x$ is the number of <u>cents</u> spent for the 5-cent stamps. Likewise $8(23 - x)$ is the number of <u>cents</u> spent for 23-cent stamps. Hence [A] is <u>not</u> correct, since 1.18 is the number of <u>dollars</u> spent. The number of <u>cents</u> spent is 118, so [B] is correct.

If you have difficulty in finding an appropriate open sentence for a "word problem", it may be helpful to guess an answer to the question asked. (This was our method in the first example in this chapter.)

Two cars start from the same point at the same time and travel in the same direction at constant speeds of 34 and 45 miles per hour, respectively. In how many hours will they be 35 miles apart?

If both cars travel for 4 hours, the faster car
43 goes 45(______) miles and the slower car — 45(4)
44 goes 34(______) miles. — 34(4)

By the conditions of the problem, the faster car
45 has gone ______ miles farther from the starting point than the slower car. — 35

We thus consider the sentence
46 45(4) = __________ 35. — $45(4) - 34(4) = 35$

47 This sentence is __________ (true, false). — false

However, the sentence

$$45(4) - 34(4) = 35$$

suggests a pattern for the open sentence we are seeking.

48	If each car travels for k hours, then an appropriate open sentence would be ________.	$45k - 34k = 35$
49	The number of hours they actually drove is an element of the ______ set of this sentence.	truth

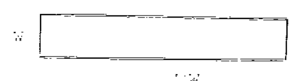
Now try your hand at writing open sentences. In each problem state what the variable will represent. Then write an open sentence, using this variable, which fits the problem. When you have finished, check your result with the one given below the problem. It is possible that your answer, while different from the one given, may be correct. If you are not completely sure that you are correct, or if you had difficulty, complete the items in the box following the problem. If you are sure, go on to the next problem.

50 A rectangle is 6 times as long as it is wide. Its perimeter is 144 inches. How wide is the rectangle?

Let w be the number of inches in the width. Open sentence: $2(w) + 2(6w) = 144$, or

$2(w + 6w) = 144$.

51	If w is the width in inches of the rectangle, then ______ is the length in inches.	$6w$



52	Since the perimeter of a rectangle is the ______ of the lengths of the four sides, an open phrase	sum
53	for the perimeter is $2(w) + 2(\_\_\_\_)$.	$2(w) + 2(6w)$
54	The perimeter is ______.	144 inches
	We can write the open sentence	
55	$2(\_\_\_) + 2(\_\_\_)$ ______	$2(w)+2(6w) = 144$

56 The measure of the largest angle of a triangle is 20° more than twice the measure of the smallest angle, and the measure of the third angle is 70°. What is the measure of the smallest angle? (Remember, the sum of the measures of the angles of a triangle is always 180°.)

Let s be the number of degrees in the measure of the smallest angle. Open sentence:

$$s + (2s + 20) + 70 = 180.$$

57	If s is the measure of the smallest angle, the measure of the largest angle is ______.	2s + 20
58	The sum of the measures of the three angles is ___ + (___) + ___.	s + (2s + 20) + 70
59	The sum of the measures of the three angles of any triangle is ______.	180
60	The open sentence is: s + (______) + 70 = ______.	s+(2s+20)+70=180

61 The length of a rectangle is 5 inches more than its width. What is the length of the rectangle if its area is 594 square inches?

Let w be the number of inches in the width.

Open sentence: $w(w + 5) = 594$;

or

Let a be the number of inches in the length.

Open sentence: $a(a - 5) = 594$.

62	If w is the number of inches in the width, the number of inches in the length is ______.	w + 5
63	The area of a rectangle is the ______ of its length and its width.	product
64	The area of this particular rectangle is 594 square inches. Hence the open sentence is w(______) = ______.	w(w + 5) = 594

65 A passenger train travels 20 miles per hour faster than a freight train. At the end of 5 hours the passenger train has traveled 100 miles farther than the freight train. How fast does the freight train travel?

Let r be the rate, in miles per hour, at which the freight train travels.

Open sentence: $5(r + 20) = 5r + 100$ or

$$5(r + 20) - 5r = 100.$$

66	If the number of miles per hour the freight train travels is r, then the rate in miles per hour of the passenger train can be represented by the open phrase ______.	$r + 20$
67	At the end of 5 hours, the distance the passenger train has gone is ______ miles.	$5(r + 20)$
68	The freight train has gone ______ miles in the 5 hours.	$5r$
69	The distance the passenger train has gone equals the distance the freight train has gone plus ______ miles.	100
70	The open sentence is: $5(r + 20) =$ ______.	$5(r+20)=5r+100$

Word problems about inequalities may also lead to open sentences:

	In six months, Mr. Adams earned more than \$7000. How much did he earn per month?	
71	If d represents the number of dollars that Mr. Adams earns per month, then he earns ______ dollars in six months.	$6d$
72	An open sentence for this problem is ______.	$6d > 7000$, or $7000 < 6d$

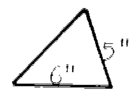
	A boat traveling downstream goes 12 miles per hour faster than the rate of the current. The boat's velocity downstream is less than 30 miles per hour. What is the rate of the current?	
73	When the rate of the current is c miles per hour, the boat travels downstream ______ miles per hour.	$c + 12$
74	An open sentence for this problem is ______.	$c + 12 < 30$

75	A teacher said, "If I had three times as many students in my class as I now have, I would have at least 26 more than I now have." He found a correct open sentence, ______, written on the blackboard beside the letter "s" is the number of students you have in your class."	$3s \geq s + 26$

Word problems sometimes require compound open sentences.

Two sides of a triangle have lengths of 5 inches and 6 inches. What is the length of the third side?

Recall that the length of any side of a triangle must be less than the sum of the lengths of the other two sides.



If n represents the number of inches in the length of the third side, this condition is expressed by the open

76 sentence $n <$ _____. — $n < 5 + 6$

At the same time, the length of the six-inch side must be less than the sum of the lengths of the other two sides.

77 Expressed as an open sentence this becomes $6 <$ _____. — $6 < n + 5$

Since both of these conditions must hold, a compound

78 English sentence with the connective _____ (and, or) is implied. — and

An open sentence which fits the problem is:

79 _____ and _____. — $n < 5 + 6$ and $6 < n + 5$

Note that there is a third condition; namely,

$$5 < n + 6.$$

But this is true for all positive values of n. Thus, $5 < n + 6$ adds nothing to the information about the sides of the triangle.

We have said previously that, unless otherwise specified, the domain of the variable will be the set of <u>numbers of arithmetic</u>. However, in statements which refer to people, coins, tickets, and the like, it is clear that the domain of the variable is the set of <u>whole numbers</u>. We cannot have two-thirds of a person, one-half of a coin, or .07 tickets. You will find that common sense will help indicate the domain for a given problem.

	If y is the number of boys who are now members of a club, the sentence "$y + 3 < 10$ and $y + 3 > 5$" can be used to state the relationship, "If three more boys join the club, we shall have more than 5 and	
80	fewer than ______ members."	10
81	The domain of the variable y is the set of ______ ______.	counting numbers

82 For which of the following word problems is the open sentence <u>not</u> adequate?

[A] The amount of $205 is to be divided among Tom, Dick and Harry. Dick is to have $15 more than Harry, and Tom is to have twice as much as Dick. How must the money be divided?

If Harry has n dollars, then Dick has $(n + 15)$ dollars and Tom has $2(n + 15)$ dollars. The open sentence is $n + (n + 15) + 2(n + 15) = 205$.

[B] John said, "It will take me more than 2 hours to mow the lawn and I must not spend more than 4 hours on the job or I won't be able to go swimming." How much time can he expect to spend on the job?

If John can spend x hours on the job, an open sentence is $x > 2$ and $x \leq 4$.

[C] Bill is 5 years older than Norman, and the sum of their ages is less than 23. How old is Norman?

If n is Norman's age, then an open sentence is $(n + 5) < 23$.

You should have selected [C] as the problem for which the suggested open sentence was not correct. An open sentence for the problem in [C] might be $n + (n + 5) < 23$.

Write an open sentence for each of these problems. Then turn to page xiii to check what you have written. Remember that any letter may be used as the variable. You may find it helpful to draw figures for some of the problems.

83. One-third of a number added to three-fourths of the same number is equal to or greater than 26. What is the number?

84. A square and an equilateral triangle have equal perimeters. A side of the triangle is five inches longer than a side of the square. What is the length of the side of the square? (Draw a figure.)

85. A student has test grades of 75 and 82. What must he score on a third test to have an average of 88 or higher? If 100 is the highest score possible on th third test, how high an average can he achieve? What is the lowest average he can achieve?

5-5. Review

	In some problems we have made such statements as "the width is x feet." This is to be understood to	
1	mean "________ is the number of feet in the width."	x
	The value that we assign to a variable is always	
2	a ______.	number
	A variable may be used to represent the number of feet, pounds, people, coins, etc.	
	Phrases such as $x + 3$, $5t + 10$, etc., also repre-	
3	sent ______.	numbers
	A boat starts on a trip of 60 miles. If it travels 10 miles per hour for t hours, then $60 - 10t$ is	
4	the phrase which represents the ______ of miles remaining to be traveled.	number
	The base of a triangle is b inches in length. The altitude to the same base has 4 less than 3 times the number of inches in the base. Then $\frac{1}{2}b(3b - 4)$ is the phrase representing the number of square	
5	______ in the ______ of the triangle.	inches, area

177

6	If Jeff earns \$15 this week and deposits it with the x dollars he now has in the bank, the phrase $x + 15$ will represent the total _____ of dollars in his account.	number
7	If one side of a rectangle is s inches long and the other side is three inches longer, then a phrase for the area of the rectangle in square inches is _____ .	$s(s + 3)$
8	$3p + 2(p - 1)$ is a phrase expressing the same id[ea] as "the total amount paid for three pencils costi[ng] _____ cents each".	1
9	If four dollars more than twice Betty's allowance is represented by the phrase $2x + 4$, then the number of dollars in Betty's allowance is _____.	x
10	If s is the number of years in Sam's age now, "Sam's age seven years from now" may be expressed by the phrase _____.	$s + 7$
11	If the number of years in Mary's age now is m, ten years ago she was _____ years old.	$m - 10$
12	If at 8 a.m. the temperature is t degrees and at noon it is 20 degrees warmer, the temperature at noon is _____ degrees.	$t + 20$
13	The cost in cents of n pencils at 5 cents each may be written _____.	$5n$
14	I have x dimes, y nickels and 6 pennies in my pocket. The value of my money in cents is $10x +$ ___ $+$ ___.	$10x + 5y + 6$
15	If n is a certain number, then a phrase for "this number increased by half the same number" is _____.	$n + \frac{n}{2}$
16	"One number n increased by a second number twice as large as n" may be written _____.	$n + 2n$
17	_____ is the number of days in w weeks.	$7w$
18	P is the population of Lakeside, Kansas. A city with one million more than twice the population of Lakeside has _____ people.	$2P + 1,000,000$

19	If the number of dollars earned each month by Mr. Kincaid is x, in one year he earns _______ dollars.	$12x$
20	If Jake travels for h hours at 40 miles per hour, then he travels _______ miles.	$40h$
21	Mr. Brown's taxes are computed at the rate of \$25 per \$1000 valuation. If y is the value of Mr. Brown's property, his tax is _______ dollars.	$\frac{y}{1000}(25)$, or $\frac{y}{40}$, $.025y$, etc.
22	If Earl's weight in pounds is p, then a phrase representing the weight of an object weighing forty pounds more than Earl is _______.	$p + 40$
23	If u, v, and w are the number of degrees in each angle of a triangle, then $u + v + w = 180$ can be interpreted as a statement that the _______ of the number of degrees in the angles of a triangle is _______.	sum 180
24	$z(z + 13) = 360$ is a sentence relating the area and the dimensions of a certain rectangle. If the width of this rectangle is z feet and its area is 360 square feet, then the length must be _______ (how many) feet more than the width.	13
25	If x is Mr. Jones' age in years, then $x < 80$ is an open sentence for the statement, "Mr. Jones is _______ _______ eighty years old."	is less than
26	"Bill bought b bananas at 9 cents each and paid 54 cents" may be translated into the sentence _______.	$9b = 54$
27	If b is the number of dollars in Cheryl's allowance, the statement, "Janice's allowance is one dollar more than twice Cheryl's and is also two dollars less than three times Cheryl's", may be written as _______.	$2b + 1 = 3b - 2$
28	If z is the population of Spillville, which is less than one million in population, we can write: _______.	$z < 1,000,000$

Two million is more than twice the population of any city in Colorado. If p is the number of people in a city in Colorado, then
29 $2{,}000{,}000 >$ ______. $2{,}000{,}000 > 2p$

The length of the side of a certain square is x feet. The length of a rectangle is $x + 1$ feet and its width is $x - 1$ feet. A sentence stating that the square has a larger area than the rectangle is:
30 x^2 ________. $x^2 > (x+1)(x-1)$

The tax on real estate is calculated at \$24.00 per \$1000 valuation. The tax assessment on property valued at y dollars is \$348.00. Translating this
31 into a sentence would give: $\frac{y}{1000}(\quad) = 348$. $\frac{y}{1000}(24) = 348$

If John added 40 pounds to his weight, he would still not weigh more than 152 pounds. If John weighs w pounds, translating the foregoing into a sentence
32 would give: $w + 40$ ___ 152. $w + 40 \leq 152$, or $w + 40 \not> 152$

"The distance from Dodge City to Oklahoma City, 260 miles, was traveled in t hours at an average speed of 40 miles per hour on July 4." In order to find the number of hours traveled, we may write the
33 sentence ______. $40t = 260$

Pike's Peak is more than 14,000 feet above sea level. If the height in feet of Pike's Peak is represented by h, then a corresponding open
34 sentence would be: h ___ $14{,}000$. $h > 14{,}000$
$(<, >, =)$

A book 1.4 inches thick has n sheets; each sheet is 0.003 inches thick, and each cover is $\frac{1}{10}$ inches thick and is blue. An open sentence we might use to
35 find the number of sheets is: $1.4 =$ ___$(n) +$ ___$(.1)$. $1.4 = 0.003(n) + 2(.1)$

Carol, who is 16 years old and has two brothers, is 4 years older than her sister. If her sister is s years old, an open sentence which may be used to find
36 her sister's age is ______. $s + 4 = 16$

Sometimes there is information in a word problem [illegible] is not needed to answer the question asked. In Item 36, the fact that the cover is blue has nothing to do with the thickness of the book. The fact that Carol has two brothers is not needed in Item 35 to write an open sentence representing her sister's age.

Item		Answer
37	If a whole number is n, then its successor is ______.	$n + 1$
38	"The sum of a whole number and its successor is 576" could be stated: ______.	$n + (n+1) = 576$
39	Of two successive whole numbers one must be even and the other odd. Hence their sum ______ (can, cannot) be even.	cannot
40	It follows that $n + (n + 1) = 576$ is ______ (true, false) for all whole numbers n.	false
41	7 is an odd number. The next consecutive odd number is $7 + 2$, or 9. If n is an odd number, the next consecutive odd number is ______.	$n + 2$
42	The sum of two consecutive odd numbers is 75. If n is an odd number, the open sentence corresponding to the given information is $n + (\quad) = 75$. Note that this sentence is false for all whole number values of n, because the sum of two odd numbers	$n + (n+2) = 75$
43	always is ______ (even, odd).	even
44	A board 16 feet long is cut in two pieces such that one piece is one foot longer than twice the other. If the shorter piece of the board is f feet long, then the other piece is ______ feet long.	$2f + 1$
45	A sentence for the problem is ______.	$f + (2f+1) = 16$
46	Polly earns \$2.25 babysitting for 3 hours at x cents an hour. A sentence for this would be: $3x =$ ______.	$3x = 225$
47	Ann has 16 more books than Beth. If Beth has k books, a sentence showing that together they have more than 28 books would be: ______.	$k + (k+16) > 28$
48	The phrase ______ represents the number of books Ann has.	$k + 16$

49 The sum of two numbers is 42. If the first number is represented by n, then a phrase for the second number would be _______. — $42 - n$

50 A number is increased by 17 and the sum is multiplied by 3. If n represents the number, we could write a sentence stating that the resulting product equals 192 as _______. — $3(n+17) = 192$

51 If 17 is added to a number and the sum is multiplied by 3; the resulting product is less than 192. To restate this as an open sentence, if n represents the number, we would have: _______. — $3(n+17) < 192$

52 One number is 5 times another. The sum of these two numbers is 15 more than 4 times the smaller. If the smaller number is n, then an open sentence is _______. — $n+5n = 4n+15$

53 Mr. Barton paid $176 for a freezer which was sold at a discount of 12% of the marked price. To restate this by a sentence, with the marked price as m dollars, we would write: 176 = m - _______. — $176 = m - \frac{12}{100}m$, or $176 = m - .12m$

54 A man's paycheck for a work week of 48 hours was $166.40. He is paid at the rate of $\frac{3}{2}$ times his normal rate for all hours worked in excess of 40 hours. If the rate of pay for one hour's work is x dollars, then _______ is the number of dollars for one hour's work at the overtime rate. — $\frac{3}{2}x$, or $\frac{3x}{2}$

55 An open sentence corresponding to the given information would be: ( )x + ($\frac{3}{2}$x) = 166.40. — $(40)x + 8(\frac{3}{2}x) = 166.40$

A farmer can plow a field in 7 hours with one tractor and in 5 hours with another tractor. Let us find out how long it would take him to plow the field using <u>both</u> tractors.

56 If he uses the first tractor, in one hour he can plow _______ of the field. — $\frac{1}{7}$

57 Using the first tractor, in two hours he can plow ______ of the field.

$2(\frac{1}{7})$, or $\frac{2}{7}$

58 But using the second tractor for two hours, he can plow ______ of the field.

$2(\frac{1}{5})$, or $\frac{2}{5}$

59 If he has both tractors going for 2 hours, he can plow (+) of the field.

$\frac{2}{7} + \frac{2}{5}$

60 If both tractors are used for x hours, the part of the field that is plowed can be represented by the phrase ______.

$\frac{x}{7} + \frac{x}{5}$, or $x(\frac{1}{7} + \frac{1}{5})$

61 The entire field can be indicated by "1". Hence if x is the number of hours it takes to plow the entire field, we can write: ______ = 1.

$\frac{x}{7} + \frac{x}{5} = 1$, or $x(\frac{1}{7} + \frac{1}{5}) = 1$

*62 Test your common sense. Then look on page xiii for the answer.

A man with five dollars in his pocket stops at a candy store on his way home with the intention of taking his wife two pounds of candy. He finds candy by the pound box selling for \$1.69, \$1.95, \$2.65, and \$3.15. He leaves the store with two one-pound boxes of candy.

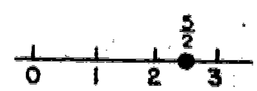
a) What is the smallest abount of change he can have?

b) What is the largest amount?

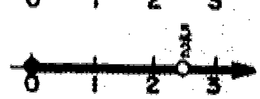
It is a good idea to think back over what you have learned in earlier chapters. In Chapter 6 we will be using the number line a great deal. If you feel that you need review, do Items 63 to 79. If not, go to Chapter 6.

Graph these open sentences:

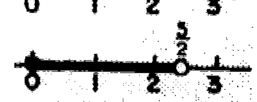
63 $x = \frac{5}{2}$ ______

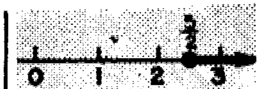


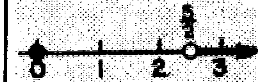
64 $x \neq \frac{5}{2}$ ______

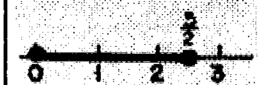


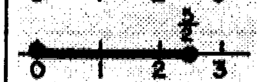
65 $x < \frac{5}{2}$ ______

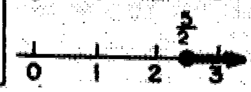


66 $x \nless \frac{5}{2}$ ________________ 

67 $x > \frac{5}{2}$ ________________ 

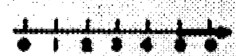
68 $x \ngtr \frac{5}{2}$ ________________ 

69 $x \leq \frac{5}{2}$ ________________ 

70 $x \geq \frac{5}{2}$ ________________ 

Given the compound open sentence: "$x = 2$ <u>or</u> $x > 5$", we would graph this as in the following steps:

the graph of $x = 2$, 

71 the graph of $x > 5$, ________________ 

72 the graph of $x = 2$ or $x > 5$, ________________ 

Given the compound open sentence: "$x = 2$ <u>and</u> $x > 5$", we would graph this as follows:

73 ________________ 

the empty set

Given the compound open sentence: "$x > 2$ or $x < 5$", we would graph this as in the following steps:

the graph of $x > 2$, 

74 the graph of $x < 5$, ________________ 

75 the graph of $x > 2$ or $x < 5$, ________________ 

Given the compound open sentence: "$x > 2$ <u>and</u> $x < 5$", we would graph this as follows:

76 ________________ 

Determine an open sentence for each of the following graphs:

77 An open sentence is ______. | $x > 2$

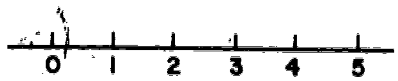
78 An open sentence is ______. | "$x \leq 6$", or "$x \not> 6$" or "$0 \leq x \leq 6$"

79 An open sentence is ______. | "$2 < x \leq 6$", or "$x > 2$ and $x \leq 6$" or "$x > 2$ and $x \not> 6$"

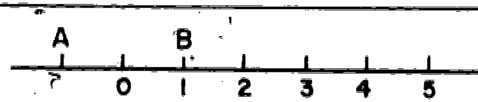
Chapter 6

THE REAL NUMBERS

6-1. The Real Number Line



The number line above reminds us that we have associated numbers with the points to the right of the point represented by zero. In this chapter we shall associate numbers with points to the left of zero and begin the study of the set of numbers corresponding to all of the points on the number line.



1 The point B is one unit to the ______ of zero and — right

2 is associated with the number ______. — 1

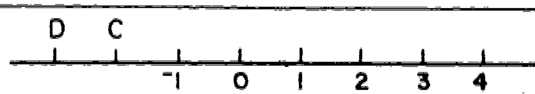
The point A is one unit to the left of zero. We shall associate the number $^-1$ with this point. $^-1$ is read "negative 1". (Notice how high the dash in the numeral $^-1$ is written.)

3 "$^-1$" is read "______ 1" and represents a new number which we shall soon find to be very useful. — negative 1

4 "$^-2$" is read "______". — negative 2

5 "Negative 3" would be written "______". — $^-3$

6 "Negative 53" is ______ (correctly, incorrectly) written "$^-53$". — correctly



Point C is located two units to the left of zero and

7 is associated with the number ______. Point D is — $^-2$

8 associated with the number ______. — $^-3$

9	The point ten units to the left of zero is associated with the number _______, which is called the coordinate of that point.	$^{-}10$
10	$^{-}10$ is read "_______".	negative 10
11 12	$^{-}1000$ is the coordinate of the point which is _______ units to the _______ of zero.	1000 left
13 14	$^{-}8$ is the _______ of a point which lies _______ units to the left of zero.	coordinate, 8

This process of locating points to the left of 0 continues without end. We may indicate this by writing these numbers as members of the set $\{\ldots, {}^{-}3, {}^{-}2, {}^{-}1\}$. This set may also be written $\{{}^{-}1, {}^{-}2, {}^{-}3, \ldots\}$.

If we form the union of this set with the set of whole numbers $\{0, 1, 2, 3, \ldots\}$ we obtain the <u>set of integers</u>:

$$\{\ldots, {}^{-}3, {}^{-}2, {}^{-}1, 0, 1, 2, 3, \ldots\}.$$

Note that the counting numbers are elements of the set of integers. These are called the <u>positive integers</u>. The elements $^{-}1$, $^{-}2$, $^{-}3$, ..., are also integers and are called the <u>negative integers</u>. 0 is an integer but is neither positive nor negative.

This is indicated in the following diagram:

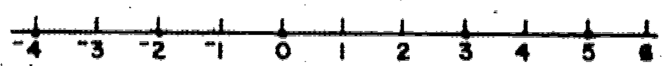
$$\{\ldots, \underbrace{{}^{-}3, {}^{-}2, {}^{-}1}_{\text{negative integers}}, \underbrace{0}_{\text{zero}}, \underbrace{1, 2, 3, \ldots}_{\text{positive integers}}\}.$$

15	The positive integers are associated with points to the _______ of 0.	right
16	The negative integers are associated with points to the _______ of 0.	left
17 18 19	0 is an _______ which is neither _______ nor _______.	integer positive(negative) negative(positive)

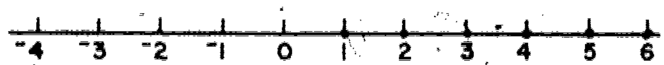
	If W is the set of whole numbers, then	
20	W = {______}.	{0,1,2,3,...}
	If P is the set of positive integers, then	
21	P = {______}.	{1,2,3,...}
22	If I is the set of integers, then I = {______}.	$\{\dots,^-2,^-1,0,1,2,\dots\}$
	If L is the set of non-negative integers, then	
23	L = {______} and L is the same as the set of	{0,1,2,3,...}
24	______ numbers.	whole
	If N is the set of counting numbers, then	
25	N = {______} and N is the same as the set of	{1,2,3,...}
26	______ integers.	positive
	If Q is the set of non-positive integers, then	
27	Q = {______}.	$\{0,^-1,^-2,^-3,\dots\}$
	If S is the set of negative integers, then	
28	S = {______}.	$\{^-1,^-2,^-3,\dots\}$

Draw the graphs of the following sets.

29 $\{0,3,5,^-2,^-4\}$ ______ See Answer below



30 The set of positive integers less than 7. ______ See answer below



Let us recall some of the things which we have learned about rational numbers:

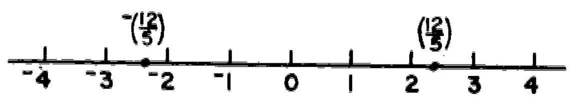
	$\frac{6}{5}$ and 4.8 are examples of rational numbers which	
31	______ whole numbers. (are, are not)	are not
32	Every whole number ______ a rational number. (is, is not)	is

If a number can be named by the indicated division of a whole number by a counting number, the number is
33 a _______ number. — rational

On the number line, the point $\frac{12}{5}$ is located $\frac{2}{5}$
34 of a unit to the $\underset{\text{(right, left)}}{\underline{\hspace{3cm}}}$ of the point 2. — right

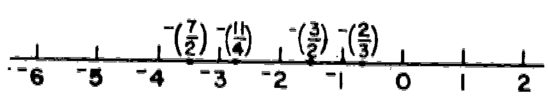
Proceeding as we did when we associated rational numbers with points to the right of zero, we can subdivide the intervals to the left of zero and label points within the intervals.

Thus, the point which lies $\frac{2}{5}$ of a unit to the left of $^{-}2$ is labeled $^{-}(\frac{12}{5})$. $^{-}(\frac{12}{5})$ is read "negative twelve-fifths". In this way $^{-}(\frac{12}{5})$ is the same distance to the left of 0 as $\frac{12}{5}$ is to the right of 0.



In general, if the rational number r is the coordinate of a point to the right of zero, ^{-}r will be the coordinate of a point which lies exactly as far to the left of zero as the point with coordinate r lies to the right of 0. This number ^{-}r is called a <u>negative rational number</u>. The union of the set of rational numbers of arithmetic and the set of negative rational numbers is the <u>set of rational numbers</u>. By the <u>positive rational numbers</u> we mean those rational numbers which correspond to points to the right of zero.

Some other rational numbers are graphed on the number line below:



35 The number $^{-}(\frac{3}{2})$ is read "_______". — negative $\frac{3}{2}$ [negative three-halves]

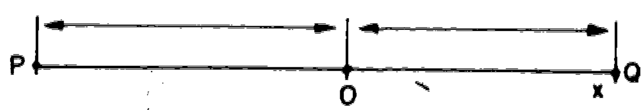
36 Negative eleven-fourths is written _______. — $^{-}(\frac{11}{4})$

37 We note that $^{-}(\frac{2}{3})$ is to the $\underset{\text{(right, left)}}{\underline{\hspace{3cm}}}$ of $^{-}(\frac{7}{2})$. — right

38	The rational numbers to the ______ of zero are called positive rational numbers.	right of zero
39	The rational numbers to the ______ of zero are called negative rational numbers.	left of zero
40	Among the rational numbers are numbers like $^-6$, $^-1$, 12, 17. These numbers are also called ______.	integers
41	We have seen that every integer ______ (is, is not) a rational	is
42	number, but numbers like $\frac{1}{2}$, $^-(\frac{3}{4})$, 1.9 are rational numbers which ______ (are, are not) integers.	are not
43	Thus, we can say that the set of integers ______ (is, is not) a subset of the set of rational numbers.	is

You recall that there are points to the right of zero with coordinates that are not rational numbers. Similarly, there are points to the left of zero with coordinates that are not rational numbers.

Let P be a point to the left of 0 which is not labeled by a negative rational numbers. We can locate a point Q which is the same distance from 0, but to the right of 0.



Let x be the coordinate of the point Q. To follow the scheme that we used for negative rational numbers, we must label the point P with the symbol

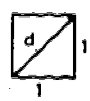
44	______.	^-x
45	Since the coordinate of P is not a ______ number,	rational
46	x ______ (is, is not) a positive rational number, but	is not
47	x ______ (is, is not) a number of arithmetic.	is
48	^-x ______ (is, is not) a rational number and	is not
49	^-x ______ (is, is not) a number of arithmetic.	is not

In this manner we can associate every point on the number line with some number. The set of all numbers associated with points on the number line is called the set of real numbers. The numbers to the left of zero are called the negative real numbers and those to the right are called the positive real numbers. Remember that zero is a real number, but is neither positive nor negative. Another name for the set of numbers of arithmetic is the set of non-negative real numbers.

We emphasize the fact that now every point on the number line has a real number as coordinate by calling the number line the real number line.

A real number which is not a rational number is called an irrational number. We have not as yet shown how to locate a point on the number line whose coordinate is not a rational number.

Let us consider the positive real number whose square is equal to 2. It is customary to represent this number by the symbol $\sqrt{2}$. How can we locate the point on the number line which corresponds to $\sqrt{2}$? Consider the square of side 1.



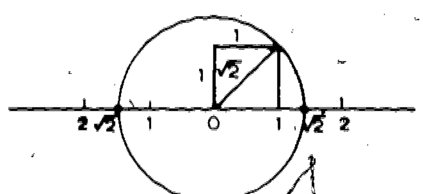
You may recall the Pythagorean Theorem, which says that in any right triangle the square of the length of the hypotenuse is equal to the sum of the squares of the lengths of the two legs. In our case this would tell us that

$$d^2 = 1^2 + 1^2$$

$$\text{or} \quad d^2 = 2 .$$

Hence, the length of the diagonal of the square is $\sqrt{2}$.

In order to locate the point with coordinate $\sqrt{2}$, all we have to do is to transfer the length of the diagonal to our number line. This we can do by drawing a circle with center at the point 0 on the number line and with radius equal to the length of the diagonal of the square. This circle cuts the number line in two points, whose coordinates are the real numbers $\sqrt{2}$ and $^-\sqrt{2}$.



50 Study the discussion above. What is its main purpose?

[A] To show that in any right triangle, $c^2 = a^2 + b^2$.

[B] To show how the point on the number line corresponding to the irrational number $\sqrt{2}$ can be determined.

[C] To show that $\sqrt{2}$ is the length of the diagonal of a square of side 1.

We stated earlier that there are points on the number line corresponding to irrational numbers. The purpose of the preceding discussion is to demonstrate how to locate the point on the number line corresponding to the irrational number $\sqrt{2}$. [B] is the correct answer.

The number 1.4 is sometimes used as an approximation for $\sqrt{2}$. Do you think 1.4 is actually equal to $\sqrt{2}$?

51 $1.4 \times 1.4 =$ ______ 1.96

Perhaps $1.41 = \sqrt{2}$!

52 $1.41 \times 1.41 =$ ______ 1.9881

53 $(1.41)^2 \underset{(<,\ >)}{\underline{\quad}} (1.4)^2$ >

54 $(1.41)^2 \underset{(<,=,>)}{\underline{\quad}} 2$ <

If you were to try $(1.414)^2$ you would get

55 1.999396 which is $\underset{(<,=,>)}{\underline{\quad}}$ 2. <

Although we could find rational numbers with squares which are near to 2, we are never able to find

56 a ______ number whose square is equal to 2. Later on you will see a proof that $\sqrt{2}$ is irrational. rational

Of course there are many other irrational numbers. It can be shown that numbers like $6 + 3\sqrt{2}$ are irrational. Also if a whole number is not the square of another whole number its square root is irrational. For example, $\sqrt{3}$, $\sqrt{5}$, etc., are irrational numbers.

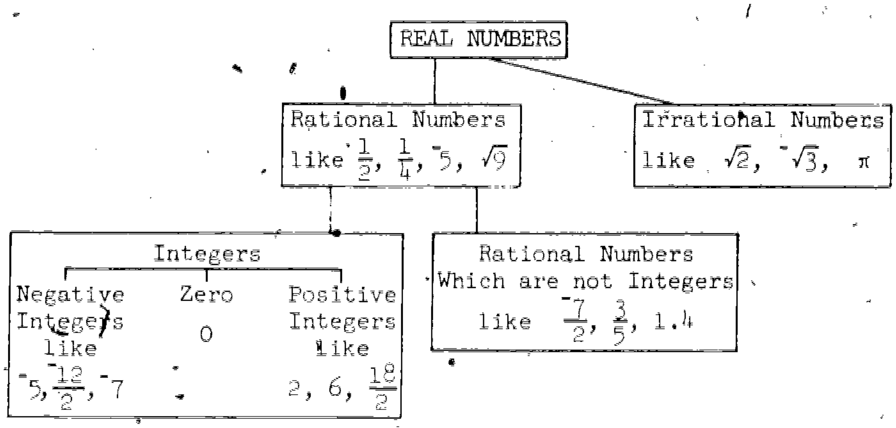
Another type of irrational number is the number π.

57 The irrational number π is the ratio of the circumference of a circle to its diameter. Thus, a circle whose diameter is of length 1 has a circumference of length π. Imagine such a circle resting on the number line at the point 0. If the circle is rolled on the line, without slipping, one complete revolution to the right it will stop on a point. What is the coordinate of this point?

[A] 1 [B] $^{-}1$ [C] π [D] $^{-}\pi$

As the circle rolls one revolution, it will roll a distance of one complete circumference of the circle. Since the circumference of the circle with diameter 1 is π, the point on which it stops is π units to the right of zero. Thus the point on which the circle stops has coordinate π. [C] is the correct choice.

The following diagram may help to review the sets of numbers we have discussed.



Refer to the diagram above to answer the following:

All of the sets of numbers we have studied are subsets

58 of the set of _______ numbers. real numbers

59 The set of integers is a subset of the set of _______ _______ as well as a subset of the set of real numbers. rational numbers

60 The set of positive integers is a _______ of the set of integers. subset

We could think of our work so far as successive stages of extending our idea of "numbers" until we finally obtain the set of real numbers. Now <u>every point</u> on the number line can be named by a real number, and <u>every real number</u> names a point on the number line.

After you have worked Items 61-80, turn to page i to check your work.

Draw the graphs of the following sets:

61. $\{0, 3, ^-5, ^-(\frac{1}{2})\}$

62. $\{^-(\frac{2}{3}), \frac{2}{3}, \frac{5}{2}, ^-(\frac{5}{2})\}$

63. $\{^-(\frac{3}{2}), 5, ^-7, ^-(\frac{11}{3})\}$

64. $\{^-(\frac{12}{7}), ^-(\frac{6}{5}), \frac{6}{5}, \frac{7}{12}\}$

65. $\{\sqrt{2}, ^-\sqrt{2}, 3, ^-3\}$

66. $\{^-1, ^-(1 + \frac{1}{2}), (1 + \frac{1}{2})\}$

67. $\{^-(\frac{1}{2}), (\frac{1}{2})^2, ^-(\frac{6}{4}), (3 - 3)\}$

Of the two points whose coordinates are given in the pairs below, which is to the right of the other?

68. $3, ^-4$

69. $5, ^-4$

70. $^-2, ^-4$

71. $^-\sqrt{2}, 1$

72. $^-(\frac{5}{2}), 0$

73. $^-(\frac{5}{2}), ^-(\frac{10}{4})$

74. $0, 3$

75. $^-4, \sqrt{2}$

76. $^-(\frac{16}{3}), ^-(\frac{21}{4})$

77. $^-(\frac{1}{2}), \frac{1}{2}$

78. Is $^-2$ a whole number? An integer? A rational number? A real number?

79. Is $^-(\frac{10}{3})$ a whole number? An integer? A rational number? A real number?

80. Is $^-\sqrt{2}$ a whole number? An integer? A rational number? A real number?

6-2. Order on the Real Number Line

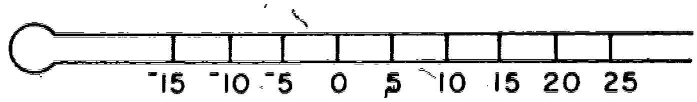
How did we describe order for the positive real numbers? Let us review, using the numbers 5 and 6.

	We agree that the two sentences: "5 is to the left of 6 on the number line," and	
1	"5 ________ 6" say the same thing about 5 and 6. (is less than, is greater than)	is less than
2.	"5 is less than 6" means that 5 is to the ________ of 6 on the real number line. (left, right)	left
3	"5 is less than 6" may be written "5 ____ 6". (<, =, >)	<
4	"5 is to the left of 6" may also be written "5 ____ 6". (<, =, >)	<
5	$5 < 6$ is a ________ sentence. (true, false)	true

Thus for a pair of positive real numbers, "is to the left of on the number line" and "is less than" both describe order.

What shall we mean by "is less than" for any two real numbers, whether they are positive, negative or 0? We agree that "is less than" will mean "is to the left of on the real number line".

Let us look for a justification in common experience. All of us are familiar with thermometers and are aware that scales on thermometers use numbers above 0 and numbers below 0, as well as 0 itself. We know that the colder the weather, the lower on the scale we read the temperature. If we place a thermometer in a horizontal position as shown, we see that it resembles part of our real number line. When we say "is less than" ("is a lower temperature than"), we mean "is to the left of" on the thermometer placed in a horizontal position, as shown.



6	Refer to the diagram of the thermometer and find which numerical sentence below describes the order of the temperatures, $^-5^\circ$ and $^-10^\circ$. [A] $^-5 < {}^-10$ [B] $^-10 < {}^-5$	

$^-10$ is to the left of $^-5$. Since "<" means "is less than" which can be interpreted as "is to the left of", $^-10 < {}^-5$ is correct.

Thus we extend our former meaning of "is less than" to the set of all real numbers. We agree that:

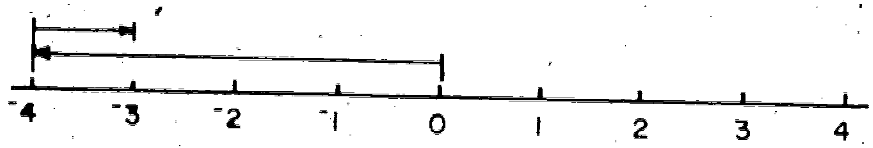
> "is less than" for real numbers means "is to the left of on the real number line". If a and b are real numbers, "a is less than b" is written
>
> $$a < b.$$

(Now and in the future a variable is understood to have as its domain the set of <u>real numbers</u>, unless otherwise stated.)

7	Since "is less than" for real numbers means "is to the left of on the real number line", then "is greater than" should mean "is to the ______ (left, right) of on the number line".	right
8	If a and b are real numbers, "a is greater than b" will be written as a ______ (<, =, >) b.	$a > b$
9	$^-4 > {}^-3$ is ______ (true, false).	false
10	$^-3$ ______ (<, =, >) $^-4$ is true.	$^-3 > {}^-4$
11 12	Since 0 <u>is</u> ______ <u>than</u> $^-4$, 0 is to the ______ (left, right) of $^-4$.	is greater than right
13 14	Since 0 <u>is</u> ______ <u>than</u> any negative real number, 0 is to the ______ (left, right) of all negative real numbers.	is greater than right
15 16	Since 0 <u>is</u> ______ <u>than</u> any positive real number, 0 is to the ______ (left, right) of all positive real numbers.	is less than left

17	0 is to the ______ of $^-5$.	right
18	Thus 0 <u>is</u> ______ <u>than</u> $^-5$.	is greater than

Just as with the set of positive real numbers, we see that since $^-4 < {}^-3$, we would go from $^-4$ to $^-3$ by moving to the right on the number line.

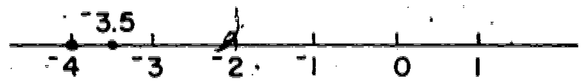


Let us extend the meaning of the other verb forms to the set of all real numbers.

19	If a and b are real numbers, "a is less than or equal to b" is written "$a \le b$" and "a is greater than or equal to b" is written "a ___ b". $(\le, \ge)$	$a \ge b$
20	$^-(\frac{3}{5}) \le {}^-(\frac{3+0}{5})$ is ______. (true, false)	true
21	3 ___ $^-1$ is true. $(\le, \ge)$	$3 \ge {}^-1$
22	We have seen that the symbol "$\le$" can be used to mean "is to the left of or is the same point as". We may conclude that the symbol "$\ge$" means "is to the ______ of or is the same point as".	right
23	"Is not to the left of", or "is not less than", is written "$\nless$". The symbol used for "is not to the right of", or "is not greater than", is written ____.	$\ngtr$
24	$^-4 \nless {}^-2.8$ is ______. (true, false)	false
25	$^-4 \neq 3.5$ is ______. (true, false)	true
26	$^-4 \ngtr 3.5$ is ______. (true, false)	true

If you found it difficult to answer the questions in Items 19-26, complete Items 27-35; otherwise, continue with Item 36.

Is the sentence $^{-}4 \nless {}^{-}3.5$ true or false? Let us locate these points on the number line.



27 Since $^{-}4$ is to the ______ of $^{-}3.5$ on the number line, then — **left**

28 $^{-}4$ ______ $^{-}3.5$. $(<, =, >)$ We can conclude that — **$^{-}4 < {}^{-}3.5$**

29 the sentence $^{-}4 \nless {}^{-}3.5$ is ______. (true, false) — **false**

For each of the following sentences determine which are true and which are false.

30 [illegible] — false

31 [illegible] — true

32 [illegible] — true

33 [illegible] — true

34 [illegible] — false

35 [illegible] — false

36 Consider a positive real number, p, and a negative real number, n. Are all of the following sentences true?

[illegible], [illegible], [illegible], [illegible]

(a) Yes (b) No

All of these sentences are true sentences. Clearly if n is any negative number and p is any positive number $n < p$ and $n \neq p$. Remember that $p \geq n$ and $p \nless n$ are just other ways of writing $n < p$.

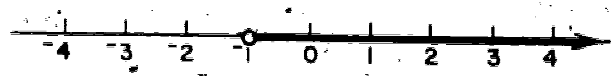
193

Previously we have graphed inequalities with the domain of the variable the set of <u>numbers of arithmetic</u>. In the same manner, we can graph the inequalities with domain of the variable the set of <u>real numbers</u>.

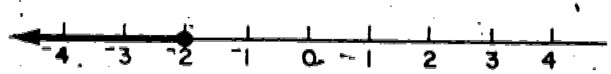
For example:

<u>Open sentence</u>: <u>Graph</u>: (the domain is the set of real numbers)

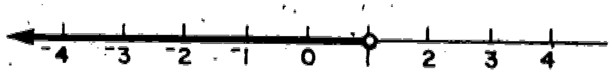
$x > {}^-1$



$x \leq {}^-2$



$x < 1$



Graph the truth set of each of the following open sentences.

37. [illegible]	42. [illegible] < 2 and $c > {}^-2$
38. [illegible]	43. $a \leq {}^-3$ and $a \geq {}^-3$
39. $v \geq$ [illegible]	44. $d \leq {}^-1$ or $d > 2$
40. $r \neq$ [illegible]	45. $a < 6$ and $a < {}^-2$
41. x [illegible] or $x < {}^-1$	46. $u > 2$ and $u < {}^-3$

The correct graphs are to be found on page

47. Which two of the open sentences below have as their truth set the set of all real numbers not equal [illegible]? ______ (b) and (d)

(a) $x \leq$ [illegible] (c) $x \ngtr$ [illegible]

(b) x [illegible] or x [illegible] (d) $x \neq$ [illegible]

48. Which two of the open sentences below have as their truth set the set of all real numbers less than or equal to [illegible]? ______ (a) and (d)

(a) $v \leq$ [illegible] (c) v [illegible] and v [illegible]

(b) $v \nless$ [illegible] (d) $v \ngtr$ [illegible]

Find the truth set for each of the following open sentences if the domain of the variable is the set of <u>integers</u>.

49 $^{-}2 < p$ and $p < 3$ ______ $\{^{-}1, 0, 1, 2\}$

50 $p \leq {}^{-}2$ and $^{-}4 < p$ ______ $\{^{-}2, {}^{-}3\}$

51 $p = 2$ or $p = {}^{-}5$ ______ $\{^{-}5, 2\}$

In Items 52-55 we shall examine pairs of real numbers. See whether you can discover the property of order suggested by these examples. It may help to use the number line to answer these questions. In each of the following groups of three sentences, select the true sentence.

52 $^{-}2 < {}^{-}1.6$; $^{-}2 = {}^{-}1.6$; $^{-}1.6 < {}^{-}2$ ______ $^{-}2 < {}^{-}1.6$

53 $0 < {}^{-}2$; $0 = {}^{-}2$; $^{-}2 < 0$ ______ $^{-}2 < 0$

54 $^{-}2 < 2$; $^{-}2 = 2$; $2 < {}^{-}2$ ______ $^{-}2 < 2$

55 $^{-}.01 < {}^{-}(\frac{1}{100})$; $^{-}.01 = {}^{-}(\frac{1}{100})$; $^{-}(\frac{1}{100}) < {}^{-}.01$ ______ $^{-}.01 = {}^{-}(\frac{1}{100})$

Observe that in each of the three groups of sentences <u>one and only one</u> sentence is true. This illustrates a simple but highly important fact about order for the real numbers. For any real numbers, a and b, either they are the same number, or the first is less than the second or the second is less than the first. (Only one of these is true!)

This is called the <u>comparison property</u>, which may be stated:

> If a and b are real numbers, then
> exactly one of the following is true:
> $a < b$, $a = b$, $b < a$.

The comparison property is true in the set of numbers of arithmetic as well as in the set of real numbers. While we did not actually state this property in so many words, we have assumed it.

56	Another way we may state the comparison property is: If $m \neq n$, then either $m < n$ or ___ < ___	$n < m$
57	Suppose the value of x is such that we know that $x = 4$ is false and that $4 < x$ is also false, we may conclude: ___ < ___ is true.	$x < 4$
58	For any two real numbers, the comparison property is stated in terms of "<". Can we state this property in terms of ">"? Think of a point with coordinate 5 on the number line. Let x be any other real number. Then exactly one of the following is true: 5 is to the right of x, x is to the ______ of 5.	right
59	The comparison property may also be stated: If a and b are real numbers, then exactly one of the following is true: a ______, ___ = ___, ___ > ___.	$a > b$, $a = b$, $b > a$
60	Which of the following is correctly stated? I. If a and b are real numbers, then exactly one of the following is true: $a \leq b$, $b \leq a$. II. If a and b are real numbers, then exactly one of the following is true: $a \geq b$, $b > a$. [A] I [B] II [C] I and II	

If $a = 3$ and $b = 3$ we see that $a \leq b$ and $b \leq a$ are both true sentences. Thus I should read:

If a and b are real numbers then exactly one of the following is true:

$a \leq b$, $b < a$.

II is correctly stated. Thus, the correct choice is [B].

61	If a and b are real numbers, and both $a \leq b$ and $b \leq a$ are true, we may conclude $a \underset{(\neq, =)}{\underline{\quad}} b$.	$a = b$

Another property of order is suggested by the following argument. Suppose a, b, and c are real numbers such that, on the number line, a is to the left of b and b is to the left of c. Then we see that a is to the left of c. This is an instance of the transitive property.

We can state the transitive property using the verb form "$<$":

If a, b, and c are real numbers and if $a < b$ and $b < c$, then $a < c$.

Although this property has not been formally stated for the numbers of arithmetic, we know from the number line and the definition of "less than" that this property is true in the set of numbers of arithmetic.

	Since $\frac{14}{5} < 3$, and $3 < \frac{19}{6}$ we may conclude that	
62	$\frac{14}{5} \underset{(<, >)}{\underline{\quad}} \frac{19}{6}$, using the _____ property.	$<$, transitive
63	If x is a negative number, then x ___ 0;	$x < 0$
64	if y is a positive number, then $0 <$ ___;	$0 < y$
65	hence, x ___ y.	$x < y$
	Suppose we know that m is less than 4 and that 4	
66	is less than n, it follows from the _____ property	transitive
67	that _____.	$m < n$

	Which is true, $\frac{4}{5} < \frac{3}{4}$ or $\frac{3}{4} < \frac{4}{5}$?	
	One way to answer this question is to apply the multiplication property of 1 to each number to get	
68	$\frac{4}{5} = \frac{4}{5} \cdot \frac{\square}{\square} = \frac{16}{20}$.	$\frac{4}{4}$
69	and $\frac{3}{4} = \frac{3}{4} \cdot \frac{\square}{\square} = \frac{15}{20}$.	$\frac{5}{5}$
70 71	Then $\frac{3}{4} \underset{(<, >)}{\underline{\quad}} \frac{4}{5}$ since _____ is to the left	$<$, $\frac{15}{20}$
72	of _____.	$\frac{16}{20}$

Suppose you wanted to compare $\frac{337}{113}$ and $\frac{165}{55}$.

Could the above method be used? The answer, of course, is "yes". However, perhaps there is a simpler way of doing this.

Recall that we can always find a number between any two given numbers. Thus we can find some number n

73 which lies ______ the numbers $\frac{167}{55}$ and $\frac{337}{113}$. between

If we can compare $\frac{337}{113}$ and this number n, and and also $\frac{167}{55}$ and the same number n, then we can say one of the following:

74 $\frac{337}{113} < n$ and $n <$ ______ $\frac{337}{113} < n$, and $n < \frac{167}{55}$

or

75 $\frac{167}{55}$ ___ n and n ___ $\frac{337}{113}$. $\frac{167}{55} < n$ and $n < \frac{337}{113}$

76 One of these must be true by the ______ property of order. comparison

If $\frac{337}{113} < n$ and $n < \frac{167}{55}$ is true, then we can

77 conclude that $\frac{337}{113} < \frac{167}{55}$ by the ______ property. transitive

If $\frac{167}{55} < n$ and $n < \frac{337}{113}$, then we can conclude

78 that $\frac{167}{55} < \frac{337}{113}$, by the ______ property. transitive

Try to find some number which will make either Item 74 or Item 75 true, and then answer the question:

79 Which is the lesser of the numbers $\frac{167}{55}$ and $\frac{337}{113}$? ______ $\frac{337}{113}$

If you had trouble completing Item 79, do Items 80-86. If not, continue with Item 87.

80 $\frac{167}{55}$ is "near to" the integer ______ since — 3

81 $3 = \frac{\square}{55}$. — $3 = \frac{165}{55}$

82 Clearly 3 ______ $\frac{167}{55}$. $(<, >)$ — $3 < \frac{167}{55}$

83 On the other hand, if we write 3 as $\frac{\square}{113}$ we can conclude that — $\frac{339}{113}$

84 $\frac{337}{113}$ ______ 3. $(<, >)$ — $\frac{337}{113} < 3$

Which of the following is a true statement?

85 $\frac{337}{113} < 3 < \frac{167}{55}$ or $\frac{167}{55} < 3 < \frac{337}{113}$ ______ — $\frac{337}{113} < 3 < \frac{167}{55}$

Finally, we can conclude

86 ______ $<$ ______. — $\frac{337}{113} < \frac{167}{55}$

Compare the following pairs of real numbers:

87 $^-1.9$ and $^-(\frac{947}{470})$ — $^-(\frac{947}{470}) < {^-1.9}$

88 $(\frac{2}{3} + \frac{7}{6})$ and $(\frac{1}{2} + \frac{7}{4})$ — $(\frac{2}{3} + \frac{7}{6}) < (\frac{1}{2} + \frac{7}{4})$

89 $\frac{22}{77}$ and $\frac{5}{3}$ — $\frac{22}{77} < \frac{5}{3}$

90 $^-(\frac{22}{77})$ and $^-(\frac{5}{3})$ — $^-(\frac{5}{3}) < {^-(\frac{22}{77})}$

91 1.9 and $\frac{947}{470}$ — $1.9 < \frac{947}{470}$

92 $^-(\frac{17}{35})$ and $^-(\frac{7}{13})$ — $^-(\frac{7}{13}) < {^-(\frac{17}{35})}$

93 $^-(\frac{145}{28})$ and $^-(\frac{104}{21})$ — $^-(\frac{145}{28}) < {^-(\frac{104}{21})}$

94 $\frac{17}{35}$ and $\frac{7}{13}$ — $\frac{17}{35} < \frac{7}{13}$

95 $\frac{145}{28}$ and $\frac{104}{21}$ — $\frac{104}{21} < \frac{145}{28}$

Art and Bob are seated on opposite ends of a see-saw (teeter-totter), and Art's end of the see-saw comes slowly down to the ground. Cal gets on and Art gets off, after which Bob's end of the see-saw comes to the ground. Who is heavier, Art or Cal? [A] Art [B] Cal	

You should have selected [A]. Art is heavier than Bob and Bob is heavier than Cal, so by the transitive property, Art is heavier than Cal.

