ED 160 455 SE 025 104

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TITLE Geometry with Coordinates, Teacher's Commentary, Part II, Unit 50. Revised Edition.
INSTITUTION Stanford Univ., Calif. School Mathematics Study Group.
SPONS AGENCY National Science Foundation, Washington, D.C.
PUB DATE 65
NOTE 314p.; For related documents, see SE 025 101-103

EDRS PRICE MF-$0.83 HC-$16.73 Plus Postage.
DESCRIPTORS *Analytic Geometry; Curriculum; *Geometry; *Instruction; Mathematics Education; Secondary Education; *Secondary School Mathematics; *Teaching Guides
IDENTIFIERS *School Mathematics Study Group

ABSTRACT

This is part two of a two-part manual for teachers using SMSG high school text materials. The commentary is organized into four parts. The first part contains an introduction and a short section on estimates of class time needed to cover each chapter. The second or main part consists of a chapter-by-chapter commentary on the text. The third part is a collection of essays on topics that cannot conveniently be dealt with in the main part of the commentary in connection with a particular passage. The fourth part contains answers to Illustrative Test Items and the solutions to the problems. Chapter topics include: coordinates in a plane; perpendicularity, parallelism, and coordinates in space; directed segments and vectors; polygons and polyhedrons; and circles and spheres. (MN)

TEACHER'S COMMENTARY UNIT NO. 50

GEOMETRY WITH COORDINATES

PART II

SCHOOL MATHEMATICS STUDY GROUP

Geometry with Coordinates

Teacher's Commentary, Part II

REVISED EDITION

Prepared under the supervision of a
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Stanford, California
Distributed for the School Mathematics Study Group
by A. C. Vroman, Inc., 367 Pasadena Avenue, Pasadena, California

Financial support for School Mathematics
Study Group has been provided by the
National Science Foundation.

Printed in the United States of America.

Contents

Chapter 8

COORDINATES IN A PLANE

In this chapter we develop coordinates as a tool for studying geometry in a plane. This development includes a sequence of basic theorems, the distance formula, midpoint formula, parametric equations for a line, the slope concept, perpendicularity and parallelism conditions, and the use of coordinates in proving several theorems about triangles and quadrilaterals.

We do not speak of synthetic geometry (or methods or proofs) versus coordinate geometry (or methods or proofs) in this course. We hope that the students do not get the idea that synthetic geometry and coordinate geometry are two different kinds of geometry but see, instead that they are ways of studying the same formal geometry. In this course we repeatedly recognize two distinct brands of geometry: (1) the geometry of physical space developed through intuition, observation, measurement, and inductive reasoning, and (2) formal geometry developed as a mathematical system which is characterized by a list of undefined terms, definitions, postulates, and theorems, and deductive reasoning. Of course, coordinate methods are used in both kinds of geometry. The major objective of the chapter is to make the student see that coordinates are a useful tool in formal geometry.

We prefer not to think of this chapter as an introduction to analytic geometry. The traditional analytic geometry course includes various standard forms of equations for lines and conic sections. It emphasizes the plotting of graphs and the finding of equations of curves from information about their graphs. It places little emphasis upon the use of coordinates in the formal development of the elementary geometry of lines, triangles, and quadrilaterals. Why should it? The students have already acquired this background before they enter the analytic geometry course. However in this chapter the use of coordinates in the formal development of elementary geometry is emphasized. This emphasis is made rather indirectly. Students see coordinates used in the proofs of several theorems which are new and (we hope) interesting to them. We try to impress the student with the idea that sometimes coordinates should be used because they make a proof easier and that sometimes they should not be used since a proof without them may be easier. We do not give any general rules as to when it is advisable to use coordinate methods. We do not give any because we do not have any. Our message is that the process of "finding a proof" should include a consideration of the possible use of coordinates.

Your geometry students have a background which provides them with a strong sense of relationships among numbers. This course takes advantage of that background and strengthens it. A coordinate system on a line is an idea which evolves easily from the notion of a number line; a simple extension yields coordinates in a plane. The students' concepts of a line and of a plane are enhanced by the introduction and use of coordinate systems.

A review of coordinate systems on a line is important for the work of this chapter. Your students should have no difficulty in seeing that the x-coordinate system and the y-coordinate system are examples of such systems. Later, in the development of parametric equations for a line, a clear understanding of the relationship between a point on a line and the value of k associated with the point depends upon a clear understanding of the concept of a coordinate system on a line.

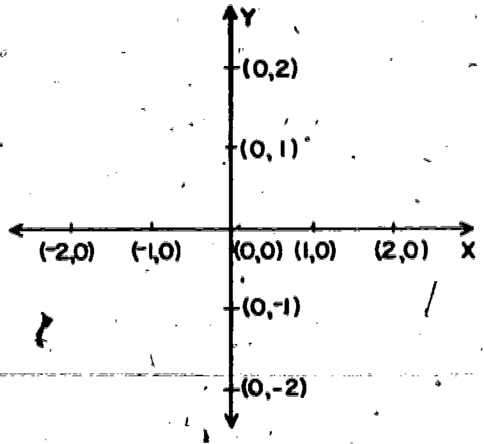
The introductory remarks in the first paragraph are not meant to be a definition. Not all one-to-one correspondences which have the properties given here are coordinate systems on a line. The definition is in Chapter 3.

The purpose of Problem Set 8-1 is two-fold: to review the notion of a coordinate system on a line and to motivate the desirability of having ordered pairs (or triples) of numbers as coordinates.

The unit-pair of points need not, of course, lie in the given plane. In the last part of this chapter we encourage the students to set up an xy-coordinate system which will fit the problem. This means that there is considerable freedom in locating the x-axis and the y-axis. But for our work there seems to be no advantage in changing the unit of distance, and so we think of it as fixed throughout.

Some students may feel that our definitions of horizontal and vertical violate the usual meanings of these words. It is convenient to have two words which have the precise technical meanings which we have given to the words horizontal and vertical. And it seems advisable to use familiar words whose non-technical meanings have a relationship to the technical meanings we want.

Although we usually label points in $\overleftrightarrow{OX}$ with their x-coordinates and points in $\overleftrightarrow{OY}$ with their y-coordinates, it should be made clear that every point in $\overleftrightarrow{OX}$ has an x-coordinate in the x-coordinate system, and an x-coordinate in the xy-coordinate system. Of course these coordinates are the same. A similar statement applies to every point in $\overleftrightarrow{OY}$. When we speak of the coordinates of a point, we are referring to its x-coordinate and its y-coordinate in the xy-coordinate system. Some may prefer to label points on the x-axis and the y-axis with their x and y-coordinates written as ordered pairs.



The statement that there is exactly one vertical line through P follows from the following considerations. The x-axis is defined to be the line in a given plane which is perpendicular to the y-axis at point O . Its existence and uniqueness follows from Theorem 4-21. In the general case, Theorems 4-21 and 5-11 assure us of the existence and uniqueness of lines through P perpendicular to the axes.

Our definitions of the x-coordinate and the y-coordinate of a point P are stated in terms of lines through P which are perpendicular to the y-axis and the x-axis, respectively. We could, of course, have worded these definitions in terms of the lines through P which are parallel to the x- and the y-axis. For the purposes of this course neither wording has any obvious advantage over the other. In other courses where "oblique coordinates" are introduced, the definition of the coordinates is most naturally given in terms of parallel lines.

In this course the concept of our ordered pair is not defined. Most of the students have been introduced to ordered pairs of numbers in their study of elementary algebra. Just as the notion of a <u>set</u> is taken as a basic term which we do not define, so ordered pair is taken as an undefined phrase. In this course parentheses, as in (5,8), are used for "ordered pairs" of elements, while braces, as in {5,8}, are used for sets of elements. The order of the names within the braces is immaterial. We should like to emphasize here that ordered pairs of numbers need not involve two distinct numbers. Thus (5,5) is an ordered pair of numbers.

Some student might ask whether {5,5} is an acceptable symbol for some set of numbers. The answer is yes. Indeed, {5,5} and {5} are names for the same set. According to the definition of equality in set theory, A = B means that every element of A is an element of B, and that every element of B is an element of A. So if A = {5,5} and B = {5}, then A = B, for the only number in A is 5, and it is also in B. And the only number in B is 5, and it is in A.

Ordered pair is not taken as undefined in some courses. Indeed, the ordered pair, (a,b), may be defined as the set {a, {a,b} } in which the order of the members of the set may be altered without actually changing the set itself. We could just as well define it as {a, {b,a} }, or as {{a,b}, a}, or as {{b,a}, a}, since these are all names for the same set. Let us see how this definition applies in an example. Suppose A = (5,8), B = (8,5). Then by definition A = {5, {5,8}}, B = {8, {8,5}}. Since 5 is an element of A, but not of B, it follows that A ≠ B. In other words (5,8) and (8,5) are different ordered pairs. (Note that A and B, considered as sets, do have a common element, since {5,8} = {8,5}.)

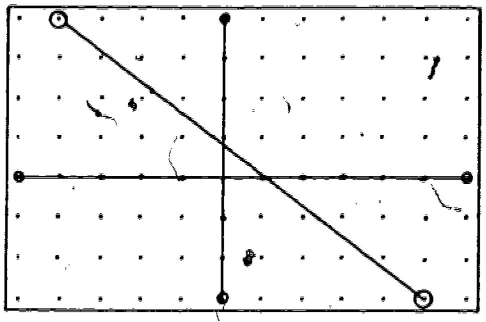
For the purpose of this course, however, it is best to use the ordered pair symbol (a,b) without any reference to the definition in terms of sets. It is important to understand clearly the concept of equality for ordered pairs. Thus, (a,b) = (c,d) if and only if

$a = c$ and $b = d$. We could prove this as a theorem if we defined ordered pair in terms of sets. Since we do not define ordered pair, we accept it as a definition of equality for ordered number pairs. According to this definition $(3,5) = (3,5)$, $(2 + 1, 5) = (3,5)$, and $(x,y) = (3,5)$ if and only if $x = 3$ and $y = 5$. Thus, $(a,b) = (c,d)$ if and only if a and c are names for the same number, and b and d are names for the same number.

With regard to using ordered pairs as names for points, it should be clear that any point could have any ordered pair of real numbers for its name--provided the xy-coordinate system is properly set up. Thus, $(0,0)$ is the name of any point whatsoever--for any point can be chosen as the origin of a coordinate system. But once a coordinate system is set up in a plane we may regard it as a "frame of reference," and every point in the plane has a unique ordered pair of real numbers as its name.

Note our use of "plot" and "graph" in this text. We use plot as a verb and graph as a noun. In drawing a graph, if the set of points is unbounded, it is impossible to "show" all of the points. Just as we use arrows to indicate a line, we may use arrows, jagged edges, or written notes to indicate the infinite extent of a graph.

In the introductory work on plotting some teachers may prefer to use the chalkboard exclusively as a visual aid. Others may prefer to use a "pegboard" with elastic materials of several colors--say white for the axes and yellow for the graph.



The purpose of Problem Set 8-2 is to help students learn the concepts of coordinates and graphs. The number of problems assigned from this set will vary greatly according to the background of the class. Some students, for which this part of the course is largely review, should work only a few problems in this set.

In connection with Theorems 8-1 and 8-2, a possible teaching problem might arise in that some students might not see the "point" of these theorems. Our Ruler Postulate tells us that $AB = |y_B - y_A|$ but it does not tell us that $PQ = |y_P - y_Q|$. The fact is that $PQ = |y_P - y_Q|$ depends upon (1) the definition of the y-coordinates of P and Q which implies that $y_P = y_A$ and $y_Q = y_B$, and (2) the property of parallelograms, which gives us $AB = PQ$.

The two examples used to introduce the Distance Formula involve finding the lengths of oblique segments. This might suggest that the Distance Formula is used only in such cases. However the formula can also be used for vertical and horizontal segments.

The properties of the y-coordinate system tell us that $y_C = \frac{1}{2}(y_A + y_B)$. The fact that $y_M = \frac{1}{2}(y_P + y_Q)$ needs proof. In the proof in the text we start with M as the midpoint of $\overline{PQ}$. The fact that $x_M = x_P$ follows immediately from the fact that M and P lie in the same vertical line. The fact that C is the midpoint of $\overline{AB}$ follows from a theorem on parallel lines cut by transversals, Theorem 7-2.

Theorem 8-9 is, of course, a locus theorem. Although the word "locus" is not used in the text it is mentioned in passing in the text in Section 8-6 after the students have had some experience with the concept. To prove the theorem we first identify a particular vertical line, and call it m . We then show that m is the set of all points in the xy-plane each of which has x-coordinate a .

We show that every point in m has x-coordinate a (Part 1) and that every point which has x-coordinate a is in m (Part 2). In other words, every point in m has the desired property, and every point which has the desired property is in m . In the last part of the proof, we show that it is impossible to have two vertical lines containing A in view of the Parallel Postulate.

Symbols are used in mathematics because they facilitate communication. The statement $x \geq 3$ is a statement which is true if x is 38 . In elementary algebra we may want to find out how old Mary is. The available information may include the fact that she is at least 3 years old. If we let x denote her age, then we write $x \geq 3$ and, using other available information, we may eventually learn Mary's age. In this situation, x in the statement, $x \geq 3$, stands for one number. In another situation we may wish to consider the set of all real numbers which are greater than or equal to 3 . We use the symbol $\{x: x \geq 3\}$ to denote this set. It is a good symbol in the sense that once we understand how the symbol is formed we know what it means without being told. What is the graph of the inequality $x \geq 3$? It depends. The set-builder symbol makes it definite. The graph of $\{x: x \geq 3\}$ is a ray. The graph of $\{(x,y): x \geq 3\}$ is the union of a halfplane and its edge. The graph of $\{(x,y,z): x \geq 3\}$ is the union of a plane and all the points which lie on the same side of it as does the point (4,0,0) .

In connection with the subject of equivalent equations some teachers may desire to relate <u>implication</u> with <u>set inclusion</u>, and "reversible steps" with set equality as in the following examples. We state first an implication in three different ways.

In words:

If $2x + 3 = 4x + 13$, then $3 = 2x + 13$.

Using "$A \longrightarrow B$" to mean "A implies B" :

$$[2x + 3 = 4x + 13] \longrightarrow [3 = 2x + 13] .$$

Using "$A \subset B$" to mean "A is a subset of B" :

$$\{(x,y): 2x + 3 = 4x + 13\} \subset \{(x,y): 3 = 2x + 13\} .$$

The "reverse step" is stated three different ways as:

If $3 = 2x + 13$, then $2x + 3 = 4x + 13$.

$$[3 = 2x + 13] \longrightarrow [2x + 3 = 4x + 13] .$$

$$\{(x,y): 3 = 2x + 13\} \subset \{(x,y): 2x + 3 = 4x + 13\} .$$

Combining the statement and its converse we can write the compound statement in three ways:

$2x + 3 = 4x + 13$ if and only if $3 = 2x + 13$.

$$[2x + 3 = 4x + 13] \longleftrightarrow [3 = 2x + 13] .$$

$$\{(x,y): 2x + 13 = 4x + 13\} = \{(x,y): 3 = 2x + 13\} .$$

Another way of saying that the five equations are equivalent is the following:

$$\{x: 2x + 3 = 4x + 13\} = \{x: 3 = 2x + 13\} .$$

$$\{x: 3 = 2x + 13\} = \{x: -10 = 2x\} ,$$

$$\{x: 2x = -10\} = \{x: x = -5\}$$

or briefly that

$\{x: 2x + 3 = 4x + 13\}$, $\{x: 3 = 2x + 13\}$, $\{x: -10 = 2x\}$ $\{x: 2x = -10\}$, $\{x: x = -5\}$ are five names for the same set of numbers. (Another name for this set is $\{-5\}$) .

If we think of each of these five equations as a condition on (x,y) , then the fact that these five equations are equivalent means that

$\{(x,y): 2x + 3 = 4x + 13\}$, $\{(x,y): 3 = 2x + 13\}$, $\{(x,y): -10 = 2x\}$, $\{(x,y): 2x = 10\}$, $\{(x,y): x = -5\}$

are five names for the same set of points in the xy-plane. Thus two sets are equal if the conditions which define them are equivalent.

In Section 8-7 we develop parametric equations for lines. This is not the traditional approach to the study of lines using coordinate methods. But we believe it is a good approach. The traditional treatment emphasizes early in the course the relationship between lines and linear equations. The student "sees" a line as a single object of thought when he reads, $y = 3x + 4$. The present treatment emphasizes the concept of a line as a set of points. The symbol

$$\{(x,y): x = 4 + 2k , y = 5 + 3k , k \text{ is real}\}$$

is, by virtue of the braces and (x,y) before the colon, first of all, a set of points. And the symbol tells us how to get the x and y coordinates of any point on the line in terms of the number k, which has an interesting geometrical significance. The relationship of x to y is clearly revealed through the "middle man" k. Although the present treatment emphasizes parametric equations for a line, we do include a two-point form and point-slope form later in the chapter.

Just as $4x + 4y = 8$, $3x + 3y = 6$, $5x + 5y = 10$, are three equations for the same line, so a line may be represented by many different parametric equations. Consider, for example, the line

(1) $p = \{(x,y): x = 1 + 4k , y = 2 + 3k , k \text{ is real}\}$.

Setting $k = 0,1$ we get $(x,y) = (1,2)$, $(5,5)$, the two points which yield the equations $x = 1 + 4k$ and $y = 2 + 3k$ if one applies Theorem 8-11, using $(x_1,y_1) = (1,2)$ and $(x_2,y_2) = (5,5)$. Using any two distinct values for k other than 0 and 1, for example, 2 and -1, we get two more points on the line p, and using them in Theorem 8-11 we get another pair of parametric equations for p.

With $k = 2$ we get $(x_1,y_1) = (9,8)$.

With $k = -1$ we get $(x_2,y_2) = (-3,-1)$.

Then p is the line through those two points. Thus,

(2) $p = \{(x,y): x = 9 - 12k , y = 8 - 9k , k \text{ is real}\}$.

In (1) k is the coordinate of the point (x,y) in the coordinate system on a line which is determined by taking the coordinate of $(1,2)$ as 0 and the coordinate of $(5,5)$ as 1 . In (2) k is the coordinate of the point (x,y) in the coordinate system on a line which is determined by taking the coordinate of $(9,8)$ as 0 and the coordinate of $(-3,-1)$ as 1 .

We have motivated Theorem 8-11 in the text by considering a particular line. The following proof for the general case of an oblique line is included here for those who may wish to see it.

THEOREM. If $P_1(x_1,y_1)$ and $P_2(x_2,y_2)$ are any two points on an oblique line, then

$$\overleftrightarrow{P_1P_2} = \{(x,y): x = x_1 + k(x_2 - x_1) , y = y_1 + k(y_2-y_1), k \text{ is real}\} .$$

Proof: On $\overleftrightarrow{P_1P_2}$ there is a coordinate system on a line in which the coordinate of P_1 is 0 and the coordinate of P_2 is 1 ; we call this the c-coordinate system. Let AB denote the measure of the distance between two points A , B in $\overleftrightarrow{P_1P_2}$ relative to the c-coordinate system. Let P be any point in $\overleftrightarrow{P_1P_2}$, let P_1', P_2', P' , be the feet of the perpendiculars from P_1, P_2 , P , respectively, to the x-axis, and P_1'', P_2'', P'' , be the feet of the perpendiculars from P_1 , P_2 , P , respectively, to the y-axis. Since perpendiculars to the x-axis are parallel, it follows, if P is distinct from P_1 and P_2 , that the betweenness relations among P_1, P_2, P are preserved among P_1', P_2', P' . For example, P_2 is between P_1 and P if and only if P_2' is between P_1' and P' . Let k be the c-coordinate of P .

We consider two cases: (1) P is in $\overrightarrow{P_1P_2}$, (2) P is in the ray opposite to $\overrightarrow{P_1P_2}$.

(1) In this case $k \geq 0$, $k = P_1P$ (in C) and $k = \frac{P_1P \text{ (in C)}}{P_1P_2 \text{(in C)}}$ since P_1P_2 (in C) $= 1$. But

$\frac{P_1P \text{ (in C)}}{P_1P_2 \text{(in C)}} = \frac{P_1P}{P_1P_2}$ since the ratio of two distances is independent of the coordinate systems used. (See Postulate 13.) Also, from the theorem regarding the proportionality of segments on two transversals of three parallel lines, it follows that

$$k = \frac{P_1P}{P_1P_2} = \frac{P_1'P'}{P_1'P_2'} = \frac{P_1''P''}{P_1''P_2''} .$$

But $P_1'P_2' = |x_2 - x_1|$, $P_1''P_2'' = |y_2 - y_1|$,

$P_1'P' = |x - x_1|$, $P_1''P'' = |y - y_1|$.

Then $k = \frac{|x - x_1|}{|x_2 - x_1|} = \frac{|y - y_1|}{|y_2 - y_1|}$. Because of the order properties mentioned on the previous page, if $x - x_1 \neq 0$, then $x - x_1$ and $x_2 - x_1$ are both positive (if P_2 lies to the right of P_1) or both negative (if P_2 lies to the left of P_1). Hence, in all cases,

$$k = \frac{x - x_1}{x_2 - x_1} \quad \text{and} \quad x = x_1 + k(x_2 - x_1) .$$

Similarly, we treat the second case.

(2) If P is in the ray opposite to $\overrightarrow{P_1P_2}$, then $k \leq 0$, P_1P (in C) $= -k$, and $-k = \frac{P_1P \text{ (in C)}}{P_1P_2 \text{(in C)}} = \frac{|x - x_1|}{|x_2 - x_1|}$

$= \frac{|y - y_1|}{|y_2 - y_1|}$.

If $x - x_1 \neq 0$, then one of the numbers, $x - x_1$, $x_2 - x_1$, is positive and the other is negative. Hence

$$-k = \frac{-(x - x_1)}{x_2 - x_1} \quad \text{and} \quad x = x_1 + k(x_2 - x_1) .$$

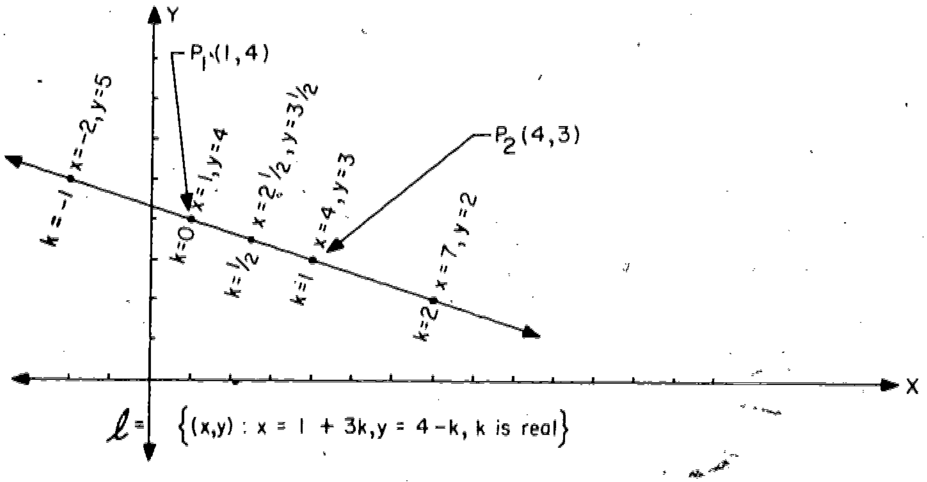
Similarly, $y = y_1 + k(y_2 - y_1)$.

It might be helpful, in the case of an oblique line, $\overleftrightarrow{P_1P_2}$, to think of three <u>coordinate systems on a line</u> as follows.

(1) The coordinate system in which the coordinate of P_1 is 0 and the coordinate of P_2 is 1 . The coordinate k of a point P tells us whether P is in $\overrightarrow{P_1P_2}$ (when $k \geq 0$) or in the opposite ray (when $k \leq 0$) . And the absolute value of that coordinate k is equal to the quotient obtained by dividing the distance between P_1 and P by the distance between P_1 and P_2 .

(2) The coordinate system in which the coordinate of P_1 is x_1 and the coordinate of P_2 is x_2 . The coordinate of an arbitrary point P on $\overleftrightarrow{P_1P_2}$ is x_P, the x-coordinate of P .

(3) The coordinate system in which the coordinate of P_1 is y_1 and the coordinate of P_2 is y_2 . The coordinate of an arbitrary point P on $\overleftrightarrow{P_1P_2}$ is y_P, the y-coordinate of P .



It might be helpful to think of the parametric equations in this theorem in relation to the results of Chapter 3 as follows. Using the notation of the previous proof it follows from Theorem 3-6 that if

$k \geq 0$, then $\frac{P_1'P'}{P_1'P_2'} = k$ if and only if

$x = x_1 + k(x_2 - x_1)$ and $\frac{P_1''P''}{P_1''P_2''} = k$ if and only if

$y = y_1 + k(y_2 - y_1)$. From the present theorem we see

that if $k \geq 0$, then $\frac{P_1P}{P_1P_2} = k$ if and only if

$x = x_1 + k(x_2 - x_1)$ and $y = y_1 + k(y_2 - y_1)$. Similarly,

if $k \leq 0$, then $\frac{P_1'P'}{P_1'P_2'} = -k$ if and only if

$x = x_1 + k(x_2 - x_1)$, $\frac{P_1''P''}{P_1''P_2''} = -k$ if and only if

$y = y_1 + k(y_2 - y_1)$, and $\frac{P_1P}{P_2P_1} = -k$ if and only if

$x = x_1 + k(x_2 - x_1)$ and $y = y_1 + k(y_2 - y_1)$.

In discussing the general case of a line it is important that we understand clearly the variables involved. Thus the symbol

$$\{(x,y): x = a + kb , y = c + kd , k \text{ is real}\} ,$$

involves the seven variables, a, b, c, d, k, x, y . For each set of numbers a, b, c, d, (with b and d not both zero) ,

$$\{(x,y): x = a + kb , y = c + kd , k \text{ is real}\}$$

is a line. It should be clear that

$$\{(x,y): x = a + kb , y = c + kd , \text{with } a, b, c, d, k \text{ real}\}$$

is the set of all points in the xy-plane. For if

(x_1,y_1) is any point there are real numbers a, b, c, d, k such that

$$x_1 = a + kb \quad \text{and} \quad y_1 = c + kd \ .$$

Just take $a = x_1$, $c = y_1$, $b = d = k = 0$.

Suppose next that a, b, c, d, k are real numbers. What is the set

$$\{(x,y): x = a + kb \ , \ y = c + kd\} \ ?$$

It should be clear that this is a set whose only element is the point $(a + kb \ , c + kd)$.

When we think of

$$\{(x,y): x = a + kb \ , \ y = c + kd \ , \ k \text{ is real}\}$$

as a line, we are thinking of a, b, c, d (b and d not both zero) as "fixed." That is the reason we say a, b, c, d are real numbers before we write the set-builder symbol. Also we are thinking of k as "taking on" real values. Each value "taken on" by k yields a point (x,y) on the line. The line is the set of <u>all</u> points (x,y) each of which can be obtained from the equations $x = a + bk$, $y = c + dk$ using <u>some</u> real number for k. (We cannot juggle the a, b, c, d ; they are fixed for a given line.)

There are situations where there are still more "flavors" among the variables, specifically, in situations involving sets, or families, of lines. For example, if x_1 and y_1 are real numbers, then

$$\{(x,y): x = x_1 + kb \ , \ y = y_1 + kd \ , \ k \text{ is real}\}$$

might be thought of the family of all lines through (x_1,y_1) . Each choice of the parameters b and d (not both zero) yields a line in the family. Once b and d are "pinned down," each value of k yields a point on that line.

Or we might think of m as a real number, and then

$$\{(x,y): x = a + k\ ,\ y = b + km\ ,\ k \text{ is real}\}$$

might be thought of as the family of all lines in the xy-plane with slope m. Each choice of values for a and b would yield a line of the family (the line with slope m passing through (a,b)). Once a and b are fixed, each value of k yields a point on that line.

In the text we have not defined parametric equations. Theorem 8-12 stands on its own feet without such a definition. But we do speak of the equations, $x = a + bk$, $y = c + dk$, which appear in the set-builder symbol as parametric equations. Of course, parametric equations appear in many places in mathematics, and there are many curves in addition to straight lines which can be represented by parametric equations. The parametric equations which represent lines are precisely those described in Theorem 8-12. Thus, if x and y are linear functions of k (not both constant), then the set of points (x,y) determined is a line. To repeat, if a, b, c, d are real numbers, with b and d not both 0, then

$$x = a + bk \quad \text{and} \quad y = c + dk$$

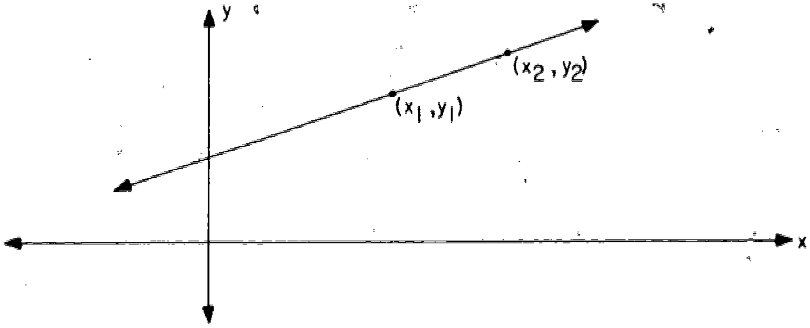
are parametric equations for a line; the variable k is the parameter and the set of all points (x,y) is the line. For further discussion of parametric equations and parameters see <u>Talks to Teachers</u>, No. 7.

In this treatment of slopes we avoid the use of directed distance. We motivate the idea of slope using the "rise over run" idea (which is a non-negative over positive situation in the physical world). Then we define the slope of a line segment in terms of the coordinates of the endpoints of the segment. We show, then, that all segments of a line have the same slope, and this permits us to define the slope of a line as we do.

Our proof that all segments of a line have the same slope does not involve a tacit assumption that the "sign" of $\frac{y_2 - y_1}{x_2 - x_1}$ is the same for all choices of two points (x_1,y_1) and (x_2,y_2) on a given line. It is not mathematically sound to prove this property by considering three cases of non-vertical lines:

(1) Those parallel to the x-axis;

(2) Those which "run uphill," that is, those in



which $\frac{y_2 - y_1}{x_2 - x_1}$ is positive for every choice of two points in the line.

(3) Those which "run downhill," that is, those for which $\frac{y_2 - y_1}{x_2 - x_1}$ is negative for every choice of two points in the line. Such a classification into three classes omits, for example those lines for which $\frac{y_2 - y_1}{x_2 - x_1}$ is positive for some choice of points, (x_1,y_1) and (x_2,y_2), and negative for some other choice. Of course, a picture "shows" that if a line is going uphill between two of its points, then it is going uphill between every pair of its points. Pictures convince us that the three cases listed on the previous page do include all lines. Our proof avoids assuming this by capitalizing on the properties of coordinate systems on a line.

In the proof of Part (1) of Theorem 8-15, p and q are two parallel lines. Hence, Q_1 and Q_2 are on the same side of line p. If they are above p, then h and k are both positive; if they are below p, then h and k are both negative.

In connection with Corollary 8-15, note that if A, B, C, D, are four points and if $m_{\overline{AB}} = m_{\overline{CD}}$ then either $\overleftrightarrow{AB} \parallel \overleftrightarrow{CD}$ or A, B, C, D are collinear. To test four points A, B, C, D for collinearity we could check to see if they all have the same x-coordinate. If not, then they are collinear if and only if $m_{\overline{AB}} = m_{\overline{BC}} = m_{\overline{CD}}$.

In traditional analytic geometry courses, the equation in Theorem 8-16 is sometimes called a symmetric equation for the line in the xy-plane. Although it is not included in this text, this result extends easily to symmetric equations for a line in an xyz-coordinate system. Thus, if $P = (x_1,y_1,z_1)$ and $Q = (x_2,y_2,z_2)$ and if $x_2 - x_1 \neq 0$, $y_2 - y_1 \neq 0$, $z_2 - z_1 \neq 0$, then

$$\overleftrightarrow{PQ} = \{(x,y,z): \frac{x - x_1}{x_2 - x_1} = \frac{y - y_1}{y_2 - y_1} = \frac{z - z_1}{z_2 - z_1}\}.$$

In the proof of Corollary 8-16-1, it is permissible to divide by $x_2 - x_1$ and by $y_2 - y_1$, since they are each different from zero. This follows from the fact that $\overleftrightarrow{PQ}$ of Theorem 8-16 is an oblique line.

A working knowledge of parametric equations for a line, the equations of the form x = a and y = a for vertical and horizontal lines, respectively, and the equations in Section 8-7 ff., will provide the student a good background for working with lines in the coordinate geometry of the plane.

If you are planning to teach Chapter 10 you may want to omit this section, along with many of the problems in the next problem set. In Chapter 10 we prove that two non-zero vectors are perpendicular if and only if the sum of the products of corresponding components is equal to zero; that is, if $a_1a_2 + b_1b_2 = 0$. The similarity between this condition and $m_1m_2 = -1$ is revealed if we note that $m_1 = -\frac{b_1}{a_1}$ and $m_2 = -\frac{b_2}{a_2}$ $(a_1, a_2 \neq 0)$.

If you plan to teach Chapter 10 you may also want to omit Theorems 8-22 and 8-24 since these are also proved as Theorems 10-15 and 10-14 respectively. Actually there are many more theorems in Chapter 8 which may be postponed to Chapter 10, where they may be proved with the aid of directed segments or vectors. However you should permit your students sufficient practice in the use of coordinates in proofs of theorems before presenting them with the vector methods.

In this section we intend to prepare the student with some basic definitions and theorems concerning parallelograms and special kinds of parallelograms. There are some good opportunities to use the concept of subsets as indicated in the diagram in Section 8-11. It is also an opportunity to explore some of the properties of parallelograms and special kinds of parallelograms. This study is continued in Section 8-13, where coordinates are used extensively as an aid in proofs. In this section proofs need not use coordinates.

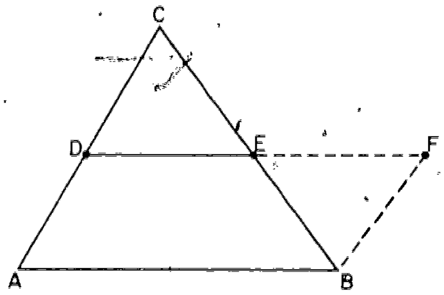
In Proof I we could, of course, set up an xy-coordinate system so that $A = (0,0)$, $B = (b,0)$, $C = (c,d)$ with $b > 0$ and $d > 0$. The proof given in the text could then be modified slightly by omitting several absolute value symbols near the end of the proof as follows:

$$DE = \left|\frac{b+c}{2} - \frac{c}{2}\right| = \left|\frac{b}{2}\right| = \frac{b}{2} ,$$

$$AB = |b - 0| = b ,$$

$$DE = \frac{1}{2}AB .$$

The proof with coordinates of Theorem 8-22 requires no ingenuity; just do what comes naturally and there is the result! The traditional proof without coordinates is a bit ingenious. For those of you who do not recall it, the plan of proof is as follows:

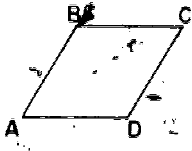


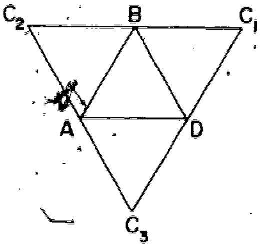
Let F be the point in $\overleftrightarrow{DE}$ such that $DE = EF$ and E is between D and F. Prove $CD = FB$, $\overline{CD} \parallel \overline{FB}$. Then $FB = AD$, $\overline{FB} \parallel \overline{AD}$, ADFB is a parallelogram, $DF = AB$, $\overline{DF} \parallel \overline{AB}$, and $DE = \frac{1}{2}AB$, $\overline{DE} \parallel \overline{AF}$.

Another proof of Theorem 8-22 appears in Chapter 10 on vectors. Some teachers may wish to omit the proofs of several theorems which are proved both in Chapter 8 and in Chapter 10. But in the case of this theorem we have used it as an example in introducing proofs with coordinates. In no case should it be omitted from this chapter.

This section, 8-13, shows how coordinates are used to advantage in producing simple proofs of theorems. The subject matter of these theorems was chosen deliberately to be parallelograms in order that students might continue the study initiated in Section 8-11, this time with coordinates.

Theorem 8-23 seems to say that a parallelogram is determined by three of its vertices. It is in one sense, and it is not in another sense. If A, B, C, D are the vertices of a parallelogram ABCD, and if A, B, D are given, then there is one position for C . If A, B, C, D are the vertices of a parallelogram, and if A, B, D are given, then there are three positions for C .



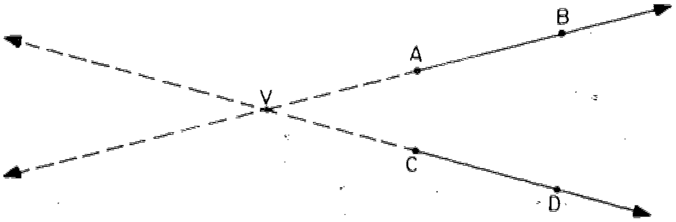


ABCD is a parallelogram. ABC_1D , AC_2BD , $ABDC_3$ are three parallelograms.

The topic of trapezoids in Section 8 - 14 is left almost entirely for students. For this reason only the formal definitions are given with no discussion of theorems or proofs of theorems. It is hoped that students will carry on their own investigations either by proposing propositions about trapezoids which could then be proved or disproved, or by doing the problems in Problem Set 8-14, and perhaps thereby possible ~~theorems~~ may be suggested to them.

In setting up the coordinate system we say that there is a real number a, $a \neq 0$, such that $A = (-a,0)$, etc. Some teachers may think it a digression to insert $a \neq 0$. But this is needed later in the proof and it seems best to insert it early. This is in keeping with the modern emphasis in elementary algebra of specifying the domain of a variable.

According to the text we note that concurrent rays lie on concurrent lines. Be sure, also, to note that not all rays which lie on concurrent lines are concurrent.



The point I is in the interior of each of the angles of the triangle, hence in the interior of the triangle. It is equidistant from the lines which contain the three sides; it is the center of the inscribed circle of the triangle and is sometimes called the <u>incenter</u> of the triangle.

Illustrative Test Items

Chapter 8

1. Given $\triangle ABC$ with $A = (6,2)$, $B = (-8,7)$, $C = (2,-4)$.
   (a) Find the coordinates of D , the midpoint of $\overline{AC}$.
   (b) What is the slope of $\overline{AC}$?
   (c) Show that $\overline{BD} \perp \overline{AC}$.
   (d) What kind of triangle is ABC ?
   (e) Check your answer to (d) by finding BC , and BA .
   (f) Express $\overleftrightarrow{AC}$ using parametric equations.
   (g) Express $\overleftrightarrow{BD}$ using the two-point form.
   (h) Find the coordinates of the point of intersection of the x-axis and $\overleftrightarrow{AC}$.
   (i) Express the line through B parallel to $\overleftrightarrow{AC}$ using the point-slope form.
   (j) Find E so that BCAE is a parallelogram.
   (k) Find F so that CABF is a parallelogram.