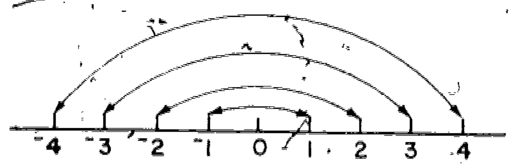
If hamburger is cheaper than steak and hot dogs are cheaper than hamburger, we may conclude that hot dogs are ______ ______ steak.	cheaper than
A man has x dollars and wishes to buy an item costing y dollars. One of the following is true: he has to borrow $y - x$ dollars; he can make the purchase and receive no change; he has $x - y$ dollars after the purchase is made. The property involved in this situation is the ______ ______.	comparison property
Consider the real number x and the real number 0; then we can say only one of the following is true: $x < 0$, $x = 0$, $x > 0$ by the ______ property.	comparison
Thus the set of real numbers consists of the set of 100 ______ real numbers (the set of numbers greater 101 than 0), the number 0, and the set of ______ real numbers (the set of numbers less than 0).	positive negative

A discussion of Items [illegible] and [illegible] will be found on page [illegible].

*102. Let us write "$\prec$" for the phrase "is nearer to 0 than" on the real number line. Does the real number system have the comparison property under this definition of order? In other words, if a and b are different real numbers, is it true that $a \prec b$ or $b \prec a$ but not both? Does the transitive property hold?

*103. If we define order for the numbers of arithmetic as we did in Item 102, do the comparison and transitive properties hold?

6-3. Opposites



The real number line above shows pairs of points at the same distance from the point 0. For example,

1 the point 1 is paired with the point ______. — $^-1$

It may be said that the numbers 1 and $^-1$ are at

2 the same ______ from 0. — distance

On the number line, 1 and $^-1$ are the same distance

3 from ______ but are on opposite sides of 0. — 0

Because 1 and $^-1$ are the same distance from 0 but are on opposite sides of it, the numbers 1 and $^-1$ are called opposites. That is, 1 is the

4 ______ of $^-1$. — opposite

5 Also, $^-1$ is the opposite of ______. — 1

The diagram shows that $^-2$ is the opposite of 2 and that 2 is the opposite of $^-2$.

6 $^-2$ is the ______ of 2. — opposite

7 $^-3$ and ______ are opposites. — 3

8 3 is the opposite of ______. — $^-3$

9 The opposite of $\frac{7}{2}$ is ______. — $^-(\frac{7}{2})$

10 $^-(\frac{2}{3})$ and ______ are opposites. — $\frac{2}{3}$

11 The opposite of $^-428$ is ______. — 428

12 1,000,000 and $^-1,000,000$ are ______. — opposites

The opposite of a non-zero real number is the other

13 real number which is at an equal distance from ______ on the real number line. — 0

14 The opposite of 0 is ______. That is, the number — 0

15 0 is its own ______. — opposite

16	If x is <0, the opposite of x is ______.	0
17	The opposite of a positive number is a ______ number. (positive, negative)	negative
18	The opposite of a ______ number is a positive number. (positive, negative)	negative
19	The opposite of -0 is ______.	0

To summarize:

1. By the opposite of a non-zero real number we mean the other real number which is at an equal distance from 0 on the real number line.
2. The opposite of 0 is 0.
3. The opposite of a positive number is negative and the opposite of a negative number is positive.

It is awkward to write "the opposite of". We will use a lower dash "-" to mean the "opposite of".

	Thus, when we write "-2", we would read this as	
20	"the ______ of 2".	the opposite of
21	$-{}^{-}(\frac{1}{2})$ is the opposite of ______.	${}^{-}(\frac{1}{2})$
22	$-$______ is the opposite of 0.	-0

23	-2 is read "the ______ ______ 2".	the opposite of 2
24	${}^{-}2$ is read as "______ 2".	negative 2
25	-2 is the number ______.	${}^{-}2$
	We may write the true sentence	
26	$-2 =$ ______.	$-2 = {}^{-}2$

	Other true sentences are	
27	$-{}^{-}(\frac{1}{2})$ ______	$-{}^{-}(\frac{1}{2}) = \frac{1}{2}$
28	-0 ______	$-0 = 0$

With this new symbolism, we recognize that $^{-}2$ and -2 are both names for the same number. Hence, it makes no difference at what height the dash is drawn. There is no need to retain both symbols. Since the upper dash refers only to negative numbers while every real number has an opposite, we shall retain the lower dash.

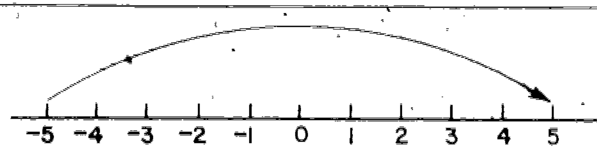
	What would we mean by the notation "$-x$"? We must understand first that x is a variable	
29	representing a real ______.	real number
30	Since x is a ______ number,	real number
31	x may be a positive ______,	positive number
32	a ______ number, or the real number, 0.	negative number
	By "$-x$" we mean the "opposite of the real number x".	
33	If x is a negative real number, $-x$ is a ______ real number.	positive
34	If x is a positive real number, $-x$ is a ______ real number.	negative
35	If x is 0, $-x$ is ______.	0

If we attach a dash to a variable x, obtaining $-x$, we are indicating that we have performed the operation of "determining the opposite" of x. We read "$-x$" as "opposite of x".

Do not confuse this with subtraction, which always involves two numbers.

Here is some practice in determining opposites. Notice that we can find the opposite of any real number.

	If x is	then $-x$ is	
36	2.3	______	-2.3
37	-2.3	______	2.3
38	(42×0)	______	0
39	$(42 + 0)$	______	-42
40	$-(3.6 - 2.4)$	______	1.2
41	$(3.6 - 2.4)$	______	-1.2



The diagram at the above shows that the opposite of -5 is 5.

42 That is, $-(-5) =$ ______. 5

43. $-(-2) =$ ______. 2

44 $-(-\frac{2}{3})$ ______ $\frac{2}{3}$

45 $-($______$) = 5.2$ $-(-5.2)$

From the above examples we can see that $-x$ may be positive, negative, or 0 depending on the value of x.

	If y is	then $-y$ is	and $-(-y)$ is	
	$\frac{1}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$	
46	1	-1	______	1
47	-1	______	-1	1
48	-4.2	4.2	______	-4.2
49	0	______	0	0
50	y	$-y$	______	y

Recalling item 50 we have:

$-(-y) = y$, or in words,

"the opposite of the opposite of a real number is the number itself."

51 Which of the following are true statements about real numbers?

Every number is the opposite of some number.

The opposite of a positive number is a negative number.

The opposite of a negative number is a positive number.

The real numbers are closed under the operation: "determine the opposite".

[A] all except one [B] all

If you believe that any of these statements is false, you should review Section 6-3 to this point. [B] is correct.

On the number line -2 is to the left of 2. That is, $-2 < 2$ is true. Is $-(-5) < -5$ also true? Since $-(-5)$ is another name for 5, we see that $-(-5) < -5$ is <u>false</u>. In fact $-a < a$ is true only if a is <u>positive</u>.

Suppose we start with a pair of real numbers, say a and b such that $a < b$. What can we conclude about the order of $-a$ and $-b$? Let's try some examples:

$$5 < 7, \text{ but } -7 < -5.$$
$$-5 < 7, \text{ but } -7 < 5.$$

$$-7 < -5, \text{ but } 5 < 7.$$
$$-7 < 5, \text{ but } -5 < 7.$$

Experiment with other examples. It is helpful to use a number line. You will see that there is a general order property for opposites:

For real numbers a and b, if $a < b$, then $-b < -a$.

52	The lesser of 2.97 and -2.97 is ______	-2.97
53	Which number has the lesser opposite, 2.97 or -2.97? ______	2.97
54	Of the two numbers 5 and [illegible], which has the greater opposite? ______	5
55	Of the two numbers -5[illegible] and -762, which has the greater opposite? ______	-762
56	Of two different numbers, the larger number will have the ______ (greater, lesser) opposite.	lesser

Use each of the following pairs of numbers in a true sentence with the relation "<":

57. [illegible], [illegible] 59. π, $\frac{22}{7}$ 61. $-\left(\frac{[illegible]}{7}\right)$, -2.1

58. $\sqrt{2}$, $-\pi$ 60. [illegible], $\frac{5}{4}(20+[illegible])$ 62. $-((3+17)0)$, $-((5+0)3)$

Turn to page iii to check your work.

We can use the order property of opposites to help in finding the truth sets and the graphs of certain open sentences.

Suppose we start with the open sentence

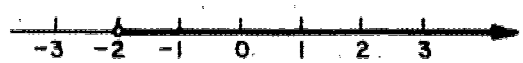
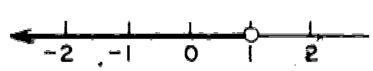
$$-x < 2.$$

Then it follows from the order property of opposites that

$$-2 < -(-x).$$

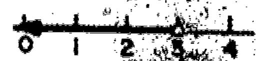
But $-(-x) = x$, hence we can rewrite this last sentence as:

$$-2 < x.$$

63	Thus, the truth set of $-x < 2$ is the same as the truth set of $-2 <$ ____.	$-2 < x$
64	This truth set is the set of real numbers which are ________ than -2. (greater, lesser)	greater than
65	$-2 < x$ may also be written $x >$ ____.	$x > -2$
66	The graph of the sentence $x > -2$ is ____.	see answer below
		
67	This is also the graph of the sentence ____ $<$ ____.	$-x < 2$ or $-2 < x$
68	The truth set of $-x < \frac{1}{2}$ is the set of real numbers which are ________ than $-\frac{1}{2}$. (greater, less)	greater than
69	The truth set of $-x < \frac{1}{2}$ is the same as the truth set of $x >$ ____.	$x > -\frac{1}{2}$
70	The truth set of $-x > 3$ is the same as the truth set of $x <$ ____.	$x < -3$
	An open sentence corresponding to the graph 	
71	is $x <$ ____, (or $1 > x$).	$x < 1$

72 Graph $-x > 3$. [Hint: See Item 70.]

73 Graph $-x > -3$. [Think: "$-3 < -x$, hence, $x < 3$".]



74 Graph $-x < 3$.

75 Graph $-x < -3$.

Complete the description of the truth set of each of the following:

Sentence	Description of truth set	
76 $-x \neq 3$	set of all real numbers except ______.	-3
77 $-x < 0$	set of all ______ (positive, negative) real numbers.	positive
78 $-x \geq 0$	set of all ______ (non-positive, non-negative) real numbers.	non-positive
79 $-x \ngtr 0$	set of all ______ (non-positive, non-negative) real numbers.	non-negative

80 Consider the graphs of the two open sentences

$$x < 2 \qquad , \qquad -x > 3$$

Which of the following is a correct statement?

[A] The graphs have no points in common.

[B] The graph of $-x > 3$ is a subset of the graph of $x < 2$.

[C] The graphs have only the points between -3 and 2 in common.

[D] The graphs have only the points between -2 and 3 in common.

If you chose [B] you are correct; go to Item 85.
If you chose any other statement, continue with Item 81.

The graph of x [illegible] is ______.

The graph of $-x$ [illegible] is the same as the graph of ______, which can be rewritten as x ______ -3.

$-3 \geq x$,
$x < -3$

Graph $-x$ [illegible] ([illegible]) ______

Comparing the graphs of x [illegible] and $-x$ [illegible] we see that every point of the graph of $-x$ [illegible] is included among the points of the graph of x [illegible]. Reread [illegible] and verify that [illegible] is the only correct statement.

Suppose we were asked to decide which of two negative rational numbers is the smaller number. We would first compare the opposites of the numbers, both of which are positive, and then apply the order property of opposites.

For example, [illegible] $\frac{1}{4}$, then ______.

$-\frac{3}{4} < -\frac{1}{2}$

Compare $-\frac{4}{7}$ and $-\frac{7}{15}$. Change $\frac{4}{7}$ and $\frac{7}{15}$ to forms having the same denominator,

[illegible] ______.

$\frac{21}{49}$

Which is the [illegible] of the numbers $\frac{4}{7}$ or $\frac{7}{15}$? ______

$\frac{4}{7} < \frac{7}{15}$

Then ______.

$-\frac{7}{15} < -\frac{4}{7}$

Use this method to compare $-\frac{13}{42}$ and $-\frac{15}{49}$.

______.

$-\frac{13}{42} < -\frac{15}{49}$

Find the open sentence suggested by the following word problems. Give the meaning of the variable in each case.

[illegible]. John's score is greater than negative 1.0. What is his score?

[illegible]. I know that I don't have any money, but I am no more than \$200 in debt. What is my financial condition?

[illegible]. Paul has paid \$1[illegible] of his bill, but still owes more than \$[illegible]. What was the amount of Paul's bill?

Turn to page 111 to check what you have written.

6-4. Absolute Value

For each of the following numbers write its opposite, and then choose the lesser of the number and its opposite.

	Number	Opposite	Lesser is:	
1	3	______	______	-3, -3
2	-7.2	______	______	7.2, -7.2
3	$-\sqrt{2}$	______	______	$\sqrt{2}$, $-\sqrt{2}$
4	-0.01	______	______	0.01, -0.01
5	$-(-2)$	______	______	-2, -2
6	$(1 - \frac{1}{4})^2$	______	______	$-(1 - \frac{1}{4})^2$, $-(1 - \frac{1}{4})^2$
7	0	______	______	0, neither

8 We know that the statement: If -3 is less than 3, then 3 is greater than -3 is ______ (true, false) — true

9 Using the symbols "<" and ">" we can rewrite the above sentence as: If $-3 < 3$ then 3 ____ -3. — $3 > -3$

10 If -3 is the lesser of -3 and 3, then 3 is the greater of -3 and ______. — -3 and 3

11 Just as "$-3 < 3$" and "$3 > -3$" mean the same thing, so "-3 is the lesser of 3 and -3" and "3 is the ______ of 3 and -3" mean the same thing. — greater

We may rewrite Items 1-7 by putting the greater of the number and its opposite in the third column:

	Number	Opposite	Greater of the Number and its Opposite	
12	3	-3	______	3
13	-7.2	7.2	______	7.2
14	$-\sqrt{2}$	$\sqrt{2}$	______	$\sqrt{2}$
15	-0.01	0.01	______	0.01

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16	$-(-2)$	-2	______	$-(-2)$
17	$(1 - \frac{1}{4})^2$	$-(1 - \frac{1}{4})^2$	______	$(1 - \frac{1}{4})^2$
18	0	0	______	neither

19. From Items 12-18 we are led to certain conclusions. Which of the following statements are true?

I. In Items 12-17, the numbers in the third column are always positive.

II. The greater of a non-zero number and its opposite is always positive.

III. The opposite of 0 is 0, hence neither is greater.

[A] I, II and III [B] I and II [C] none are true

[A] is correct. Don't forget that $-(-2)$ is the number 2, and is a positive number! All of the statements are true.

From the definition of "opposites" and "greater than" we are able to make the following general statement:

The greater of a number and its opposite is always positive or zero.

In Section 6-2 we considered the operation of "taking opposites" which meant that for every real number, we could find another number an equal distance from 0 on the number line. Now we are ready to define another operation on a <u>single</u> real number. This operation will relate a unique real number to every real number.

The <u>absolute value</u> of a non-zero real number is defined to be the greater of that number and its opposite. The absolute value of 0 is 0.

Complete the following table:

	Number	Opposite of Number	Greater of Number and its Opposite	Absolute Value of Number	
20	4	-4	______	______	4, 4
21	25	-25	______	______	25, 25
22	______	$\frac{3}{2}$	$\frac{3}{2}$	______	$-\frac{3}{2}$, $\frac{3}{2}$
23	______	-5.06	5.06	______	5.06, 5.06
24	(6 - 4)	______	______	(6 - 4) or 2	-(6-4), (6-4)
25	0	______	neither	______	0, 0

Since the absolute value of a non-zero number is defined to be the greater of the number and its opposite, we can say that the absolute value of a

26 non-zero number is always ______. positive

27 The absolute value of 0 is ______. 0

As usual, we find that it is convenient to agree on a symbol to indicate the operation. We write

$$|n|$$

to mean the absolute value of n.

28 The symbol $|4|$ is read the ______ value of 4. absolute value

29 $|4| =$ ______ 4

Write "the absolute value of -4" using absolute

30 value symbols. ______ $|-4|$

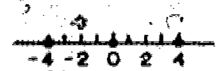
31 $|-4| =$ ______ 4

32 $|1.5| =$ ______ 1.5

33 $-|-3| =$ ______ -3

34 $|-(-3)| =$ ______ 3

35 $|0| =$ ______ 0

Frame	Item	Answer
36	$\lvert -(14 + 0)\rvert =$ _______	14
37	$-\lvert 1.2 + .3\rvert =$ _______	-1.5
38	$\lvert -\frac{2}{3}\rvert =$ _______	$\frac{2}{3}$
	Remember that "$\lvert -\frac{2}{3}\rvert$" is read:	
39	"The _______ _______ of $-\frac{2}{3}$".	absolute value
40	Graph -4, 0, and 4.	
41	What is the distance between -4 and 0? _______	4
42	What is the distance between 4 and 0? _______	4
43	Graph $-\frac{3}{2}$, 0, and $\frac{3}{2}$.	
	The distances between $-\frac{3}{2}$ and 0, and between $\frac{3}{2}$	
44	and 0 are both _______.	$\frac{3}{2}$
45	The absolute value of $-\frac{3}{2}$ is _______,	$\frac{3}{2}$
46	and the absolute value of $\frac{3}{2}$ is _______.	$\frac{3}{2}$.
	The distance between $\frac{3}{2}$ and 0 is the same number	
47	as the _______ _______ of $\frac{3}{2}$.	absolute value
48	The _______ between $-\frac{3}{2}$ and 0 is the same number as the absolute value of $-\frac{3}{2}$.	distance
	You have noticed in these examples that the <u>distance</u>	
49	between a real number and _______ is equal to the	0
50	_______ _______ of the number.	absolute value
	In fact, we could have defined absolute value for	
51	every real number n as "$\lvert n\rvert$ is the _______ between n and 0".	distance

Keep in mind that the <u>distance between</u> two points on a number line is a non-negative real number. Therefore, we can say:

> The <u>distance</u> between a real number and 0 on the real number line is the <u>absolute value</u> of that real number.

There is still another way that we could have defined the absolute value of a real number. Let us try to develop this alternate definition.

First we examine the absolute value of some non-negative numbers:

52 $|\frac{4}{3}| =$ ______ $\frac{4}{3}$

53 $|\frac{1}{2}| =$ ______ $\frac{1}{2}$

54 $|10| =$ ______ 10

55 $|0| =$ ______ 0

56 We conclude that the ______ ______ of a non-negative number is the number itself. **absolute value**

In symbols,

57 $|x| =$ ______, if x is non-negative. $|x| = x$

Now let us consider the absolute value of some negative numbers.

58 $|-\frac{4}{3}| = -(-\frac{4}{3}) =$ ______ $\frac{4}{3}$

59 $|-\frac{1}{2}| = -(-\frac{1}{2}) =$ ______ $\frac{1}{2}$

60 $|-10| = -(-10) =$ ______ 10

The absolute value of a negative number is the opposite of the number.

In symbols,

61 $|x| =$ ______, if x is negative. $|x| = -x$

If we combine the statements which we made in Items 57 and 61, we have a statement that could be used as the definition for the absolute value of a real number. We can state:

$|x| = x$, if x is non-negative, that is, $x \geq 0$.

$|x| = -x$, if x is negative, that is $x < 0$.

62 For any real number, it is always true that

[A] $|x| = x$ [B] $|x| \geq 0$ [C] $|x| > 0$

If $x = -7$, $|x| = 7$. Thus $|x| = x$ is not always true.
If $x = 0$, $|x| = 0$. Thus $|x| > 0$ is not always true.
The correct choice is [B].

Let us summarize:

(a) The absolute value of a number is the greater
63 of that number and its ______. — opposite

(b) The distance between 0 and n on the
64 number line is equal to the ______ ______ of n. — absolute value

65 (c) For every real number a, $|a|$ ___ 0. — $\geq$

66 (d) If $y \geq 0$, $|y| =$ ______. — If $y \geq 0$, $|y| = y$

67 (e) If $y < 0$, $|y| =$ ______. — If $y < 0$, $|y| = -y$

Let $S = \{-2,-1,0,1,2\}$. Let A be the set of the absolute values of the elements of S, then

68 A = { ______ }. — {0,1,2}

69 Is A a subset of S? ______ (yes,no) — yes

70 Hence S ______ (is,is not) closed under the operation of taking the absolute value. — is

Is the set of all real numbers closed under the
71 operation of taking the absolute value? ______ (yes,no) — yes

Here is some practice. Write a common numeral for each of the following.

72 $|2| + |3| = 2 + 3 =$ ____ (The sum of the absolute values) — $2 + 3 = 5$

73 $|2 + 3| = |5| =$ ____ (The absolute value of the sum) — $|5| = 5$

74 $|-2| + |3| =$ ____ — 5

75 $|-2| + |-3| =$ ____ — 5

Write a common numeral for each of the following:

76 $|-5|2 =$ ____ — 10

77 $|-5||-2| =$ ____ — 10

78 $-(|-2||-5|) = -((2)(\quad)) =$ ____ — $-((2)(5)) = -10$

79 $-(|-5||-2|) =$ ____ — -10

80 $3 - |3 - 2| =$ ____ — 2

81 $-(|-7| - 6) =$ ____ — -1

Judge each of the following as true or false.

82 $|-7| < 3$ (Remember $|-7| = 7$) — false

83 $|-2| \leq |-3|$ — true

84 $2 \nless |-3|$ — false

85 $|-5| \nless |2|$ — true

86 $-2 < |-3|$ — true

87 $|-2|^2 = 4$ — true

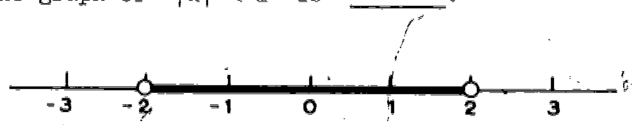
We may have open sentences involving the absolute value of a variable.

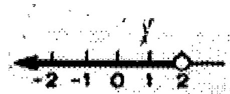
For example, consider the open sentence $|x| \leq 2$.

88 Since $|2| = 2$, 2 ____ (is, is not) in the truth set of $|x| < 2$. — is not

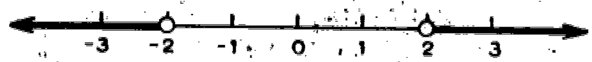
89 The truth set ____ (can, cannot) include numbers greater than 2. — cannot

90	Since $\|-2\| = 2$, -2 ________ (is, is not) in the truth set of $\|x\| < 2$.	is not
91	The absolute value of a number less than -2 is ________ (less than, greater than) 2. (Remember that, for example $\|-3\| = 3$.)	greater than
92	Therefore the truth set ________ (can, cannot) include numbers less than -2.	cannot
93	The absolute value of a number between 0 and 2 ________ (is, is not) less than 2.	is
94	The absolute value of 0 is ________.	0
95	The absolute value of a number between -2 and 0 ________ (is, is not) less than 2.	is
96	The truth set of $\|x\| < 2$ consists of all numbers between -2 and ________.	2
97	The graph of $\|x\| < 2$ is ________.	see answer below



98	The graph of $x < 2$ is ________.	
99	The graph of $x > -2$ is ________.	
100	Another way to describe the truth set of $\|x\| < 2$ is the set of elements in the truth set of ________	$x < 2$
101	and in the truth set of ________.	$x > -2$
102	The truth set of $\|x\| < 2$ is the ________ (intersection, union) of the truth sets of $x < 2$ and $x > -2$.	intersection

103 Graph $|x| > 2$. ______ — see answer below



If you were unable to graph $|x| > 2$ or if you are not sure of how to find the truth set of $|x| > 2$, continue with Item 104. If you are sure of these ideas, go on to Item 108.

104 $|3| = 3$, therefore $|3|$ ____ 2. (>, ≯) — $|3| > 2$

105 If $x > 2$, $|x|$ ____ 2. (>, ≯) — $|x| > 2$

106 Therefore, the truth set of $|x| > 2$ ______ (does, does not) include all numbers greater than 2. — does

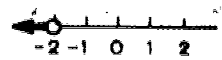
107 The graph of $x > 2$ is ______. —

108 $|-3| = 3$, therefore $|-3|$ ____ 2. (>, ≯) — $|-3| > 2$

109 If $x < -2$, $|x|$ ____ 2. (>, ≯) — $|x| > 2$

110 Therefore, the truth set of $|x| > 2$ ______ (does, does not) include all numbers less than -2. — does

111 The graph of $x < -2$ is ______. —



The truth set of $|x| > 2$ is the set of numbers greater than 2 and the set of numbers less than -2.

Another way of saying this is:

The truth set of $|x| > 2$ is the union of the

112 truth sets of $x > 2$ and ______. — $x < -2$

113 The graph of $|x| > 2$ is ______. — see answer below

222

Find the truth set of:

114 $|x| = 0$ — $\{0\}$

115 $|x| = 2$ — $\{-2, 2\}$

116 $|x| = -1$ — $\emptyset$

117 $|x| < 0$ — $\emptyset$

Did you respond correctly to Items 116 or 117? Remember that the absolute value of any number can never be less than 0.

118 $|x| \leq 8$ has the same truth set as

[A] $x \leq -8$ or $x \geq 8$

[B] $x \leq -8$ and $x \geq 8$

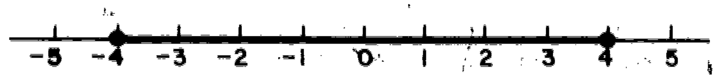
[C] $x \geq -8$ and $x \leq 8$

[D] $x \geq -8$ or $x \leq 8$

If $x = 9$, $|x| > 8$, if $x = -9$, $|x| > 8$.

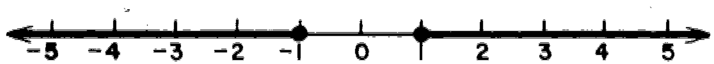
[C] is the correct choice.

*119 Using the absolute value symbol write an open sentence whose graph is:



$|x| \leq 4$, or $|x| \ngtr 4$

*120 Using the absolute value symbol write an open sentence whose graph is:



$|x| \geq 1$, or $|x| \nless 1$

121 Which of the following open sentences is suggested by the English sentence, "The temperature stayed within 5 degrees of 0 today"?

[A] $x < 5$

[B] $x < 5$ and $x > -5$

[C] $x < 5$ or $x > -5$

[D] $|x| < 5$

[E] $|x| > 5$

You are correct if you selected either [B] or [D]; they have the same truth set. [C] is true for all real numbers x. The truth sets of [A] and [E] both contain, for example, -15, and 15 degrees below zero is not within five degrees of zero.

*122 Are the sentences: $x \leq |x|$

$-x \leq |x|$

$-|x| \leq x$

all true for all real values of x?

[A] yes [B] no

All three sentences are true. Notice:

If $x \geq 0$, then $x = |x|$; if $x < 0$, then $x < |x|$;

hence, $x \leq |x|$ is true for all real values of x.

If $x \geq 0$, then $-x \leq |x|$; if $x < 0$, then $-x = |x|$;

hence, $-x \leq |x|$ is true for all real values of x.

If $x \geq 0$, then $-|x| \leq x$; if $x < 0$, then $-|x| = x$;

hence, $-|x| \leq x$ is true for all real values of x.

6-5. Summary and Review

1. Points to the left of 0 on the number line are labeled with negative numbers; the set of real numbers consists of all numbers of arithmetic and their opposites.

2. Many points on the number line are not assigned rational number coordinates. These points are labeled with irrational numbers. The set of real numbers consists of all rational and irrational numbers.

3. "Is less than" for real numbers means "to the left of" on the number line.

4. Comparison Property. If a is a real number and b is a real number, then exactly one of the following is true: $a < b$, $a = b$, $b < a$.

5. Transitive Property. If a, b, c are real numbers and if $a < b$ and $b < c$, then $a < c$.

6. The opposite of 0 is 0 and the opposite of any other real number is the other number which is at an equal distance from 0 on the real number line.

   If x is a positive number, then $-x$ is a negative number.
   If x is a negative number, then $-x$ is a positive number.

7. The absolute value of a non-zero number is the greater of that number and its opposite. The absolute value of 0 is 0.

   The absolute value of the real number n is denoted by $|n|$. Also, $|n|$ is a non-negative number which is the distance between 0 and n on the number line.

   If $n \geq 0$, then $|n| = n$;
   if $n < 0$, then $|n| = -n$.

8. If $a > 0$,

   the truth set of $|x| < a$ is the same as the truth set of the compound sentence $-a < x < a$.

   the truth set of $|x| > a$ is the same as the truth set of the compound sentence $x < -a$ or $x > a$.

The correct answers for the following review problems will be found on page iii.

1. Which of the following sentences are true?

   (a) $-2 < -5$
   (b) $-(5 - 3) = -(2)$
   (c) $-(5 - 3) < -|-2|$
   (d) $-\frac{1}{2} \geq -(\frac{6 + 1}{14})$
   (e) $-2 < |-5|$
   (f) $|-2| < |-5|$

2. Which of the following sentences are false?

   (a) $-(-3) + 5 > -(-2) + 6$
   (b) $3 \neq -3$ or $3 > -3$
   (c) $3 \neq -3$ and $3 > -3$
   (d) $\frac{2}{3} < \frac{3}{4}$ and $-\frac{4}{5} < -\frac{5}{6}$
   (e) $|\frac{3}{4}| < -(-|-1|)$
   (f) $-(|-3| \cdot 2) > -(|-5| \div |3|)$

3. Draw the graph of each of the following sentences.

   (a) $v \geq 1$ and $v < 3$
   (b) $|r| = 2$
   (c) $x < 4$ or $x > 2$
   (d) $|x| = x$

4. Describe the truth set of each sentence.

   (a) $y \leq 3$ and $y > 4$
   (b) $-|u| < 2$
   (c) $-3 < x < 2$
   (d) $|x| = -x$
   (e) $|y| < -2$
   (f) $|v| \geq 0$

5. Consider the open sentence "$|x| < 3$". Draw the graph of its truth set if the domain of x is the set of:

   (a) real numbers
   (b) integers
   (c) non-negative real numbers
   (d) negative integers

6. Describe the variable used and write an open sentence suggested by the following:

   On Sunday the temperature remained within 6° of -5°.

7. If R is the set of all real numbers, P the set of all positive real numbers, F the set of all rational numbers, I the set of all integers, which of the following are true statements?

   (a) F is a subset of R.
   (b) Every element of I is an element of F.
   (c) There are elements of I which are not elements of R.
   (d) Every element of I is an element of P.
   (e) There are elements of R which are not elements of F.

8. Draw the graph of the set of integers less than 6 whose absolute values are greater than 3. Is -8 an element of this set?

9. When a certain integer and its successor are added, the result is the successor itself.

   (a) Write an open sentence suggested by the English sentence above.
   (b) Find the truth set of this sentence.

10. The perimeter of a square is less than 10 inches.

    (a) What do you know about the number of units, s, in the side of this square? Graph this set.
    (b) What do you know about the number of units, A, in the area of this square? Graph this set.

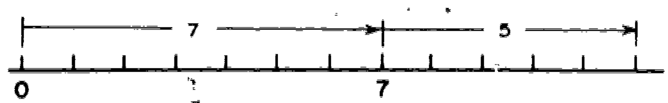
Chapter 7

PROPERTIES OF ADDITION

7-1. Addition of Real Numbers

You have known for a long time what it means to add numbers of arithmetic; that is, non-negative real numbers. Now we are dealing with the set of real numbers, of which the set of numbers of arithmetic is a proper subset. Your experience in adding non-negative numbers, both in arithmetic and on the number line, should give you clues about the addition of real numbers.

Let us consider the daily profits and losses of an imaginary ice cream vendor during his 10 days in business. These profits and losses are summarized in the left-hand column of Panel 7-1, which you find on the response sheet for this section.

1	We could use positive numbers to represent gains. Since a loss is the opposite of a gain, we would use _______ numbers to represent losses.	negative
2	During his first two days of business, the ice cream vendor recorded profits of \$7 and \$_____. See Panel 7-1.	\$5
3	For this period, a sentence expressing the net gain in dollars is: ___ + ___ = ___.	7 + 5 = 12
4	He had a net gain of \$_____.	\$12
5	To show on the number line a gain of \$7 followed by a gain of \$5, we use our usual idea of addition on the number line. We first draw an arrow of length 7, starting at 0 and extending to the _________ (right, left).	right
6	We then draw a second arrow of length _______, starting at 7 and also extending to the right. Thus we have	5
		
7	The second arrow ends at the point _______.	12

Look at the first entries in Column 2 and in Column 3 of Panel 7-1. Note how they exhibit your responses to Items 5, 6 and 7.

During the next two days of business, the vendor
8 recorded a profit of $6 (on Wednesday) and a ______ loss
of $4 (on Thursday).

During this two-day period he made in all a
9 net ______ (gain, loss). gain

10 In fact, his net gain was ______. $2

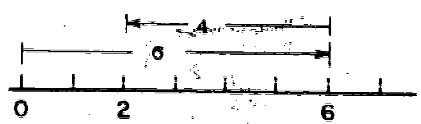
11 A loss of $4 is represented by the number ______. -4
Moreover, the net income for two days is the sum of each day's income. Hence, it is reasonable to write:

12 $6 + (\quad) = \_\_\_$. $6 + (-4) = 2$

Note to the student: The correct responses to certain items are underlined with a wavy line: ~~~~~~~~~. This wavy line is a signal to you that you are to record your response in the appropriate space on the panel.

Consider the number line illustration of these two days of business.

13 To show a gain of $6, we draw an arrow of length ______, 6
14 starting at ______ and extending to the right. Then, 0
to indicate a loss of $4, we draw a second arrow of
15 length ______, starting at 6 and extending to the 4
left. We have:



(Enter this diagram in the panel)

16 The second arrow ends at the point ______. 2

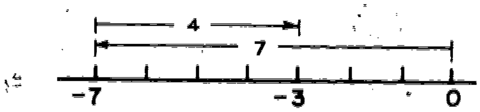
For the next two-day period (Friday and Saturday) the
17 vendor reported a loss of $______ and then a profit $7
of $4.

18 For these two days, he had a net ______ (gain, loss) of $3. loss

19 We thus write the sentence: ______. $(-7) + 4 = -3$

20 To illustrate on the number line the results of Friday's and Saturday's business, we first show the $7 loss by an arrow of length 7 which starts at 0 and extends to the ______. | left

We then complete the drawing. It looks like this:



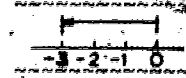
(Enter in the panel)

Can you complete the rest of Panel 7-1? If you can, do so and skip Items 21-24. If not, review Items 8-20 and continue.

On the next line of the panel, referring to Sunday and Monday, you should enter, in Column 2, the sentence:

21 ______ | $0 + (-3) = -3$

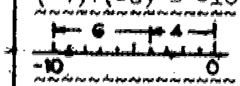
22 Draw the diagram for this line of the panel.



Corresponding to the net income for the last two days,

23 we write ______. | $(-4)+(-6) = -10$

24 Draw the appropriate number line diagram.



Now that you have finished Panel 7-1, compare your results with the completed panel on page v. You will see that these accounts illustrate almost every possible sum of real numbers: a positive plus a positive, a positive plus a negative, a negative plus a positive, 0 plus a negative, and a negative plus a negative.

Notice how our experience with adding non-negative numbers has given us a clue as to how to add real numbers.

We were able, just as before, to illustrate addition on the number line. We used arrows extending to the right

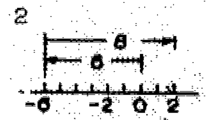
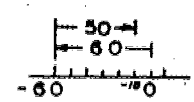
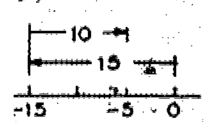
25 to represent ______ numbers, while negative numbers | positive

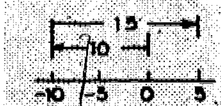
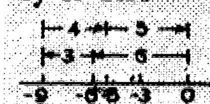
26 were represented by arrows extending to the ______. | left

The number line illustrations agreed with our common sense ideas about combining gains and losses.

Item	Frame	Answer
27	We used positive numbers to represent _______ (gains, losses),	gains
28	and we represented losses by _______ numbers.	negative
	To add two real numbers a and b on the number line, we first go to the point corresponding to a. We move	
29	from a to the right if b is _______. If b is	positive
30	negative, we move to the _______.	left
31	No motion from point a occurs if b is _______.	0
	Item 31 illustrates the fact that for any real number	
32	a, $a + 0 =$ _______.	a

Let us consider some further illustrations.

Item	Frame	Answer
	If a football team lost 6 yards on the first play and gained 8 yards on the second play, its net	
33	yardage gained for the two plays is _______ yards.	2
34	Draw the appropriate number line diagram.	
35	Write the appropriate numerical sentence.	$(-6) + 8 = 2$
	Suppose John paid Jim the 60 cents he owed, and then John collected the 50 cents that Al owed him. After	
36	these transactions, John had _______ cents less than the amount he started with.	10
37	Draw the appropriate number line diagram.	
38	Write the appropriate numerical sentence.	$(-60) + 50 = -10$
	At sunset on a wintry day the temperature is zero degrees. By midnight it has fallen to -15 degrees. Then, between midnight and sunrise, the temperature rises 10°. The temperature at sunrise	
39	is _______ degrees _______ (above, below) zero.	5, below
40	Draw the appropriate number line diagram.	
41	Write the appropriate numerical sentence.	$(-15) + 10 = -5$

	Suppose the temperature was zero degrees at sunset, dropped to -10° at midnight, and rose 15° by	
42	sunrise. The temperature at sunrise was ______	5
43	degrees ______ zero. (above, below)	above
44	Draw the appropriate number line diagram.	
45	Write the appropriate numerical sentence.	$(-10) + 15 = 5$
	Miss Jones lost 6 pounds during the first week of her dieting, lost 3 pounds the second week, gained 4 pounds the third week, and gained 5 pounds the last	
46	week. Her net gain (or loss) was ______ pounds.	0, or zero
47	Draw the appropriate number line diagram.	
48	Write the appropriate numerical sentence.	$(-6)+(-3)+4+5=0$
	In performing the indicated operations on real numbers, thinking about the number line may help you.	
49	$(4 + (-6)) + (-5) =$ ______ $+ (-5)$	-2
50	$=$ ______	-7
51	$4 + ((-6) + (-5)) = 4 + ($ ______ $)$	(-11)
52	$=$ ______	-7
53	$-(4 + (-6)) = -($ ______ $)$	$-(-2)$
54	$=$ ______	2
55	$(-2) + 2 =$ ______	0
56	The sum of the number -2 and its ______, 2,	opposite
57	is ______.	0
58	$3 + ((-2) + 2) =$ ______	3
59	$2 + (0 + (-2)) = 2 + ($ ______ $)$	(-2)
60	$=$ ______	0
61	$((-3) + 0) + (-2.5) =$ ______	-5.5
62	$\lvert -2 \rvert + (-2) =$ ______, since $\lvert -2 \rvert = 2$.	0
63	$(-3) + (\lvert -3 \rvert + 5) =$ ______	5

64 To go from a to a + b on the number line we move b units to the right if b is positive. If b is negative, we move to the left. How far do we move if b is negative?

[A] We go b units to the left.

[B] We go -b units to the left.

The number of units is a number of arithmetic; that is, a non-negative number. If b is negative, then -b is positive. [B] is the correct choice.

We can summarize our process of adding real numbers: To find a + b on the number line,

start at point a, and

if b is positive, go b units to the right,
if b is negative, go -b units to the left,
if b is 0, go 0 units.

Let us see whether we have a notation that will state this more concisely.

When b is added to a on the number line, if b is non-zero, the direction from point a depends upon
65 whether b is positive or ______; for every b, the — **negative**
66 distance from point a is a non-______ number. — **non-negative**

We saw in Chapter 6 that for every real number n the
67 distance between n and 0 is ______. — **$|n|$**

68 We can use $|b|$ to describe the ______ we go from a in adding b. — **distance**

Let us verify that this says the same thing about b as the last three lines in the summary of the process of adding real numbers:

69 if b is positive, $|b| =$ ______, — **b**
70 if b is negative, $|b| =$ ______, — **-b**
71 if b is 0, $|b| =$ ______. — **0**

Now we can restate the process of adding real numbers in terms of absolute value: If a and b are any real numbers, then to find $a + b$ on the number line

start at point a, and

if b is non-negative, go $|b|$ units to the right;

if b is negative, go $|b|$ units to the left.

	To find $7 + (-10)$ on the number line, we begin at 7	
72	and move ______ units to the left.	$\|-10\|$
	To find $-10 + 7$ on the number line, we begin at -10	
73	and move $\|7\|$ units to the ______.	right
74	To find $-5 + 0$ on the number line we begin at ______	-5
75	and move ______ units.	0

76 Here are several numerals involving 0, 7, 10 and their opposites:

P. $7 + 10$ U. $(-7) + (-10)$

Q. $7 + (-10)$ V. $(-7) + 10$

R. $10 + (-7)$ W. $(-10) + 7$

S. $(-10) + (-7)$ X. $(-10) + 0$

T. $10 + 7$ Y. $0 + 7$

Which of the following statements is <u>not</u> true?

[A] The process of addition in P, T, and Y is exactly the same as for the numbers of arithmetic.

[B] In S, U, and X you can do the addition by finding the sum of two numbers of arithmetic and then taking the opposite of that sum.

[C] The numbers represented in Q, R, V, and W are negative.

The numbers represented by Q and W are negative, but those represented by R and V are positive, hence, [C] is the only false statement. [A] is true. Notice that [B] is also true. For example, $(-10) + (-7)$ and $-(10 + 7)$ both name the number -17. We shall return to this point.

In this section we have illustrated the addition of real numbers on the number line. Now we are ready for an algebraic definition of real numbers.

7-2. <u>Definition</u> <u>of</u> <u>Addition</u>

Use these problems to test yourself.

1 $(-7) + 4 =$ ______ -3

2 $(-6) + 0 =$ ______ -6

3 $8 + (-2) =$ ______ 6

4 $(-8) + (-7) =$ ______ -15

(If you missed any of Items 1-4, review Section 7-1, Items 63-75.)

In computing sums like these you can use the number line. You have learned:

- to find $a + b$ on the number line

  first, move to a on the number line

  second, from a, move

5 $|b|$ units to the <u>right</u> if b is ______, positive

6 $|b|$ units to the ______ if b is negative, left

0 units if b is 0.

For example, to find $5 + (-8)$ on the number line we

7 move to 5 and, from 5, move $|-\_\_\_|$ units to $|-8|$

8 the ______. left

In other words, you know how to translate the sum $a + b$ into addition on the number line when a and b are given. In this section we are going to restate the process of addition on the number line in algebraic language. When we have finished, we shall have an algebraic definition for the sum of two real numbers.

We want to express <u>distances</u> and <u>directions</u> on the number line in algebraic symbols. It will help you if you remember: The distance between a number and zero

9 is the absolute ______ of the number. value

10 Where is -4 on the number line? It is to the ______ left of 0.

11	The distance between -4 and 0 is $\lvert - \quad \rvert$.	$\lvert -4 \rvert$
12	We realize that $\lvert -4 \rvert$ = ______, and of course 4 is easier to write than $\lvert -4 \rvert$. However, writing $\lvert -4 \rvert$ reminds us that we are thinking of the distance between	4
13	-4 and ______.	0, or zero
14	In general, for any real number n, the distance between n and 0 is ______.	$\lvert n \rvert$

We are now ready to proceed with our task of stating in algebraic language precisely what we mean by $a + b$, when a and b are any real numbers.

First of all, if a and b are non-negative, then we know the meaning of $a + b$ from our experience with the numbers of arithmetic. Next, we examine some cases where both numbers are negative.

15	$(-4) + (-6)$ ______	-10
16	$(-2) + (-3)$ ______	-5
17	$(-6) + (-7)$ ______	-13
18	$(-7) + (-5)$ ______	-12
	The sentences above have certain things in common:	
19	Each has to do with the sum of two ______ numbers.	negative
20	In each, the result is a ______ number.	negative
21	In each, we can find the sum by first adding two numbers of arithmetic and then taking the ______ of the result.	opposite
	When you were performing the indicated addition $(-4) + (-6)$ you added the numbers of arithmetic 4 and 6 and took the opposite of the sum. 4 is the absolute value of -4; likewise,	
22	______ $= \lvert -6 \rvert$.	6
	You found that $(-4) + (-6) = -(\lvert -4 \rvert + \lvert -6 \rvert)$. Since $\lvert -2 \rvert = 2$ and $\lvert -3 \rvert = 3$,	
23	$(-2) + (-3) = -(\lvert \quad \rvert + \lvert \quad \rvert)$	$-(\lvert -2 \rvert + \lvert -3 \rvert)$

Can you see a general rule for adding two <u>negative</u> numbers? Try to state it in words for yourself, then compare your statement with this one:

> The sum of two negative numbers is negative. The absolute value of this sum is the sum of the absolute values of the numbers.

	An algebraic statement of this fact is: If $a < 0$ and $b < 0$, then	
24	$a + b = -(\lvert a\rvert + \underline{\quad})$.	$a+b = -(\lvert a\rvert + \lvert b\rvert)$

Now go to Panel 7-2 on the response sheet. Part 1 has already been completed; you are now ready to complete Part 2.

25	If $a < 0$ and $b < 0$, then ______.	$a+b = -(\lvert a\rvert + \lvert b\rvert)$ [A wavy line means to enter response on the panel.]
	In Items 26-34, use the definitions in Panel 7-2 to find a common name for each sum:	
26	$(-2) + (-7)$ $\quad -(\lvert -\underline{\quad}\rvert + \lvert -7\rvert)$ $-(2 + 7)$	$-(\lvert -2\rvert + \lvert -7\rvert)$
27	______	-9
28	$(-4.6) + (-1.4)$ $\quad -(\lvert -\underline{\quad}\rvert + \lvert -1.4\rvert)$ $-(4.6 + 1.4)$	$-(\lvert -4.6\rvert + \lvert -1.4\rvert)$
29	______	-6
30	$(-2[illegible]) + (-7[illegible])$ $\quad -(\lvert -2[illegible]\rvert + \lvert -7[illegible]\rvert)$ ______	-98
31	$(-\frac{10}{[illegible]}) + (-\frac{8}{[illegible]})$ ______	-6
32	$\frac{11}{2} + \frac{5}{2} =$ ______. [Look at Part 1 of the definition on Panel 7-2.]	8
33	$\lvert -7\rvert + \lvert -6\rvert$ ______	13
34	$\lvert -7\rvert$ and $\lvert -6\rvert$ ______ (are, are not) numbers of arithmetic.	are

35 All of the following statements are true. Which one best illustrates Part 2 of Panel 7-2?

[A] $(-6) + (-4) = -(6 + 4)$

[B] $(-6) + 4 = -2$

[C] $(-6) + (-4) = -(|-6| + |-4|)$

[C] is the best illustration. Note that [B], though a true statement, cannot illustrate Part 2, which begins, "If $a < 0$ and $b < 0$". [A] is really a consequence of [C].

Now that we have completed and discussed Part 2 of Panel 7-2, let us go on to consider the sum of two real numbers when one number is non-negative and the other is negative. In Part 3 of the panel we have, "If $a \geq 0$ and $b < 0$", while in Part 4 we have, "If $b \geq 0$ and $a < 0$".

Suppose one of the numbers is 0 and the other number is negative. Using the number line, we can see that.

36 $0 + (-7) =$ ______ -7

37 $0 + b =$ ______, for any negative number b. b

38 $(-\frac{8}{3}) + 0 =$ ______ $-\frac{8}{3}$

39 $a + 0 =$ ______, for any negative number a. a

(Of course, $0 + b = b$ and $a + 0 = a$ are also true if a and b are non-negative; but these cases are included in Part 1.)

Consequently, in considering Parts 3 and 4 of Panel 7-2, we may concentrate our attention on those cases where either a or b is negative and the other is positive.

Consider the following examples of profits and losses.

40 Profit of \$7 and loss of \$3: $7 + (-3) =$ ______ 4

41 Profit of \$3 and loss of \$7: $3 + (-7) =$ ______ -4

42 Loss of \$7 and profit of \$3: $(-7) + 3 =$ ______ -4

43 Loss of \$3 and profit of \$7: $(-3) + 7 =$ ______ 4

The four examples above suggest the following statements:

Each has to do with the sum of two numbers, one
44 positive and one ______. negative

45 In each, the absolute value of the result is ______. 4

In each, we find the absolute value of the result by
46 subtracting 3 from ______. 7

In fact, in each the absolute value of the sum is the <u>difference</u> of the absolute values of two numbers.

The sum is positive if the positive number has the
47 greater ______ value. absolute

The sum is negative if the negative number has the
48 ______ absolute value. greater

Find the sum of the following pairs of real numbers. Notice that each pair includes one positive and one negative number. *If you have difficulty, you may wish to refer to Items 40-48, where you will find examples and suggestions.*

49	$(-4) + 6$	______	2
50	$6 + (-4)$	______	2
51	$(-6) + 4$	______	-2
52	$4 + (-6)$	______	-2
53	$(-6) + 6$	______	0
54	$4 + (-4)$	______	0
55	$(-8) + 5$	______	-3
56	$5 + (-3)$	______	2
57	$\frac{1}{4} + (-\frac{1}{2})$	______	$-\frac{1}{4}$
58	$(-4.7) + (3.2)$	______	-1.5
59	$18 + (-14)$	______	4
60	$(-\frac{2}{3}) + 5$	______	$\frac{13}{3}$
61	$3 + (-8)$	______	-5
62	$(-13) + 2$	______	-11

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Now that we have gained some skill in adding two real numbers, we shall think about how to complete Panel 7-2. This panel, when completed, will provide a precise statement of the definition of addition for real numbers.

63	$7 + (-3) =$ ______	4
64	$7 > 0$ and (-3) ______ 0. $(>, <)$	$(-3) < 0$
65	$\lvert 7\rvert$ ______ $\lvert -3\rvert$ $(>, <)$	$\lvert 7\rvert > \lvert -3\rvert$
66	The sum $7 + (-3)$ has the form $a + b$, where ______ > 0 and ______ < 0, and $\lvert a\rvert \geq \lvert b\rvert$. (a, b) (a, b)	$a > 0$ and $b < 0$
67	In general, if $a > 0$ and $b < 0$, and if $\lvert a\rvert \geq \lvert b\rvert$, then $a + b = \lvert a\rvert - \lvert\ \rvert$.	$a + b = \lvert a\rvert - \lvert b\rvert$

Notice that $|a|$ and $|b|$ are numbers of arithmetic. If $|a| \geq |b|$, we are sure from our knowledge about subtraction for the numbers of arithmetic that $|a| - |b|$ names exactly one number of arithmetic.

You could determine the following sums at a glance. Let us, however, use the definition in Item 67 in order to convince ourselves that the definition gives the results that we feel sure are correct.

$18 + (-14) = |18| - |-14|$

$= 18 - 14$

68	$=$ ______	4
69	$8 + (-3) = \lvert 8\rvert - \lvert\ \ \rvert$	$\lvert 8\rvert - \lvert -3\rvert$
70	$=$ ____ $-$ ____	$8 - 3$
71	$=$ ______	5
72	$\frac{1}{2} + (-\frac{1}{4}) = \lvert\ \ \rvert - \lvert\ \ \rvert$	$\lvert\frac{1}{2}\rvert - \lvert -\frac{1}{4}\rvert$
73	$=$ ____ $- \frac{1}{4}$	$\frac{1}{2} - \frac{1}{4}$
74	$=$ ______	$\frac{1}{4}$
75	$6 + (-4) =$ ____ $-$ ____	$\lvert 6\rvert - \lvert -4\rvert$
76	$=$ ____ $-$ ____	$6 - 4$
77	$=$ ______	2

We have a definition of addition which we can use when we add a negative to a non-negative number in cases such as $7 + (-3)$, where the non-negative number has the greater absolute value.

Now suppose that the negative number has the greater absolute value.

78	$3 + (-7) =$ _____.	-4
79	Similarly, $5 + (-11) =$ _____	-6
80	and $2 + (-3) =$ _____.	-1
81	In these examples the sum is _____ (positive, negative) and	**negative**
82	its absolute value is equal to the _____ (sum, difference) of the absolute values of the numbers being added.	**difference**
	Hence, Part 3 in Panel 7-2 should read:	
	If $a \geq 0$ and $b < 0$, and	
83	(a) if $\|a\| \geq \|b\|$, then $a + b =$ _____.	$a + b = \|a\| - \|b\|$
84	(b) if $\|b\| > \|a\|$, then $a + b =$ _____.	$a+b=-(\|b\| - \|a\|)$

By now we have a pattern for completing the definition in the panel.

	Part 4, in the panel, should read:	
	If $b \geq 0$ and $a < 0$, and	
85	(a) if $\|b\| \geq \|a\|$, then $a + b =$ _____.	$a+b = \|b\| - \|a\|$
86	(b) if $\|a\| > \|b\|$, then $a + b =$ _____.	$a+b=-(\|a\| - \|b\|)$

Compare your completed panel with the one below. Your panel should be exactly like this one.

PANEL 7-2

Definition of Addition for Real Numbers

Part 1. If $a \geq 0$ and $b \geq 0$, then a and b are numbers of arithmetic and $a + b$ has its usual meaning.

Part 2. If $a < 0$ and $b < 0$, then $a + b = -(|a| + |b|)$

Part 3. If $a \geq 0$ and $b < 0$, and

(a) if $|a| \geq |b|$, then $a + b = |a| - |b|$

(b) if $|b| > |a|$, then $a + b = -(|b| - |a|)$

Part 4. If $b \geq 0$ and $a < 0$, and

(a) if $|b| \geq |a|$, then $a + b = |b| - |a|$

(b) if $|a| > |b|$, then $a + b = -(|a| - |b|)$

You are not expected to memorize this definition. Given two real numbers, you should be able to add them without referring to the definition. On the other hand, we hope you understand what has been accomplished in this definition of addition. If a and b are *any* real numbers, then we can compute $a + b$ by applying the appropriate part of Panel 7-2. In order to do it you need know only:

1) how to find the sum of two numbers of arithmetic, for Parts 1 and 2;

2) how to subtract one number of arithmetic from another which is at least as large, for Parts 3 and 4;

3) how to form the absolute value and the opposite of a real number.

	Let us summarize Parts 3(a), 3(b), 4(a), 4(b) of the definition. The sum of a negative and a non-negative number:	
87	is equal in absolute value to the ________ (sum, difference) of the absolute values;	difference
88	is non-negative if the non-negative number has the ________ (greater, lesser) absolute value;	greater
89	is negative if the negative number has the greater ______ ______.	absolute value

If a and b are particular real numbers, we can find exactly one part of Panel 7-2 that applies. From the appropriate part, we find exactly one number $a + b$. Thus, we have succeeded in giving a complete definition of the binary operation of addition in the set of real numbers.

90	Since the sum of any two real numbers is a unique _____ number, the set of all real numbers is closed under the	real
91	operation of ______.	addition

	For each of the following open sentences, find a real number which will make the sentence true. If you have trouble, you may find it helpful to think about the number line.	
92	$x + 2 = 7$ ______	5
93	$3 + y = -7$ ______	-10
94	$8 = n + 10$ ______	-2

95	$a + 5 = 0$	______	-5
96	$b + (-7) = 3$	______	10
97	$(-\frac{5}{6}) + x = -\frac{5}{6}$	______	0
98	$c + (-3) = -7$	______	-4
99	$7 = (-5) + a$	______	12
100	$y + \frac{2}{3} = -\frac{5}{6}$	______	$-\frac{3}{2}$
101	$\frac{3}{4} = \frac{1}{2} + m$	______	$\frac{1}{4}$
102	$\frac{x}{2} + (-4) = 6$	______	20
103	$\left(y + (-2)\right) + 2 = 3$	______	3
104	$(3 + x) + (-3) = -1$	______	-1

Bill earned 40¢ on Tuesday, spent 60¢ on Wednesday, but doesn't remember how much he had on Monday. He had 30¢ left Wednesday night. To determine what amount he had on Monday, we can write an open sentence with the variable x representing the money he had Monday. The open sentence is:

105 $x + 40 + (-60) =$ ______. — 30

106 Bill began with ______ ¢ on Monday. — 50

107, 108 If you drive 40 miles north and then 55 miles south, you are then ______ (how many) miles ______ (north, south) of your starting point. — 15, south

For each sentence below, tell whether it is true or is false. After you have recorded all of your responses, turn to page v and check your answers.

109. $(-4) + 0 = 4$
110. $-(|-1.5| - |0|) = -1.5$
111. $(-3) + 5 = 5 + (-3)$
112. $\left(4 + (-6)\right) + 6 = 4 + \left((-6) + 6\right)$
113. $(-5) + \left(-(-5)\right) = -10$
114. $(-7) + \left((-5) + (-3)\right) = \left((-7) + (-5)\right) + 3$
115. $-\left(6 + (-2)\right) = (-6) + (-2)$
116. $(-7) + (-9) = -(7 + 9)$
117. $(-3) + 7 = -\left(3 + (-7)\right)$

Look at Items 111, 112, and 116, and see if you can guess some of the ideas that will be discussed in the following sections.

7-3. Properties of Addition

In Chapters 2 and 4 we described in detail and then listed the properties of addition for the numbers of arithmetic.

We now have a definition for addition that we can use to add any two real numbers. (Remember, the set of real numbers includes the negative numbers as well as the numbers of arithmetic.)

We want to see whether the properties of addition already discussed for the numbers of arithmetic are also true in the set of real numbers. Further, we want to see whether there are any new properties of addition for this set.

Complete the following statements. They will help you to recall the properties of addition we have studied.

1 $7 + 3 = 3 + 7$ is a true sentence that illustrates the _______ property of addition. **commutative**

2 $6 + 0 = 6$ is a true sentence that illustrates the _______ property of _______. **addition, 0**

3 The true sentence

$$(7 + 5) + 4 = 7 + (5 + 4)$$

illustrates the _______ property of addition. **associative**

We have seen that the properties of addition can often be used to simplify computation. Here is a detailed illustration. Tell which property of addition is used in each step.

4 $(\frac{3}{5} + \frac{7}{8}) + \frac{2}{5} = \frac{3}{5} + (\frac{7}{8} + \frac{2}{5})$ _______ property **associative**

5 $= \frac{3}{5} + (\frac{2}{5} + \frac{7}{8})$ _______ property **commutative**

6 $= (\frac{3}{5} + \frac{2}{5}) + \frac{7}{8}$ _______ property **associative**

Hence,

7 $(\frac{3}{5} + \frac{2}{5}) + \frac{7}{8} =$ _______ $+ \frac{7}{8}$. **$1 + \frac{7}{8}$**

Often, we do not think through each step in such detail. We simply think:

$$(\frac{3}{5} + \frac{7}{8}) + \frac{2}{5} = (\frac{3}{5} + \frac{2}{5}) + \frac{7}{8}.$$

In other words, we have convinced ourselves, by many examples, that the associative and commutative properties of addition for the numbers of arithmetic permit us to regroup and reorder the numbers in a sum in any way that we like.

	Compute each sum, by any method that you like.	
8	$4 + (-3) =$ ______, and	1
9	$(-3) + 4 =$ ______.	1
10	$(-1) + 5 =$ ______, and	4
11	$5 + (-1) =$ ______.	4
12	$(-2) + (-6) =$ ______, and	-8
13	$(-6) + (-2) =$ ______.	-8
14	$(-18) + 4 =$ ______, and	-14
15	$4 + (-18) =$ ______.	-14

It appears that when we find the sum of any two real numbers, the sum is the same regardless of the order in which we add.

This is the <u>Commutative Property of Addition</u>:

For any two real numbers a and b,

$$a + b = b + a.$$

	Since $(5 + (-3))$ and 7 are both real numbers, the commutative property of addition shows us that	
16	$(5 + (-3)) + 7 = 7 + ($______$)$.	$(5 + (-3))$
	If x is any real number, then $x + (-3) = (-3) + x$	
17	would follow from the ______ property of addition.	commutative

Let us now consider whether or not, in a sum involving three real numbers, we can group (or associate) the numbers to suit our convenience.

We know that if a, b, and c are any numbers of arithmetic, then $(a + b) + c = a + (b + c)$. This is the associative property of addition.

Now let us see what happens when one or more of the numbers a, b, c is negative.

Frame		Answer
18	$(7 + (-9)) + 3 = (\_\_\_) + 3$	$(-2) + 3$
19	$= \_\_\_\_$ (common name)	1
20	$7 + ((-9) + 3) = 7 + (\_\_\_)$	$7 + (-6)$
21	$= \_\_\_\_$	1
22	Are $(7 + (-9)) + 3$ and $7 + ((-9) + 3)$ names for the same number? ____	yes
23	$((-8) + 5) + (-4) = \_\_\_ + \_\_\_$	$(-3) + (-4)$
24	$= \_\_\_\_$	-7
25	$(-8) + (5 + (-4)) = \_\_\_ + \_\_\_$	$(-8) + 1$
26	$= \_\_\_\_$	-7
27	$((-8) + 5) + (-4)$ and $(-8) + (5 + (-4))$ ____ (are, are not) names for the same number.	are
28	$(4 + 5) + (-6) = 4 + (5 + (-6))$ ____ (is, is not) a true sentence.	is

We have seen that the following sentences are all true:

Frame		Answer
29	$(7 + (-9)) + 3 = 7 + (\_\_\_ + \_\_\_)$	$7 + ((-9) + 3)$
30	$((-8) + 5) + (-4) = (-8) + (\_\_\_ + \_\_\_)$	$(-8) + (5 + (-4))$
31	$(4 + 5) + (-6) = \_\_\_ + (\_\_\_\_)$	$4 + (5 + (-6))$

We could look at many examples of sums of three real numbers involving different combinations of negative and non-negative numbers. Such examples would indicate that addition is associative for all real numbers.

This is the <u>Associative Property of Addition</u>:

For all real numbers a, b, and c,

$$(a + b) + c = a + (b + c).$$

Of course, the fact that the associative and commutative properties hold true in several instances is not enough to guarantee that they hold true in every instance. In mathematics this guarantee is achieved by giving a <u>proof</u> that the properties always hold. The commutative and associative properties could be proved to hold for the <u>real numbers</u> as a consequence of two things: the way in which we defined addition for real numbers, and the familiar properties of the <u>numbers of arithmetic</u>. However, because these proofs are

very detailed and very lengthy, they will be omitted here. At this point, we shall simply take for granted that the properties in question do hold in the set of real numbers.

Very soon--in Section 7-5--you will begin to learn how we can <u>prove</u> that certain relationships must follow from others. As you become familiar with the idea of proof, you will gain a better understanding of the real number system and of how its properties are related to those of the numbers of arithmetic.

	Let us now investigate another property of addition.	
32	$4 + (-4) =$ ______	0
33	We note that -4 is the ______ of 4.	opposite
	$-(-4)$ is the opposite of (-4).	
34	$(-4) + \left(-(-4)\right) =$ ______	0
35	What is the opposite of 5? ______	-5
36	$5 + (-5) =$ ______	0
37	$\frac{3}{2} + (-\frac{3}{2}) =$ ______	0
38	$(-12) + \left(-(-12)\right) =$ ______	0
	The preceding responses indicate that the following statement is true: "The sum of a number and	
39	its ______ is zero."	opposite
40	$0 +$ ______ $= 0$	0
41	$8 +$ ______ $= 0$	(-8)
42	$\frac{3}{5} +$ ______ $= 0$	$(-\frac{3}{5})$
43	______ $+ \left(-(-1)\right) = 0$	(-1)

The preceding examples lead us to state the <u>Addition Property of Opposites</u>:

For every real number a,

$$a + (-a) = 0 \quad \text{and} \quad (-a) + a = 0.$$

The addition property of opposites is of special interest because it is a property of <u>real numbers</u> but it is not a property of <u>numbers of arithmetic</u>. When we consider the set of numbers of arithmetic, the idea of opposite does not occur.

As in the case of the associative and commutative properties of addition, the addition property of opposites for <u>real numbers</u> follows from our definition of addition and from properties of the <u>numbers of arithmetic</u>.

	Let us now investigate still another property of addition.	
	Refer to the definition of addition to complete the following:	
44	$(-5) + 0$ ______	-5
45	______ $+ 0 = 4$	4
46	$(-9) +$ ______ $= -9$	0
47	$0 + (-7) =$ ______	-7
48	______ $+ 4 = 4$	0
	We observe: adding 0 to any number gives, as the sum, that same number. Likewise, adding any number to 0 gives the number.	

We may state our conclusion as the <u>Addition Property of 0</u>:

For every real number a,

$$a + 0 = a \quad \text{and} \quad 0 + a = a.$$

This property for <u>real numbers</u> follows from our definition of addition and from the addition property of 0 for <u>numbers of arithmetic</u>.

	We can use the properties of addition to simplify phrases and sentences.	
	In order to simplify the phrase $\left(7 + (-7)\right) + 6$, we note:	
49	$7 + (-7) =$ ______ because of the addition property	0
50	of ______.	opposites
51	$0 + 6 =$ ______ because of the addition property	6
52	of ______.	0
53	Thus, we see that $\left(7 + (-7)\right) + 6 =$ ______.	6

Here is another example:

54 $3 + \left((-3) + 4\right) = (\underline{\qquad}) + \underline{\quad}$ by the associative property of addition — $\left(3 + (-3)\right) + 4$

55 $= 0 + 4$ by the addition property of _____ — opposites

56 $= 4$ by the addition property of _____ — 0

57 Hence, $3 + \left((-3) + 4\right) = \underline{\qquad}$. — 4

58 $5 + \left(3 + (-5)\right) = 5 + \left((-5) + 3\right)$ by the _____ property of addition — commutative

59 $= \left(5 + (-5)\right) + 3$ by the _____ property of addition — associative

60* $= 0 + 3$ by the _____ property of opposites — addition

61 $= 3$ by the _____ property of 0 — addition

62 Thus, $5 + \left(3 + (-5)\right) = \underline{\qquad}$. — 3

Consider the open sentence

$$x = x + \left((-x) + 3\right)$$

Let us look at the right side of the equation. We know that if x is *any* real number, then

63 $x + \left((-x) + 3\right) = \left(x + (-x)\right) + 3$ by the _____ property of addition — associative

64 $= \underline{\qquad} + 3$ — 0

65 by the addition property of _____ — opposites

66 $= 3$ by the addition property of _____ — 0

67 We have found: If x is *any* real number, then a simpler name for $\left(x + (-x) + 3\right)$ is _____. Now suppose that the open sentence — 3

$$x = x + \left((-x) + 3\right)$$

is true for some x. Then the sentence

68 $x = \underline{\qquad}$ — $x = 3$

is true for this same x, since 3 is another name for $x + \left((-x) + 3\right)$.

69	But the truth set of the sentence $x = 3$ is seen to be the set containing only the number ______.	3
70	We can easily verify that 3 is also an element of the truth set of the original equation. Hence, the truth set of $x = x + \left((-x) + 3\right)$ is ______.	$\{3\}$

	Consider the open sentence: $m + \left(7 + (-m)\right) = m$. We can apply some of the properties of addition to obtain the simpler sentence:	
71	______ $= m$	$7 = m$
72	What is the truth set of $m + \left(7 + (-m)\right) = m$?	$\{7\}$

	Find the truth set of each sentence:	
73	$(-7) + (-4) = (y + 4) + (-4)$ ______	$\{-11\}$
74	$n + (n + 2) + (-n) = 0$ ______	$\{-2\}$

The associative property of addition guarantees that

$$\left(4 + (-1)\right) + 3 = 4 + \left((-1) + 3\right).$$

Thus, we can write the sum

$$4 + (-1) + 3$$

without indicating how the numbers are to be grouped for addition, since the way they are grouped does not affect the final sum.

The commutative property of addition enables us to change the order of adding two numbers if we wish.

	In computing $4 + (-1) + 3$, we may think, if we like: $4 + (-1) + 3 = 4 + 3 + (-1)$,	
75	since $(-1) + 3 = 3 + (-1)$ by the ______ property of addition.	commutative
	$4 + (-1) + 3 = 4 + 3 + (-1)$ $= 7 + (-1)$	
76	$=$ ______	6

As the previous example suggests, the associative and commutative properties of addition, taken together, mean that the sum of several numbers is independent both of the order in which the numbers are written and of the way in which they are grouped.

This general conclusion is exactly like the one we have already used in many examples with the numbers of arithmetic. Just as we found for the numbers of arithmetic that reordering and regrouping numbers in a sum helped in computing, so we find for real numbers that reordering and regrouping is often helpful.

In each of the following items, consider various ways of doing the computation and then perform the addition in the easiest way:

77 $\frac{5}{16} + 28 + (-\frac{5}{16}) =$ ______ 28

78 $0.27 + (-18) + .73 + 3 =$ ______ -14

79 $(-\frac{1}{2}) + 7 + (-2) + (-\frac{3}{2}) + 2 =$ ______ 5

80 $w + (w + 2) + (-w) + 1 + (-3) =$ ______ w

81 $|-\frac{3}{2}| + \frac{5}{2} + (-7) + |-4| =$ ______ 1

82 $253 + (-67) + (-82) + (-133) =$ ______ -29

Find the truth set of each open sentence.

83 $x + 5 + (-x) = 12 + x + (-12)$ ______ {5}

84 $n + (n + 2) + (-n) + 1 + (-3) = 0$ ______ {0}

7-4. Addition and Equality

Consider the true numerical sentence

$$6 + 2 = 8.$$

In this sentence, the symbol "=" signifies that $6 + 2$ and 8 are names for the same number. Suppose we add -2 to this number. Then

$$6 + 2 + (-2) \text{ and } 8 + (-2)$$

are again names for one number, which has the common name 6.

Here is another example.

1 $4 + (-5) = (-1)$ is a $\underset{\text{(true, false)}}{\underline{\hspace{6em}}}$ sentence. true

2 $4 + (-5)$ and (-1) are two names for the same ______. number

Let us add 3 to this number. Then the phrases $4 + (-5) + 3$ and $(-1) + 3$ are again names for

3 one number, whose common name is ______. 2

Hence, we may write the true sentence

4 $$4 + (-5) + 3 = (-1) + \underline{\hspace{4em}}.$$ 3

In a similar way,

$$4 + (-5) + 5 = (-1) + 5$$
$$4 + (-5) + (-2) = (-1) + (-2)$$
$$4 + (-5) + \frac{1}{2} = (-1) + \frac{1}{2}$$

5 are all $\underset{\text{(true, false)}}{\underline{\hspace{6em}}}$ sentences. true

In fact, if c is any real number, we can conclude

6 that $4 + (-5) + c = (-1) +$ ______ is a true sentence, c

since we began with the true sentence

$$4 + (-5) = (-1).$$

Let us generalize from these examples.

Suppose a and b are names for the same real number.

7 Then $a + 4$ and $\underline{b + \hspace{3em}}$ are also names for one b + 4

number. Hence, if $a = b$, then

8 $$a + 4 \underset{(<, =, >)}{\underline{\hspace{4em}}} b + 4.$$ =

In fact, if c is any real number, we can conclude from $a = b$ that

9 $$\underline{a + \hspace{3em}} = b + c.$$ a + c

We restate this as follows:

For any real numbers a, b, and c,

if $a = b$

10 then $\underline{a + \hspace{3em} = b + \hspace{3em}}$. a + c = b + c

Similarly, for any real numbers a, b, and c

if $a = b$

then $c + a = c + b$.

This generalization is an important fact about addition. It is a consequence of the fact that if a and c are two numbers, there is exactly one number which is their sum. This number does not depend on the particular numeral we choose as a name for a. (Remember "$a = b$" means "a and b name the same number".)

We shall use this generalization so often that it is useful to have a special name for it. We will call it the <u>addition property of equality</u>.

For any real numbers a, b, and c,

$$\text{if } a = b$$
$$\text{then } a + c = b + c \text{ and } c + a = c + b.$$

11	Which of the following expresses the idea of the addition property of equality? [A] If a and b are names of numbers, then $a + c$ and $b + c$ are also names of numbers. [B] If a and b name the same number, then the number named by $a + c$ is the same as the number named by $b + c$.

[B] is the correct choice. Although [A] is a correct sentence, it is not the generalization which we have called the addition property of equality.

12	If $x = y$, then $x + a = y +$ ______.	a
13	If $m =$ ______, then $m + p = n + p$.	n
14	If $v = w$, then $5 + v = 5 +$ ______.	w
15	If $m = 6$, then $m +$ ______ $= 8$.	2
16	If $x + 4 = 7$, then $(x + 4) + (-4) =$ ______ $+ (-4)$.	7

Item 16 leads us to wonder if the addition property of equality might be useful in finding truth sets of open sentences. Up to now we have found truth sets by intelligent "guesswork". Using the addition property of equality we can determine the truth sets of certain open sentences in a more systematic way.

Since we are dealing with a property of <u>equality</u>, we shall examine open sentences such as $x + 4 = 7$ where the verb symbol is "$=$". Open sentences having this verb form are often called <u>equations</u>. In the equation $x + 4 = 7$ we call $x + 4$ the <u>left side</u> and 7 the <u>right side</u>.

17	$x + \frac{3}{5} = (-2)$ ______ (is, is not) an equation.	is
18	The open sentence $2 + x = 5 + (-3)$ is an ______.	equation
19	$x < 5$ ______ (is, is not) an equation.	is not
20	$4 + (-7) \neq 5 + (-x)$ ______ (is, is not) an equation.	is not
21	An ______ is an open sentence that contains only the verb form "$=$".	equation

Let us return to the equation

$$x + 4 = 7.$$

Since this is an open sentence, it is neither true nor false as it stands. In order to find the truth set of this equation, we can reason as in the following items:

22	If (the "if" is important) there is some number, x, that makes $x + 4 = 7$ a true sentence, then that same number also makes $(x + 4) + (-4) = 7 + (\_\_\_\_)$ a true sentence.	(-4)
23	We are able to make this conclusion by using the ______ property of equality.	addition
	[Can you guess why we chose -4 as the number to add?]	
	Now we can proceed as follows:	
24	If $(x + 4) + (-4) = 7 + (-4)$ is true for some x then $x + \left(4 + (-4)\right) = 7 + (-4)$ is ______ for that same x,	true
25	and $x +$ ______ $= 3$ is true for that same x	0
26	and, finally ______ $= 3$ is true for that same x.	x
27	The truth set of $x = 3$ is obvious. It is ______.	{3}
28	Hence, <u>if</u> there is some number x which makes $x + 4 = 7$ true, this number must be ______.	3
	<u>Does</u> 3 make the original sentence true?	
29	Yes, since ______ $+ 4 = 7$.	$3 + 4 = 7$

We have shown that <u>if</u> there is a number making $x + 4 = 7$ true, then the <u>only</u> number which x can be is 3. After we check and find that 3 <u>does</u> make the sentence true, we have found the one and only number which belongs to the truth set.

Let us simplify our language a bit by agreeing to the following:

Instead of "determine the truth set of the open sentence" we shall often say "<u>solve</u> the equation".

Instead of "truth set" we may say "<u>solution set</u>".

Instead of "a member of the truth set of the open sentence" we shall often say "a <u>solution</u> of the equation".

30	The solution set of $x + 4 = 7$ is ______.	$\{3\}$
31	The only solution of $x + 4 = 7$ is ______.	3
32	3 is a ______ of the equation $x + 4 = 7$.	solution
33	$\{3\}$ is the ______ set or truth set of $x + 4 = 7$.	solution set

In solving an equation, as in Items 22-29, we may use many different properties. The first one that was used here was the addition property of equality. We were very careful to say "If $x + 4 = 7$ is true for some x," etc. In our next example we shall continue to make this statement and at the same time we shall examine the use of the various properties.

	Solve the equation $5 + \frac{3}{2} = x + (-\frac{1}{2})$.	
	If $5 + \frac{3}{2} = x + (-\frac{1}{2})$ is true for some x,	
34	then $(5 + \frac{3}{2}) + \frac{1}{2} = \left(x + (-\frac{1}{2})\right) +$ ______ is true	$\frac{1}{2}$
35	for the same x, using the addition property of ______.	equality
	$5 + (\frac{3}{2} + \frac{1}{2}) = x + \left((-\frac{1}{2}) + \frac{1}{2}\right)$ is true for	
36	the same x, using the ______ property of addition.	associative
37	$5 + 2 = x +$ ______ is true for	0
38	the same x, using the ______ ______ of opposites.	addition property
	$7 = x$ is true for the same x,	
39	using $5 + 2 = 7$ and the ______ ______ of 0.	addition property

If x is 7, then in the original equation

the left side is $5 + \frac{3}{2}$

and $5 + \frac{3}{2} = \frac{13}{2}$.

40 The right side is $(\_\_\_\_) + (-\frac{1}{2})$ (7)

and $7 + (-\frac{1}{2}) = \frac{13}{2}$.

41 Hence, the solution set is $\{\_\_\_\_\}$. $\{7\}$

In solving $x + 4 = 7$ we began by adding -4 to each side of the equation. Of course, we could have added some other number, such as 5, to each side. If we had added 5 we would have obtained the equation $x + 9 = 12$. By adding -4 we obtain $x = 3$, in which the left side contains x alone. $x = 3$ has an obvious solution set.

In solving $5 + \frac{3}{2} = x + (-\frac{1}{2})$ we began by adding $\frac{1}{2}$ to each side of the equation, obtaining $7 = x$. The truth set of $7 = x$ is obviously $\{7\}$.

	In solving	we begin by adding to both sides	
	$x + 9 = -11$	(-9)	
42	$x + (-2) = 5$	____	2
43	$-3 + x = -1$	____	3
44	$5 = 6 + x$	____	(-6)

45 Consider the following example:

Solve the equation $x + \frac{3}{5} = (-2)$.

If, $x + \frac{3}{5} = (-2)$ is true for some x,

then $(x + \frac{3}{5}) + (-\frac{3}{5}) = (-2) + (-\frac{3}{5})$ is true for the same x,

$x + \left(\frac{3}{5} + (-\frac{3}{5})\right) = -\frac{13}{5}$ is true for the same x,

$x + 0 = -\frac{13}{5}$ is true for the same x.

y, $x = -\frac{13}{5}$ is true for the same x.

Check: If x is $-\frac{13}{5}$, then in the original equation

the left side is $(-\frac{13}{5}) + \frac{3}{5}$

and $(-\frac{13}{5}) + \frac{3}{5} = -2$

and the right side is -2.

Hence, the truth set of $x + \frac{3}{5} = (-2)$ is $\{-\frac{13}{5}\}$.

Which property was not used in the above example?

[A] Commutative property of addition

[B] Addition property of equality

[C] Associative property of addition

[D] Addition property of opposites

[E] Addition property of 0

All of the properties except [A] were used. Review each step in the example and note that the properties [B] to [E] are listed in the order in which they were used.

It is important in studying mathematics to learn how to organize written work. Well organized and neat work helps in two ways. First, the chance of making careless errors is reduced. Second, good organization is an aid to understanding and reasoning.

Examine Item 45. Notice that we have not stated the properties used to obtain each successive equation. In solving equations, it is not necessary to write down each reason but it is essential that you understand each reason and that you are prepared to explain each step.

The first line of your written work should begin: "If ... is true for some x", and the second line should begin: "then ...". By writing this we emphasize the logic of our argument. You may wish to omit writing "is true for the same x" at the end of each line.

As you gain experience, you may be able to do certain steps mentally. For example, you may wish to proceed from $(x + \frac{3}{5}) + (-\frac{3}{5}) = (-2) + (-\frac{3}{5})$ to $x = -\frac{13}{5}$ in one step.

Finally, in simple equations, the "checking" may be done mentally, but it is important to understand that the "check" is essential.

<u>Solve</u> and <u>check</u> each of the following. Organize and show your work on the response sheet in a form similar to that shown in Item 45. Since we wish to become familiar with the use of the addition property of equality, be sure to show each step that makes use of this property. Compare with pages v and vi.

46. $x + 5 = 13$

47. $(-6) + 7 = (-8) + x$ [Hint: Begin by finding a simple numeral for the left side.]

48. $(-1) + 2 + (-3) = 4 + x + (-5)$

49. $(x + 2) + x = (-3) + x$ [Hint: Nothing prevents us from adding $(-x)$ to both sides.]

Here are two easy examples of how the properties we have developed can help us to solve problems. For each we shall write a related open sentence, find its truth set, and then answer the question in the problem.

After Bill added six books to those on his desk, he had 24 books in all. How many did he have at first?

50 If b represents the number of books he had at first, then _______ represents the number on the desk after 6 are added. $b + 6$

51 A related open sentence is _______ = 24. $b + 6 = 24$

52 The truth set is _______. $\{18\}$

53 Bill had _______ books. 18

54 To verify: If Bill had 18 books on his desk and then added 6, he would have _______ books in all. 24

Two numbers differ by 2. The sum of the numbers is 10 more than the smaller number. What are the numbers?

55 If n represents the smaller number, then _______ represents the larger number. $n + 2$

56 Their sum can be represented by _______. $n + (n + 2)$

An open sentence which says that the sum of the two numbers is 10 more than the smaller number is

57 $n + (n + 2) =$ _______. $n+(n+2)=n+10$

	If $n + (n + 2) = n + 10$ is true for some n,	
58	then $(-n) + n + (n + 2) = (\underline{\quad\quad}) + n + 10$ is true for the same n	$(-n)$
59	and $0 + (n + 2) = \underline{\quad + \quad}$ is true for the same n,	$0 + 10$
60	$\underline{n + 2 = \quad\quad}$ is true for the same n.	$n + 2 = 10$
61	The truth set is ______.	$\{8\}$
	The smaller number is 8 and the other number	
62	is ______.	10
63	To verify: $10 - 8 =$ ______, and	2
64	$10 + 8$ is 10 more than ______.	8

In this problem we reasoned: <u>If</u> there is a number which fits the conditions of the problem, then that number is an element of the truth set of

$$n + (n + 2) = n + 10.$$

In solving the open sentence, we concluded: <u>If</u> there is a number in the truth set of the equation, then it is 8. Since we are interested in solving the word problem, we checked 8 <u>in the original problem</u>.

Suppose we try to solve the equation

$$(-2) + x + (-3) = x + (-\tfrac{5}{2}).$$

If $(-2) + x + (-3) = x + (-\frac{5}{2})$ is true for some x,

then $x + (-2) + (-3) = x + (-\frac{5}{2})$ is true for the same x,

and $x + (-5) = x + (-\frac{5}{2})$ is true for the same x.

$(-x) + x + (-5) = (-x) + x + (-\frac{5}{2})$ is true for the same x.

$(-5) = (-\frac{5}{2})$ is true for the same x.

Remember, our method depends on the assumption that there are numbers for which the given equation is true. But there is no real number x which will make this last sentence true. Therefore, our assumption is false and the solution set of the original sentence is $\emptyset$. We also express this by saying that the equation has no solution.

Here is a slightly different example.

Solve $(-2) + |x| = 5$.

If $(-2) + |x| = 5$ is true for some x,

65 then using the _____ property of _____ we have — addition; equality

66 $2 + (-2) + |x| =$ _____ $+ 5$. — 2

By using the appropriate properties of addition,

$|x| = 7$ is true for the same x.

67 The solution set of this last equation is _____. — $\{-7, 7\}$

Do both solutions of $|x| = 7$ make the original

68 sentence true? _____ — yes

69 Hence, the truth set of $(-2) + |x| = 5$ is _____. — $\{-7, 7\}$

Solve the following equations.

70 $|x| + (-3) = |-2| + 5$ — $\{-10, 10\}$

71 $(-\frac{3}{8}) + |x| = (-\frac{3}{4}) + (-1)$ (Careful!) — $\emptyset$

72 $x + (-3) = |-4| + (-3)$ — $\{4\}$

73 $(-\frac{4}{3}) + (x + \frac{1}{2}) = x + (x + \frac{1}{2})$ — $\{-\frac{4}{3}\}$

74 $(-1) + 4 = 3 + |x|$ — $\{0\}$

In this section we have stated the addition property of equality and have discussed briefly how this property together with properties already familiar to us may be used to find the truth sets ("solution sets") of certain open sentences. In the process of finding the truth sets of ("solving") these sentences ("equations") we have laid stress on the logic of replacing one open sentence with another until we obtain a sentence whose truth set is obvious. You will encounter this same pattern of procedure many more times. Later on we shall examine more carefully the need for checking whether or not the solutions of the final equation are also solutions of the original equation.

7-5. The Additive Inverse

We have pointed out that 0 is the identity element for addition. This is simply another way of expressing the addition property of 0: the sum of any number and 0 is equal to the number.

Suppose we have two numbers whose sum is 0. These two numbers are related in a special way.

1	What number when added to 3 yields the sum 0? ____	-3
2	What number when added to -4 yields the sum 0? ____	4
	In general, if x and y are real numbers and if $x + y = 0$, we say that y is an <u>additive inverse</u> of x and that x is an additive inverse of y.	
3	Since $(-2) + 2 = 0$, 2 is an ______ inverse of -2.	**additive**
4	$(-\frac{1}{2})$ is an ______ ______ of $\frac{1}{2}$.	**additive inverse**
5	If y is an additive inverse of x, then $x + y =$ ____.	0
6	An additive inverse of $(-\frac{5}{8})$ is ______.	$\frac{5}{8}$
7	An additive inverse of 0 is ______.	0
8	If t is an additive inverse of s, is it also true that s is an additive inverse of t? ______ (yes, no)	yes
9	Given two real numbers, each is the additive inverse of the other if their sum is ______.	0

10 Consider the following pairs of real numbers. Which pairs are pairs of additive inverses?

-7 and $|7|$

$\frac{4}{3}$ and $-\frac{4}{3}$

0 and 0

$4 + (-5)$ and $(-4) + 5$

(-5) and $-(-5)$

[A] all but one [B] all [C] all but two

Since for each pair we can verify that the sum is 0, [B] is the correct choice. If you answered incorrectly, convince yourself that the sum of each pair is 0.

Perhaps you have noticed that we have carefully referred to an additive inverse of a number rather than the additive inverse. If we think of some particular number, say 3, we quickly recall that $3 + (-3) = 0$ and, therefore, -3 is an additive inverse of 3. All our experience with numbers suggests that -3 is the only additive inverse.

Previously in this course, whenever our experience strongly suggested a property, we have assumed the truth of that property. This was the case with the various properties of the numbers of arithmetic which we have asserted. Mathematicians have found, however, that many properties need not be assumed because they can be shown to be logical consequences of properties which have already been assumed.

Let us try to show ("prove") that 3 has just one additive inverse.

	We wish to show that 3 has a unique (meaning "just one") additive inverse.	
11	First of all, we know one additive inverse of 3, namely, ______.	-3
12	If any number is an additive inverse of 3, it must be a solution of the equation $3 + x =$ ______. (Refer to the meaning of additive inverse given in Item 2.	0
	We need to show that the number -3 is the only solution of $3 + x = 0$.	
	If $3 + x = 0$ is true for some x,	
13	then $(-3) + (3 + x) = (-3) + 0$ by the ______ ______ of equality	addition property
14	and $((-3) + 3) + x = (-3) + 0$ by the ______ property of addition.	associative
15	Hence, $0 + x = (-3) + 0$ by the ______ property of opposites.	addition
16	Finally, $x = -3$ by the addition property of ______ must also be true for that same x.	0
17	Any number x which makes $3 + x = 0$ true, must also make $x = -3$ true. But there is just one number in the truth set of $x = -3$, namely ______.	-3
18	Therefore, -3 is not just an additive inverse of 3, it is the additive ______ of 3.	inverse

Thus we did not need to assume that the number 3 has only one additive inverse--we were able to reach that conclusion by a logical argument that depended only on facts which we already knew.

Surely, there is nothing special about 3. Do you think that 5 has a unique additive inverse? Does 0? Does (-6.3)? We suspect that the answer is yes in each case. Perhaps you see that we could check by repeating the reasoning of Items 11-18. We certainly do not wish to check all numbers. What we need is a result which is true for any real number x. We know that -x is one additive inverse of x. We would like to state: any real number x has exactly one additive inverse, namely -x.

	When we say that 3 has a unique additive inverse we really imply two ideas.	
19	First, 3 has an additive ______.	inverse
	Second, 3 has only one additive inverse.	
20	3 has a u______ additive inverse means:	unique
	There is exactly one number which is the additive inverse of 3.	
21	That is, unique conveys the idea of exactly ______.	one

	We have already stated, in the addition property	
22	of ______, that for every real number a,	opposites
23	a + (______) = 0.	(-a)
	The addition property of opposites thus states that	
24	every number has one additive ______.	inverse

Now we are going to prove: Any real number has exactly one additive inverse. Statements of new facts or properties which can be shown to follow from previously established facts are frequently called "theorems". The argument by which a theorem is shown to be a consequence of other facts or theorems is called a proof of the theorem.

Let us, then, state and prove a theorem. For this proof, we shall parallel the argument of Items 11-18.

Theorem 7-5a. Any real number x has exactly one additive inverse, namely $-x$.

Proof:

First, we already know that $-x$ is an additive inverse

25 of x, since $x + (\underline{\qquad}) = 0$. — $(-x)$

Suppose z is any additive inverse of x, then

26 $x + z = 0$, by the definition of additive ______. — inverse

27 We wish to show that $z =$ ______. — $-x$

If $x + z = 0$ is true for some z, then

28 $(-x) + (x + z) =$ ______ by the addition property of equality. — $(-x) + 0$

29 ______ $+ z = (-x) + 0$, by the associative property. — $\big((-x) + x\big)$

30 ______ $= (-x) + 0$, by the addition property of opposites. — $0 + z$

$z = -x$ by the addition property of 0.

Therefore, $-x$ is the only additive inverse.

We have completed the proof of Theorem 7-5a. It would be natural for you to ask, "Why bother with a proof of something which we strongly expected to be true all along?"

The answer to this question, for the present, is that we wish you to become familiar with the idea of a mathematical proof. From time to time we shall prove more theorems in order to help you to become more accustomed to mathematical reasoning. Eventually we hope you will be able to prove statements whose truth is not obvious.

Note that in the proof of Theorem 7-5a we added $(-x)$ to each side of the equation $x + z = 0$. Why did we select $(-x)$ to add? If you recall your work with solving equations, you will see that adding $(-x)$ to each side enabled us to obtain z alone on the left-hand side. We often need to "rig" the proof of a theorem in this way by using previous knowledge in a manner which leads towards our goal.

Theorem 7-5a may be used to solve some equations. You should be able to find the solution sets of the following at a glance.

31 $(-2) + a = 0$ $\{2\}$

32 $3 + 5 + y = 0$ $\{-8\}$

33 $x + (-\frac{1}{2}) = 0$ $\{\frac{1}{2}\}$

34 $-(-\frac{2}{3}) + y = 0$ $\{-\frac{2}{3}\}$

35 $2 + x + (-5) = 0$ $\{3\}$

36 $3 + (-x) = 0$ (Be careful!) $\{3\}$

37 $|-4| + 3 + (-4) + c = 0$ $\{-3\}$

38 $3 + x = 0$ $\{-3\}$

39 $\left(-(2 + \frac{1}{3})\right) + a = 0$ $\{\frac{7}{3}\}$, or $\{2 + \frac{1}{3}\}$

On two successive plays the Redskins gained 3 yards and 4 yards, respectively. After the next play the ball was back where it started at the beginning of the succession of plays. What happened on the third down?

If the number of yards the ball moved on the third down is represented by y, then an appropriate open sentence is

40 $3 + \underline{\quad} + \underline{\quad\quad}$. $3 + 4 + y = 0$

$7 + y = 0.$

41 The solution set is ______. $\{-7\}$

42 On the third down there was a ______ (gain, loss) of 7 yards. loss

To verify: Gains of 3 and 4 yards, followed by a

43 loss of ______ yards brings the ball back where it started. 7

Mary had $2 in her purse before Mrs. Jones paid her for baby-sitting. After she got paid she went shopping and bought a sweater for $5, which took all the money she had. How much was she paid for baby-sitting?

If x is the number of dollars she received for baby-sitting, an appropriate open sentence is

44 $2 + x + (\quad) = \underline{\quad}$. — $2 + x + (-5) = 0$

Then $x + 2 + (-5) = 0$,

45 or $x + (\underline{\quad}) = 0$. — $(\quad)$

46 The truth set is ____. — { }

47 Mary was paid $____ for baby-sitting. — $

To verify: After she added the $ for baby-sitting

48 to the $2 she had, the $____ for the sweater took all her money. — $5

Now we shall try to discover another general property and then to prove that this property is valid for all real numbers. We can't prove a theorem until we suspect one; let us find one to suspect.

Compare $-(3 + 5) = -8$ and

$(-3) + (-5) = -8$.

Have we discovered a general property?

Try some more examples:

49 The common name for $-(2 + 9)$ is ____. — -11

50 The common name for $(-2) + (-9)$ is ____. — -11

51 $-\left(4 + (-2)\right) = (-4) + \left(-(-2)\right)$ is a ____ (true, false) sentence. — true

52 $-\left((-1) + (-4)\right)$ and $1 + 4$ both ____ (do, do not) name the same number. — do

In each of the examples we have seen that the <u>opposite</u> of the <u>sum</u> of two numbers is equal to the <u>sum</u> of

53 the ____. — opposites

54 In symbols $-(a + b) = \underline{\quad} + \underline{\quad}$. — $(-a) + (-b)$

We now state our suspected property as a theorem.

Theorem 7-5b. For any real numbers a and b,

$$-(a + b) = (-a) + (-b).$$

Before we begin the proof of this theorem, let us think about what we already know and what we must prove.

How may we prove that $-(a + b) = (-a) + (-b)$?

We need to show that $(-a) + (-b)$ names the same
55 number as $-(\_\_\_\_)$. $-(a + b)$

56 $-(a + b)$ names the ______ of $(a + b)$. opposite

We have just seen that the opposite of a number is the
57 unique additive ______ of that number. inverse

That is, if $(a + b) + z = 0$, then z names the same
58 number as $-(\_\_\_\_)$. $-(a + b)$

Thus, if we can show that

59 $(a + b) + (-a) + (-b)$ ______, 0

then we will know that $(-a) + (-b)$ and $-(a + b)$ both name the additive inverse of $(a + b)$.

Theorem 7-5b. For any real numbers a and b

$$-(a + b) = (-a) + (-b).$$

Proof: Consider $(a + b) + \big((-a) + (-b)\big)$. We shall show that this is equal to zero.

$(a + b) + \big((-a) + (-b)\big) = a + b + (-a) + (-b)$

$= \big(a + (-a)\big) + \big(b + (-b)\big)$

60 using the associative and ______ properties of addition commutative

61 $= 0 + 0$ using the addition property of ______ opposites

62 $= 0$ using the ______ property of 0. addition

Therefore, $(-a) + (-b)$ names the additive inverse
63 of ______. $(a + b)$

64	But $-(a + b)$ also names the additive inverse of _____, since it is the opposite of $(a + b)$. However, by Theorem 7-5a, we know that $(a + b)$ has only <u>one</u> additive inverse. It follows that $(-a) + (-b)$ and $-(a + b)$ name the same number.	$(a + b)$
65	Therefore, $-(a + b) = (-a) +$ ______.	$-(a+b)=(-a)+(-b)$

Did you notice that we used Theorem 7-5a in the proof of Theorem 7-5b? It is quite usual to prove one result by using other results which have already been proved.

	We can use Theorem 7-5b to decide whether or not the following open sentences are true for <u>all</u> real numbers. Respond true or false to each.	
66	$-(x + y) = (-x) + (-y)$	true
67	$-(x + z) = (-x) + z$	false
68	$-(x + z) = (-x) + (-z)$	true
69	$-\big(x + (-z)\big) = (-x) + z$ [Recall that $-(-z) = z$.]	true
70	$-\big(x + (-y)\big) + (-y) = -x$	true

Since $-(a + b) = (-a) + (-b)$, it seems reasonable to suspect that $-(a + b + c) = (-a) + (-b) + (-c)$.

It is in fact true that the opposite of any finite sum of real numbers equals the sum of the opposites. We will not prove this extension of Theorem 7-5b, but we will use it whenever it helps us write numerals in a different form.

Thus, $-\big((-2) + 6 + (-5)\big) = 2 + (-6) + 5$.

	Similarly,	
71	$-(2 + 3 + 5) = (-2) + (-3) +$ ______	(-5)
72	$-\big(2x + (-3y)\big) =$ ______	$(-2x) + 3y$
73	$-\big(x + (-y) + (-1)\big) =$ ______	$(-x) + y + 1$
74	$-\big(a + (-b) + (-3c)\big) =$ ______	$(-a) + b + 3c$

In this section we have introduced the idea of the additive inverse of a real number and have used additive inverses together with the addition

property of equality to solve some equations. Equally important, we have stated and proved two theorems. We are not going to prove rigorously everything we assert from now on--we cannot at this stage--but we are trying to give you a little experience with the kind of thinking we call "proof". Don't be discouraged if you did not follow every detail of the arguments. We hope that by the end of the course you will have some feeling for the nature of proof, and hence a better idea of algebra and a greater interest in mathematics. Then you will be able, not only to use algebra in solving problems, but also to understand something about its logical development.

Complete the following proof of the statement

For any real number a, and any real number b, and any real number c, if $a + c = b + c$, then $a = b$.

Proof:

$a + c = b + c$

75. $(a + c) + (-c) =$ ______ Addition property of equality

$(a + c)+(-c) = (b + c)+(-c)$

76. $a + (c + (-c)) = b + (c + (-c))$ ______ property of ______

Associative, addition

77. ______ $= b + 0$ Addition property of opposites

$a + 0 = b + 0$

78. $a = b$ ______ property of ______

Addition, zero

79. Which of the following statements is the better description of what we mean by a theorem?

[A] A theorem is an assertion which, from experience, we know to be true, but since we can "prove" it, we do so.

[B] A theorem is an assertion which, from experience, we suspect to be true and which may be proved on the basis of facts already established.

[B] is not a precise definition, but it serves to give some "feel" for what a theorem is, and [B] is the better choice.

80 Do you agree or disagree with the following statement?

> A proof of an assertion is an argument which proceeds step by step, each step being justified by previously established facts, until the desired conclusion is reached.
>
> [A] agree [B] disagree

Again, this statement is not a formal definition but it at least presents the idea of what we mean by "proof". If you disagree, perhaps you should restudy some of the proofs of this section before proceeding to Section 7-6. You should have chosen [A].

7-6. Summary and Review Problems

We have defined addition of real numbers as follows:

The sum of two non-negative numbers is familiar from arithmetic.

The sum of two negative numbers is negative; the absolute value of this sum is the sum of the absolute values of the numbers.

The sum of two numbers, of which one is positive (or 0) and the other is negative, is obtained as follows:

The absolute value of the sum is the difference of the absolute values of the numbers.

The sum is positive if the positive number has the greater absolute value.

The sum is negative if the negative number has the greater absolute value.

The sum is 0 if the positive and negative numbers have the same absolute value.

We have satisfied ourselves that the following properties hold for addition of real numbers:

<u>Commutative Property of Addition</u>: For any two real numbers a and b,

$$a + b = b + a.$$

<u>Associative Property of Addition</u>: For any real numbers a, b, and c,

$$(a + b) + c = a + (b + c).$$

<u>Addition Property of Opposites</u>: For every real number a,

$$a + (-a) = 0 \quad \text{and} \quad (-a) + a = 0.$$

<u>Addition Property of 0</u>: For every real number a,

$$a + 0 = a \quad \text{and} \quad 0 + a = a.$$

We also stated a fact, already clear from our earlier ideas, which we called the <u>addition property of equality</u>:

For any real numbers a, b, and c,

if $a = b$, then $a + c = b + c$ and $c + a = c + b$.

This idea supplied us with a useful procedure for determining the truth sets of open sentences.

We have proved that the additive inverse is unique--that is, that each number has exactly one additive inverse, which we call its opposite.

We have discovered and proved the fact that the opposite of the sum of two numbers is the same as the sum of their opposites.

Review Problems

The correct answers for these Review Problems are on page vii.

1. Find a common name for each of the following:

(a) $3(8 + (-6))$

(b) $(-3) - 2 \times 3$

(c) $2 \times 7 + (-14)$

(d) $(-\frac{2}{5}) + \frac{3}{5}$

(e) $|6||3| + (-3)$

(f) $6(1 + |-4|)$

2. Which of the following sentences are true?

(a) $3 + (-3) = (-3) + 3$

(b) $|-3| + (-3) = 0$

(c) $6 \times 3 - 3 = 0$

(d) $(2 + (-3)) + 6 = 2 + ((-3) + 6)$

(e) $5 - 3 \leq |-5| - |-3|$

(f) $5(|10 - 7|) < 3 \times 3 \times 2$

(g) $6 - 6 = 6 + (-6)$

3. Show how the properties of addition can be used to explain why the following sentence is true:

$$\frac{2}{3} + \left(7 + \left(-\frac{2}{3}\right)\right) = 7$$

4. Find the truth set of each of the following:

(a) $\frac{5}{9} + 32 = x + \frac{5}{9}$

(b) $x + 5 + (-x) = 12 + (-x) + (-3)$

(c) $x + \frac{15}{2} + x = 10 + x + (-\frac{7}{2})$

(d) $|x| + 3 = 5 + |x|$

5. For what set of numbers is each of the following sentences true?

(a) $|3| + |a| > |-3|$

(b) $|3| + |a| = |-3|$

(c) $|3| + |a| < |-3|$

6. Two numbers are added. What do you know about these numbers if

(a) their sum is negative?

(b) their sum is 0?

(c) their sum is positive?

7. A salesman earned a basic salary of $80 a week. In addition, he received a commission of 3% of his total sales. During one week he earned $116. What was the amount of his sales for the week? Write an open sentence for this problem.
(If you had trouble writing an open sentence for this problem, do the following steps:)

If n is the number of dollars in his sales for the week, then his commission on the sales, at 3%, would be (.03)____ dollars.	$(.03)n$
Since his earnings of $116 include his basic salary of $____ and his commission of .03n dollars, an appropriate open sentence is 80 + ____ = ____.	$80 $80 + (.03n) = 116$

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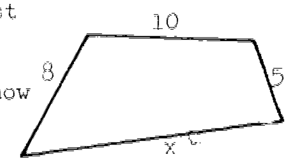
8. A figure has four sides. Three of them are 8 feet, 10 feet, and 5 feet, respectively. How long is the fourth side?

   (a) Write a compound open sentence for this problem.

   (b) Graph the truth set of the open sentence.

   (If you need help in writing this compound open sentence, here are some things to think about:)

Since each of the four sides must have *some* length, if x is the length of the fourth side, we know that $x >$ ______. 	$x > 0$
Is there any positive number x that would be too great? If $x = 8 + 10 + 5$, the figure would look like this: 	
In other words, the figure would be a line segment, instead of a figure of ______ sides.	four
Can the length of the fourth side be greater than 23? ______ (yes, no)	no
Hence, we see that for this problem $x <$ ______.	$x < 23$
These two restrictions on x are stated in the compound open sentence $x >$ ______ and ______,	$x > 0$ and $x < 23$
or $0 < x < 23$.	

9. If a, b, and c are numbers of arithmetic, write each of the indicated sums as an indicated product, and each of the indicated products as an indicated sum:

(a) $(2b + c)a$	(e) $x^2y + xy$
(b) $3a(b + c)$	(f) $6a^2b + 9ab^2$
(c) $3a + 3b$	(g) $ab(ac + 3b)$
(d) $5x + 10ax$	(h) $3a(a + 2b + 3c)$

10. Given the set $\{-5, 0, \frac{3}{4}, -.75, 5\}$

(a) Is this set closed under the operation of taking the opposite of each element of the set?

(b) Is this set closed under the operation of taking the absolute value of each element?

(c) If a set is closed under the operation of taking the opposite, is it closed under the operation of taking the absolute value? Why?

11. Given the set $\{-5, 0, \frac{3}{4}, 5, 7\}$

(a) Is this set closed under the operation of taking the absolute value of each element of the set?

(b) Is this set closed under the operation of taking the opposite of each element?

(c) If a set is closed under the operation of taking the absolute value, is it closed under the operation of taking the opposite? Why?

Chapter 8

PROPERTIES OF MULTIPLICATION

8-1. Multiplication of Real Numbers

Until we were introduced to the negative numbers, all our work was with numbers of arithmetic. However, quite often we are faced with problems which call for the use of negative as well as non-negative numbers. For example, in the launching of a missile, the upward force of the missile is counteracted by the force of gravity. If one of the forces is considered positive, the other would be considered negative. Problems involving such concepts sometimes require operations with negative numbers as well as with numbers of arithmetic.

As another illustration, let's consider the case of a merchant who is offering weekly "leaders". These are items which are sometimes offered at a loss to attract customers. Let's say the merchant is handling record albums.

1	Suppose the merchant makes a \$3 profit on each album of brand X and a \$2 profit on each of brand Y. If he sells 15 albums of brand X and 20 of brand Y, then his net profit may be represented by $\underline{(15)(3) + (20)(\quad)}$.	$(15)(3)+(20)(2)$

On the other hand, suppose the merchant makes a \$3 profit on brand X and takes a \$2 loss on brand Y. What were his proceeds from the sale of 15 sets of brand X and 20 sets of brand Y?