2. (a) Find the length of the side of a square whose diagonal measures $12\sqrt{2}$.
   (b) How long is the diagonal of a square whose side measures $12\sqrt{2}$?

3. For each of the properties listed below, tell whether a parallelogram having that property is best classified as a <u>rectangle</u>, as a <u>rhombus</u>, or as a <u>square</u>.
   (a) Its diagonals are congruent.
   (b) Its diagonals bisect its angles.
   (c) Its diagonals are perpendicular to each other.
   (d) Its diagonals are congruent and mutually perpendicular.
   (e) It is equiangular.
   (f) It is equilateral.
   (g) It is both equiangular and equilateral.

4. Prove that a parallelogram is a rectangle if and only if its diagonals are congruent.

5. $A = (-1,2)$ and $B = (3,0)$. Find the coordinates of P if:

   (a) $AP = 3AB$ and P is in $\overrightarrow{AB}$.
   (b) $AP = 3AB$ and P is in $\overrightarrow{BA}$.
   (c) $AP = 100AB$ and P is in $\overrightarrow{AB}$.

6. Tell whether p is a vertical, a horizontal, or an oblique line.

   (a) $p = \{(x,y) : x = 3 + 2k,\ y = 2,\ k \text{ is real}\}$.
   (b) $p = \{(x,y) : x = 3 + 2k,\ y = 2 + k,\ k \text{ is real}\}$.
   (c) $p = \{(x,y) : x = 3,\ y = 2 + k,\ k \text{ is real}\}$.
   (d) $p = \{(x,y) : x + y = 7\}$.
   (e) $p = \{(x,y) : y = 3\}$.

7. Prove: The line containing the median of a trapezoid bisects each of its altitudes.

8. In rectangle $ABCD$, $AB = 20$, $BC = 15$. If P is in $\overline{AC}$ and $\overline{BP} \perp \overline{AC}$, find $\frac{AP}{AC}$.

Chapter 9

PERPENDICULARITY, PARALLELISM, AND COORDINATES IN SPACE

The main objective of this chapter is to help the student develop his concepts of spatial relationships. We want him to be able to think in terms of three dimensions and to be able to visualize and sketch three-dimensional configurations.

No attempt has been made to present a completely formal approach to space geometry. We agree with the Commission Report that there is neither the time nor is there "virtue in so doing." We want the student to "discover" the essential space relationships and we have therefore used an intuitive approach in the form of exploratory problems. These are followed by formal theorems, with some deductive proofs included just to convince the student that there is nothing very peculiar about proofs in three-dimensional geometry. One could easily spend a great deal of class time on these proofs, but this would not be economical. Most teachers will aim for comprehension of the theorems rather than facility in proving them.

Problems requiring proofs are optional. Those proofs not developed in the text have been included in this Commentary for teachers who wish them.

It is to be hoped that teachers will extend the intuitive presentation at the beginning of each section. The lists of exploratory problems are far from exhaustive. Likewise, the suggested physical models to be used in experimentation are the most readily available ones--pencils paper, books. Many other frequently used aids can prove most helpful. These include wire coat hangers, thin wires, straws, string, cardboard, toothpicks, and balsam wood. Standard classroom equipment such as yardsticks, pointers, and window poles can serve as good demonstration models. In addition, some excellent materials for constructing models are available commercially from suppliers of scientific and mathematical equipment. Models constructed by the student are preferred to those ready-made.

At the beginning of the first unit, it might be well to review related properties studied in the plane. It will be helpful if the students recall the simple relationships mentioned in Chapter 2, and then discuss the postulates stated in later chapters which assure them of the existence of infinitely many points. Refer to the Review Problems at the end of Chapter 2, concerning the number of different planes that might contain (a) one point, (b) a certain pair of points, and (c) a certain set of three points. Contrast the answers given at the end of Chapter 2 with acceptable answers at this point in the development of our logical system.

The set of Exploratory Problems is designed to capitalize on the students' intuitive ideas of parallel, intersecting, or perpendicular lines and planes. This is also an opportunity for the students to practice sketching figures in space. At this point, the diagrams should be carefully checked and, when necessary, the students should be referred to the suggestions offered in Appendix V .

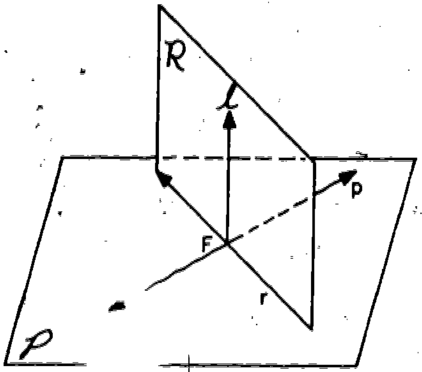
At the beginning of Section 9-2, the experiments may be performed individually or as a class activity. In either case, try to make certain that all of your students have an intuitive idea of perpendicularity in space before proceeding to the formal definitions and theorems.

Discussion of a spoked wheel and axle should make Theorem 9-1 plausible to the students: Any line perpendicular to the axle at the hub must be in the plane of the wheel.

THEOREM 9-3. There is a unique line which is perpendicular to a given plane at a given point in the plane.

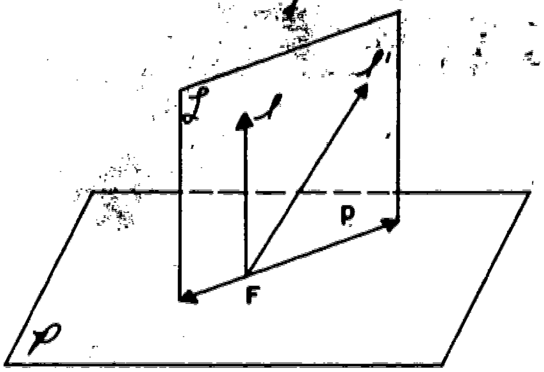
Proof: To prove this theorem we must show two things: first, that there is at least one line perpendicular to the given plane at the given point, and second, that there is no more than one such line.

To show the existence of a perpendicular line, let F be a given point in a given plane $\mathcal{P}$ and let p be any line in $\mathcal{P}$ which contains F. According to Postulate 24 there is a unique plane, say $\mathcal{R}$, which is perpendicular to p at F. Let r be the intersection of the planes $\mathcal{P}$ and $\mathcal{R}$ and let ℓ be the line in $\mathcal{R}$ which is perpendicular to r at the point F.



Then ℓ is perpendicular to both p and r. Hence, by Theorem 9-2 it is perpendicular to the plane $\mathcal{P}$ at the point F, as required.

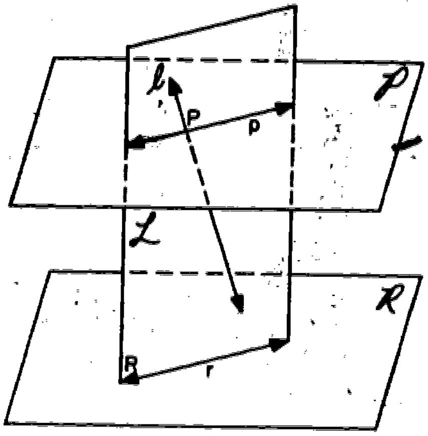
To show that there is no other line which is perpendicular to $\mathcal{P}$ at the point F, suppose that there were such a line, say ℓ'. Let $\mathcal{L}$ be the plane determined by ℓ and ℓ', and let p be the line in which $\mathcal{L}$ and $\mathcal{P}$ intersect (Postulate 9).



Then in the plane $\mathcal{L}$ both ℓ and ℓ' are perpendicular to p at the point F . But in a given plane there is exactly one line which is perpendicular to a given line at a given point. Hence the assumption of a second line, ℓ' , perpendicular to the plane $\mathcal{P}$ at the point F leads to a contradiction and must be rejected. In other words, there is exactly one line which is perpendicular to the given plane $\mathcal{P}$ at the given point F in $\mathcal{P}$.

THEOREM 9-7. If a line intersects one of two distinct parallel planes in a single point, it intersects the other plane in a single point also.

Proof: Let $\mathcal{P}$ and $\mathcal{R}$ be two distinct parallel planes and let ℓ be a line which intersects $\mathcal{P}$ in a single point, say P . Let R be any point in the plane $\mathcal{R}$ not on ℓ , and let $\mathcal{L}$ be the plane determined by ℓ and R .



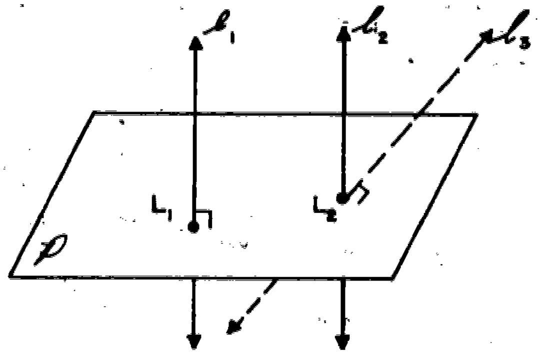
Now $\mathcal{L}$ has the point P in common with the plane $\mathcal{P}$ and the point R in common with the plane $\mathcal{R}$. Hence by Theorem 9-6, $\mathcal{L}$ must intersect $\mathcal{P}$ and $\mathcal{R}$ in two parallel lines, say p and r, respectively, each of which is clearly different from ℓ. Thus in $\mathcal{L}$ we have two parallel lines, p and r, one of which, namely p, is met by ℓ. The other line r must also be met by ℓ. Since ℓ meets r, it certainly meets $\mathcal{R}$, as asserted.

<u>THEOREM 9-8</u>. If a line is parallel to one of two parallel planes, it is parallel to the other also.

<u>Proof</u>: By hypothesis, the line ℓ and the planes $\mathcal{P}$ and $\mathcal{P}_1$ satisfy the conditions that $\ell \| \mathcal{P}$ and $\mathcal{P} \| \mathcal{P}_1$. We wish to prove that $\ell \| \mathcal{P}_1$. Suppose ℓ were not parallel to $\mathcal{P}_1$. Then ℓ intersects $\mathcal{P}_1$ in a single point. Thus $\mathcal{P}$, which by hypothesis does not intersect ℓ in a single point, is distinct from $\mathcal{P}_1$. We now apply Theorem 9-7 to find that ℓ intersects $\mathcal{P}$ in a single point. Contradiction! Hence $\ell \| \mathcal{P}_1$.

THEOREM 9-11. If a plane is perpendicular to one of two distinct parallel lines, it is perpendicular to the other line also.

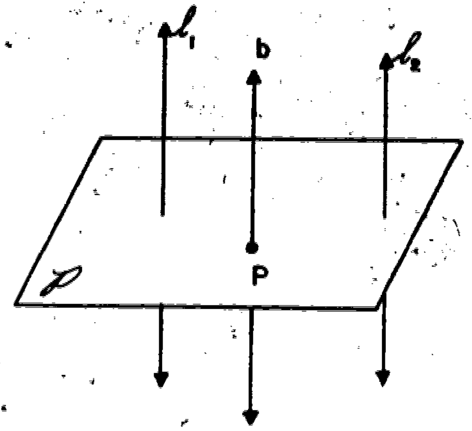
Proof: Let ℓ_1 and ℓ_2 be two parallel lines and let $\mathcal{P}$ be a plane which is perpendicular to one of these lines, say ℓ_1, at the point L_1. Then by Theorem 9-4, $\mathcal{P}$ must also intersect ℓ_2 in a point, say L_2.



Now at L_2 there is, by Theorem 9-3, a line which is perpendicular to $\mathcal{P}$, say ℓ_3. By Postulate 25 this line is parallel to ℓ_1. But according to the Parallel Postulate, there is a unique line parallel to a given line through a given point. Hence, since both ℓ_2 and ℓ_3 are parallel to ℓ_1 and pass through L_2, it follows that ℓ_2 and ℓ_3 are the same line. In other words, ℓ_2 is also perpendicular to $\mathcal{P}$, as asserted.

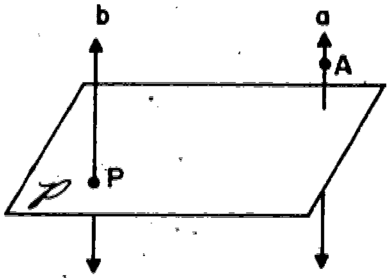
THEOREM 9-12. If two lines are each parallel to a third line, they are parallel to each other.

Proof: Let ℓ_1 and ℓ_2 be two lines each of which is parallel to a third line, b. Let P be any point of b and let $\mathcal{P}$ be the plane which by Postulate 24 is perpendicular to b at P. Then by Theorem 9-11, ℓ_1 and ℓ_2 are each perpendicular to $\mathcal{P}$. Hence, by Postulate 25, they are parallel, as asserted.



THEOREM 9-13. Given a plane and a point not in the plane, there is a unique line which passes through the point and is perpendicular to the plane.

Proof: Let $\mathcal{P}$ be the given plane and let A be the given point, not in $\mathcal{P}$. Let P be any point in $\mathcal{P}$ and let b be the unique perpendicular to $\mathcal{P}$ at P , which is guaranteed by Theorem 9-3. If b passes through A , it is the required perpendicular. If b does not pass through A , let a be the line parallel to b through A which is guaranteed by Theorem 6-3.

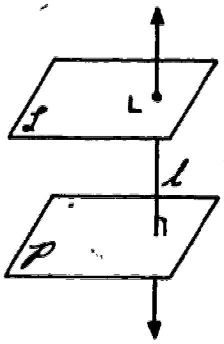


Then by Theorem 9-11, a is also perpendicular to $\mathcal{P}$, and hence is the desired line.

However we must still show that there is only one line through A which is perpendicular to $\mathcal{P}$. To do this, let a' be any line through A which is perpendicular to $\mathcal{P}$. Then by Postulate 25, a' is parallel to b . However, according to the Parallel Postulate, there is only one line through A which is parallel to b . Hence a' and a are the same line, and our proof is complete.

THEOREM 9-14. There is a unique plane parallel to a given plane through a given point.

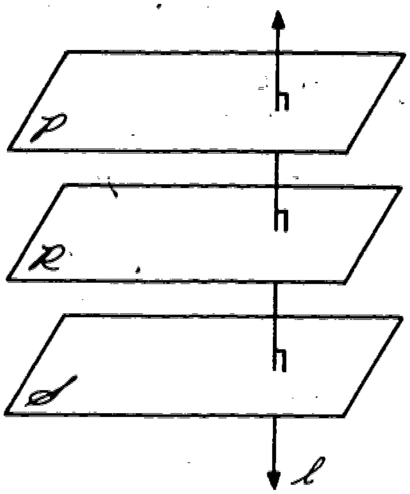
Proof: Let $\mathcal{P}$ be the given plane and let L be the given point. Then there is a unique line ℓ which passes through L and is perpendicular to $\mathcal{P}$. Let $\mathcal{L}$ be the plane which is perpendicular to ℓ at L . Then by



Theorem 9-9, $\mathcal{L}$ is parallel to $\mathcal{P}$. Moreover, there can be no other plane through L parallel to $\mathcal{P}$. In fact, if $\mathcal{L}'$ is any plane through L parallel to $\mathcal{P}$, then by Theorem 9-10, $\mathcal{L}'$ is perpendicular to ℓ . But by Postulate 24, there is exactly one plane perpendicular to a given line at a given point. Hence $\mathcal{L}'$ and $\mathcal{L}$ are the same plane, and our proof is complete.

<u>THEOREM</u> <u>9-15</u>. If two planes are each parallel to a third plane, they are parallel to each other.

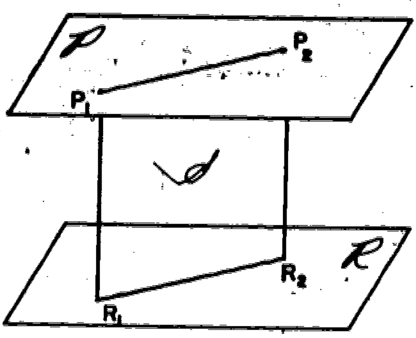
<u>Proof</u>: Let $\mathcal{P}$ and $\mathcal{R}$ be two planes each of which is parallel to a third plane $\mathcal{S}$. Let ℓ be a line perpendicular to $\mathcal{S}$. Then by Theorem 9-10, ℓ is also perpendicular to $\mathcal{P}$ and $\mathcal{R}$. Since $\mathcal{P}$ and $\mathcal{R}$ are both perpendicular to ℓ, then by Theorem 9-9 they are parallel, as asserted.



Notice the remarks that parallelism for lines and parallelism for planes are equivalence relations: they have the the reflexive, symmetric, and transitive properties. This feature is one persuasive argument in favor of adopting the convention that a line or a plane is parallel to itself.

<u>THEOREM</u> <u>9-17</u>. All segments which are perpendicular to each of two distinct parallel planes and have their endpoints in the planes have the same length.

<u>Proof</u>: Let $\mathcal{P}$ and $\mathcal{R}$ be two distinct parallel planes. Let the points P_1 and P_2 in $\mathcal{P}$ and the points R_1 and R_2 in $\mathcal{R}$ be such that each of the distinct segments $\overline{P_1R_1}$ and $\overline{P_2R_2}$ is perpendicular to each of the planes $\mathcal{P}$ and $\mathcal{R}$ (Theorem 9-10).



By Postulate 25, the two lines $\overleftrightarrow{P_1R_1}$ and $\overleftrightarrow{P_2R_2}$ are parallel; hence they lie in the same plane, say $\mathcal{A}$. By Theorem 9-6 $\overleftrightarrow{P_1P_2}$ and $\overleftrightarrow{R_1R_2}$ are parallel. Therefore $P_1P_2R_2R_1$. is a parallelogram (in fact, it is a rectangle), and hence, by Theorem 6-6, $P_1R_1 = P_2R_2$, as asserted.

Although the theory of projections is important in engineering, particularly in drafting, it was deemed not necessary to devote a section of the text to this particular concept. Instead, definitions of the projection of a point into a plane and of the projection of a set of points into a plane are stated in Problem Set 9-4, followed by some problems based upon these concepts. It would be well to precede the assignment of these problems with a brief discussion of this geometrical interpretation of the word projection.

The conventional phrase is to project a point or figure "onto" a plane rather than "into" a plane. We have preferred "into," in order to be consistent with mathematical usage in the theory of mappings or transformations. A mapping is a correspondence which associates with each point of a given set S a unique point of a set S' . We describe this by saying that each point of S is "mapped into" its associated point of S' and that S is "mapped into" S' . We say S is "mapped onto" S' only when the whole of S' is involved, that is when each point of

S' is the associated point of some point of S . Since this distinction between "into" and "onto" is quite firmly established in higher mathematics we thought it wise to use the appropriate technical term "into" even at this elementary level.

Review Section 4-13 with the students to recall the definition of a dihedral angle. Note that we cannot just speak of the union of two halfplanes, but that we must include their common edge in the union. This is because a halfplane does not contain its edge. Similarly, the face of a dihedral angle is defined, not as a halfplane, but as the union of a halfplane and its edge. (This is sometimes called a "closed" halfplane to emphasize that the halfplane has been "closed up" by adjoining its bounding line; in contrast, a halfplane in our sense is called an "open" halfplane.) Observe that the intersection of the two faces is their common edge, just as the intersection of the two sides of an (ordinary) angle is their common endpoint.

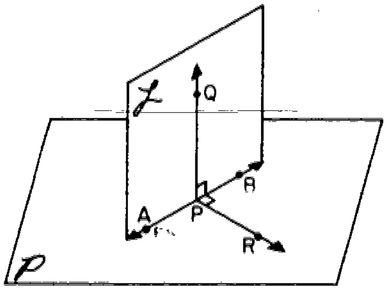
Illustrate dihedral angles by using the covers, or two pages, of a book. From this physical model, try to give the students a feeling for the relative size of dihedral angles, bisection, perpendicularity of planes, etc.

Suggested definitions: Dihedral angles $\angle A\text{-}PQ\text{-}B$ and $\angle A'\text{-}PQ\text{-}B'$ are <u>vertical</u> if A and A' are on opposite sides of $\overleftrightarrow{PQ}$, and B and B' are on opposite sides of $\overleftrightarrow{PQ}$.

The <u>interior</u> of dihedral angle $\angle A\text{-}PQ\text{-}B$ consists of all points which are on the same side of [illegible]e APQ as B and are on the same side of plane BPQ [illegible] A . The <u>exterior</u> of a dihedral angle consists of all points which are not in the interior of the dihedral angle and are not in the dihedral angle itself.

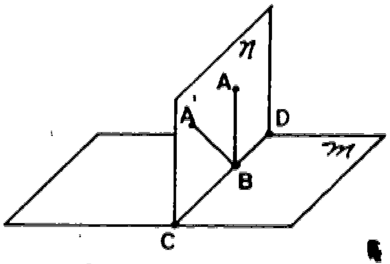
THEOREM 9-20. If a line is perpendicular to a plane, then any plane containing this line is perpendicular to the given plane.

Proof: Let $\overleftrightarrow{QP}$ be the line perpendicular to a given plane $\mathscr{P}$ at the point P, and let $\mathscr{L}$ be any plane containing $\overleftrightarrow{QP}$. Let $\overleftrightarrow{AB}$ be the intersection of $\mathscr{L}$ and $\mathscr{P}$; and in $\mathscr{P}$ let $\overleftrightarrow{PR}$ be the line which is perpendicular to $\overleftrightarrow{AB}$ at P. Since $\overleftrightarrow{QP}$ is perpendicular to $\mathscr{P}$, the line $\overleftrightarrow{AB}$ is also perpendicular to $\overleftrightarrow{QP}$. Hence, by Theorem 9-2, the plane determined by $\overleftrightarrow{QP}$ and $\overleftrightarrow{PR}$ is perpendicular to $\overleftrightarrow{AB}$. Therefore $\angle$QPR is a plane angle of the dihedral angle $\angle$Q-AB-R. Moreover, since $\overleftrightarrow{QP}$ is perpendicular to the plane $\mathscr{P}$, it follows that $\angle$QPR is a right angle. Hence $\angle$Q-AB-R is a right dihedral angle, and $\mathscr{L}$ is perpendicular to $\mathscr{P}$, as asserted.



THEOREM 9-22. If two planes are perpendicular, then any line perpendicular to one of the planes at a point on their line of intersection lies in the other plane.

Proof: By hypothesis, $\mathscr{n} \perp \mathscr{m}$, intersecting in $\overleftrightarrow{CD}$, and $\overleftrightarrow{AB} \perp \mathscr{m}$ at B on $\overleftrightarrow{CD}$. We are asked to prove that $\overleftrightarrow{AB}$ lies in the plane $\mathscr{n}$. In $\mathscr{n}$, there is a line $\overleftrightarrow{A'B}$ which is perpendicular to $\overleftrightarrow{CD}$ at B. Then $\overleftrightarrow{A'B} \perp \mathscr{m}$ at B by Theorem 9-21.



Therefore $\overleftrightarrow{AB}$ and $\overleftrightarrow{A'B}$ coincide by Theorem 9-3. Since $\overleftrightarrow{A'B}$ lies in plane π, $\overleftrightarrow{AB}$ lies in plane π.

Sections 9-6 through 9-9, concerning a three-dimensional coordinate system, are not considered part of the minimum course. Inclusion of these sections in a course for your students should depend upon the time available, the degree of success they enjoyed in studying Chapter 8, and the feeling for spatial relationships they were able to develop in Section 9-1 through 9-5.

Our treatment of a three-dimensional coordinate system is brief, but not rigorous. Rather, it is an extension of the concepts of a two-dimensional coordinate system as developed in Chapter 8. We have tried to capitalize upon some intuitive notions through the use of illustrative diagrams. If your students are not capable of doing all of the work, you might use the diagrams and charts in Section 9-6 as a basis for an informal discussion of a coordinate system in space, omitting the remaining sections.

Some classes may be able to benefit only from Sections 9-6 and 9-7, including the distance formula, but omitting the description of a line or a plane by means of equations. For classes of superior students, not only are all sections strongly recommended, but the treatment of coordinates in space might well be extended.

This is the first time the students have encountered a family of lines presented in set builder notation. Remind them that $\{m: m \parallel \text{z-axis}\}$ reads "the set of all lines m such that m is parallel to the z-axis." Spend some time discussing with your students the pictorial representation of the family of lines.

Using diagrams or physical model[illegible] develop with the students the concepts summarized in t[illegible] charts. A very helpful physical model can be made easily from three pieces of pegboard. If the pegboard is painted with green slate paint and colored elastic is used for lines, the model is effective as well as attractive. Such aids are also available commercially.

Illustrative Test Items

Chapter 9

1. For each of the following, write + if the statement is true (true in every case); write 0 if the statement is false (false in some or all cases).

   (a) Given $P_1(2,0,-3)$ and $P_2(-2,-5,0)$, the length of $\overline{P_1P_2}$ is $5\sqrt{2}$.

   (b) If a line is perpendicular to each of two distinct lines in a plane it is perpendicular to the plane.

   (c) Through a point in a plane only one plane can be passed.

   (d) There are infinitely many lines perpendicular to a given line at a given point on the line.

   (e) Two distinct lines perpendicular to the same plane are coplanar.

   (f) Through a point on a line two distinct planes can be passed perpendicular to the line.

   (g) All points that are equidistant from the endpoints of a given segment are coplanar.

   (h) In a three-dimensional coordinate system, $y = 0$ is an equation of the yz-plane.

   (i) Given a plane $\mathcal{E}$, a line which is perpendicular to a line in $\mathcal{E}$ is perpendicular to $\mathcal{E}$.

   (j) If $\overleftrightarrow{AB}$ and plane $\mathcal{E}$ are each perpendicular to $\overleftrightarrow{FH}$ at point P, then $\overleftrightarrow{AB}$ lies in plane $\mathcal{E}$.

   (k) If a plane intersects two other planes in parallel lines, then the two planes are parallel.

   (l) Two planes perpendicular to the same line are parallel.

   (m) If each of two planes is parallel to a line, the planes are parallel to each other.

   (n) The projection of a line into a plane is a line.

   (o) Two lines are parallel if they have no point in common.

(p) The length of the projection of a segment into a plane is less than the length of the segment.

(q) Two lines parallel to the same plane are parallel to each other.

(r) If two planes are each perpendicular to a third plane, they are parallel to one another.

(s) If a plane bisects a segment, every point of the plane is equidistant from the ends of the segment.

(t) Through a point not in a plane there is exactly one line perpendicular to the plane.

(u) If plane $\mathcal{E}$ is perpendicular to $\overleftrightarrow{AB}$ and $\overleftrightarrow{AB} \parallel \overleftrightarrow{CD}$, then $\mathcal{E} \perp \overleftrightarrow{CD}$.

(v) A plane perpendicular to one of two perpendicular planes is not perpendicular to the other plane.

(w) If plane $\mathcal{M}$ is perpendicular to plane $\mathcal{N}$ and $\triangle ABC$ lies in plane $\mathcal{M}$, then the projection of $\triangle ABC$ into plane $\mathcal{N}$ is a segment.

(x) It is possible for the (degree) measure of a plane angle of an acute dihedral angle to be 90.

(y) Given $A(4,-3,0)$ and $B(-2,-1,6)$, the coordinates of the midpoint of $\overline{AB}$ are $(1,-2,3)$.

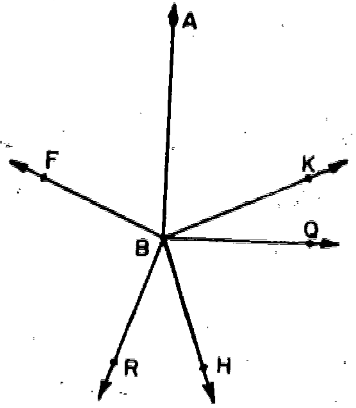
(z) If a line is not perpendicular to a plane, then each plane containing this line is not perpendicular to the plane.

2. Given in this figure that $\overleftrightarrow{BK} \perp \overleftrightarrow{AB}$, $\overleftrightarrow{QB} \perp \overleftrightarrow{AB}$, $\overleftrightarrow{HB} \perp \overleftrightarrow{AB}$, $\overleftrightarrow{RB} \perp \overleftrightarrow{AB}$ and $\overleftrightarrow{BF} \perp \overleftrightarrow{AB}$.

   (a) $\overleftrightarrow{BK}$ and $\overleftrightarrow{AB}$ determine a plane ABK. Is $\overleftrightarrow{BQ}$ perpendicular to plane ABK ?

   (b) Do $\overleftrightarrow{FB}$, $\overleftrightarrow{RB}$, $\overleftrightarrow{HB}$ all lie in plane KBQ ? Why?

   (c) There are at most _____ different planes determined by pairs of the given lines.

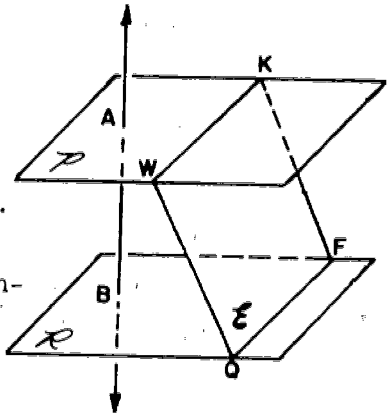


3. In the figure, plane $\mathcal{P} \perp \overleftrightarrow{AB}$ and plane $\mathcal{R} \perp \overleftrightarrow{AB}$.

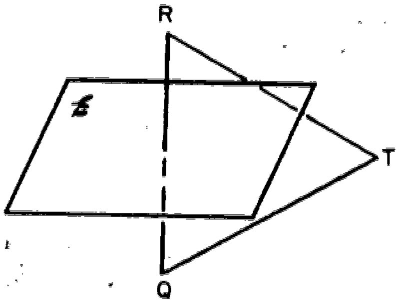
   (a) Is $\mathcal{P} \parallel \mathcal{R}$? Why?

   (b) Plane $\mathcal{E}$ intersects $\mathcal{P}$ and $\mathcal{R}$ in $\overleftrightarrow{WK}$ and $\overleftrightarrow{QF}$, respectively. Is $\overleftrightarrow{WK} \parallel \overleftrightarrow{QF}$? Why?

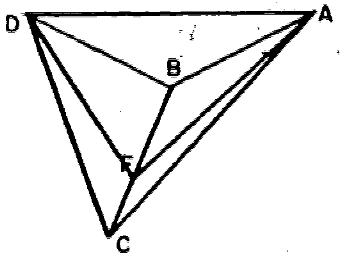
   (c) If a line m is perpendicular to $\overleftrightarrow{WK}$ and intersects $\overleftrightarrow{QF}$, what kind of angles does m make with $\overleftrightarrow{QF}$? Justify your answer.



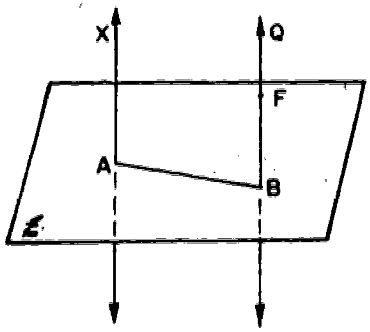
4. In this figure, plane $\mathcal{E}$ bisects $\overline{RQ}$ and $\mathcal{E} \perp \overleftrightarrow{RQ}$. Also RT = QT . Explain why T lies in plane $\mathcal{E}$.



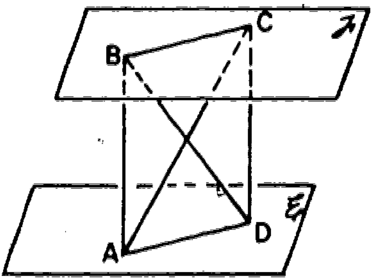
5. Points A, B, C, and D are not coplanar. $\triangle ABC$ is isosceles with $AB = AC$. $\triangle DBC$ is isosceles with $DB = DC$. F is the midpoint of $\overline{BC}$. In the figure at least one segment is perpendicular to a plane. What segment? What plane? Justify your answers.



6. Given: $\overleftrightarrow{XA} \perp \mathcal{E}$ at A. $\overleftrightarrow{QB} \perp \mathcal{E}$ at B. F is a point on $\overleftrightarrow{QB}$. Are X, A, B, F coplanar? State a theorem to support your conclusion.



7. Given: $\mathcal{E} \parallel \mathcal{F}$.
$\overleftrightarrow{AB} \perp \mathcal{E}$ at A.
B is in $\mathcal{F}$.
$\overleftrightarrow{CD} \perp \mathcal{E}$ at D.
C is in $\mathcal{F}$.
Prove: $AC = BD$.



8. The following sets of lines (m) and planes ($\mathscr{P}$) are described in reference to a three-dimensional coordinate system, having x, y, z-axes. By pairing those on the left with those on the right, match the equivalent sets.

(a) $\{m : m \parallel \overleftrightarrow{OY}\}$
(b) $\{\mathscr{P} : \mathscr{P} \parallel \text{yz-plane}\}$
(c) $\{\mathscr{P} : \mathscr{P} \parallel \text{xy-plane}\}$
(d) $\{m : m \parallel \overleftrightarrow{OX}\}$
(e) $\{m : m \parallel \overleftrightarrow{OZ}\}$
(f) $\{\mathscr{P} : \mathscr{P} \parallel \text{xz-plane}\}$

(r) $\{\mathscr{P} : \mathscr{P} \perp \text{z-axis}\}$
(s) $\{m : m \perp \text{xy-plane}\}$
(t) $\{\mathscr{P} : \mathscr{P} \perp \text{y-axis}\}$
(u) $\{\mathscr{P} : \mathscr{P} \perp \text{x-axis}\}$
(v) $\{m : m \perp \text{yz-plane}\}$
(w) $\{m : m \perp \text{xz-plane}\}$

9. Find the point in which the line m intersects the xz-plane if

$$m = \{(x, y, z): x = 2 + k , y = 4 - 2k , z = 3k , k \text{ is a real number}\} .$$

10. Show that $\triangle ABC$ is isosceles if its vertices are A(2,3,-5) ; B(-2,4,-2) ; C(-1,0,1) .

11. Find an equation of the line in the xy-plane which is the line of intersection of the xy-plane and the plane whose equation is $2x - y + z = 7$.

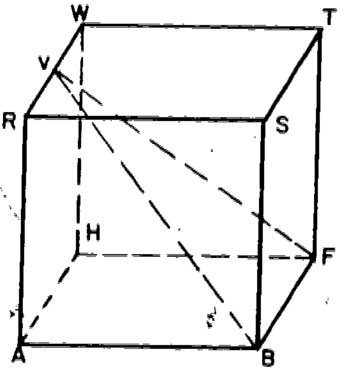
12. (a) Given points M(6,-2,3) and N(-5,1,4) . Find the coordinates of the midpoint P of $\overline{MN}$.

(b) Given points A(2,0,-2) and B(x,-1,z) . Find x and z , so that the midpoint of $\overline{AB}$ is the same point P as in Part (a).

13. Find an equation of the plane determined by the points A(1,2,5) ; B(0,1,6) ; C(2,0,1) .

14. V is the midpoint of edge $\overline{RW}$ of the cube shown in this figure.

Prove, with or without coordinates, that VB = VF .



Chapter 10

DIRECTED SEGMENTS AND VECTORS

10-1. Introduction.

This is an optional chapter and should be omitted where the ability level of the class or the lack of time makes its omission necessary. For this reason relatively few geometrical theorems are proved and no new ones are presented.

The purpose of this chapter is to introduce the student of above average ability to another mathematical system, one which has wide application in physics and engineering, as well as in mathematics. Moreover we feel that the work in this chapter will help to solidify the ideas of closure, commutativity, associativity, and the other properties of real numbers.

The treatment of an entire set of directed line segments as a single entity, called a vector, will probably seem an unnecessary departure from the common notion of directed line segments being vectors. However this is the modern concept and we believe that the ideas stressed in this chapter will make it easier for the student to proceed to a more advanced study of vector analysis with relatively little difficulty.

Some of the problems in the problem sets and in the review set deal with the use of vectors in solving certain problems of physics. The student does not need an extensive background in physics to handle the problems. It is sufficient that he knows that when two or more forces or velocities act on a body the resultant force or velocity can be found by the rules of vector addition.

10-2. Directed Segments.

677 The main ideas of this section are equivalence of directed segments, addition of directed segments, and multiplication of directed segments by real numbers. The student is required to translate statements of geometric relation into algebraic language.

By this time equivalence relations should be familiar to the students. The fact that directed segment equivalence is reflexive, symmetric and transitive enables us to consider all directed segments that are equivalent to a given directed segment as a set that is well defined. It is this fact that paves the way for the definition of a vector in Section 3.

As a matter of vocabulary, it is worth noting that directed segments, which are often inaccurately referred to simply as vectors, are sometimes called bound vectors, since in a sense they are "bound" to a particular point as origin. When this terminology is used, the entities which we call vectors are then referred to as free vectors. Since it is very convenient to be able to denote a vector (in our sense) by the symbol $\overrightarrow{AB}$, we have introduced the symbol $(\overrightarrow{A,B})$ for a directed segment (or bound vector).

We have tried to stress that the addition of directed segments is not commutative. This may be the students' first encounter with non-commutative addition and, as such, should not be "glossed" over.

10-3. Vectors.

695 The main topic of this section is the algebra of vectors as ordered pairs, [p,q] . The transition from coordinates of points to components of vectors is a little subtle and may present difficulty to the student, but once the changeover is made, the algebraic properties are easily established.

Property 3 states $\vec{u} + \vec{0} = \vec{u}$. In other words $\vec{0}$ plays the role in vector addition comparable to 0 in the addition of real numbers. Hence $\vec{0}$ is often called the identity element for vector addition. Similarly Property 4 indicates that each vector has an additive inverse.

Properties of Vectors

1. If $\vec{u}, \vec{v}$ are vectors then $\vec{u} + \vec{v}$ is a vector.
2. If $\vec{u}, \vec{v}, \vec{w}$ are any three vectors then
   $$(\vec{u} + \vec{v}) + \vec{w} = \vec{u} + (\vec{v} + \vec{w}) .$$
3. There is a vector $\vec{0}$ such that for any $\vec{u}$
   $$\vec{u} + \vec{0} = \vec{u} .$$
4. For every vector $\vec{u}$ there is a vector $-\vec{u}$ such that
   $$\vec{u} + (-\vec{u}) = \vec{0} .$$
5. If $\vec{u}, \vec{v}$ are any two vectors then
   $$\vec{u} + \vec{v} = \vec{v} + \vec{u} .$$
6. If $\vec{u}, \vec{v}$ are any two vectors and k is any scalar then
   $$k(\vec{u} + \vec{v}) = k\vec{u} + k\vec{v} .$$
7. If $\vec{u}$ is any vector then $k\vec{u} = \vec{u}$ when $k = 1$.
8. If $\vec{u}$ is any vector and p,q are any two scalars then
   $$(p + q)\vec{u} = p\vec{u} + q\vec{u} .$$
9. If $\vec{u}$ is any vector and p,q are any two scalars then
   $$p(q\vec{u}) = (pq)\vec{u} .$$
10. If $\vec{u}$ is any vector and k is any scalar then
    $$|k\vec{u}| = |k| \cdot |\vec{u}| .$$

Proof of 1: Obvious from definition.