2	If we represent the \$3 profit as (3) and the \$2 loss as (-2), then the result of the sale may be represented by $\underline{(15)(3) + (20)(\quad)}$.	$(15)(3)+(20)(-2)$

Furthermore, if the dealer has to accept some returned merchandise, say 5 of brand X and 7 of brand Y, then the resulting expression may look still more involved:

$$(15)(+3) + (20)(-2) + (-5)(+3) + (-7)(-2).$$

We may need to know how to handle the multiplication of two real numbers. All that we can say at present is that we know how to multiply non-negative numbers (that is, numbers of arithmetic). We want to give meaning to products which involve negative numbers.

Let us recall some of the things we know about multiplication for the numbers of arithmetic.

	We noted that among the properties of multiplication that hold for non-negative real numbers, that is, for	
3	the numbers of ______, are	arithmetic
	the commutative property of multiplication,	
	the associative property of multiplication,	
	the multiplication property of 0,	
	the multiplication property of 1.	
4	Moreover, for the numbers of arithmetic, the ______	distributive
5	property connects the operations of ______ and m______.	addition multiplication

We use these properties repeatedly when we use the numbers of arithmetic.

	Furthermore, in Chapter 7 we found that for real numbers, just as for the numbers of arithmetic, we have the commutative and associative properties	
6	of ______.	addition

We found that we were able to define addition of real numbers in such a way that certain properties holding for non-negative numbers were maintained for the real numbers. We shall find that we can also define multiplication of real numbers so that the properties in Items 3-5 are preserved. To be useful, the definition should be such that when it is applied to actual problems, the interpretation is realistic. We will see that our definition will fulfill this requirement. In particular, since all numbers of arithmetic are real numbers, the definition of multiplication should give the familiar results when we multiply numbers of arithmetic. Thus, the definition applied to (3)(2) must give 6 to agree with our knowledge of the numbers of arithmetic.

However, at this point the symbols (-2)(0), (0)(-2), (3)(-2), (-2)(3) and (-2)(-3) are meaningless to us. Each has the form of an indicated product and each has at least one negative number. We are going to define such products in a way that preserves the basic properties mentioned above.

7	We know that if a is any non-______ real number,	non-negative
8	then $a \cdot 0 =$ ______ by the	0
9	______ property of zero.	multiplication

As we extend the set of numbers from the non-negative real numbers to <u>all</u> real numbers, we shall want the multiplication property of zero to hold. That is, our definition of multiplication must insure that $a \cdot 0 = 0$, and $0 \cdot a = 0$ for every real number a.

10	If the multiplication property of ______ is to hold,	zero, or 0
11	then $(-2)(0) =$ ______	0
12	$(0)(-2) =$ ______	0
13	$(-\sqrt{2})(0) =$ ______	0
14	$(0)(-\pi) =$ ______	0

Now let us consider $(3)(-2)$. What real number shall this represent?

15	-2 is the additive ______ of 2, that is,	inverse
16	$2 + (-2) =$ ______.	0
	Examine $3\big(2 + (-2)\big)$.	
17	If the multiplication property of ______ is to hold,	0
18	then $3\big(2 + (-2)\big) =$ ______.	0
	But we also want to define multiplication so that the distributive property holds true.	
	Hence, we want	
19	$3\big(2 + (-2)\big) =$ ______ $= 0$	$3 \cdot 2 + 3(-2)$
	Now we are getting somewhere!	
	If multiplication of real numbers is to have both the distributive property and the multiplication property of zero, then:	
20	$3 \cdot 2 + 3(-2) =$ ______ must be true.	0
21	Now $3 \cdot 2 =$ ______, so we see that	6
22	$6 + 3(-2) =$ ______ must be true.	0
	Hence, to be consistent, $3(-2)$ must be the additive	
23	______ of 6.	inverse
24	But ______ is the additive inverse of 6 and a real	-6
25	number has only one ______ ______.	additive inverse
26	Therefore, our definition of ______ must guarantee that $3(-2) = -6$.	multiplication

Note to the student: Be careful in your thinking! We did [illegible]rove that $3(-2)$ must be -6. We showed only that if we want to define multiplication in a way that preserves certain properties, then we must define $3(-2)$ as -6. Note, also, that this agrees with our intuitive notion of $3(-2)$.

27 If record albums are sold at a loss of $2 each, then the sale of 3 albums results in a ______ (gain, loss) of $6 to the dealer. — loss

Observe that if we define multiplication so that the distributive property is to hold, then

$$3x = (1 + 1 + 1)x = x + x + x.$$

If x is -2, then

$$3(-2) = (-2) + (-2) + (-2) = -6.$$

It should be clear that when we state our definition of multiplication, it must guarantee not only that $3(-2) = -6$ but that

28 $7(-8) =$ ______ — -56

29 $5(-\frac{1}{2}) =$ ______ — $-\frac{5}{2}$

30 $(\frac{1}{2})(-\frac{7}{10}) =$ ______ — $-\frac{7}{20}$

We are ready now to consider the product $(-3)(-2)$.

Let us proceed in this manner:

	$0 = (0)(-2)$	If the multiplication property of 0 is to hold for real numbers.	
31	$0 = (3 + (-3))(-2)$	We write $(3 + (-3))$ for 0, using the addition property of ______.	opposites
32	$0 =$ ___ + ___	If the distributive property is to hold for real numbers.	$(3)(-2)+(-3)(-2)$
33	$0 =$ ___ $+ (-3)(-2)$	We have previously found that our definition of multiplication had to guarantee that $(3)(-2) = -6$.	(-6)
34	Now $(-3)(-2)$ must be the additive inverse of ______.		-6
35	Hence, the only possibility is that $(-3)(-2)$ is ___.		6

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36	The same line of reasoning would lead to the true sentence $(-\frac{1}{2})(-\frac{7}{3}) =$ ______.	$\frac{7}{6}$
37	It would appear that we should define multiplication for all real numbers in such a way that the product of two negative numbers is ______ (positive, negative).	positive

So far we have been concerned with preserving the multiplication property of 0 and the distributive property. We also want to preserve the commutative property of multiplication.

38	We desire that for any real numbers a and b $a \cdot b =$ ______.	$b \cdot a$
39	Thus, if $3(-2) = -6$, then we must have also $(-2)(3) =$ ______.	-6
40	The product of a positive number and a negative number should be ______ (positive, negative).	negative
41	The product of 3 and the opposite of 2 is the ______ of the product of 3 and 2.	opposite

We have now covered all the possible cases of multiplication of real numbers:

- one or both of the numbers are zero,
- both numbers are positive,
- one number is positive and one is negative,
- both numbers are negative.

We have reasoned as to what the products must be, if certain properties are to hold. How should we formulate our definition of the product of two real numbers? Let's look at the various products we obtained above:

$$(2)(3) = 6$$
$$(-2)(-3) = 6$$
$$(2)(-3) = -6$$
$$(-2)(3) = -6$$

Each result is 6 or its opposite, -6. Both 6 and -6 have the same absolute value. This absolute value can be obtained by multiplying the absolute values of the numbers on the left. These examples suggest that we might think of formulating our definition in terms of absolute value. We must notice,

however, that the product is -6 when one of the numbers is positive and the other is negative.

	First we review some facts about absolute value. If the real number a is non-negative, it is either	
42	positive or ______.	0, or zero
43	If a is 0, then $\|a\| =$ ______.	0
44	If a is non-negative, then $\|a\| = \underset{(a,-a)}{\underline{\qquad}}$.	a
45	If a is negative, then $\|a\| =$ ______.	$-a$
	Applying the definition of absolute value, we have	
46	$\|7\| =$ ______	7
47	$\|-7\| = -(\quad) =$ ______	$-(-7) = 7$
48	$\|0\| =$ ______.	0
49	Also, $\|0\| = -(\quad) =$ ______.	$-(0) = 0$

Let us now try to express our definition of products of real numbers in terms of absolute value.

	It follows at once from the definition of absolute value that $(2)(3) = \|2\|\|3\|$. Thus, if we formally define	
50	$ab = \|a\|\|\quad\|$ for positive numbers a and b, we shall not conflict with what we already know about the product of numbers of arithmetic.	$ab = \|a\|\|b\|$

	Now consider $(-2)(-3)$. The preservation of basic	
51	properties requires that $(-2)(-3)$ ______.	6
52	Notice that $\|-2\|\|-3\| = (\quad)(\quad) =$ ______.	$(2)(3) = 6$
	Hence, $(-2)(-3) = \|-2\|\|-3\|$. It would appear that for any pair, a, b, of negative real numbers we want it to be true that	
53	$ab = \|\quad\|\|\quad\|$.	$ab = \|a\|\|b\|$

	Let us now try the same approach on the product $(-2)(3)$. We found that our definition had to give	
54	the result $(-2)(3) =$ ______.	-6
55	However, $\|-2\|\|3\| = (\ \ )(\ \ ) =$ ______.	$(2)(3) = 6$
	Thus, $(-2)(3) = -(\|-2\|\|3\|)$.	
	From this example it appears that for a pair of real numbers of which one is positive and the other is negative, we want it to be true that	
56	$ab = -(\|\ \ \|\|\ \ \|)$.	$ab = -(\|a\|\|b\|)$

	Finally, consider the product $(0)b$, where b is any real number. We have said that our definition of multiplication must insure that	
57	$(0)b =$ ______.	0
	Since $\|b\|$ is a number of arithmetic, and since $\|0\| = 0$,	
58	we can state $\|0\|\|b\| =$ ______	0
59	and $-(\|0\|\|b\|) =$ ______.	0

Remember that for <u>every</u> real number a, $|a|$ is a number of arithmetic. The principal achievement of the above discussion is that every possible product of real numbers has been expressed in terms of a product of numbers of arithmetic. This enables us to define the new operation in terms of one which is already known.

<u>Definition</u>: Let a and b be any two real numbers. In case a and b are both negative or are both non-negative,

$$ab = |a||b|.$$

In case one of the numbers is negative and the other is non-negative,

$$ab = -(|a||b|).$$

The second statement of this definition may appear awkward. Let's pause to analyze what it means. We are given any two real numbers, a, b, of which one is negative and the other non-negative. First, the definition tells us to find the absolute value of these numbers. Having done this for each of two

real numbers, a, b, we now have numbers of arithmetic corresponding to them, and we can form the product $|a||b|$. Finally, $-(|a||b|)$ tells us to find the opposite of the product.

From the definition and from Items 58 and 59, we see that for a pair a, b, of real numbers of which at least one is zero, both

$$ab = |a||b| \quad \text{and}$$
$$ab = -(|a||b|).$$

You should convince yourself that the above definition properly handles all of the cases which were discussed above.

Examples:

$$(5)(-3) = -(|5||-3|) = -15$$
$$(-5)(-3) = (|-5||-3|) = 15$$
$$(-5)(3) = -(|-5||3|) = -15$$
$$(-7)(0) = -(|7||0|) = -0 = 0$$

	Using the definition, find the simplest names for the following products:	
60	$(-8)(4) = -(\|\ \|\|\ \|) =$ ____	$-(\|-8\|\|4\|) = -32$
61	$(\frac{1}{2})(\frac{7}{9}) = \|\ \|\|\ \| =$ ____	$\|\frac{1}{2}\|\|\frac{7}{9}\| = \frac{7}{18}$
62	$(-\frac{1}{3})(\sqrt{5}) = -(\|\ \|\|\ \|) =$ ____	$-(\|-\frac{1}{3}\|\|\sqrt{5}\|) = -\frac{\sqrt{5}}{3}$
63	$(\frac{1}{6})(-\sqrt{2}) = -(\|\ \|\|\ \|) =$ ____	$-(\|\frac{1}{6}\|\|-\sqrt{2}\|) = -\frac{\sqrt{2}}{6}$
64	$(-1.1)(-3) = \|\ \|\|\ \| =$ ____	$(\|-1.1\|\|-3\|) = 3.3$

You should make very sure that you understand the method of computing products which involve negative numbers. Here are some questions to help you.

65	For all real numbers a and b, $\|a\|$ and $\|b\|$ are non-negative. Consequently, $\|a\|\|b\|$ is a non-____ number. This non-negative number can be computed from what we know about the numbers of arithmetic.	non-negative
	To compute the product of any two real numbers you need to know:	
66	how to compute products of numbers of ____.	arithmetic
67	how to decide whether the product is ____ or non-negative.	negative

	From the first part of the definition, we see that:	
68	the product of two positive numbers is a ______ number, and that	positive
69	the product of two negative numbers is a ______ number.	positive
	From the second part of the definition, we see that:	
70	the product of a positive and a negative number is a ______ number.	negative
	We can also tell from the definition that	
71	the product of a real number and 0 is ______.	0
72	The product of two non-negative numbers is a ______ number.	non-negative

73	If $7w$ is positive, w is ______. (positive, negative, 0)	positive
74	If $(-9)t$ is positive, t is ______.	negative
75	If $8u$ is negative, u is ______.	negative
76	If $(-2)s$ is negative, s is ______.	positive
77	If $(-117)q$ is zero, q is ______.	0

	From Items 68 and 69 we get a simple but important fact about x^2 for all real numbers x.	
78	By the ______ property, if x is a real number, exactly one of the following is true.	comparison
79	$x > 0$, $x = 0$, or ___ $<$ ___.	$x < 0$
	If $x \neq 0$, then either	
80	x ______ or x ______.	$x > 0$ or $x < 0$ or $x < 0$ or $x > 0$
	Since x^2 means $x \cdot x$, if $x > 0$, x^2 is the	
81	product of a ______ number and a positive number,	positive
82	and this product is a ______ number.	positive
83	If $x < 0$, x^2 is the ______ of a negative number	product
84	and a ______ number, and this product is	negative
85	a ______ number.	positive

From Items 79-84, we have the following:

If x is a real number and $x \neq 0$, then

86 x^2 ____ 0. $x^2 > 0$

87 Consider the products:

J. $(-7)(-8)$ M. $(-\frac{2}{5})(-\frac{3}{4})$

K. $(\frac{2}{3})(-12)$ N. $(-3)(0)$

L. $(-18)(\frac{3}{5})$.

Which of these statements about the products is correct?

[A] Only J and M are non-negative real numbers.

[B] Only N is a non-negative real number.

[C] Only K and L are negative real numbers.

If you selected [C], you made the correct choice. $(-7)(-8)$ and $(-\frac{2}{5})(-\frac{3}{4})$ are non-negative, and so is $(-3)(0)$.

Calculate the following. The answers are on page viii.

88. $(-\frac{1}{2})((2)(-5))$

89. $(-3)(-4) + (-3)(7)$

90. $(-3)((-4) + 7)$

91. $(-3)(-4) + 7$

92. $|-3|(-4) + 7$

93. $|3||-2| + (-6)$

94. $(-3)|-2| + (-6)$

95. $(-3)(|-2| + (-6))$

96 Which of the following sentences are not true when x is -2, y is 3, f is -10, m is 2, and a is -4?

P. $2x + 7y = 17$

Q. $x^2 + (3|a| + (-4)y) = 4$

R. $2f + 8 = -12$

S. $|m + 3| + (-2)(|m + (-4)|) \geq 1$

T. $x + y + f = 15$.

[A] P and Q are not true sentences.

[B] Only R is not a true sentence.

[C] Only S is not a true sentence.

[D] Only T is not a true sentence.

You should have selected [D] as your choice, since they are all true except T. $x + y + f = 15$ becomes $(-2) + 3 + (-10) = 15$, or $-9 = 15$, which is a false sentence.

Items 97-100 give you additional practice in using the definition of multiplication of real numbers.

	Label each of the following true or false.	
97	$2(-y) + 8 = 28$, if y is -10.	true
98	$(-3)\big((2)(-x)\big) + 8 \neq 20$, if x is 2.	false
99	$(-5)\big((-b)(-4)\big) + 30 < 0$, if b is 2.	true
100	$\lvert x + 2\rvert + (-5)\lvert(-3) + 2x\rvert > (-40)$, if x is -2.	true

Recall the procedure which was developed in Section 7-4 to find the truth sets of open sentences. Let us apply this procedure to open sentences which have products involving negative numbers. In working through the first example, we shall supply all the reasoning, in order to refresh your memory. As you work on through these problems you may wish to drop some of the explanatory words, following the pattern of work suggested in Section 7-4.

<u>Example</u>: Find and graph the truth set of

$$(3)(-3) + y = 3(-4)$$

We can proceed as follows:

If	$(3)(-3) + y = 3(-4)$	is true for some y,
then	$-9 + y = -12$	is true for that same y.
Likewise,	$-9 + y + 9 = -12 + 9$	is true for that same y.
Also,	$y + \big((-9) + 9\big) = -12 + 9$	is true for that same y.
Finally,	$y = -3$	is true for that same y.

But $y = -3$ is true only if y is -3. Note that from above, we have only claimed that <u>if</u> $(3)(-3) + 3(-4)$ is true for <u>some</u> y, <u>then</u> after a chain of reasoning, finally, $y = -3$ is true for the same y. There has been no point during this process where we have determined whether the original sentence is true if y is -3, and this must be verified if -3 is to be in the truth set. If y is -3, then the left side is $(3)(-3) + (-3)$, or -12, and the right side is $3(-4)$, [illegible] -12. When we say that if the first statement is true for some y, then [illegible]inal statement is true for the same y, we mean

that no other y will do. Hence, the truth set is $\{-3\}$.

The graph of the truth set of $(3)(-3) + y = 3(-4)$ is

101 102	Find the truth set of $x + (-3)(-4) = 8$ and draw its graph. The truth set is ______ and the graph is ______.	$\{-4\}$ 

Omitting words which show the reasoning, the work for Item 101 might appear like this:

$$
\begin{aligned}
\text{If}\quad x + (-3)(-4) &= 8 \qquad \text{is true for some } x\\
x + 12 &= 8\\
x + 12 + (-12) &= 8 + (-12)\\
x &= -4 \qquad \text{is true for the same } x.
\end{aligned}
$$

Check: The left side is $(-4) + (-3)(-4) = 8$.

The right side is 8.

Thus, since the original statement is true if x is -4, the truth set is $\{-4\}$.

Find the truth sets of each of the following open sentences. Then draw their graphs. When you have finished, turn to page viii for the correct results.

103. $x + (-\frac{3}{2})(-2) = 3$

104. $2(-2) + y = 3(-2)$

105. $x + 2 = 3(-6) + (-4)(-5)$

106. $x + (-2)(4) = |-2|(-3)$

107. $|x| = (-\frac{2}{3})7 + (-1)(-5)$

Let us examine some products of real numbers, giving our attention to the absolute values of the numbers and of their products.

108	$\|7\|\|-3\| =$ ______	21
109	$\|(7)(-3)\| =$ ______	21
110	$\|7\|\|-3\|$ ______ $\|(7)(-3)\|$ $(<,=,>)$	$\|7\|\|-3\| = \|(7)(-3)\|$

111	$\lvert -8 \rvert \lvert -6 \rvert =$ ______	48
112	$\lvert (-8)(-6) \rvert =$ ______	48
113	$\lvert -8 \rvert \lvert -6 \rvert$ ______ $\lvert (-8)(-6) \rvert$	$\lvert -8 \rvert \lvert -6 \rvert = \lvert (-8)(-6) \rvert$
114	The examples above might lead us to wonder whether it is true for all real numbers a and b: $\lvert ab \rvert = \lvert\ \rvert \lvert\ \rvert$.	$\lvert ab \rvert = \lvert a \rvert \lvert b \rvert$

As a matter of fact, the statement is true for all real numbers, and can be stated as a property of multiplication:

For all real numbers a and b,

$$|ab| = |a||b|.$$

When we take many numerical examples and arrive at the same conclusion each time, we still do not have a proof that our conclusion is true for all real numbers. However, the property stated above can be proved as a theorem. Items 115-119 are starred, but we hope that you are curious about the proof and will work through the starred items.

Theorem 8-1. For all real numbers a and b,

$$|ab| = |a||b|.$$

*	Proof: Suppose a and b are any two real numbers.	
	Case I: If a and b are either both negative or both non-negative, then	
*115	$ab =$ ______, from the first case of the definition of multiplication for real numbers, page 273.	$\lvert a \rvert \lvert b \rvert$
*116	Thus, $\lvert ab \rvert = \lvert \lvert a \rvert \lvert b \rvert \rvert = \lvert a \rvert \lvert b \rvert$, since the absolute value of a non- ______ number is the number itself.	non-negative
	Case II: If one of the numbers a and b is negative and the other is non-negative, then	
*117	$ab =$ ______, from Case II of the definition for multiplication of real numbers.	$ab = -(\lvert a \rvert \lvert b \rvert)$
*118	Thus, $\lvert ab \rvert = \lvert -(\lvert a \rvert \lvert b \rvert) \rvert = -\left(-(\lvert a \rvert \lvert b \rvert)\right) = \lvert a \rvert \lvert b \rvert$, since ab is a ______ number, or 0, and its absolute value is the opposite of the number.	negative
*119	Hence, in either case, $\lvert ab \rvert =$ ______.	$\lvert ab \rvert = \lvert a \rvert \lvert b \rvert$

8-2. The Commutative Property and the Multiplication Properties of 1 and 0

In the last section we defined the operation of multiplication in the set of real numbers.

If a and b are any two real numbers, the product ab is defined as follows:

$ab = |a||b|$ if a and b are both
1 non-negative or both ______. negative

$ab = -(|a||b|)$ if one of the numbers a and b
2 is non-negative and the other is ______. negative

Thus, for any two real numbers a and b:

<u>either</u> ab is equal to the product of $|a|$
3 and $|\ \ |$, <u>or</u> ab is equal to the $|b|$
4 ______ of the product of $|a|$ and $|b|$. opposite

In the last section we saw that <u>if</u> multiplication for the set of real numbers is to have certain properties, <u>then</u> the definition given is the only possible one.

Now you should understand the task which lies before us. Beginning with our definition of multiplication we shall investigate whether or not certain familiar properties that hold for the set of <u>non-negative</u> real numbers also hold for the set of <u>all</u> real numbers.

To accomplish this, we shall first test a given property by trying out some examples. Doing this will enable us to <u>suspect</u> that a certain theorem is true; at times we shall also see that the reasoning used in each of a given group of examples fits into a certain pattern. In some instances we shall organize this reasoning into a careful, step-by-step proof.

In this chapter you will see many proofs. Do your best to understand the main ideas of each one, but do <u>not</u> try to memorize proofs. This is a waste of time. Eventually you will learn how to construct some proofs for yourself. This is an important skill, and one that takes time and patience to achieve. Fortunately, the idea of proof gets easier as you go along.

We are familiar with the following properties of non-negative real numbers.

If a, b, and c are non-negative real numbers, then

$ab = ba$ Commutative property of multiplication

5 $(ab)c = a(bc)$ ______ property of multiplication Associative

6 $a \cdot 1 =$ ______ and Multiplication property a
7 $1 \cdot a = a$ of ______ 1

$a \cdot 0 = 0$ and Multiplication property
8 $0 \cdot a = 0$ of ______ 0

9 $a(b + c) = ab + ac$ and $(b + c)a = ba + ca$ ______ property Distributive

10 We are going to find out that these five properties hold true in the set of all ______ numbers. (real, non-negative) real

In the proofs which follow, notice how we use the fact that the properties stated above hold for all non-negative numbers.

Here are several products involving the numbers $\frac{18}{5}$ and 17 and their opposites. The first product has been computed for you. Compute the other products as quickly as you can.

$(\frac{18}{5})(17) = \frac{306}{5}$

11 $(17)(\frac{18}{5}) =$ ______ $\frac{306}{5}$

12 $(-17)(-\frac{18}{5}) =$ ______ and $(-\frac{18}{5})(-17) =$ ______ $\frac{306}{5}$, $\frac{306}{5}$

13 $(-17)(\frac{18}{5}) =$ ______ and $(\frac{18}{5})(-17) =$ ______ $-\frac{306}{5}$, $-\frac{306}{5}$

Did you use paper and pencil, or did you do some intelligent guessing?

If you guessed intelligently, then you must suspect
14 that the ______ property of multiplication holds for all real numbers. commutative

15 We know that this property holds for all non-______ numbers. non-negative

16	The fact that the commutative property holds for non-negative real numbers guarantees that $\|-17\|\|-\frac{18}{5}\| = \|-\frac{18}{5}\|\|-17\|$, because $\|-17\|$ and $\|-\frac{18}{5}\|$ are non-________ numbers.	non-negative
17	Can we conclude without using pencil and paper that the following numerical sentence is true? $(-53)(-47) = (-47)(-53)$ ________ (yes, no)	yes
	In fact, we can reason:	
	$(-53)(-47) = \|-53\|\|-47\|$ — Definition of multiplication	
18	$= \|-47\|\|-53\|$ — ________ property of multiplication for non-negative real numbers	Commutative
19	$= (-47)(-53)$ — ________ of multiplication	Definition
20	Can we conclude without actually carrying out the multiplication that the sentence $(-53)(47) = (47)(-53)$ is true? ________ (yes, no)	yes
21	The first step in the reasoning is to note that $(-53)(47) = -(\|\quad\|\|\quad\|)$ from the definition of multiplication. Try to complete the reasoning.	$-(\|-53\|\|47\|)$

We can now organize the reasoning needed in the preceding examples to construct a step-by-step proof that the commutative property holds for all real numbers.

22	Theorem. For all real numbers a and b, $ab =$ ________.	$ab = ba$
	Proof.	
23	Case I. Suppose a and b are both non-negative or both ________.	negative

24	ab = |a||b|	Definition of ______	multiplication
25	= |b||a|	______ property of multiplication for non-negative real numbers	Commutative
	= ba	Definition of multiplication	
	Case II. Suppose one of the numbers a and b is		
26	non-negative and the other is ______.		negative
27	ab = -(| || |)	Definition of multiplication	-(|a||b|)
	= -(|b||a|)	Commutative property of	
28		multiplication for ______ real numbers	non-negative
29	= ______	Definition of multiplication	ba

Read the proof carefully. Try to see how it translates the reasoning you used in Items 11-16.

*Another way of proving this theorem is to recall first that at the end of the preceding section, we proved that for all real numbers a and b, $|ab| = |a||b|$. A similar statement can be made for $|ba|$. Next, we consider and compare the products ab and ba for the various cases of a and b:

if a and b are both positive,
if they are both negative,
if one is positive and the other is negative,
if one or both of these are zero.

Try to construct a proof along these lines and compare your results with those on page ix.

* The reasoning in both of these proofs is similar. In both ways of reasoning, the essential ideas we need are:

the fact that the commutative property is known
*30 to hold for the ______ real numbers; and — non-negative
the definition of multiplication.

*31 The moral is: A theorem can ______ (sometimes, never) be proved in more than one way. — sometimes

Before we take up the multiplication property of 1, try these examples.

32 $4 \cdot 1 =$ ______ 4

33 $1 \cdot \frac{11}{8} =$ ______ $\frac{11}{8}$

34 $(-3) \times 1 =$ ______ -3

35 $1 \times (-\sqrt{5}) =$ ______ $-\sqrt{5}$

We suspect: If a is any real number,

36 $a \cdot 1 =$ ______ and $1 \cdot a = a$. a

Let us see the pattern we would follow in working out these examples.

The first two examples (Items 32 and 33) involve only non-negative numbers. We already know that the mul-

37 tiplication property of ______ holds for non-negative real numbers. 1

What about $(-3) \times 1$? We find a simpler name for this product by using our definition of multiplication:

$$(-3) \times 1 = -(|-3| \cdot |1|).$$

38 We know that $|-3| =$ ______. 3

39 Since $3 \times 1 = 3$ (______ property of 1), and multiplication

40 since the opposite of 3 is ______, we have: -3

41 $(-3) \times 1 =$ ______. -3

There is a point to considering $(-3) \times 1$ in so much detail. We could have used exactly the same reasoning with any other negative number: for instance, we could use it to explain why $(-5) \times 1 = -5$. Likewise, the example $4 \cdot 1 = 4$ illustrates a general property of non-negative numbers. In other words, the reasoning we have used in talking about these examples can be used in a proof that if a is any real number, $a \cdot 1 = a$ and $1 \cdot a = a$.

On Panel 8-2 on your response sheet, try to complete for yourself the proof of the theorem:

If a is any real number,
then $a \cdot 1 = a$.

Do not refer to Items 42-60 below unless you find blanks in the proof which you cannot complete. In such a case, refer to that item below which is numbered to correspond to the one which gave you trouble.

In Case I, we use at once the fact that the
42 ______ property of ______ holds for — multiplication, 1
43 all non-______ real numbers. — non-negative

44 In Case II, we assume that a is ______. — negative

Since a is a negative real number, the fact that
$a \cdot 1 = -(|a||1|)$ is an application of the
45 ______ of ______ for real numbers. — definition, multiplication

Since the absolute value of a non-negative number is
46 the number itself, $|1| =$ ______. — 1

Since $|a|$ is non-negative, we know that
47 $|a| \cdot 1 = |a|$, hence, $-(|a| \cdot 1) =$ ______. — $-|a|$

Since a is negative and the absolute value of any
negative number is the opposite of the number,
48 $|a| =$ ______. — $-a$

Since the opposite of the opposite of a real number is
the number itself, we see that $-|a|$, which is $-(-a)$,
49 is ______. — a

We have shown that $a \cdot 1$, $-|a|$, and a are all
50 names for the same number. Hence, $a \cdot 1 =$ ______. — a

The proof of the theorem is now complete, because each
real number is either non-negative or negative and in
51 both cases we have shown that ______ a. Check your — $a \cdot 1 = a$
completed proof with that shown on the following page.

PANEL 8-2

Theorem. If a is any real number, then $a \cdot 1 = a$.

Proof.

Case I: a is non-negative.

If a is non-negative, then $a \cdot 1 = a$, because the multiplication property of 1 holds for all non-negative real numbers.

Case II: a is negative.

Since a is negative and 1 is non-negative,

$a \cdot 1 = -(\lvert a\rvert \lvert 1\rvert)$	Definition of multiplication of real numbers
$= -(\lvert a\rvert 1)$	since $\lvert 1\rvert = 1$
$= -\lvert a\rvert$	Multiplication property of 1 for non-negative real numbers
But, since a is negative, $\lvert a\rvert = -a$	Definition of absolute value
Hence, $-\lvert a\rvert = -(-a)$	
$= a$	The opposite of the opposite of a real number is the number itself.
Hence, $a \cdot 1 = a$	Since $a \cdot 1 = -\lvert a\rvert$ [Item 47] and $-\lvert a\rvert = a$ [Item 50]

We could prove, by exactly the same steps, that for any real number a we have $1 \cdot a = a$. You should be able to see how this proof would look. However, we have already proved that the commutative property of multiplication holds for the real numbers. Then it follows at once that since $a \cdot 1 = a$, it is also true that $1 \cdot a = a$. This again illustrates the fact that a proof can sometimes be given in more than one way.

The proof of a mathematical statement helps us see why the statement is true. In studying a proof, the goal is to understand the ideas and the reasoning, not simply to commit the proof to memory. It is useless to memorize a proof that you don't understand; and if you really understand a proof, there is no need to memorize it.

	One important idea used in the proof in Panel 8-2 is	
52	that the multiplication property of ______ holds for all non-negative real numbers.	1
	This property is used in each of the two cases.	
53	If a is non-negative (Case I) $a \cdot 1 =$ ______ from this property.	a
	In Case II, where a is negative, we need this	
54	property to explain why $\lvert a \rvert \cdot 1 =$ ______.	$\lvert a \rvert$

Now you can try your hand at constructing a proof.

Try to prove the multiplication property of 0. Do your best to write a complete proof for yourself. If you are successful you can use Items 55-63 as a check of your work. If you have trouble, use these items to help you complete the proof.

	We want to construct the proof that if a is any real	
55	number, then $a \cdot 0 =$ ______.	0
	As in the previous proof, we can separate this proof	
56	into ______ cases.	two
	We first suppose that a is non-negative, and then	
57	that a is ______.	negative
	Case I. If a is non-negative, then $a \cdot 0 = 0$	
58	because the ______ ______ of ______ holds for	multiplication property of 0
59	______ real numbers.	non-negative
60	Case II. If a is ______, then $a \cdot 0 = -(\lvert a \rvert \cdot 0)$	negative
61	from the definition of ______.	multiplication
	The next step is to note that $\lvert a \rvert \cdot 0 = 0$.	
	Since $\lvert a \rvert$ is a non-negative real number, $\lvert a \rvert \cdot 0 = 0$	
62	because of the ______ ______ of ______.	multiplication property of 0
	So we have, from Items 60 and 62, $a \cdot 0 = -(\lvert a \rvert \cdot 0) = -0$.	
63	The proof is complete, since we know that $-0 =$ ______	0

We proved that if a is any real number, then $a \cdot 0 = 0$.	
We could prove in exactly the same way:	
64 If a is any real number, then $0 \cdot a =$ ______.	0
However, we could also prove this at once, since	
65 $a \cdot 0 = 0 \cdot a$ because of the ______ property	commutative
66 of ______.	multiplication
Thus, we have both	
$a \cdot 0 = 0$ and $0 \cdot a = 0$.	
We speak of the multiplication property of 0 whenever we want to refer to either of these statements.	

In this and in the next section, the starred items are designed to give you more practice in thinking about proofs.

*67 In proving that the multiplication property of 0 holds for all real numbers, which, if any, of the following ideas were not used?

[A] The multiplication property of 0 for non-negative real numbers.

[B] The definition of multiplication in the set of all real numbers.

[C] The multiplication property of 1.

[D] The property that $-0 = 0$.

Look carefully at [C]. We proved the multiplication property of 0 <u>in the same way</u> as we proved the multiplication property of 1. However, we did not <u>use</u> the multiplication property of 1 in our proof. We could prove the multiplication property of 0 even if we did not know the multiplication property of 1. Thus, [C] was <u>not</u> used in the proof, but [A], [B] and [D] <u>were</u> used. (Look again at Items 55 to 63, if you weren't sure.)

Apply the multiplication properties of 1 and 0 (which we have just shown to be true in the set of all real numbers) to compute the following products:

68	$(-2) \cdot 0 =$ ______	0
69	$(-\sqrt{3})((-7) + 8) =$ ______	$-\sqrt{3}$
70	$(-\sqrt{3})((-8) + 8) =$ ______	0
71	$(-3\sqrt{5}) \cdot 1 =$ ______	$-3\sqrt{5}$
72	$(7 + (-9))(\sqrt{6} + (-\sqrt{6})) =$ ______	0
73	$(x + (-y))(y + (-y)) =$ ______	0

8-3. The Associative and Distributive Properties

Since the associative property of multiplication holds for the numbers of arithmetic, we are led to ask whether this property holds for the real numbers.

1 If the associative property for multiplication holds for real numbers a, b, and c, $(ab)c =$ ______.	$(ab)c = a(bc)$

First, we can argue that if a is 0, and b is a real number, then $ab = 0$, and hence, $(ab)c = 0$. Similarly, we can show that if one or more of the numbers a, b, c, is 0, then $(ab)c = 0$ and also $a(bc) = 0$. Thus, $(ab)c = a(bc)$ if one or more of the numbers a, b, c is 0.

To prove that the associative property holds for non-zero real numbers, we might show that $(ab)c = a(bc)$ is true if all of the numbers a, b, c are positive, and also true if one or more of them is negative. There are eight different cases, three of which are listed below:

a is positive, b is positive, and c is positive;

a is positive, b is positive, and c is negative;

a is positive, b is negative, and c is positive.

The detailed proof of one such case, namely, if a and b are positive and c is negative, is given in the starred Items 91-101 at the end of this section.

You may want to try constructing this proof for yourself and compare it with the one given there.

Instead of proceeding with the proof by considering each of the eight cases separately, we are going to indicate a more condensed version in which several cases are sometimes combined in one. As usual, the consideration of a few examples will help.

	Compute the following:	
2	$(-7)\big((4)(-3)\big) = $ ______	84
3	$7\big((4)(-3)\big) = $ ______	-84
4	$\big((-7)(4)\big)(-3) = $ ______	84
5	$\big((-7)(-4)\big)(-3) = $ ______	-84
6	Not all of your answers were equal, but all had the same ______ value.	absolute
7	This suggests that it might be helpful in comparing the products $(ab)c$ and $a(bc)$, to compute $\lvert(ab)c\rvert$ and $\lvert\ \ \ \rvert$.	$\lvert a(bc)\rvert$
8	In doing this, remember that if x and y are any real numbers, $\lvert xy\rvert = \lvert x\rvert\lvert\ \ \rvert$.	$\lvert xy\rvert = \lvert x\rvert\lvert y\rvert$
9	Applying this to $\lvert(ab)c\rvert$, $\lvert(ab)c\rvert = \lvert ab\rvert\lvert c\rvert = \lvert\ \rvert\lvert\ \rvert\lvert c\rvert$,	$\lvert(ab)c\rvert = \lvert a\rvert\lvert b\rvert\lvert c\rvert$
10	which we can write without parentheses because $\lvert a\rvert$, $\lvert b\rvert$, $\lvert c\rvert$, are numbers of arithmetic, and the ______	associative
11	property of multiplication holds for non-______ real numbers.	non-negative
12	In the same way, we can write $\lvert a(bc)\rvert = \lvert a\rvert\lvert b\rvert$ ______.	$\lvert a(bc)\rvert = \lvert a\rvert\lvert b\rvert\lvert c\rvert$
13	Combining Items 10 and 13, since $\lvert(ab)c\rvert$ and ______	$\lvert a(bc)\rvert$
14	are both equal to $\lvert a\rvert\lvert b\rvert\lvert c\rvert$, we can conclude that $\lvert(ab)c\rvert = $ ______	$\lvert(ab)c\rvert = \lvert a(bc)\rvert$

Having shown that $(ab)c$ and $a(bc)$ have the same absolute value for all real numbers a, b, c, we will be sure that $(ab)c = a(bc)$ if we can prove that--whatever the numbers a, b, and c represent--$(ab)c$ and $a(bc)$ are either both positive, both negative, or both 0. If you tried the second

proof for the commutative property in the last section, you noticed that the line of reasoning followed there parallels the reasoning outlined here.

We shall not have to consider the case if all of the numbers are positive since the associative property holds for all numbers of arithmetic.

	If one of the numbers a, b, c, is negative and the others are positive, either ab or c is negative and the other is positive. So the product (ab)c is	
15	______ (positive, negative)	negative
16	Similarly, either a or ______ is negative, and the other is positive.	bc
17	So the product a(bc) is ______.	negative
18	Hence, in this case, (ab)c and a(bc) are both ______.	negative

Thus, we have proved: If one of the numbers a, b, c, is negative and the others are positive, (ab)c and a(bc) are both negative. Moreover, we have proved that the absolute value of (ab)c is the same as the absolute value of a(bc). Hence, in this case (ab)c = a(bc).

For the other cases, we can proceed in the same way.

*If two of the numbers a, b, c, are negative and the other positive, either

c is positive and both a and b are negative, or
c is negative, one of the numbers a, b, is negative, and the other is positive.

*19	In the first case, ab is ______ since both a and b are negative.	positive
*20	Then (ab)c is ______ (positive, negative).	positive
*21	In the second case, ab is ______, and	negative
*22	(ab)c is ______.	positive
*23	In either case, (ab)c is ______.	positive
	Continuing in the same way, we can show that if two of the numbers a, b, c, are negative and the other is	
*24	positive, then (ab)c and a(bc) are both ______.	positive
	Finally, if a, b, c, are all negative, then	
*25	(ab)c and a(bc) are both ______.	positive

Since we are now sure that the associative property holds for all real numbers, we can write the product of the numbers a, b, and c as abc, omitting parentheses, without fear of confusion.

	The commutative and associative properties of multiplication permit us to compute the product of three or more numbers in any convenient order and grouping.	
	It is convenient to find $(-5)(17)(-20)$ as follows:	
26	$((-5)(-20))(17) = (\_\_\_\_)(17)$	$(100)(17)$
27	$= \_\_\_\_$	1700
	Perform the computations in the easiest way. In each indicate the grouping.	
28	$(\frac{2}{3})(\frac{7}{5})(-\frac{3}{4}) = (-\frac{1}{2})(\frac{7}{5}) = \_\_\_\_$	$(-\frac{1}{2})(\frac{7}{5}) = -\frac{7}{10}$
29	$(-7)(-25)(3)(-4) = (\quad)(100) = \_\_\_\_$	$(-21)(100) = -2100$
30	$(-\frac{2}{3})(\frac{4}{5})(\frac{3}{2})(-\frac{5}{4}) = (\quad)(\quad) = \_\_\_\_$	$(-1)(-1) = 1$
31	$(\frac{1}{5})(-19)(-3)(50) = (\quad)(\quad) = \_\_\_\_$	$(10)(57) = 570$
32	$(-1500)(\frac{1}{13})(-\frac{1}{15})(52) = (\quad)(\quad) = \_\_\_\_$	$(100)(4) = 400$

Recall that for any real number a, $1 \cdot a = a$. We may also ask: "What is the product $(-1)a$ for any real number, a?" It will turn out that the answer to this question will be very useful in our later work.

	Complete, using the simplest name:	
33	The opposite of 6 is ____.	-6.
34	The opposite of -3 is ____.	3
	Find the values of the following:	
35	$(-1)6 = \_\_\_\_$	-6
36	$(-1)(-3) = \_\_\_\_$	3
37	$(-1)(-\frac{2}{7}) = \_\_\_\_$	$\frac{2}{7}$
38	$(-1)(8) = \_\_\_\_$	-8
	The last item can be read:	
39	The product of -1 and 8 is the ____ of 8.	opposite

Similarly, the product of -1 and -3 is the

40 ______ of -3. — **opposite**

41 If a is any real number, $-a$ is the ______ of a. — **opposite**

The preceding responses should help you suspect a useful theorem: for every real number a

42 $(-1)a =$ ______. — **$-a$**

Theorem 8-3. For every real number a,

$$(-1)a = -a.$$

After we have discussed the distributive property for real numbers we shall give an interesting proof of this theorem. However, you may want to prove it now, using methods similar to those of the other proofs we have studied. We can start by expressing the absolute value of $(-1)a$ as a product of two absolute values. Try the proof if you are curious. If you try it, use Items 43 to 48 as a help or a check. Otherwise, proceed to Item 49.

Although your proof need not be exactly like this, it probably uses the same main ideas.

*43 $|(-1)a| = |-1||\quad|$ Theorem 8-1. — **$|-1||a|$**

*44 $= 1 \cdot |a|$,   since $|-1| =$ ______ — **1**

$= |a|$,   from the multiplication property of 1.

Moreover, $(-1)a$ and $-a$ are either both positive, both negative, or both ______.

*45 — **0**

In fact, if a is positive, then $(-1)a$ and $-a$

*46 are both ______, while if a is negative, then — **negative**

*47 $(-1)a$ and $-a$ are both ______. — **positive**

Hence, for all a we have: $(-1)a$ and $-a$ have the same absolute value and are either both positive, both

*48 negative, or both 0. Hence, they are ______. — **equal**

Use Theorem 8-3 to rewrite the following expressions as indicated:

49 $(-1)r =$ ______ — $-r$

50 $(-1)(-7) = -(-7) =$ ______ — 7

51 $-t = (-1)$______ — $(-1)t$

Here are some examples that suggest another theorem.

52	$(-3)(8) =$ ____ and $-(3 \cdot 8) =$ ____	-24, -24
53	$(-5)(-6) =$ ____ and $-(5(-6)) =$ ____	30, 30
54	$(7)4 =$ ____ and $-((-7)4) =$ ____	28, 28

We might guess that for all real numbers a and b

$$(-a)b = -(ab).$$

We could prove this in the same way as we proved Theorem 8-3. But we can also prove it in a different way, using Theorem 8-3. Try to construct the proof, using this method. Then read on.

We would like to prove: For any real numbers a and b,

55 $(-a)b = -(\quad)$. — $(-a)b = -(ab)$

From Theorem 8-3, we know that $-a = (-1)a$.

56 Hence, $(-a)b = (\quad)b$ — $((-1)a)b$

57 $= (-1)(ab)$ by the ____ property of multiplication — associative

$= -(ab)$ from Theorem 8-3

By the commutative and associative properties and Theorem 8-3, we can similarly show that

$$a(-b) = -(ab).$$

In practice, the expression $-(ab)$ is frequently written as $-ab$. In summary, we may state that for any real numbers a and b,

$$-ab = -(ab) = (-a)b = a(-b).$$

Write each product without parentheses.

58	$(3x)(-7y) =$ ____	$-21xy$
59	$(2x)(-4) =$ ____	$-8x$
60	$(-3a)(5bc) =$ ____	$-15abc$

	Let us now consider the product $(-a)(-b)$.	
	If a is 5 and b is 3, then	
61	ab is ______ and $(-a)(-b)$ is ______.	15, 15
	If a is (-5) and b is 3, then	
62	ab is ______ and $(-a)(-b) =$ ______.	-15, -15
	If a is (-5) and b is (-3), then	
63	ab is ______ and $(-a)(-b) =$ ______.	15, 15
	Even from these few examples, ~~you~~ probably suspect	
64	that $(-a)(-b) =$ ______.	ab

You can easily show that you are correct--that

$$(-a)(-b) = ab.$$

Try to construct the proof for yourself and compare your proof with the one shown on page ix.

	From the statement $$(-a)(-b) = ab,$$ can we say that if a and b are real numbers, then the	
65	product of their opposites is positive? ______	no, see Item 62

	Write each product without parentheses.	
66	$(-2a)(-5c) =$ ______	10ac
67	$(-xy)(-z) =$ ______	xyz
68	$(-x)^2 = (-x)(-x) =$ ______	x^2
69	$(-5c)(\frac{3}{5}d) =$ ______	-3cd
70	$(\frac{9}{2}bc)(-6a) =$ ______	-27abc
71	$(-3x)^2 =$ ______	$9x^2$

Let's look into the distributive property. This property holds in the set of all real numbers. The proof, however, requires many cases. A few examples will show you why.

The distributive property is: For all real numbers a, b, and c,

$$a(b + c) = ab + ac \quad \text{and} \quad (b + c)a = ba + ca.$$

We know already that this property holds for all non-negative real numbers. For example,

72 $7 \cdot 99 + 7 \cdot 1 = 7(\_\_ + \_\_) = 700$ — 99 + 1

Here are a few examples that suggest that the distributive property holds for all real numbers.

73 $(-3)(5 + 6) = (-3) \cdot 11 = \_\_\_\_$, and — -33

74 $(-3) \cdot 5 + (-3) \cdot 6 = -15 + (\_\_\_\_) = -33$ — (-18)

75 $-3((-4) + (-6)) = -3(-10) = \_\_\_\_$, and — 30

76 $(-3)(-4) + (-3)(-6) = 12 + 18 = \_\_\_\_$ — 30

77 $4((-8) + (-7)) = 4(\_\_\_\_) = -60$, and — -15

78 $4(-8) + 4(-7) = (-32) + (-28) = \_\_\_\_$ — -60

The distributive property is also illustrated by these examples:

79 $3((-5) + 8) = 3 \cdot 3 = \_\_\_\_$, and — 9

80 $3(-5) + 3(8) = -15 + 24 = \_\_\_\_$ — 9

81 $(-4)((-3) + 1) = (-4)(-2) = \_\_\_\_$, and — 8

82 $(-4)(-3) + (-4)(1) = 12 + (\_\_\_\_) = 8$ — -4

In order to give a formal proof that the distributive property holds in all cases it is necessary to use certain familiar properties of subtraction of the numbers of arithmetic. These properties have not been dealt with so far. (The whole matter of subtraction will be explored in Chapter 11.) However, it is a fact that the distributive property does hold true for all possible cases. We shall use this property of the set of all real numbers from now on without further justification.

Use the distributive property where helpful to perform the indicated operations with the minimum amount of work.

83 $(-9)(-97) + (-9)(-3)$ ____ — 900

84 $(.63)(6) + (-1.63)(6)$ ____ — -6

85 $(-\frac{3}{4})((-93) + (-7))$ ____ — 75

36	$(-\frac{3}{2})((-4) + 6) = $ ______	-3
37	$(-7)(\frac{2}{3}) + (-5)(\frac{2}{3}) = $ ______	-8

We have made frequent use of Theorem 8-:

For every real number a,

$$(-1)a = -a.$$

It is interesting to see how we can use the distributive property in proving this theorem.

	We want to prove that $(-1)a = -a$; that is, that the	
38	product of -1 and a is the ______ inverse of a.	additive
	If we can show that $a + (-1)a = 0$, then we will	
39	know that ______ is the additive inverse of a.	$(-1)a$
	Then since $-a$ is the only additive inverse of a, we shall be able to conclude:	
40	$(-1)a = $ ______.	$-a$

Following these suggestions, try to give a complete proof of the theorem. Compare your completed proof with the one below.

Theorem 8-. For every real number a,

$$(-1)a = -a.$$

Proof: $a + (-1)a$	$= 1 \cdot a + (-1)a$	Multiplication property of 1
	$= (1 + (-1))a$	Distributive property
	$= 0 \cdot a$	Since $1 + (-1) = 0$
	$= 0$	Multiplication property of 0
Hence, $(-1)a$ is a name for the additive inverse of a.		
But $-a$ is the only additive inverse of a.		The opposite of a number is the unique additive inverse of the number.
Hence, $(-1)a = -a$.		From the preceding two statements.

Notice that in this proof it is not necessary to study separate cases. Notice, too, that the proof demonstrates the fact that the distributive property ties together addition and multiplication.

*The following items suggest a proof for one case of the associative property that uses the same method of proof as was used in Section 8-2. It is a clumsy proof compared to that carried out in Items 2-25. It is included, however, to show you how you might have proceeded in working out the proof for yourself. When you come to a proof it is always a good idea to think about how you would construct it. Don't worry, though, if sometimes you don't succeed.

Item	Frame	Answer
	We want to consider the products $(ab)c$, and $a(bc)$, where a and b are positive and c is negative.	
	For example, if a is 7, b is 4, and c is -3, we have:	
*91	$(ab)c = (7\cdot4)(-3) = 28(-3) =$ _____	-84
*92	$a(bc) = 7\big(4(-3)\big) = 7($_____$) = -84$	$7(-12)$
*93	If a and b are positive, then ab is also _____.	positive
*94	Since c is _____,	negative
*95	$(ab)c = -(\|\ \|\|\ \|)$ from the definition of multiplication.	$-(\|ab\|\|c\|)$
	Since b is non-negative and c is negative, we have, again from the definition of multiplication,	
*96	$bc = -(\|b\|$ _____$)$.	$-(\|b\|\|c\|)$
*97	$\|b\|\|c\|$, of course, is _____, and $\big\|\,\|b\|\|c\|\,\big\| = \|b\|\|c\|$.	non-negative
	Consequently, once more using the definition of multiplication,	
*98	$a(bc) = -(\|\ \|(\|b\|\|c\|))$	$-\big(\|a\|(\|b\|\|c\|)\big)$

Let us compare our last result,

$$a(bc) = -\Big(|a|(|b||c|)\Big)$$

with our result in Item 95:

$$(ab)c = -(|ab||c|)$$

We see that we can replace $|ab|$ by $|a||b|$, and that when we do this we have:

$$(ab)c = -\Big((|a||b|)|c|\Big).$$

Now we are almost finished. $|a|$, $|b|$, and $|c|$ are absolute values of numbers. Hence, they are all <u>non-negative</u>. We know that

$$(|a||b|)|c| = |a|(|b||c|).$$

*99 since the ______ property of multiplication holds for — **associative**

*100 ______ real numbers. — **non-negative**

Thus, $(ab)c$ and $a(bc)$ are equal; because they are

*101 ______ of equal numbers. — **opposites**

8-4. Use of the Multiplication Properties

Recall that in Section 4-3 we mentioned some conventions of writing products. We want to emphasize that these are not hard and fast rules, but another convention that we might mention here is that we prefer to write ab instead of $(-a)(-b)$.

1 The expression $\big(11 + (-x)\big)(-c)$ is an indicated ______. — **product**

By the use of the distributive property,

2 $$\big(11 + (-x)\big)(-c) = \text{______}.$$ — **$11(-c)+(-x)(-c)$**

The expression $11(-c) + (-x)(-c)$ is an

3 indicated ______. — **sum**

4 $11(-c) + (-x)(-c)$ can also be written $\underline{-11 \quad + c \quad}$, which is the preferred way of writing the indicated sum. — **$-11c + cx$**

30_

In arithmetic, of course, you do not think of the reason for each step in a computation. You could find $3 \cdot 21$ by thinking:

$$\begin{aligned} 3 \cdot 21 &= 3 \cdot \left((2 \cdot 10) + 1\right) \\ &= 3(2 \cdot 10) + 3 \cdot 1 \\ &= (6 \cdot 10) + 3 \end{aligned}$$

If you are asked to write the indicated product as an indicated sum, often you can perform the manipulation without stopping to supply the reasons. For example, you may merely write:

$$\left(11 + (-x)\right)(-c) = -11c + cx.$$

While you are learning the technique, however, it is wise to refer back frequently to the properties of real numbers. This will help you to avoid errors.