Proof of 2: If $\vec{u} = [a,b]$, $\vec{v} = [c,d]$, $\vec{w} = [e,f]$, then

$$(\vec{u} + \vec{v}) + \vec{w} = ([a,b] + [c,d]) + [e,f]$$

$$= [a + c,\ b + d] + [e,f] = [a + c + e,\ b + d + f]$$

$$= [a,b] + [c + e,\ d + f] = [a,b] + ([c,d] + [e,f]) .$$

Proof of 3: Let $\vec{u} = [a,b]$ and $\vec{0} = [0,0]$. Then

$$\vec{u} + \vec{0} = [a,b] + [0,0] = [a,b] = \vec{u} .$$

Proof of 4: Let $\vec{u} = [a,b]$, $-\vec{u} = [-a,-b]$. Then

$$\vec{u} + (-\vec{u}) = [a,b] + [-a,-b] = [0,0] = \vec{0} .$$

Proof of 5: Let $\vec{u} = [a,b]$ and $\vec{v} = [c,d]$. Then

$$\vec{u} + \vec{v} = [a,b] + [c,d]$$

$$= [a + c,\ b + d] = [c + a,\ d + b]$$

$$= [c,d] + [a,b]$$

$$= \vec{v} + \vec{u} .$$

Proof of 6: Let $\vec{u} = [a,b]$ and $\vec{v} = [c,d]$. Then

$$k(\vec{u} + \vec{v}) = k([a,b] + [c,d]) = k[a + c,\ b + d]$$

$$= [k(a + c) ,\ k(b + d)]$$

$$= [ka + kc,\ kb + kd] = [ka,kb] + [kc,kd]$$

$$= k[a,b] + k[c,d]$$

$$= k\vec{u} + k\vec{v} .$$

Proof of 7: Let $\vec{u} = [a,b]$. Then

$k\vec{u} = k[a,b] = [ka,kb]$. But if $k = 1$, then

$$[ka,kb] = [1 \cdot a ,\ 1 \cdot b] = [a,b] .$$

Proof of 8: Let $\vec{u} = [a,b]$. Then

$$(p + q)\vec{u} = (p + q)[a,b] = [(p + q)a,(p + q)b]$$

$$= [pa + qa,\ pb + qb] = [pa,pb] + [qa,qb]$$

$$= p[a,b] + q[a,b]$$

$$= p\vec{u} + q\vec{u} .$$

Proof of 9: Let $\vec{u} = [a,b]$. Then

$$p(q\vec{u}) = p(q[a,b]) = p[qa,qb] = [pqa,pqb]$$
$$= (pq)[a,b]$$
$$= (pq)\vec{u} .$$

Proof of 10: Let $\vec{u} = [a,b]$. Then

$$k\vec{u} = k[a,b] = [ka,kb] .$$

Thus $|k\vec{u}| = |[ka,kb]| = \sqrt{k^2a^2 + k^2b^2} = |k|\sqrt{a^2 + b^2}$.

But $|k| \cdot |\vec{u}| = |k| \cdot |[a,b]| = |k|\sqrt{a^2 + b^2}$.

10-4. The Two Fundamental Theorems.

702 Theorem 2. One point in this proof is the assertion that if $k\vec{v} = \vec{0}$ then $k = 0$. The students may have trouble following this and therefore we suggest that the following be discussed prior to the discussion of this theorem.

If $k\vec{v} = \vec{0}$ and $\vec{v} \neq \vec{0}$ then $k = 0$.

Proof: $k\vec{v} = \vec{0}$ means $k\vec{v} = \vec{0} = [0,0]$ also $\vec{v}$ is some vector of the type $[a,b]$ where, by hypothesis, a and b are not both zero.

But $k\vec{v} = k[a,b] = [ka,kb]$,

therefore $[ka,kb] = [0,0]$

and it follows that $k = 0$.

Incidentally, the occurrence in this proof of both the zero vector, $\vec{0}$, and the scalar quantity, 0 , should be carefully noted, and the difference should be made clear to the student.

10-5. Geometrical Application of Vectors.

706 This section has two main topics. The first is that vectors can be manipulated according to most of the usual rules of algebra; the second is that certain problems of elementary geometry can be solved by such manipulations.

Each of the examples is worked out as an isolated problem. No hint is given about a general approach to any type of problem. However, there is a general approach which teachers may want to discuss. Each problem can be solved by:

1. Choosing two directed line segments on non-parallel lines.
2. Expressing each of the other directed line segments in terms of the ones originally selected.

10-6. The Scalar Product of Two Vectors.

709 The scalar product is often called the inner product or dot product. We chose the terminology scalar product to emphasize that the result of this operation is a number (or scalar). However, great care must be exercised so that the student does not confuse the scalar product with multiplication of a vector by a scalar.

To prevent a careless student from mistaking $\vec{a} \cdot \vec{b}$, a scalar product of two vectors, for $x \cdot y$, a product of two numbers, we recommend that the dot not be used for the product of two numbers, if there is a possibility of confusion.

The symbol $\vec{0}$ was used to represent the zero vector. Students should be cautioned not to use the single letter 0 to name a vector as it may lead to unnecessary errors. Moreover it should be made quite clear that if we have $\vec{u} \cdot \vec{v} = 0$, the result is a scalar and is different from $\vec{u} - k\vec{v} = \vec{0}$.

Proof of Property 3.

$$\text{Let } \vec{u} = [a,b] \text{ , } \vec{w} = [c,d] \text{ ,}$$

$$\begin{aligned} \vec{u} \cdot (k\vec{w}) &= [a,b] \cdot [kc,kd] \\ &= akc + bkd = (k\vec{u}) \cdot \vec{w} \\ &= k(ac + bd) \\ &= k(\vec{u} \cdot \vec{w}) \text{ .} \end{aligned}$$

Proof of Property 4.

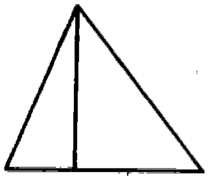
$$\text{Let } \vec{u} = [a,b]$$

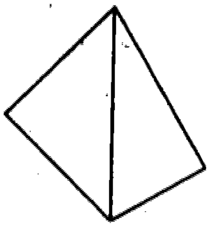
$$\begin{aligned} \vec{u} \cdot \vec{u} &= [a,b] \cdot [a,b] \\ &= a^2 + b^2 \\ &= |\vec{u}|^2 \text{ .} \end{aligned}$$

The scalar product enriches vector algebra to the point that it can be used to prove many more geometric theorems. In Problem 19 of the sample test questions the student is asked to prove the diagonals of a rhombus are perpendicular to one another. The teacher may wish to present this in class to show an application of the scalar product to geometric proofs.

The student should be made aware of the fact that the scalar product does not obey the law of closure. He might well be asked to give other examples which do not obey the law of closure. Some examples of this are:

1. The product of two negative numbers is not in the set of negative numbers.
2. The product of two irrationals is not always irrational.
3. The region formed by two triangles with a side in common is not always a triangular-region.





Leaving the world of mathematics we have,

4. Combining two gases does not always produce a gas. Hydrogen and oxygen may combine to form water.

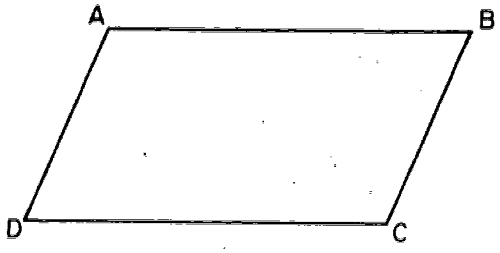
Illustrative Test Items for Chapter 10

1. If A, B, C are three collinear points such that B is between A and C , list all the directed line segments they determine.

2. If A, B, C are the points (4,2) , (3,7) , (-2,1) , respectively, list all the vectors represented by the directed segments joining these points.

3. If $\vec{a}$ is [3,7] and $\vec{b}$ is [-2,1] find
   (a) $\vec{a} + \vec{b}$.
   (b) $\vec{a} - \vec{b}$.
   (c) $\vec{a} \cdot \vec{b}$.
   * (d) $|\vec{a}|$.
   (e) $|\vec{a} + \vec{b}|$.

4. If ABCD is a parallelogram, as indicated below,



   (a) $(\overrightarrow{A,B}) \doteq ?$
   (b) $(\overrightarrow{A,D}) \doteq ?$
   (c) $(\overrightarrow{D,A}) \doteq ?$
   (d) $(\overrightarrow{D,C}) + (\overrightarrow{C,B}) \doteq ?$
   (e) $(\overrightarrow{D,C}) + (\overrightarrow{C,B}) + (\overrightarrow{B,A}) \doteq ?$

5. What is the negative of [-2,3] ?

6. What is the vector $\vec{f}$ such that for all $\vec{v}$ we have $\vec{v} + \vec{f} = \vec{v}$?

7. If $\vec{a} = [2,1]$, $\vec{b} = [3,6]$, $\vec{c} = [-1,-3]$, find
   (a) $\vec{a} + \vec{b} + \vec{c}$.
   (b) $|\vec{a} + \vec{b} + \vec{c}|$.

8. Determine x and y so that

   (a) $[5,6] + [x,y] = [3,2]$.
   (b) $[-2,3] + [x,y] = [4,5]$.
   (c) $[5,2] + [x,y] = [0,0]$.
   (d) $[9,7] + [x,y] = [14,-3]$.
   (e) $[-6,-2] + [x,y] = [-6,-2]$.

9. Determine x and y so that
   $x[2,-3] + y[3,-1] = [5,3]$.

10. If $\vec{a} = [2,1]$ and $\vec{b} = [3,6]$, find $\vec{a} \cdot \vec{b}$.

11. If $AM = MB$, must $(\overrightarrow{A,M}) \doteq (\overrightarrow{M,B})$?
    Explain your answer.

12. If $AM = MB$, must $|\overrightarrow{AM}| = |\overrightarrow{MB}|$?
    Explain your answer.

13. If $\vec{a} = [3,4]$ and $\vec{b} = [6,8]$,

    (a) express $\vec{b}$ in terms of $\vec{a}$;
    (b) express $\vec{a}$ in terms of $\vec{b}$.

14. Determine x and y so that
    $[5,6] + [-2,3] + [x,y] = [6,4]$.

15. What conditions must hold for two directed line segments to be equivalent?

16. Show that $P(0,0)$, $Q(6,8)$, $R(15,18)$ and $S(9,10)$ are the vertices of a parallelogram.

17. Show that $P(2,1)$, $Q(5,3)$, $R(3,6)$, $S(0,4)$ are the vertices of a rectangle.

18. Show that the line determined by $P(6,6)$ and $Q(8,0)$ is perpendicular to the line determined by $R(3,5)$ and $S(9,7)$.

19. Prove that the diagonals of a rhombus are perpendicular to each other.

20. If ABC is a triangle and D , E are points on $\overline{AB}$ and $\overline{AC}$ respectively such that $AD = \frac{1}{3}AB$ and $AE = \frac{1}{3}AC$, prove that $\overline{DE}$ is parallel to $\overline{BC}$ and $DE = \frac{1}{3}BC$.

A Note on Chapters 11 and 12

There are several adaptations that the teacher might make in using the present text without loss of continuity of subject matter. Four alternate plans are outlined briefly for consideration by the teacher. Each teacher should study the plans carefully and decide which one, if any, is more desirable than the present sequence of the text for use with a particular class.

Plan A

Sections 12-1 through 12-5 ,
11-1 through 11-12,
12-6 through 12-9 .

Plan B

Sections 12-1 through 12-5 ,
11-1 through 11-8 ,
12-6 through 12-9 ,
11-9 through 11-12.

Plan C

Sections 11-1 through 11-3 ,
12-1 through 12-5 ,
11-4 through 11-8 ,
12-6 through 12-9 ,
11-9 through 11-12.

Plan D

Sections 11-1 through 11-8 ,
12-1 through 12-9 ,
11-9 through 11-12.

Any one of the above plans may be modified by placing Section 11-3 immediately before Section 11-8.

Teachers who are pressed for time should consider omitting entirely the sections (11-6 through 11-9) on polyhedrons, to gain time for Chapter 12. In any case, Sections 12-1 through 12-5 should not be omitted.

Chapter 11

POLYGONS AND POLYHEDRONS

This chapter treats the conventional subject matter of polygons and polyhedrons. The viewpoint is essentially that of Euclid, and many of the theorems in this chapter have proofs similar to the proofs of corresponding theorems in other geometry texts. However, there are several differences. First, the introduction of the term polygonal-region; and second, the study of area by postulating the properties of area rather than by deriving the properties from a definition of area based on the measurement process. Actually both of these treatments are implicit in the conventional treatment. We have only brought them to the surface, sharpened, and clarified them. After this basis is laid, our methods of proof are simple and conventional. However, the placement of topics and the order of theorems may differ from the conventional sequence.

In the work with polygonal-regions we are restricting ourselves to the relatively simple case of a polygonal-region whose boundary is rectilinear, that is, whose boundary is a union of segments. Our theory for polygonal-regions will be extended in a later chapter to include more general configurations such as circles.

Although we have previously defined polygon, convex polygon, and the interior of a convex polygon (see Section 4-12 of text), difficulty arises when an attempt is made to define the interior of a non-convex polygon. Since any triangle is a convex polygon, our definition of polygonal-region avoids this difficulty. We merely take the simplest and most basic type of region, the triangular-region, and use it as a building block to define a polygonal-region.

You should also note that we have not defined region as a single word, and that our use of the term polygonal-region differs from the usual mathematical usage which requires that a region be connected or "appear in one piece." Since our definition of a polygonal-region does not require connection, we avoid confusion by placing a hyphen between the words polygonal and region.

The following pictures illustrate three polygonal-regions which represent:

(1) The union of two triangular-regions with no points in common;

(2) The union of two triangular-regions with one point in common; and

(3) The union of two triangular-regions with a segment in common.

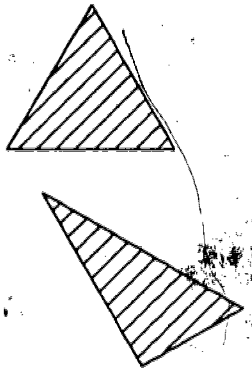


Figure 1

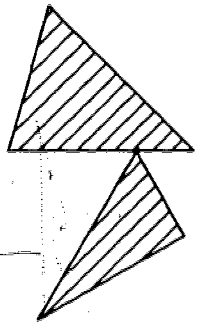


Figure 2

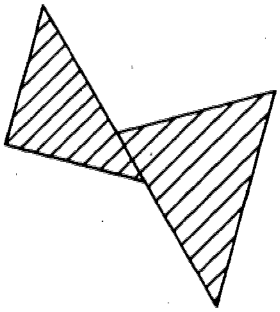


Figure 3

Our definition of polygonal-region, which allows a disconnected portion of the plane, implies that the boundary of a polygonal-region is not necessarily a polygon. This should cause no trouble; it simply means that our theory has broader coverage than the usual mathematical use of the word.

In many geometry texts the theorem and corollaries pertaining to the sum of the measures of the angles of a polygon follow Theorem 6-9. We did not include this material in Chapter 6 because our primary objective in Chapters 1 - 7 was to develop a sequence of theorems essential to the establishment of the Pythagorean Theorem, which would permit us to make early use of a coordinate system in a plane. This was not essential to that development. Since the theorem on the sum of the measures of the angles of a polygon and the subject of areas require an understanding of polygonal-regions, we achieve unity of subject matter by treating polygonal-regions, the sum of the measures of the angles of a polygon, and the area of polygons in the same chapter.

Prior to Theorem 11-1, there is a set of exploratory problems which should help the students understand the proof of this theorem. The answers to these problems are given below:

Number of sides of convex polygon	Number of diagonals from A	Number of triangular regions	Sum of measures of angles of the polygon
4	1	2	2 × 180 = 360
5	2	3	3 × 180 = 540
6	3	4	4 × 180 = 720
7	4	5	5 × 180 = 900
8	5	6	6 × 180 = 1080
n	n - 3	n - 2	(n - 2) 180

Sometimes in a mathematical discussion we give an explicit definition of area for a certain type of figure. For example, the area of a rectangle is the number of unit squares into which the corresponding rectangular-region can be separated. This is a difficult thing to do in general terms for a wide variety of figures. Thus the suggested definition of area of a rectangle (rectangular-region) is applicable only if the rectangle has sides whose lengths are integers. Literally how many unit squares are contained in a rectangular region whose dimensions are $\frac{1}{2}$ and $\frac{1}{3}$? The answer is none! Clearly the suggested definition must be modified for a rectangle with rational dimensions. To formulate a suitable definition when the dimensions are irrational numbers, for example $\sqrt{2}$ and $\sqrt{3}$, is still more complicated and involves the concept of limits. Furthermore, it would also be necessary to define the area concept for triangles, quadrilaterals, circles, and so on. The complete study of area along these lines involves integral calculus and finds its culmination in the branch of mathematics called the Theory of Measure.

Since this is too sophisticated an approach for our purposes, we do not attempt to give an explicit definition of a polygonal-region by means of a measurement process using unit squares. Rather we study area in terms of its basic properties as stated in Postulates 26, 27, 28, 29. On the basis of these postulates we prove the familiar formula for the area of a triangle. Consequently we get an explicit procedure for obtaining areas of triangles and of polygonal-regions in general.

<u>Some remarks on the postulates</u>. Observe that our treatment of area is similar to that for distance and the measure of angles. Instead of giving an explicit definition of area (or distance, or angle measure) by means of a measurement process, we postulate its basic properties which are intuitively familiar from study of the measurement process.

Postulate 26 is analogous to Postulate 10 for distance. The "given polygonal-region" plays the same role as the unit-pair. However, the difference lies in the fact that in area we soon restricted ourselves to one unit of area, which does not necessitate an additional postulate for a change of the unit of area. Postulate 26 can also be considered an analog of Postulate 16 for angles.

Postulate 27 is analogous to the definitions of betweenness for points on a line and betweenness for angles. It is a precise formulation, for the study of area, of the vague statement "The whole is the sum of its parts." This statement is open to several objections. It seems to mean that the measure of a figure is the sum of the measures of its parts. Even in this form it is not acceptable, since the terms "figure" and "part" need to be sharpened in this context, and it permits the "parts" to overlap. Postulate 27 makes clear that the "figures" are to be polygonal-regions, the "measures" are areas, and that the "parts" are to be polygonal-regions whose union is the "whole" and which do not overlap.

In Chapter 3 we defined congruence of segments in terms of segments having the same length. Here, our situation is different. We already have the notions of congruence, and we try to make our idea of area come into line with that of congruence. Hence, we formulate Postulate 28 which states if two triangles are congruent, then the triangular regions associated with them have the same area with respect to any given unit of area.

These three postulates seem to give the essential properties of area, but they are not quite complete. We pointed out that Postulate 26 presupposes that a unit has been chosen, but we have no way of determining such a unit, that is, a polygonal-region whose area is unity. For example, these postulates permit a rectangle of dimensions 3 and 7 to have area unity.

Postulate 29 takes care of this by guaranteeing that a square whose edge has length 1 shall have area 1 . In addition, Postulate 29 gives us an important basis for further reasoning by assuming the formula for the area of a rectangle.

Since we are introducing a block of postulates concerning area, this may be a good time to remind your students of the significance and purpose of postulates.

They are precise formulations of the basic intuitive judgments suggested by experience, from which we derive more complex principles by deductive reasoning.

To make the postulates on area more significant for the students, discuss the measuring process for area concretely, using simple figures like rectangles or right triangles with integral or rational dimensions. Have them subdivide regions into congruent unit squares, so that the students get the idea that every "figure" has a uniquely determined area number. Then present the postulates as simple properties of the area number which can be verified concretely in diagrams.

The problems in Section 11-4 emphasize the relations that exist when a set of rectangles have equal bases, equal altitudes, or equal areas. They are introduced early in the study of area and serve as exploratory problems for the development of the theorems in Sections 11-6 and 11-7. Similar exploratory problems should be included in daily reviews along with the development of the theorems in Section 11-5.

The formulas for the area of triangles and quadrilaterals are developed in junior high school mathematics courses and most of them are familiar to high school students. We develop them in rapid sequence so that the thread of continuity is maintained in proceeding from one theorem to another. Teachers of superior students will probably want to teach these theorems in a single day. Teachers who need to use a slower pace will find the problems in the Problem Set organized in a sequence which will make convenient day by day assignments relating to any specific theorem or combination of theorems. However, it would be helpful to students in these classes if they would reread the section after all formulas have been developed in order that they can more fully appreciate the continuous thread of development.

Large cardboard models of triangles and quadrilaterals should be helpful in demonstrating the various theorems in this section. Use the figures accompanying the theorems as patterns in constructing the models.

After postulating the area of a rectangle we proceed to develop our formulas for areas in the following manner: right triangle, triangle, rhombus, parallelogram and trapezoid. The right triangle permits us to work with any triangle. This in turn gives us the machinery to find the area of any polygonal-region by chopping it up into a number of triangular regions and finding the sum of the areas of these triangular regions. The proofs of these theorems illustrate the fact that Postulate 27 is a sort of separation theorem, in which a given region R is the union of the regions R_1 and R_2 .

The problems in Problem Set 11-4 following the postulate for the area of a rectangle give pupils early opportunity to explore the relations that exist for sets of rectangles with equal bases, equal altitude, and equal area. Students should be given similar numerical exercises for the triangle and the parallelogram, and questions should be asked which lead students to make the generalizations which are proved as theorems in Section 11-6. Visual aids such as sets of cardboard triangles with equal bases, equal altitudes, and equal areas should help students understand these relations.

Some of the problems in Problem Set 11-4 deal with similar rectangles. A procedure similar to that used for developing the generalizations in Section 11-6 should be used in developing the generalizations in Section 11-7. By means of student drawings and informal discussions students should come to the following understandings: (1) Corresponding linear measurements of similar polygons have the same ratio, and (2) Corresponding areas of similar polygons have the same ratio as the squares of any two corresponding linear measures.

In conventional texts the area of a regular polygon is developed in the chapter on circles. In order to unify the work on area of polygons, we include it at this time. Hence our definitions of center, radius, and apothem must be independent of inscribed and circumscribed circles. Theorem 11-8 serves as a basis for these definitions.

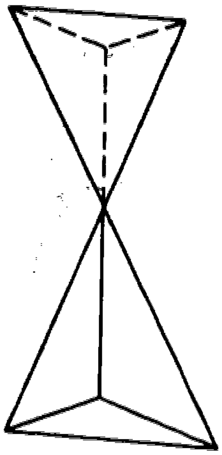
The work in the remainder of the chapter is informal in nature. Models, experiments, and exploratory problems serve as the basis for student discovery of many important theorems. By this time students should realize that this is not a part of our formal development of geometry, but an informal extension of two dimensional concepts into three dimensional space. Students should be encouraged to make models of the regular polyhedrons in studying Euler's famous formula and Theorem 11-19.

The exploratory problems in Section 11-10 help the student to discover the ideas embodied in Theorems 11-17 and 11-18. They also help the student to visualize the difference between congruence and symmetry of models in the physical world. The solutions to the problems are as follows:

1. 8 ; 90 .
2. Yes; yes; yes; no; the vertex would lie in the plane of the determining polygon.
3. No, same reason as Problem 2.
4. No, same reason as Problem 2.
5. Less than 360 .
6. Yes.
7. Yes.
8. No.

9. They exhibit a correspondence such that the corresponding face angles are congruent, but they are arranged in "reverse" order. Common examples of symmetrical models in the physical world are a pair of shoes, a pair of gloves, and the reverse plan of a house.

10. No; yes.



Theorems 11-17 and 11-18 are interesting to students, and the proofs of the theorems are easy to demonstrate to the class when large physical models are used in the demonstration. However, the notation in writing these proofs becomes very involved. For this reason the proofs are omitted from the text. A sketch of the proof of each is included in the Commentary for teachers who wish to use them in class demonstrations.

Before the details of the Proof of Theorem 11-17 can be supplied, you will need two theorems in order to establish the following property for triangles: In $\triangle VBH$ and $\triangle VGH$, if $VB = VG$ and $BH > GH$, then $m \angle BCH > m \angle GVH$. These two theorems will be designated as Theorem 11-17A and Theorem 11-17B. However, many teachers may wish to do only Theorem 11-17B in an informal manner and thus avoid a break in the continuity of the subject matter.

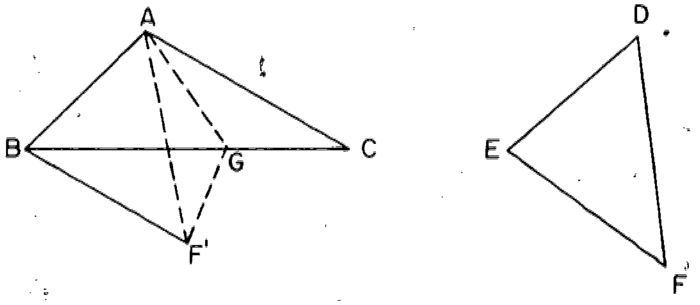
Prisms and pyramids are introduced and problems for finding the area of the lateral surface and the area of the total surface are included. The volume of prisms and pyramids is discussed in the appendix. In working with prisms, pyramids, and frustums, teachers will find that large models similar to those pictured in the text will be useful in demonstrating theorems and explaining problems in the problem set. These models can easily be constructed from various media such as D-stix, balsa wood, pieces of wire, and cardboard.

The answers to the experiment in Section 11-11 are:

1. Perimeter; base; lateral edge; the perimeter of the base; the length of a lateral edge.
2. Yes; the lateral area is the sum of the areas of the parallelograms, each of which has a base equal to a lateral edge and the sum of whose altitudes is RS ; right section.

<u>Theorem</u> <u>11-17A</u>. If two sides of one triangle are equal respectively to two sides of another triangle, but the measures of the included angles are unequal, then the sides opposite the unequal angles are unequal in the same order.

<u>Proof</u>: We are given $\triangle ABC$ and $\triangle DEF$ with $AB = DE$; $AC = DF$; and $m\angle BAC > m\angle D$. We are required to prove $BC > EF$.



Consider $\overrightarrow{AF'}$ so that $m \angle BAF' = m \angle EDF$ and let $AF' = DF = AC$. Then $\triangle BAF' \cong \triangle EDF$ by S.A.S. and $BF' = EF$.

The bisector of $\angle F'AC$ intersects $\overline{BC}$ at G. Then $BG + GC = BC$, and $\triangle AGF' \cong \triangle AGC$ by S.A.S.

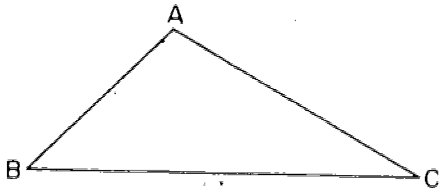
Therefore, $F'G = CG$.

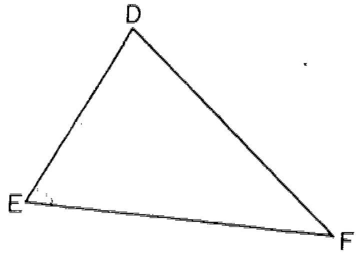
But $BG + GF' > BF'$ by the Triangle Inequality Theorem.

Therefore, $BG + GC > EG$, and $BC > EF$.

Theorem 11-17B. If two sides of one triangle are equal respectively to two sides of another, but the third sides are unequal, then the measures of the angles opposite the unequal sides are unequal in the same order.

Proof: We are given $\triangle ABC$ and $\triangle EDF$ with $AB = DE$; $AC = DF$, and $BC > EF$. We are required to prove that $m \angle A > m \angle D$.





We will use the indirect method to prove $m \angle A > m \angle D$.

The possibilities are:

(1) $m \angle A = m \angle D$; (2) $m \angle A < m \angle D$; (3) $m \angle A > m \angle D$.

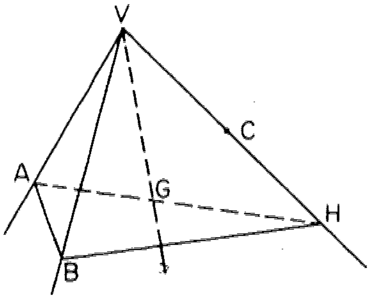
If $m \angle A = m \angle D$, then $\triangle ABC \cong \triangle DEF$, and $BC = EF$. But $BC \neq EF$. Therefore $m \angle A > m \angle D$.

If $m \angle A < m \angle D$, then by the previous theorem $BC < EF$. But BC is not less than EF, and hence $m \angle A$ is not less than $m \angle D$.

Therefore, $m \angle A > m \angle D$.

Theorem 11-17. The sum of the measures of any two face angles of a trihedral angle is greater than the measure of the third face angle.

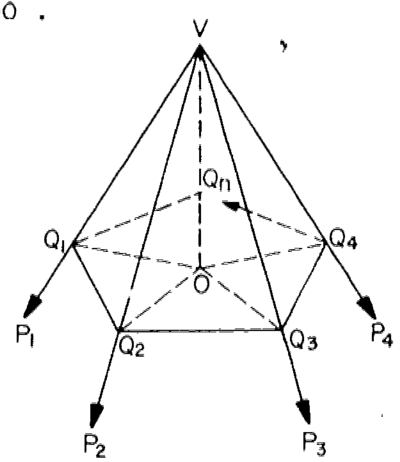
We give only a sketch of the proof of this theorem. Let the given trihedral angle be $\angle V - ABC$. Suppose that $\angle AVC$ has the greatest measure of any of the three face angles. If we can show that $m\angle AVB + m\angle BVC > m\angle AVC$, then the theorem is proved. Why?



In the interior of $\angle AVC$, consider the point G such that $m\angle AVG = m\angle AVB$ and $VG = VB$. Since $\triangle AVG \cong \triangle AVB$, we conclude that $AG = AB$. Let H be the intersection of plane ABG and ray $\overrightarrow{VC}$. Then $BH > GH$. The next step is to show that $m\angle BVH > m\angle GVH$, and finally we conclude that $m\angle AVB + m\angle BVC > m\angle AVC$.

Theorem 11-18. The sum of the measures of the face angles of any convex polyhedral angle is less than 360 .

Proof: We are given polyhedral $\angle V\text{-}P_1P_2P_3 \ldots P_n$. We are required to prove $m\angle P_1VP_2 + m\angle P_2VP_3 + \ldots + m\angle P_nVP_1 < 360$.



Let a plane intersect the faces of the polyhedral angle to form section $Q_1Q_2Q_3 \ldots Q_n$. Take a point O in the interior of $Q_1Q_2Q_3 \ldots Q_n$ and draw segments from O to each of the vertices of the polygon. These segments form with the sides of the polygon n triangles with a common vertex O . We will designate these triangles as the "O triangles." Therefore, the sum of the measures of the angles of the "O triangles" is 180n .

We will designate the triangles with vertex V as the "V triangles." There are n "V triangles." Hence the sum of the measures of the angles of the "V triangles" is 180n .

In trihedral $\angle Q_1 - Q_nVQ_2$, $m \angle Q_nQ_1V + m \angle VQ_1Q_2 > m \angle Q_nQ_1Q_2$

In trihedral $\angle Q_2 - Q_1VQ_3$, $m \angle Q_1Q_2V + m \angle VQ_2Q_3 > m \angle Q_1Q_2Q_3$
etc.

Therefore, the sum of the measures of the base angles of the "V triangles" is greater than the sum of the measures of the base angles of the "O triangles," and the sum of the measures of the vertex angles of the "V triangles" is less than the sum of the measures of the vertex angles of the "O triangles."

But the sum of the measures of the vertex angles of the "O triangles" is 360 .

Therefore, the sum of the measures of the vertex angles of the "V triangles" is less than 360 , or

$$m \angle P_1VP_2 + m \angle P_2VP_3 + \ldots + m \angle P_nVP_1 < 360 .$$

The volume of prisms and pyramids is discussed in the appendix.

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Illustrative Test Items for Chapter 11

A. Measure of the Angles of a Polygon.

Each of the questions or incomplete statements in 1 - 12 is followed by three or four suggested answers. Choose the answer you consider correct.

1. The measure of each interior angle of a regular octagon is: (a) 120 , (b) 108 , (c) 135 , (d) 45 .
2. If the sum of the measures of the interior angles of a polygon is 720 , the number of sides of the polygon is: (a) 8 , (b) 6 , (c) 5 , (d) 4 .
3. If the measure of each interior angle of a polygon is 165 , the number of sides of the polygon is: (a) 10 , (b) 12 , (c) 15 , (d) 24 .
4. If the measure of each exterior angle of a regular polygon is x, the number of sides of the polygon is: (a) $\frac{360}{x}$, (b) $180(x - 2)$, (c) $\frac{180(x - 2)}{x}$, (d) $180 - \frac{360}{x}$.
5. The sum of the measures of the interior angles of a polygon of nine sides is: (a) 1620 , (b) 360 , (c) 1080 , (d) 1260 .
6. If a regular polygon has ten sides, the measure of each exterior angle is: (a) 36 , (b) 144 , (c) 45 , (d) 135 .
7. If the sum of the measures of the interior angles of a polygon is 1620 , the number of sides of the polygon is: (a) 7 , (b) 9 , (c) 11 , (d) 13 .

8. If the sum of the measures of seven angles of an octagon is 980 , the measure of the eighth angle is: (a) 135 , (b) 140 , (c) 100 (d) $122\frac{1}{2}$.

9. Consider a set of polygons of n sides. As n is replaced by greater integers, the sum of the measures of the interior angles: (a) increases, (b) decreases, (c) remains the same.

10. Consider a set of polygons of n sides. As n is replaced by greater integers, the sum of the measures of the exterior angles: (a) increases, (b) decreases, (c) remains the same

11. Consider a set of regular polygons of n sides. As n is replaced by greater integers, the measure of each exterior angle: (a) increases, (b) decreases, (c) remains the same.

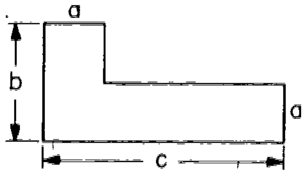
12. Consider a set of regular polygons of n sides. As n is replaced by greater integers, the measure of each interior angle: (a) increases, (b) decreases, (c) remains the same.

B. Area Formulas.

1. The perimeter of a square is 20 . Find its area.

2. The area of a square is n . Find its side.

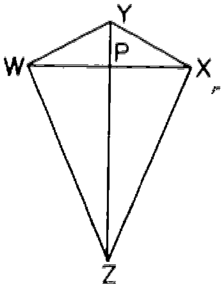
3. Find the area of the figure in terms of the lengths indicated.



4. The base of a rectangle is three times as long as the altitude. The area is 147 square inches. Find the base and the altitude.

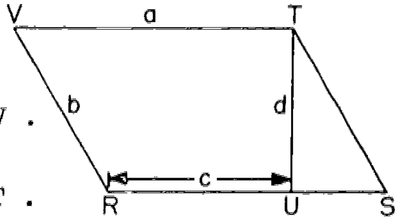
5. The area of a triangle is 72 . If one side is 12 , what is the altitude to that side?

6. In the figure WY = XY and WZ = XZ . WX = 8 and YZ = 12 . Find the area of WZXY .

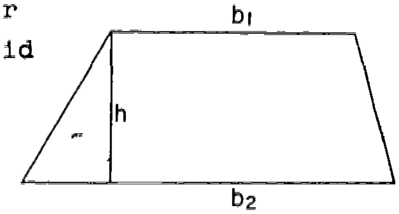


7. RSTV is a parallelogram. If the lower case letters in the drawing represent lengths, give the area of:

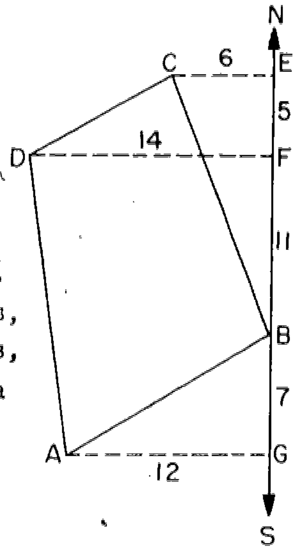
(a) Parallelogram RSTV .
(b) $\triangle STU$.
(c) Quadrilateral VRUT .



8. Show how a formula for the area of a trapezoid may be obtained from the formula $A = \frac{1}{2}bh$ for the area of a triangle.



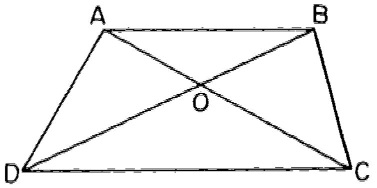
9. In surveying field ABCD shown here a surveyor laid off north and south line $\overleftrightarrow{NS}$ through B and then located the east and west lines $\overleftrightarrow{CE}$, $\overleftrightarrow{DF}$ and $\overleftrightarrow{AG}$. He found that CE = 6 rods, DF = 14 rods, AG = 12 rods, BG = 7 rods, BF = 11 rods, FE = 5 rods. Find the area of the field.



C. Comparison of Areas.

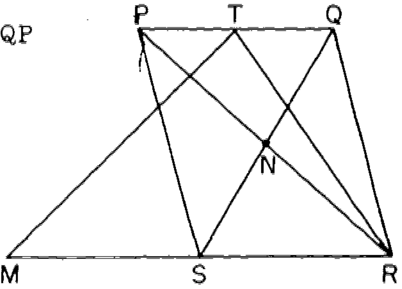
1. Given: ABCD is a trapezoid. Diagonals $\overline{AC}$ and $\overline{BD}$ intersect at O .

   Prove: Area $\triangle AOD$ = Area $\triangle BOC$.



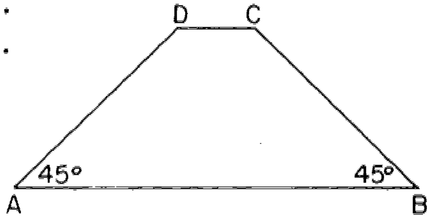
2. In this figure PQRS is a parallelogram with PT = TQ and MS = SR . In (a) through (e) below compare the areas of the two figures listed.

   (a) Parallelogram SRQP and $\triangle SQR$.
   (b) Parallelogram SRQP and $\triangle MTR$.
   (c) $\triangle PNS$ and $\triangle MTR$.
   (d) $\triangle SQR$ and $\triangle SPR$.
   (e) $\triangle MTR$ and $\triangle RQT$.

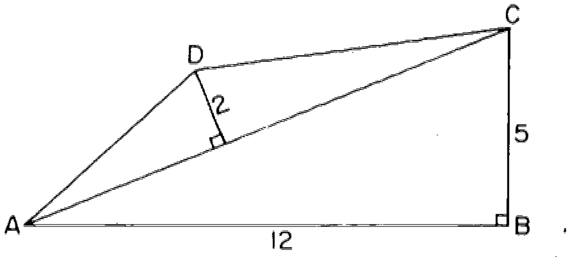


D. Miscellaneous Problems.

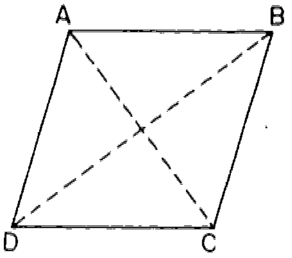
1. ABCD is a trapezoid. CD = 1 and AB = 5 . What is the area of the trapezoid?



2. What is the area of ABCD ?

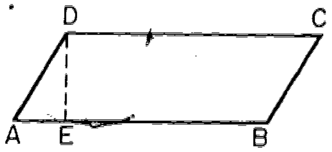


3. ABCD is a rhombus with AC = 24 and AB = 32 .
   - (a) Compute its area.
   - (b) Compute the length of the altitude to $\overline{DC}$.



4. Find the area of a triangle whose sides are 9" , 12" , and 15" .

5. ABCD is a parallelogram with altitude $\overline{DE}$. Find the area of the parallelogram if:
   - (a) DE = $2\frac{1}{2}$ and AB = $6\frac{1}{3}$.
   - (b) AB = 10 , AD = 4 , and m $\angle A$ = 30 .



6. Find the area of an isosceles triangle which has congruent sides of length 8 and base angles of 30° .

E. Coordinates.

1. The coordinates of the vertices of a triangle are: A(-2,-3) , B(4,5) , and C(-4,1) . Find the area of the triangle.

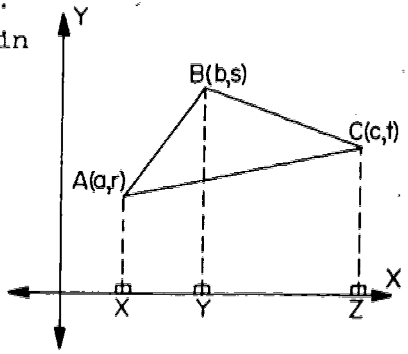
2. The coordinates of the vertices of a quadrilateral are: A(3,2) , B(0,6) , C(-3,2) and D(0,-2) .
   - (a) What is the name of the quadrilateral? Explain.
   - (b) Find the area of the quadrilateral.

3. The coordinates of the vertices of $\triangle RST$ are (-5,1) , (-3,-3) , and (4,6) . Find the area of $\triangle RST$. Hint: The area can be found by subtracting areas of right triangles from the area of a rectangle.

4. Prove the area of $\triangle ABC$ is

$$\frac{a(t - s) + b(r - t) + c(s - r)}{2}$$

where $A = (a,r)$, $B = (b,s)$, and $C = (c,t)$. <u>Hint</u>: Find three trapezoids in the figure.



F. <u>Area Relations</u>.

1. Two similar polygons have corresponding sides of lengths 5 and 9 . The area of the larger is 567 . What is the area of the smaller?

2. If the ratio of the bases of two parallelograms is 2:3 , and the ratio of the corresponding altitudes is 3:2 , the ratio of the areas is ________ .

3. Two triangles have equal areas. If the ratio of the bases is 2 and 3 , then the ratio of the corresponding altitudes is ________ .

4. If the side of one square is double the side of a given square, the area of the square is ________ the area of the given square.

5. If the side of one square is double the side of a given square, the perimeter of the square is ________ the perimeter of the given square.

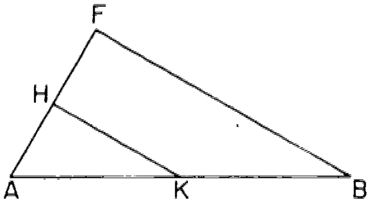
6. Two triangles have equal bases. If the ratio of the altitudes is 2:3 , then the ratio of the areas is ________ .

7. If the area of a square is double the area of a given square, then each side of the square is ________ a side of the given square.

8. What is the ratio of the areas of two similar triangles whose bases are 3 inches and 4 inches? x inches and y inches?

9. A side of one of two similar triangles is 5 times the corresponding side of the other. If the area of the first is 6 , what is the area of the second?

10. In the figure if H is the midpoint of $\overline{AF}$ and K is the midpoint of $\overline{AB}$, the area of $\triangle ABF$ is how many times as great as the area of $\triangle AKH$? If the area of $\triangle ABF$ is 15 , find the area of $\triangle AKH$.



11. The area of the larger of two similar triangles is 9 times the area of the smaller. A side of the larger is how many times the corresponding side of the smaller?

12. The areas of two similar triangles are 225 sq. in. and 36 sq. in. Find the base of the smaller if the base of the larger is 20 inches.

G. <u>Regular Polygons</u>.

1. Find the area of a regular polygon if the perimeter of the polygon is 36 inches and the apothem is $3\sqrt{3}$ inches.

2. The apothems of two equilateral triangles are 3 and 7 . What is the ratio of the sides? the perimeters? the areas?

3. Find the area of a regular hexagon if the radius of the hexagon is 10 .

8. What is the ratio of the areas of two similar triangles whose bases are 3 inches and 4 inches? x inches and y inches?

9. A side of one of two similar triangles is 5 times the corresponding side of the other. If the area of the first is 6 , what is the area of the second?

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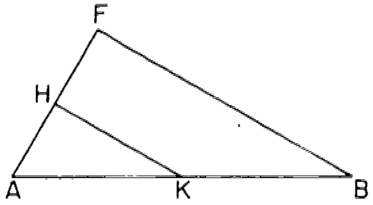

11. The area of the larger of two similar triangles is 9 times the area of the smaller. A side of the larger is how many times the corresponding side of the smaller?

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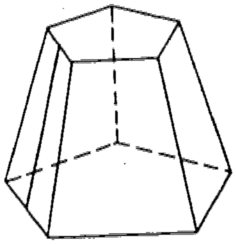
2. The apothems of two equilateral triangles are 3 and 7 . What is the ratio of the sides? the perimeters? the areas?

3. Find the area of a regular hexagon if the radius of the hexagon is 10 .

7. Find the area of the lateral surface of the frustum of a regular pentagonal pyramid. Each edge of the upper base is 12 and each edge of the lower base is 14 . The altitude of one of the faces of the frustum is 15 . Find the area of the lateral surface.



8. The edges of one cube are double those of another.

   (a) What is the ratio of the sums of their edges?

   (b) What is the ratio of their total surface areas?

Chapter 12

CIRCLES AND SPHERES

The depth of treatment of the material of this chapter must depend upon factors which include the caliber of the class and the time remaining in the school year. Generally speaking, there are three levels of treatment: (a) the minimal, which strives only for basic appreciation of the relations between angles and arcs; of tangents, chords, and secants; and of the measures of sector area and arc length; (b) the average treatment, which adds to the minimal some deeper analysis of the above relations by use of coordinates; and (c) the thorough treatment, which involves considering every problem in the chapter. The minimal treatment is recommended only where time permits nothing more; the thorough treatment is strongly recommended for high-ability classes.

813 Observe that in the proof of Theorem 12-1 we do not assert that $\{(x,y,z): x^2 + y^2 + z^2 = r^2 \text{ and } z = 0\}$ is the same set of points as $\{(x,y): x^2 + y^2 = r^2\}$. To make such an assertion would be to say that a set of ordered triples of numbers is a set of ordered pairs of numbers! It would be correct to say that the following two sets are equal:

$$\{(x,y,z): x^2 + y^2 + z^2 = r^2 \text{ and } z = 0\}$$

$$\{(x,y,0): x^2 + y^2 = r^2\}$$

and this is, in fact, what we asserted when we "recognized" $\{(x,y,0): x^2 + y^2 = r^2\}$ to be the set of points <u>in the xy-plane</u> given by $\{(x,y): x^2 + y^2 = r^2\}$.

815 The equations of circle and sphere developed in the text keep the centers at the origin. In the problem set that follows, notably in Problems 4, 5, 18, equations of circles or spheres whose centers are not at the origin are introduced. This is not a difficult concept and should be part of the average treatment.

822 In the minimal treatment, Theorem 12-4 and its corollaries may be asserted without proof. In the average treatment, the teacher should lead the class through the proof of one of the cases of the theorem. All classes should understand the proof of Theorem 12-5.

823 Note in the proof of Theorem 12-4 the assertion $\ell = \{(a,y)\}$. We ask that, in this chapter, the curly brackets be taken to signify sets of points rather than sets of ordered pairs (triples) of numbers. In other words, we ask that $\{(a,y)\}$, $\{(x,y): x = a\}$, $\{(a,y):\ y \text{ real}\}$ and $\{(x,y): x = a ,\ y \text{ real}\}$ all be read alike, as the set of *all points whose coordinates are* ordered pairs of real numbers such that the first number is a and the second is not restricted.

827 Note that our definition of tangent circles requires them to be coplanar. Noncoplanar circles can, of course, intersect in a single point. However, we have chosen not to apply the word "tangent" in such cases. In other texts, tangent circles may be defined in such a way as not to require them to be coplanar.

Some teachers may search the text in vain for a method of "constructing" a tangent to a given circle from a given point outside the circle. It may be useful and appropriate at some point to demonstrate, not how to "construct" the tangent, but how to find the coordinates of the point of tangency. A sample of such a demonstration is here indicated: Choose a coordinate system such that O , the center of the given circle has coordinates (0,0) and A , the given point, has coordinates (a,0) . Then, P , the point of tangency must have coordinates (x,y) such that $\frac{y}{x} \cdot \frac{y}{x - a} = -1$ and $x^2 + y^2 = r^2$. That is, $m_{\overline{OP}} \cdot m_{\overline{AP}} = -1$; and P is on the given circle. These yield

$$\left(\frac{r^2}{a^2} , \frac{r}{a}\sqrt{a^2 - r^2}\right) \text{ and } \left(\frac{r^2}{a^2} , -\frac{r}{a}\sqrt{a^2 - r^2}\right)$$

as coordinates of P .

835 Only high ability classes should be required to master the proofs of Theorem 12-6 and its corollaries, although all students should understand the statements.

836 Note the symbolic expression $\{(x,y,a): x^2 + y^2 = 0\}$. Students may need help initially in translating such expressions. The intended meaning is: The set of all points whose coordinates are ordered triples of real numbers, the first two of which are the ordered pair (x,y) such that $x^2 + y^2 = 0$, and the third is the number a. Clearly this is the set whose only element is the point with coordinates $(0,0,a)$.

839 Note that, whereas the (degree) measure of an angle is, by definition, restricted to positive numbers less than 180, the degree measure of an arc may be 180 or greater. The degree measure of an arc is a positive number less than 360.

840 The symbols $m\,\widehat{AXB}$ denote the "degree measure" of $\widehat{AXB}$, and we read it "measure" of $\widehat{AXB}$, for brevity. In Section 12-8, we consider the length, another kind of measure, of an arc. We distinguish length from degree measure by using $\ell\,\widehat{AXB}$ to denote length of $\widehat{AXB}$.

845 The proof of Theorem 12-7 should be required of average high-ability classes.

846 A proof, using coordinates, of Corollary 12-7-1 is indicated in Problem 15 of the next problem set.

848 The meanings of the terms "secant ray," "tangent ray," and "chord ray," used in Problem 2, are obvious.

Some bright student may observe the general relationship between the measure of angles and the measures of arcs they intercept. If no student makes the discovery on his own, the class should be led to it. The measure of the angle is half of either the sum or the difference of the measures of the intercepted arcs depending upon the location of the vertex. If the vertex is inside the circle (special case, at the center) the measure of the angle is half the sum; if outside, half the difference; if on the circle, it doesn't matter, for one "intercepted arc" has zero measure.

861 It may be advisable to introduce Section 12-5 with an informal, intuitive discussion of "segments joining a point to a circle." Problem 5 of Problem Set 1-4 should be reviewed. A well-drawn circle on the blackboard, a meter stick, and a selection of situations, each analyzed numerically, should lead the students to generalize somewhat as follows: the point can be inside, on, or outside the circle; for a given point, the product of the lengths of the two segments joining it to the circle remains constant, regardless of the slope of the line containing the segments. Of course, if the point is on the circle, one segment has zero length; and if the line is tangent to the circle, the "two segments" have the same length; but the generalization still holds.

Teachers wishing to give a test at this point will find a number of suitable questions in the Illustrative Test Items for the chapter.

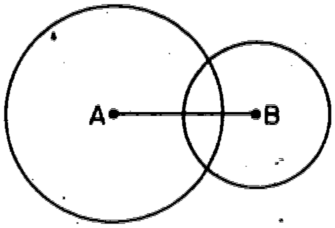
The following section is devoted to a discussion of the constructibility of a triangle whose sides are to be congruent respectively to three given line segments. It is presented as an example of a method one might use to investigate such problems when the only available drawing instruments are straightedge and compass.

Suppose we wish to make a copy of a certain triangle. One way of doing it would be to follow these steps.

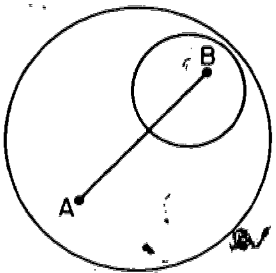
(1) Measure the sides of the triangle you wish to copy.

(2) Draw a segment $\overline{AB}$ whose length is one of the lengths you found in Step 1. A•————————•B

(3) With a compass, draw a circle with center at A , whose radius is another of the lengths you found in Step 1, and draw a circle with center at B whose radius is the third of the lengths you found in Step 1. Your diagram should now look like this.



Then, if C and C' are the intersection points of your circles, each of triangles ABC and ABC' is congruent to the original triangle (by S.S.S.), and therefore a copy of it. This method of construction guarantees that all the triangles it produces are copies of the original one. Does it necessarily produce any triangles? Could the construction lead to a diagram like this?



It is certainly possible to draw two circles, such as those of the last diagram, which have no common points. Our question is whether it is possible to draw non-intersecting circles with centers A and B , if it [illegible] given that the radii of the circles are the lengths of two sides of a triangle whose third side has length AB . A theorem asserts that this is not possible.

<u>Theorem</u>. Let a, b, c be positive numbers for which

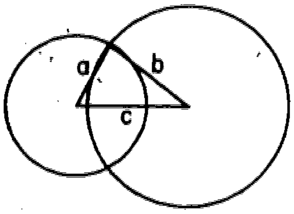
$$a + b > c ,$$
$$b + c > a ,$$
$$c + a > b .$$

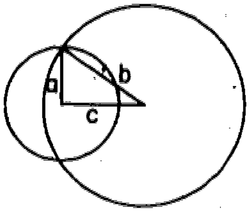
Then

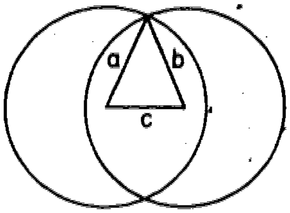
I. There is a triangle whose sides have lengths a, b, and c .

II. If the distance between the centers of two coplaner circles is c and if the circles have radii a and b , then the circles intersect in two points, one on each side of the line of centers.

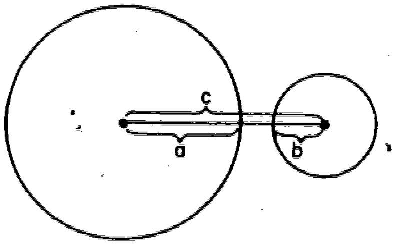
<u>Proof</u>: Here are some situations in which the inequalities stated in the theorem are all satisfied:

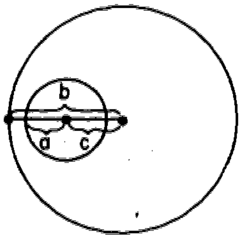


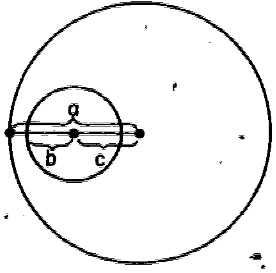




Here are some situations in which the inequalities stated in the theorem are not all satisfied. It appears that the circles do not intersect and that no triangle is formed.







We prove Parts I and II together, using coordinates. We consider the points A(0,0) and B(0,c) , and try to find the coordinates (x,y) of a point C where the circle with center A and radius b intersects the circle with center B and radius a . By finding such a C we show that there is a triangle ABC whose sides have lengths a, b, and c . It will turn out that there are two such points, one on each side of the x-axis. We use the distance formula to express the conditions that AB = b and BC = a .

We have

$$x^2 + y^2 = b^2$$

and

$$(x - c)^2 + y^2 = a^2 .$$

We try to find values of x and y which satisfy both these equations. We rewrite the second as

$$x^2 - 2cx + c^2 + y^2 = a^2$$

and then subtract the first, obtaining

$$-2cx + c^2 = a^2 - b^2 .$$

This shows that the only possible value for x is

$$\frac{c^2 + b^2 - a^2}{2c} .$$

We return to the first equation with this value for x and try to find y . We have

$$\left(\frac{c^2 + b^2 - a^2}{2c}\right)^2 + y^2 = b^2$$

or

$$y^2 = b^2 - \left(\frac{c^2 + b^2 - a^2}{2c}\right)^2 .$$

We can solve this equation for y if and only if the right side is not negative. Let us try therefore to show that

$$b^2 - \left(\frac{c^2 + b^2 - a^2}{2c}\right)^2$$

is not negative. By means of algebraic manipulation it can be derived that this expression equals

$$\frac{(a + b + c)(a + b - c)(b + c - a)(c + a - b)}{4c^2} .$$

That this latter expression is positive follows from the facts that

$$a + b + c > 0$$
$$a + b - c > 0$$
$$b + c - a > 0$$
$$c + a - b > 0$$
$$4c^2 > 0$$

Let us call this expression u . Now that we know that u is positive, we know that the symbol $\sqrt{u}$ is a meaningful one, and that the possible values for y are $\sqrt{u}$ and $-\sqrt{u}$. We conclude that there are two such points (x,y) . They are

$$\left(\frac{c^2 + b^2 - a^2}{2c} , \sqrt{u}\right)$$

and

$$\left(\frac{c^2 + b^2 - a^2}{2c} , -\sqrt{u}\right) .$$

Notice that $\sqrt{u}$ is positive, $-\sqrt{u}$ is negative, so that these points lie on opposite sides of the x-axis, which is the line joining the centers of the two circles.

Illustrative Test Items for Chapter 12

Part A. Indicate by "+" if each of the following is always true, by "0" if it is possibly false.

1. If two chords of a circle bisect each other, both are diameters of the circle.

2. If one chord contains one endpoint of a diameter and another chord contains the other endpoint of the diameter, the two chords are equidistant from the center of the circle.

3. If a tangent and a chord intersect at a point of the circle, and if the measure of the tangent chord-angle is 30 , the length of the chord is equal to the length of the radius of the circle.

4. If two chords of a circle are perpendicular to each other then at least one of the chords is a diameter.

5. If two chords of equal length, but not diameters, intersect in the interior of the circle, the quadrilateral whose vertices are the endpoints of the chords is an isosceles trapezoid.

6. If two chords of equal length intersect in the interior of a circle, the radius containing the point of intersection bisects one of the pairs of vertical angles containing the two chords.

7. A diameter which bisects a chord is perpendicular to the chord.

8. If a chord intersects a tangent at the point of contact, the chord is a diameter.

9. If a rhombus is inscribed in a circle, the rhombus is a square.

10. If a parallelogram is circumscribed about a circle, then it is a square.

11. The set of centers of all possible circles tangent to a given line at a given point is contained in the line perpendicular to the given line at the given point.

12. A trapezoid inscribed in a circle is isosceles.

13. If a pair of opposite angles of a quadrilateral are supplementary, a circle exists which contains all four of the vertices of the quadrilateral.

14. If a tangent to a circle contains a vertex of an inscribed triangle, at least one of the tangent-chord angles is congruent to one of the angles of the triangle.

15. The measure of an inscribed angle is equal to one-half of the degree measure of the arc in which it is inscribed.

16. If one side of an inscribed triangle is a diameter of the circumscribing circle, two of the angles of the triangle are complementary.

17. If an inscribed angle contains two chords of equal length, its midray contains the center of the circle.

18. An angle inscribed in a major arc is obtuse.

19. If a circle is circumscribed about a regular hexagon, the radius of the circle is congruent to a side of the hexagon.

20. An inscribed angle that intercepts a minor arc is acute.

21. If two chords intersect within a circle forming pairs of non-adjacent angles, and if the non-adjacent arcs intercepted by these angles are congruent, then the chords are diameters of the circle.

22. If a right triangle is inscribed in a circle, its hypotenuse is the diameter of the circle.

23. The quotient of the circumference divided by the radius is the same number for all circles.

24. If two regular polygons are inscribed in a circle, the one with the greater number of sides has an apothem which is more nearly equal to the radius of the circumscribing circle.

25. If the radius of one circle is three times that of a second circle, the circumference of the first is three times that of the second.

26. The area of a square inscribed in a given circle is half the area of one circumscribed about the circle.

27. In a given circle, the areas of two sectors are proportional to the degree measures of their arcs.

28. The quotient of the area of a circle divided by the square of its radius is π .

29. The length of an arc of a circle can be obtained by dividing its degree measure by π .

30. The areas of two circles are proportional to their respective circumferences.

Part B.

1. Chords $\overline{CD}$ and $\overline{BA}$ intersect at P . The degree measures of nonadjacent arcs $\overset{\frown}{AD}$ and $\overset{\frown}{BC}$, respectively, are 32 and 40 . What is the measure of $\angle APD$?

2. Chords $\overline{AB}$ and $\overline{CD}$ are perpendicular. The degree measures of adjacent arcs $\overset{\frown}{BD}$ and $\overset{\frown}{DA}$, respectively are 50 and 40 . What are the degree measures of $\overset{\frown}{AC}$ and $\overset{\frown}{CB}$?

3. Chords $\overline{AC}$ and $\overline{BD}$ are equal in length and they intersect at P . The degree measures of adjacent arcs $\widehat{BC}$ and $\widehat{CD}$, respectively, are 50 and 80 . What are the measures of $\angle CPD$, $\angle ADC$, and $\angle DCB$?

4. Parallel chords $\overline{AE}$ and $\overline{BD}$ intersect chord $\overline{AC}$ in two points A and P , respectively. m $\widehat{CD} = \frac{1}{3}$ m $\widehat{AB}$. m $\widehat{DE} = 84$. D and E are on the same side of $\overleftrightarrow{AC}$. What is the measure of $\angle CAE$?

5. Find the measure of an interior angle of a regular nine-sided polygon.

6. Into how many triangular regions would a convex polygonal region, with a polygon of 100 sides as the boundary, be separated by all possible diagonals which connect a given vertex of the boundary with other vertices of the boundary?

7. If the circumference of a circle is a number C such that $16 < C < 24$, and the radius of the circle is an integer, find the radius.

8. If the number of sides of a regular polygon inscribed in a given circle is increased indefinitely, what is the limit of the length of one side? Of its perimeter?

9. Write a formula for the area of a circle in terms of its circumference instead of in terms of its radius.

10. The area of a circle is 2π ; what is its radius?

11. If the areas of two circles have the ratio $\frac{1}{100}$, what is the ratio of their diameters?

12. Two sectors of a circle are such that the measures of their angles are 50 , 100 respectively. What is the ratio of the lengths of their respective arcs? What is the ratio of their respective areas?

13. A circular lake is approximately 2 miles in diameter. About how many hours will it take to walk around it if you walk at 3 miles per hour? (Give the answer to the nearest whole number.)

14. What is the least possible value of the difference between the area of a semicircular region and the area of a triangle inscribed in the semicircle, if the radius of the semicircle is 6 ?

15. Sphere $S = \{(x,y,z): x^2 + y^2 + z^2 = 36\}$. Tell whether each of the following points is on the sphere, in its interior, or in its exterior.

(a) $(-6,0,0)$
(b) $(-6,1,0)$
(c) $(-6,-1,-1)$
(d) $(5,2,2)$
(e) $(-\sqrt{27},-3,0)$
(f) $(5,5,-5)$
(g) $(4,-4,2)$
(h) $(3,1,5)$

16. Chord $\overline{AD}$ intersects diameter $\overline{AC}$ at A . AC = 50 . AD = 30 . What is the distance from the center to $\overline{AD}$?

17. In a circle whose center is O , the chord $\overline{XY}$ is the perpendicular bisector of radius $\overline{OA}$. OA = 6 . Find m $\widehat{XAY}$, $\ell\,\widehat{XAY}$, the area of sector XOY , and the area of the segment of the circle bounded by $\overline{XY}$ and $\widehat{XAY}$.

18. A regular hexagon is circumscribed about a circle. The perimeter of the hexagon is 12 . Find the circumference and the area of the circle.

19. On an aerial photograph the surface of a reservoir appears as a circular-region of diameter 7/8 inch. If the scale of the photograph is 2 miles to 1 inch, find the approximate (nearest one-half square mile) area of the surface of the reservoir.

20. Circle $C = \{(x,y): x^2 + y^2 = 25\}$. Find the slope of each of the two chords whose endpoints are, respectively, the point $P(3,4)$ and an endpoint of the diameter in the x-axis. Find the slope of each of the two chords whose endpoints are, respectively, P and an endpoint of the diameter in the y-axis.

21. If a plane is 8 inches from the center of a sphere whose radius is 17 inches, what is the length of the radius of the circle which is the intersection of the plane and the sphere? What is the ratio of the area of this circle to the area of a great circle of the sphere?

22. $\overleftrightarrow{CD}$ is tangent at C to the circle whose center is B . $\overline{AB}$ is perpendicular to the plane of the circle. $BC = 6$. $AB = 8$. $CD = 24$. Find AD .

23. A continuous belt runs around two wheels of radius $\sqrt{2}$ and $9\sqrt{2}$ feet, respectively. The centers of the wheels are 16 feet apart. Find the approximate length of the belt (to the nearest foot). ($\sqrt{2}$ is approximately 1.414 ; π is approximately 3.142 .)

24. Circle $C = \{(x,y): x^2 + y^2 = 25\}$ and line $\ell = \{(x,y): x + y = 5\}$. Find the length of the chord of C which is contained in ℓ .

25. Circle $C = \{(x,y): x^2 + y^2 = 5\}$ and line $t = \{(x,y): 2x + y = 5\}$. Find the coordinates of a point of intersection of C and t . How many such points of intersection exist? What is the relation between t and C ?

26. Circle $C = \{(x,y): x^2 + y^2 = 10\}$ and line $\ell = \{(x,y): x + 2y = 5\}$:

(a) Find the coordinates of the points of intersection of C and ℓ.

(b) Find the midpoint of the chord of C contained in ℓ.

(c) Find the slope of this chord.

(d) Write an equation of the line containing the midpoint of the chord and the center of the circle.

(e) Find the distance from the chord to the center of the circle.

27. Circle $C = \{(x,y): x^2 + y^2 = 4\}$.

(a) What is the x-coordinate of each point of C whose y-coordinate is 1 ?

(b) Does the point $T (\sqrt{2}, \sqrt{2})$ lie on the circle?

(c) Does the point $S(2,3)$ lie on the circle?

28. Find the coordinates of the points of intersection (if any exist) of the circle $C = \{(x,y): x^2 + y^2 = 25\}$ and each of the following sets of points:

(a) $A = \{(x,y): y = -4\}$

(b) $B = \{(x,y): y - x = 7\}$

(c) $C = \{(x,y): x = 2 + k , y = 9 + k , k \text{ real}\}$

(d) $D = \{(x,y): x^2 + y^2 = 9\}$

LINEAR AND PARAMETRIC EQUATIONS

In this Talk we investigate further the equations of lines and planes discussed in Chapters 8 and 9.

1. Lines in the xy-plane.

Consider first a line ℓ in the xy-plane which is not parallel to the y-axis. Then it has a slope m and if (x_1,y_1) is any point on it we may write:

$$\ell = \{(x,y)\colon y - y_1 = m(x - x_1)\} ,$$

in which the equation has the familiar point-slope form. Using some elementary algebra we get:

$$\ell = \{(x,y)\colon y - y_1 = mx - mx_1\} ,$$

$$\ell = \{(x,y)\colon mx - y + (-mx_1 - y_1) = 0\} ,$$

and if we set $a = m$, $b = -1$, $c = -mx_1 - y_1$, then

$$\ell = \{(x,y)\colon ax + by + c = 0\}$$

in which the equation has the form of the general first degree equation in x and y . An equation of the form $ax + by + c = 0$ is a first degree equation in x and y if a, b, c are real numbers and a and b are not both zero.

Consider next a vertical line v in the xy-plane. Then v does not have a slope and if (x_1,y_1) is any point on it, we have

$$v = \{(x,y)\colon x = x_1\} .$$

If we set $a = 1$, $b = 0$, $c = x_1$, then

$$v = \{(x,y)\colon ax + by + c = 0\} .$$

Hence a vertical line has an equation which is a special case of the general first degree equation.

2. General First Degree Equations in x and y.

In Section 1 we observed that every line ℓ in the xy-plane can be represented by an equation of the form $ax + by + c = 0$ in which a and b are not both zero. The representation is in the following sense:

Given a line ℓ, there exist real numbers a, b, c with a and b not both 0, such that

$$\ell = \{(x,y): ax + by + c = 0\} .$$

In this section we consider the question: Does every first degree equation $ax + by + c = 0$ represent a line?

Note that

$$\{(x,y): 0x + 0y + 0 = 0\}$$

is the entire xy-plane and that

$$\{(x,y): 0x + 0y + 1 = 0\}$$

is the null set. This shows that there are at least two equations of the form $ax + by + c = 0$ which are not equations of lines. Indeed,

$$\{(x,y): 0x + 0y + c = 0\}$$

is either the null set (if $c \neq 0$) or the entire xy-plane (if $c = 0$). But note also that $0x + 0y + c = 0$ is not a first degree equation.

Consider now any general first degree equation. To be definite, suppose we are given three real numbers, a, b, c with a and b not both 0, and that S is the following set:

$$S = \{(x,y): ax + by + c = 0\} .$$

We wish to show that the set S is a line.

There are two cases to consider: either $b = 0$ or $b \neq 0$.

If $b = 0$, then $a \neq 0$, and

$$S = \{(x,y): x = -\frac{c}{a}\} .$$

But if ℓ is the vertical line through $(-\frac{c}{a},0)$ we know that

$$\ell = \{(x,y): x = -\frac{c}{a}\} .$$

It follows that $S = \ell$ and hence that S is a line.

If $b \neq 0$, then

$$S = \{(x,y): y = -\frac{a}{b}x - \frac{c}{b}\} .$$

Let p be the line with slope $-\frac{a}{b}$ which contains the point $(0,-\frac{c}{b})$. Then

$$p = \{(x,y): y + \frac{c}{b} = -\frac{a}{b}(x - 0)\} .$$

It follows that $S = p$ and that S is a line.

This shows that every equation $ax + by + c = 0$ in which a and b are not both 0 is the equation of a line in the xy-plane. Summarizing Sections 1 and 2, we see that every equation $ax + by + c = 0$, with a and b not both 0, is the equation of a line in the xy-plane, and conversely, that every line in the xy-plane can be represented by an equation $ax + by + c = 0$ in which a and b are not both 0. Of course, this is why $ax + by + c = 0$ is called a linear equation.

3. Parametric versus Linear Form.

In Section 1 we derived the general linear equation starting from the point-slope form. In this and the next sections we show two other derivations of the general linear equation, one using parametric equations and one using the Pythagorean Theorem.

Let (x_1,y_1) and (x_2,y_2) be two distinct points and ℓ the line which contains them. Then

$$\ell = \{(x,y): x = x_1 + k(x_2 - x_1),\ y = y_1 + k(y_2 - y_1),\ k \text{ is real}\}.$$

We consider two cases: ℓ is vertical, or it isn't. If ℓ is vertical, then $x_2 = x_1$, $y_2 \neq y_1$, and

$$\ell = \{(x,y): x = x_1, \; y = y_1 + k(y_2 - y_1), \; k \text{ is real}\} .$$

We know from our work in Chapter 3 that the set of all real numbers y such that $y = y_1 + k(y_2 - y_1)$ for some real number k is the set of all real numbers. Therefore the last two conditions on x and y in the set-builder symbol for ℓ, are equivalent to the condition that y be real. Therefore

$$\ell = \{(x,y): x = x_1\} ,$$

and

$$\ell = \{(x,y): ax + by + c = 0\}$$

in which $a = 1$, $b = 0$, $c = - x_1$ (and hence a is not 0) .

If ℓ is not vertical, then $x_2 \neq x_1$ and

$$\ell = \left\{(x,y): k = \frac{x - x_1}{x_2 - x_1} , \; y = y_1 + k(y_2 - y_1) , \; k \text{ is real}\right\},$$

and

$$\ell = \left\{(x,y): k = \frac{x - x_1}{x_2 - x_1}, \; y = y_1 + \frac{x - x_1}{x_2 - x_1}(y_2 - y_1), \; k \text{ is real}\right\}.$$

Since every real number x can be obtained from some real number k by using the formula $k = \dfrac{x - x_1}{x_2 - x_1}$, the first and third conditions in the set-builder symbol above are equivalent to the condition that x be real. Therefore

$$\ell = \left\{(x,y): y = y_1 + \frac{x - x_1}{x_2 - x_1}(y_2 - y_1) \right\} ,$$

$$\ell = \{(x,y): (y_2 - y_1)x + (x_1 - x_2)y + (x_2y_1 - x_1y_2) = 0\} ,$$

and

$$\ell = \{(x,y): ax + by + c = 0\}$$

in which $a = y_2 - y_1$, $b = x_1 - x_2$, and $c = x_2y_1 - x_1y_2$ (and hence b is not zero.)

This shows that if we start with any line in the xy-plane and accept the fact that it can be represented parametrically as in Chapter 8, then it has a first degree equation $ax + by + c = 0$.

Suppose, now, that we start with a general first degree equation. Can we get parametric equations for a line from it? Let a, b, c be real numbers with a and b not both 0 and let S be the following set.

$$S = \{(x,y): ax + by + c = 0\} .$$

Then either $b = 0$ or $b \neq 0$. If $b = 0$, then $a \neq 0$, $(-\frac{c}{a},0) \in S$ and $(-\frac{c}{a},1) \in S$. Let ℓ be the line:

$$\ell = \{(x,y): x = -\frac{c}{a} + k(-\frac{c}{a} + \frac{c}{a}),\ y = 0 + k(1 - 0),\ k \text{ is real}\}.$$

Then

$$\ell = \{(x,y): x = -\frac{c}{a}\} = \{(x,y): ax + c = 0\} = S .$$

If, on the other hand, $b \neq 0$, then $(0,-\frac{c}{b}) \in S$ and $(1, \frac{-a-c}{b}) \in S$. Let q be the line:

$$q = \{(x,y): x = 0 + k(1 - 0),\ y = -\frac{c}{b} + k(\frac{-a}{b}),\ k \text{ is real}\} .$$

Then

$$q = \{(x,y): y = \frac{-c}{b} + x(\frac{-a}{b})\}$$

$$q = \{(x,y): ax + by + c = 0\} ,$$

and $q = S$. This shows that if we accept the parametric equations for a line in the xy-plane and if $ax + by + c = 0$ is any first degree equation, then there are two distinct points (x_1,y_1) and (x_2,y_2) such that the set

$$S = \{(x,y): ax + by + c = 0\}$$

is the same as the set

$$\ell = \{(x,y): x = x_1 + k(x_2 - x_1),\ y = y_1 + k(y_2 - y_1), k \text{ is real}\}.$$

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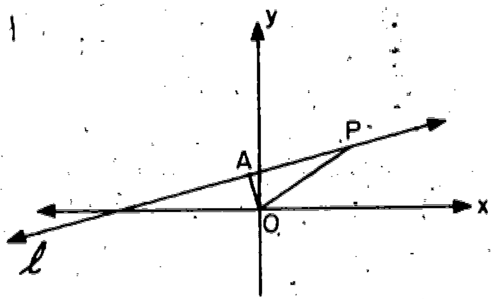
4. Derivation of the Linear Equation Using the Pythagorean Theorem.

Let ℓ be any line in the xy-plane which does not contain $(0,0)$. Let $A(a,b)$ be the foot of the perpendicular from O to ℓ. Note, since ℓ does not contain O, that a and b are not both zero. Then $P(x,y)$ is a point on ℓ if and only if



$$(OP)^2 = (OA)^2 + (AP)^2,$$

$$x^2 + y^2 = a^2 + b^2 + (x - a)^2 + (y - b)^2,$$

$$0 = 2a^2 + 2b^2 - 2ax - 2bx,$$

$$ax + by = a^2 + b^2.$$

Therefore

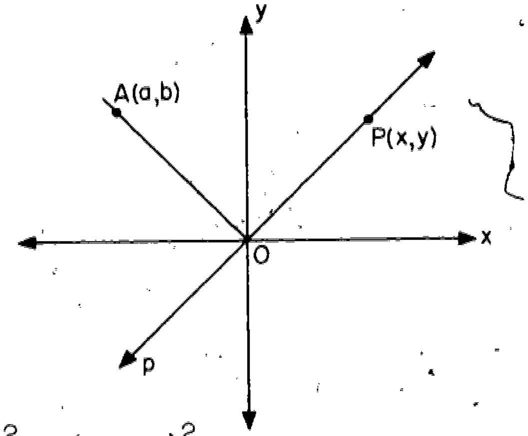
$$\ell = \{(x,y): ax + by = a^2 + b^2\}.$$

Thus

$$\ell = \{(x,y): ax + by + c = 0\}$$

where (a,b) is the foot of the perpendicular from O to ℓ and $c = -a^2 - b^2$.

Suppose next that p is a line in the xy-plane through the point $(0,0)$ and that $A(a,b)$ is a point different from $O(0,0)$ on the line through $(0,0)$ in the xy-plane which is perpendicular to p. Note that a and b are not both 0. Then $P(x,y)$ is a point in p if and only if



$$(OA)^2 + (OP)^2 = (AP)^2,$$

$$a^2 + b^2 + x^2 + y^2 = (x - a)^2 + (y - b)^2,$$

$$0 = -2ax - 2by,$$

$$ax + by = 0.$$

Therefore

$$p = \{(x,y): ax + by + c = 0\}$$

where (a,b) is a point, not 0, on the line through 0 and perpendicular to p, and where $c = 0$.

In this section we not only have derived the first degree equation for a line using the Pythagorean Theorem but we have obtained a useful by-product. It is: If a and b are not both 0; then the line from $(0,0)$ to (a,b) is perpendicular to the line $ax + by + c = 0$, whether this latter line is through the origin or not. Stated in another way: If $ax + by + c = 0$ is a first degree equation for a line ℓ, then $[a,b]$ is a vector perpendicular to ℓ (i.e. perpendicular to every vector which can be represented by a directed line segment contained in ℓ), briefly, $[a,b]$ is a normal vector to ℓ.