	Which of the following sentences is true for all real numbers x? (Copy the true sentence.)	
	$3x(x + 1) = 3x^2 + 1$	
5	$3x(x + 1) = 3x^2 + 3x$ __________.	$3x(x+1)=3x^2+3x$
	What two properties of the real numbers make you sure it is true?	
6	(1) The ______ property and	distributive
7	(2) the ______ property of ______.	multiplication, 1

Now you will have a chance to practice some manipulations. Try to write your answers in the preferred form. Do not become discouraged at once if your results are written differently from the responses given. Perhaps your result is correct but it is not in the same form. Some of these will be applications of the distributive property and some will be applications of the associative and commutative properties.

	Give the simplest name for	
8	$-mp(2aq) =$ ______	$-2ampq$
9	$-4a(-7b) =$ ______	$28ab$
10	$-a(-ab) =$ ______	a^2b

Write these as indicated sums.

11	$m(5 + n) = $ ______ + ______	$5m + mn$
12	$(a + (-9))(-2)$ ______	$-2a + 18$
13	$-mnp(17 + y)$ ______	$-17mnp + (-mnpy)$
14	$(3 + ef)(-cd)$ ______	$-3cd + (-cdef)$
15	$2(a + b + c) = $ ______	$2a + 2b + 2c$
16	$-m(n + (-p))$ ______	$-mn + mp$
17	$-h(b + (-4))$ ______	$-bh + 4h$
18	$-d(-a + 5 + (-c))$ ______	$ad + (-5d) + cd$
19	$-1(y + (-z) + 5)$ ______	$-y + z + (-5)$
20	$-8((-4) + (-m))$ ______	$32 + 8m$

21 The ______ property can be used, as in the preceding items, to express an indicated product as an — distributive

22 indicated ______. — sum

Thus, we use the distributive property when we write the

23 indicated product $6(x + y)$ as an indicated sum ______ + ______. — $6x + 6y$

But we can do the reverse. The indicated sum $5x + 5y$ can be written as the indicated product $5(x + y)$.

24 Whether we prefer to use $5x + 5y$ or $5($ ______ $)$ depends on the particular situation. — $5(x + y)$

Use the distributive property to write the following indicated sums as indicated products.

25	$7x + 7y = 7($ ______ $)$	$7(x + y)$
26	$3a + 3(-b) = $ ______	$3(a + (-b))$
27	$(-9)b + (-9)c = $ ______	$-9(b + c)$
28	$3x + 3y + 3z$ ______	$3(x + y + z)$
29	$km + kp$ ______	$k(m + p)$
30	$(a + b)x + (a + b)y$ ______	$(a + b)(x + y)$

(Hint: Think of $(a + b)$ as a single number.)

303

Consider the indicated sum $7x + 14y$. We can express it as an indicated product. To do so, we need to notice that $14 = 7 \cdot 2$.

$$7x + 14y = 7x + 7 \cdot 2y = 7(x + 2y)$$

Use the distributive property to write the following indicated sums as indicated products:

31 $2a + 6b = 2(\_\_\_\_)$ — $2(a + 3b)$

32 $6x + 12y = \_\_\_\_$ — $6(x + 2y)$

33 $4x^2 + 10y = \_\_\_\_$ — $2(2x^2 + 5y)$

34 $26m^2 + 39(-n) = \_\_\_\_$ — $13(2m^2 + (-3n))$

35 Since $a = a \cdot 1$, $ax + a = ax + a \cdot 1 = a(\_\_\_\_)$. — $a(x + 1)$

Write the following indicated sums as indicated products:

36 $rx + r = \_\_\_\_$ — $r(x + 1)$

37 $5x + 5 = \_\_\_\_$ — $5(x + 1)$

38 $ac + bc + c = \_\_\_\_$ — $(a + b + 1)c$

Consider the following ways of writing $6x + 30y$ as an indicated product:

39 $6x + 30y = 2(3x + \_\_\_\_)$ — $2(3x + 15y)$

40 $6x + 30y = 3(2x + \_\_\_\_)$ — $3(2x + 10y)$

41 $6x + 30y = 6(\_\_\_\_)$ — $6(x + 5y)$

The ways listed are all correct; $1(6x + 30y)$, $30(\frac{1}{5}x + y)$, ..., are also correct ways of writing $6x + 30y$ as an indicated product. Usually we prefer the form in Item 41. This form, $6(x + 5y)$ has the advantage that the numbers used are all integers, and the indicated sum $x + 5y$ cannot be expressed as a product, without the use of fractions.

Let us try writing $3x + (-6y)$ as an indicated product. We see:

42 $$3x + (-6y) = 3x + (3)(\_\_\_\_) = 3(x + (-2y)).$$ — $(3)(-2y)$

We might also have written for the same indicated sum:

43 $$3x + (-6y) = (-3)(-x) + (-3)(2y) = (-3)(\_\_\_\_).$$ — $(-3)(-x + 2y)$

The indicated products in the last two items are $3(x + (-2y))$ and $(-3)(-x + 2y)$. In both of these indicated products, the numbers used are all integers, and the indicated sums, $(x + (-2y))$ and $(-x + 2y)$, cannot be expressed as a product without the use of fractions. Here, the indicated product which would usually be preferred is $3(x + (-2y))$.

44 $4a + 2b + (-6c) = 2(\_\_\_\_\_)$. — $2(2a + b + (-3c))$

Express the following indicated sums as indicated products:

45 $2m + (-14p) = \_\_\_\_\_$ — $2(m + (-7p))$

46 $(-7c) + 21 = \_\_\_\_\_$ — $7(-c + 3)$

47 $(-3x) + (-3) = \_\_\_\_\_$ — $(-3)(x + 1)$

48 $ax + bx = \_\_\_\_\_$ — $(a + b)x$

Of particular interest is this example:

> Express the indicated sum $3x + 2x$ as an indicated product.

49 $3x + 2x = (3 + 2)x = \_\_\_\_\_$. — $5x$

50 The fact that $3x + 2x = \_\_\_\_\_$ is a consequence of — $5x$

51 the _____ property. — distributive

Use the distributive property to write each indicated sum as an indicated product in simpler form.

52 $14x + x = (14 + \_\_)x = 15x$ — $(14 + 1)x$

53 $5x + (-3)x = \_\_\_\_\_x$ — $2x$

54 $3x + (-5)x = \_\_\_\_\_$ — $-2x$

55 $-12t + t = \_\_\_\_\_$ — $-11t$

56 $4a + (-10)a$ _____ — $-6a$

57 $\frac{3}{4}a + \frac{1}{2}a$ _____ — $\frac{5}{4}a$

In the examples above, the indicated product is simpler in form than the indicated sum.

We often say we simplify the expression $3x + x$ when we

58 write it in the form _____. — $4x$

59 We simplify $x + (-2x)$ when we write it as _____. — $-x$

60 $3(a + b) + 7(a + b) = \_\_\_\_(a + b)$ — $10(a + b)$

The word "simplify" is used quite generally in mathematics to refer to the process of changing an expression to a more compact or preferred form.

We can simplify $2x + x$ by using the distributive property. In fact, $2x + x = 3x$.

Can we simplify $2x + y$? No, there is no way to apply the distributive property to the problem of simplifying $2x + y$. Hence, it is customary to say that $2x + y$ is in its simplest form.

	Simplify each of the following, if this is possible. If it is not possible, copy the given expression on the response sheet.	
61	$(1.6)b + (2.4)b =$ ______	4b
62	$-5x + 2x + 11x =$ ______	8x
63	$8r + (-14r) + 6r =$ ______	0
64	$-2y + (-y) =$ ______	-3y
65	$7(x + y) + (-3)(x + y) =$ ______	4(x + y)
66	$7x + 8y =$ ______	7x + 8y
67	$-3a + 5a + 33a =$ ______	35a
68	$3x + 5y + 7z =$ ______	3x + 5y + 7z

In the indicated sum $u + v$, "u" and "v" are called terms; in a phrase of the form $u + v + w$, "u", "v", and "w" are called terms, etc. The terms in $xy + z$ are "xy" and "z". The terms in the phrase $3a + 2b$ are "3a" and "2b".

The expression $4x + (-2y) + \frac{1}{2}xy$ has three terms; these terms are: $4x$, $-2y$, and $\frac{1}{2}xy$.

	The terms in the phrase, $(-3a) + 5a$ are $-3a$ and $5a$. To simplify this phrase, we might note that	
69	$-3a =$ (______)a, so that	(-3)a
	$-3a + 5a = (-3)a + 5a.$	
70	We then use the ______ property to find that	distributive
71	$-3a + 5a =$ ______.	2a
	We say, in this example, that we have collected or combined terms.	

311

Collect terms in the following indicated sums:

72	$3x + 10x =$ ____	$13x$
73	$11k + (-2k) =$ ____	$9k$
74	$-9a + (-4a) =$ ____	$-13a$
75	$17n + (-16n) =$ ____	n
76	$x + 8x =$ ____	$9x$
77	$2x + 35x + (-15x) =$ ____	$22x$
78	$\frac{7}{8}a + \frac{9}{8}a =$ ____	$\frac{16}{8}a$ (or $2a$)

Sometimes we can apply the distributive property to part of an indicated sum. For example,

79	$3x + 8x + 2y = (3 + 8)x +$ ____ $= 11x + 2y$	$2y$

Since the last expression cannot be written in simpler form, $11x + 2y$ is the simplest form of $3x + 8x + 2y$.

Simplify the following expressions:

80	$3a + 5a + (-7c) =$ ____	$8a + (-7c)$
81	$6a + (-4a) + 5b + 14b =$ ____	$2a + 19b$
82	$(m + n) + m + (-n) =$ ____	$2m$
83	$x + 4 + 3x + 1 =$ ____	$4x + 5$
84	$5x + (-10x) + 5x =$ ____	0
85	$7x + (-10x) + 3y =$ ____	$-3x + 3y$
86	$6a + 5b =$ ____	$6a + 5b$

In the example,

$$x^2 + 5x + 7x^2 = (x^2 + 7x^2) + 5x$$
$$= 8x^2 + 5x,$$

we can go a step further and write $8x^2 + 5x$ as an indicated product:

$$x(8x + 5).$$

There will be times in our work when we shall prefer the form $8x^2 + 5x$ to $x(8x + 5)$. Furthermore, an expression such as $8x^2 + 5x + 3$ would usually be left as it is and not rewritten as $x(8x + 5) + 3$.

	Collect terms (where possible) in the following indicated sums:	
87	$12a + 5c + (-2c) + 3c^2 =$ ______	$12a + 3c + 3c^2$
88	$6a + 5b + c =$ ______	$6a + 5b + c$
89	$9p + 4q + (-3)p + 7q =$ ______	$6p + 11q$
90	$4x + (-2x^2) + (-5x) + 5x^2 + 1 =$ ______	$-x + 3x^2 + 1$ or $3x^2 + (-x) + 1$
91	$x + y + y + x =$ ______	$2x + 2y$

The process of collecting terms is often useful in solving open sentences. For example, consider the open sentence: $6x + 9x = 30$. If we collect terms in the expression $6x + 9x$, we can simplify the open sentence to: $15x = 30$.

Now consider the open sentence:

$$2x + 5 + 6x + (-5) + 11x = -38.$$

By combining terms, we can simplify the open sentence to:

$$19x = -38.$$

Both of these sentences can be solved by intelligent guesswork at this time. You will see more equations of this form in Chapter 9. There you will learn how to solve such sentences without guessing. Moreover, the method given there can be applied to sentences that may not be as easily obtained by guesswork.

Although multiplication is originally defined for two real numbers, we have seen how the commutative and associative properties of multiplication can be used to simplify certain expressions in the form of indicated products. For example, we can use these properties to express $(3xy)(7x)$ in the following simpler form:

$$(3xy)(7x) = (3\cdot 7)(x)(xy) = 21x^2y.$$

In an earlier section we saw how the distributive property could be used to write indicated products as indicated sums.

92	One form of the ______ property is: For all real numbers a, b, c:	distributive
93	$(a + b)c =$ ______	$ac + bc$
	If we apply this form of the distributive property to the indicated product $(x + 3)(x + 2)$ we have:	
94	$(x + 3)(x + 2) = \underline{x(x + 2) + \quad (\quad)}$	$x(x+2) + 3(x+2)$

95 We can apply the ______ property three more times: — distributive

96 Twice here: $x^2 + 2x +$ ______ + ______ — $3x$, 6

97 Once here: $x^2 + ($ ______ + ______ $)x + 6$, — 2, 3

and obtain $x^2 + 5x + 6$.

Let's try another indicated product,

$$(a + 7)(a + (-3)).$$

98 $(a + 7)(a + (-3)) = a(a + (-3)) +$ ______ — $7(a + (-3))$

$= a^2 + a(-3) + 7a + 7(-3)$

99 $= a^2 +$ ______ $+ (-21)$. — $a^2 + 4a + (-21)$

Write as indicated sums in the simplest form possible:

100	$(x + 8)(x + 2) =$ ______	$x^2 + 10x + 16$
101	$(y + (-3))(y + (-5)) =$ ______	$y^2 + (-8y) + 15$
102	$(6a + (-5))(a + (-2)) =$ ______	$6a^2 + (-17a) + 10$
103	$(a + 2)(a + 2) =$ ______	$a^2 + 4a + 4$
104	$(x + 6)(x + (-6)) =$ ______	$x^2 + (-36)$
105	$(3m + 2)(3m + (-2)) =$ ______	$9m^2 + (-4)$
106	$(2x + 5)(3x + (-7)) =$ ______	$6x^2 + x + (-35)$

The items below will give you more practice. The answers are to be found on page ix.

Write as indicated sums in the simplest form possible:

107. $(y + (-4))(3y + (-2)) =$ ______

108. $(m + (-3))(m + (-3)) =$ ______

109. $(a + (-c))(b + d) =$ ______

110. $(a + b)(a + b) =$ ______

111. $(3a + (-2))(a + 1) =$ ______

112. $(x + (-5))(4x + (-3)) =$ ______

113. $(c + (-3))\ (5 + c) =$ ______

114. $(-2 + y)(2y + (-3)) =$ ______

115. $(1 + (-n))(8 + (-5n)) =$ ______

116. $(x + 3)(x^2 + 2x + 1) =$ ______

117. $(2pq + (-8))(3pq + 7) =$ ______

118. $(5y + (-2x))(3y + (-x)) =$ ______

Write in the simplest form that you can:

119. $(-3xy)(2x^2) =$ ______

120. $100(-ab^2)(-\frac{1}{100}a^2b) =$ ______

8-5. Summary and Review

We have defined the product of two real numbers a and b as follows:

If a and b are both negative or both non-negative, then $ab = |a||b|$.

If one of the numbers a and b is positive or zero, and the other is negative, then $ab = -(|a||b|)$.

From this definition and the properties of operations upon numbers of arithmetic, the following properties of operations can be proved:

1. Multiplication property of 0: For any real number a,

$$(a)(0) = 0 \text{ and } (0)(a) = 0.$$

2. Multiplication property of 1: For any real number a,

$$(a)(1) = a \text{ and } (1)(a) = a.$$

3. Commutative property of multiplication: For any real numbers a and b,

$$ab = ba.$$

4. Associative property of multiplication: For any real numbers a, b, c,

$$(ab)c = a(bc).$$

5. Distributive property: For any real numbers, a, b, and c,

$$a(b + c) = ab + ac,$$

and $$(b + c)a = ba + ca.$$

Review Problems

The answers for these review problems are on page x.

1. Change to indicated sums:

(a) $3a(a + (-2))$

(b) $(x + 1)(x + 6)$

(c) $(a + b)(a + (-b))$

(d) $(m + (-5))^2$

(e) $(x + (-4))(2x + 3)$

2. Write each of the following as an indicated product:

(a) $2ax + 2ay$

(b) $ac + (-bc) + c$

(c) $c(a + b)x + (a + b)y$

(d) $10x^2 + (-15x) + (-5)$

3. Collect terms in the following phrases:

(a) $3a + b + a + (-2b) + 4b$

(b) $7x + b + (-3x) + (-3b)$

(c) $6a + (-7a) + 13.2 + (-5)a + (-8.6)$

(d) $|x| + 3|-x| + (-2)|-x|$

4. A record dealer makes a \$3 profit on records of brand X and takes a \$2 loss on brand Y. He sold 15 of brand X and 20 of brand Y; he also accepted 5 returned records of brand X and 7 of brand Y. What were his proceeds?

5. A mutual fund corporation has the following assets in companies A, B, C, D, E, and F.

500 shares in A; 300 shares in B; 125 shares in C;
500 shares in D; 400 shares in E; 175 shares in F.

At the end of the year, the gain or loss in points (dollars) per share for each of the companies were as follows:

A, $+\frac{3}{4}$; B, $-\frac{3}{8}$; C, $+\frac{3}{8}$; D, $+\frac{1}{4}$; E, $-\frac{7}{8}$; F, $+\frac{3}{8}$.

What was the profit or gain from this investment program?

6. A man's paycheck for a work week of 48 hours was \$166.40. He is paid at the rate of $\frac{3}{2}$ times his normal rate for all hours worked in excess of 40 hours. If the rate of pay for one hour's work is x dollars, write an open sentence corresponding to the given information.

Chapter 9

MULTIPLICATIVE INVERSE

9-1. Multiplicative Inverse

1	$3 + \_\_\_ = 3$	$3 + 0 = 3$
2	$3 + (\ \ ) = 0$	$3 + (-3) = 0$
3	$3(\ \ ) = 3$	$3(1) = 3$
4	$3(\ \ ) = 1$	$3(\frac{1}{3}) = 1$
5	$3 + 0 = 3$ is true by the ______ property of 0.	addition
	For any real number a,	
6	$a + 0 = \_\_\_\_$,	$a + 0 = a$
7	and $0 + a = \_\_\_\_$.	$0 + a = a$
	The number 0 is called the identity element for addition.	
	Another way of saying this is that 0 is the additive identity.	
	Since 0 is the only real number such that $a + 0 = a$ and $0 + a = a$, we can say that 0 is the only	
8	additive ______.	identity
9	$3 + (-3) = 0$ is true by the ______ property of opposites.	addition
10	For any real number a, there exists ______ (exactly, more than) one real number $(-a)$ such that	exactly
11	$a + (-a) = \_\_\_\_$	$a + (-a) = 0$
12	and $(-a) + a = \_\_\_\_$.	$(-a) + a = 0$
	The number $-a$, the opposite of a, is also called	
13	the additive ______ of a.	additive inverse
	The number a, the opposite of $-a$, is the	
14	______ ______ of $-a$.	additive inverse

15	$3 \cdot 1 = 3$ is true by the ______ property of 1.	multiplication
	For any real number a,	
16	$a \cdot 1 =$ ______	$a \cdot 1 = a$
17	and, $1 \cdot a =$ ______.	$1 \cdot a = a$
	The number 1 is called the identity element for	
18	multiplication, or the multiplicative ______.	identity

	$3(\frac{1}{3}) = 1$	
19	The product of 3 and $\frac{1}{3}$ is the ______ identity.	multiplicative
20	We called 3 and (-3) additive ______, since	inverses
21	$3 + (-3) =$ ______.	$3 + (-3) = 0$
	In the same way, we call 3 and $\frac{1}{3}$ multiplicative	
22	inverses, since $3 \cdot \frac{1}{3} =$ ______.	$3 \cdot \frac{1}{3} = 1$
23	$\frac{1}{4}$ and 4 are ______ inverses, since $\frac{1}{4} \cdot 4 = 1$.	multiplicative
24	$(1)(\ \ ) = 1$. Therefore, 1 is its own	$(1)(1) = 1$
25	multiplicative ______.	inverse

26	The sum of a number and its additive ______ is	inverse
27	the ______ identity.	additive
28	The product of a number and its ______ inverse is	multiplicative
29	the multiplicative ______.	identity

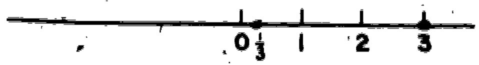
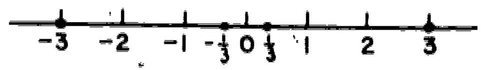
If a and b are real numbers such that

$$ab = 1$$

then b is called the multiplicative inverse of a. a is also called the multiplicative inverse of b.

Does every real number have a multiplicative inverse? At this point you do not know the answer to this question. However, we do know that every non-zero number of arithmetic has a multiplicative inverse. This inverse is also a non-zero number of arithmetic.

Let us try to discover something about the existence of multiplicative inverses for negative numbers.

30	Locate the numbers 3 and $\frac{1}{3}$ on the number line. 	See answer below.
31	$3 \cdot \frac{1}{3} =$ ______, that is, 3 and $\frac{1}{3}$ are	$3 \cdot \frac{1}{3} = 1$
32	multiplicative ______.	inverses
33	Now locate the opposite of 3, and the opposite of $\frac{1}{3}$. 	See answer below.
34	We know that $(-3)(-\frac{1}{3}) = (\ \ )(\ \ )$ $= 1.$	$(-3)(-\frac{1}{3}) = (3)(\frac{1}{3})$
35	Since $(-3)(-\frac{1}{3}) = 1$, we can say that -3 and $-\frac{1}{3}$ are multiplicative ______.	inverses

From the fact that 3 and $\frac{1}{3}$ are multiplicative inverses, we could conclude that -3 and $-\frac{1}{3}$ are also multiplicative inverses. This suggests a general theorem about negative numbers and multiplicative inverses.

36	We know that ______ (every, not every) positive number has a multiplicative inverse.	every
	We would like to prove that every negative number has a multiplicative inverse.	
37	We also know that, if a is a positive number, $-a$ is a ______ number.	negative
38	Indeed, every negative number is the ______ of a positive number.	opposite (additive inverse)
39	If we can show that the opposite of every positive number has a multiplicative inverse, then we can conclude that every ______ number has a	negative
40	______ inverse.	multiplicative

In order to show that every negative number has a multiplicative inverse, we first prove the following theorem.

Theorem 9-1. If a is a positive number and $ab = 1$, then $(-a)(-b) = 1$.

41 $(-a)(-b) = (\quad)(\quad)$,

(for all real numbers $(-a)(-b) = (a)(b)$).

$(-a)(-b)=(a)(b)$

42 $(-a)(-b) =$ ______

since $(a)(b) = 1$ by assumption.

$(-a)(-b) = 1$

We have proved that if a and b are positive real

43 numbers and a and b are ______ inverses, then

44 $(-a)$ and $(-b)$ are also multiplicative ______.

multiplicative

inverses

Now let us consider any non-zero negative number.

45 Every negative number is the opposite of some ______ number a.

positive

Since every positive number a has a multiplicative inverse, there is a positive number b such

46 that ______ $= 1$.

$ab = 1$

By Theorem 9-1, if $ab = 1$, then $(-a)(-b) =$ ______.

$(-a)(-b) = 1$

Therefore, a multiplicative inverse for $(-a)$ is ______

$-b$

We can conclude that every negative number has a multiplicative inverse.

	Number	Multiplicative Inverse	
49	4	______	$\frac{1}{4}$
50	-4	______	$-\frac{1}{4}$
51	$\frac{1}{3}$	______	3
52	$-\frac{1}{3}$	______	-3
53	$\frac{10}{3}$	______	$\frac{3}{10}$
54	$-\frac{10}{3}$	______	$-\frac{3}{10}$
55	$.2$	______	5
56	$-.2$	______	-5

57 Are we sure that every negative number has a multiplicative inverse?

[A] yes [B] no

For any negative number, such as $-.025$, we find the multiplicative inverse by finding the inverse of $.025$ and taking its opposite. Thus, since $(.025)(40) = 1$, the inverse of $-.025$ is -40. Any negative number has a multiplicative inverse. The correct choice is [A].

We have been careful to consider inverses of positive numbers and negative numbers. What about the number 0? Does 0 have an inverse?

If 0 is to have an inverse, then there must be a

58 number b such that $0 \cdot b =$ ______. — $0 \cdot b = 1$

59 But, by the multiplication property of 0, $0 \cdot b =$ ______ for every real number b. — $0 \cdot b = 0$

Therefore, for any real number b, $0 \cdot b \neq 1$.

0 cannot have a multiplicative inverse.

60 Decide whether or not each of the following numbers has a multiplicative inverse: $\frac{1}{2}$, $(-\frac{5}{4})$, 10, 0, 1.

[A] Each has a multiplicative inverse.

[B] Each of the numbers listed, except 0, has a multiplicative inverse.

[B] is correct. Each number listed except 0 has a multiplicative inverse. $\frac{1}{2} \cdot 2 = 1$; $(-\frac{5}{4})(-\frac{4}{5}) = 1$; $10 \cdot (\frac{1}{10}) = 1$; and $1 \cdot 1 = 1$. But does 0 have a multiplicative inverse? In other words, is there a number b such that $0 \cdot b = 1$? No!

Consider the following example:

$$(-2)(-3) = 6.$$

(-2) and (-3) and 6 are different names for the

61 same ______. — number

Let us multiply $(-2)(-3)$ by 4 and also multiply 6 by 4. We have

$$(4)\big((-2)(-3)\big) = 4 \cdot 6$$

62 $24 =$ ______ — 24

Since $(-2)(-3)$ and 6 are different names for the same number, when we multiply this number by 4, we obtain

$$4\big((-2)(-3)\big) \quad \text{and} \quad (4)(6)$$

as different names for the same new number.

63 $(5)(-3)$ and -15 are different names for the ______ number. — same

Therefore, if we multiply this number by $(-\frac{1}{5})$, we can write

64 $$\big((3)(-5)\big)(-\tfrac{1}{5}) = (-15)(\underline{\quad}).$$ — $-\frac{1}{5}$

$\big((3)(-5)\big)(-\frac{1}{5})$ and $(-15)(-\frac{1}{5})$ are different names for the same new number since

$$\big((3)(-5)\big)(-\tfrac{1}{5}) = (-15)(-\tfrac{1}{5})$$

$$3 = 3.$$

These examples illustrate a very useful property of multiplication and equality. Since we shall use this property a great deal, let us call this the <u>multiplication property of equality</u> which we state as follows:

For any real numbers a, b and c,
if $a = b$ then $ac = bc$ and $ca = cb$.

The multiplication property of equality follows, as did the addition property of equality, from the idea that "$a = b$" means "a and b are names for the same number".

65 Which of the following statements is <u>not</u> an illustration of the multiplication property of equality?

[A] if $2x = 6$ is true for some x,
then $2x(\frac{1}{2}) = 6(\frac{1}{2})$ is true for the same x.

[B] If $\frac{1}{3}a = 7$ is true for some a,
then $3(\frac{1}{3}a) = 3(7)$ is true for the same a.

[C] If $\frac{1}{4}n = 12$ is true for some n,
then $\frac{1}{4}n(4) = 12(\frac{1}{4})$ is true for the same n.

[D] If $24 = \frac{3}{5}m$ is true for some m,
then $(\frac{5}{3})24 = (\frac{5}{3})\frac{3}{5}m$ is true for the same m.

If $\frac{1}{4}n = 12$ is true for some n, is $\frac{1}{4}n(4) = 12(\frac{1}{4})$ also true? We must multiply $\frac{1}{4}n$ and 12 by the <u>same</u> number to be sure we obtain the <u>same</u> <u>new</u> number. [C] is <u>not</u> an illustration of the multiplication property of equality.

In Chapter 7 we found the addition property of equality of great importance in solving equations. Later in this chapter we shall learn more about the solution of equations.

The multiplication property of equality will help us prove an important theorem about multiplicative inverses.

66	If a is a real number, then there is ______ (exactly, at least) one additive inverse.	exactly
67	What is a multiplicative inverse of 4? ______	$\frac{1}{4}$
	Write the multiplicative inverse for each of the following:	
68	$(-\frac{3}{5})$ ______	$-\frac{5}{3}$
69	2 ______	$\frac{1}{2}$
70	$\frac{8}{5}$ ______	$\frac{5}{8}$
71	From our experience we would guess that each non-zero real number has ______ (exactly, at least) one multiplicative inverse.	exactly

Let us prove that each non-zero real number has <u>exactly</u> one multiplicative inverse.

	If $a \neq 0$, then a has at least one multiplicative inverse, that is, there is a b such that $ab = 1$ and $ba = 1$. Assume there is also a number c such that $ac = 1$ and $ca = 1$. We shall prove that $b = c$. Write: $ab = ab$		
72	$c(ab) = c(ab)$	by multiplication property of equality.	$c(ab) = c(ab)$
73	$c(ab) = (\ \ )b$	by associative property of multiplication.	$c(ab) = (ca)b$

74	$c \cdot 1 = \underline{\quad} b$	since $ab = 1$ and $ca = 1$.	$c \cdot 1 = 1 \cdot b$
75	$c = b$	multiplication property of ______.	1

We have proved that the multiplicative inverse is <u>unique</u>; that is, every non-zero real number has <u>exactly</u> one multiplicative inverse.

9-2. Reciprocals

We know that every non-zero negative number has a <u>unique</u> multiplicative inverse. That is, there is <u>exactly</u> one real number b such that

$$ab = 1 \quad \text{and} \quad ba = 1.$$

We shall find it convenient to use a shorter name for the multiplicative inverse. We use the name <u>reciprocal</u>. This parallels the use of the name <u>opposite</u> for the additive inverse.

1	The additive inverse of the number a is written (____).	$(-a)$
2	Another name for the additive inverse of a is the ______ of a.	opposite
3	The <u>reciprocal</u> of the number a is the ______ inverse of a.	multiplicative
	We agree to write the <u>reciprocal</u> of a as $\frac{1}{a}$.	
4	Thus if a is a non-zero real number, its multiplicative inverse may be written ______.	$\frac{1}{a}$
5	For example, $\frac{1}{5}$ is the reciprocal of ______.	5
6	$\frac{1}{3}$ is the ______ of 3.	reciprocal
7	______ is the reciprocal of 7.	$\frac{1}{7}$
	The number $\frac{1}{5}$ is also a rational number. From our experience we know that	
8	$5 \cdot \frac{1}{5} =$ ______.	$5 \cdot \frac{1}{5} = 1$

Thus calling $\frac{1}{5}$ the multiplicative inverse of 5 agrees with out knowledge of arithmetic.

The definition of $\frac{1}{a}$ as the "multiplicative inverse of a" agrees with our experience from arithmetic. What about other real numbers?

9 Can we say that every non-zero real number a has a reciprocal?

[A] yes [B] no

Every non-zero real number a, has exactly one multiplicative inverse. We have called this multiplicative inverse of a "the reciprocal of a". The number a must have a reciprocal which we can write as $\frac{1}{a}$. The correct answer is yes!

10 Does 0 have a reciprocal?

[A] yes [B] no

Does 0 have a multiplicative inverse? Since we have proved that 0 does not have a multiplicative inverse, we know it does not have a reciprocal. Remember that "reciprocal" is a simpler name for the multiplicative inverse. The correct answer is no!

Since 0 has no reciprocal, the symbol $\frac{1}{0}$ has no meaning.

11 If a is (-1) which of the following numbers have no reciprocals?

R. $a + 1$ S. $(a + (-1))a$ T. $a^2 + (-1)$

[A] R and S
[B] R and T
[C] S and T

[B] is the correct choice. R and T have no reciprocals if a is (-1). For if a is (-1), then $a + 1$ and $a^2 + (-1)$ are both 0, and 0 has no reciprocal. However $(a + (-1))a$ is not 0 if a is -1, and hence, it has a reciprocal.

12	If a is ______ then $\frac{a}{a+1}$ is meaningless.	-1
13	When a is 0, $\frac{a}{a+1}$ = ______.	0
14	Hence $\frac{a}{a+1}$ ______ (has, does not have) a reciprocal when a is 0.	does not have
15	The set of values for which $a^2 + (-1)$ has no reciprocal is { ______ }.	$\{1, -1\}$
16	Since $a^2 + 1$ is different from 0 for all values of a, the set of values for which $a^2 + 1$ has no reciprocal is ______.	$\emptyset$
17	$\frac{1}{a^2+1}$ has a reciprocal for ______ (all, no) values of a.	all
18	We saw that every real number except 0 has exactly one multiplicative ______.	inverse
19	We can restate this: "Every real number except 0 has exactly ______ reciprocal(s).	one
20	$\frac{1}{\frac{1}{3}}$ is the ______ of $\frac{1}{3}$.	reciprocal
21	We know that $\frac{1}{3} \cdot 3$ = ______, so that 3 is also the reciprocal of $\frac{1}{3}$.	$\frac{1}{3} \cdot 3 = 1$
22	Since $\frac{1}{3}$ has only ______ (how many) reciprocal(s), this means that $\frac{1}{\frac{1}{3}} = 3$.	one
	We can use the same reasoning to find a simpler name for $\frac{1}{-\frac{3}{4}}$.	
23	By definition, $\frac{1}{-\frac{3}{4}}$ means the ______ of $-\frac{3}{4}$.	reciprocal
24 25	Since $(-\frac{3}{4})(-\frac{4}{3})$ = ______, $-\frac{4}{3}$ is the ______ of $-\frac{3}{4}$.	$(-\frac{3}{4})(-\frac{4}{3}) = 1$, reciprocal
26	Since $-\frac{3}{4}$ has only 1 reciprocal(s), $\frac{1}{-\frac{3}{4}} = -\frac{4}{3}$.	one

The reciprocal of 0.25 can be written $\frac{1}{0.25}$.

The common name for $\frac{1}{0.25}$ is 4.

27 This is true since $0.25 = \frac{1}{4}$, and $4(\frac{1}{4}) =$ ______. 1

28 Hence, 4 is the ______ of 0.25. reciprocal

Try this problem:

What number is equal to its reciprocal? If n represents the number, we may write its reciprocal

29 as ______. $\frac{1}{n}$

An open sentence suggested by this problem is

30 n = ______. $n = \frac{1}{n}$

31 The truth set of this sentence is {-1, }. {-1,1}

---

32 The reciprocal of -9 is ______. $\frac{1}{-9}$

33 However $(-9)(-\frac{1}{9}) = (9)(\frac{1}{9}) =$ ______. 1

Thus, $-\frac{1}{9}$ is the multiplicative inverse of -9.

34 That is, $-\frac{1}{9}$ is the ______ of -9. reciprocal

35 Since the reciprocal is unique, $\frac{1}{-9} =$ ______. $\frac{1}{-9} = -\frac{1}{9}$

Similarly, we could show that $\frac{1}{-6} = -\frac{1}{6}$, or

$\frac{1}{-3} = -\frac{1}{3}$.

We suspect that for any non-zero number a, $\frac{1}{-a} = -\frac{1}{a}$.

Let us prove that for every real number a except 0,

$\frac{1}{-a} = -\frac{1}{a}$.

36 Since $(-\frac{1}{a})(-a) = (\frac{1}{a})(a) =$ ______, $-\frac{1}{a}$ is the 1

reciprocal of -a.

37 But -a has exactly one ______, $\frac{1}{-a}$. reciprocal

38 Therefore: $\frac{1}{-a}$ ______ $-\frac{1}{a}$. $\frac{1}{-a} = -\frac{1}{a}$

The fact that every number except 0 has exactly one reciprocal was used to prove $\frac{1}{-a} = -\frac{1}{a}$.

A number of other useful theorems can be proved easily in a similar way. Suppose we wish to find a simpler expression for $\frac{1}{\frac{1}{a}}$, where a is any real number except 0. As usual, trying a few examples helps us to state the theorem we would like to prove.

39	$\frac{1}{9}$ is the reciprocal of ______.	9
40	$\frac{1}{\frac{1}{9}}$ is the reciprocal of ______.	$\frac{1}{9}$
41	Since $\frac{1}{9}(9) = 1$, we see that ______ is also the reciprocal of $\frac{1}{9}$. But $\frac{1}{9}$ has only one reciprocal, hence, $\frac{1}{\frac{1}{9}} = 9$.	9
42	Since $\frac{1}{\frac{1}{9}}$ is the ______ of $\frac{1}{9}$, which is the reciprocal of 9; "$\frac{1}{\frac{1}{9}} = 9$" can be read:	reciprocal
43	"the reciprocal of the reciprocal of 9 is ______".	9
44	Is it true that for any a except 0, $\frac{1}{\frac{1}{a}} =$ ______?	a
45	Is the reciprocal of the ______ of a the number a?	reciprocal
46	Of course, we must be sure that $\frac{1}{a} \neq$ ______ if $\frac{1}{a}$ is to have a reciprocal. Since $a \neq 0$, $\frac{1}{a}$ cannot be 0. Thus, for any real number a except 0, $\frac{1}{\frac{1}{a}}$ is a real number.	$\frac{1}{a} \neq 0$

We are ready to prove that the reciprocal of the reciprocal of a non-zero number is the number itself.

47	For $a \neq 0$, $a \cdot \frac{1}{a} =$ ______, $\frac{1}{a}$ is the reciprocal of	1
48	of ______ and a is the reciprocal (multiplicative inverse) of $\frac{1}{a}$.	a
49	By definition, the reciprocal of $\frac{1}{a}$ is written ______.	$\frac{1}{\frac{1}{a}}$

50	Since $\frac{1}{a}$ has only one ______	reciprocal
51	$\frac{1}{\frac{1}{a}} =$ ______.	$\frac{1}{\frac{1}{a}} = a$

We have proved that the reciprocal of a number is the number itself.

52	From arithmetic we know: $\frac{1}{3} \cdot \frac{1}{4} =$ ______.	$\frac{1}{12}$
	How shall we compute $\frac{1}{-3} \cdot \frac{1}{-4}$?	
53	Remember, $\frac{1}{-3}$ means "the ______ of -3".	reciprocal
54	Can we be sure that a rule which works for the numbers of arithmetic holds for numbers such as $\frac{1}{-3}$ and $\frac{1}{-4}$? ______ (yes, no)	no
	In computing $\frac{1}{-3} \cdot \frac{1}{-4}$ we note that $\frac{1}{-3} = -\frac{1}{3}$,	
55	and $\frac{1}{-4} =$ ______.	$-\frac{1}{4}$
56	$\frac{1}{-3} \cdot \frac{1}{-4} = (-\frac{1}{3})(-\frac{1}{4})$ ______.	$\frac{1}{12}$
57	$\frac{1}{-2} \cdot \frac{1}{3} = (\quad)(\frac{1}{3}) =$ ______.	$(-\frac{1}{2})(\frac{1}{3}) = -\frac{1}{6}$
	From the previous theorem we know that $-\frac{1}{6} = \frac{1}{-6}$.	
	Therefore $\frac{1}{-2} \cdot \frac{1}{3} = \frac{1}{-6}$.	
	In general, for all numbers a and b different from 0, is $\frac{1}{a} \cdot \frac{1}{b} = \frac{1}{ab}$?	

	Let us prove that, for all real numbers a and b different from 0, $\frac{1}{a} \cdot \frac{1}{b} = \frac{1}{ab}$.	
58	We remember that $\frac{1}{ab}$ is the ______ of ab.	reciprocal
59	We shall again use our knowledge that any number except 0 has exactly ______ reciprocal(s).	one

60	We see that $(\frac{1}{a} \cdot \frac{1}{b})(ab) = (\frac{1}{a} \cdot a)(\frac{1}{b} \cdot$ ____ $)$ from the	b
61	associative and ______ properties of multiplication.	commutative
62	But, $\frac{1}{a} \cdot a =$ ______ and $\frac{1}{b} \cdot b = 1$.	1
63	Hence, $(\frac{1}{a} \cdot \frac{1}{b})(ab) =$ ______.	1
64	Hence, $\frac{1}{a} \cdot \frac{1}{b}$ is the ______ of ab.	reciprocal
65	But $\frac{1}{ab}$ is the ______ of ab, and we know that ab	reciprocal
66	has only ______ reciprocal.	one
	Hence $\frac{1}{a} \cdot \frac{1}{b} = \frac{1}{ab}$.	

Express the following indicated products in their simplest form (assume all variables are different from 0).

[illegible]	$\left(\frac{1}{[illegible]a}\right)\left(\frac{1}{[illegible]b}\right)$ ______	$\frac{1}{6ab}$
[illegible]	$\left(\frac{1}{x}\right)\left(\frac{1}{3ax}\right)$ ______	$\frac{1}{3ax^2}$

(Remember that we have agreed to write $3ax^2$ rather than $x \cdot 3ax$ or $3 \cdot x \cdot a \cdot x$.)

[illegible]	$\left(\frac{1}{[illegible]}\right)\left(\frac{1}{[illegible]}\right)$ ______	$\frac{1}{21yz}$
[illegible]	$\left(\frac{1}{[illegible]}\right)\left(\frac{1}{[illegible]}\right)$ ______	$\frac{1}{3a^2b}$
[illegible]	$\left(\frac{1}{[illegible]}\right)\left(\frac{1}{[illegible]}\right)$ ______	$\frac{1}{6m^3n^3}$
[illegible]	$\left(\frac{[illegible]}{[illegible]}\right)\left(\frac{1}{[illegible]}\right)$ ______	$\frac{1}{1}$, or 1

73	$87 \times (-5) \times 0 \times \frac{2}{3} \times 642 =$ ______	0
74	$8 \cdot 17 = 0$ is ______ (true, false)	false
75	If $p = 5$, $q = 7$, then $p \cdot q$ ___ 0. $(=, \neq)$	$p \cdot q \neq 0$
76	If $p = 0$, $q = 0$, then $p \cdot q$ ___ 0.	$p \cdot q = 0$
77	If $n \cdot 50 = 0$, then we suspect that $n =$ ______.	$n = 0$
78	If $3p = 0$, then we suspect that $p =$ ______.	$p = 0$

We know that if $a = 0$ or $b = 0$, $ab = 0$. If $ab = 0$, what can we say about a or b? Items 73-78 suggest the following.

Theorem: For real numbers a and b, if $ab = 0$ then $a = 0$ or $b = 0$.

Proof:

79	For any number a, $a = 0$ or $a \neq 0$ but not ______.	both
80	If $a = 0$, then "$a = 0$ or $b = 0$" is ______ and the theorem is proved.	true
81	If $a \neq 0$, then a has a ______.	reciprocal
	If $ab = 0$	
82	$\frac{1}{a}(ab)$ = ______ $\cdot 0$ by the multiplication property of equality.	$\frac{1}{a}(ab) = \frac{1}{a} \cdot 0$
83	$(\frac{1}{a} \cdot a)b$ ______, by the associative property and multiplication property of 0.	$(\frac{1}{a} \cdot a)b = 0$
84	$1 \cdot b = 0$ by the definition of ______.	reciprocal
	$b = 0$ by the multiplication property of 1.	

Thus, if $a = 0$, "$a = 0$ or $b = 0$" is true and the theorem is proved. If $a \neq 0$, b must be 0 and "$a = 0$ or $b = 0$" is true and the theorem is proved.

Look carefully at the wording of the theorem we have just proved:

If $ab = 0$ then $a = 0$ or $b = 0$.

85	When is "$n \cdot 10 = 0$" true? <u>Only if</u> $n =$ _____.	0
86	Similarly $5 \cdot x = 0$ only if $x =$ _____.	0
	$xy = 0$ <u>only if</u> $x = 0$ or $y = 0$.	
	And in fact we could state the theorem this way:	
87	$ab = 0$ _____ _____ $a = 0$ or $b = 0$.	only if

We already know the multiplication property of 0; that is, that:

$ab = 0$ <u>if</u> $a = 0$ or $b = 0$.

We have proved:

$ab = 0$ <u>only if</u> $a = 0$ or $b = 0$.

A quick way of combining these two statements is:

$ab = 0$ <u>if and only if</u> $a = 0$ or $b = 0$.

	To be sure that $ab = 0$ <u>if and only if</u> $a = 0$ or $b = 0$ we had to know <u>two</u> things:	
88	if $a = 0$ or $b = 0$ then _____.	$ab = 0$
89	if $ab = 0$ then _____ or _____.	$a = 0$ or $b = 0$
	In proving our "if and only if" statement we had to	
90	prove _____ (how many) parts.	two
91	We have proved: $ab = 0$ _____ _____ _____ _____ $a = 0$ or $b = 0$.	if and only if

The statement that $ab = 0$ only if $a = 0$ or $b = 0$ is useful in finding truth sets of certain equations.

92	The truth set of $p \cdot 50 = 0$ is _____.	$\{0\}$
93	The truth set of $7(x + (-5)) = 0$ is _____.	$\{5\}$
94	The truth set of $9 \cdot y \cdot 17 \cdot 13 = 0$ is _____.	$\{0\}$

Consider the equation $(x + (-3))(x + (-8)) = 0$.

The left side of this equation is a product.

95 It is 0 <u>if and only if</u> $x + (-3) = 0$ or $x +$ ___ $= 0$. — (-8)

96 The truth set of $x + (-3) = 0$ is _______. — $\{3\}$

97 The truth set of $x + (-8) = 0$ is _______. — $\{8\}$

Hence, the truth set of $(x + (-3))(x + (-8)) = 0$
98 is the union of $\{3\}$ and $\{8\}$, or _______. — $\{3,8\}$

The open sentence $(x + (-20))(x + (-100)) = 0$ has the same truth set as the compound sentence: $x + (-20) = 0$
99 or _______. — $x + (-100) = 0$

100 The truth set is _______. — $\{20,100\}$

The open sentence $(x + 6)(x + 9) = 0$ is equivalent
101 to "$x + 6 = 0$ or _______." — $x + 9 = 0$

102 The truth set is _______. — $\{-6,-9\}$

Find the truth set for each equation: Turn to page x for the correct answers.

103. $(x + (-1))(x + (-2))(x + (-3)) = 0$

104. $2(x + \frac{1}{2})(x + (-2)) = 0$

105. $x(x + 4) = 0$

106. $x^2(x + (-1))^2 = 0$

107. $9|x + (-6)| = 0$

108. $x(x + 4) = x^2 + 8$

---

109. Try to construct a proof of the following theorem:

If a, b, c are real numbers, and if $ac = bc$, and $c \neq 0$, then $a = b$.

Compare your proof with the proof on page x.

9-3. Solution of Equations

Consider the equation

$$\frac{1}{2}x = 5.$$

If there is a number x for which $\frac{1}{2}x = 5$ is true, then, using the multiplication property of equality,

1 $2 \cdot \frac{1}{2}x = 2 \cdot 5$ is a true for the same x, $2 \cdot \frac{1}{2}x = 2 \cdot 5$

2 $2 \cdot \frac{1}{2}x = 10$ and $x =$ _____ are also true for the same x. 10

The reasoning used in Items 1 and 2 above is similar to that developed in Section 7-4, where we used the addition property of equality to solve some equations.

Suppose we consider another example showing in detail the steps in solving the equation.

Find the truth set of $\frac{5}{2}x = 60$.

If $\frac{5}{2}x = 60$ is true for some x, then

$$\frac{2}{5}\left(\frac{5}{2}x\right) = \frac{2}{5}(60)$$

$$\left(\frac{2}{5} \cdot \frac{5}{2}\right)x = \left(\frac{2}{5}\right)(60)$$

$$1 \cdot x = 24$$

$x = 24$ is true for that same x.

Check: If x is 24,
the left side is $\left(\frac{5}{2}\right)(24) = 60$.
The right side is 60.

Hence, the truth set of $\frac{5}{2}x = 60$ is $\{24\}$.

Look carefully at the steps in the example, then complete the following items.

In the first part, we suppose that there is some number x such that the sentence $\frac{5}{2}x =$ ______ is true. 60

We want to obtain a simpler sentence in which the left side is x. Hence, we choose to multiply both sides

4 of the equation by ______ because $\frac{2}{5}$ is the multi- — $\frac{2}{5}$

5 plicative ______ of $\frac{5}{2}$. That is, — inverse

6 $\frac{2}{5} \cdot \frac{5}{2}$ ______ — $\frac{2}{5} \cdot \frac{5}{2} = 1$

7 $\frac{2}{5}(\frac{5}{2}x) = (\frac{2}{5} \cdot \frac{5}{2})x$ because of the ______ property of multiplication. — associative

8 Furthermore, $1 \cdot x =$ ______ because of the — x

9 ______ property of 1. — multiplication

Once you understand the steps you might write:

10 If $\frac{5}{2}x = 60$ is ______ for some x, — true

11 $x =$ ______ is true for the same x. — 24

Similarly, you may wish to write:

12 Check. If x is 24, the left side is ______, — 60

13 and the right side is ______. — 60

Hence, the truth set of $\frac{5}{2}x = 60$ is {24}.

Of course, you should show all the additional steps you need, and in more difficult problems there may be several steps.

The first part shows that *if* a number is an element of the truth set, then the number is 24.

14 The first part ______ (does, does not) show that 24 is an element of the truth set. — does not

15 The first part ______ (does, does not) show that the only — does

16 number that can belong to the truth set is ______. — 24

The check shows that if a number is in the truth set

17 of $x = 24$, then it is also in the ______ ______ of $\frac{5}{2}x = 60$. — truth set

18 (Of course, it is obvious that ______ is the only element in the truth set of $x = 24$.) — 24

19 Remember, both parts ______ (are, are not) necessary. — are

How would we solve

$$2x + 7 = 15?$$

We know that if there is <u>some</u> number x such that $2x + 7 = 15$ is true, then perhaps we can obtain a simpler sentence that is true for the <u>same</u> number x.

Let us consider the following approach to this problem:

Find the truth set of

$$2x + 7 = 15.$$

By the addition property of equality we may add -7 to both sides to get

20 $(2x + 7) + (\quad) = 15 + (\quad)$. — $(2x + 7) + (-7) = 15 + (-7)$

21 Since $7 + (-7) =$ ______ — 0

22 and $15 + (-7) =$ ______ — 8

23 we obtain ______. — $2x = 8$

The equations

$$2x + 7 = 15$$

and $$2x = 8$$

24 have ______ (the same, different) truth sets. — the same

Now, by the multiplication property of equality, we may multiply both sides of $2x = 8$ by $(\frac{1}{2})$ to get

25 $\frac{1}{2}(\quad) = \frac{1}{2} \cdot 8$ — $\frac{1}{2}(2x) = \frac{1}{2} \cdot 8$

or $x = 4$.

Thus, if there is a number x such that $2x + 7 = 15$, then the sentence $x = 4$ is true for the <u>same</u> x.

26 The truth set of $x = 4$ is ______. — $\{4\}$

We must check to be sure that <u>if</u> x is 4 then

27 $2x + 7 = 15$ is ______. — **true**

If x is 4, then the left side of $2x + 4$ is

28 $\underline{2(\quad) + 7}$ or 15. — $2(4) + 7$

29 The right side is ______. — 15

30 Hence, the truth set of $2x + 7 = 15$ is _____. — $\{4\}$

We have used both the addition property of equality and the multiplication property of equality to find a simple sentence which has the same truth set as $2x + 7 = 15$.

Suppose, instead of beginning by <u>adding</u> the <u>opposite</u> of 7, we begin by <u>multiplying</u> by the reciprocal of 2:

Find the truth set of

$$2x + 7 = 15.$$

By the multiplication property of equality we may <u>multiply</u> both sides to get

31 $$\underline{\tfrac{1}{2}(2x + 7) = \qquad\qquad}.$$ — $\frac{1}{2}(2x+7) = \frac{1}{2} \cdot 15$

or $$\left(\tfrac{1}{2} \cdot 2x\right) + \left(\tfrac{1}{2} \cdot 7\right) = \tfrac{1}{2} \cdot 15$$

or $$x + \tfrac{7}{2} = \tfrac{15}{2}.$$

32 This is true by the ______ property and the definition of reciprocal. — **distributive**

By the addition property of equality we may add $-\frac{7}{2}$ to both sides to get

33 $$\underline{\left(x + \tfrac{7}{2}\right) + (\qquad) = \tfrac{15}{2} + (\qquad)}$$ — $\left(x+\frac{7}{2}\right)+\left(-\frac{7}{2}\right) = \frac{15}{2}+\left(-\frac{7}{2}\right)$

or $$x = \tfrac{8}{2}$$

34 or $$\underline{x = \qquad}.$$ — $x = 4$

Using this method, we again get the sentence

$$x = 4.$$

From the check shown in Items 27-30 we know that $x = 4$ and $2x + 7 = 15$ have the same truth sets.

Clearly, both of these methods are correct. Which you should use depends on the equation you wish to solve. A "rule of thumb" that might be helpful would be to try to keep the "arithmetic" as simple as possible.

In the course of finding the truth set of $2x + 7 = 15$, we found a simpler equation which has $\{4\}$ as its truth set. This simpler equation is "$x = 4$". Thus, although the equations $2x + 7 = 15$ and $x = 4$ look quite different, they have the same 35 ______ set.	truth

Note that throughout this discussion we have assumed that the domain of the variable is the same for both open sentences.

Definition. Two open sentences which have the same truth set are called equivalent open sentences. In particular, two equations which have the same truth set are called equivalent equations.

Below is a chain of three equivalent equations:

$$2x + 7 = 15$$

$$2x = 8$$

$$x = 4$$

Let us review the way in which we constructed this chain of equivalent equations.	
36 We began by applying the ______ property of equality.	addition
Our next step was to use properties of real numbers to write simpler names for each side of the equation. Doing this resulted in the equation $2x = 8$.	
37 We continued by applying the ______ property of equality.	multiplication
Again we used certain number properties to simplify both sides of the equation. We thus arrived at the equation 38 x ______.	$x = 4$

Through this chain of steps we have shown that <u>if</u> $2x + 7 = 15$ is true for some x, <u>then</u> $x = 4$ is true for the same x.

39. If $x = 4$ is true for some x, then $2x$ ____ is true for the same x from the ____ property of equality. — $2x = 8$

40. — multiplication

41. If $2x = 8$ is true, then we can use the ____ property of equality to conclude that $2x + 7 = 8 +$ ____ is true for the <u>same</u> x. — addition

42. — 7

43. Thus we see: If $x = 4$ is true for some x, then $2x + 7 = 15$ is true for the ____ x. — same

44. Our last statement tells us: <u>If</u> a number is in the truth set of $x = 4$, <u>then</u> it is in the ____ ____ of $2x + 7 = 15$. — truth set

On the other hand, our earlier steps told us: <u>If</u> a number is in the truth set of $2x + 7 = 15$, <u>then</u> it is in the truth set of $x = 4$.

45. The two parts, considered together, tell us: The equations $2x + 7 = 15$ and $x = 4$ have the same ____ ____. — truth set

46. In other words, $2x + 7 = 15$ and $x = 4$ are ____ equations. — equivalent

Now let us stop and think. Suppose we <u>know</u> that

$$2x + 7 = 15$$

and $$x = 4$$

are equivalent equations. {4} is obviously the truth set of the equation $x = 4$. Hence, we can be sure that {4} is the truth set of $2x + 7 = 15$, since equivalent equations have the same truth sets.

In constructing the equivalent chain

$$2x + 7 = 15$$
$$2x = 8$$
$$x = 4$$

47. we multiplied both sides of $2x = 8$ by ____. — $\frac{1}{2}$

On the other hand, to go from

$$x = 4$$

to $\quad 2x = 8$

48 we multiplied both sides of $x = 4$ by ____.

2

49 We note that 2 and $\frac{1}{2}$ are multiplicative ______ of each other.

inverses

To go from

$$2x + 7 = 15$$

to $\quad 2x = 8$

50 we add (____) to both sides of $2x + 7 = 15$.

(-7)

The corresponding step in the chain of equivalent equations

$$x = 4$$
$$2x = 8$$
$$2x + 7 = 15$$

51 is to add ____ to both sides of $2x = 8$.

7

52 7 is the ______ inverse of (-7).

additive

We can summarize.

In the first case we "<u>climbed</u> <u>down</u>", by steps involving

53 the addition and multiplication properties of ______,

equality

from the equation $2x + 7 = 15$ to the simpler

54 equation ______.

$x = 4$

In the second case we "<u>climbed</u> <u>up</u>" from the simpler equation to the original one, by steps that were the <u>reverse</u> of those in the first part.

The reverse step to multiplying both sides of $2x = 8$

55
56 by $\frac{1}{2}$ was ______ both sides of $x = 4$ by ____.

multiplying, 2

We could have saved the actual work of the last part by recognizing at each step, as we "climbed down" in the first part, that the steps could be reversed. By <u>reasoning</u> alone, without going through the actual work, we could have convinced ourselves that we could "climb" back from the equation $x = 4$ to the equation $2x + 7 = 15$.

Let us summarize the kinds of (reversible) steps we used in going from our first equation to $x = 4$. The major steps were adding the same number to both sides of the equation and multiplying both sides of the equation by the same (non-zero) number. Adding the same number to both sides of an equation can be reversed by adding the additive inverse of the number. Multiplying both sides by a non-zero number can be reversed by multiplying by the multiplicative inverse of the number.

We also applied various properties of the real numbers which are true for all real numbers. The application of these properties is obviously reversible: For example, we can replace $1 \cdot a$ by a or a by $1 \cdot a$ whenever we like. For any real number a, we know that a and $1 \cdot a$ name the same number.

57	Thus, we could have concluded from our first steps alone that $2x + 7 = 15$ and $x = 4$ have the same _______ set.	truth
58	We could have concluded without checking that $2x + 7 = 15$ and $x = 4$ are _______ equations.	equivalent
	Our conclusion is based on the fact that each step used in obtaining $x = 4$ from $2x + 7 = 15$ is reversible.	

The foregoing discussion was long. Once you have understood the reasoning however, you can apply it quickly.

Example. Find the truth set of $5y = 2y + (-18)$.

$$5y = 2y + (-18)$$
$$5y + (-2y) = 2y + (-18) + (-2y)$$
$$3y = -18$$
$$y = -6$$

The truth set is $\{-6\}$.

59	In this example, we began by adding _______ to both sides of the equation.	$-2y$
	We then simplified, applying properties which hold for all real numbers, and obtained the equation $3y = -18$.	
60	We multiplied both sides of this equation by _______,	$\frac{1}{3}$
61	obtaining the still simpler equation _____ = _____.	$y = -6$

Since our steps were all reversible, we are sure that all the equations $5y = 2y + (-18)$, $5y + (-2y) = 2y + (-18) + (-2y)$, $5y = -18$, and

62 $y = -6$ are ______. equivalent

63 Because they are equivalent, their ______ sets must be the same. truth

64 Since ______ is obviously the truth set of $y = -6$, it is also the truth set of the original equation. (-6)

We can thus reason, without further checking, that we have found the truth set. This should not convince you that checking is a waste of time. People make mistakes in arithmetic, and a check is a good way to catch such mistakes. Keep on checking for accuracy!

The check for the example just completed is:

If y is -6, the left side is $5(-6) = -30$ and the right side is $2(-6) + (-18) = -30$.

Remember,

Two open sentences are equivalent if they have the

65 same ______ ______. truth set

If we multiply each side of an equation by the same

66 non-______ number, non-zero

67 we obtain an ______ equation. equivalent

If we add the same number to each side of an equation,

68 we obtain an equivalent ______. equation

The reason is that both of these steps are reversible.

Suppose we have the open sentence $x = x$.

This sentence is true for all real values of x.

If we add $(-x)$ to both sides, we obtain:

$$x + (-x) = x + (-x)$$

69 ______ $0 = 0$

	By adding $(-x)$ to both sides of our open sentence we obtain the sentence $0 = 0$.	
70	This is ______ (true, false) for all values of x.	true
	We shall never get into logical inconsistencies if we regard this as an open sentence which is true for all values of x.	

	Consider the equation $$x = 1 + x.$$ Adding $-x$ to both sides, we have:	
71	$0 = $ ______.	$0 = 1$
	$0 = 1$ and $x = 1 + x$ are equivalent open sentences.	
72	The truth set of $0 = 1$ is ______.	$\emptyset$
	This may be thought of as an open sentence which	
73	is ______ (true, false) for all values of x.	false
74	The truth set of $x = 1 + x$ is ______.	$\emptyset$

Sometimes we use the terms <u>right member</u> and <u>left member</u> of an equation instead of right side and left side.

	"The right member of an equation" is another way of	
75	saying "the right ______ of an equation".	side
	The left side of an equation is also called the	
76	left ______.	member

77 In which of the following are the two equations not equivalent?

[A] $x + (-3) = 5$, and $x = 8$

[B] $\frac{1}{2}b = 8$, and $b = 16$

[C] $z + (-7) = -z + 3$, and $2z = 10$

[D] $6x = 7$, and $(6x)0 = (7)0$

The sentences in [D],

$$6x = 7 \text{ and } (6x)0 = (7)0$$

are not equivalent.

$(6x)0 = (7)0$ is the same as $0 = 0$

which has as truth set all real numbers. Clearly, we do not have equivalent sentences because we have multiplied by 0 and 0 does not have a reciprocal--we cannot reverse our steps.

78	$4y + (-6) = 5y + (-6)$ and $0 = y$ ______ (are, are not) equivalent.	are
79	$-3a = -6$ and $a = 2$ ______ (are, are not) equivalent.	are
80	$5m + 5 = -m + (-7)$ and $12 = -6m$ ______ (are, are not) equivalent.	are
81	$x = 3$ and $\|x\| = \|3\|$ ______ (are, are not) equivalent.	are not
82	The truth set of $x = 3$ is ______.	$\{3\}$
83	The truth set of $\|x\| = \|3\|$ is ______.	$\{3,-3\}$

Find the truth set of

$$6x + 12 = 2y + (-8).$$

There are several approaches to this. Here is one approach.

$$6y + 12 + (-12) = 2y + (-8) + (-12)$$

$$6y = 2y + (-20)$$

$$6y + (-2y) = 2y + (-20) + (-2y)$$

$$4y = -20$$

$$y = -5$$

Check: If $y = -5$, the left side is $6(-5) + 12 = -18$.
The right side is $2(-5) + (-8) = -18$.

Remember that there are other approaches to this problem that are equally correct.

	Find the truth set of each of the following equations:		
84	$2a + 5 = 17$	______	{6}
85	$4y + 3 = 3y + 5 + y + -2$ (Collect terms first)	______	the set of all real numbers
86	$12x + (-6) = 7x + 24$	______	{6}
87	$8x + (-3x) + 2 = 7x + 8$	______	{-3}
88	$6y + 9 + (-4y) = 18 + 2y$	______	$\emptyset$
89	$12n + 5n + (-4) = 3n + (-4) + 2n$	______	{0}
90	$15 = 6x + (-8) + 17x$	______	{1}
91	$5y + 8 = 7y + 3 + (-2y) + 5$	______	the set of all real numbers

Find the truth set of each of the following equations.

Answers are on page xi.

92. $-6a + (-4) + 2a = 3 + (-a)$

93. $0.5 + 1.5x + (-1.5) = 2.5x + 2$

94. $\frac{1}{2} + (-\frac{1}{2}c) + (-\frac{5}{2}) = 4c + (-2) + (-\frac{7}{2}c)$

95. $2y + (-6) + 7y = 8 + (-9y) + (-10) + 18y$

96. $(x + 1)(x + 2) = x(x + 2) + 3$

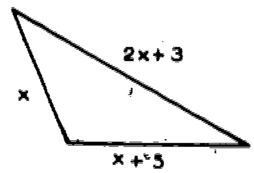
Equations arise in the study of physics, chemistry, engineering, etc. As you saw in Chapter 5, many everyday situations also give rise to open sentences. In that chapter we learned how to find the open sentence suggested by a given word problem. Now we are able to find the truth set of the open sentence.

Example. The perimeter of a triangle is 44 inches. The length of the second side is 3 inches more than twice the length of the third side, and the first side is 5 inches longer than the third side. Find the lengths of the three sides of this triangle.

	If x is the length of the third side in inches	
97	then ______ is the length of the second side in inches	$2x + 3$
98	and ______ is the length of the first side in inches.	$x + 5$
	The open sentence suggested by this problem is $$x + (2x + 3) + (x + 5) = 44.$$	

We now find the truth set of this equation.

$x + (2x + 3) + (x + 5) = 44$



99 ________ $4x + 8 = 44$

$4x = 36$

100 ________ $x = 9$

101 The truth set is { }. {9}

The solution of this equation corresponds to the length of the third side.

If 9 is the length of the third side, then

102 ( ) + ____ is the length of the second side. $(2\cdot 9) + 3$

103 ________ is the length of the first side. $9 + 5$

The lengths of the sides are 9 inches, 21 inches, and 14 inches.

Check: $9 + 21 + 14 = 44$

(Note that we go back to the original problem, not the equation, and check our solution with the conditions of the problem.)

You will find it much easier to do these problems if you use good form in setting up the problems and in your solutions.

Now try your hand at solving some word problems. For each of the following problems, write your work carefully, using the general form of the preceding example. When you have finished, check your answer with the one given. If your answer is not correct, or if you are not sure that you understand, complete the items below the problem.

104 Plant A grows 2 inches each week and it is now 20 inches tall. Plant B grows 3 inches each week and it is now 12 inches tall. How many weeks from now will they be equally tall? ______ In 8 weeks.

Let x = number of weeks from now until the plants are equally tall.

105 In x weeks plant A grows ______ inches. 2x

106	Plant B grows ______ inches.	$3x$
107	At the end of x weeks, plant A is ______ inches tall.	$2x + 20$
108	Plant B is ______ inches tall.	$3x + 12$
109	An open sentence which states that at the end of x weeks the plants are equally tall is ____ = ____	$2x+20 = 3x+12$
110	The truth set of this open sentence is ______.	{8}

111 If a counting number and its successor are added, the result is one more than twice the number. What is the number? (Remember to specify the domain of the variable in this open sentence. The statement of the problem will usually indicate the domain of the variable.)

Any counting number.

112	If n is a counting number, its successor is ______.	$n + 1$
113	The number that is 1 more than $2n$ is ______.	$2n + 1$
114	An open sentence which fits the problem is: $n + (n + 1)$ ______.	$n+(n+1)=2n+1$
115	The domain of the variable is the set of ______ numbers.	counting
116	Since $n + (n + 1) = 2n + 1$ ______ (is, is not) true for all numbers in the domain, the number may be any counting number.	is

117 The sum of two consecutive odd integers is 11. What are the integers?

The set of pairs of consecutive odd integers that fit this problem is the null set.

118	1, 3, 5, 7, ... are consecutive odd integers. If n stands for an odd integer, the next odd integer is ____ + 2.	$n + 2$

An open sentence which fits the problem is
119 $n + (n + 2) =$ ______. | $n + (n + 2) = 11$

The domain of the variable is the set of integers.

An open sentence equivalent to $n + (n + 2) = 11$
120 is $2n =$ ______. | $2n = 9$

121 This sentence has truth set { }. | $\{\frac{9}{2}\}$

Since $\frac{9}{2}$ is not an integer, the truth set of the
122 open sentence, $n + (n + 2) = 11$ is the ______ set. | null, or empty

123 A radiator contains only water. Two quarts of alcohol are added. The mixture then contains 20 per cent alcohol. How many quarts of water were in the radiator? ______ | 8 quarts

124 Let x = the number of quarts of ______ in the radiator. | water

After the alcohol is added, the radiator
125 contains ______ quarts of liquid. | $x + 2$

126 Of this, ______ quarts is alcohol. | 2

We know that 20 per cent of the liquid is alcohol.

We may find 20 per cent of the number $x + 2$
127 by computing either $.20(x + 2)$ or $\frac{1}{5}($____$)$. | $x + 2$

An open sentence appropriate to this problem is
128 $\frac{1}{5}(x + 2) =$ ______. | 2

129 An equivalent open sentence is: $x + 2 =$ ______. | 10

Turn to page xi for the solutions.

130. One number is 5 times another and their sum is 15 more than 4 times the smaller. What is the smaller number?

131. A number is increased by 17 and the sum is multiplied by 3. If the resulting product is 192, what is the number?

132. In an automobile race, one driver, starting with the first group of cars, drove for 5 hours at a certain speed and was then 120 miles from the finish line. Another driver, who set out with a later heat, had traveled at the same rate as the first driver for 3 hours, and was 250 miles from the finish. How fast were these men driving?

133. Four times a counting number is 10 more than twice the successor of that number. What is the number?

134. "Jake has $1.65 in his pocket, all in nickels, dimes and quarters. He has one more quarter than he has dimes, and the number of nickels he has is one more than twice the number of dimes. How many dimes has he?"

9-4. Summary and Review

We have defined multiplicative inverse as follows:

> For each real number c different from 0, there exists a real number d such that
>
> $$cd = 1 \text{ and } dc = 1.$$

We have proved that for each non-zero real number c there is exactly one multiplicative inverse.

We have agreed to use the name reciprocal for the multiplicative inverse, and to represent the reciprocal of any number a by "$\frac{1}{a}$".

We have also proved the following theorems:

1. The number 0 has no reciprocal.

2. The reciprocal of the reciprocal of a non-zero real number a is a.

3. For any non-zero real numbers a and b,

$$\frac{1}{a} \cdot \frac{1}{b} = \frac{1}{ab}.$$

4. For any non-zero real number, a,

$$\frac{1}{-a} = -\frac{1}{a}.$$

5. For real numbers a and b, $ab = 0$ if and only if $a = 0$ or $b = 0$.

Review Problems

The answers to these review problems are on page xiii.

1. Find the truth set of each of the following equations:
   (a) $4a + 7 = 2a + 11$
   (b) $8x + (-18) = 3x + 17$
   (c) $7x + 2 + (-5x) = 3 + 2x + (-1)$
   (d) $|-2| + 2x = (-3) + 3x + 5$
   (e) $3x^2 + (-2)x = x^2 + 2 + 2x^2$

2. Find the truth set of each of the following:
   (a) $(x)(x + 1)(x + 6) = 0$
   (b) $(x + (-4))(2x + 3) = 0$
   (c) $(3x + (-5))(2x + (-1)) = 0$

3. We have proved that the opposite of the sum is the sum of the opposites. That is, for any real numbers a and b, $-(a + b) = (-a) + (-b)$. Is it true that for non-zero real numbers a and b, the sum of the reciprocals is the reciprocal of the sum?

4. For what values of b does each of the following have no reciprocal?
   (a) $2b + 6$
   (b) $b(b + (-1))$
   (c) $b^2 + 1$

5. For each of the following problems, write an open sentence, find its truth set, and answer the question asked in the problem.

   (a) Jim and I plan to buy a basketball. Jim is working, so he agrees to pay $2 more than I pay. If the basketball costs $11, how much does Jim pay?

   (b) The sum of two consecutive odd integers is 40. What are the integers?

   (c) The length of a rectangle is 27 yards more than the width. The perimeter is 398 yards. Find the length and the width.

   (d) Mary and Jim added their grades on a test and found the sum to be 170. Mary's grade was 14 points higher than Jim's. What were their grades?

   (e) A man worked 4 days on a job and his son worked half as long. The son's daily wage was $\frac{2}{5}$ that of his father. If they earned a total of $96, what were their daily wages?

   (f) In a farmer's yard were some pigs and chickens, and no other creatures except the farmer himself. There were, in fact, sixteen more chickens than pigs. Observing this fact, and further observing that there were 74 feet in the yard, not counting his own, the farmer exclaimed happily to himself--for he was a mathematician as well as a farmer, and was given to talking to himself--"Now I can tell how many of each kind of creature there are in my yard." How many were there? (Hint: Pigs have 4 feet, chickens 2 feet.)

   (g) At the target shooting booth at a fair, Montmorency was paid 10¢ for each time he hit the target, and was charged 5¢ each time he missed. If he lost 25¢ at the booth and made ten more misses than hits, how many hits did he make?

Chapter 10

PROPERTIES OF ORDER

10-1. The Fundamental Properties of Order

We have used the concept of order to make certain statements about numbers. In Chapter 3, when we were considering only numbers of arithmetic, we used "is less than" to mean "is to the left of" on the number line. In Chapter 6, we extended this meaning to the set of real numbers. The statement "a is less than b" is written in symbols as "$a < b$".

If we are given a first number a, and a second number b, then the statement $a < b$ is either true or false. In this sense, "<" is called a <u>relation</u> in the set of real numbers.

We have discovered two fundamental properties of the relation "<". Let us review these properties.

If a and b are real numbers and if $a < b$ is false, then either

$a = b$ is true

1 or b ____ is true. — $b < a$

2 The above is a statement of the ______ property. — comparison

If a, b, c are real numbers and if $a < b$ and

3 $b < c$, then a ______. — $a < c$

4 The above is a statement of the ______ property of the relation "<". — transitive

Recall that we have called this the "transitive property of order".

5 If $x < -6$ and $-6 < -5$, which of the following relations is a consequence of the transitive property of order?

[A] $-5 < x$ [B] $x < -5$ [C] $x = -5$

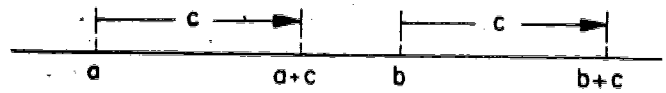
"If $x < -6$ and $-6 < -5$, then $x < -5$" is a direct application of the transitive property. Thus, [B] is correct.

There is one other property of the relation "<" that we shall develop in this section.

6	To find the truth set of $3 + x = 3$, we use the ______ property of equality to write:	addition
7	If $(-3) + (3 + x) = (-3) + 3$ is true for some x, then $x =$ ____ is true for the same x.	$x = 0$
8	Thus, we determine that the truth set of $3 + x = 3$ is ______.	$\{0\}$
9	The sentences "$3 + x = 3$" and "$x = 0$" are equivalent sentences, since they have the same ______ set.	truth
	Now let us consider the open sentence $3 + x < 3$.	
10	-1 ______ (is, is not) in the truth set of $3 + x < 3$, since $3 + (-1) < 3$ is a true sentence.	is
11	In fact, the truth set of $3 + x < 3$ must include all ______ (negative, positive) real numbers.	negative
12	5 ______ (is, is not) in the truth set of $3 + x < 3$, since $3 + 5 < 3$ is a false sentence.	is not
13	In fact, no ______ (negative, positive) real number is in the truth set of $3 + x < 3$.	positive
14	Is 0 in the truth set of $3 + x < 3$? ______ (yes, no)	no
	Hence, the truth set of $3 + x < 3$ is the set of all negative real numbers.	
15	A simple open sentence having this truth set is $x <$ ____.	$x < 0$
16	The sentences "$3 + x < 3$" and "$x < 0$" are ______ sentences, since they have the same truth set.	equivalent

Perhaps you suspect the property of "$<$" which we wish to discuss.

It is helpful to use the number line. Let us fix <u>two</u> points a and b on the number line with $a < b$.



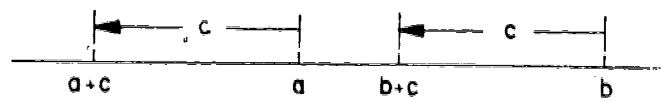
On the number line above, a is to the left of b.

17 a $\underset{(<,=,>)}{\underline{\qquad}}$ b. <

We note also that if we [illegible] c, a positive number, to a and to b, a + c is to the ______ of b + c. left

19 a + c ______ b + c. [illegible]

Now let us look at the same [illegible] c < 0.



20 On the number line [illegible] a [illegible] the ______ [illegible]. [illegible]

[illegible]

We also note that [illegible]

[illegible]

21 a + c is to the ______ of b + c. left

22 a + c ______ b + c. <

[illegible]

[illegible]

	It is true that $2 < 5$.	
27	If 4 is added to 2 and also to 5, the resulting sentence will be $2 + 4 < 5 + 4$ or $6 <$ ______,	$6 < 9$
28	which is a ______ (true, false) sentence.	true
	Again, starting with the true sentence $2 < 5$, if -4 is added to 2 and also to 5, the resulting sentence	
29	will be $2 + (-4) < 5 + (-4)$ or ______ < 1, which	$-2 < 1$
30	is a ______ (true, false) sentence.	true
	Since $(-7) < (-3)$,	
31	then $(-7) + 2$ ______ $(<, =, >)$ $(-3) + 2$,	$<$
32	or ______ is to the left of (-1).	-5
	Since $(-7) < (-3)$,	
33	then $(-7) + (-3)$ ______ $(<, =, >)$ $(-3) + (-3)$,	$<$
34	or ______ $< (-6)$.	$-10 < (-6)$

We can now state another fundamental property of order.

> <u>Addition property of order</u>: If a, b, c are real numbers and if $a < b$, then
>
> $$a + c < b + c \quad \text{and} \quad c + a < c + b.$$

	Use the addition property of order to determine which of the following sentences are true and which are false.		
35	$(-\frac{6}{7}) + 4 < (-\frac{3}{4}) + 4$	______	true
36	$(-\frac{2}{3})(\frac{6}{5}) + (-5) < (-\frac{1}{4}) + (-5)$	______	true
37	$(-5.2) + (-2)(-\frac{4}{7}) < (-0.4) + \frac{8}{7}$	______	true
38	$(-\frac{15}{5}) + 2 < (\frac{6}{2})(-\frac{3}{4}) + 2$	______	false

Recall the addition property of equality:

> For any real numbers a, b, c, if $a = b$, then $a + c = b + c$ and $c + a = c + b$.

39 How would we write an addition property which combines the addition property of equality with the addition property of order?

[A] If $a = b$ or $a < b$, then $a + c = b + c$ <u>or</u> $a + c < b + c$.

[B] If $a = b$ or $a < b$, then $a + c = b + c$ <u>and</u> $a + c < b + c$.

[C] If $a \leq b$, then $a + c \leq b + c$.

Since $a + c = b + c$ <u>and</u> $a + c < b + c$ is always false, [B] is incorrect. While either [A] or [C] is a correct choice, [C] is the more concise form.

Is $\frac{68}{19} < \frac{65}{18}$ a true sentence? Let us use the addition property of order to decide.

40 $\frac{68}{19} = 3 + \frac{\square}{19}$. $\frac{11}{19}$

41 $\frac{65}{18} = 3 + \frac{\square}{18}$. $\frac{11}{18}$

42 Now, $\frac{11}{19} \underset{(<,>)}{\underline{\qquad}} \frac{11}{18}$. $<$

Hence, we may conclude that

43 $\frac{68}{19} < \frac{65}{18}$ is a $\underset{(\text{true, false})}{\underline{\qquad\qquad}}$ sentence. true

Here is a harder one. Let's determine whether $\frac{5}{3} + \frac{11}{8}$ is less than $\frac{7}{4} + \frac{13}{7}$.

44 First, we see that $\frac{5}{3} \underset{(<,>)}{\underline{\qquad}} \frac{7}{4}$. (We would use the argument of Items 40-43, if necessary.) $<$

So

45 $\frac{5}{3} + \frac{11}{8} < \frac{7}{4} + \frac{11}{8}$, using the ______ property of order. addition

46 Now, $\frac{11}{8} \underset{(<,>)}{\underline{\qquad}} \frac{13}{7}$. $<$

47 Therefore, $\frac{7}{4} + \frac{11}{8} < \frac{7}{4} + \frac{13}{7}$, using the ______. addition property of order

We have $\frac{5}{3} + \frac{11}{8} < \frac{7}{4} + \frac{11}{8}$ and $\frac{7}{4} + \frac{11}{8} < \frac{7}{4} + \frac{13}{7}$.

Hence, we may conclude that

$\frac{5}{3} + \frac{11}{8} < \frac{7}{4} + \frac{13}{7}$ is a true statement,

48 using the ______ property of order. transitive

10

addition

(-c)

a < b

is

equivalent

Let us summarize the three fundamental properties of "<" which we are assuming to be true.

> <u>Comparison Property</u>: If a and b are real numbers then exactly <u>one</u> of the following is true:
>
> $$a < b, \quad a = b, \quad b < a.$$
>
> <u>Transitive Property of Order</u>: If a, b, c are real numbers and if $a < b$ and $b < c$, then $a < c$.
>
> <u>Addition Property of Order</u>: If a, b, c are real numbers and if $a < b$, then $a + c < b + c$.

There are, of course, other "properties" of order. These, however, can be shown to be consequences of our fundamental properties. We have already stated the order property of opposites:

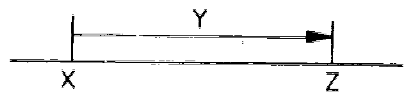
> If a and b are real numbers and if $a < b$, then $-b < -a$.

We can now <u>prove</u> this property, using the addition property of order. Before we start the proof, let us see how we will proceed. We start with the inequality $a < b$ and wish to obtain an inequality having $-a$ to the right of the symbol "<". This suggests that we add $-a$ to both sides of $a < b$.

Prove: "If $a < b$, then $-b < -a$".	
Complete the statements and the reasons, as indicated.	
(1) If $a < b$, then	
(2) $a + (-a) < \underline{b + (\quad)}$	$b + (-a)$
(3) $0 < \underline{\qquad}$	$b + (-a)$
(4) $(-b) + 0 < (-b) + b + (-a)$	
(5) $-b < \underline{\qquad} + (-a)$	$0 + (-a)$
(6) $-b < -a$	
Reasons for steps (2) - (6).	
(2) addition property of ______	order
(3) ______ property of ______	addition, opposites
(4) ______ property of order	addition
(5) addition property of ______, and	zero
______ property of opposites	addition
(6) ______ property of ______	addition, zero

10-2. Further Properties of Order

In Section 7-1 we began our discussion of addition of real numbers by considering addition in terms of the number line. We found that to add a positive number, we move to the right on the number line. Thus, the diagram



may be interpreted as $x + y = z$ and y is positive.

Position on the number line has been interpreted in terms of order. In our diagram x is to the left of z, or $x < z$.

Notice that we have two statements

1 "$x + \_\_\_ = z$ and y positive" — $x + y = z$

and

2 "$x \_\_\_ z$". — $x < z$

3 The first statement says that if a ____ number is added to x, we obtain z. — positive

The second statement is an order statement that

4 x is ____ than z. — is less than

5 If we know that $a + c = b$, then we know that __ < __. — $a < b$

Let us state and prove the theorem which we have illustrated on the number line.

Theorem 10-2a. If x, y, and z are real numbers such that $x + y = z$ and y is positive, then $x < z$.

Proof:

6 $0 < y$, since we are given that y is ____ — positive

7 $x + 0 < x + y$, by the addition property of ____ — order

8 $x < x + y$, by the addition property of ____ — 0

Now let $x + y = z$ and the proof is complete.

positive

$-5 < -4$

order

$r < t$

$(q + r) < p$

positive

is less than

adding

lesser

[illegible] Try to complete the proof of Theorem 10-[illegible] by yourself. Notice that [illegible] to be proved:

First, we must find [illegible] y [illegible] that $x + y$ [illegible] is true.

[illegible], we must prove that [illegible] the [illegible] of y that we [illegible].

It is not hard to find a value of y [illegible] that $x + y$ [illegible] is true. [illegible] from your experience with solving equations that you could [illegible] $x + y$ [illegible].

For the [illegible] part, [illegible] show that [illegible] y. Start with $x < z$, [illegible] and try to obtain [illegible].

This proof [illegible] written [illegible] page xv.

The following exercises will help you check your understanding of Theorem 10-[illegible].

[illegible]

$-\frac{5}{4} < \frac{63}{4}$

$\frac{68}{4}$ or 17

$\frac{7}{10} < \frac{6}{5}$

$\frac{5}{10}$ or $\frac{1}{2}$

$-\frac{1}{2} < \frac{1}{3}$

$\frac{5}{6}$

$-345 < -254$

91

$-0.21 < 1.47$

1.68

$-\frac{33}{13} < -\frac{98}{39}$

$\frac{1}{39}$

The preceding examples show that for each two different numbers with which we started we found a positive number which added to the smaller yielded the larger. This is what Theorem 10-2b states we should be able to do.

Theorems 10-2a and 10-2b taken together might be stated as follows: $x < z$ is true if and only if there is a positive number y such that $x + y = z$. If we had chosen to do so, we could have used this statement as a "definition" of "<". This would have been consistent with our knowledge of the numbers of arithmetic. Refer to our brief discussion in Section 4-5. Had we adopted this definition, we could have proved the comparison, transitive, and addition properties of "<".

Let us turn to another theorem. In solving equations such as

$$3x + 100 = 1000$$

we make use of what we have referred to as the addition and multiplication property of equality.

	If Jake had \$100 more than three times the amount he now has, he would still have less than \$1000 dollars. How much money does he now have?	
	This situation leads to the open sentence	
35	______ < 1000.	$3x + 100 < 1000$
	An equivalent sentence is	
36	$3x <$ ______.	$3x < 900$
37	We can guess that Jake now has ______ \$300.	less than \$300
38	Is it true that if $a \neq 0$ then "$ax < b$" and "$x < b(\frac{1}{a})$" are equivalent open sentences? ______ (yes, no)	no

Were you surprised by the answer to Item 38? Let's discover the reason for that answer.

	Consider the true sentence $5 < 8$.	
	If each member of this inequality is multiplied by 2,	
39	we get $(5)(\ ) < (8)(\ )$.	$(5)(2) < (8)(2)$
40	Is this new sentence true? ______ (yes, no)	yes

41	$7 < 10$ is true; $(7)(6) < (10)(6)$ is also ______.	true
42	$-9 < 4$ and $(-9)(5) < (4)(5)$ are both ______ (true, false) sentences.	true
43	Given that, $-7 < -2$. If we multiply each member of this inequality by 2, we get the inequality ___ < ___ which is a true sentence.	$(-7)(2)<(-2)(2)$ or $-14 < -4$
44	Did you notice that in each example we started with a true inequality and ______ both numbers by a positive number?	multiplied
45	If $a < b$ is true, then $(a)(3)<(b)(3)$ is also ______.	true
46	It appears that if $a < b$ and if c is positive, then $ac < b$______.	$ac < bc$

Still, the answer to Item 38 was "no". Can you guess why?

47 $2 < 6$ is true. Which of the following numbers cannot be inserted in the parentheses so that the resulting sentence is true?

$$(2)(\ \ ) < (6)(\ \ )$$

[A] $\frac{3}{5}$ [C] -2

[B] 68 [D] Each will make the sentence true.

$(2)(\frac{3}{5}) < (6)(\frac{3}{5})$ and $(2)(68) < (6)(68)$ are both true sentences, but $(2)(-2) < (6)(-2)$ is false. [C] is the correct choice.

	$2 < 3$ is a true sentence. Let us write a true sentence using "<" that shows the order of $(2)(-4)$ and $3(-4)$.	
48	$2(-4) =$ ______.	-8
49	$3(-4) =$ ______.	-12
50	So $3(-4)$ ___ $2(-4)$ is a true sentence.	$3(-4) < 2(-4)$
	$-1 < 8$ is a true sentence. Write a true sentence, using "<", that shows the idea of $(-1)(-3)$ and $8(-3)$.	
51	____________.	$8(-3) < (-1)(-3)$
52	$-5 < 0$ and $0(-1)$ ___ $(-5)(-1)$ are both true sentences.	$0(-1) < (-5)(-1)$
53	If we know that $a < b$, what may we conclude about the order of $-2a$ and $-2b$? ___ < ___	$-2b < -2a$

[illegible]

[illegible]

[illegible]	positive
[illegible]	negative

[illegible]

[illegible]

[illegible]

Case I. [illegible]	
[illegible]	
[illegible]	distributive
[illegible]	positive
[illegible]	

Case II. [illegible]	
[illegible]	positive
[illegible]	$a(-c) < b(-c)$
[illegible]	$-(ac) < -(bc)$
Therefore, [illegible]	because

[illegible] property in finding the truth sets of [illegible].

[illegible] Try to construct a proof for yourself. [illegible]

[illegible]

[illegible] ?	no [Try $a = -5$, $b = 3$]
[illegible] ?	no [Try $a = -5$, $b = -3$]

[illegible]

[illegible]

[illegible] ______.	$y > x$
[illegible] ______.	$x > z$
[illegible] ______.	$x + z > y + z$

If we assume the truth of the properties stated in Items 1-3, we could proceed to prove a number of theorems about " $>$ ". Fortunately, this is not really necessary.

When we are dealing with two different real numbers it really does not matter whether we assert, for example, $\frac{1}{2} < \frac{3}{4}$, or $\frac{3}{4} > \frac{1}{2}$. We may move, with complete freedom, from one order relation to the other.

	In particular, we may state the order property of opposites as follows:	
4	if $a < b$, then __ $>$ __.	$-a > -b$
	Similarly, if $a < b$, and c is negative,	
5	then __ $>$ __.	$ac > bc$
6	$-3 < 2$, and 3 __ -2. (<,>)	$3 > -2$
7	$\frac{4}{5} < \frac{7}{3}$, and $(-15)(\frac{4}{5})$ ____ $(-15)(\frac{7}{3})$.	$>$
8	$-x < 3$ is equivalent to x ____.	$x > -3$

9 Examine the following inequalities:

P. $y < 5$ R. $y > 5$

Q. $y < -5$ S. $-y > -5$

Which two are equivalent open sentences?

[A] P and R [B] P and S [C] Q and R [D] Q and S

The truth sets of these sentences are:

P. the set of real numbers less than 5

Q. the set of real numbers less than -5

R. the set of real numbers greater than 5

S. the set of real numbers less than 5 (if $-y > -5$ is true, it follows that $y < 5$ is true)

The correct response is [B].

10	$5 < 7$ is true, hence __ > -7 is also true.	$-5 > -7$
11	If $x < y$, then $-x >$ ____.	$-x > -y$
12	If $x < 5$, what is the order of -5 and $-x$? ____	$-5 < -x$, or $-x > -5$
13	If $-x < 2$, then x ____.	$x > -2$

Here is some practice in finding truth sets and in drawing graphs. Determine the truth set and graph each of the following open sentences. The answers are on page xvi.

14. $y < 3$

15. $-y < 3$

16. $y < -3$

17. $-y < -3$

18. $|y| < 3$ [Recall that $|y|$ means the distance between 0 and y.]

19. $-|y| < 3$

20. $|y| < -3$

21. $-|y| < -3$

Here are some practice exercises in finding truth sets of inequalities. Answers are on page xvii.

22. $x + (-2) > -3$

23. $(-\frac{3}{2}) + x < (-5) + \frac{3}{2}$

24. $2x < (-5) + x$

25. $3x > \frac{4}{3} + 2x$

26. $(-\frac{3}{4}) + \frac{5}{4} \geq x + |-\frac{3}{2}|$

The multiplication property of order can be used to help find truth sets of open sentences. We shall apply this property in the form which is the more helpful.

27 Which of the following pairs of sentences are equivalent?

[A] $-\frac{2}{3}x > -4$ and $x > 6$

[B] $-7x < -3$ and $x > \frac{3}{7}$

If we multiply both sides of $-\frac{2}{3}x > -4$ by $-\frac{3}{2}$ we obtain the equivalent sentence $x < 6$. So [A] does not contain two equivalent sentences. We obtain $x > \frac{3}{7}$ by multiplying both sides of $-7x < -3$ by $-\frac{1}{7}$, hence [B] is the correct choice.

If we use the addition and multiplication properties of order, we can find the truth set of an open sentence such as the following:

If	$(-3x) + 4 < -5$	
then	$(-3x) + 4 + (-4) < (-5) + (-4)$	(addition property of order)
or	$(-3x) < (-9)$	
hence	$(-\frac{1}{3})(-3x) > (-\frac{1}{3})(-9)$	(multiplication property of order)
finally	$x > 3$.	

Therefore the truth set of the given open sentence is the set of real numbers greater than [illegible], since $(-[illegible]x) + [illegible]$ [illegible] and $x >$ [illegible] are equivalent open sentences. Notice that all of the operations are reversible.

Of course, the order in which the steps are made may be changed. For example, our work might appear as

$(-[illegible]x) +$ [illegible]

$(-[illegible]x)$ [illegible]

[illegible] (What property is used?)

$x >$ [illegible]

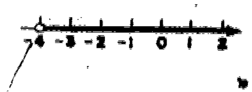
Find the truth set for the inequality:

28 [illegible] ______

The set of all numbers greater than [illegible]

Graph the truth set for the inequality.

29 [illegible] ______ (This is not the same problem as in Item [illegible].)



30 Solve this inequality: [illegible] ______

The set of all numbers greater than [illegible]

[illegible]

[illegible] $(-[illegible]x)$ [illegible] $(-$ [illegible] $)$ [illegible]

[illegible] (What properties [illegible]?)

[illegible] [illegible]

Thus [illegible] truth set is ______.

The set of all numbers greater than $\frac{4}{[illegible]}$

[illegible] $-([illegible] + x)$ [illegible] $(-[illegible])$ [illegible]

[illegible] (What properties [illegible]?)

[illegible] x ______.

$x >$ [illegible]

[illegible]

[illegible]

[illegible] ______.

$x >$ [illegible]

Again, starting with $(-2x) + 1 < 5$ one might
first write $2x + (-1) > -5$
then $2x > -4$
34 finally, $x >$ ____. | $x > -2$

Perhaps, starting with $(-2x) + 1 < 5$ one might
first write $2x + (-1) > -5$
then $x + (-\frac{1}{2}) > -\frac{5}{2}$
35 finally, $x >$ ____. | $x > -2$

36 The truth set of $(-2) + 5 + (-3x) < 4x + 7 + ($[illegible] set of real numbers which are:

[A] less than $\frac{4}{5}$ [C] less than $-\frac{5}{4}$

[B] greater than $-\frac{4}{5}$ [D] greater than $\frac{5}{4}$

The proper choice is [B]. If you had difficulty in determining the truth set, continue with Items 37-41, otherwise go to Item 42.

If $(-2) + 5 + (-3x) < 4x + 7 + (-2x)$

37 then $3 + (-3x) <$ ____ $+ 7$ | $2x + 7$

38 $3 <$ ____ $+ 7$ | $5x + 7$

39 ____ $-4 <$ ____ | $-4 < 5x$

40 ____ $< x$ | $-\frac{4}{5} < x$

41 or $x >$ ____. | $x > -\frac{4}{5}$

42 Find the truth set of $2x < 3 + |-2||-\frac{4}{3}|$. ____ | The set of all numbers less than $\frac{17}{6}$

43 Find the truth set of $\frac{1}{2}x + (-2) < (-5) + \frac{5}{2}x$. ____
[As a first step you might write: $3 < 2x$] | The set of all numbers greater than $\frac{3}{2}$

44 Find the truth set of $2\left(x + (-3)\right) > 5$. ____
[As a first step you might either multiply both sides by $\frac{1}{2}$ or apply the distributive property.] | The set of all numbers greater than $\frac{11}{2}$

45 Find the truth set of $2x + 1 \geq -3x + 9$. ____
[Be careful!] | The set of all numbers greater than or equal to $\frac{8}{5}$

Let us construct an open sentence for the following problem:

Carol has 16 more books than Patricia.
Together they have more than 28 books.
How many books does Patricia have?

46 Open sentence: ______ and x is a whole number. — $x + (x + 16) > 28$

47 Find the truth set. ______ — $\{7, 8, 9, 10, \ldots\}$

48 Patricia has at least ______ books? — 7
(how many)

If a number of flower bulbs of a certain type are planted, it is known that fewer than $\frac{5}{8}$ of them will grow. However, with proper care, more than $\frac{3}{8}$ of them will do well. If a careful gardener grows 15 of these bulbs, how many did he probably plant?

49 He must have planted more than _____ and less than _____. — 24, 40

If you did not see how to do the question above, do Items 50-57. If the problem gave you no trouble, go on with Section 10-4.

If the number he planted is represented by the variable b,

50 then $\frac{5}{8}$ of them can be represented by ______, and — $\frac{5}{8}b$

51 $\frac{3}{8}$ of them can be represented by ______. — $\frac{3}{8}b$

Since less than $\frac{5}{8}$ will grow, and we are told that 15 do grow, we can write the open sentence

52 $\underline{\qquad} < \frac{5}{8}b$ — $15 < \frac{5}{8}b$

The truth set of this open sentence is the set of

53 all numbers greater than ______. — 24

Since with proper care more than $\frac{3}{8}$ grow, we can

54 write the open sentence $15 >$ ______. — $15 > \frac{3}{8}b$

The truth set of this is the set of all numbers

55 ______ than 40. — less than

56 So we know that the gardener planted more than ______ — 24

57 bulbs and fewer than ______. — 40

10-4. Review

Here is a list of problems which will give you more practice in using the ideas developed in this chapter. Answers are on page xvii.

1. For each pair of numbers, determine their order.

   (a) $-100, -99$
   (b) $0.2, -0.1$
   (c) $|-3|, |-7|$
   (d) $\frac{6}{7}, \frac{5}{8}$
   (e) $3(4) + (-4), 3(4 + (-4))$
   (f) $x^2 + 1, 0$

2. If $p > 0$ and $n < 0$, determine which sentences are true and which are false.

   (a) If $5 > 3$, then $5n < 3n$.
   (b) If $a > 0$, then $ap < 0$.
   (c) If $3x > x$, then $3px > px$.
   (d) If $(\frac{1}{n})x > 1$, then $x > n$.
   *(e) If $p > n$, then $\frac{1}{p} < \frac{1}{n}$.
   *(f) If $\frac{1}{p} > \frac{1}{x}$ and $\frac{1}{x} > 0$, then $p < x$ and $x > 0$.

3. Which of the following pairs of sentences are equivalent?

   (a) $3a > 2$, $(-3)a > (-2)$
   (b) $3x > 2 + x$, $2x > 2$
   (c) $3y + 5 = y + (-1)$, $2y = (-6)$
   (d) $-x < 3$, $x > (-3)$
   (e) $-p + 5 < p + (-1)$, $6 > 2p$
   (f) $\frac{1}{m} < \frac{1}{2}$ and $m > 0$, $m < 2$

4. If $p > 0$ and $n < 0$, determine which represent positive numbers, which represent negative numbers.

   (a) $-n$
   (b) n^2
   (c) $-n^2$
   (d) pn
   (e) $(-p + (-n))^2$
   (f) $|n|$

5. Solve each of the following inequalities.

   (a) $-x > 5$
   (b) $(-1) + 2y < 3y$
   (c) $(-\frac{1}{2})z < 3$
   (d) $(-4) + (-x) > 3x + 8$
   (e) $b + b + 5 + 2b + 12 \leq 381$
   (f) $x(x + 1) < x$

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6. Find the truth set of each of the following sentences.

(a) $\frac{1}{x} < \frac{1}{2}$ and $x > 0$

(b) $\frac{1}{x} = \frac{1}{2}$

(c) $\frac{1}{x} < \frac{1}{2}$ and $x < 0$

(d) $(\frac{1}{x})^2 > 0$ and $x \neq 0$

(e) $0 \leq 2x < 180$

(f) $x^2 + 1 = 0$

7. If the domain of the variable is the set of integers, find the truth sets of the following sentences.

(a) $3x + 2x = 10$

(b) $x + (-1) = 3x + 1$

(c) $2x + 1 = -3x + (-9)$

(d) $2(x + (-3)) = 5$

(e) $3x + 5 < 2x + 3$

(f) $(\frac{1}{2}) + (-x) > (-\frac{1}{2}) + (-2x)$

8. Solve the following equations.

(a) $3x = 5$

(b) $3 + x = 5$

(c) $2n + n + (-2) = 0$

(d) $7y + 3 = y + (-3)$

(e) $3x = 7x + (-2)x$

(f) $3q + (-q) + 5 + q = (-2)$

9. Solve the following equations.

(a) $3(x + 5) = (x + 3) + x$

(b) $x(x + 3) = (x + (-4))(x + 3)$

(c) $\frac{1}{2}y + (-\frac{1}{3}) = (-\frac{1}{2})y + (-\frac{1}{3})$

(d) $a^2 = a(a + 1)$

(e) $(x + 2)(x + 3) = x(x + 5) + 6$

(f) $2q^2 + 2q + q^2 = (3q + 5)(q + 1)$

10. The length of a rectangle is known to be greater than or equal to 6 units and less than 7 units. The width is known to be 4 units. Find the area of the rectangle.

11. The length of a rectangle is known to be at least 6 units and less than 7 units. The width is known to be at least 4 units and less than 5 units. Find the area of the rectangle.

*12. The length of a rectangle is known to be greater than or equal to 6.15 inches and less than 6.25 inches. The width is known to be greater than or equal to 4.15 inches and less than 4.25 inches. Find the area of the rectangle.

*13. (a) A certain variety of Iowa corn plant yields 240 seeds per plant. Not all the seeds will grow into new plants when planted. Between $\frac{3}{4}$ and $\frac{5}{6}$ of the seeds will produce new plants. Each new plant will also yield 240 seeds. From a single corn plant whose seeds are harvested in 1964 how many seeds can be expected in 1965?

(b) Suppose instead that a corn plant did not yield exactly 240 seeds, but between 230 and 250 seeds. Under this condition how many seeds can be expected in 1965 from the 240 seeds planted at the beginning of the season?

14. Write open sentences and find the solution to each of the questions which follow.

(a) A square and an equilateral triangle have equal perimeters. A side of the triangle is 3.5 inches longer than a side of the square. What is the length of the side of the square?

(b) A boat traveling downstream goes 10 miles per hour faster than the rate of the current. Its velocity downstream is not more than 25 miles per hour. What is the rate of the current?

(c) Mary has typing to do which will take her at least 3 hours. If she starts at 1 P.M. and must finish by 6 P.M., how much time can she expect to spend on the job?

(d) Jim receives \$1.75 per hour for work which he does in his spare time, and is saving his money to buy a car. If the car will cost him at least \$75, how many hours must he work?

10-5. The Real Numbers - Summary

Let us summarize the material of Chapters 6-10.

We began by introducing the set of negative numbers, as suggested by our knowledge of the numbers of arithmetic and the number line. We called the union of the set of negative numbers and the set of numbers of arithmetic the <u>set of real numbers</u>.

Addition and multiplication of real numbers were then defined in such a way that

(1) the usual meanings of addition and multiplication in the set of numbers of arithmetic were preserved,

and (2) many of the properties of these operations, which were true for the set of numbers of arithmetic, were also true in the set of real numbers.

We discussed an order relation in the set of real numbers, again based on our ideas of order in the set of numbers of arithmetic.

In the course of this development we stated many properties of addition, multiplication, and order.

Here is a list of certain basic properties which we considered:

For any real numbers a, b, and c,

1. $a + b$ is a unique real number.
2. $a + b = b + a$
3. $(a + b) + c = a + (b + c)$
4. There is a special real number 0 such that $a + 0 = a$ and $0 + a = a$.
5. For each a there is a unique real number $(-a)$ such that $a + (-a) = 0$ and $(-a) + a = 0$.

1'. ab is a unique real number.

2'. $ab = ba$

3'. $(ab)c = a(bc)$

4'. There is a special real number 1 such that $a \cdot 1 = 1$ and $1 \cdot a = a$.

5'. For each a, different from 0, there is a unique real number $(\frac{1}{a})$ such that
$$a \cdot \frac{1}{a} = 1 \quad \text{and} \quad \frac{1}{a} = 1.$$

6. $a(b + c) = ab + ac$
$(b + c)a = ba + ca$

7. For a and b, exactly one of the following is true:
$$a < b, \quad a = b, \quad b < a.$$

8. If $a < b$ and $b < c$, then $a < c$.

9. If $a < b$, then $a + c < b + c$.

These are by no means all of the properties we discovered, but we will interrupt our listing at this point to make the following remark. If we add just one more basic property, we have a list of basic properties that could be used to prove everything about the real numbers. Unfortunately, this additional property would involve us in mathematics beyond the scope of this course.

Practically all the algebra of this course may be developed from the fourteen properties listed above. Let us now list some of the more important properties which may be proved as consequences of the basic properties. All of these have been discussed in the text. As you study more mathematics you will learn much more about the real numbers.

1. For real numbers a, b, and c, if $a + c = b + c$, then $a = b$.
2. For real numbers a, b, and c with $c \neq 0$, if $ac = bc$, then $a = b$.
3. For real numbers a and b, $ab = 0$ if and only if $a = 0$ or $b = 0$.
4. For any real number a, $(-1)a = -a$.
5. For any real numbers a and b, $-(a + b) = (-a) + (-b)$.
6. For any real numbers a and b, $(-a)b = -(ab)$ and $(-a)(-b) = ab$.
7. The opposite of the opposite of a real number a is a.
8. The reciprocal of the reciprocal of a non-zero real number a is a.
9. For any non-zero real numbers a and b, $\frac{1}{a} \cdot \frac{1}{b} = \frac{1}{ab}$.
10. If a and b are real numbers, then $a < b$, if and only if there is a positive number c such that $b = a + c$.
11. For any real numbers a, b, and c,

$$\text{if } a < b \text{ and } 0 < c, \text{ then } ac < bc,$$

$$\text{if } a < b \text{ and } c < 0, \text{ then } bc < ac.$$

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12. For any real numbers a and b, if $a < b$, then $-b < -a$.

13. If $0 < a < b$, then $\frac{1}{b} < \frac{1}{a}$.

14. If $x \neq 0$, then x^2 is positive.

15. If $0 < a < b$, then $a^2 < b^2$.

If we consider the set of real numbers together with the operations "+" and "$\cdot$", the relation "<", and the basic properties we may call this the <u>real number system</u>.

Chapter 11

SUBTRACTION AND DIVISION

In the preceding chapters we have succeeded in extending the order relation and the operations of addition and multiplication from the numbers of arithmetic to the set of all real numbers. We were then able to state certain fundamental properties of the real number system.

However, in stating these fundamental properties we have never mentioned subtraction and division. We remember from arithmetic that subtraction is closely related to addition and that division is closely related to multiplication. It is therefore natural to define subtraction of real numbers in terms of addition and to define division of real numbers in terms of multiplication.

11-1. Definition of Subtraction

Suppose you make a purchase which amounts to 83 cents, and give the cashier one dollar. How does she count out your change? She might hand you 2 cents and say "85", one nickel and say "90", and one dime and say "one dollar".

The clerk wants to find the difference between 83 and 100. But in subtracting 83 from 100, she has found what she has to add to 83 to obtain 100. She has mentally changed the wording of the problem from "One hundred minus eighty-three equals what?" to "Eighty-three plus what equals one hundred?"