5. First Degree Equations for Planes.

Let p be a plane not containing $(0,0,0)$ and suppose $A = (a,b,c)$ is the foot of the perpendicular from 0 to p. Note that a, b, c are not all zero. Then $P(x,y,z)$ is in p if and only if

$$(OP)^2 = (OA)^2 + (AP)^2,$$

$$x^2 + y^2 + z^2 = a^2 + b^2 + c^2 + (x - a)^2 + (y - b)^2 + (z - c)^2,$$

$$2ax + 2by + 2cz = 2a^2 + 2b^2 + 2c^2,$$

$$ax + by + cz = a^2 + b^2 + c^2$$

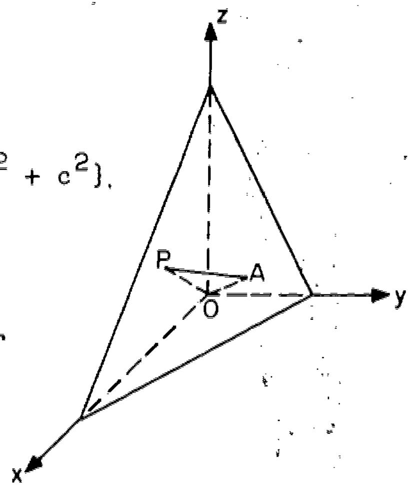
and

$$p = \{(x,y,z): ax + by + cz = a^2 + b^2 + c^2\}.$$

Observe that

$$p = \{(x,y,z): ax + by + cz = d\}$$

where $[a,b,c]$ is a normal vector for the plane.



Let q be a plane through $O(0,0,0)$ and suppose $A = (a,b,c)$ is a point other than O on the line through O and perpendicular to q. Note that a, b, c are not all zero. Then $P(x,y,z)$ is a point in q if and only if

$$(OA)^2 + (OP)^2 = (AP)^2 ,$$

$$a^2 + b^2 + c^2 + x^2 + y^2 + z^2 = (x - a)^2 + (y - b)^2 + (z - c)^2 ,$$

$$ax + by + cz = 0 .$$

Observe that

$$q = \{(x,y,z): ax + by + cz = 0\} .$$

and that $[a,b,c]$ is a normal vector for q.

Next we start with an arbitrary first degree equation in x, y, z. Suppose $ax + by + cz = d$ is any equation with a, b, c not all 0.

Is this an equation for a plane? On the basis of our development above it would seem so if either $d = 0$ or $d = a^2 + b^2 + c^2$. What is the situation for an equation like $3x + 4y + 5z = 6$ for which neither of the equations, $d = 0$, $d = a^2 + b^2 + c^2$, is true? (We are identifying $a = 3$, $b = 4$, $c = 5$, $d = 6$ in this example.) Is

$$S = \{(x,y,z): 3x + 4y + 5z = 6\}$$

a plane? Multiplying through by $\dfrac{6}{3^2 + 4^2 + 5^2}$ $\left(\text{i.e.} \dfrac{d}{a^2 + b^2 + c^2}\right)$ we see that

$$S = \left\{(x,y,z): \frac{18}{50}x + \frac{24}{50}y + \frac{30}{50}z = \frac{36}{50}\right\}$$

and that

$$\left(\frac{18}{50}\right)^2 + \left(\frac{24}{50}\right)^2 + \left(\frac{30}{50}\right)^2 = \frac{36}{50} .$$

If we set

$$a' = \frac{18}{50} ,\ b' = \frac{24}{50} ,\ c' = \frac{30}{50} ,\ d' = \frac{36}{50} ,$$

then

$$d' = a'^2 + b'^2 + c'^2 .$$

Thus S is the plane which is perpendicular at $\left(\frac{18}{50}, \frac{24}{50}, \frac{30}{50}\right)$ to the directed segment from the origin to the point $\left(\frac{18}{50}, \frac{24}{50}, \frac{30}{50}\right)$.

In the general case, if a, b, c are not all zero, and if

$$S = \{(x,y,z): ax + by + cz = d\},$$

then

$$S = \{(x,y,z): ax + by + cz = 0\},$$

if $d = 0$; and

$$S = \{(x,y,z): a'x + b'y + c'z = d'\}$$

if $d \neq 0$, where

$$a' = \frac{ad}{a^2 + b^2 + c^2}, \quad b' = \frac{bd}{a^2 + b^2 + c^2}, \quad c' = \frac{cd}{a^2 + b^2 + c^2}.$$

and

$$d' = \frac{d^2}{a^2 + b^2 + c^2}.$$

Note that $d' = a'^2 + b'^2 + c'^2$

and that $[a',b',c']$ and $[a,b,c]$ are parallel vectors. Thus it follows, regardless of whether or not d is zero, that S is a plane with normal vector $[a,b,c]$. If $d = 0$ then S is the plane containing the origin and perpendicular to the segment from $(0,0,0)$ to (a,b,c). If $d \neq 0$ and (i) $a \neq 0$ [or (ii) $b \neq 0$, or (iii) $c \neq 0$] then S is the plane containing (i) $(\frac{d}{a},0,0)$ [or (ii) $(0,\frac{d}{b},0)$ or (iii) $(0,0,\frac{d}{c})$] and perpendicular to the segment from $(0,0,0)$ to (a,b,c).

In the development above we made direct use of the Pythagorean Theorem in developing the first degree equation for a plane. We present now another development using vector ideas. (Elementary properties of vectors are discussed in the Text in Chapter 10 and in Appendix XI.) Recall that two vectors are perpendicular if and only if their scalar (or dot)

$\left(\overline{50}, \overline{50}, \overline{50}\right)$.

In the general case, if a, b, c are not all zero, and if

$$S = \{(x,y,z) \colon ax + by + cz = d\},$$

then

$$S = \{(x,y,z) \colon ax + by + cz = 0\},$$

if $d = 0$; and

$$S = \{(x,y,z) \colon a'x + b'y + c'z = d'\}$$

if $d \neq 0$, where

$$a' = \frac{ad}{a^2 + b^2 + c^2}, \quad b' = \frac{bd}{a^2 + b^2 + c^2}, \quad c' = \frac{cd}{a^2 + b^2 + c^2}$$

and

$$d' = \frac{d^2}{a^2 + b^2 + c^2}.$$

Note that $d' = a'^2 + b'^2 + c'^2$

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In the development above we made direct use of the Pythagorean Theorem in developing the first degree equation for a plane. We present now another development using vector ideas. (Elementary properties of vectors are discussed in the Text in Chapter 10 and in Appendix XI.) Recall that two vectors are perpendicular if and only if their scalar (or dot)

The text development leading to this result rests in a very essential way upon the Ruler Postulate and the theorem regarding proportionality of the segments formed when three parallel lines are cut by two transversals. The following alternate development is based on vector ideas.

Given a line $\overleftrightarrow{AB}$ where $A = (x_1,y_1,z_1)$ and $B = (x_2,y_2,z_2)$, then $P(x,y,z)$ is on $\overleftrightarrow{AB}$ and if and only if there is a real number k such that $\overrightarrow{AP} = k \cdot \overrightarrow{AB}$, and

$$[x - x_1 , y - y_1 , z - z_1] = k[x_2 - x_1 , y_2 - y_1 , z_2 - z_1] .$$

But this vector equation is true if and only if all three following scalar equations hold:

$$x - x_1 = k(x_2 - x_1)$$

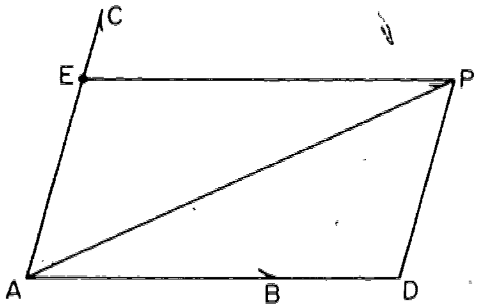
$$y - y_1 = k(y_2 - y_1)$$

$$z - z_1 = k(z_2 - z_1) .$$

It follows that

$$\ell = \left\{ (x,y,z) : \begin{array}{l} x = x_1 + k(x_2 - x_1) , \\ y = y_1 + k(y_2 - y_1) , \text{and } k \text{ is real.} \\ z = z_1 + k(z_2 - z_1) , \end{array} \right\}$$

A similar development yields a parametric equation representation for a plane. Let $A(x_1,y_1,z_1)$, $B(x_2,y_2,z_2)$, $C(x_3,y_3,z_3)$ be three noncoplanar points and p the plane which contains them. Then $P(x,y,z)$ is in p if and only if



$\overrightarrow{AD} = h \cdot \overrightarrow{AB}$

$\overrightarrow{AE} = k \cdot \overrightarrow{AC}$

there are two real numbers h and k such that

$$\overrightarrow{AP} = h \cdot \overrightarrow{AB} + k \cdot \overrightarrow{AC} \; ;$$

that is,

$$[x - x_1 , y - y_1 , z - z_1]$$
$$= h[x_2 - x_1, y_2 - y_1, z_2 - z_1] + k[x_3 - x_1, y_3 - y_1 , z_3 - z_1] ,$$

or

$$[x - x_1 , y - y_1 , z - z_1]$$
$$= [h(x_2 - x_1) + k(x_3 - x_1), h(y_2 - y_1) + k(y_3 - y_1), h(z_2 - z_1)$$
$$+ k(z_3 - z_1)] \; .$$

But this vector equation is true if and only if the following three scalar equations are all true:

$$(*) \left\{ \begin{array}{l} x - x_1 = h(x_2 - x_1) + k(x_3 - x_1) \\ y - y_1 = h(y_2 - y_1) + k(y_3 - y_1) \\ z - z_1 = h(z_2 - z_1) + k(z_3 - z_1) \end{array} \right\} .$$

Therefore

$$p = \left\{ (x,y,z) : \begin{array}{l} x = x_1 + h(x_2 - x_1) + k(x_3 - x_1) \\ y = y_1 + h(y_2 - y_1) + k(y_3 - y_1) \\ z - z_1 + h(z_2 - z_1) + k(z_3 - z_1) \end{array} \text{ and } h \text{ and } k \text{ are real.} \right\}$$

It is of interest to note that if h and k are "eliminated" from the set of three equations (*) above, that a first degree equation in x, y, z results. One way to show this is to rewrite the equations as

$$-(x - x_1) + h(x_2 - x_1) + k(x_3 - x_1) = 0$$
$$-(y - y_1) + h(y_2 - y_1) + k(y_3 - y_1) = 0$$
$$-(z - z_1) + h(z_2 - z_1) + k(z_3 - z_1) = 0$$

and to think of them as three equations in the "unknowns," -1 , h and k . The corresponding determinant of the coefficients is

$$\Delta = \begin{vmatrix} x - x_1 & x_2 - x_1 & x_3 - x_1 \\ y - y_1 & y_2 - y_1 & y_3 - y_1 \\ z - z_1 & z_2 - z_1 & z_3 - z_1 \end{vmatrix}.$$

Since the system of equations (*) has a "solution" other than (0,0,0) , it follows that Δ must be 0 . Expanding the determinant we get

$$(1) \quad ax + by + cz = ax_1 + by_1 + cz_1$$

where

$$a = \begin{vmatrix} y_2 - y_1 & y_3 - y_1 \\ z_2 - z_1 & z_3 - z_1 \end{vmatrix} \quad b = \begin{vmatrix} x_3 - x & x_2 - x_1 \\ z_3 - z_1 & z_2 - z_1 \end{vmatrix} \quad c = \begin{vmatrix} x_2 - x_1 & x_3 - x_1 \\ y_2 - y_1 & y_3 - y_1 \end{vmatrix}$$

and (1) is the desired equation.

7. Why Parametric Equations?

Consider the sets

$$(1) \begin{cases} S_1 = \{(x,y,z): 7x - 2y - 2z = 3\} \\ S_2 = \{(x,y,z): x - 2y + z = 0\} \\ S_3 = \{(x,y,z): 7x - 2y - 2z = 3 \text{ and } x - 2y + z = 0\}. \end{cases}$$

Then S_1 and S_2 are planes and S_3 is the line of intersection of S_1 and S_2 . What information about line S_3 is revealed by the equations in the set-builder symbol for S_3 ? It is easy to see that S_3 lies in S_1 and S_2 , and hence that S_3 is perpendicular to each of the normal vectors [7,-2,-2] and [1,-2,1] . But what is the direction of S_3 , and what points does it contain?

Set $x = 1$ in the equations $7x - 2y - 2z = 3$ and $x - 2y + z = 0$ and solve the resulting equation for y and z. Do this over again with $x = 3$. We find that $(1,1,1)$ and $(3,4,5)$ are two points in S_3 and hence that

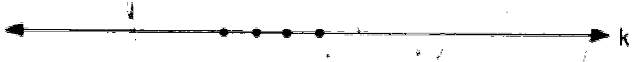
$$(2) \quad S_3 = \left\{ (x,y,z) : \begin{array}{l} x = 1 + k(3 - 1) , \\ y = 1 + k(4 - 1) , \text{ and } k \text{ is real} \\ z = 1 + k(5 - 1) , \end{array} \right\}$$

$$(3) \quad S_3 = \left\{ (x,y,z) : \begin{array}{l} x = 1 + 2k , \\ y = 1 + 3k , \text{ and } k \text{ is real} \\ z = 1 + 4k , \end{array} \right\}$$

The parametric equations in (2) seem to reveal more information about S_3 than the equations in (1). An inspection of (2) reveals that S_3 passes through $A(1,1,1)$ and $B(3,4,5)$, using $k = 0$ and $k = 1$. By taking $k = -1$, ± 2, ± 3, ... we get other points along S_3 with a minimum expenditure of effort.

The parametric equations in (3) show that S_3 contains $(1,1,1)$, by taking $k = 0$, and that it is parallel to the vector $[2,3,4]$, by looking at the coefficients of k.

One way to think of the parametric equations is as a "mapping" from the k-axis to a set of points in xyz-space. As a point "marches along" the k axis,



the corresponding point (x,y,z) "marches along" the line S_3.

$k \longrightarrow (x,y,z)$
$0 \longrightarrow (1,1,1)$
$1 \longrightarrow (3,4,5)$
$2 \longrightarrow (5,7,9)$
$3 \longrightarrow (7,10,13)$
. .
. .
. .

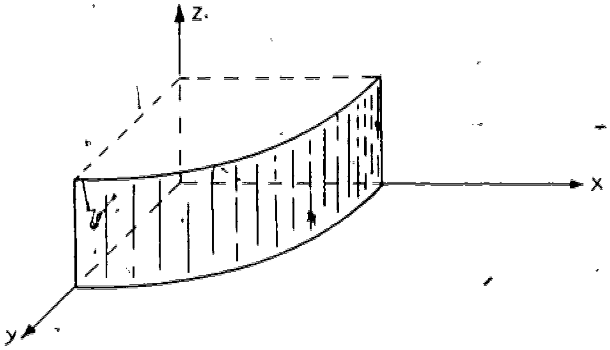
As another example let us consider the sets

$$S_4 = \{(x,y,z):\ x^2 + y^2 = 9\ ,\ x \geq 0\ ,\ y \geq 0\}$$

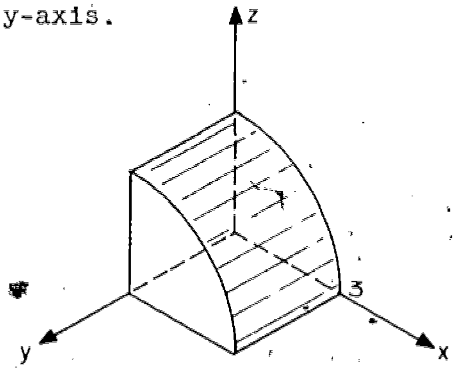
$$S_5 = \{(x,y,z):\ x = 3 \cos z\ ,\ 0 \leq z \leq \tfrac{\pi}{2}\}$$

$$S_6 = \left\{\begin{array}{l}(x,y,z):\ x^2 + y^2 = 9\ ,\ x = 3 \cos z\ , \\ \qquad\qquad x \geq 0\ ,\ y \geq 0\ ,\ 0 \leq z \leq \frac{\pi}{2}\end{array}\right\}$$

S_4 is a portion of a right circular cylinder.



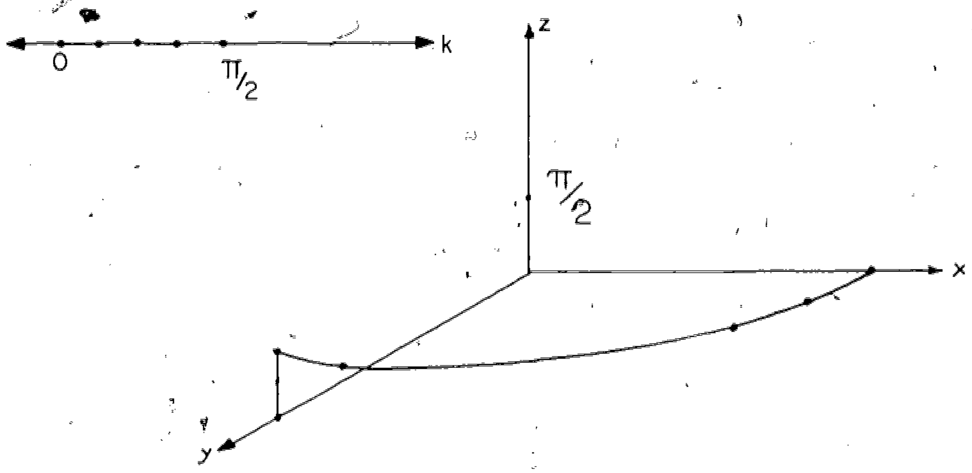
S_5 is a cylindrical surface which is the union of lines parallel to the y-axis.



S_6 is the intersection of S_4 and S_5, actually an arc of a helix. The equations in the set-builder symbol for S_6 above tells us that the curve S_6 is the intersection of two surfaces; it seems to emphasize the surfaces unnecessarily if the object of one's attention is really the curve in which they intersect.

Compare the above representation as the intersection of two surfaces with the following parametric representation.

$$S_6 = \left\{ (x,y,z):\ \begin{array}{l} x = 3\cos k\ , \\ y = 3\sin k\ , \\ z = k\ , \end{array} \quad \text{and} \quad 0 \le k \le \frac{\pi}{2} \right\}$$



Imagine a "particle" moving along the k-axis from 0 to $\frac{\pi}{2}$. As it does, the corresponding "particle" (x,y,z) moves continuously from $(3,0,0)$ to $(0,3,\frac{\pi}{2})$ along the curve S_6 . If k denotes the number of time units (minutes for example) since the particle departed from $(3,0,0)$ on its flight along the helix, then the parametric equations for S_6 may be used to find easily the position of the particle at any given instant.

Two problems in differential geometry are (1) to find the line which is tangent to a curve at a given point, and (2) to find the plane which is perpendicular to a curve at a given point. The parametric equations for S_6 may be used to solve these problems easily for the arc of the helix in the example. Thus, corresponding to $k = \frac{\pi}{4}$, we have the point

$P_1\left(\frac{3}{\sqrt{2}}\ ,\frac{3}{\sqrt{2}}\ ,\frac{\pi}{4}\right)$ on S_6 . Using a bit of elementary calculus as indicated below we find the components of a tangent vector to S_6 at P_1 . (The dots indicate differentiation with respect to the parameter k .)

$$\begin{array}{lll|l} x = 3\cos k & \dot{x} = -3\sin k & & = \frac{-3}{\sqrt{2}} \\ y = 3\sin k & \dot{y} = 3\cos k & \Big) & = \frac{3}{\sqrt{2}} \\ z = k & \dot{z} = 1 & & = 1 \\ & & & \text{at } k = \frac{\pi}{4} \end{array}$$

Then $\left[-\frac{3}{\sqrt{2}}, \frac{3}{\sqrt{2}}, 1\right]$ is a tangent vector to S_6 at P. It then follows easily that the tangent line and the normal plane are as follows:

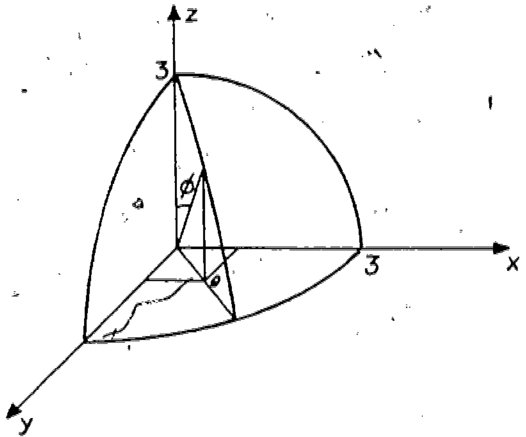
$$\text{T.L.} = \left\{ (x,y,z)\colon \begin{array}{l} x = \frac{3}{\sqrt{2}} - \frac{3}{\sqrt{2}}k, \\ y = \frac{3}{\sqrt{2}} + \frac{3}{\sqrt{2}}k, \text{ and } k \text{ is real} \\ z = \frac{\pi}{4} + k, \end{array} \right\}$$

$$\text{N.P.} = \{(x,y,z)\colon -\frac{3}{\sqrt{2}}\left(x - \frac{3}{\sqrt{2}}\right) + \frac{3}{\sqrt{2}}\left(y - \frac{3}{\sqrt{2}}\right) + 1\left(z - \frac{\pi}{4}\right) = 0\}$$

As another example let us consider the sphere $S_7 = \{(x,y,z)\colon x^2 + y^2 + z^2 = 9\}$. This non-parametric equation reveals clearly that S_7 is the set of all points (x,y,z) the square of whose distance from $(0,0,0)$ is 1. But there are other things about the sphere not clearly revealed by this equation--things having to do with "latitude" and "longitude," for example. Also this form of representation has the disadvantage that the relationships among x, y, z are implicit; the equation does not give us any one of the variables explicitly as a function of the others. If we solve for z in terms of x and y we get $z = \pm\sqrt{9 - x^2 - y^2}$, a "double-valued function." Neither of the two functions included in this "double function" possesses partial derivatives for values of x and y such that $x^2 + y^2 = 9$. Of course this is a disadvantage if one is interested in normal vectors or tangent planes, or one of the host of applications which use these vectors and planes as tools.

Another representation of this sphere S_7 is the following parametric one based on the spherical coordinates θ , ϕ , r , with $r = 3$.

$$S_7 = \left\{ (x,y,z): \begin{array}{ll} x = 3 \cos \theta \sin \phi , & \\ y = 3 \sin \theta \sin \phi , & 0 \leq \theta < 2\pi , \\ z = \cos \phi , & 0 \leq \phi \leq \pi \end{array} \right\}$$



The parameters θ and ϕ are called the longitude and the colatitude, respectively. The three equations define x, y, z explicitly as single-valued, differentiable functions of the parameters θ and ϕ . As one might expect, the parametric equations for S_7 are more fruitful and easier to use for certain purposes than is the nonparametric equation for S_7 . Furthermore the variables θ and ϕ seem to belong to a coordinate representation of the sphere. The θ and ϕ values at a point are more useful for many purposes than are the x, y, and z values at the point.

A final example is the cycloid arch given parametrically as follows:

$$S_8 = \{(x,y): x = 3(\theta - \sin \theta), y = 3(1 - \cos \theta), 0 \leq \theta \leq 2\pi\}.$$

These parametric equations are derived in a natural way from the definition of the cycloid. These equations express x and y in terms of the radian measure θ of the angle through which the generating circle has rotated. For the details see any of the traditional college analytic geometry texts. This same cycloid arch may be given in non-parametric form as follows:

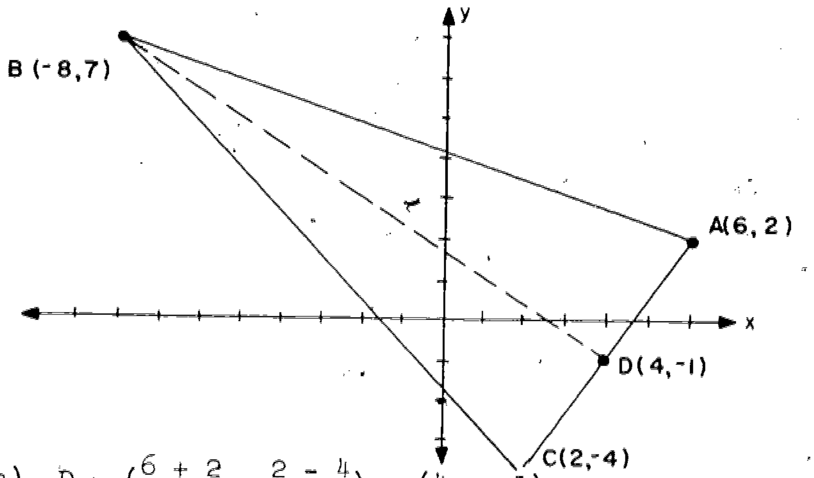
$$S_8 = \left\{(x,y): \begin{array}{c} x = 3\cos^{-1}\dfrac{3-y}{3} - \sqrt{6y-y^2} \\ \text{or} \\ x = 6\pi - 3\cos^{-1}\dfrac{3-y}{3} - \sqrt{6y-y^2}, \end{array} \text{ and } 0 \le y \le 6\right\}$$

Of course this is less useful and more difficult to handle than the parametric equations. And who in the world would ever discover these non-parametric equations without first finding the parametric equations?

Answers to Illustrative Test Items

Chapter 8

1.



(a) $D = \left(\frac{6 + 2}{2}, \frac{2 - 4}{2}\right) = (4, -1)$

(b) $m_{\overline{AC}} = \frac{2 - (-4)}{6 - 2} = \frac{6}{4} = \frac{3}{2}$

(c) $m_{\overline{BD}} = \frac{-1 - 7}{4 - (-8)} = \frac{-8}{12} = -\frac{2}{3}$ and $m_{\overline{AC}} \cdot m_{\overline{BD}} = -1$

(d) Δ ABC is isosceles, because $\overleftrightarrow{BD}$ is the perpendicular bisector of $\overline{AC}$.

(e) $BA = \sqrt{(6 - (-8))^2 + (2 - 7)^2} = \sqrt{196 + 25} = \sqrt{221}$

$BC = \sqrt{(2 - (-8))^2 + (-4 - 7)^2} = \sqrt{100 + 121} = \sqrt{221}$

(f) $\overleftrightarrow{AC} = \{(x, y): x = 2 + 4k,\ y = -4 + 6k,\ k \text{ is real}\}$ or any equivalent form.

(g) $\frac{x + 8}{4 + 8} = \frac{y - 7}{-1 - 7}$ or any equivalent form.

(h) If $y = 0$ for a point of $\overleftrightarrow{AC}$, then $-4 + 6k = 0$, so $k = \frac{2}{3}$. Then $x = 2 + 4 \cdot \frac{2}{3} = \frac{14}{3}$. Intersection point is $(\frac{14}{3}, 0)$.

(i) $y - 7 = \frac{3}{2}(x - (-8))$.

(j) $(-4, 13)$

(k) $(-12, 1)$

2. (a) 12

   (b) 24

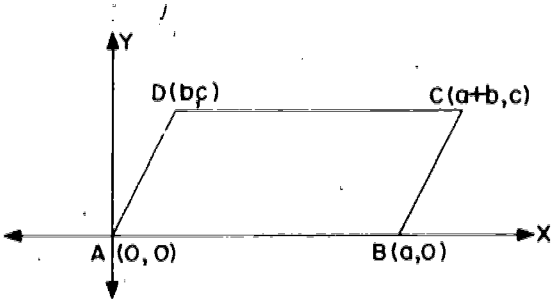
3. (a) Rectangle (e) Rectangle

   (b) Rhombus (f) Rhombus

   (c) Rhombus (g) Square

   (d) Square

4. There is a coordinate system which assigns to parallelogram ABCD the coordinates as shown.



Part I. Given $DB = AC$, to prove ABCD is a rectangle.

Proof: $DB = AC$ implies

$\sqrt{(a - b)^2 + c^2} = \sqrt{(a + b)^2 + c^2}$ or

$a^2 - 2ab + b^2 + c^2 = a^2 + 2ab + b^2 + c^2$ or

$-ab = ab$

or, since $a \neq 0$, $-b = b$,

or $b = 0$,

or D is on the y-axis,

or $\angle BAD$ is a right angle,

or ABCD is a rectangle.

Part II. Given ABCD a rectangle, to prove $BD = AC$.

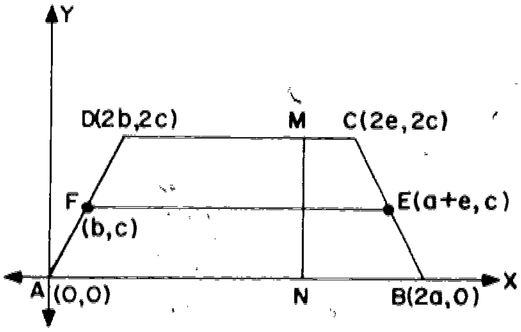
Proof: ABCD is a right angle. Therefore D is in the y-axis, and $b = 0$.

Therefore $AC = \sqrt{a^2 + c^2}$ and $BD = \sqrt{a^2 + c^2}$, so $AC = BD$.

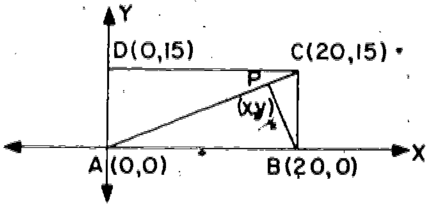
5. (a) $P = (-1 + 4 \cdot 3\ ,\ 2 - 2 \cdot 3) = (11,-4)$
   (b) $P = \big(-1 + 4(-3)\ ,\ 2 - 2(-3)\big) = (-13,8)$
   (c) $P = (-1 + 4 \cdot 100\ ,\ 2 - 2 \cdot 100) = (399,-198)$

6. (a) Horizontal
   (b) Oblique
   (c) Vertical
   (d) Oblique
   (e) Horizontal

7. The line containing the median of a trapezoid bisects each of its altitudes. Proof. Let ABCD be the trapezoid. There is a coordinate system which assigns coordinates to A, B, C, D as indicated. Then the coordinates of the endpoints F and E of the median are as indicated. If $\overline{MN}$ is perpendicular to $\overline{AB}$ and has its endpoints in the lines containing the parallel sides of ABCD , then $\overline{MN}$ is an altitude of ABCD . In terms of coordinates, $\overline{MN}$ is an altitude if and only if $M = (x,2c)$ and $N = (x,0)$, for some x . The midpoint of $\overline{MN}$ is therefore $(\frac{x}{2},c)$. Since $\overleftrightarrow{FE}$ is $\{(x,y): y = c\ ,\ x \text{ is real}\}$, it follows that the midpoint is on $\overleftrightarrow{FE}$.



8. ABCD is a rectangle with coordinates as given. BP $\perp$ AC. We are required to find how far along $\overline{AC}$, P lies.



$\overline{AC} = \{(x,y):\ x = 20k\ ,\ y = 15k \text{ and } 0 \leq k \leq 1\}$.

Slope of $\overline{BP} = -\frac{20}{15}$.

$\overleftrightarrow{BP} = \{(x,y):\ x = 20 - 15h\ ,\ y = 20h \text{ and } h \text{ is real}\}$.

Then there are real numbers k and h such that $P = (x,y) = (20k,15k) = (20 - 15h,20h)$.

$$\text{Then } 20k = 20 - 15h \text{ and } 15k = 20h ,$$
$$\text{or } 16k = 16 - 12h \text{ and } 9k = 12h .$$

$$\text{Then } 25k = 16$$
$$k = \frac{16}{25}$$
$$\text{and } \frac{AP}{AC} = \frac{16}{25} .$$

Alternate solution, using slopes.

$$m_{\overline{AP}} = m_{\overline{AC}} ; \frac{y}{x} = \frac{15}{20} , y = \frac{3x}{4} ,$$
$$m_{\overline{BP}} = -\frac{1}{m_{\overline{AC}}} ; \frac{y}{x - 20} = -\frac{20}{15} = -\frac{4}{3} ,$$
$$3y = -4(x - 20)$$
$$3\left(\frac{3x}{4}\right) = -4x + 80 ,$$
$$9x = -16x + 320$$
$$25x = 320$$
$$x = \frac{64}{5} ,$$
$$k = \frac{x_P}{x_C} = \frac{\frac{64}{5}}{20} = \frac{16}{25} . \quad \text{Thus } \frac{AP}{AC} = \frac{16}{25} .$$

Alternate solution, using Pythagorean Theorem and proportions of a right triangle.

(1) $AC = \sqrt{(AB)^2 + (BC)^2} = \sqrt{(20)^2 + (15)^2} = 25$

(2) $(AB)^2 = (AC)(AP)$

$(20)^2 = 25 \cdot AP$

$16 = AP$

(3) $\frac{AP}{AC} = \frac{16}{25}$.

Answers for Illustrative Test Items

Chapter 9

1. (a) +
   (b) 0
   (c) 0
   (d) +
   (e) +
   (f) 0
   (g) +
   (h) 0
   (i) 0
   (j) +
   (k) 0
   (l) +
   (m) 0
   (n) 0
   (o) 0
   (p) 0
   (q) 0
   (r) 0
   (s) 0
   (t) +
   (u) +
   (v) 0
   (w) +
   (x) 0
   (y) +
   (z) 0

2. (a) Not necessarily. $\overleftrightarrow{BQ}$ cannot be proved perpendicular to plane ABK on the basis of information given.
   (b) Yes, by Theorem 9-1.
   (c) Six planes: ABK , ABQ , ABH , ABR , ABF , and the plane perpendicular to $\overleftrightarrow{AB}$ at B .

3. (a) $\mathcal{P} \parallel \mathcal{R}$, since planes perpendicular to the same line are parallel. (Theorem 9-9)
   (b) $\overleftrightarrow{WK} \parallel \overleftrightarrow{QF}$ by Theorem 9-6.
   (c) Right angles. In a plane, if a line is perpendicular to one of two parallel lines, it is perpendicular to the other.

4. This follows from Theorem 9-18.

5. Points F, A, D determine a plane; for if they were collinear, the line containing them and the line $\overleftrightarrow{BC}$ would determine a plane containing all four of the noncoplanar points A, B, C, D . Then $\overline{BC}$ is perpendicular to plane DFA , by Theorem 9-2 (or, by Theorem 9-18).

6. Two lines perpendicular to the same plane are parallel, and any two parallel lines are coplanar.

7. Since $\overleftrightarrow{AB} \perp \mathcal{E}$ and $\overleftrightarrow{CD} \perp \mathcal{E}$ by hypothesis, $\overleftrightarrow{AB} \parallel \overleftrightarrow{CD}$ by Postulate 25. Thus A, B, C, D are coplanar, and BADC is a quadrilateral. Since $\overleftrightarrow{AB} \perp \mathcal{E}$ and $\overleftrightarrow{CD} \perp \mathcal{E}$, each of the angles $\angle BAD$ and $\angle CDA$ is a right angle. Since $\mathcal{E} \parallel \mathcal{F}$ by hypothesis, $\overleftrightarrow{AB} \perp \mathcal{F}$ and $\overleftrightarrow{CD} \perp \mathcal{F}$ by Theorem 9-10. Hence each of the angles $\angle ABC$ and $\angle DCB$ is a right angle. By Theorem 8-20, the quadrilateral BADC is a rectangle. By Theorem 8-25, $AC = BD$.

8. (a) (w)
(b) (u)
(c) (r)
(d) (v)
(e) (s)
(f) (t)

9. A point is in the xz-plane if and only if its y-coordinate is 0 . Therefore, $4 - 2k = 0$, or $k = 2$. Hence the required point has coordinates (4,0,6) .

10. $AB = \sqrt{26}$ and $BC = \sqrt{26}$. Therefore, $\triangle ABC$ is isosceles by definition.

11. A point is in the xy-plane if and only if its z-coordinate is 0 . Therefore, points in the xy-plane which also lie in the plane whose equation is $2x - y + z = 7$ lie on the line of intersection, represented by the equation $2x - y = 7$.

12. (a) $P\left(\frac{1}{2}, -\frac{1}{2}, \frac{7}{2}\right)$

(b) $\frac{x + 2}{2} = \frac{1}{2}$, $x = -1$

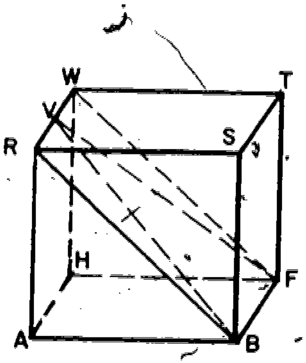
$\frac{z - 2}{2} = \frac{7}{2}$, $z = 9$

13. Using the equation of a plane, $ax + by + cz = d$, and coordinates of points A, B, C, we have the following equations to solve for a, b, and c in terms of d .

$$\begin{aligned} a + 2b + 5c &= d \\ b + 6c &= d \\ 2a + c &= d \end{aligned}$$

$a = \frac{2}{5}d$, $b = -\frac{1}{5}d$, $c = \frac{1}{5}d$ and an equation of the plane becomes $2x - y + z = 5$.

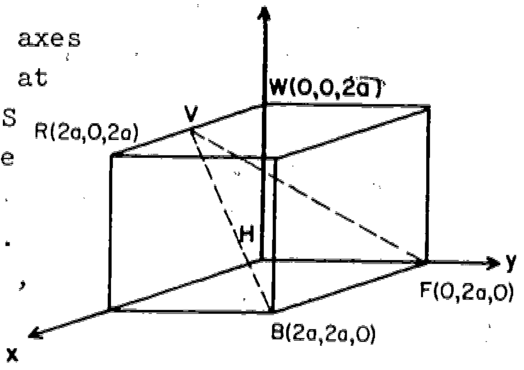
14. Proof, without coordinates.



Use auxiliary segments $\overline{RB}$ and $\overline{WF}$.

Statements	Reasons
1. $\triangle RAB \cong \triangle WHF$.	1. S.A.S.
2. $RB = WF$.	2. Definition of congruence for triangles.
3. $RV = VW$.	3. Definition of midpoint.
4. $\angle VRB$ and $\angle VWF$ are right angles.	4. Definition of line perpendicular to a line.
5. $\triangle VRB \cong \triangle VWF$.	5. S.A.S.
6. $VB = VF$.	6. Definition of congruence for triangles.

Proof, with coordinates.



Choose the coordinate axes so that vertex H is at (0,0,0) and vertex S at (2a,2a,2a) , where 2a is the length of each edge of the cube. The coordinates of V , the midpoint of $\overline{WR}$, will be (a,0,2a) . Using the distance formula,

$$VB = \sqrt{a^2 + 4a^2 + 4a^2} = 3a ,$$

$$VF = \sqrt{a^2 + 4a^2 + 4a^2} = 3a .$$

Therefore, $VB = VF$.

Answers to Illustrative Test Items

Chapter 10

1. $\overrightarrow{AB}$, $\overrightarrow{BA}$,
   $\overrightarrow{BC}$, $\overrightarrow{CB}$,
   $\overrightarrow{AC}$, $\overrightarrow{CA}$.

2. $\overrightarrow{AB} = [-1,5]$, $\overrightarrow{BA} = [1,-5]$.
   $\overrightarrow{AC} = [-6,-1]$, $\overrightarrow{CA} = [6,1]$.
   $\overrightarrow{BC} = [-5,-6]$, $\overrightarrow{CB} = [5,6]$.