The open sentence $100 - 83 = x$

1 has become $83 + \_\_\_\_ = 100.$ $83 + x = 100$

Recall how we have solved the equation

$$83 + x = 100.$$

In order to find the truth set, we add the opposite

2 of ______ to get 83

$$(83 + x) + (-83) = 100 + (-83).$$

Using the properties of addition, we get

3 $x = 100 + (\_\_\_\_).$ -83

4	Thus, "100 - 83" and "___ + (___)" are names for the same number. In this case, the number	100 + (-83)
5	is ______.	17

	In the same way, $27 - 8 = 19$	
6	and $8 +$ ______ $= 27$ are both true sentences.	19
7	$6 - 1 =$ ______	5
8	and $1 +$ ______ $= 6$ are both true sentences.	5

In the set of numbers of arithmetic, <u>if</u> $a - b$ names the number n, then $b + n = a$. In the set of <u>real numbers</u>, we are able to solve the equation $b + n = a$ no matter what number a and b represent. We are led to state the following definition:

> In the set of real numbers, the sentence
> $$a - b = n$$
> means exactly the same thing as the sentence
> $$b + n = a.$$

Given two real numbers, a and b, we use the definition to find "$a - b$" by solving the equation $b + n = a$.

$$\begin{array}{lr} \text{If} & b + n = a, \\ \text{then} & b + n + (-b) = a + (-b), \\ \text{and} & n = a + (-b). \end{array}$$

Therefore, "$a - b$" and "$a + (-b)$" name the same number. Furthermore, the closure property of addition assures us that "$a + (-b)$" names exactly one number.

9	"a - b" and "a + ___" name the same number.	$a + (-b)$
10	"a minus b" and "a plus the ______ of b" have the same meaning.	opposite
11	In order to subtract a real number, we ______ its opposite.	add
12	In order to subtract 2, we would add ______.	-2

13	In order to subtract 0, we would add ______.	0 [-0 = 0]
14	In order to subtract -2, we would add ______.	2 [-(-2) = 2]
15	5 - 2 = 5 + ___ = 3	5 + (-2) = 3
16	5 - 0 = 5 + ___ = ___	5 + 0 = 5
17	5 - (-2) = 5 + ___ = ___	5 + 2 = 7
18	-5 - 2 = -5 + (-2) = ___.	-5 + (-2) = -7
19	-5 - 0 = ___ + ___ = ___	-5 + 0 = -5
20	-5 - (-2) = ___ + ___ = ___	-5 + 2 = -3
	How do we read the expression 5 - (-2)?	
21	We read this as 5 minus the ______ of 2.	opposite
	In the expression 5 - (-2), the symbol " - " is used in two different ways.	
22	The first " - " indicated the operation of ______.	subtraction
23	The second " - " indicates the ______ of 2.	opposite

To help keep these uses of the symbol clear, we make the following parallel statements about them.

In "a - b",
" - " stands between two numerals and indicates the operation of subtraction. We read the above as "a minus b".

In "a + (-b)",
" - " is part of one numeral and indicates the opposite of. We read the above as "a plus the opposite of b".

Here is some more practice. The answers are on page xviii.

24. 15 - (-8) = ______
25. (-7) - 2 = ______
26. 0 - 5 = ______
27. 3 - 0 = ______
28. -9 - (-7) = ______
29. 8 - (-8) = ______
30. 8 - 8 = ______
31. (-5000) - (-2000) = ______
32. $\frac{3}{4}$ - ($\frac{1}{2}$) = ______
33. (- $\frac{9}{2}$) - (-6) = ______
34. (-0.631) - (0.631) = ______
35. (-1.79) - (-1.22) = ______
36. 75 - (-85) = ______

Instead of saying "to subtract b, add the opposite of b", we could say, "to subtract b, add the additive inverse of b". The language is somewhat awkward but it indicates why mathematicians sometimes speak of subtraction as being as operation which is the <u>inverse</u> of the operation of addition.

	To find the truth set of $y - 725 = 25$ we might use our definition of subtraction to write	
37	$y =$ ____ $+ 25$.	725
38	The truth set is ____.	{750}
	Solve:	
39	$z - 34 = 76$	{110}
40	$2x + 8 = -16$	{-12}
41	$3y - 2 = -14$	{-4}
42	$x + 23.6 = 7.2$	{-16.4}
43	$z + (-\frac{3}{4}) = -\frac{1}{2}$	$\{\frac{1}{4}\}$
44	$5 - z = 7$	{-2}

The operation of subtraction can be used to answer questions which may be stated in several ways.

45	The question "What number is 5 less than -3?" is answered by writing (-3) - ____ = ____.	(-3) - 5 = -8
46	"How much greater is 5 than -3?" is answered by writing 5 - (____) = ____.	5 - (-3) = 8
47	"-3 is how much greater than -5?" We can find the answer by writing (____) - (____) = ____.	(-3)-(-5) = 2
48	The question "How much less than 8 is -10?" has as its answer 8 - (____), or ____.	8-(-10), 18

49	Subtract -8 from 15. 15 - (-8) = ____	23
50	From -25, subtract -4. ____	-21
51	What number is 6 less than -9? ____	-15

	-12 is how much greater than -17?	
52	$(-12) - (-17) =$ ____	5
53	How much greater is 8 than -5? ____	13

11-2. Properties of Subtraction

Included in the list of properties for the real number system were the commutative and associative properties for addition and multiplication.

Is subtraction commutative or associative? If we look at some examples we might find the answer to this question.

	Examine	
1	$(-8) - (-3) =$ ____	-5
2	and $(-3) - (-8) =$ ____.	5
3	This one example is sufficient evidence to enable us to conclude that subtraction ____ (is, is not) commutative.	is not

On the other hand, we must remember that addition is commutative.

	$x - y = x + (-y)$	
	$= (-y) + (x)$ by the commutative property	
4	of ____	addition [not subtraction]
	$= -y + x$	
5	$a - b$ may be written as $-b +$ ____.	$-b + a$
6	$-7 + x =$ ____	$x - 7$
7	$-3 - x = -x$ ____	$-x - 3$
8	In each of these examples we made use of the ____ property of ____.	commutative, addition

Examine

9 $8 - (2 - 7) = 8 - (-5) =$ _______ 13

10 and $(8 - 2) - 7 = 6 - 7 =$ _______. -1

Again, this one example is enough to allow us to

11 assert that subtraction is not _______. associative

Let us recall that addition is an operation involving two numbers. Hence, $8 + (2 + 7)$ and $(8 + 2) + 7$ are both meaningful expressions. The associative property of addition assures us that these numerals name the same number. Therefore, we may omit the parentheses and write $8 + 2 + 7$ without confusion as to its meaning.

Since subtraction does not have some of the properties to which we have become accustomed, we need to be careful in handling expressions involving subtraction. For example, since subtraction is an operation involving two numbers, both $8 - (2 - 7)$ and $(8 - 2) - 7$ are meaningful. $8 - 2 - 7$ does not name a specific number, hence is not a numeral. However, it is convenient to agree on a single meaning for this expression.

We must decide whether it is to mean

$$8 - (2 - 7) = 13$$

or

$$(8 - 2) - 7 = -1 .$$

The second of these is the meaning we decide upon. That is, we agree that, for any real numbers a, b, and c,

$$a - b - c \text{ means } (a - b) - c.$$

By our agreement,

$a - b - c = (a - b) - c$

12 $= (a - b) + ($ ____ $)$, $(a - b) + (-c)$

13 $= (a +$ ____ $) + (-c)$, $(a+(-b)) + (-c)$

$= a + (-b) + (-c)$, since addition has the

14 _______ property. associative

$a - b - c$ means $a + (-b) + (-c)$. Since addition is commutative, we notice that $a + (-c) + (-b)$, $(-b) + a + (-c)$, etc., all name the same number.

Express each of the following in terms of addition and taking of opposites.

15	$x - y =$ ______	$x + (-y)$
16	$x - y - z =$ ______	$x + (-y) + (-z)$
17	$3x + 4y - 2 =$ ______	$3x + 4y + (-2)$
18	$x^2 - x + 1 =$ ______	$x^2 + (-x) + 1$
19	$2a - b - c - 5d =$ ______	$2a+(-b)+(-c)+(-5d)$

Express each of the following in terms of subtraction.

20	$a + (-b) =$ ______	$a - b$
21	$x + (-y) + (-3) =$ ______	$x - y - 3$
22	$(-3x) + y + (-4) =$ ______	$-3x + y - 4$

Multiplication is distributive over addition. We are able to write, for example, $3(x + 2) = 3x + 6$ and $5x + 3x = (5 + 3)x = 8x$. We now ask, is multiplication distributive over subtraction? Is it true that $3(x - 2) = 3x - 6$ and that $5x - 3x = (5 - 3)x = 2x$?

23 $3(x - 2)$ may be written as $3(x + \ \ )$. — $3(x + (-2))$

$3(x + (-2)) = 3x + 3(-2)$, by the distributive property of multiplication over

24 ______. — addition [not subtraction]

$= 3x + (-6)$

25 $= 3x -$ ____ — $3x - 6$

26 $5x - 3x$ may be written as $5x + (\ \ \ \ )$. — $5x + (-3x)$

$5x + (-3x) = 5x + (-3)x$

27 $= (5 + (-3))x$, by the ______ property of multiplication over addition. — distributive

$= (5 - 3)x$

$= 2x$

Since our method (in Items 23-27) is entirely general, we are able to state:

If a, b, c are real numbers, then

$$a(b - c) = ab - ac$$

and $$(b - c)a = ba - ca$$

Multiplication is distributive over subtraction. This is not a new basic property. It is a consequence of our definition of subtraction in terms of addition and of the distributive property of multiplication over addition.

	Apply the distributive property.	
28	$5(x - 1) =$ ______	$5x - 5$
29	$x(x - 2) =$ ______	$x^2 - 2x$
30	$-5(x - 1) = \underline{-5x}$ ______	$-5x + 5$
31	$-7(-m + n) =$ ______	$7m - 7n$
32	$-3(2x - 5y) =$ ______	$-6x + 15y$
33	$4(x^2 - 2x - 1) =$ ______	$4x^2 - 8x - 4$

	Apply the distributive property.	
34	$6x - 2x = (6 - 2)x =$ ______	$4x$
35	$2x - 6x = (\quad)x = -4x$	$(2 - 6)x$
36	$-7x - x =$ ______ [Recall $-x = (-1)x$.]	$-8x$
37	$x^2 - 2x =$ ______	$x(x - 2)$ or $(x - 2)x$
38	$x^2 - x =$ ______	$x(x - 1)$ [since $-x = (-1)x$]
39	$3y - 3x =$ ______	$3(y - x)$

We already know that the opposite of a sum of two real numbers equals the sum of the opposites: $-(a + b) = (-a) + (-b)$.

Using the notation of subtraction, we see that

40 $-(a + b) = \underline{-a \qquad}$. $-a - b$

Let us see how $-(a - b)$ might be written.

41 $a - b$ means $\underline{a + \qquad}$. $a + (-b)$

42 So $-(a - b)$ means $\underline{-(a + \qquad)}$. $-(a + (-b))$

Since $-(a + (-b))$ is the opposite of a sum,

$$-(a + (-b)) = (-a) + (-(-b))$$

43 $= \underline{-a \qquad}$ $-a + b$

---

44 $-(x - y) = \underline{\quad + \quad}$ $-x + y$

45 $-(x - 5) = \underline{\qquad}$ $-x + 5$

46 $-(b - a) = \underline{\quad + \quad} = a - b$ $-b + a$

47 $x - (x - y) = x - x + y = \underline{\qquad}$ y

48 $5 - (a - 2) = \underline{\qquad}$ $-a + 7$

---

49 $(x - 3) + (x - 2) = \underline{2x \qquad}$ $2x - 5$

50 $(x - 3) - (x - 2) = \underline{\qquad}$ -1

$3(x + 1) - 2(x - 1) = 3x + 3 - 2x + 2$

51 $= \underline{\qquad}$ $x + 5$

52 $(3a + 2b - c) - (a - b + 3c) = \underline{\qquad}$ $2a + 3b - 4c$

If you were able to respond correctly to Item 52, you have developed a great deal of skill in working with subtraction. Although you should be able to take many steps mentally, you should also be able to explain each step. Can you explain each step below?

$$
\begin{aligned}
(x - 3) - (x - 2) &= \big(x + (-3)\big) + \big(-(x + (-2))\big) \\
&= \big(x + (-3)\big) + \big((-x) + 2\big) \\
&= x + (-3) + (-x) + 2 \\
&= x + (-x) + (-3) + 2 \\
&= 0 + (-1) \\
&= -1
\end{aligned}
$$

If you wish to check your reasons, see page xix.

53 A simpler numeral for $(5x - 3y) - (2 + 5x) + (3y - 2)$ is

- [A] 0
- [B] $10x - 4$
- [C] -4
- [D] $10x - 6y - 4$
- [E] none of these

The correct choice is [C]. If you feel that you need more practice, or if you responded incorrectly, continue with Item 54. If you had no difficulty, proceed to Item 61.

Simplify each of the following.

54	$(3x - 6) + (7 - 4x) =$ ______	$-x + 1$
55	$(3x - 6) + (6 - 3x) =$ ______	0
56	$(5a - 17b) - (4a - 6b) =$ ______	$a - 11b$
57	$-(3x - 4y) =$ ______	$-3x + 4y$
58	$-(7 - x) =$ ______	$-7 + x$
59	$(3a + 2b - 4c) - (5a - 3b + c) =$ ______	$-2a + 5b - 5c$
60	$-(7x + 5) + (-3 + 2x) =$ ______	$-5x - 8$

3

61 $7x^2 - 4x^2 - 11x^2 =$ ______ $-8x^2$

62 $3a^2 - 5a^2 + 6a =$ ______ $+ 6a$ $-2a^2 + 6a$

63 $(x^2 - x + 1) + (x^2 + 3x - 4) =$ ______ $2x^2 + 2x - 3$

64 $(x^2 - x + 1) - (x^2 + 3x - 4) =$ ______ $-4x + 5$

65 $(7x^2 - 3x) + (4x^2 - 7x - 8) =$ ______ $11x^2 - 10x - 8$

66 $(7x^2 - 3x) - (4x^2 - 7x - 8) =$ ______ $3x^2 + 4x + 8$

67 From $11a + 13b - 7c$ subtract $8a - 5b - 4c$. ______ $3a + 18b - 3c$

What is the result of subtracting $-3x^2 + 5x - 7$

68 from $-3x + 12$? ______ $3x^2 - 8x + 19$

What must be added to $3s - 4t + 7u$ to obtain

69 $-9s - 3u$? ______ $-12s + 4t - 10u$

70 $5a - a = (5 - 1)a =$ ______ $4a$

71 $3(x - y) - (x - y) = 2($ ______ $)$ $2(x - y)$

72 $-(x^2 + 2x + 1) - (x^2 + 2x + 1) = -2($ ______ $)$ $-2(x^2 + 2x + 1)$

73 $4(x + y) - (x + y) - 3(x + y) =$ ______ 0

74 A simpler expression for $3x(1 - x) - x(x - 2)$ is

[A] $-4x^2 + x$

[B] $-4x^2 + 5x$

[C] $2x^2 + 5x$

[D] $2x^2 + x$

The correct response is [B]. If you had difficulty continue with Item 75, otherwise go to Item 79.

75 $3x(1 - x) = 3x -$ ______ $3x - 3x^2$

76 $-x(x - 2) = (-x)(x) + (-x)(-2) = -x^2$ ______ $-x^2 + 2x$

$3x(1 - x) - x(x - 2) = 3x(1 - x) - x(x - 2)$

77 $= 3x - 3x^2 -$ ______ $+$ ______ $3x - 3x^2 - x^2 + 2x$

$= (-3x^2 - x^2) + (3x + 2x)$

78 $=$ ______ $-4x^2 + 5x$

79	$4u(2u + 3) - 3(2u + 3) =$ ______	$8u^2 + 6u - 9$
80	$x(x + y) - y(x + y) =$ ______	$x^2 - y^2$
81	$a(b + c + 1) - 2a(2b + c - 1) =$ ______	$-3ab - ac + 3a$
82	$-x(4x - y) + 2x(-x - y) =$ ______	$-6x^2 - xy$

Formerly in writing open sentences we have been careful to write, for example,

$$3x + (-4) = 5.$$

We are now able to write this as $3x - 4 = 5$.

Our method of solving, of course, remains the same: we add the opposite of -4 to both sides. Since $-(-4) = 4$, we have

$$3x - 4 = 5$$
$$3x - 4 + 4 = 5 + 4$$
$$3x = 9$$
$$x = 3$$

The truth set is $\{3\}$.

Solve

83	$3x - [illegible] = x + [illegible]$	$\{4\}$
84	$-2y = 2 + y - [illegible]$	$\{2\}$
85	$3a + [illegible] - 7a + [illegible] = a - [illegible]$	$\{0\}$
86	$4u + [illegible] > -5u - [illegible]$	set of numbers greater than $-\frac{2}{3}$
87	$1.2 - 2.5x < -1.3 - [illegible]$	set of numbers greater than 3

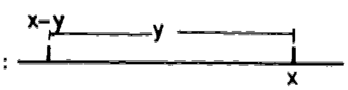
88	The width of a rectangle is [illegible] inches less than its length. What is its length if its perimeter is [illegible] inches? ______	12 inches
89	If 17 is subtracted from a number, and the result is multiplied by 3, the product is 102. What is the number? ______	51

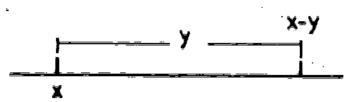
90 A teacher says, "If I had 3 times as many students in my class as I do have, I would have less than 46 more than I now have." What is the largest number of students that he could have in his class? ______ 22

If you had trouble finding an open sentence for any of these, you will find suggestions on page xix.

11-3. Subtraction and Distance

We already know how to illustrate the addition of two real numbers on the number line. Remember that to add a positive number y to a real number x, we start at x and move $|y|$ units to the right. Similarly, to add a negative number y, we move $|y|$ units to the left. Since subtraction is defined in terms of addition, we therefore know how to illustrate subtraction.

If $y > 0$: 

If $y < 0$: 

Although "subtraction on the number line" as shown above offers us nothing new, we might try a somewhat different approach.

1 We begin by noticing that, since $(a - b) = \underline{-(\qquad)}$, -(b - a)

$$|a - b| = |-(b - a)|.$$

Now, for any real number x, $|-x| = |x|$.

It follows that

$$|-(b - a)| = |b - a|.$$

2 Therefore, $|a - b| = |b \quad\quad|$. — $|b - a|$

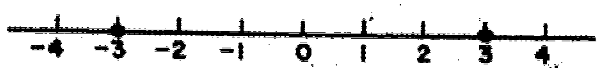
Whatever statement we make about $|a - b|$ will also apply to $|b - a|$.

Next, we recall that the absolute value of any real number represents a distance on the number line.

3 $|a|$ represents the distance between a and ____. — 0

4 The truth set of $|x| = 3$ is ______, — $\{-3, 3\}$

5 and its graph is ______. — See answer below.



$|x|$ does not seem to indicate subtraction, but we

6 <u>could</u> write $|x| = |x - 0| = |\quad - x|$. — $|0 - x|$

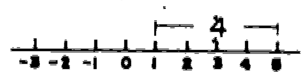
That is, $|x - 0|$ and $|0 - x|$ each represents the

7 ______ between x and 0. — distance

Given any two real numbers, it is reasonable to ask: may we interpret $|a - b|$ in terms of distance on the number line? We start by examining some particular pairs of numbers.

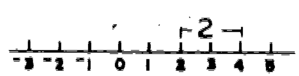
8 $|5 - 1| = |1 - 5| =$ ____ — 4

On the number line,



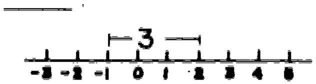
9 $|4 - 2| = |2 - 4| =$ ____ — 2

On the number line,



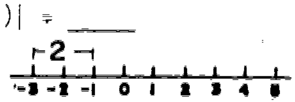
10 $|2 - (-1)| = |(-1) - 2| =$ ____ — 3

On the number line,



11 $|(-1) - (-3)| = |(-3) - (-1)| =$ ____ — 2

On the number line,



Our examples lead us to the general statement:

> If a and b are any real numbers, then the distance between a and b is given by $|a - b|$. (Of course, $|a - b| = |b - a|$.)

12 The distance between 5 and 1 is ______. 4
13 $|5 - 1|$ = ______ 4

14 The distance between 7 and 0 is ______. 7
15 $|7 - 0|$ = ______ 7

16 The distance between -3 and 5 is ______. 8
17 $|-3 - 5|$ = ______ 8

18 The distance between 6 and -2 is ______. 8
19 $|6 - (-2)|$ = ______ 8

20 The distance between -1 and -3 is ______. 2
21 $|-1 - (-3)|$ = ______ 2

For any real number x, the distance between 2 and x may be written either $|2 - x|$ or $|x - 2|$. We usually prefer the form $|x - 2|$.

The distance between

22 x and 5 may be written as |___ - 5|. $|x - 5|$
23 2 and x may be written as |x ___|. $|x - 2|$
24 x and -2 may be written as |x ___|. $|x-(-2)|$, or $|x+2|$
25 0 and x may be written as ______. $|x - 0|$, or $|x|$.

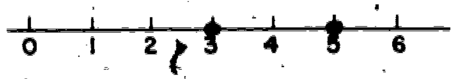
$|x - 1|$ may be interpreted as the distance between
26 x and ______. 1

$|x + 1|$ may be interpreted as the distance between
27 x and ______. -1

We have seen that "distance between" is related to subtraction and absolute value. An understanding of this relationship will help give us one way of finding truth sets of equations such as $|x - 4| = 1$.

The expression $|x - 4|$ may be interpreted as "the
28 _______ between x and 4". distance

The sentence "$|x - 4| = 1$", therefore, may be interpreted as "the distance between x and 4
29 is _______". 1

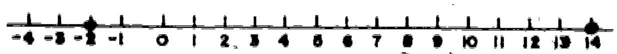


The distance between 4 and each of the points
30 indicated by heavy dots on the number line is _______. 1

The distance between 3 and 4, and between
31 _______ and 4 is 1. 5

The truth set of the sentence $|x - 4| = 1$ is
32 { _______ }. {3,5}

$|x - 6| = 8$ may be interpreted as, "the distance
33 between x and 6 is _______". 8.



The distance between 6 and each of the points
34 indicated on the number line is _______. 8

35 The distance between _______ and 6, and between _______ and 6 is 8. -2, 14

36 The truth set of $|x - 6| = 8$ is _______. {-2,14}

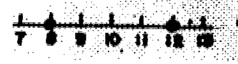
$|10 - a| = 2$ may be interpreted as, "the distance
37 between _______ and _______ is 2". 10 and a

The distance between 12 and 10 is 2, also the
38 distance between _______ and 10 is 2. 8

39 The truth set of $|10 - a| = 2$ is _______. {8,12}

Since $|10 - 8| = |2|$ and $|2| = 2$, and since
40 $|10 - 12| = |$ _______ $|$ and _______ $= 2$, we are certain that 8 and 12 are members of the truth set of $|10 - a| = 2$. |-2|, |-2|

41 Draw the graph of $|10 - a| = 2$. ______

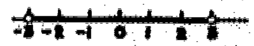


In graphing $|x| < 3$ it may be convenient to think of $|x| < 3$ as $|x - 0| < 3$.

This may be interpreted as: "the distance between
42 ______ and 0 is less than 3". x

Since the distance between x and 0 is less than 3, the graph of $|x| < 3$ includes all points to
43 the right of -3 and to the ______ of 3. left

44 Graph the truth set of $|x| < 3$.



45 Which of the following sentences is <u>not</u> equivalent to the other three?

[A] $|x| < 3$ [C] $-3 < x$ or $x < 3$

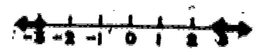
[B] $-3 < x$ and $x < 3$ [D] $-3 < x < 3$

The truth sets of [A], [B] and [D] are all the same: the set of real numbers between -3 and 3. The truth set of [C] is the set of all real numbers. Thus, [C] is the correct choice.

46 Graph the truth set of $|x| \geq 3$.

(This may be read, "the distance between x and
47 0 is greater than or equal to ______".) 3

48 Graph the truth set of $x \leq -3$ or $x \geq 3$.

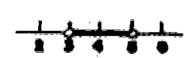


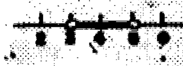
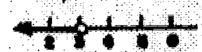
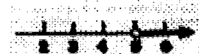
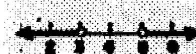
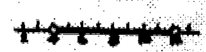
The graphs of "$|x| \geq 3$", and of "$x \leq -3$ or $x \geq 3$"
49 are ______. the same
(the same, different)

$|x - 4| < 1$ may be interpreted as "the distance
50 between x and 4 is less than ______". 1

51 Draw the graph of the truth set of $|x - 4| < 1$.

52 Graph the truth set of $x > 3$ and $x < 5$.



	"$x > 3$ and $x < 5$" is usually written "$3 < x < 5$".	
53	Graph the truth set of $3 < x < 5$.	
	The graph of the truth set of $\|x - 4\| < 1$ is the	
54	same as the graph of the truth set of ____ < x ____.	$3 < x < 5$
	The truth set of $\|x - 4\| > 1$ is the set of all	
55	numbers x such that the distance between ________	x and 4
56	is greater than ______.	1
57	Graph the truth set $x < 3$.	
58	Graph the truth set of $x > 5$.	
59	Graph the truth set of $x < 3$ or $x > 5$.	
60	Graph the truth set of $\|x - 4\| > 1$.	
	Notice that the truth set of $x < 3$ or $x > 5$ is the	
61	same as the truth set of ______.	$\|x - 4\| > 1$
	"$\|y - 8\| < 4$" may be read:	
62	"The distance between ____ and ____	y and 8
63	is less than ______."	4
	The truth set of $\|y - 8 < 4\|$ is all real numbers y	
64	such that ___ < y < ___.	$4 < y < 12$
65	Graph $\|y - 8\| < 4$.	
	"$\|x - (-9)\| = 3$" may be read, "The distance between	
66	x and -9 is ______."	3
67	The truth set of $\|x - (-9)\| = 3$ is ______.	{-12, -6}
68	Graph $\|x - (-9)\| = 3$.	

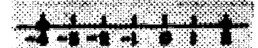
$|y + 5| = 9$ can be written $|y - (-5)| = 9$.
Written in this form it is now easy to see that the

69 distance between y and ____ is 9. — (-5)

70 The truth set of $|y - (-5)| = 9$ is ______. — $\{-14, 4\}$

71 The truth set of $|x + 3| = 5$ is ______. — $\{-8, 2\}$

72 Graph the truth set of $|x + 1| = 3$.



73 Graph $|x| > -3$. (Be careful!)

[For any x, $|x|$ is non-negative.]

Here are some more exercises on finding truth sets and their graphs. If you need more practice, work these exercises. If you have had no trouble with this section, go to Section 11-4, or to starred Item 82.

Find the truth set of each of the following. Graph each of these sets. Answers will be found on page xix.

74. $y + |-6| = 10$	78. $|x + 4| = 3$
75. $|y| + 12 = 13$	79. $|x - 3| < 5$
76. $|z| + 12 = 6$	80. $|x - 2| > 3$
77. $|x - 3| = 2$	81. $|x + 3| < 4$

* In later mathematics courses you will encounter absolute value in a variety of situations. By examining some special cases, perhaps you can discover a result which is useful in more advanced mathematics.

We shall compare the absolute value of a sum of two numbers and the sum of the absolute value of the two numbers.

*82	$|7 + 5| =$ ____	$|7| + |5| =$ ____	12, 12
*83	$|-7 + 5| =$ ____	$|-7| + |5| =$ ____	2, 12
*84	$|7 + (-5)| =$ ____	$|7| + |-5| =$ ____	2, 12
*85	$|-7 + (-5)| =$ ____	$|-7| + |-5| =$ ____	12, 12

We might guess that

if a and b are positive or both negative, then

*86 $|a + b|$ ______, $|a| + |b|$

if one of a or b is positive and the other negative then

*87 $|a + b|$ ______ $|a| + |b|$. $<$
(?, ?, ?)

Combining Items 86 and 87, our conclusion is:

For any real numbers a and b

*88 $|a + b| \leq$ ______ $|a+b| \leq |a| + |b|$

A similar, although somewhat less important, result may be discovered by comparing $|a - b|$ with $|a| - |b|$.

*89 $|7 - 5| =$ ______ $|7| - |5| =$ ______ 2, 2

*90 $|7 - (-5)| =$ ______ $|7| - |-5| =$ ______ 12, 2

*91 $|-7 - 5| =$ ______ $|-7| - |5| =$ ______ 12, 2

*92 $|-7 - (-5)| =$ ______ $|-7| - |-5| =$ ______ 2, 2

Our conclusion;

*93 $|a - b|$ ______ $|a| - |b|$ $\geq$

You may wish to verify for yourself that it is also true that

$$|a - b| \geq |b| - |a|.$$

In fact, since $|a| - |b|$ and $|b| - |a|$ are opposites, and since $|a - b|$ is greater than or equal to both of these, we have

*94 $|a - b| \geq \big||a|$ ______ $\big|$ $|a-b| \geq \big||a| - |b|\big|$

The result

$$|a - b| \geq \big||a| - |b|\big|$$

may be interpreted as: "The distance between any two real numbers a and b is at least as great as the distance between their absolute values." This is apparent if you recall that $|a|$ and $|b|$ are both numbers of arithmetic, while one of a or b may be to the right of 0, the other to the left.

11-4. Division

In arithmetic, we often refer to the four "fundamental" operations of addition, multiplication, subtraction, and division. We have developed a set of properties of addition and multiplication. Having these properties, we then defined subtraction in terms of addition. No really "new" properties were needed, since our definition allowed us to apply all the previously developed properties of addition.

It would seem reasonable that we could define the fourth operation, division, in terms of multiplication.

$a - b = a + (-b)$ implies: "to subtract means to add the opposite" or "to subtract means to add the additive inverse."

Let us develop the definition of division in a parallel manner.

First of all, let us agree to write "a divided by b" as $\frac{a}{b}$, rather than as $a \div b$ or $b\overline{)a}$.

1.	$\frac{10}{2} = 5$ and $5(2) =$ ____.	10
2.	$\frac{12}{3} =$ ____ and $4(3) = 12$.	4
3.	$20(.08) = 1.6$ and $\frac{1.6}{.08} =$ ____.	20

These examples, taken from our knowledge of arithmetic, illustrate the relationship between division and multiplication. We are led to state the following definition:

In the set of real numbers, if $b \neq 0$, the sentence

$$\frac{a}{b} = n$$

means exactly the same thing as the sentence

$$a = nb.$$

If we start with two real numbers a and b, $(b \neq 0)$, we may find the number named by "$\frac{a}{b}$" by solving the equation $a = nb$. Since $b \neq 0$, b has a unique reciprocal (multiplicative inverse).

$$\text{if} \quad a = nb$$

$$\text{then} \quad a \cdot \frac{1}{b} = nb \cdot \frac{1}{b},$$

$$a \cdot \frac{1}{b} = n.$$

Therefore "$\frac{a}{b}$" and "$a \cdot \frac{1}{b}$" name the same number. Furthermore, the closure property of multiplication assures us that "$a(\frac{1}{b})$" names exactly one number, remembering that $b \neq 0$.

4	$\frac{a}{b}$ and a_____ name the same number.	$a \cdot \frac{1}{b}$
5	"a divided by b" and "a multiplied by the _____ of b" have the same meaning.	reciprocal, or multiplicative inverse
6	In order to divide by a real non-zero number we _____ by its reciprocal.	multiply
7	In order to divide by 5, we would multiply by _____.	$\frac{1}{5}$
8	In order to divide by $\frac{1}{5}$ we would multiply by _____.	5 $[\frac{1}{\frac{1}{5}} = 5]$
9	In order to divide by -3 we would multiply by _____.	$-\frac{1}{3}$ $[\frac{1}{-3} = -\frac{1}{3}]$
10	In order to divide by $-\frac{1}{3}$ we would multiply by _____.	-3 $[\frac{1}{-\frac{1}{3}} = -3]$

11 In the statement of the definition of division it is important to recognize that division is <u>not</u> defined for all pairs of numbers a and b.

$\frac{x}{y}$ is not defined if,

[A] x is 0. [B] y is 0. [C] either x or y is 0.

To determine "$\frac{x}{y}$" we multiply "x times the reciprocal of y". Since 0 has no reciprocal, y cannot be zero. On the other hand, $\frac{0}{2}$, for example, means $0 \cdot \frac{1}{2}$ which <u>is</u> the name for a number. $0 \cdot \frac{1}{2} = 0$. Thus [B] is correct.

As in arithmetic, we shall call "a" the <u>numerator</u> and "b" the <u>denominator</u>. When there is no possibility of confusion we shall call the <u>number</u> named by "a" the numerator, and the number named by "b" the denominator.

39

12	In an indicated quotient, the _______ must not be a name for the number 0.	denominator
13.	In $\frac{6}{2+1}$, the denominator is _______,	2 + 1, or 3
14	and the numerator is _______.	6
15	If $y =$ ____, then $\frac{1}{y}$ does not name a number.	$y = 0$
16	$\frac{4x}{2x} = 2$, provided _______.	$x \neq 0$
17	$\frac{4x}{2} =$ _______, for any number x.	2x

Write the common name for each indicated quotient. Answers are on page xx.

18. $\frac{75}{5} =$ _______

19. $\frac{570}{570} =$ _______

20. $\frac{2500}{1} =$ _______

21. $\frac{-30}{5} = (-30)(\frac{1}{5}) =$ _______

22. $\frac{30}{-5} = (30)(-\frac{1}{5}) =$ _______

23. $\frac{-30}{-5} =$ _______

24. $\frac{\frac{2}{3}}{\frac{1}{6}} = (\frac{2}{3})\left(\frac{1}{\frac{1}{6}}\right) = (\frac{2}{3}) \cdot 6 =$ _______

25. $\frac{4}{\frac{3}{5}} =$ _______

26. $\frac{\frac{7}{8}}{2} =$ _______

27. $\frac{-\frac{2}{3}}{-\frac{1}{3}} =$ _______

28. $\frac{0}{7} =$ _______

Two special cases of division are immediate consequences of our definition:

29	For any real number x, $\frac{x}{1} =$ ____, since $x = x \cdot 1$.	x
30	For any non-zero real number x, $\frac{x}{x} =$ ____, since $x = 1 \cdot x$.	1

If you need more practice, complete Items 31-37. Otherwise, go to Item 38.

31	$\frac{-45x}{5} =$ ______	$-9x$
32	$\frac{3\sqrt{5}}{3} =$ ______	$\sqrt{5}$
33	$\frac{35p}{7p} =$ ______ , $(p \neq 0)$	5
34	$\frac{-\frac{2}{3}}{3y} =$ ______ , $(y \neq 0)$	$-\frac{2}{9y}$
35	$\frac{12}{\frac{1}{3x}} =$ ______ , $(x \neq 0)$	$36x$
36	$\frac{-\frac{5x}{3}}{-\frac{7x}{3}} =$ ______ , $(x \neq 0)$	$\frac{5}{7}$
37	$\frac{0}{15} =$ ______	0

Notice that, since division by a real number is defined as multiplication by the reciprocal of that number,

38	if a positive number is divided by a negative number the result is ______.	**negative**
39	if a negative number is divided by a positive number the result is ______.	**negative**
40	if a negative number is divided by a negative number the result is ______.	**positive**

Instead of saying

"to divide by b, multiply by the reciprocal of b",

we could say

"to divide by b, multiply by the multiplicative inverse of b".

As was the case for subtraction, this language is awkward but it indicates why division is referred to as an operation which is the <u>inverse</u> operation of multiplication.

Notice particularly that "divide by b" is only possible if b has a reciprocal. Every real number except 0 has a reciprocal. 0 has no reciprocal, hence division by 0 is not defined. In an indicated division such as $\frac{y}{x}$, we must be careful that the domain of x does not include 0.

We have solved equations such as

$$6y = 42$$

by using the multiplication property of equality.

Thus,

$$6y = 42$$

$$(\tfrac{1}{6})6y = (\tfrac{1}{6})42$$

$$y = 7.$$

We may now solve such equations by applying our definition of division.

Thus, $6y = 42$ is true for some y,

if and only if $y = \frac{42}{6}$ is true for the same y,

$y = 7$.

By either method, we obtain the truth set $\{7\}$.

Find the truth set of:

41.	$-6y = 42$	______	$\{-7\}$
42.	$6y = -42$	______	$\{-7\}$
43.	$-6y = -42$	______	$\{7\}$
44.	$42y = 6$	______	$\{\frac{1}{7}\}$
45.	$42y = 42$	______	$\{1\}$
46.	$\frac{1}{5}x = 20$	______	$\{100\}$
47.	$\frac{x}{5} = 15$	______	$\{75\}$
48.	$\frac{1}{3}a = 4$	______	$\{12\}$
49.	$\frac{b}{3} = 0$	______	$\{0\}$
50.	$6x = \frac{12}{3}$	______	$\{\frac{2}{3}\}$

327

Find the truth set of:

51	$5a - 8 = -53$	$\{-9\}$
52	$\frac{3}{4}y + 13 = 25$	$\{16\}$
53	$x + .30x = 6.50$	$\{5\}$
54	$n + (n + 2) = 58$	$\{28\}$
55	$\frac{1}{3}a = \frac{1}{9}a + 4$	$\{18\}$

Solve:

56	$\frac{1}{x} = 3\ ,\quad x \neq 0 \qquad (\frac{1}{x} = 3$ implies $1 = 3x)$	$\{\frac{1}{3}\}$
57	$\frac{2}{y} = 4\ ,\quad y \neq 0$	$\{\frac{1}{2}\}$
58	$\frac{-3}{x} = \frac{9}{-4}\ ,\quad x \neq 0$	$\{\frac{4}{3}\}$
59	$-\frac{2}{5} = \frac{7}{a}\ ,\quad a \neq 0$	$\{-\frac{35}{2}\}$

60.	If six times a number is decreased by 5, the result is -37. Find the number.	$-\frac{16}{3}$
61.	If two-thirds of a number is added to 32, the result is 38. What is the number?	9
62.	Find two consecutive even integers whose sum is 46.	22 and 24
63.	Find two consecutive odd positive integers whose sum is less than or equal to 83.	Any two positive consecutive odd numbers less than 42.
64.	On a 20% discount sale, a chair cost $30. What was the price of the chair before the sale?	$37.50
65.	Mary bought 17 four-cent stamps and some five-cent stamps. She paid $1.86. Was she charged the correct amount?	no
66.	A syrup manufacturer made 160 <u>gallons</u> of syrup worth $608 by mixing maple syrup worth $2 per <u>quart</u> with corn syrup worth 60 cents per <u>quart</u>. How many <u>gallons</u> of maple syrup did he use?	40 gallons

In case you had difficulty solving any of the problems in Items 60-66 you will find open sentences and suggestions on page xx .

11-5. Summary and Review

Subtraction

For real numbers a, b, n
$a - b = n$ has the same meaning as $b + n = a$.

To subtract a real number b from a real number a, we add the opposite of b to a.

We agree that $a - b - c = a + (-b) + (-c)$.

$|a - b| = c$ may be interpreted on the number line as "the distance between points with coordinates a and b is c units".

Division

For real numbers a, b, n $(b \neq 0)$
$\frac{a}{b} = n$ has the same meaning as $a = nb$.

To divide a real number a by a non-zero real number b, we multiply a by the reciprocal of b.

Review Problems

Answers are on page xxi.

1. Which of the following name real numbers?

   (a) $(6 - 6) \cdot 7$ (d) $3 \cdot \frac{1}{8}$

   (b) $\frac{0}{-3}$ (e) $3 \cdot \frac{1}{0}$

   (c) $\frac{|0|}{0}$ (f) $3 \cdot 0$

2. Find the value of the phrase $b^2 - 4ac$ if

   (a) $a = 2$, $b = -1$, $c = 5$

   (b) $a = 1$, $b = -3$, $c = -2$

   (c) $a = -9$, $b = 6$, $c = -1$

3. Solve the following open sentences.

   (a) $7x + 4 - x = 3x - 8$ (d) $|x - 1| = 1$

   (b) $2a - 3 < a + 4$ (e) $|x + 2| < 3$

   (c) $3|x| \leq 6$

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4. Apply the distributive property to each of the following.

   (a) $5(x - [illegible])$

   (b) $-2(w - 1)$

   (c) $-\frac{1}{2}(\frac{1}{2} - x)$

   (d) $-3(-[illegible])$

5. Write each phrase as an indicated product.

   (a) $7 + 14y = 7(\qquad)$

   (b) $7 - 14y = 7(\qquad)$

   (c) $-7 + 14y = -7(\qquad)$

   (d) $-7 - 14y = -7(\qquad)$

6. Find simpler expressions for each of the following.

   (a) $(x - [illegible]) + (x + [illegible])$

   (b) $(x - 1) - (2x - [illegible])$

   (c) $[illegible](3x - 2) - 2(2x + 1)$

   (d) $-\frac{1}{2}(5x + 7) - [illegible](\frac{[illegible]}{[illegible]} - \frac{1}{[illegible]}x)$

7. Graph the truth set of:

   (a) $|x - 2| [illegible]$

   (b) $|x - 2| [illegible]$

   (c) $|x - 2| < [illegible]$

   (d) $|x - 2| > [illegible]$

8. An airplane which flies at an average speed of 200 m.p.h. (when no wind is blowing) is held back by a head wind and takes $[illegible]\frac{1}{2}$ hours to fly [illegible]0 miles. What is the average speed of the wind?

9. A clothing store owner sold two shirts for $5.75 each. On the first he lost [illegible]% of the cost and on the second he gained 25% of the cost. How much did he gain or lose, or did he break even on the two sales?

10. The temperature in a certain manned satellite must be kept within a range less than [illegible]° from 70° F. Write an open sentence, using absolute values. Find the truth set.

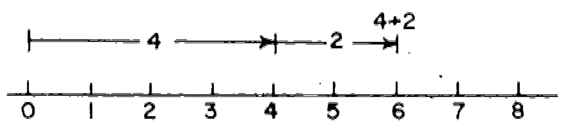
ANSWER KEY

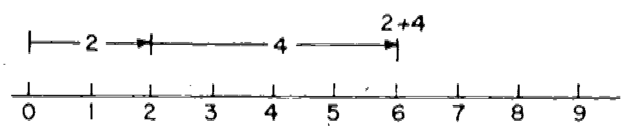
Answers for Chapter [illegible] SETS AND THE NUMBER LINE

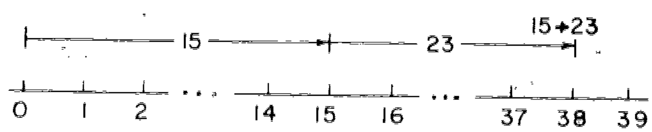
Section 1-1

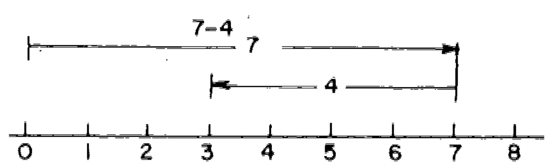
[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	...	n, a counting number
[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	...	
[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	...	2^n

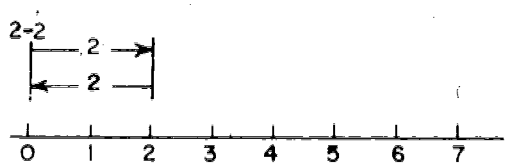
Section 1-2

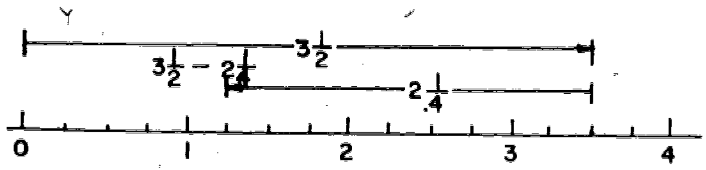


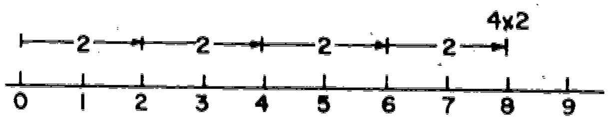








10. 

13. 

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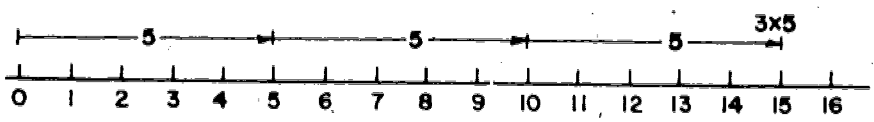
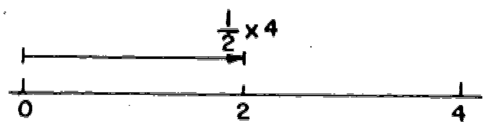
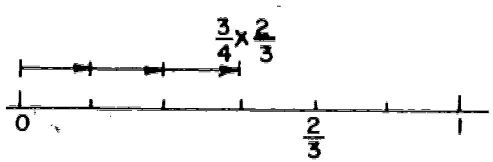
15. 

*18. 

*19. 

*20. 

Answers for Chapter 2 NUMERALS AND VARIABLES

Section 2-4

54. 185 55. 16 56. 560 57. 92 58. 1080

59. 3 60. 6 61. $\frac{7}{4}$ 62. .5 or $\frac{1}{2}$

Answers for Chapter 3 SENTENCES

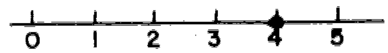
Section 3-2

25. $c = \frac{5}{9}(86 - 32)$; truth set is {30}; temperature is 30 degrees Centigrade.

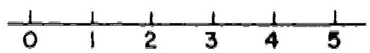
26. $120 = (1000)(.04)t$; truth set is {3}; time is 3 years.

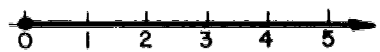
27. $(15)(600) = (75)V$; truth set is {120}; volume is 120 cubic units.

28. $20 = \frac{1}{2}(B + 4)(4)$; truth set is {6}; missing base has a length of 6 units.

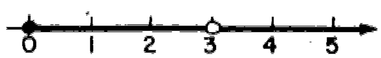
36. {4}


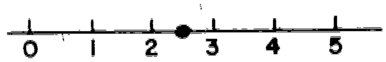
37. {3}


38. $\emptyset$


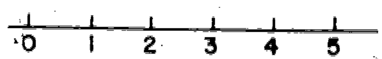
39. the set of all numbers of arithmetic


40. {0}


41. the set of all numbers except 3


42. $\{\frac{5}{2}\}$


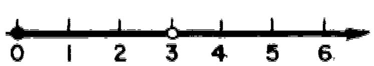
43. the set of all numbers except $\frac{4}{3}$

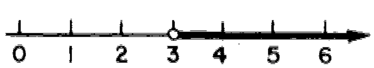

44. $\emptyset$


45. {0,2}


Section 3-3

62.


63.


64.


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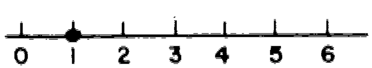

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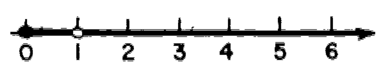
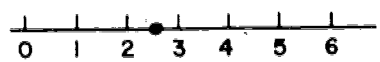
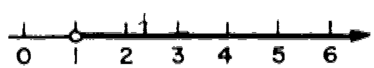
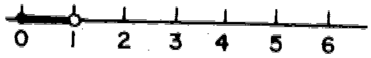


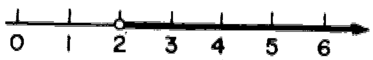
67.


68.

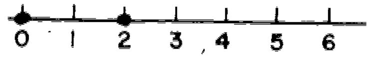

69.


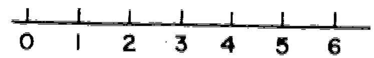
70. 

71. 

72. 

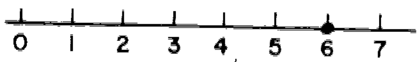
73. 

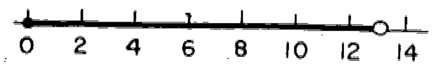
74. 

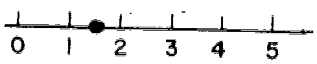
75. 

∅

Section [illegible]

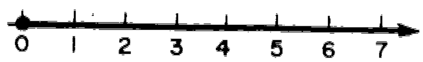
1. {6} 

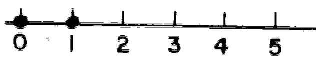
2. the set of all numbers less than [illegible] 

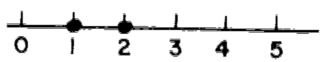
3. {[illegible]} 

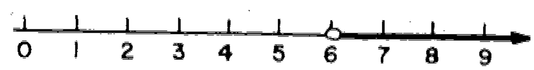
4. ∅ 

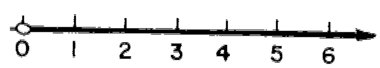
(empty set)

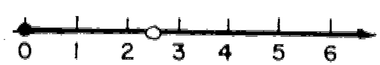
5. the set of all numbers of arithmetic 

6. {0, 1} 

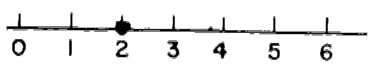
7. {1, 2} 

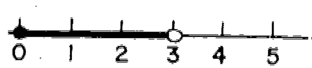
8. all numbers greater than [illegible] 

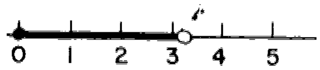
9. the set of all numbers except 0 

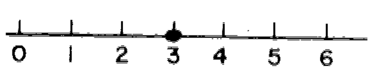
10. the set of all numbers except [illegible] 

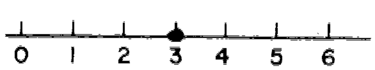
11. {[illegible]} 

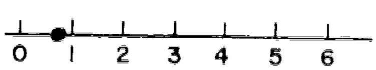
12. {[illegible]} 

13. the set of all numbers less than [illegible] 

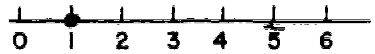
14. the set of all numbers less than [illegible] 

15. {[illegible]} 

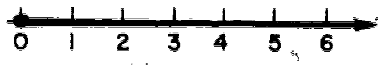
16. {[illegible]} 

17. {[illegible]} 

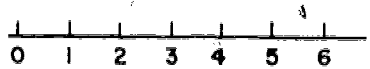
18. {1}



19. the set of all numbers



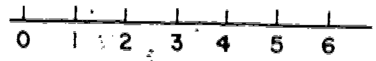
20. $\emptyset$



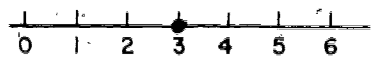
21. {2,3}

22. $\{\frac{1}{2},3\}$

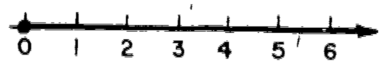
23. $\emptyset$



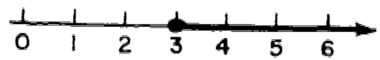
24. {3}



25. the set of all numbers



26. the set consisting of 3 and all numbers greater than 3

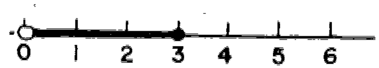


27. the set consisting of 1 and all numbers greater than 3

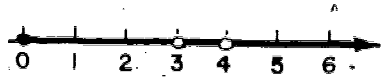
28. the set of numbers between 3 and 6

29. the set consisting of all numbers less than 2, the number 5, and all numbers greater than 5

30. the set consisting of and all numbers less than 3

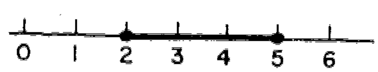


31. the set of all numbers except 3 and 4

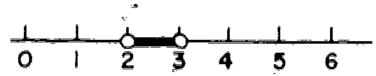


32. the set consisting of all numbers less than 2, the number 3, and all numbers greater than 4

33. the set consisting of the number 2 and 5 and all numbers between 2 and 5



34. the set of all numbers between 2 and 3



Section 4-1

17. $\frac{7}{8}\left((3.7 + 0.3) - 4\right) = \frac{7}{8}(4 - 4)$

$= (\frac{7}{8})(0)$

$= 0$ by the multiplication property of zero

19. $\left(\frac{2}{3} - \frac{1}{3}(2)\right) + 17 = (\frac{2}{3} - \frac{2}{3}) + 17$

$= 0 + 17$

$= 17$ by the addition property of zero

21. $\left((\frac{16}{3} - 5)(3)\right)(5280) = \left((\frac{1}{3})(3)\right)(5280)$

$= (1)(5280)$

$= (5280)(1)$

$= 5280$ by the multiplication property of one

Section 4-4

116. $(y + 2)(y + 9) = (y + 2)y + (y + 2)9$

$= y^2 + 2y + 9y + 18$

$= y^2 + (2 + 9)y + 18$

$= y^2 + 11y + 18$

117. $(x + 1)(x + 5) = (x + 1)x + (x + 1)5$

$= x^2 + x + 5x + 5$

$= x^2 + (1 + 5)x + 5$

$= x^2 + 6x + 5$

118. $(2x + a)(x + a) = (2x + a)x + (2x + a)a$

$= 2x^2 + ax + 2ax + a^2$

$= 2x^2 + (a + 2a)x + a^2$

$= 2x^2 + ((1 + 2)a)x + a^2$

$= 2x^2 + 3ax + a^2$

119. $(3x + 4)(4x + 3) = (3x + 4)4x + (3x + 4)3$

$= 12x^2 + 16x + 9x + 12$

$= 12x^2 + (16 + 9)x + 12$

$= 12x^2 + 25x + 12$

120. $(x + y)(x + y) = (x + y)x + (x + y)y$

$= x^2 + xy + xy + y^2$

$= x^2 + (1 + 1)xy + y^2$

$= x^2 + 2xy + y^2$

Section 4-6, Review - Chapters 1-4

1. (a) H is the set of odd integers greater than 19 and less than 51, or the set of odd integers from 21 to 49 inclusive.

   (b) H is a subset of A.

   (c) Set H is finite. Set A is infinite.

2. Since $\frac{3}{4}$ is $\frac{18}{24}$ and $\frac{5}{6}$ is $\frac{20}{24}$, the coordinate of one point between the two is $\frac{19}{24}$. There are infinitely many points between $\frac{3}{4}$ and $\frac{5}{6}$.

3. (a) Set T is closed under addition, since the sum of two elements gives an integral multiple of 3.

   (b) Set T is not closed under "averaging" since the average of two elements (such as 3 and 6) is not necessarily an integer. $\frac{3+6}{2} = \frac{9}{2}$.

4. (a)

   (b) π, $\frac{17}{4}$, $4\frac{1}{3}$ are in the domain of the variable t.
   $\frac{3}{2}$, $\frac{11}{2}$, $\sqrt{2}$ are not in the domain of the variable t.

5. In (c).

6. (a) {12} (c) {4} (e) {2}
   (b) {6} (d) {3} (f) {1}

7. If m is a number of arithmetic, the truth sets are

   (a) {1} (c) {0}
   (b) the set of all numbers of arithmetic (d) $\emptyset$

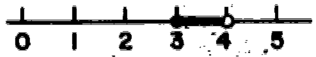
8. If the domain of m is the set of counting numbers, the truth sets are

   (a) {1} (c) $\emptyset$
   (b) the set of all counting numbers (d) $\emptyset$

9. (a) 3 is an element of T.

   (b) 2 is an element of T.

   (c) $\emptyset$ is a subset of T. (Note: $\emptyset$ is a subset of every set.)

10. 

11. (a) $\emptyset$ (b) $\{0\}$ (c) The set consisting of 0, $\frac{1}{2}$, and all numbers between 0 and $\frac{1}{2}$.

12. (a) The sentence is true.

(b) All that is needed is to note that $(8 + 1)$ and $(4 + 5)$ are names for the same number.

13. (a) true (d) true

(b) false (e) true

(c) true (f) true

14.
$$\frac{\frac{3}{5}+\frac{2}{3}}{\frac{3}{4}} = \frac{\frac{3}{5}+\frac{2}{3}}{\frac{3}{4}}\left(\frac{60}{60}\right) = \frac{(\frac{3}{5}+\frac{2}{3})(60)}{\frac{3}{4}(60)} = \frac{36+40}{45} = \frac{76}{45}$$

15. $3x + y + 2x + 3y = (3x + 2x) + (y + 3y)$ by the associative and commutative properties of addition

$= (3 + 2)x + (1 + 3)y$ by the distributive property

Since the associative, commutative, and distributive properties are true for all numbers, $3x + y + 2x + 3y = 5x + y$ is true for all numbers.

16. $(x + 1)(x + 1) = (x + 1)x + (x + 1)1$ distributive property

$= x^2 + x + x + 1$ distributive property

$= x^2 + (1 + 1)x + 1$ distributive property

$= x^2 + 2x + 1$

(b) $(x + 2)(x + 2) = x^2 + 4x + 4$ as above

17. $5 + 5x$

18. $\frac{1}{4}a(3 + 2)$ or $a(\frac{3}{4} + \frac{2}{4})$

19. $3(11) + 15(11) = (3 + 15)11$
$= (18)(11)$
or
$11(3) + 11(15) = 11(3 + 15)$
$= 11(18)$

20. $(1 + 2)c$ or $3c$

21. $\frac{1}{2}y + \frac{1}{3}y$

22. $xy + x$

23. $x(y + 1)$

24. $a(3x + 2y)$

25. $(5 + y + 2)y$ or $(7 + y)y$

26. $(1 + 3a)b$ or $b(1 + 3a)$

27. $2a^2 + 2ab + 2ac$

28. $(u + 2v)(u + v) = u^2 + 2uv + uv + 2v^2$
$= u^2 + 3uv + 2v^2$

29. $((a + 1)a + (a + 1)1 = a^2 + 1a + 1a + 1$
$= a^2 + 2a + 1$

30. $17x + x = (17 + 1)x$
$= 18x$

31. $2x + y + 3x + y = 2x + 3x + 1y + 1y$
$= (2 + 3)x + (1 + 1)y$
$= 5x + 2y$

32. $3(x + 1) + 2x + 7 = 3x + 3 + 2x + 7$
$= 3x + 2x + 3 + 7$
$= (3 + 2)x + 10 = 5x + 10$

33. $1.6a + .7 + .4a + .3b = 1.6a + .4a + .7 + .3b$
$= (1.6 + .4)a + .7 + .3b$
$= 2.0a + .7 + .3b$

34. $by + 2by = (1 + 2)by$
$= 3by$

35. $9x + 3 + x + 2 + 11x = 9x + 1x + 11x + 3 + 2$
$= (9 + 1 + 11)x + 5 = 21x + 5$

*36. Since $2 + 3 + 5 + 7$ is not divisible by 9, we cannot use the distributive property to write 2357 as the product of 9 and a whole number, so we find that 2357 is not divisible by 9.

$$\begin{aligned}
35874 &= 3(10,000) + 5(1000) + 8(100) + 7(10) + 4(1) \\
&= 3(9999 + 1) + 5(999 + 1) + 8(99 + 1) + 7(9 + 1) + 4(1) \\
&= 3(9999) + 3(1) + 5(999) + 5(1) + 8(99) + 8(1) + 7(9) + 7(1) + 4(1) \\
&= (3(9999) + 5(999) + 8(99) + 7(9)) + 3(1) + 5(1) + 8(1) + 7(1) + 4(1) \\
&= (3(1111) + 5(111) + 8(11) + 7(1))9 + (3 + 5 + 8 + 7 + 4) \\
&= (3333 + 555 + 88 + 7)9 + (27) \\
&= (3333 + 555 + 88 + 7 + 3)9
\end{aligned}$$

Hence, 35874 is divisible by 9.

The rule which becomes apparent is that if the sum of the digits of a number is divisible by 9, the number is divisible by 9.

*37. distributive property; distributive property; associative property of addition; commutative property of multiplication and distributive property.

$15 \times 14 = (15 + 4)10 + 20 = 210$

$13 \times 17 = (13 + 7)10 + 21 = 221$

$11 \times 12 = (11 + 2)10 + 2 = 132$

Answers for Chapter 5 ENGLISH AND MATHEMATICAL SENTENCES

Section 5-1

	Open Phrase	Meaning of Variable	English Phrase
	$3x + 25$	x is number of cents Tom earns in one hour.	The number of cents Tom earned in three hours if he gets a bonus of 25 cents.
34	$n + 7$	n is John's age now, in years.	Mary's age in years if she is seven years older than John; or John's age in years seven years from now.
35	$n - 7$	n is Joan's original weight in pounds.	Joan's weight in pounds after she lost seven pounds.
36	$\frac{y}{2}$	y is the number of dollars Larry earned in two weeks.	Larry's wages in dollars for one week.
37	$2r + 5$	r is the cost in cents of one bunch of rhubarb.	5 more than the cost in cents of 2 bunches of rhubarb.
38	$a + b$	a is my age in years and b is my sister's age in years.	the combined age in years of my sister and me

48. If I bought n bracelets, and the same number of necklaces, then the phrase is: The total cost in dollars if I bought some bracelets at \$1 each and the same number of necklaces at \$7 each.

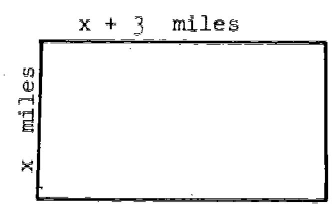
49. If there are r paper clips in each box, then the phrase is: The number of paper clips I now have if I had two boxes of them and my roommate took away five paper clips and then brought back seven.

50. If the first side of a triangle is x inches long, then the phrase is: The number of inches in the perimeter of a triangle if the second side is three times the first one and the third side is one inch longer than twice the first side.

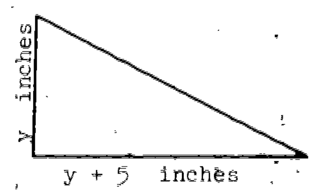
51. If y is the number of people in each of the new families, then the phrase is: The population of a village which originally had 5,000 people after four more families move in.

52. If y is the number of dollars I put in the bank, then the phrase is: The number of dollars in the bank after simple interest for one year at 4 percent has been added.

*53. If a rectangular National Forest is x miles wide, then the phrase is: The number of square miles in the area of the forest if it is 3 miles longer than it is wide.



*54. If the altitude of a triangle is y inches, then the [illegible] is: The number of squ[illegible]es in the area of the tri[illegible] its base is 5 inches lo[illegible] than its altitude.



55. If the first side of a triangle is 5 inches long, then the phrase is $5 + 3$.

56. If a people bought tickets, then the phrase is $2a$.

57. $r + 12$

If x is the number of dollars in the inheritance, then the phrase is:

58. (a) $\frac{1}{2}x$ or $\frac{x}{2}$

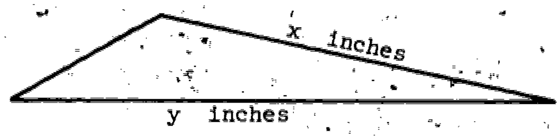
(b) $\frac{1}{10}x + 50$ or $\frac{x}{10} + 50$

(c) $x - \left(\frac{1}{2}x + \left(\frac{1}{10}x + 50\right)\right)$ or $x - \left(\frac{x}{2} + \left(\frac{x}{10} + 50\right)\right)$

(d) x. Of course, you might write $\left(\frac{1}{2}x\right) + \left(\frac{1}{10}x + 50\right) + \left(x - \left(\frac{1}{2}x + \left(\frac{1}{10}x + 50\right)\right)\right)$. You might find it interesting to show that this phrase represents the same number as x.

59. (a) $\frac{1}{2}(x + y)$

(b) $x + y + \frac{1}{2}(x + y)$



60. If the plant grows g inches per week, then the height in 5 weeks is $20 + 5g$.

*61. If t is the number of minutes after his arms are immersed, and $t \geq 10$, then the phrase is $t - 10$.

Section 5-3

	Open Sentence	Meaning of Variable	English Sentence
7	$3x - 2 = 784$	x is the number of books in the library.	The number of books in the library is tripled. Two books are lost. There are 784 books left.
8	$n + 7 = 82$	Jonathan weighs n pounds.	Sammy, who is 7 pounds heavier than Jonathan, weighs 82 pounds. (Or a similar sentence.)
9	$\frac{c}{2} = 17$	c is the number of students in the class which consists of the same number of boys as girls.	A class that has the same number of boys as girls has 17 boys (girls).

11. In order to buy 500 envelopes, I had to buy two boxes of envelopes.

12. I have three pieces of chain. The second piece has twice as many links as the first and the third piece has three times as many links as the first. I get a single chain with the same number of links whether I fasten the second and third pieces together and then fasten them to the first, or whether I fasten the first and second pieces together and then fasten the third to them.

13. James bought some 5 cent stamps and the same number of 8 cent stamps. The total cost was 65 cents.

14. The perimeter of a square is 100 feet.

15. I have to travel just as far whether I go from here to Fairwood and then the five miles to Middlebury, or whether I go first from Middlebury to Fairwood and then back here.

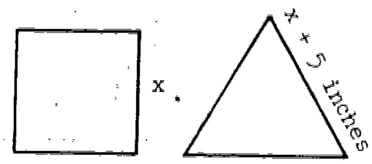
16. We received 6 gallons of milk, some in quart bottles and some in half-gallon bottles.

Section 5-4

83. If the number is n, then $\frac{3}{4}n + \frac{1}{3}n \geq 26$.

84. If a side of the square is x inches long, then a side of the triangle is $x + 5$ inches long.

Open sentence: $4x = 3(x + 5)$



85. If his score on the third test is t, then

$$\frac{75 + 82 + t}{3} \geq 88$$

$$\frac{75 + 82 + 100}{3} = \frac{257}{3} = 85\frac{2}{3}$$

$$\frac{75 + 82 + 0}{3} = \frac{157}{3} = 52\frac{1}{3}$$

Section 5-5

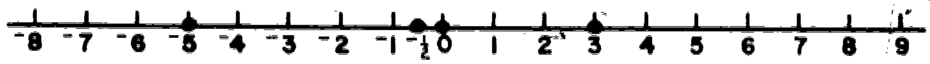
*62. (a) The smallest amount of change the man could have is \$.16.

(b) The largest possible amount of change is \$1.62.

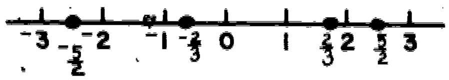
ANSWER KEY

Section 6-1. THE REAL NUMBERS

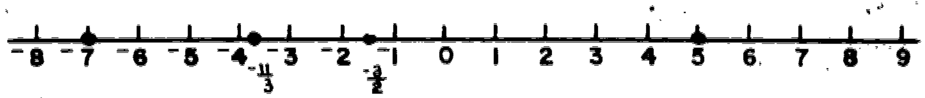
61.



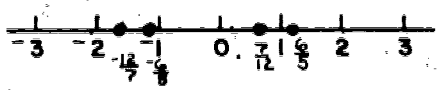
62.



63.



64.



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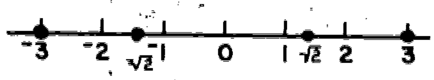
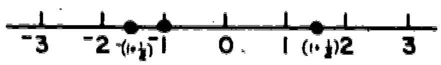
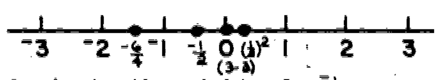


66.



67.



68. 3 is to the right of $^-4$

69. 5 is to the right of $^-4$

70. $^-2$ is to the right of $^-4$

71. 1 is to the right of $^-\sqrt{2}$

72. 0 is to the right of $^-(\frac{5}{2})$

73. $^-(\frac{5}{2})$ and $^-(\frac{10}{4})$ are names for the same number and so name the same point on the number line.

74. 3 is to the right of 0

75. $\sqrt{2}$ is to the right of $^-4$.

76. $^-(\frac{21}{4})$ is to the right of $^-(\frac{16}{3})$ $\quad ^-(\frac{21}{4} \times \frac{3}{3}) = {}^-(\frac{63}{12})$ and $^-(\frac{16}{3} \times \frac{4}{4}) = {}^-(\frac{64}{12})$

$^-(\frac{63}{12})$ is to the right of $^-(\frac{64}{12})$

77. $\frac{1}{2}$ is to the right of $^-(\frac{1}{2})$

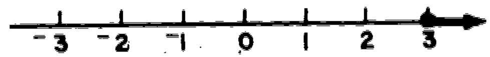
78. $^-2$ is an integer, a rational number, a real number. It is not a whole number.

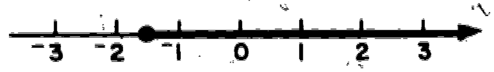
79. $^{-}(\frac{10}{3})$ is a rational number and a real number. It is neither a whole number nor an integer.

80. $^{-}\sqrt{2}$ is a real number. It is neither a whole number, nor an integer, nor a rational number.

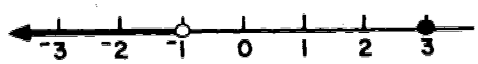
Section 6-2

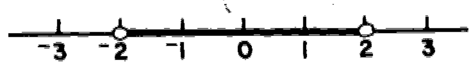
37. $y < 2$ 

38. $u \nless 3$ 

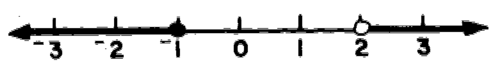
39. $v \geq {}^{-}(\frac{3}{2})$ 

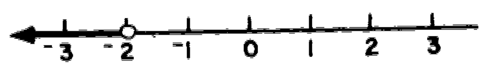
40. $r \neq {}^{-}2$ 

41. $x = 3$ or $x < {}^{-}1$ 

42. $c < 2$ and $c > {}^{-}2$ 

43. $a \leq {}^{-}3$ and $a \geq {}^{-}3$ 

44. $d \leq {}^{-}1$ or $d > 2$ 

45. $a < 6$ and $a < {}^{-}2$ 

46. $u > 2$ and $u < {}^{-}3$. The truth set in this case is the empty set, $\emptyset$, and hence has no graph. You should observe that there is no number both less than $^{-}3$ and greater than 2.

*102. and *103.

Under the ordering "$\prec$", $3 \nprec {}^{-}3$ and $^{-}3 \nprec 3$. Since $3 \neq {}^{-}3$ we see that the comparison property does not hold. Clearly the transitive property is true for the real numbers. This would not be a useful extension of order although in the set of numbers of arithmetic, "$\prec$" and "$<$" actually do have the same meaning.

Section 6-3

57. $-\frac{1}{6} < \frac{2}{7}$

58. $-\pi < \sqrt{2}$

59. $\pi < \frac{22}{7}$ (The value of π, to 4 decimal places, is 3.1416.)

60. $3(\frac{4}{3} + 2) < \frac{5}{4}(20 + 8)$

61. $-2.1 < (\frac{8 + 6}{7})$

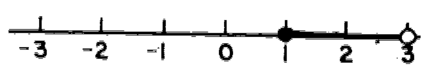
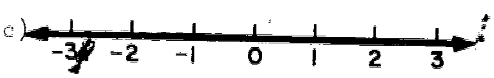
62. $-((5 + 0)3) < -((3 + 17)0)$

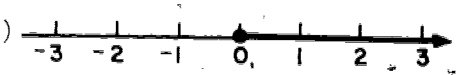
90. If s is the number representing John's score, the open sentence is $s > -100$.

91. If n is the number representing my financial condition in dollars, the open sentence is $n \leq 0$ and $n \geq -200$ or $-200 \leq n \leq 0$.

92. If d is the number of dollars in Paul's original debt, the open sentence is $d - 10 > 25$, (or $d > 35$).

Section 6-5

1. Sentences (b), (d), (e), (f) are true.
2. Sentences (a), (d), (f) are false.
3. (a)  (c) 

   (b)  (d) 

4. (a) $\emptyset$ (No numbers are in the intersection of the truth sets of $y \leq 3$ and $y > 4$.)

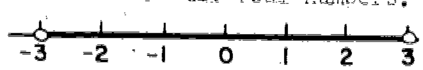
   (b) The set of all real numbers. ($-|u| < 2$ is the same as $|u| > -2$. Since $|u|$ is always non-negative, $|u|$ is always greater than -2.)

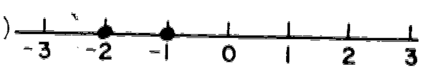
   (c) The set of all real numbers greater than -3 and less than 2.

   (d) The set of all non-positive numbers. (Look at the sentence $|x| = -x$. $|x|$ is always non-negative. "$-x$" is read "the opposite of x". What is the set of numbers whose opposites are positive?)

   (e) $\emptyset$ ($|x|$ is never negative.)

   (f) The set of all real numbers.

5. (a)  (c) 

   (b)  (d) 

6. If we let "t" represent the number degrees of the temperature on Sunday, and if we interpret "within" to mean a variation up to 6° but not including 6°, then the open sentence can be written: $t > -11$ and $t < 1$, or $-11 < t < 1$.

You might look at the number line. If you go 6 to the left, you are at -11 and if you go 6 to the right, you are at 1.

7. (a), (b), and (e) are true statements.

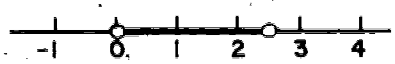
8. 

Yes, -8 is an element of this set. $|-8| = 8$ and 8 is greater than 3.

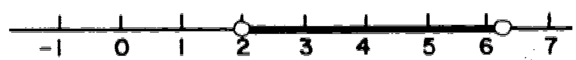
9. (a) If n is the integer, $n + 1$ is its successor and $n + (n + 1) = n + 1$.

(b) The truth set of this sentence is $\{0\}$.

10. (a) If s is the number of units in the side of this square, s is positive, $4s$ is the perimeter of the square. A sentence for this is $s > 0$ and $4s < 10$.



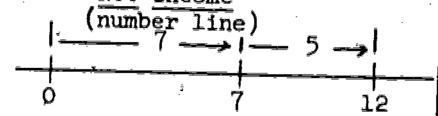
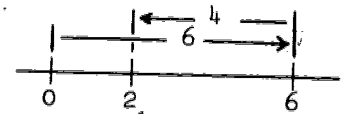
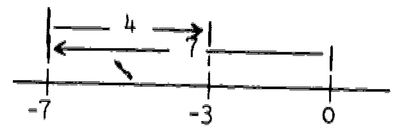
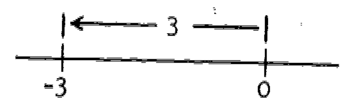
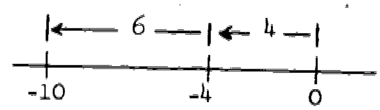
(b) If A is the number of units in the area of the square, then $A = s^2$, where $s > 0$ and $4s < 10$, as in part (a). Since A is s^2, and s is a number from the set of numbers between 0 and 2.5, the truth set of A is the set of numbers between $(0)^2$ and $(2.5)^2$ or between 0 and 6.25.



PROPERTIES OF ADDITION

Section 7-1

PANEL 7-1

Column 1 Business	Column 2 Net income (by arithmetic)	Column 3 Net income (number line)
Mon.: Profit of $7 Tues.: Profit of $5	$7 + 5 = 12$	
Wed.: Profit of $6 Thurs.: Loss of $4 (Tire trouble)	$6 + (-4) = 2$	
Fri.: Loss of $7 (Another tire) Sat.: Profit of $4	$(-7) + 4 = -3$	
Sun.: Day of rest Mon.: Loss of $3 (cold day)	$0 + (-3) = -3$	
Tues.: Loss of $4 (colder) Wed.: Loss of $6 (gave up)	$(-4) + (-6) = -10$	

Section 7-2

109. false
110. true
111. true
112. true
113. false
114. false
115. false
116. true
117. true

Section 7-4

46. If $x + 5 = 13$ is true for some x,

then $(x + 5) + (-5) = 13 + (-5)$

$x = 8$ is true for the same x.

Check: If x is 8, in the original question

the left side is: $8 + 5$

and: $8 + 5 = 13$

the right side is: 13

Hence, the truth set is $\{8\}$.

47. If $(-6) + 7 = (-8) + x$ is true for some x,

then $1 = (-8) + x$

$$8 + 1 = 8 + \big((-8) + x\big)$$

$9 = x$ is true for the same x.

Check: if x is 9,

the left side is: $(-6) + 7$

and $(-6) + 7 = 1$

the right side is: $(-8) + 9$

and $(-8) + 9 = 1$

Hence, the truth set is $\{9\}$.

48. If $(-1) + 2 + (-3) = 4 + x + (-5)$ is true for some x,

then $-2 = x + (-1)$

$$(-2) + 1 = \big(x + (-1)\big) + 1$$

$-1 = x$ is true for the same x.

Check:

the left side is: $(-1) + 2 + (-3)$

and $(-1) + 2 + (-3) = -2$

the right side is: $4 + (-1) + (-5)$

and $4 + (-1) + (-5) = -2$

Hence, the truth set is $\{-1\}$.

Unacceptable:

$$(-1) + (2) + (-3) = 4 + x + (-5)$$
$$(-1) + 2 + (-3) = -2 = x + (-1) =$$
$$(-2) + 1 = \big(x + (-1)\big) + 1$$
$$-1 = x$$

49. If $(x + 2) + x = (-3) + x$ is true for some x,

then $\big((x + 2) + x\big) + (-x) = \big((-3) + x\big) + (-x)$

$$x + 2 = -3$$

$$(x + 2) + (-2) = (-3) + (-2)$$

$x = -5$ is true for the same x.

Check: If x is -5

the left side is: $\big((-5) + 2\big) + (-5)$

and $\big((-5) + 2\big) + (-5) = -8$

the right side is: $(-3) + (-5)$

and $(-3) + (-5) = -8$

Hence, the truth set is $\{-5\}$.

Section 7-6

1. (a) 6 (d) $\frac{1}{5}$

   (b) 3 (e) 15

   (c) 0 (f) 30

2. All the sentences are <u>true</u> except for 2(c).

3. The left numeral is

$$\frac{2}{3} + \left(7 + \left(-\frac{2}{3}\right)\right) = \frac{2}{3} + \left(\left(-\frac{2}{3}\right) + 7\right)$$ Commutative property of addition

$$= \left(\frac{2}{3} + \left(-\frac{2}{3}\right)\right) + 7$$ Associative property of addition

$$= 0 + 7$$ Addition property of opposites

$$= 7$$ Addition property of 0

The right numeral is also 7; hence, the sentence is true.

4. (a) $\{32\}$ (b) $\{4\}$ (c) $\{-1\}$ (d) $\emptyset$

5. (a) Set of all numbers except 0 (b) $\{0\}$ (c) $\emptyset$

6. (a) Either both are negative; or one is negative and the other is either positive or 0, and the negative number has the greater absolute value.

   (b) One is the opposite of the other.

   (c) Either both are positive; or one is positive and the other is either negative or 0, and the positive number has the greater absolute value.

7. If x is the number of dollars in the week's sales, $80 + .03x = 116$.

8. (a) If x is the length of the fourth side, $x > 0$ and $x < 23$, or $0 < x < 23$.

   (b)

9. (a) $2ab + ac$ (e) $xy(x + 1)$

   (b) $2ab + 2a^2$ (f) $2ab(3a + b)$

   (c) $3(a + b)$ (g) $a^2bc + 3ab^2$

   (d) $5x(1 + 2a)$ (h) $3a^2 + 6ab + 2ac$

10. (a) Yes, the set is closed under the operation of "opposite".

    (b) Yes, the set is closed under the operation of "absolute value".

    (c) Yes, if a set is closed under "opposite" it is closed under "absolute value", since either the number or its opposite is the absolute value of the number.

11. (a) Yes, the set is closed under "absolute value".

    (b) No, the set is not closed under the operation of "opposite".

    (c) No, if a set is closed under "absolute value", it is not necessarily closed under "opposite", since even if the absolute value of a number is in the set, the opposite of the number may not be.

Section 8-1

38. $(-\frac{1}{2})(2)(-5)$ [illegible] $(-1)(-5) = 5$

39. $(-3)(-4)+(-3)(7)$ [illegible] $12+(-21) = -9$

40. $(-3)((-4)+7)$ [illegible] $(-3)(3) = -9$

41. $(-3)(-4)+7$ [illegible] $12+7$ [illegible]

42. $|-3|(-4)+7 = 3(-4)+7 = -5$

43. $|3||-2|+(-6)$ [illegible] $(3)(2)+(-6) = 0$

44. $(-3)|-2|+(-6)$ [illegible] $(-3)(2)+(-6) = -12$

45. $(-3)(|-2|+(-6))$ [illegible] $(-3)(-4) = 12$

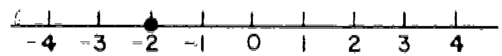
103. $x + (-\frac{2}{3})(-3)$ [illegible]

$x + 2$ [illegible]

$x = 0$

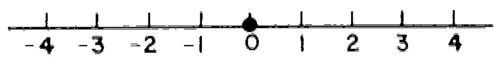
Check: $0 + (-\frac{2}{3})(-3)$ [illegible]. Hence, the truth set is $\{0\}$.

104. [illegible]



Check: [illegible]. Hence, the truth set is [illegible].

105. [illegible]



Check: [illegible]. Hence, the truth set is [illegible].

106. [illegible]

Check: [illegible]. Hence, the truth set is [illegible].

107. [illegible]

Check: [illegible]

[illegible]. Hence, the truth set is [illegible].

Section 8-2

*Theorem. For all real numbers a and b, $ab = ba$.

Proof. By Theorem 8-1, for all real numbers a and b,

$$|ab| = |a||b| \quad \text{and} \quad |ba| = |b||a|.$$

$|a||b|$ and $|b||a|$ are equal because the commutative property of multiplication holds for non-negative numbers. Consequently, ab and ba have the same absolute value. Moreover, ab and ba are both positive if a and b are both positive or both negative.

Also, if one of the numbers a and b is positive and the other negative, then both ab and ba are negative by the definition of multiplication of real numbers.

And if one or both of the numbers a and b are zero, ab and ba are both 0.

In short, if a and b are any real numbers, ab and ba have the same absolute value and they are either both positive, both negative, or both 0. Hence, $ab = ba$.

Section 8-3

Proof.	
$(-a)(-b) = \big((-1)a\big)\big((-1)b\big)$	Theorem 8-3.
$= \big((-1)(-1)\big)(ab)$	Associative and commutative property of multiplication
$= 1(ab)$	Definition of multiplication of two negative numbers
$= ab$	Multiplication property of 1

Section 8-4

107. $3y^2 + (-14y) + 8$

108. $m^2 + (-6m) + 9$

109. $ab + (-bc) + ad + (-cd)$

110. $a^2 + 2ab + b^2$

111. $3a^2 + a + (-2)$

112. $4x^2 + (-23x) + 15$

113. $c^2 + 2c + (-15)$, or $(-15) + 2c + c^2$

114. $2y^2 + (-7y) + 6$ or $6 + (-7y) + 2y^2$

115. $8 + (-13n) + 5n^2$

116. $x^3 + 5x^2 + 7x + 3$

117. $6p^2q^2 + (-10pq) + (-56)$

118. $15y^2 + (-11xy) + 2x^2$

119. $-6x^3y$

120. a^2b^3

Section 8-5

1. (a) $3a^2 + (-6a)$
   (b) $x^2 + 7x + 6$
   (c) $a^2 + (-b^2)$
   (d) $m^2 + (-10m) + 25$
   (e) $2x^2 + (-5x) + (-12)$

2. (a) $2a(x + y)$
   (b) $c(a + (-b) + 1)$
   (c) $(a + b)(cx + y)$
   (d) $5\left(2x^2 + (-3x) + (-1)\right)$

3. (a) $4a + 3b$
   (b) $4x + (-2b)$
   (c) $(-6a) + 4.6$
   (d) $|x| + |-x|$, or $2|x|$

4. The dealer made \$4 on the transactions,

$$15(3) + 20(-2) + (-5)(3) + (-7)(-2) = 45 + (-40) + (-15) + 14 = 4.$$

Notice that by the commutative and associative properties of addition and by the distributive property,

$$\begin{aligned} &15(3) + 20(-2) + (-5)(3) + (-7)(-2) \\ &\quad = \left(15 + (-5)\right)(3) + \left(20 + (-7)\right)(-2) \\ &\quad = 10(3) + 13(-2) = 4. \end{aligned}$$

5. \$150.

$$\begin{aligned} &500(\tfrac{3}{4}) + 300(-\tfrac{3}{8}) + 125(\tfrac{3}{8}) + 500(\tfrac{1}{4}) + 400(-\tfrac{7}{8}) + 175(\tfrac{3}{8}) \\ &\quad = 500(\tfrac{3}{4} + \tfrac{1}{4}) + 300(-\tfrac{3}{8}) + (125 + 175)(\tfrac{3}{8}) + 400(-\tfrac{7}{8}) \\ &\quad = 500(1) + 300(-\tfrac{3}{8}) + 300(\tfrac{3}{8}) + 400(-\tfrac{7}{8}) \\ &\quad = 500 + 300\left((-\tfrac{3}{8}) + \tfrac{3}{8}\right) + (-350) \\ &\quad = 150. \end{aligned}$$

6. If the rate of pay for one hour's work is x dollars, then $\frac{3}{2}x$ is the number of dollars for one hour's work at the overtime rate. Since he worked 48 hours, he was paid at the normal rate for 40 hours and at overtime rate for 8 hours.

$$40x + 8(\tfrac{3}{2}x) = 166.40$$

or

$$52x = 166.40.$$

Section 9-2

103. $\{1,2,3\}$
104. $\{-\frac{1}{2}, 2\}$
105. $\{0,-4\}$
106. $\{0,1\}$
107. $\{6\}$
108. $\{2\}$

109. Theorem: If a, b, c are real numbers, and if $ac = bc$ and $c \neq 0$, then $a = b$.

Proof: $ac = bc$

$ac(\frac{1}{c}) = bc(\frac{1}{c})$; Multiplication property of equality

$a(c \cdot \frac{1}{c}) = b(c \cdot \frac{1}{c})$ Associative property

$a \cdot 1 = b \cdot 1$

$a = b$

Section [illegible].

32. $(-[illegible]a) + (-[illegible]) + 2a [illegible] = (-a)$

$(-[illegible]a) + (-[illegible]) [illegible] + (-a)$

$-3a \quad 7$

$a = -\frac{7}{[illegible]}$ The solution set is $\{-\frac{7}{3}\}$.

[illegible] $0.[illegible] + 1.[illegible]x + (-[illegible]) [illegible]x + [illegible]$

$1.[illegible]x + (-1) [illegible]x + [illegible]$

$-[illegible] \; x$ The solution set is $\{-3\}$.

[illegible] [illegible] The solution set is $\{0\}$.

[illegible] (-10) [illegible] The solution set is $\emptyset$.

[illegible] $(x + [illegible])(x + [illegible]) \quad x(x + [illegible]) + [illegible]$

[illegible] The solution set is $\{1\}$.

[illegible] Let x represent the smaller number.

Then [illegible] represent the larger number.

[illegible]

[illegible] The number is [illegible].

[illegible] This is equal [illegible]

[illegible] $x - 141$ [illegible]

[illegible] The number is [illegible].

132. Let r represent the rate of each man.
Then [illegible]r is the number of miles the first man went.
[illegible]r + 120 is the total length of the course.
3r is the number of miles the second man went.
3r + 2[illegible]0 is another expression for the total length of the course.

[illegible]r + 120 = [illegible]r + 2[illegible]0

[illegible]

[illegible] The truth set is [illegible]. Each man was traveling [illegible] miles per hour.

Check: [illegible]([illegible]) + 120 = [illegible]

3([illegible]) + [illegible] = [illegible]

13[illegible]. Let x be the number.
x + [illegible] is the [illegible] of the number.

The open sentence is

[illegible]x + [illegible](x + [illegible]) + [illegible]0

[illegible]

The truth set is [illegible]. The number is [illegible].

Check: If the number is [illegible], [illegible]

[illegible] + 10

[illegible]

13[illegible]. If x is the number of [illegible], then [illegible] is the number of [illegible] and [illegible] is the number of [illegible].

[illegible]

[illegible]

[illegible]

[illegible] The open sentence is

[illegible]

The truth set is [illegible].
The number of [illegible] is [illegible].
The number of quarters is [illegible].
The number of [illegible] is [illegible].

Check: [illegible]

Section 9-4

1. (a) $\{2\}$ (d) $\{0\}$
   (b) $\{7\}$ (e) $\{-1\}$
   (c) set of all real numbers

2. (a) $\{-6,-1,0\}$
   (b) $\{-\frac{3}{2}, 4\}$
   (c) $\{\frac{5}{3}, \frac{1}{2}\}$

3. No. $\frac{1}{a} + \frac{1}{b} \neq \frac{1}{a+b}$ for non-zero real numbers a and b. For example, if $a = 2$ and $b = 3$,

$$\frac{1}{2} + \frac{1}{3} = \frac{5}{6}$$
$$\frac{1}{2+3} = \frac{1}{5}$$

Remember that if a statement is false in a single instance, it is <u>not</u> a true statement for all numbers.

4. (a) $b = -3$
   (b) $b = 0$, $b = 1$
   (c) $b^2 + 1$ is never equal to 0 since $b^2 > 0$ for all non-zero values of b. Thus, $b^2 + 1$ always has a reciprocal.

5. (a) If x is the number of dollars I pay,
   then $x + 2$ is the number of dollars Jim pays.
   $x + (x + 2) = 11$
   The truth set is $\{4\frac{1}{2}\}$. Jim pays \$6.50.

   (b) If x is the first odd integer,
   then $x + 2$ is the next odd integer.
   $x + (x + 2) = 40$
   The truth set is $\{19\}$.
   The integers are 19 and 21.

   (c) If w is the width of the rectangle in yards,
   then $w + 27$ is the length of the rectangle in yards.
   $2w + 2(w + 27) = 398$
   The truth set is $\{86\}$. The rectangle is 86 yards wide and 113 yards long.

   (d) If x is Jim's grade,
   then $x + 14$ is Mary's grade.
   $x + (x + 14) = 170$
   The truth set is $\{78\}$. Jim's grade was 78, and Mary's grade was 92.

5. (continued)

(e) If x is the father's wage in dollars per day,
then $\frac{2}{5}x$ is the son's wage in dollars per day.

$4x + 2(\frac{2}{5}x) = 96.$

The truth set is $\{20\}$. The father earned \$20 per day and the son earned \$8 per day.

(f) If x is the number of pigs,
then $x + 16$ is the number of chickens.

$4x + 2(x + 16) = 74$

The truth set is $\{7\}$. There were 7 pigs and 23 chickens.

(g) If x is the number of hits,
then $x + 10$ is the number of misses.

$10x + (-5)(x + 10) = -25$

The truth set is $\{5\}$. He made 5 hits.

Section 10-1

65.
$$x + 5 < -1$$
$$(x + 5) + (-5) < -1 + (-5)$$
$$x < -6$$

Truth set: the set of all numbers less than -6.

66. Truth set: set of numbers less than -1.

67. Truth set: set of numbers less than 1.2.

68. Truth set: set of numbers greater than 5.

69.
$$3x < 5$$
$$3x + (-2x) < (5 + 2x) + (-2x)$$
$$x < 5$$

Truth set: set of all numbers less than 5.

70.
$$(-\tfrac{2}{3}) + 2x < \tfrac{5}{3} + x$$
$$\left((-\tfrac{2}{3}) + 2x\right) + (-x) < (\tfrac{5}{3} + x) + (-x)$$
$$(-\tfrac{2}{3}) + x < \tfrac{5}{3}$$
$$\tfrac{2}{3} + \left((-\tfrac{2}{3}) + x\right) < \tfrac{2}{3} + \tfrac{5}{3}$$
$$x < \tfrac{7}{3}$$

Since each of the steps is reversible, we conclude that the truth set of the original sentence is the set of all numbers less than $\frac{7}{3}$. Can you explain which properties you used in each step?

71. Truth set: the set of all numbers less than 1.

72. Truth set: the set of all numbers less than 2.

73. Truth set: the set of all numbers greater than -4.

74. Truth set: the set of all numbers greater than 4.

Section 10-2

*22.

Theorem 10-2b. If x and z are two real numbers such that $x < z$, then there is a positive real number y such that $x + y = z$.

Proof: Let x and z be two real numbers such that $x < z$, as is assumed in the statement of the theorem. We then must show that there is a real number y, which satisfies two conditions, namely:

(1) $x + y = z$

and (2) y is positive.

We choose $y = z + (-x)$ in order to lead us toward our goal; then y is a real number by the closure property of addition, and

(1) $x + y = x + z + (-x)$ by the way we have chosen y,

$= x + (-x) + z$ by the commutative property of addition,

$= \big(x + (-x)\big) + z$ by the associative property of addition

$= 0 + z$ by the addition property of opposites

$= z$ by the addition property of zero; and indeed, the first condition is satisfied.

(2) Further, of course, $x < z$ is assumed, thus

$x + (-x) < z + (-x)$ by the addition property of order

or $0 < z + (-x)$ by the addition property of opposites.

Recall, however, that $y = z + (-x)$; thus, we have $0 < y$, or in other words, y is positive.

*64. Since $0 < a < b$, we know that both a and b are positive.

From $a < b$, we have $a^2 < ab$ (multiplying by a)

and also $ab < b^2$ (multiplying by b)

Since $a^2 < ab$ and $ab < b^2$, we may conclude that $a^2 < b^2$ using the transitive property of order.

*67. Given $a \neq 0$, $b \neq 0$, and $a < b$.

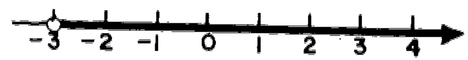
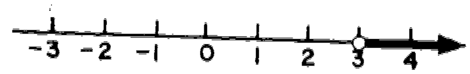
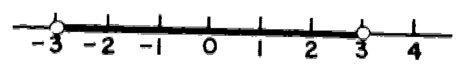
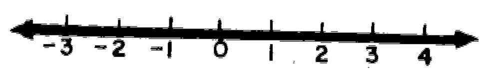
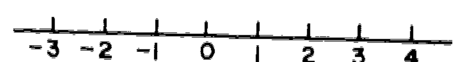
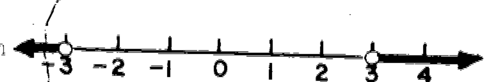
Case 1. If a and b are either both positive or both negative, then $\frac{1}{a}$ and $\frac{1}{b}$ are both positive or both negative. In either event, $\frac{1}{a} \cdot \frac{1}{b}$ is positive. Since $a < b$, we have $a(\frac{1}{a} \cdot \frac{1}{b}) < b(\frac{1}{a} \cdot \frac{1}{b})$ and the conclusion is $\frac{1}{b} < \frac{1}{a}$.

*67. (continued)

Case 2. If a is negative, b positive, then $\frac{1}{a}\cdot\frac{1}{b}$ is negative. Since $a < b$, we have $b(\frac{1}{a}\cdot\frac{1}{b}) < a(\frac{1}{a}\cdot\frac{1}{b})$ and the conclusion is $\frac{1}{a} < \frac{1}{b}$.

Note: Is it possible that a is positive, b negative? Why not?

Section 10-3

	Sentence	Truth Set (See notes below)	Graph
14.	$y < 3$	Set of numbers less than 3	
15.	$-y < 3$	Set of numbers greater than -3	
16.	$y < -3$	Set of numbers less than -3	
17.	$-y < -3$	Set of numbers greater than 3	
18.	$\|y\| < 3$	Set of numbers between -3 and 3	
19.	$\|y\| < -3$	Set of all real numbers	
20.	$\|y\| < -3$	$\emptyset$	
21.	$-\|y\| < -3$	Set of numbers less than -3 or greater than 3	

Notes for Item 15: $-y < 3$ may be written as either $-3 < y$ or as $y > -3$.

Item 17: $-y < -3$ may be written as $y > 3$.

Item 19: $-|y| < 3$ may be written as $|y| > -3$. Since $|y| \geq 0$ for all real numbers y, surely $|y| > -3$ is true for all real numbers y.

Item 20: $|y|$ cannot be negative.

Item 21: $-|y| < -3$ may be written $|y| > 3$. Compare the graph of $|y| < 3$ in Item 18.

22. The set of all numbers greater than -1.

23. The set of all numbers less than -2.

24. The set of all numbers less than -5.

25. The set of all numbers greater than $\frac{4}{3}$.

26. The set of all numbers less than or equal to -1.

Section 10-4

1. (a) $-100 < -99$
   (b) $0.2 > -0.1$
   (c) $|-3| < |-7|$
   (d) $\frac{6}{7} > \frac{5}{8}$
   (e) $3(4) + (-4) > 3(4 + (-4))$
   (f) $x^2 + 1 > 0$

2. (a) true
   (b) false
   (c) true
   (d) false
   *(e) false
   *(f) true

3. (a) not equivalent
   (b) equivalent
   (c) equivalent
   (d) equivalent
   (e) not equivalent
   (f) not equivalent

4. (a) positive
   (b) positive
   (c) negative
   (d) negative
   (e) positive, or 0 if $n = -p$.
   (f) positive

5. (a) The set of all numbers less than (-5)
   (b) The set of all numbers greater than (-1)
   (c) The set of all numbers greater than (-6)
   (d) The set of all numbers less than (-3)
   (e) The set of all numbers less than or equal to 91
   (f) $\emptyset$

6. (a) The set of all numbers greater than 2
   (b) {2}
   (c) The set of all negative real numbers
   (d) The set of all real numbers except zero
   (e) The set of all non-negative numbers less than 90
   (f) $\emptyset$

7. (a) {2}
   (b) {-1}
   (c) {-2}
   (d) $\emptyset$
   (e) The set of integers less than -2
   (f) The set of integers greater than -1

8. (a) $\{\frac{5}{3}\}$
(b) $\{2\}$
(c) $\{\frac{2}{3}\}$
(d) $\{-1\}$
(e) $\{0\}$
(f) $\{-\frac{7}{3}\}$

9. (a) $\{-12\}$
(b) $\{-3\}$
(c) $\{0\}$
(d) $\{0\}$
(e) The set of real numbers
(f) $\{-\frac{5}{6}\}$

10. If A is the number of square units in the area, $24 \leq A < 28$.

11. If A is the number of square units in the area, $24 \leq A < 35$.

*12. If A is the number of square units in the area, $25.5225 \leq A < 26.5625$.

*13. (a) If p is the number of plants at the beginning of the second year, $p > \frac{3}{4}(240)$ and $p < \frac{5}{6}(240)$; that is, $180 < p < 200$.

If n is the number of seeds at the end of the second year, $n > (180)(240)$ and $n < (200)(240)$; that is, $43,200 < n < 48,000$.

(b) If s is the number of seeds at the end of the second year, $s > (180)(230)$ and $s < (200)(250)$; that is, $41,400 < s < 50,000$.

14. (a) If the side of a square is x inches long, then the side of the triangle is $x + 3.5$ inches long, and $4x = 3(x + 3.5)$. The length of the side of the square is 10.5 inches.

(b) If the rate of the current is x miles per hour, then the rate of the boat downstream is $x + 10$ miles per hour and $x + 10 \leq 25$. The rate of the current is equal to or less than 15 miles per hour.

(c) If x is the number of hours spent on the job, then $3 \leq x \leq 5$. Mary can expect to spend from 3 to 5 hours on the job.

(d) If x is the number of hours Jim must work, $1.75x \geq 75$. Jim must work at least 43 hours, to the nearest hour.

Section 11-1

24. 23
25. -9
26. -5
27. 3
28. -2
29. 16
30. 0
31. -3000
32. $\frac{5}{4}$
33. $\frac{3}{2}$
34. -1.262
35. -0.57
36. 160

Section 11-2

$$(x - 3) - (x - 2) = \big(x + (-3)\big) + \big(-(x + (-2))\big), \quad \text{definition of subtraction}$$
$$= \big(x + (-3)\big) + \big((-x) + 2\big), \quad \text{opposite of sum equals sum of opposites}$$
$$= x + (-3) + (-x) + 2, \quad \text{associative property of addition}$$
$$= x + (-x) + (-3) + 2, \quad \text{commutative property of addition}$$
$$= 0 + (-1), \quad \text{addition property of opposites}$$
$$= -1, \quad \text{addition property of } 0.$$

88. If the length of the rectangle is y inches, the width is $y - 5$ inches. $y + (y - 5)$ is an open phrase for one-half of its perimeter. If the perimeter is 38 inches, an open sentence for the problem is $y + (y-5) = 19$.

89. If the number is n, an open sentence for the problem is $6(n-17) = 102$.

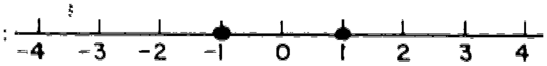
90. If x is the number of students in the class now, an open sentence for the problem is $3x < x + 40$.

Section 11-3

74. Truth set: $\{4\}$ Graph:

75. $|y| = 1$

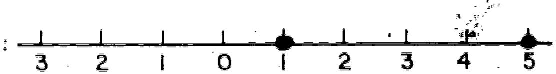
Truth set: $\{-1, 1\}$ Graph:



76. $|z| = -6$

Truth set: $\emptyset$

77. Truth set: $\{1, 5\}$ Graph:



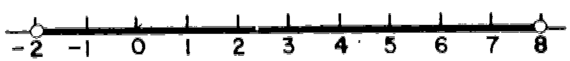
78. $|x + 4| = 3$

$|x - (-4)| = 3$ Truth set: $\{-7, -1\}$

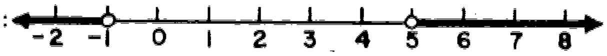
Graph:

79. Truth set: The set of all real numbers such that $-2 < x < 8$.

Graph:

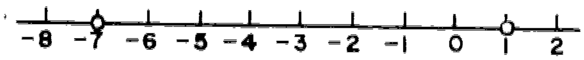


80. Truth set: The set of all real numbers less than -1 or greater than 5.

Graph: 

81. $|x + 3| < 4$

$|x - (-3)| < 4$ Truth set: The set of all real numbers such that $-7 < x < 1$.

Graph: 

Section 11-[illegible]

13. [illegible]
1[illegible]. [illegible]
20. [illegible].00
21. [illegible]
22. [illegible]
2[illegible]. [illegible]
24. [illegible]
2[illegible]. $\frac{20}{[illegible]}$
2[illegible]. $\frac{[illegible]}{1[illegible]}$
27. [illegible]
2[illegible]. 0

[illegible]0. If the number is n, an open sentence is [illegible] = [illegible] = [illegible]7.

[illegible]1. If the number is y, an open sentence is [illegible] + [illegible]y [illegible]3.

[illegible]2. If x is the smaller even integer, then $x + 2$ is the next larger even integer. An open sentence is $x + (x + [illegible])$ [illegible].

[illegible]3. If n is the smaller odd integer, then $n + 2$ is the next larger odd integer. An open sentence is $n + (n + [illegible]) \leq [illegible]$.

[illegible]4. If p is the price of the chair before the sale, then $.20p$ is the amount of the discount. An open sentence is $p - .20p$ [illegible]0 or $.80p$ [illegible] [illegible]0.

[illegible]. If x is the number of five-cent stamps, then an open sentence which says she was charged $1.[illegible], or 1[illegible] cents, is [illegible]([illegible]) [illegible] x [illegible] 13[illegible].

The truth set is [illegible], which is not an integer; hence, she was not charged the correct amount. Since x is the number of five-cent stamps, the domain of x is clearly the set of non-negative integers.

[illegible]. If m is the number of gallons of maple syrup, then [illegible]0 $- m$ is the number of gallons of corn syrup. The maple syrup, at $[illegible] a quart, was worth $[illegible] a gallon, while the corn syrup was worth $[illegible].[illegible]0 a gallon. Since the worth of the whole mixture was $[illegible], an open sentence for the problem is [illegible] + [illegible]([illegible]0 $- m$) [illegible] [illegible]0[illegible].

Section 11-5

1. (a), (b), (d), (f)

2. (a) $(-1)^2 - 4(2)(5) = -39$

 (b) $(-3)^2 - 4(1)(-2) = 17$

 (c) $(6)^2 - 4(-9)(-1) = 0$

3. (a) $\{-4\}$

 (b) Set of real numbers less than 7

 (c) If $3|x| \leq 6$ is true for some x, then $|x| \leq 2$ is true for the same x.

 Truth set: Set of real numbers consisting of -2, 2, and the real numbers between -2 and 2.

 (d) $\{0, 2\}$

 (e) If $|x + 2| < 3$ is true for some x, then $|x-(-2)| < 3$ is true for the same x.

 Truth set: set of real numbers between -5 and 1.

4. (a) $5x - 15$ (c) $-\frac{1}{4} - \frac{1}{2}x$

 (b) $-2b + 2$ (d) $ab - ac$

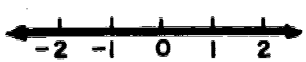
5. (a) $7(1 + 2y)$ (c) $-7(1 - 2y)$

 (b) $7(1 - 2y)$ (d) $-7(1 + 2y)$

6. (a) $2x$ (c) $5x - 8$

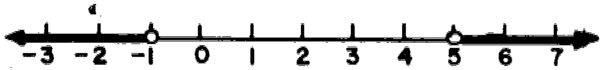
 (b) $-x + 2$ (d) $-x - 5$ or $-(x + 5)$

7. (a) No graph. The truth set is $\emptyset$ since no matter what number x represents, $|x - 2|$ is non-negative.

 (b) 

 The truth set is the set of all real numbers.

 (c) 

 (d) 

8. If w represents the average wind speed, then $200 - w$ represents the average airplane speed. An open sentence is

$$\frac{7}{2}(200 - w) = 630$$
$$7(200 - w) = 1260$$
$$200 - w = 180$$
$$-w = -20$$
$$w = 20$$

The average wind speed is 20 mph.

9. If the first shirt cost x dollars, then $x - .25x = 3.75$

$$.75x = 3.75$$
$$x = 5.00$$

Hence, the first shirt cost \$5.00. Since he sold it for \$3.75, he lost \$1.25 on it.

If the second shirt cost y dollars, then $y + .25y = 3.75$

$$1.25y = 3.75$$
$$y = 3.00$$

Hence, the second shirt cost \$3.00. Since he sold it for \$3.75, he gained \$.75 on it.

$(-1.25) + .75 = -.50$. Thus, he lost \$.50 on the two sales.

10. $|t - 70| < 15$, where t is the temperature in degrees.

Truth set: The set of real numbers between 55 and 85.

SYMBOLS

{ } braces--indicating a set

$\emptyset$ the empty set, the null set

$=$ equals, names the same number as

$\neq$ does not equal, is different from

$>$ is greater than

$<$ is less than

$\geq$ is greater than or equal to

$\leq$ is less than or equal to

$\ngtr$ is not greater than

$\nless$ is not less than

$\cup$ union

$\cap$ intersection

... and so forth

( ) parentheses

| | absolute value

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