3. (a) $[1,8]$.
   (b) $[5,6]$.
   (c) 1 .
   (d) $\sqrt{58}$.
   (e) $\sqrt{65}$.

4. (a) $(\overrightarrow{D,C})$.
   (b) $(\overrightarrow{B,C})$.
   (c) $(\overrightarrow{C,B})$.
   (d) $(\overrightarrow{D,B})$.
   (e) $(\overrightarrow{D,A})$.

5. $[2,-3]$.

6. $[0,0] = \vec{0}$.

7. (a) $[4,4]$.
   (b) $4\sqrt{2}$.

8. (a) $[-2,-4]$.
   (b) $[6,2]$.
   (c) $[-5,-2]$.
   (d) $[5,-10]$.
   (e) $[0,0]$.

9. $x = -2$ and $y = 3$.

10. 12 .

11. No, this does not imply that $\overline{AM} \parallel \overline{MB}$.

12. Yes, the magnitude is equal to the length of the segment.

13. (a) $\vec{b} = 2\vec{a}$,
(b) $\vec{a} = \frac{1}{2}\vec{b}$.

14. [3, 5] .

15. (a) They must determine parallel rays.
(b) Their magnitudes must be equal.

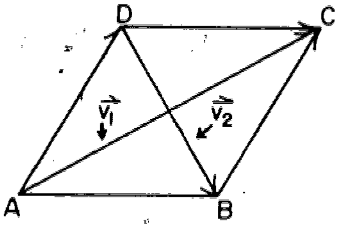
16. $\overrightarrow{PQ} = [6,8]$, $\overrightarrow{SR} = [6,8]$.

Therefore, we must have a parallelogram since a pair of opposite sides are parallel and equal.

17. $\overrightarrow{PQ} = [3,2]$, $\overrightarrow{SR} = [3,2]$, thus we have a parallelogram. Also $\overrightarrow{QR} = [-2,3]$ and $[3,2] \cdot [-2,3] = -6 + 6 = 0$. Therefore two adjacent sides are perpendicular. Hence we have a rectangle, since we have a parallelogram with a right angle.

18. $\overrightarrow{PQ} = [2,-6]$, $\overrightarrow{RS} = [6,2]$, and since $[2,-6] \cdot [6,2] = 12 - 12 = 0$, the lines are perpendicular.

19. Let the rhombus be lettered with directed segments as shown in the figure.



Let $\overrightarrow{AD}$, $\overrightarrow{BC}$ equal $[m,n]$,
and $\overrightarrow{AB}$, $\overrightarrow{DC}$ equal $[x,y]$.
Then $\vec{v}_1 = [m,n] + [x,y] = [m + x , n + y]$,
and $\vec{v}_2 = [x,y] - [m,n] = [x - m , y - n]$.
Therefore $\vec{v}_1 \cdot \vec{v}_2 = [m + x , n + y] \cdot [x - m , y - n]$,
$= x^2 - m^2 + y^2 - n^2$.

Since the magnitudes of [m,n] and [x,y] are equal, we can say

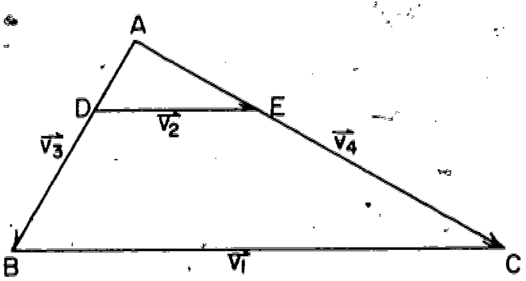
$$\sqrt{m^2 + n^2} = \sqrt{x^2 + y^2}$$

$$m^2 + n^2 = x^2 + y^2 .$$

$$x^2 - m^2 + y^2 - n^2 = 0 ,$$

and thus the diagonals are perpendicular.

20. Let triangle ABC have D, E points on $\overline{AB}$ and $\overline{AC}$ respectively such that $AD = \frac{1}{3}AB$ and $AE = \frac{1}{3}AC$, and let the segments be directed as shown. The segments represent the indicated vectors.



$$\tfrac{1}{3}\vec{v}_3 + \vec{v}_2 = \tfrac{1}{3}\vec{v}_4 \quad \text{and} \quad \vec{v}_3 + \vec{v}_1 = \vec{v}_4$$

or $\vec{v}_2 = \frac{1}{3}(\vec{v}_4 - \vec{v}_3)$ and $\vec{v}_1 = \vec{v}_4 - \vec{v}_3$;

therefore $\vec{v}_2 = \frac{1}{3}\vec{v}_1$,

which implies that $DE = \frac{1}{3}BC$ and $\overline{DE} \parallel \overline{BC}$.

Answers to Illustrative Test Items

Chapter 11

A. Measures of the Angles of a Polygon.

1. c .
2. b .
3. d .
4. a .
5. d .
6. a .
7. c .
8. c .
9. a .
10. c .
11. b .
12. a .

B. Area Formulas.

1. 25 .

2. $\sqrt{n}$.

3. $ab + a(c - a)$, or $ac + a(b - a)$, or $ab + ac - a^2$,

4. Let a be the length of the altitude and 3a the length of the base. Then

$$3a^2 = 147$$
$$a^2 = 49$$
$$a = 7 .$$

The altitude is 7 . The length of the base is 21 .

5. 12 .

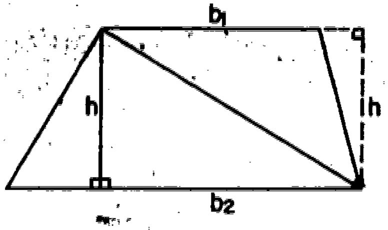
6. Consider the figure to be the union of triangular regions WYZ and XYZ . It can be proved that $\overline{YZ}$ is the perpendicular bisector of $\overline{WX}$. Hence $\overline{WP}$ and $\overline{XP}$ are altitudes of triangle WYZ and XYZ respectively. The area of each of these triangles is 24 . Hence the area of WZXY is 48 .

*7. (a) ad .

(b) $\frac{1}{2}d(a - c)$.

(c) $\frac{1}{2}d(a + c)$.

8. Separate the figure into triangular regions by a diagonal. The areas of the respective triangles are $\frac{1}{2}b_1h$ and $\frac{1}{2}b_2h$. The sum of these two areas is $\frac{1}{2}b_1h + \frac{1}{2}b_2h = \frac{1}{2}h(b_1 + b_2)$.



9. Area ABCD = Area AGFD + Area DFEC - Area AGB - Area CEB .

Area ABCD = 234 + 50 - 42 - 48 .

Area ABCD = 194 .

The area of the field is 194 square rods.

C. Comparison of Areas.

1. Area $\triangle ADC$ = Area $\triangle BCD$ because the triangles have the same base $\overline{DC}$ and equal altitudes.

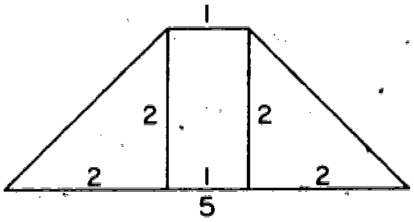
Area $\triangle DOC$ = Area $\triangle DOC$.

Therefore, by the addition property of equality, we have Area $\triangle AOD$ = Area $\triangle BOC$.

2. (a) Area parallelogram SRQP = 2 Area $\triangle SQR$.
   (b) Area parallelogram SRQP = Area $\triangle MTR$.
   (c) Area $\triangle PNS = \frac{1}{4}$ Area $\triangle MTR$.
   (d) Area $\triangle SQR$ = Area $\triangle SPR$.
   (e) Area $\triangle MTR$ = 4 Area $\triangle RQT$.

D. Miscellaneous Problems.

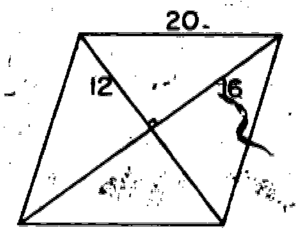
1. 6 . (See figure at right.)



2. 43 . (AC = 13 .)

3. (a) 384 . (See figure at right.)

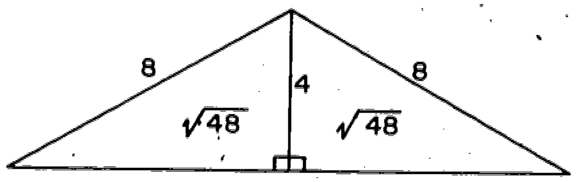
(b) 19.2 (384 ÷ 20 .)



4. 54 . ($\frac{1}{2} \cdot 9 \cdot 12$. The triangle is a right triangle.)

5. (a) $15\frac{5}{6}$. (b) 20 .

6. $16\sqrt{3}$. (or $4\sqrt{48}$.)



E. <u>Coordinates</u>.

1. Slope of BC = $\frac{1}{2}$

Slope of AC = -2 .

Therefore, BC $\perp$ AB , and $\triangle ABC$ is a right triangle.

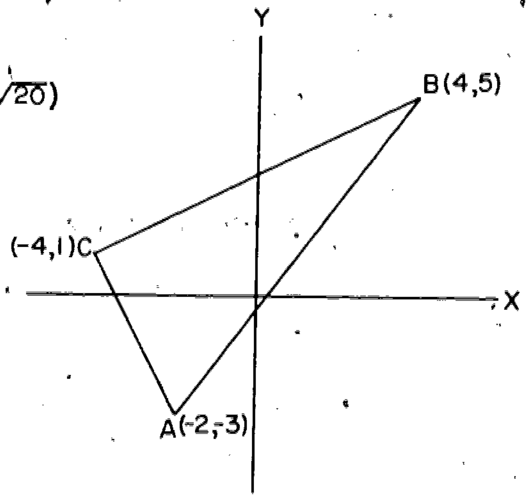
BC = $\sqrt{80}$; AC = $\sqrt{20}$.

Therefore,

$A = \frac{1}{2}(\sqrt{80} \cdot \sqrt{20})$

$A = \frac{1}{2} \cdot \sqrt{1600}$

$A = 20$.



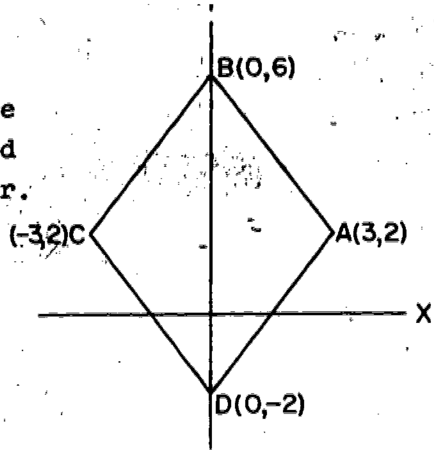
2. (a) Rhombus

The diagonals are perpendicular and bisect each other.

(b) 24 .

$A = \frac{1}{2} d \cdot d'$

$A = \frac{1}{2} \cdot 8 \cdot 6$



3. A = 23 .

The vertices of the rectangle are designated by the following coordinates:

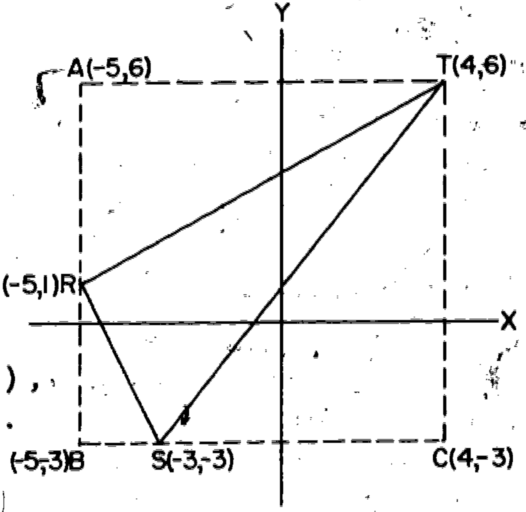
A(-5,6) , B(-5,-3) , C(4,-3) , T(4 6) .

Area of rectangle ABCT = 81 .

Area of $\triangle$RAT = $22\frac{1}{2}$

Area of $\triangle$TCS = $31\frac{1}{2}$

Area of $\triangle$RSB = 4 .



4. K = area of $\triangle$ABC

= Area of XYBA + Area of YZCB - Area of XZCA.

$$K = \frac{(b-a)(r+s)}{2} + \frac{(c-b)(s+t)}{2} - \frac{(c-a)(t+r)}{2}$$

$$K = \frac{br+\cancel{bs}-\cancel{ar}-as+cs+\cancel{ct}-\cancel{bs}-bt-\cancel{ct}-cr+at+\cancel{ar}}{2}$$

$$K = \frac{a(t-s)+b(r-t)+c(s-r)}{2} .$$

F. **Area Relations.**

1. 175 .
2. 1 to 1 .
3. 3 to 2 .
4. 4 times.
5. 2 times.
6. 2 to 3 .
7. $\sqrt{2}$ times.
8. $\frac{9}{16}$; $\frac{x^2}{y^2}$.
9. $\frac{6}{25}$.
10. 4 ; $\frac{15}{4}$.
11. 3 .
12. 8 .

G. **Regular Polygons.**

1. $54\sqrt{3}$.
2. $\frac{3}{7}$; $\frac{3}{7}$; $\frac{9}{49}$.
3. $p = 60$; $a = 5\sqrt{3}$
   $A = 150\sqrt{3}$.

H. **Polyhedrons.**

1. 180 .
2. $40 < x < 160$.
3. (a) 5 .

(b) Tetrahedron	(c)	4
Hexahedron		6
Octahedron		8
Dodecahedron		12
Icosahedron		20

4. $F + V - E = 2$.
5. $210 + 25\sqrt{3}$.
6. $p = 5.2$ in.
7. 975 .
8. 2 to 1 ; 4 to 1 .

Answers to Illustrative Test Items

Chapter 12

Part A.

1. +
2. 0
3. +
4. 0
5. +
6. +
7. 0
8. 0
9. +
10. 0
11. +
12. +
13. +
14. 0
15. 0
16. +
17. +
18. 0
19. +
20. +
21. +
22. +
23. +
24. +
25. +
26. +
27. +
28. +
29. 0
30. 0

Part B.

1. m $\angle APD = 36$
2. m $\widehat{AC} = 130$; m $\widehat{BC} = 140$
3. m $\angle CPD = 80$; m $\angle ADC = 65$; m $\angle DCB = 115$
4. m $\angle CAE = 56$
5. 140
6. 98
7. Radius is 3
8. Zero. The circumference of the circumscribing circle
9. $A = \frac{C^2}{4\pi}$
10. Radius is $\sqrt{2}$
11. $\frac{1}{10}$
12. $\frac{1}{2}$; $\frac{1}{2}$

13. 2

14. $18\pi - 36$

15. (a) on S (e) on S
(b) exterior (f) exterior
(c) exterior (g) on S
(d) interior (h) interior

16. 20

17. $m\ \widehat{XAY} = 120$; $\ell\ \widehat{XAY} = 4\pi$; area of sector XOY $= 12\pi$; area of segment $= 12\pi - 9\sqrt{3}$

18. Circumference is $2\pi\sqrt{3}$; area is 3π .

19. $2\frac{1}{2}$ square miles

20. Slopes of first pair are -2 and $\frac{1}{2}$, of the second pair 3 and $-\frac{1}{3}$.

21. Radius is 15 . Ratio of areas is $\frac{225}{289}$.

22. $AD = 26$

23. 85 feet

24. $5\sqrt{2}$

25. One point. (2,1) . t is tangent to C .

26. (a) (3,1) , (-1,3)
(b) (1,2)
(c) $-\frac{1}{2}$
(d) $y = 2x$
(e) $\sqrt{5}$

27. (a) $\sqrt{3}$, $-\sqrt{3}$
(b) Yes.
(c) No.

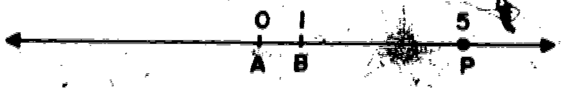
28. (a) (3,-4) , (-3,-4)
(b) (-3,4) , (-4,3)
(c) Same as (b)
(d) No points of intersection

Chapter 8

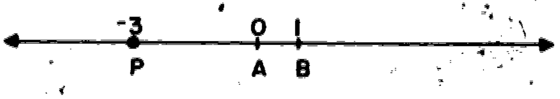
ANSWERS AND SOLUTIONS

Problem Set 8-1

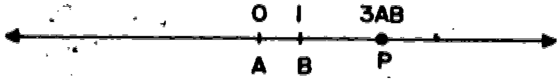
1. (a)



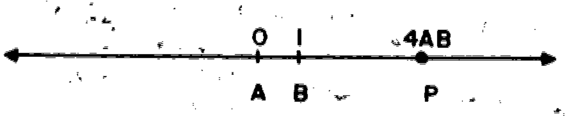
(b)



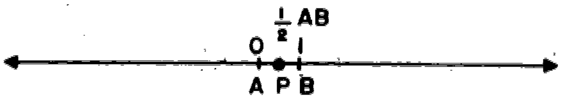
(c)



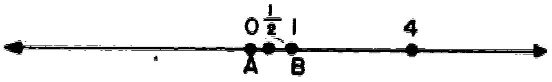
(d)



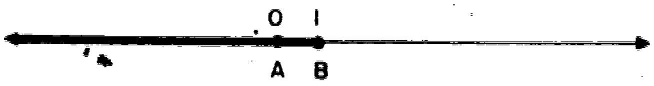
(e)



(f)



(g)

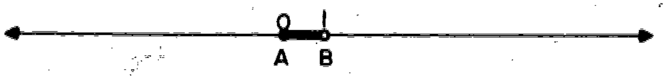


All points in this ray.

(h)

All points in this ray.

(i)

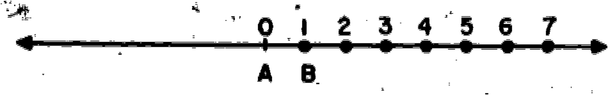


A and all points of the interior $\overline{AB}$.

(j)

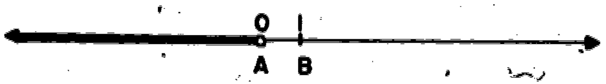
All points in the interior of this ray.

2. (a)



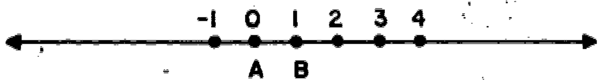
An infinite set.

(b)



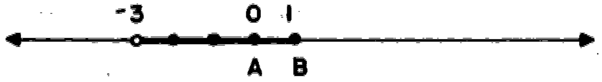
All interior points of the ray opposite to $\overrightarrow{AB}$.

(c)



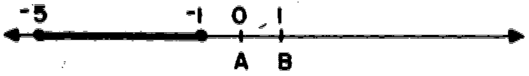
6 points

(d)



The interior of a segment and one endpoint.
The set is infinite.

(e)



(f) 1. The set in (b) is the interior of a ray.
2. The set in (d) consists of one endpoint and the interior of a segment.
3. The set in (e) is a segment.

3.

(a) 6
(b) 15
(c) 3k for $k \geq 0$. No values of x for $k < 0$.
(d) 7
(e) $2k + 1$ for $k \geq \frac{1}{2}$.

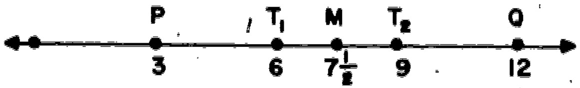
$2k + 1$ or $1 - 2k$ for $0 \leq k \leq \frac{1}{2}$.

(f) 7

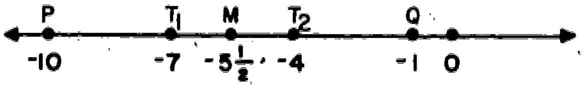
4. (a) 3
(b) 1
(c) 8
(d) 13
(e) 6
(f) 3
(g) 2a
*(h) a - b if $a \geq b$
b - a if $b < a$.

5. M $\qquad$ T_1 $\qquad$ T_2

(a) $7\frac{1}{2}$, 6, 9

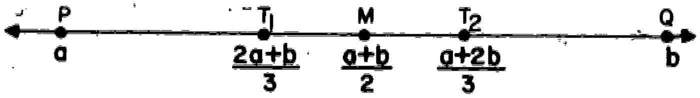


(b) $-5\frac{1}{2}$, -7, -4



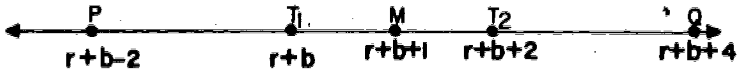
(c) 3, $5\frac{1}{2}$, 8

(d) $\frac{a+b}{2}$, $a + \frac{b-a}{3}$, $a + \frac{2(b-a)}{3}$



(e) r, $(r+a) - \frac{2a}{3}$, $(r+a) - \frac{4a}{3}$

(f) r + b + 1, (r + b - 2) + 2, (r + b - 2) + 4,
r + b + 1, r + b, r + b + 2



6. (a)

Letter	Row, Seat
A	N , 3
B	M , 4
C	L , 5
D	N , 7
E	M , 2
F	L , 6
G	N , 5
H	L , 1

(b)

Letter	Street, Avenue
A	11 , 8
B	10 , 6
C	9 , 9
D	8 , 5
E	11 , 4
F	7 , 4
G	8 , 10
H	6 , 7

(c)

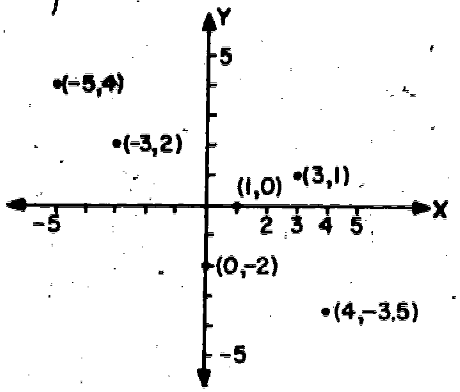
Letter	Floor, Row, Table
A	3 , 1 , 3
B	2 , 2 , 2
C	2 , 3 , 3
D	1 , 2 , 2
E	1 , 2 , 4
F	3 , 3 , 2
G	2 , 1 , 1
H	1 , 1 , 4

(d) and (e)

Letter	E or W , N or S	Problem (e) adds Elevation
A	0° , 0°	5000 ft.
B	45° E , 45° N	5200 ft.
C	45° W , 70° S	5400 ft.
D	20° W , 45° N	5600 ft.
E	90° E , 45° N	5800 ft.
F	20° W , 0°	6000 ft.
G	45° E , 0°	6200 ft.
H	90° E , 70° S	6400 ft.

Problem Set 8-2

1.



2. (a) (5,0)
   (b) (0,6)

3. I , III , II , IV , II .

4. It means that for every ordered pair of real numbers there corresponds a unique point and for every point there corresponds a unique ordered pair of numbers.

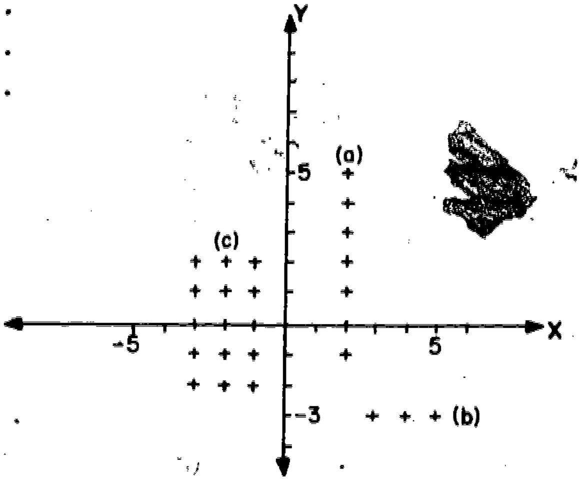
5. (3,2) , (3,5) , (3,8) .
   P , R , Q .

6. The set is a vertical line which intersects the x-axis in a point whose coordinate is 3 .

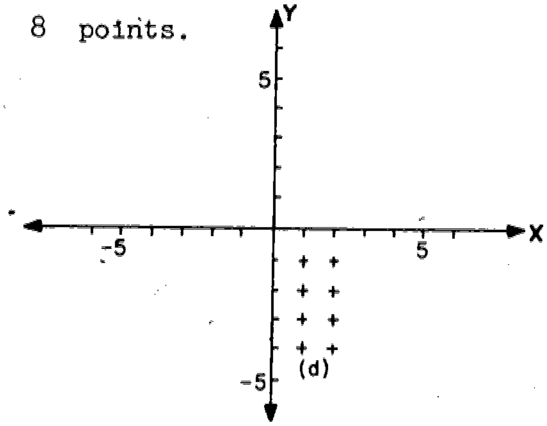
   The set is a horizontal line which intersects the y-axis in a point whose coordinate is -5 .

   They intersect in the point (3,-5) .

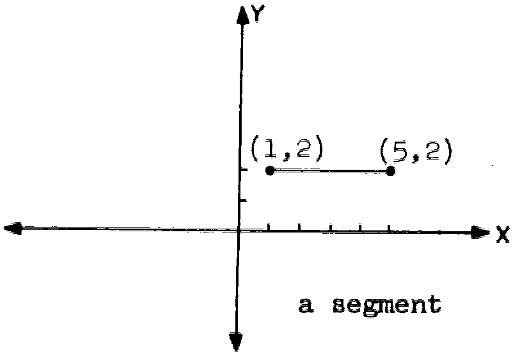
7. (a) 7 points.
   (b) 3 points.
   (c) 15 points.



(d) 8 points.

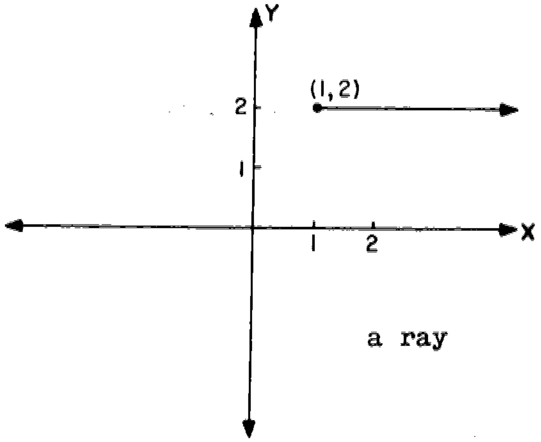


8. (a)



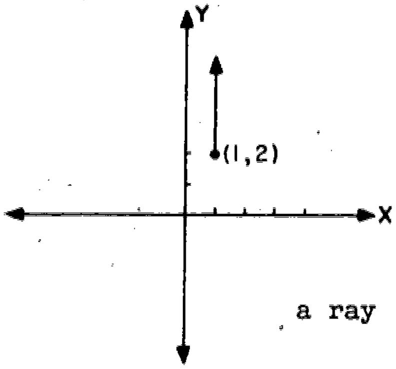
a segment

(b)



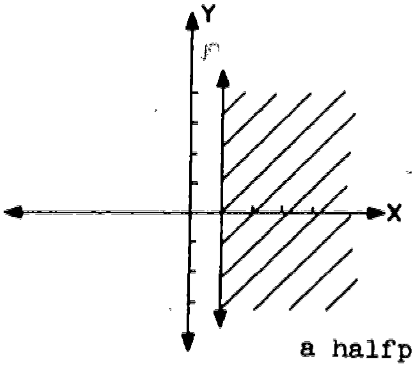
a ray

(c)



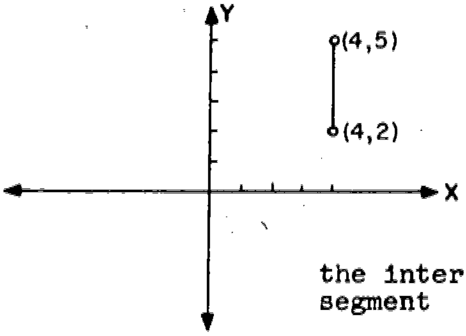
a ray

(d)



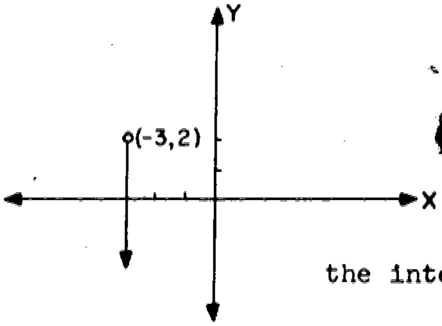
a halfplane with edge

(e)



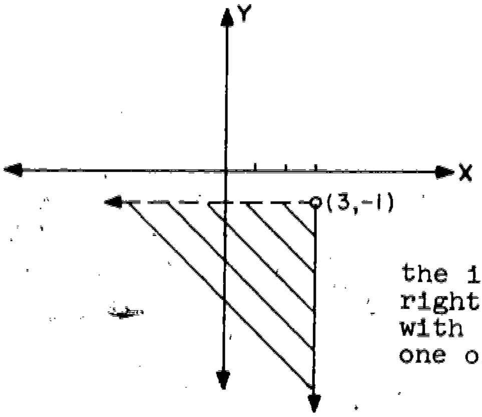
the interior of a segment

(f)



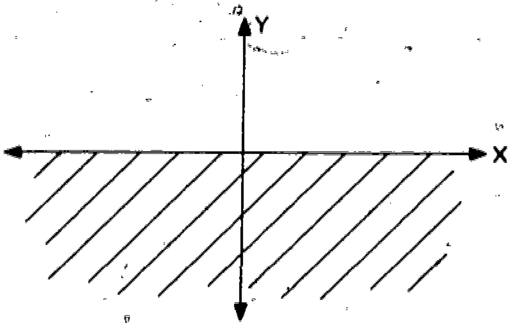
the interior of a ray

(g)



the interior of a
right angle togethe
with the interior o
one of its sides

(h)



halfplane

*9. (a) 4 units. With respect to the x-coordinate system on the x-axis the Ruler Postulate may be applied.

(b) CD = 4 . Consider A = (3,0) and B = (7,0) the respective projections of C and D into the x-axis. Quadrilateral ABDC is a parallelogram. Therefore CD = AB = 4 .

10. (2,-3) , (-1,-1) , (3,0) , (0,1) , (-5,4) , (8,6) .

11. $(-\pi,6)$, (-3,4) , (0,8) , (2,0) , $(\pi,-2)$, (4,-3) .

*12. (a) 13 (d) $b - a$ if $b \geq a$, $a - b$ if $b < a$
(b) 13 (e) $t - 5$ if $t \geq 5$, $5 - t$ if $t < 5$
(c) 4

13. (a) The set of points in Quadrant IV.

(b) Points in Quadrant I or on x-axis and to the right of the origin.

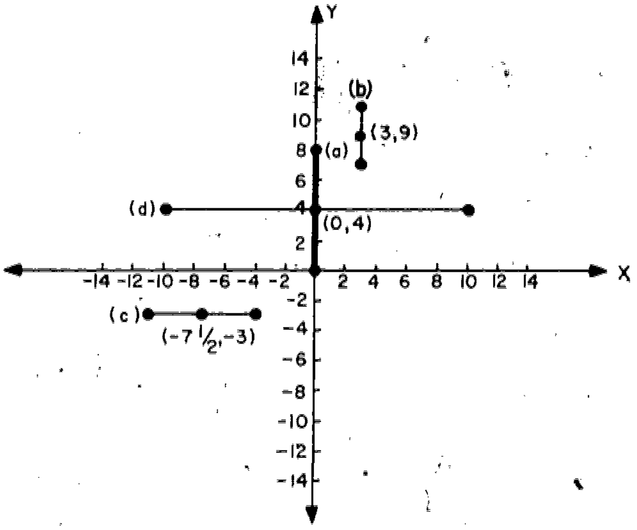
(c) Quadrant III.

(d) The Right halfplane whose edge is a vertical line 2 units to the left of the y-axis.

(e) At an intersection of a line $x = a$ and a line $y = b$, where a and b are integers.

(f) Any point in the xy-plane.

*14.



*15. (a) $(3,a)$

(b) $(1,\frac{a+b}{2})$

(c) $(\frac{3a}{2},c)$

(d) $(\frac{x_1 + x_2}{2},y_1)$

16. The point is in Quadrant III, 7 units from the y-axis and 8 units from the x-axis.

Problem Set 8-3

1. (a) $\sqrt{36 + 100} = \sqrt{136}$

(b) $\sqrt{36 + 100} = \sqrt{136}$

(c) $\sqrt{25 + 144} = 13$

(d) $\sqrt{49 + 576} = 25$

(e) $\sqrt{64 + 225} = 17$

(f) $\sqrt{1 + 1} = \sqrt{2}$

(g) $\sqrt{1521 + 6400} = 89$

(h) $\sqrt{100 + 25} = \sqrt{125}$

(i) $\sqrt{25 + 16} = \sqrt{41}$

(j) $\sqrt{16 + 9} = 5$

(k) $\sqrt{4.84 + 1.21} = \sqrt{6.05}$

(l) $\sqrt{25\pi^2 + 4\pi^2} = \pi\sqrt{29}$

2. (a) $(3,5)$
   (b) $(-3,5)$
   (c) $(3\frac{1}{2},8)$
   (d) $(-\frac{1}{2},\frac{7}{2})$
   (e) $(-5,2)$
   (f) $(2a,2)$
   (g) $(-r,3s)$

3. (a) $(x_2 - x_1)^2 + (y_2 - y_1)^2$

   (b) $(x - 0)^2 + (y - 0)^2 = 25$

4. (a) $\sqrt{9 + 16} = \sqrt{25}$
   $\sqrt{9 + 0} = \sqrt{9}$ $\quad 25 = 9 + 16$
   $\sqrt{0 + 16} = \sqrt{16}$

   (b) $\sqrt{121 + 9} = \sqrt{130}$
   $\sqrt{1 + 25} = \sqrt{26}$ $\quad 130 = 26 + 104$
   $\sqrt{100 + 4} = \sqrt{104}$

   (c) $\sqrt{36 + 64} = \sqrt{100}$
   $\sqrt{64 + 16} = \sqrt{80}$ $\quad 100 = 80 + 20$
   $\sqrt{4 + 16} = \sqrt{20}$

   (d) $\sqrt{16 + 9} = \sqrt{25}$
   $\sqrt{1 + 49} = \sqrt{50}$ $\quad 50 = 25 + 25$
   $\sqrt{9 + 16} = \sqrt{25}$

   (e) $\sqrt{484 + 16} = \sqrt{500}$
   $\sqrt{36 + 144} = \sqrt{180}$ $\quad 500 = 180 + 320$
   $\sqrt{256 + 64} = \sqrt{320}$

   (f) $\sqrt{16 + 36} = \sqrt{52}$
   $\sqrt{81 + 36} = \sqrt{117}$ $\quad 169 = 52 + 117$
   $\sqrt{169 + 0} = \sqrt{169}$

5. (a) $AC = \sqrt{25 + 16}$; $BD = \sqrt{25 + 16}$

   (b) Midpoint of $\overline{AC}$ is $(\frac{5}{2},2)$.

   Midpoint of $\overline{BD}$ is $(\frac{5}{2},2)$.

6. Midpoint of $\overline{BC}$ is $(1,2)$.

Length of median to $\overline{BC}$ is $\sqrt{10}$.

7. (a) Length of median to $\overline{ST}$ is $\frac{1}{2}\sqrt{433}$.

(b) Length of median to $\overline{RS}$ is $\frac{1}{2}\sqrt{193}$.

8. $(\frac{-1 + 7}{2} , \frac{0 + 4}{2}) = (3,2) = C$

$AC = \sqrt{4^2 + 2^2} = \sqrt{20}$

$CB = \sqrt{4^2 + 2^2} = \sqrt{20}$ $\qquad \sqrt{80} = 2\sqrt{20}$; $AC = CB = \frac{1}{2}AB$.

$AB = \sqrt{8^2 + 4^2} = \sqrt{80}$

9. (a) $AB = \sqrt{49 + 81} = \sqrt{130}$

$AC = \sqrt{9 + 121} = \sqrt{130}$

(b) Length of median to $\overline{AB}$ is $\sqrt{(3 - \frac{5}{2})^2 + (0 - \frac{13}{2})^2} = \sqrt{\frac{170}{4}}$

Length of median to $\overline{AC}$ is $\sqrt{(\frac{9}{2} + 1)^2 + (\frac{11}{2} - 2)^2} = \sqrt{\frac{170}{4}}$

10. $AB = \sqrt{16 + 36} = \sqrt{52} = 2\sqrt{13}$

$BC = \sqrt{4 + 9} = \sqrt{13}$

$AC = \sqrt{36 + 81} = \sqrt{117} = 3\sqrt{13}$

$AB + BC = AC$; A, B, C are collinear since they cannot be the vertices of a triangle. See Theorem 6-21.

11. $(0 - 6)^2 + (y + 2)^2 = 100$

$(y + 2)^2 = 64$

$(y + 2) = 8$ or $y + 2 = -8$

$y = 6$ or $y = -10$

12. $(x - 1)^2 + (0 + 6)^2 = 100$

$(x - 1)^2 = 64$

$x - 1 = +8$ or $x - 1 = -8$

$x = 9$ or $x = -7$

Two points satisfy the requirements: (0,9) and (0,-7) .

*13. $AD = \sqrt{b^2 + c^2}$

$BC = \sqrt{(a + b - a)^2 + (c - 0)^2} = \sqrt{b^2 + c^2}$.

14. One diagonal $= \sqrt{[a - (-a)]^2 + [a - (-a)]^2}$

$= \sqrt{4a^2 + 4a^2}$

Other diagonal $= \sqrt{(-a - a)^2 + [a - (-a)]^2}$

$= \sqrt{4a^2 + 4a^2}$

Therefore the diagonals are congruent.

15. xy-system: $P(-8,2)$, $Q(4,-3)$

$PQ = \sqrt{144 + 25} = 13$

x'y'-system: $P(-6,-4)$, $Q(6,1)$

$PQ = \sqrt{144 + 25} = 13$.

Yes, as long as the coordinate system on each of the axes is established with reference to the same (or equivalent) unit-pair.

Problem Set 8-5

1. (a) IV
(b) III
(c) II
(d) I
(e) II
(f) I, III, the empty set
(g) II, IV, the empty set
(h) I, II, III, IV
(i) I, IV
(j) IV
(k) II intersected with the lin through the origin bisecting the angle formed by the side of Quadrant II.

2. (a) $(0,-3\frac{1}{2})$, $(0,3\frac{1}{2})$.
(b) $(-4,0)$, $(8,0)$ or $(-8,0)$, $(4,0)$.
(c) $(0,0)$, $(0,r)$; $(0,0)$, $(r,0)$; $(0,0)$, $(-r,0)$; $(0,0)$, $(0,-r)$.
(d) $(-4,5)$, $(4,5)$.

(e) $(-\frac{5}{\sqrt{2}},0)$, $(0, \frac{-5}{\sqrt{2}})$; $(\frac{-5}{\sqrt{2}},0)$, $(0, \frac{5}{\sqrt{2}})$;

$(\frac{5}{\sqrt{2}},0)$, $(0, \frac{-5}{\sqrt{2}})$; $(\frac{5}{\sqrt{2}},0)$, $(0, \frac{5}{\sqrt{2}})$

Note that $\frac{5}{\sqrt{2}} = \frac{5\sqrt{2}}{2}$.

(f) Endpoints of $\overline{AB}$: $(0,0)$, $(0,6)$ or
$(0,0)$, $(0,-6)$.
Endpoints of $\overline{CD}$: $(3,2)$, $(3,8)$ or
$(3,2)$, $(3,-4)$.

3. (a) $(-3,0)$, $(3,0)$, $(0,4)$.
(b) $(0,2.5)$, $(0,-2.5)$, $(3,0)$.
(c) $(-3,0)$, $(3,0)$, $(0,4)$.
(d) $A = (0,0)$, $B = (7,0)$, $C = (10,5)$,
$D = (e,5)$, or
$A = (0,0)$, $B = (-7,0)$, $C = (-7 + e,5)$,
$D = (e,5)$.

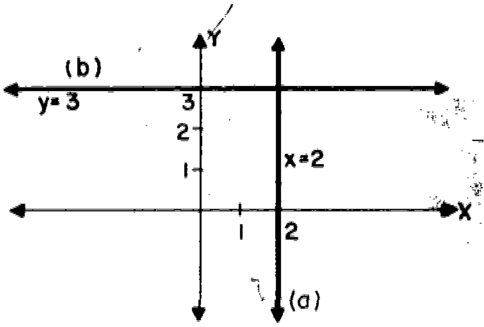
4. (a) $C = (0,0)$, $B = (-10,0)$, $A = (0,21)$ or $(0,-21)$
(b) $A = (0,0)$, $B = (4,0)$, $C = (2,3)$ or $(2,-3)$.
(c) $A = (3,2)$, $B = (3,-2)$, $C = (0,0)$.
(d) $A = (-5,0)$, $B = (5,0)$, $C = (0,5\sqrt{3})$; or
$A = (5,0)$, $B = (-5,0)$, $C = (0,5\sqrt{3})$.

5. (a) $A = (0,a)$, $B = (-b,0)$, $C = (0,0)$.
(b) $A = (0,0)$, $B = (b,0)$, $C = (\frac{b}{2},a)$.
(c) $A = (a,\frac{b}{2})$, $B = (a, -\frac{b}{2})$, $C = (0,0)$; or
$A = (a, -\frac{b}{2})$, $B = (a,\frac{b}{2})$, $C = (0,0)$.
(d) $(-\frac{s}{2},0)$, $(\frac{s}{2},0)$, $(0,\frac{s}{2}\sqrt{3})$.

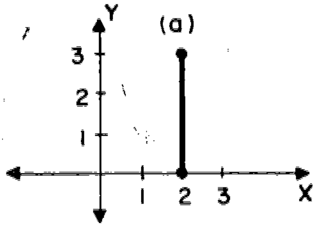
Problem Set 8-6

Problem 1 is an exploratory problem designed to introduce the work in the next section. It should not be omitted.

1. The lines are both vertical, hence parallel, and $4 - (-4) = 8$ units apart.
2. $(0,3)$, $(\pi,3)$, $(-2,3)$, for instance.
3.



4. The union is the set of all points each of which lies in one or both of the two lines. Yes.
5. The intersection is the set whose only element is the point $(2,3)$. Yes. Yes.
6. (a)

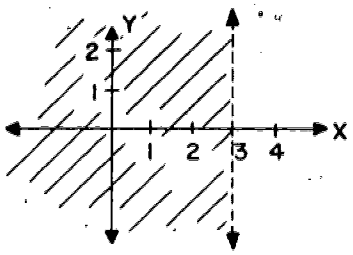


(b) a line segment

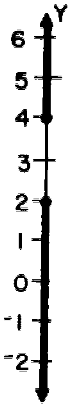
(c) an infinite number

7. (a) A ray

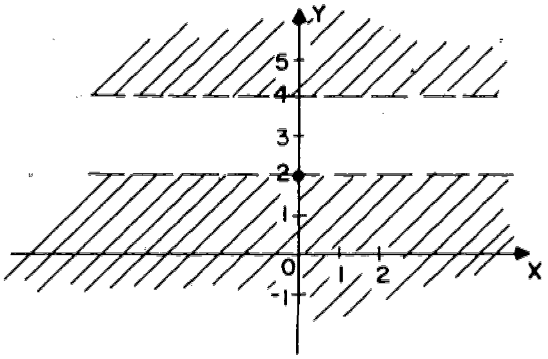
(b) A halfplane



(c) Two rays

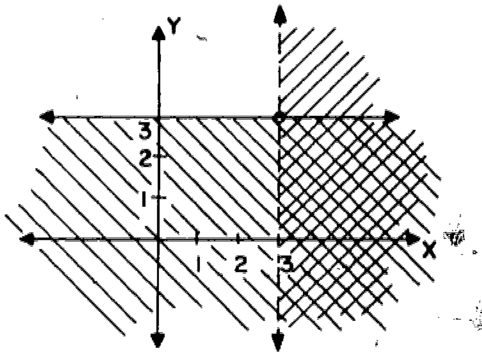


(d) Two halfplanes



8. (a) The union of two intersecting halfplanes and the edge of one of them.

$\{(x,y): x > 3 \text{ or } y \leq 3\}$.

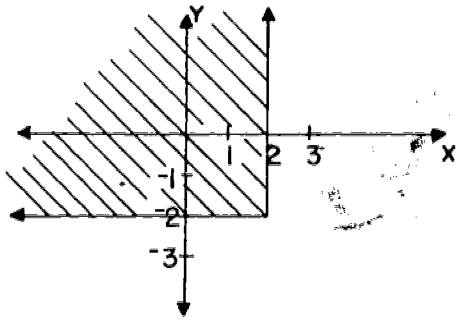


(b) The union of a right angle and its interior.

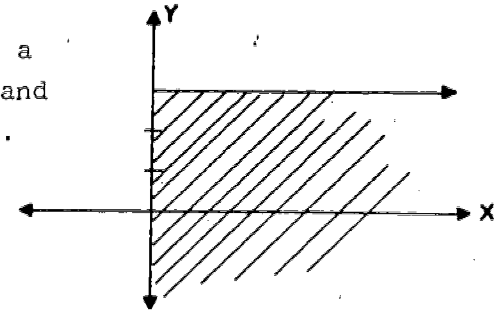
$\{(x,y): x \leq 2 \text{ and } y \geq -2\}$.

or

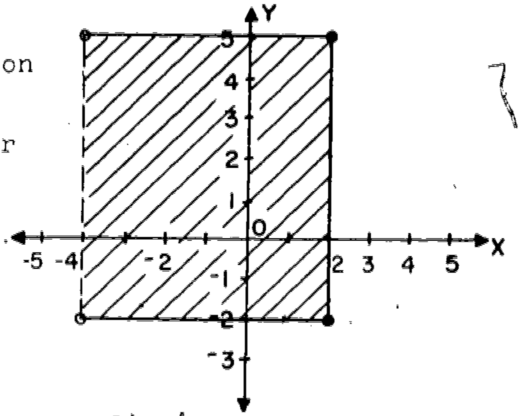
$\{(x,y): x \leq 2 , y \geq -2\}$.



(c) The union of a right angle and its interior.



(d) The union of a rectangular region and three of its sides (except for two endpoints).



9\. (a) $\{(x,y):\ y = 7 \text{ or } y = -3\}$.

(b) $\{(x,y):\ |x| = 4\}$ or $\{(x,y):\ x = -4 \text{ or } x = 4\}$

(c) $\{(x,y):\ x = -5\ ,\ y = 3\}$.

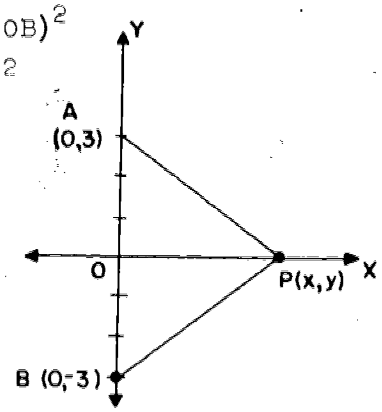
(d) $\{(x,y):\ y = 0\}$.

Let $P(x,0)$ be in $\{(x,y):\ y = 0\}$.

Then $(PA)^2 = (PO)^2 + (OA)^2$
$= x^2 + (-3)^2 = x^2 + 9$

and $(PB)^2 = (PO)^2 + (OB)^2$
$= x^2 + (-3)^2$
$= x^2 + 9$

and $PA = PB$.



Converse: Let $P(x,y)$ be so located that $PA = PB$. Then $\triangle OAP = \triangle OBP$ by S.S.S. Then $\angle POA \cong \angle POB$. Hence $\overline{OP} \perp \overline{AB}$, $y = 0$, and $(x,y) \in \{(x,y): y = 0\}$.

10. Sets are equal if and only if their conditions are equivalent.

    (a) The two sets are equal since their conditions are equivalent.

    (b) The two sets are equal since, using properties of order, the conditions can be shown to be equivalent.

    (c) The two sets are not equal since proper use of order properties indicates that the conditions are not equivalent.

    (d) The two sets are not equal. The conditions are not equivalent because $-2x + 4 < 8$ is equivalent to $x > -2$.

    (e) The two sets are not equal. Every negative number is an element of $\{x: 6 \geq 3x\}$ while no negative number is an element of $\{x: \frac{6}{x} \geq 3\}$.

*11. (a) $k = 2$; $t = 2$. Parallel lines cut off proportional segments on two transversals. $k' = 3$; $MO = 3MN$.

    (b) 4, 4, 12.

    (c) 5, 2, $A'C' = \frac{5}{2}A'B'$, $\frac{5}{2}$.

    (d) 4.

    (e) Parallel lines intercept proportional segments on two transversals, and the definition of the length of a segment.

    $OP' = OA' + A'P' = x = 2 + 2k$.

    (f) Same as (e). $OP'' = OA'' + A''P'' = y$; $y = 3 + 2k$.

(g) (1) P lies in ray $\overrightarrow{AB}$, such that B is between A and P .

(2) $P = B$.

(3) P lies in $\overline{AB}$, but $P \neq A$ and $P \neq B$.

(4) P lies in the ray opposite to $\overrightarrow{AB}$.

(5) $P = A$.

Problem Set 8-7

1. (a) $\overleftrightarrow{AB} = \{(x,y): x = 1 + k , y = 4 + 2k , k \text{ is real}\}$

$\overline{AB} = \{(x,y): x = 1 + k , y = 4 + 2k , 0 \leq k \leq 1\}$,

$\overrightarrow{AB} = \{(x,y): x = 1 + k , y = 4 + 2k , k \geq 0\}$.

Ray opposite $\overrightarrow{AB} = \{(x,y): x = 1 + k ,$
$y = 4 + 2k, k \leq 0\}$.

(b) $\overleftrightarrow{AB} = \{(x,y): x = -1 + 3k , y = 3 - 3k , k \text{ is real}\}$

$\overline{AB} = \{(x,y): x = -1 + 3k , y = 3 - 3k , 0 \leq k \leq 1\}$

$\overrightarrow{AB} = \{(x,y): x = -1 + 3k , y = 3 - 3k , 0 \leq k\}$.

Ray opposite $\overrightarrow{AB} = \{(x,y): x = -1 + 3k ,$
$y = 3 - 3k , k \leq 0\}$.

(c) $\overleftrightarrow{AB} = \{(x,y): x = 3k , y = 2k , k \text{ is real}\}$, etc.

(d) $\overleftrightarrow{AB} = \{(x,y): x = 1 + 3k , y = 1 + 3k ,$
$k \text{ is real}\}$, etc.

(e) $\overleftrightarrow{AB} = \{(x,y): x = -1 + 2k , y = 3 - 5k ,$
$k \text{ is real}\}$, etc.

(f) $\overleftrightarrow{AB} = \{(x,y): x = -3 + 3k , y = -2 + 3k ,$
$k \text{ is real}\}$, etc.

(g) $\overleftrightarrow{AB} = \{(x,y): x = a + (c - a)k , y = b + (d - b)k ,$
$k \text{ is real}\}$, etc.

(h) $\overleftrightarrow{AB} = \{(x,y): x = a + 2ak , y = 2a + 2ak ,$
$k \text{ is real}\}$, etc.

2. The midpoint of $\{(x,y): x = 1 + k , y = 4 + 2k , 0 \le k \le 1\}$ is $(1 + k , 4 + 2k)$ with $k = \frac{1}{2}$, or $(\frac{3}{2}, 5)$.

The midpoint of $\{(x,y): x = -1 + 3k , y = 3 - 3k , 0 \le k \le 1\}$ is $(-1 + 3k , 3 - 3k)$ with $k = \frac{1}{2}$, or $(\frac{1}{2}, \frac{3}{2})$.

3. (a) $(8,12)$

(b) $(-\frac{11}{2}, -\frac{3}{2})$

(c) $(\frac{7}{12}, 0)$

(d) $(2.6,-6)$

(e) $(0,0)$

(f) $(\frac{s}{2}, \frac{r}{2})$

4. (a) $(16,0)$

(b) $(7,4)$

(c) $(8,7)$

(d) $(-2,-4)$

(e) $(9a,5b)$

(f) $(-3r,7s)$

5. $(1, \frac{2}{3})$; $(2,2)$

6. $(\frac{1}{3}, -\frac{1}{3})$.

7. (a) $x = -1 + 3 \cdot 2 = 5$, $y = 3 - 3 \cdot 2 = -3$,
$(5,-3)$.

(b) $x = -1 + 3 \cdot 100 = 299$, $y = 3 - 3 \cdot 100 = -297$,
$(299,-297)$.

(c) $x = -1 + 3\sqrt{3} = 3\sqrt{3} - 1$, $y = 3 - 3\sqrt{3}$,
$(3\sqrt{3} - 1 , 3 - 3\sqrt{3})$.

(d) $x = -1 + 3\pi$, $y = 3 - 3\pi$, $(-1 + 3\pi , 3 - 3\pi)$.

8. (a) $x = -1 + 2(-2) = -5$, $y = 3 - 5(-2) = 13$,
$(-5,13)$.

(b) $x = -1 + 2(-20) = -41$, $y = 3 - 5(-20) = 103$,
$(-41,103)$.

(c) $x = -1 + 2(-3.5) = -8$, $y = 3 - 5(-3.5) = 20.5$,
$(-8,20.5)$.

(d) $x = -1 + 2(-\frac{1}{2}) = -2$, $y = 3 - 5(-\frac{1}{2}) = 5\frac{1}{2}$,
$(-2,5\frac{1}{2})$.

9. (a) If P is between A and B then $AP + PB = AB$, that is $PB = AB - AP$. Since $AP = 3PB$, we have in this case $AP = 3(AB - AP)$, that is $AP = \frac{3}{4}AB$. Thus, using the method of solution of Problems 7 and 8, $P = (2, -\frac{1}{4})$.

If B is between A and P then $AB + BP = AP$, that is $PB = AP - AB$. Since $AP = 3PB$, we have in this case $AP = 3(AP - AB)$, that is $AP = \frac{3}{2}AB$. Thus, using the method of solution of Problems 7 and 8, $P = (5,-5.5)$.

If A is between B and P then $PA + AB = PB$. Since $AP = 3PB$, we have in this case $AP = 3(AP + AB)$, that is $AP = -\frac{3}{2}AB$. But this is impossible, since the distance AP cannot be negative. Hence there are two solutions $P = (2, -\frac{1}{4})$ or $(5, -5.5)$.

(b) Using the same analysis as in Part (a) we find two solutions: $P = (-\frac{1}{5}, \frac{18}{5})$ or $(-\frac{7}{3}, \frac{22}{3})$.

(c) $P = (11,-16)$ or $(-5,12)$.

(d) $P = (19,-30)$ or $(-21,40)$.

10. (a) $\overleftrightarrow{CD}$ is horizontal.

$$\overleftrightarrow{CD} = \{(x,y): x = -1 + \big(5 - (-1)\big)k, \; y = 2 + (2 - 2)k, \; k \text{ is real}\}$$
$$= \{(x,y): x = -1 + 6k, \; y = 2, \; k \text{ is real}\}.$$

If $k = 0$, $(x,y) = (-1,2)$. If $k = 1$, $(x,y) = (5,2)$. If $k = -2$, $(x,y) = (-13,2)$.

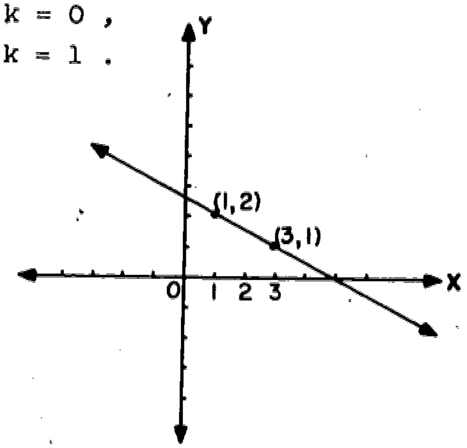
(b)
$$\{(x,y): x = x_1 + k(x_2 - x_1), \; y = a + k(a - a), \; k \text{ is real}\}$$
$$= \{(x,y): x = x_1 + k(x_2 - x_1), \; y = a, \; k \text{ is real}\}$$
$$= \{(x,y): y = a\},$$
which is the horizontal line $\overleftrightarrow{CD}$.

(c) $\{(x,y)\colon x = a + k(a - a) , y = y_1 + k(y_2 - y_1) ,$
k is real$\}$
$= \{(x,y)\colon x = a , y = y_1 + k(y_2 - y_1) ,$
k is real$\}$
$= \{(x,y)\colon x = a\}$, which is the
vertical line $\overleftrightarrow{EF}$.

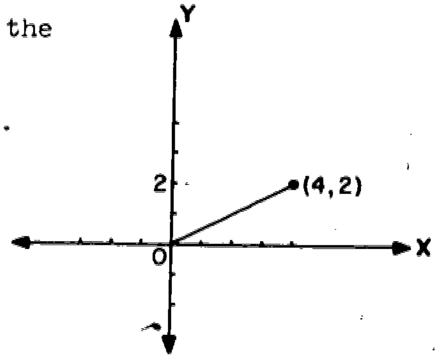
11. (a) $\overline{AB} = \{(x,y)\colon x = 0 , y = 3k , 0 \leq k \leq 1\}$
$\overline{AC} = \{(x,y)\colon x = 4k , y = 0 , 0 \leq k \leq 1\}$
$\overline{BC} = \{(x,y)\colon x = 4k , y = 3 - 3k , 0 \leq k \leq 1\}$.

(b) $\overline{DE} = \{(x,y)\colon x = -3 + 3k , y = 3k , 0 \leq k \leq 1\}$
$\overline{DF} = \{(x,y)\colon x = -3 + 6k , y = 0 , 0 \leq k \leq 1\}$
$\overline{EF} = \{(x,y)\colon x = 3k , y = 3 - 3k , 0 \leq k \leq 1\}$.

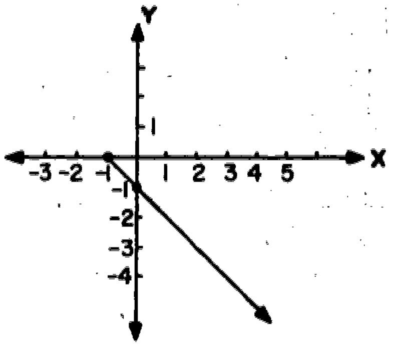
12. (a) $\{(x,y)\colon x = 1 + 2k , y = 2 - k , k \text{ is real}\}$.
$(1,2)$ for $k = 0$,
$(3,1)$ for $k = 1$.



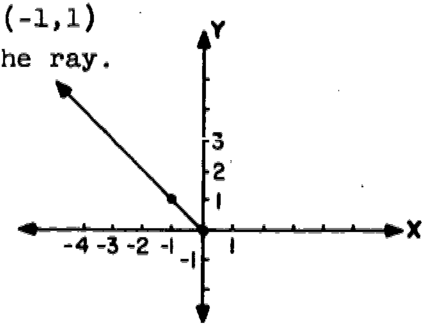
(b) $\{(x,y)\colon x = 2k , y = k , 0 < k < 2\}$
$(0,0)$ for $k = 0$, and
$(4,2)$ for $k = 2$.
These are the
endpoints of the
segment.



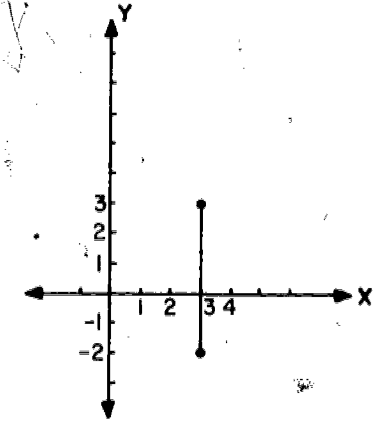
(c) $\{(x,y): x = -1 + k \ , \ y = -k \ , \ k \geq 0\}$ yields for $k = 0$, $(-1,0)$ and for $k = 1$, $(0,-1)$ and these are points in the ray, the first being its endpoint.



(d) $\{(x,y): x = k \ , \ y = -k \ , \ k \leq 0\}$ yields for $k = 0$, $(0,0)$ the endpoint of the ray, and for $k = -1$, $(-1,1)$ another point in the ray.



(e) $\{(x,y): x = 3 \ , \ y = k \ , \ -2 \leq k \leq 3\}$ yields for $k = -2$, $(3,-2)$, and for $k = 3$, $(3,3)$, and these are the endpoints of the segment.



13. $\{(x,y): x = 3k - 1, y = 3 - 5k, k \text{ is real}\}$.

(a) If $x = 5$, $k = 2$, $y = -7$.

(b) If $y = 8$, $k = -1$, $x = -4$.

(c) If $x = 29$, $k = 10$, $y = -47$.

(d) If $y = 0$, $k = \frac{3}{5}$, $x = \frac{4}{5}$.

(e) If $x = 0$, $k = \frac{1}{3}$, $y = \frac{4}{3}$.

14. $D = (\frac{9}{2},0)$, $E = (6,3)$, $F = (\frac{3}{2},3)$.

$\overline{FB} = \{(x,y): x = \frac{3}{2} + \frac{15}{2}k , y = 3 - 3k , 0 \leq k \leq 1\}$,

$\overline{EA} = \{(x,y): x = 6 - 6k , y = 3 - 3k , 0 \leq k \leq 1\}$,

$\overline{DC} = \{(x,y): x = \frac{9}{2} - \frac{3}{2}k , y = 6k , 0 \leq k \leq 1\}$.

Each of these segments contains the point $(4,2)$. Take $k = \frac{1}{3}$ in each case.)

15. $p = \{(x,y): x = a + ck , y = b + dk , k \text{ is real}\}$.

(a) If $c = 0$, $p = \{(x,y): x = a , y = b + dk , k \text{ is real}\}$
$= \{(x,y): x = a\}$, which is a vertical line.

(b) If $d = 0$, $p = \{(x,y): x = a + ck , y = b , k \text{ is real}\}$
$= \{(x,y): y = b\}$, which is a horizontal line.

(c) If $a = 0 = b$, then
$p = \{(x,y): x = ck , y = dk , k \text{ is real}\}$.

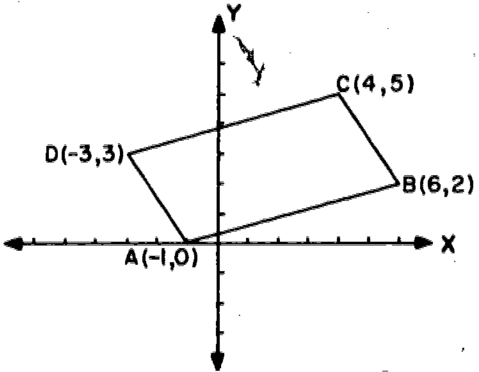
If $k = 0$, $(x,y) = (0,0)$, which is a point in p .

Problem Set 8-8a

1. (a) $\frac{1}{3}$
   (b) $-\frac{1}{3}$
   (c) $\frac{7}{4}$
   (d) $\frac{3}{4}$
   (e) $-\frac{15}{8}$
   (f) -1
   (g) $-\frac{5}{7}$
   (h) $\frac{3}{4}$
   (i) 1
   (j) -1

2. (a) 6
   (b) $\frac{9}{2}$
   (c) -1
   (d) Any real number except 5 .

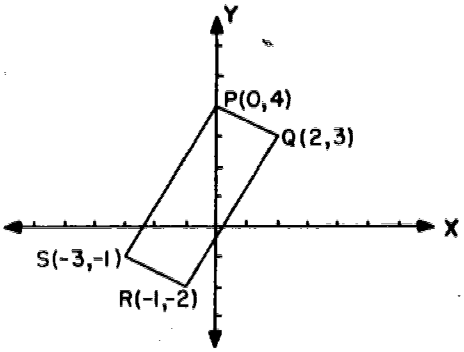
3.



Slope of $\overline{AB}$ = slope of $\overline{DC}$ = $\frac{2}{7}$

Slope of $\overline{DA}$ = slope of $\overline{CB}$ = $-\frac{3}{2}$.

4.



Slope of $\overline{RQ}$ = $\frac{5}{3}$ = slope of $\overline{SP}$.

Slope of $\overline{SR}$ = $-\frac{1}{2}$ = slope of $\overline{PQ}$.

5. (a)	Negative	(d)	Negative
(b)	Zero	(e)	Negative
(c)	Positive	(f)	Positive

Positive slope indicates "uphill" from left to right; negative slope "downhill" from left to right.

6. $\frac{101}{100} > \frac{100}{101}$, so segment from (0,0) to (100,101) is steeper.

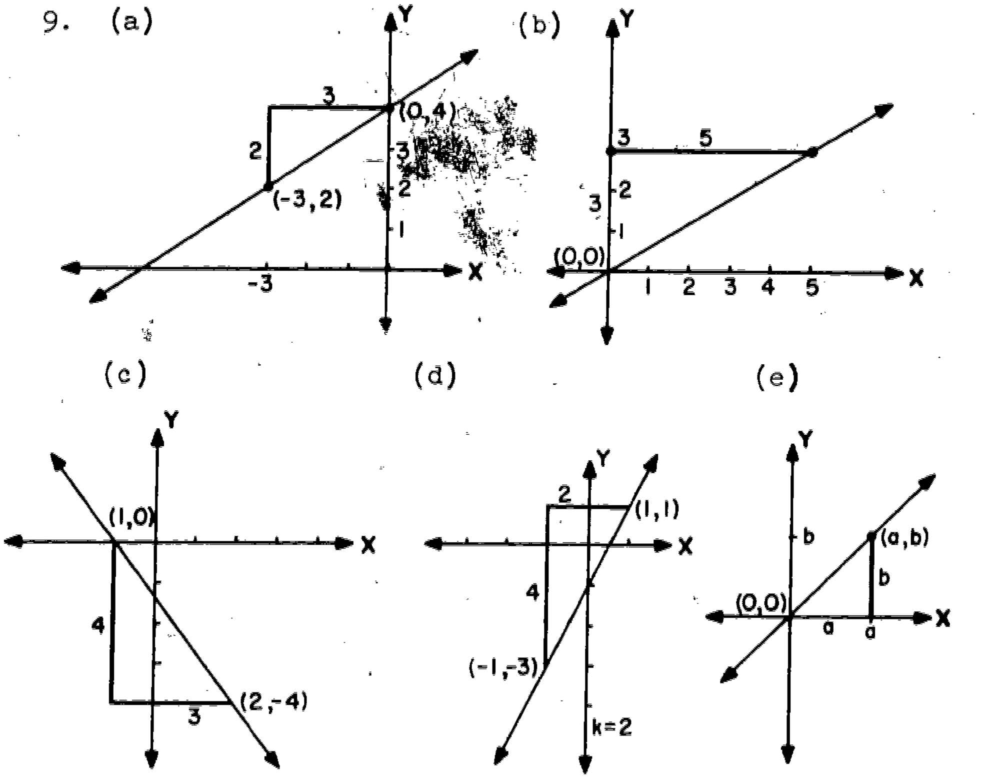
7. slope $= \dfrac{\frac{b}{a} - \frac{a}{b}}{b - a} = \dfrac{\frac{b^2 - a^2}{ab}}{b - a} = \dfrac{b + a}{ab}$.

8. $\overleftrightarrow{AB} = \{(x,y): x = 3 - 2k , y = -1 + 3k , k \text{ is real}\}$

contains $P_1(x_1,y_1) = (3,-1)$ and $P_2(x_2,y_2) = (1,2)$.

Then $\dfrac{y_2 - y_1}{x_2 - x_1} = \dfrac{3}{-2} = -\dfrac{3}{2}$ is the slope of $\overleftrightarrow{AB}$.

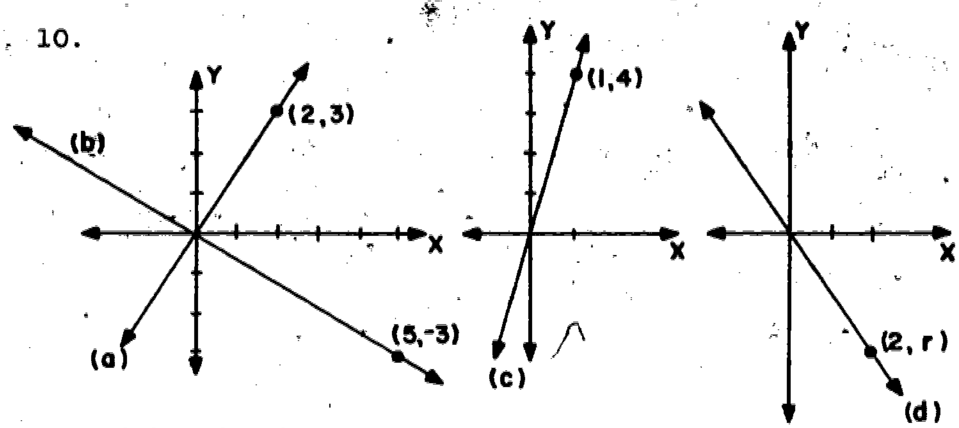
9.



(a) (b) (c) (d) (e)

The second point in each of the preceding graphs could have been located in several ways. Of course, the same line would result in every case.

10.



11. (a) $\{(x,y): x = 2k , y = 3k\}$

(b) $\{(x,y): x = 5k , y = -3k\}$ or
$\{(x,y): x = -5k , y = 3k\}$

(c) $\{(x,y): x = k , y = 4k\}$

(d) $\{(x,y): x = 2k , y = rk\}$.

Problem Set 8-8b

1. (a) Slope of $\overleftrightarrow{AB} = \frac{3}{8}$; slope of $\overleftrightarrow{CD} = \frac{3}{8}$.

Slope of $\overleftrightarrow{AD} = 5$; slope of $\overleftrightarrow{BC} = 5$.

(b) Slope of $\overleftrightarrow{AB} = \frac{1}{10}$; slope of $\overleftrightarrow{CD} = \frac{1}{10}$.

Slope of $\overleftrightarrow{AD} = 0$; slope of $\overleftrightarrow{BC} = 0$.

(c) Slope of $\overleftrightarrow{AB} = \frac{5}{4}$; slope of $\overleftrightarrow{CD} = \frac{5}{4}$.

Slope of $\overleftrightarrow{AD} = -\frac{1}{3}$; slope of $\overleftrightarrow{BC} = -\frac{1}{3}$.

2. Slope of $\overleftrightarrow{AB} = -\frac{2}{7}$; slope of $\overleftrightarrow{CD} = -\frac{2}{9}$.

3. (a) Slope of $\overleftrightarrow{AB} = 4$; slope of $\overleftrightarrow{BC} = 4$; yes.

(b) $\frac{4 + 1}{-5 - 2} \neq \frac{-8 - 4}{16 + 5}$. No.

(c) Slope of $\overleftrightarrow{AB} = \frac{96}{96} = 1$; slope of $\overleftrightarrow{BC} = \frac{-100}{-100} = 1$; yes.

(d) Slope of $\overleftrightarrow{AB} = \frac{96}{96} = 1$; slope of $\overleftrightarrow{CD} = 1$;

$\therefore \overleftrightarrow{AB} \parallel \overleftrightarrow{CD}$.

Slope of $\overleftrightarrow{BC} = \frac{197}{197} = 1$; $\therefore \overleftrightarrow{AB} = \overleftrightarrow{CD}$.

4. (a) $p = \{(x,y): x = 3 + 3k , y = 8 + 2k , k \text{ is real}\}$

$k = 0$ yields $(3,8)$
$k = 1$ yields $(6,10)$
$k = -1$ yields $(0,6)$
$k = 2$ yields $(9,12)$.

(b) $q = \{(x,y): x = -1 + 4k , y = 0 - 3k , k \text{ is real}\}$

$k = 0$ yields $(-1,0)$
$k = 1$ yields $(3,-3)$
$k = -1$ yields $(-5,3)$.
$k = 2$ yields $(7,-6)$.

5. (a) $\{(x,y): x = 3 + 3k , y = 4 + 2k , k \text{ is real}\}$

Parametric equations: $\begin{cases} x = 3 + 3k , & k \text{ is real.} \\ y = 4 + 2k , & k \text{ is real.} \end{cases}$

(b) $\{(x,y): x = -1 + k , y = 3 - k , k \text{ is real}\}$

Parametric equations: $\begin{cases} x = -1 + k , & k \text{ is real.} \\ y = 3 - k , & k \text{ is real.} \end{cases}$

6. $m_{\overleftrightarrow{AB}} = -\frac{3}{2}$.

$\overleftrightarrow{CD} = \{(x,y): x = 2k , y = -3k , k \text{ is real}\}$, or
$\overleftrightarrow{CD} = \{(x,y): x = -2k , y = 3k , k \text{ is real}\}$.

7. Slope of $a = \frac{-k}{2k} = -\frac{1}{2}$; slope of $b = \frac{-h}{2h} = -\frac{1}{2}$.

However, $k = 0$ yields $(1,2)$ in a ,
$h = 0$ yields $(3,-1)$ in b ,
and $\frac{-1 - 2}{3 - 1} = -\frac{3}{2} \neq -\frac{1}{2}$.

Hence $a \neq b$.

8. Slope of p is $\frac{4}{2}$. Slope of $q = \frac{8}{4}$. These are equal. Also $(1,3)$ is on p and on q . So, $p = q$.

9. (a) Slope of $m = \frac{3}{2}$; slope of $n = -\frac{3}{2}$; m , n not parallel. Therefore m and n intersect in one point.

(b) We seek $\{(x,y): x = 1 + 2k \ , \ y = 2 + 3k$ and
$x = 1 - 2h \ , \ y = 2 + 3h \ ; \ h \ , \ k$ real$\}$

Hence, $1 + 2k = 1 - 2h$

and

$2 + 3k = 2 + 3h$.

That is, $k = -h$ and $k = h$.
So, $k = 0 = h$, which gives
$\{(x,y): x = 1 \ , \ y = 2\}$.
So, the point of intersection is $(1,2)$.

10. Slope $\overline{AB} = \frac{3}{2}$; slope $\overline{BC} = -\frac{7}{3}$; slope $\overline{CD} = \frac{3}{2}$,
slope $\overline{AC} = -\frac{4}{5}$; slope $\overline{AD} = -\frac{7}{3}$; slope $\overline{BD} = -10$.
$\overline{AB} \parallel \overline{CD}$; $\overline{BC} \parallel \overline{AD}$. The segments are distinct since their endpoints are different.

11. Slope $\overline{AB} = -\frac{2}{3} =$ slope $\overline{CD}$.

Slope $\overline{BC} = -3 =$ slope $\overline{AD}$ and B, C, A are not collinear. Hence ABCD is a parallelogram.

12. By definition of vertical lines, if m is a vertical line, it is parallel to the y-axis. If $n \parallel m$, then n is also parallel to the y-axis and hence is a vertical line. (Recall that, in this text, a line is parallel to itself.)

13. $D = (\frac{4 - 2}{2} , \frac{2 - 4}{2}) = (1,-1)$.

$E = (\frac{4 + 6}{2} , \frac{2 + 0}{2}) = (5,1)$.

$\overleftrightarrow{AC} = \frac{4}{8} = \frac{1}{2}$,

Slope $\overleftrightarrow{DE} = \frac{2}{4} = \frac{1}{2}$, so $\overleftrightarrow{AC} \parallel \overleftrightarrow{DE}$.

14. (a) Slope $\overline{AB} = 4$,

slope $\overline{BC} = \frac{1}{2}$,

slope $\overline{CD} = 5$,

slope $\overline{AD} = \frac{3}{8}$. It is false that ABCD is a parallelogram.

(b) Slope $\overline{PQ} = \frac{2}{3}$,

slope $\overline{QR} = -\frac{1}{5}$,

slope $\overline{RS} = \frac{2}{3}$,

slope $\overline{SP} = -\frac{1}{5}$. It is true that PQRS is a parallelogram.

15. $\frac{n - 0}{0 - 3n} = -\frac{1}{3} = \frac{2n - 0}{0 - 6n}$.

16. If $\overleftrightarrow{PQ} \| \overleftrightarrow{RS}$ then either they both have the same slope or both are vertical. In case they are vertical $a = 3$ and $b = 4$, so $a = b - 1$. In case they have the same slope, $\frac{1}{3 - a} = \frac{1}{4 - b}$; that is $4 - b = 3 - a$ or $a = b - 1$. On the other hand if $a = b - 1$ and $4 - b \neq 0$ then $\frac{1}{4 - b} = \frac{1}{4 - (a + 1)} = \frac{1}{3 - a}$, and $\overleftrightarrow{PQ} \| \overleftrightarrow{RS}$. Further, if $a = b - 1$ and $4 - b = 0$, then $b = 4$ and $a = 3$ and both lines are vertical and again $\overleftrightarrow{PQ} \| \overleftrightarrow{RS}$. Furthermore, in case $\overleftrightarrow{PQ} \| \overleftrightarrow{RS}$, $\overleftrightarrow{PQ} = \overleftrightarrow{RS}$ if and only if $\overleftrightarrow{QR} \| \overleftrightarrow{PQ}$. Now, slope $\overleftrightarrow{QR} = \frac{1}{3 - b}$. So $\overleftrightarrow{QR} \| \overleftrightarrow{PQ}$ implies $\frac{1}{3 - b} = \frac{1}{3 - a}$ or $a = b$. But $a = b - 1$, so $\overleftrightarrow{PQ} \neq \overleftrightarrow{RS}$.

Problem Set 8-9

1. (a) $\frac{x - 1}{4 - 1} = \frac{y - 4}{3 - 4}$

(b) $\frac{x - 0}{-3 - 0} = \frac{y - 5}{0 - 5}$

(c) $\frac{x - 0}{3 - 0} = \frac{y - (-5)}{0 - (-5)}$

(d) $\frac{x - (-3)}{5 - (-3)} = \frac{y - 2}{-4 - 2}$

(e) $\frac{x - 0}{7 - 0} = \frac{y - 0}{-8 - 0}$

(f) $\frac{x - (-1)}{1 - (-1)} = \frac{y - 1}{-1 - 1}$

2. (a) $y - 0 = \frac{1}{2}(x - 0)$

(b) $y - 5 = \frac{2}{3}\left(x - (-3)\right)$

(c) $y - 7 = -\frac{3}{4}\left(x - (-2)\right)$

(d) $y - (-2) = 2\left(x - (-3)\right)$

(e) $y - 2 = -1\left(x - (-3)\right)$

(f) $y - (-5) = 3(x - 0)$

3. $y - 8 = -\frac{3}{4}(x - 5)$

4. (a) $\frac{x - 0}{1 - 0} = \frac{y - 0}{6 - 0}$, or $y = 6x$

(b) $\frac{x - 0}{5 - 0} = \frac{y - 0}{2 - 0}$, or $y = \frac{2}{5}x$

(c) $\frac{x - 0}{3 - 0} = \frac{y - 0}{4 - 0}$, or $y = \frac{4}{3}x$ (Midpoint of $\overline{BC} = (3,4)$)

(d) $\frac{x - \frac{1}{2}}{\frac{5}{2} - \frac{1}{2}} = \frac{y - 3}{1 - 3}$, or $y = -x + \frac{7}{2}$ (Midpoint of $\overline{AB} = (\frac{1}{2},3)$; midpoint of $\overline{AC} = (\frac{5}{2},1)$)

(e) $\frac{x - 1}{5 - 1} = \frac{y - 6}{2 - 6}$, or $y = -x + 7$

(f) $\overleftrightarrow{AD} = \{(x,y): y = x\}$

$\overleftrightarrow{BC} = \{(x,y): y - 2 = -(x - 5)\}$

Solving these equations

(1) $y = -x + 7$

(2) $y = x$

$y = \frac{7}{2}$, $x = \frac{7}{2}$, therefore the point of intersection is $(\frac{7}{2}, \frac{7}{2})$.

5. a, c, d

6. (a) $y - 4 = 2(x + 2)$

   (b) $y - 4 = x + 2$

   (c) $y - 4 = 0$

   (d) $y - 4 = -(x + 2)$

   (e) $y - 4 = \frac{1}{2}(x + 2)$

   (f) $y - 4 = -\frac{3}{2}(x + 2)$

7. $m_p = \frac{1}{2}$, $m_q = -2$, $m_r = -2$, $m_s = \frac{1}{2}$

   Hence, $p \parallel s$ and $q \parallel r$.

8. (a) intersect in one point since $p \neq q$ and $p \nparallel q$.

   (b) $p = q$ since $p \parallel q$ and the point $(8,0)$ is on both p and q .

   (c) $p \parallel q$ since slopes are the same but $p \neq q$ since $(8,0)$ is on p but not on q

9. It is the equation of a line since it is linear. Further it contains $(a,0)$ and $(0,b)$ since

$$\frac{a}{a} + \frac{0}{b} = 1 \quad \text{and} \quad \frac{0}{a} + \frac{b}{b} = 1 .$$

10. If $x = 0$, $y = b$. Hence the point $(0,b)$ is on the line and this is the point of the y-axis which is on the line. If $x = 1$, $y = m + b$. Hence, $(1,m + b)$ is on the line. The slope of the line is determined from $(0,b)$ and $(1,m + b)$ as

$$\frac{m + b - b}{1 - 0} = m .$$

Problem Set 8-10

1. $p \perp r$; $q \perp s$.

2. (a) $-\frac{2}{7}$ (c) $\frac{2}{9}$

   (b) $\frac{7}{2}$ (d) $-\frac{9}{2}$

3. $\frac{2-0}{3-0} = \frac{2}{3}$; $\frac{3-0}{-2-0} = -\frac{3}{2}$; $(\frac{2}{3})(-\frac{3}{2}) = -1$.

4. $m_{\overline{AB}} = \frac{b}{a}$; $m_{\overline{BC}} = \frac{a}{-b}$; $(\frac{b}{a})(-\frac{a}{b}) = -1$.

5. $m_{\overline{PQ}} = \frac{-6-2}{5-1} = -2$; $m_{\overline{QR}} = \frac{1}{2} = \frac{b+6}{b-5}$

which yields $b - 5 = 2(b + 6)$ or $b = -17$.

6. (1) For $k = 0$, both $\overleftrightarrow{AB}$ and $\overleftrightarrow{CD}$ contain $(1,2)$.

(2) $m_{\overline{AB}} = \frac{3}{2}$, $m_{\overline{CD}} = \frac{2}{-3}$. Hence $(m_{\overline{AB}})(m_{\overline{CD}}) = -1$.

So $\overleftrightarrow{AB} \perp \overleftrightarrow{CD}$.

7. $\overleftrightarrow{CD} = \{(x,y): x = -2 + 3k \text{ , } y = 2 + 4k \text{ , } k \text{ is real}\}$.

8. (a) $\{(x,y): x = 3 + 4k \text{ , } y = 2 - k \text{ , } k \text{ is real}\}$

(b) $\{(x,y): x = 3 + k \text{ , } y = 2 + 4k \text{ , } k \text{ is real}\}$

(c) $\{(x,y): x = k \text{ , } y = -3k \text{ , } k \text{ is real}\}$

(d) $\{(x,y): x = 3k \text{ , } y = k \text{ , } k \text{ is real}\}$

(e) $m_{\overline{BC}} = -3$, $\overleftrightarrow{BC} = \{(x,y): y - 2 = -3(x - 3)\}$

$m_{\overline{AD}} = \frac{1}{3}$, $\overleftrightarrow{AD} = \{(x,y): y = \frac{1}{3}x\}$

$\{D\} = \{(x,y): y = -3x + 11 \text{ and } 3y = x\}$

$\{D\} = \{(x,y): y = -3(3y + 11) \text{ and } 3y = x\}$

$\{D\} = \{(x,y): y = \frac{11}{10} \text{ and } x = \frac{33}{10}\}$

$D = \left(\frac{33}{10}, \frac{11}{10}\right)$

9. $m_{\overline{AB}} = -2 = m_{\overline{CD}}$; $m_{\overline{BC}} = \frac{1}{2} = m_{\overline{AD}}$;

$(m_{\overline{AB}})(m_{\overline{BC}}) = -1$; therefore $\angle B$ is a right angle.

$(m_{\overline{BC}})(m_{\overline{CD}}) = -1$; therefore $\angle C$ is a right angle.

$(m_{\overline{CD}})(m_{\overline{AD}}) = -1$; therefore $\angle D$ is a right angle.

$(m_{\overline{AD}})(m_{\overline{AB}}) = -1$; therefore $\angle A$ is a right angle.

10. (a) $\{(x,y): x = 2 + k \ , \ y = 2 - k \ , \ k \text{ is real}\}$

(b) $\{(x,y): x = k \ , \ y = \frac{1}{2} + 2k \ , \ k \text{ is real}\}$

(c) $\{(x,y): x = k \ , \ y = 2 - k \ , \ k \text{ is real}\}$

(d) $\{(x,y): x = \frac{a}{2} + bk \ , \ y = \frac{b}{2} - ak \ ; \ k \text{ is real}\}$

11. $m_p = \frac{d - b}{c - a}$ if $a \neq c$.

$m_q = \frac{a - c}{d - b} = -\frac{c - a}{d - b} = -\frac{1}{m_p}$ if $a \neq c$, $d \neq b$.

If $a = c$, p is a vertical line and q is a horizontal line.

If $d = b$, p is a horizontal line and q is a vertical line.

12. (a) $x = \frac{5}{2}$

(b) $x = 5$

13. Plot $A(3,5)$. Through A draw a horizontal line and on it locate $D(10,5)$. Through D draw a vertical line and on it locate $E(10,1)$. $\overleftrightarrow{AE}$ is the required line. ($m_{\overleftrightarrow{AE}} = \frac{-4}{7}$; $m_{\overleftrightarrow{AB}} = \frac{7}{4}$.)

14. (a) $m_{\overline{PQ}} = -2$. Plot $S(6,-2)$. $\overleftrightarrow{RS}$ is the line.

(b) Plot $T(7,1)$. $\overleftrightarrow{RT}$ is the line.

15. $p = \{(x,y): x = 2 + 3k \ , \ y = 3 + 2k \ , \ k \text{ is real}\}$

(a) $k = 1$ yields $(5,5)$
$k = -1$ yields $(-1,1)$

(b) $q \perp p$ and $(2,3)$ on q ;
$q = \{(x,y): x = 2 + 2k \ , \ y = 3 - 3k \ , \ k \text{ is real}\}$
$k = 1$ yields $(4,0)$
$k = -1$ yields $(0,6)$.

16. (a) $AC = \sqrt{(a + c - a)^2 + (b + c - b)^2}$;

$BD = \sqrt{(a + c - a)^2 + (b - b - c)^2}$; $AC = BD$;

$\overline{AC} \cong \overline{BC}$.

(b) $m_{\overline{AC}} = \frac{b + c - b}{a + c - a} = 1$, $m_{\overline{BD}} = \frac{b - b - c}{a + c - a} = -1$.

Hence $\overline{AC} \perp \overline{BD}$.

(c) Midpoint of $\overline{AC} = (\frac{a + c + a}{2} , \frac{b + b + c}{2})$;

Midpoint of $\overline{BD} = (\frac{a + c + a}{2} , \frac{b + b + c}{2})$.

Problem Set 8-11

1. Yes. A rhombus has all the properties of a parallelogram, since it is a parallelogram. No. A parallelogram is not necessarily a rhombus and therefore would not always have all the properties of a rhombus.

2. (a) A rhombus is a square if and only if it has a right angle.

   (b) A rectangle is a square if and only if two consecutive sides are congruent.

3. (a) If a quadrilateral is equiangular, then it is a rectangle.

   If a quadrilateral is a rectangle, then it is equiangular.

   (b) In quadrilateral ABCD , $\angle A \cong \angle B \cong \angle C \cong \angle D$. Therefore ABCD is a parallelogram by Theorem 8-19 since opposite angles are congruent. Since it is a parallelogram, consecutive angles must be supplementary. If $\angle A$ and $\angle B$ are both congruent and supplementary, then each must be a right angle. Since a parallelogram with a right angle is a rectangle, ABCD must be a rectangle.

Conversely, if ABCD is a rectangle, then ABCD is a parallelogram with at least one right angle, say at A .

By definition of a parallelogram, $\overline{AD} \parallel \overline{BC}$.

Therefore, the consecutive interior angles are supplementary; $m \angle A + m \angle B = 180$. Since it is given that $m \angle A = 90$, then $m \angle B = 90$. Opposite angles of a parallelogram are congruent by Theorem 8-19; therefore $m \angle C = 90$ and $m \angle D = 90$. Therefore, ABCD is equiangular.

4. (a) If a quadrilateral is equilateral, then it is a rhombus.

If a quadrilateral is a rhombus, then it is equilateral.

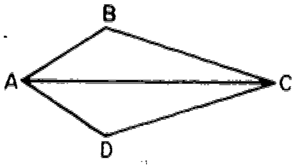
(b) In quadrilateral ABCD , all 4 sides are congruent. Since opposite sides are congruent then by Theorem 8-18, we know that ABCD is a parallelogram. Since this parallelogram has two consecutive sides congruent, it is a rhombus, by the definition of a rhombus.

Conversely, if ABCD is a rhombus, then ABCD is a parallelogram with two consecutive sides congruent, say $\overline{AB} \cong \overline{BC}$. By Theorem 6-6, we know that $\overline{AB} \cong \overline{DC}$ and $\overline{BC} \cong \overline{DA}$. By the transitive property of congruence, we know that all four sides are congruent and therefore ABCD is equilateral.

5. (a) True. Theorem 8-20.
(b) True. Theorem 8-20.
(c) True. Theorem 8-21.
(d) True. Theorem 8-21.
(e) True. If regular, then it is equilateral and equiangular. This makes it a rectangle and a rhombus both. When both, it is called a square.

(f) True. In parallelogram ABCD , $\triangle ABC \cong \triangle CDA$, by S.S.S. Congruence Postulate or A.S.A. Congruence Postulate.

(g) False. Consider the case in quadrilateral ABCD when we know only that $\triangle ABC \cong \triangle ADC$.



Problem Set 8-12

1. In $\triangle ABC$, $A = (0,0)$, $B = (2b,0)$, $C = (2c,2d)$. Then we find the midpoint of $\overline{AC}$, $D = (c,d)$, and the midpoint of $\overline{BC}$, $E = (b + c, d)$.

   Since D and E have the same y coordinates, $\overleftrightarrow{DE}$ is a horizontal line and parallel to the x-axis which is $\overleftrightarrow{AB}$. Also, by the distance formula,

   $DE = |b + c - c| = |b|$.

   $AB = |2b - 0| = |2b| = 2|b|$.

   Therefore $DE = \frac{1}{2}AB$.

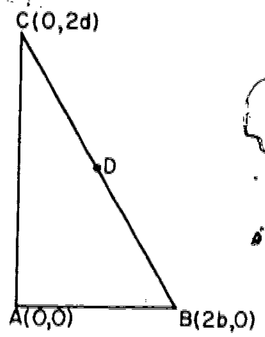
   The advantage is that the coordinates of D and E are simplified.

2. 12. Since each side of $\triangle DEF$ is half of a side of $\triangle ABC$, the perimeter is half that of $\triangle ABC$.

3. $XY = 2MN$ by Theorem 8-22.

4. right triangle. $\overline{AB}$ is on the x-axis. $\overline{AC}$ would be on the y-axis. Thus two sides of the triangle would be perpendicular and $\angle A$ would be a right angle.

5.



Using the coordinates suggested in Problem 4, we find the coordinates of D, the midpoint of $\overline{BC}$, the hypotenuse.

$$D = (b,d) .$$

Then by the distance formula,

$$CD = \sqrt{b^2 + d^2}$$
$$BD = \sqrt{b^2 + d^2}$$
$$AD = \sqrt{b^2 + d^2} .$$

Thus D is equally distant from A, B and C .

6. Let $\overline{AB}$ be the base of isosceles triangle ABC . Then, A = (-2a,0) , B = (2a,0) , and C = (0,2b) .

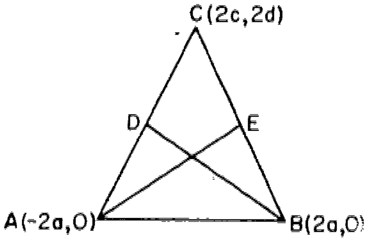
7. Use coordinates suggested in Problem 6. Let D be the midpoint of $\overline{AC}$ and E be the midpoint of $\overline{BC}$. Then D = (-a,b) and E = (a,b).

Therefore $DB = \sqrt{9a^2 + b^2}$ = the length of one median and $EA = \sqrt{9a^2 + b^2}$ = the length of the other median. Therefore the medians are congruent.

8. Let A = (-2a,0) , B = (2a,0) , C = (2c,2d) . Let $\overline{AE}$ and $\overline{BD}$ be the two congruent medians. Then D = (c - a, d)

E = (c + a, d) .

We are given that $\overline{AE} \cong \overline{BD}$, therefore



since $AE = \sqrt{(c + 3a)^2 + d^2}$

and $BD = \sqrt{(c - 3a)^2 + d^2}$

we know that

$(c + 3a)^2 + d^2 = (c - 3a)^2 + d^2$ and thus

$(c + 3a)^2 = (c - 3a)^2$.

Therefore, either (1) $c + 3a = c - 3a$

or (2) $c + 3a = -(c - 3a)$.

In other words, either (1) $a = 0$, or (2) $c = 0$.

Since a cannot be 0 , c must be 0 .

Thus $AC = \sqrt{4a^2 + 4d^2}$ and $BC = \sqrt{4a^2 + 4d^2}$.

Thus $AC = BC$ and the triangle is isosceles.

9. The medians to two sides of a triangle are congruent if and only if the triangle is isosceles.

10. $(CB)^2 = (b - a)^2 + c^2$

$(AC)^2 = b^2 + c^2$

$(AB)^2 = a^2$

$AB = |a| = a$ since a is taken in the positive x-axis.

$AR = |b| = b$ since b is positive, $\angle A$ being acute.

Therefore $(CB)^2 = (AC)^2 + (AB)^2 - 2AB \cdot AR$, since

$(b - a)^2 + c^2 = (b^2 + c^2) + (a^2) - 2 \cdot a \cdot b$.

11. $(AC)^2 = b^2 + c^2$ and $(BC)^2 = a^2$.

Thus $(AC)^2 + (BC)^2 = a^2 + b^2 + c^2$.

$(AB)^2 = (b - a)^2 + c^2$.

$M = (\frac{a + b}{2} , \frac{c}{2})$.

$(MC)^2 = (\frac{a + b}{2})^2 + (\frac{c}{2})^2 = \frac{(a + b)^2 + c^2}{4}$.

Thus $\frac{(AB)^2}{2} + 2(MC)^2$

$$= \frac{b^2 - 2ab + a^2 + c^2}{2} + \frac{a^2 + 2ab + b^2 + c^2}{2}$$

which simplifies to $\frac{2a^2 + 2b^2 + 2c^2}{2} = a^2 + b^2 + c^2$.

Therefore, $(AC)^2 + (BC)^2 = \frac{(AB)^2}{2} + 2(MC)^2$.

Problem Set 8-13

1. If ABCD is a parallelogram, then $\overline{AC}$ and $\overline{BD}$ bisect each other.

   Proof: $A = (0,0)$, $B = (a,0)$, $C = (a + b, c)$, $D = (b,c)$ by Theorem 8-23.

   Then the midpoint of $\overline{AC}$ is $(\frac{a + b}{2} , \frac{c}{2})$ and the midpoint of $\overline{BD}$ is $(\frac{a + b}{2} , \frac{c}{2})$.

   Since the midpoint of $\overline{AC}$ is also the midpoint of $\overline{BD}$, the diagonals bisect each other.

2. Part (1). If the diagonals of a parallelogram ABCD are congruent, it is a rectangle.
   $A = (0,0)$, $B = (a,0)$, $C = (a + b, c)$, $D = (b,c)$.
   By Theorem 8-23, we must prove $b = 0$.

   We know that $AC = BD$, therefore $(AC)^2 = (BD)^2$, and

   $$(a + b)^2 + c^2 = (b - a)^2 + c^2 .$$

   Therefore, $(a + b)^2 = (b - a)^2$ and
   either $a + b = b - a$ or $a + b = -(b - a)$.
   Since $a \neq 0$, $b = 0$.

   Part (2). If ABCD is a rectangle, the diagonals are congruent. $A = (0,0)$, $B = (a,0)$, $C = (a,c)$, $D = (0,c)$. $AC = \sqrt{a^2 + c^2}$ and $BD = \sqrt{(-a)^2 + c^2}$.
   Thus $\overline{AC} \cong \overline{BD}$.

3. Part (1). If the diagonals of a parallelogram are perpendicular, it is a rhombus.
   $A = (0,0)$, $B = (a,0)$, $C = (a + b, c)$, $D = (b,c)$

   where $a > 0$. We must prove $a = \sqrt{b^2 + c^2}$.

   Slope of $\overleftrightarrow{AC} = \frac{c}{a + b}$; slope $\overleftrightarrow{BD} = \frac{c}{b - a}$.

   Since $m_{\overleftrightarrow{AC}} \cdot m_{\overleftrightarrow{BD}} = -1$,

   $$\frac{c}{(a + b)} \cdot \frac{c}{(b - a)} = -1 \text{ and } c^2 = -(b^2 - a^2) .$$

   Therefore $a^2 = b^2 + c^2$ and $a = \sqrt{b^2 + c^2}$.

Part (2). If ABCD is a rhombus, $\overline{AC} \perp \overline{BD}$.

Let $A = (0,0)$, $B = (\sqrt{b^2 + c^2}, 0)$,

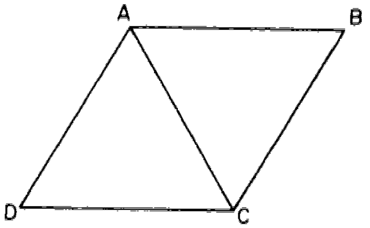
$C = (\sqrt{b^2 + c^2} + b, c)$, $D = (b,c)$ where B is on the x-axis to the right of A .

Therefore, the slope of $\overleftrightarrow{AC} = \frac{c}{\sqrt{b^2 + c^2} + b}$ and the slope of $\overleftrightarrow{BD} = \frac{-c}{\sqrt{b^2 + c^2} - b}$ and

$$m_{\overline{AC}} \cdot m_{\overline{BD}} = \frac{-c^2}{(b^2 + c^2) - b^2} = -1 .$$

Therefore, $\overline{AC} \perp \overline{BD}$.

4.



Prove:

(1) If ABCD is a parallelogram and $\overrightarrow{AC}$ bisects $\angle DAB$, then ABCD is a rhombus.

(2) If ABCD is a rhombus, $\overrightarrow{AC}$ bisects $\angle DAB$.

(Note: We do not use coordinates because students need trigonometry before they can write the equation of an angle bisector.)

Proof:

Part (1). $\angle DAC \cong \angle BAC$ by definition of angle bisector. Since $\overline{AB} \parallel \overline{DC}$, $\angle BAC \cong \angle DCA$ because they are alternate interior angles. Therefore, by the transitive property of congruence, $\angle DAC \cong \angle DCA$. Then, since two angles in $\triangle ADC$ are congruent, the sides opposite those angles are congruent. $\overline{AD} \cong \overline{DC}$. Thus ABCD is a rhombus.

Part (2). ABCD is a rhombus. Therefore, $AD = AB$, $DC = BC$. Also, $AC = AC$. $\triangle ADC \cong \triangle ABC$ by S.S.S. Therefore $\angle DAC \cong \angle BAC$ since they are corresponding angles. $\overrightarrow{AC}$ is the midray.

5. A rectangle is equiangular and all angles are right angles. Its diagonals are congruent.

6. A rhombus is equilateral. Its diagonals are perpendicular and bisect the angles.

7. Yes. Yes. A parallelogram, a rectangle, and a rhombus.

8.

	Parallelogram	Rectangle	Rhombus	Square
opp. sides $\cong$	✓	✓	✓	✓
opp. $\angle$s $\cong$	✓	✓	✓	✓
consec. $\angle$s supp.	✓	✓	✓	✓
diags. bisect	✓	✓	✓	✓
diags. $\cong$	no	✓	no	✓
diags. $\perp$	no	no	✓	✓
diags. bisect $\angle$s	no	no	✓	✓
equilateral	no	no	✓	✓
equiangular	no	✓	no	✓
regular	no	no	no	✓

9. Q = set of quadrilaterals

P = set of parallelograms

R_1 = set of rectangles

R_2 = set of rhombuses

S = set of squares

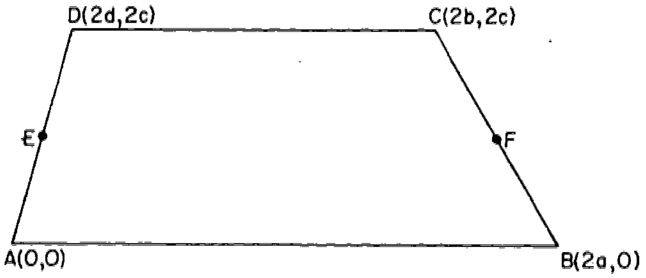
$\subset$ means "is contained in", or "is a subset of," and has the transitive property.

$S \subset R_1 \subset P \subset Q$.

$S \subset R_2 \subset P \subset Q$.

Problem Set 8-14

1.



We may assume without loss of generality that $a > 0$, $b > 0$, $b > d$, $c > 0$. $E = (d,c)$, $F = (a + b, c)$.

Thus $\overleftrightarrow{EF}$ is a horizontal line and is parallel to $\overleftrightarrow{DC}$ and $\overleftrightarrow{AB}$.

Also, $EF = a + b - d$

$AB = 2a$ and $DC = 2b - 2d$.

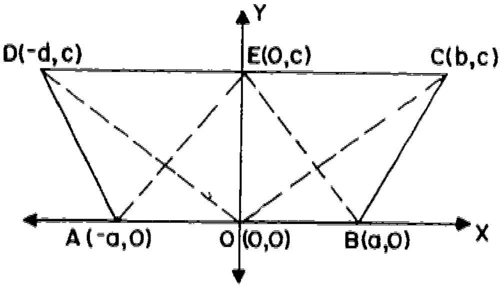
$\frac{1}{2}(AB + DC) = a + b - d$.

Therefore, $EF = \frac{1}{2}(AB + DC)$.

2. (a) $x = 11$
   (b) $x = 10$
   (c) $x = 6$, $y = 8$.

3. In ABCD, $\overline{AB} \parallel \overline{DC}$.
   If $m\angle A = 100$, $m\angle D = 80$ we would not know the measures of the other two angles.
   If $m\angle A = 100$ and $m\angle C = 70$, then $m\angle D = 80$ and $m\angle B = 110$.

4.



Given a trapezoid, label it ABCD and set up an xy-coordinate system so that $A = (-a,0)$, $B = (a,0)$, $C = (b,c)$, $D = (-d,c)$, with $a > 0$, $b > 0$, $c > 0$, $b > -d$. Then $b + d \neq 2a$. (For if $b + d = 2a$, then $AB = CD$ and ABCD is a parallelogram, not a trapezoid.) We are to prove two statements:

(1) If $AD = BC$, then $\angle A = \angle B$.

(2) If $\angle A \cong \angle B$, then $AD = BC$.

(1) If $AD = BC$ then $(AD)^2 = (BC)^2$,

$$(-d + a)^2 + c^2 = (b - a)^2 + c^2 ,$$

$$(-d + a)^2 = (b - a)^2 ,$$

$$-d + a = b - a \quad \text{or} \quad -d + a = -b + a ,$$

$$2a = b + d \quad \text{or} \quad b = d .$$

But $2a \neq b + d$,

therefore $b = d$.

Then $AO = a = OB$,

$$AD = \sqrt{(-b + a)^2 + c^2} = \sqrt{(b - a)^2 + c^2} = BC ,$$

$$DO = \sqrt{(-d)^2 + c^2} = \sqrt{b^2 + c^2} = OC ,$$

and $\triangle DAO \cong \triangle CBO$ by S.S.S.

Then $\angle DAO \cong \angle CBO$.

(2) Let $E = (0,c)$. If $\angle A \cong \angle B$, then it is easy to show that $\angle EBC \cong \angle EAD$, $EB \neq EA$, $\angle BEC \cong \angle AED$, hence that $\Delta EBC \cong \Delta EAD$ and $BC = AD$.

5. Given a trapezoid, label it ABCD and set up an xy-coordinate system so that $A = (-a,0)$, $B = (a,0)$, $C = (b,c)$, $D = (d,c)$, with $a > 0$, $c > 0$, $b > d$, $b - d \neq 2a$ (compare with solution to Problem 4). We must prove two statements:

(1) If $AD = BC$, then $AC = BD$.

(2) If $AC = BD$, then $AD = BC$.

(1) If $AD = BC$, then $d = -b$ (compare with solution to Problem 4) and

$$(AC)^2 = (-a - b)^2 + (0 - c)^2 = (a + b)^2 + c^2 ,$$

$$(BD)^2 = (a - d)^2 + (0 - c)^2 = (a + b)^2 + c^2 ,$$

and $AC = BD$.

(2) If $AC = BD$, then

$$(-a - b)^2 = (0 - c)^2 = (a - d)^2 + (0 - c)^2 ,$$

$$(b + a)^2 = (-d + a)^2 ,$$

$$b - a = -d + a \quad \text{or} \quad b + a = d - a .$$

$$b = -d \quad \text{or} \quad b - d = -2a .$$

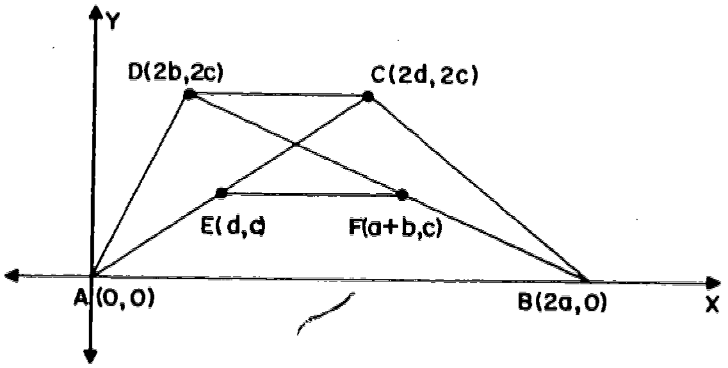
But $b - d \neq -2a$, since $b > d$ and $a > 0$.

Therefore $b = -d$.

Then $(BC)^2 = (b - a)^2 + c^2 = (-a - d)^2 + c^2$

$= (AD)^2$, and $BC = AD$.

6.



Let $A(0,0)$, $B(2a,0)$, $C(2d,2c)$, and $D(2b,2c)$ be vertices of a trapezoid, with $a > 0$, $c > 0$, and $d > b$. Then $a + b \neq d$ (see solution to Problem 4).

Let E be midpoint of $\overline{AC}$. $E = (d,c)$.

Let F be midpoint of $\overline{BD}$. $F = (a + b, c)$.

Then $m_{\overline{EF}} = \frac{c - c}{a + b - d} = 0$ and $\overline{EF} \,||\, \overline{DC} \,||\, \overline{AB}$.

Also $EF = |a + b - d|$,

and $|AB - DC| = |2a - (2d - 2b)| = 2|a + b - d|$.

Therefore $EF = \frac{1}{2}|AB - DC|$.

Problem Set 8-15

1. Theorem 8-28 and Corollary 8-28-1. Find midpoints of $\overline{AB}$ and $\overline{BC}$, and draw perpendicular bisectors. Their intersection is the desired point.

2. Theorem 8-29 and Corollary 8-29-1. Draw midray of each angle. Their intersection is the desired point.

3. (a) Draw perpendicular bisector of $\overline{AB}$ and midray of $\angle ACB$. Their intersection is the required point.
   (b) Use the ruler to plot the midpoint $(1,0)$.
   (c) $(0,4)$.

4. (a) By Theorem 8-22, $DE = \frac{1}{2}(12) = 6$, $EF = \frac{9}{2}$, $FD = 5$.

(b) By Theorem 8-22, $\overline{DE} \parallel \overline{AB}$, $\overline{EF} \parallel \overline{BC}$, $\overline{FD} \parallel \overline{AC}$.

(c) $\overline{DE}$, $\overline{EF}$, $\overline{DF}$.

(d) The perpendicular bisectors of the sides of $\triangle ABC$ are the altitudes of $\triangle DEF$. Since the former are concurrent by the corollary to Theorem 8-28, the latter are concurrent.

5. Proof I. Since the perpendicular bisectors of $\triangle PQR$ are concurrent (by corollary to Theorem 8-28), and these are the altitudes of $\triangle ABC$, the latter are concurrent.

Proof II. $m_{\overline{CB}} = \frac{0 - b}{c - 0} = -\frac{b}{c} = m_{\overline{AB}} = \frac{b - 0}{0 - a} = -\frac{b}{a}$.

$m_{h_a} = -\frac{1}{m_{\overline{BC}}} = \frac{c}{b}$, $m_{h_c} = \frac{1}{m_{\overline{AB}}} = \frac{a}{b}$.

Two non-vertical lines are perpendicular if and only if the product of their slopes is -1 .

$(0, -\frac{ac}{b})$ is contained in h_a , because

$h_a = \{(x,y): x = a + kb , y = kc , k \text{ is real}\}$ and

$k = -\frac{a}{b}$ yields $(0, -\frac{ac}{b})$.

$(0, -\frac{ac}{b})$ is contained in h_c , because

$h_c = \{(x,y): x = c + bp , y = ap , p \text{ is real}\}$ and

$p = -\frac{c}{b}$ yields $(0, -\frac{ac}{b})$.

Thus h_a , h_c intersect in $(0, -\frac{ac}{b})$. Also h_b contains $(0, -\frac{ac}{b})$, because h_b is the y-axis and contains all points whose x-coordinate is 0 . So h_a , h_c , h_b are concurrent.

Note: In Proof II we chose $k = -\frac{a}{b}$ and $p = -\frac{c}{b}$.

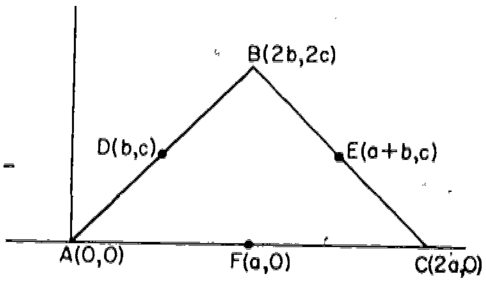
These were "happy choices" which showed that h_a and h_c had a point of intersection in the y-axis. But these "happy" choices were not accidental. They required some ingenuity.

In Proof I, which did not use coordinates, there was ingenuity displayed in considering $\triangle PQR$.

Ingenuity can be exercised no matter which type of proof is used.

6. In $\triangle ABC$, $A = (0,0)$, $B = (2b,2c)$ and $C = (2a,0)$.

(a) Plan: Find the point of intersection of two of the perpendicular bisectors and then test to see if that point is contained in the third perpendicular bisector.



(1) $m_{\overline{AB}} = \frac{2c}{2b} = \frac{c}{b}$, $m_{\overline{AC}} = 0$,

$m_{\overline{BC}} = \frac{2c}{2b - 2a} = \frac{c}{b - a}$.

Let p, q, r be the perpendicular bisectors of $\overline{AB}$, $\overline{BC}$ and $\overline{AC}$ respectively.

(2) $p = \{(x,y): y - c = -\frac{b}{c}(x - b)\}$

$r = \{(x,y): x = a\}$.

By substitution $y - c = -\frac{b}{c}(a - b)$

$$y = \frac{b^2 + c^2 - ab}{c}$$

$$x = a.$$

Therefore, $(a, \frac{b^2 + c^2 - ab}{c})$ is the point of intersection of p and r. Call it O.

(3) $q = \{(x,y): y - c = -\frac{b-a}{c}\left(x-(a+b)\right)\}$.

The point O is also contained in q since by substitution

$$\frac{(b^2 + c^2 - ab)}{c} - c = -\left(\frac{b-a}{c}\right)(a - a - b)$$

$$\frac{b^2 + c^2 - ab}{c} = \frac{b^2 - ab}{c} + c$$

$$= \frac{b^2 + c^2 - ab}{c}.$$

Since p, q, r each contain the point $O = \left(a, \frac{b^2 + c^2 - ab}{c}\right)$ the lines are concurrent.

(b) Use distance formula to show that OA = OB = OC.

$$(OA)^2 = (a - 0)^2 + \left(\frac{b^2 + c^2 - ab}{c}\right)^2$$

$$= (a)^2 + \left(\frac{b^2 + c^2 - ab}{c}\right)^2$$

$$(OC)^2 = (a - 2a)^2 + \left(\frac{b^2 + c^2 - ab}{c}\right)^2$$

$$= (a)^2 + \left(\frac{b^2 + c^2 - ab}{c}\right)^2$$

$$(OB)^2 = (a - 2b)^2 + \left(\frac{b^2 + c^2 - ab}{c} - 2c\right)^2$$

$$= (a - 2b)^2 + \left(\frac{b^2 - c^2 - ab}{c}\right)^2 .$$

Expanding

$$(OA)^2 = (OC)^2 = \frac{b^4 + c^4 - 2ab^3 - 2abc^2 + a^2c^2 + 2b^2c^2 + a^2b^2}{c^2}$$

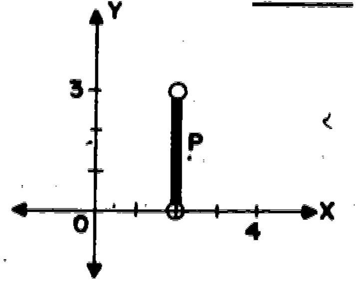
$$(OB)^2 = \frac{b^4 + c^4 - 2ab^3 - 2abc^2 + a^2c^2 + 2b^2c^2 + a^2b^2}{c^2}$$

Thus OA = OB = OC.

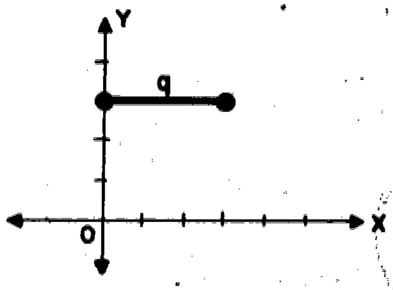
Chapter 8

Review Problems

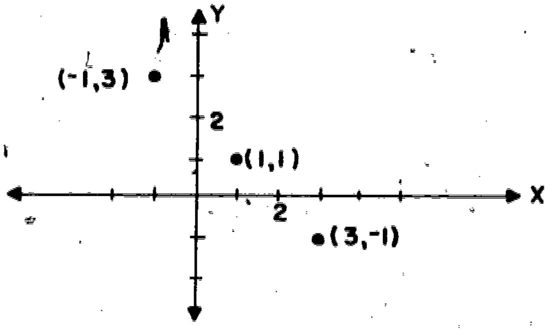
1.



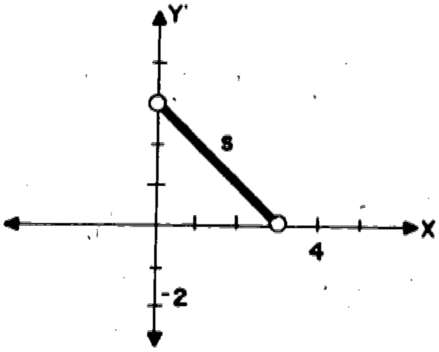
$p = \{(x,y): x = 2,$
$0 < y < 3\}$



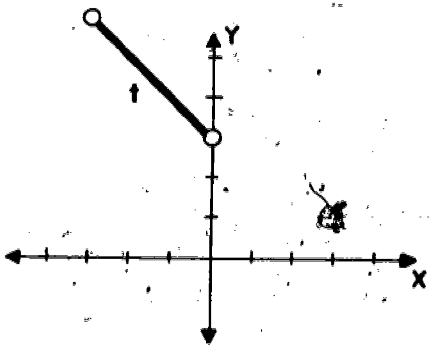
$q = \{(x,y): y = 2,$
$0 \leq x \leq 3\}$



$r = \{(-1, 3),(1, 1), (3, -1)\}$



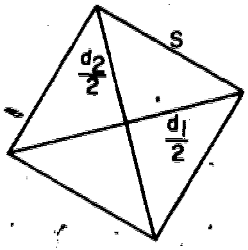
$s = \{(x,y): x + y = 3,$
$0 < x < 3, 0 < y < 3\}$



$t = \{(x,y): x + y = 3,$
$-3 < x < 0\}$

2. (a) Yes
(b) Yes
(c) Yes
(d) No
(e) Yes
(f) No
(g) Yes
(h) Yes
(i) No
(j) Yes
(k) Yes
(l) No

3. Since the diagonals of a rhombus are perpendicular and bisect each other



$$\left(\frac{d_1}{2}\right)^2 + \left(\frac{d_2}{2}\right)^2 = s^2$$

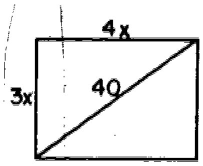
$$(8)^2 + (15)^2 = s^2$$

$$289 = s^2$$

$$17 = s .$$

Therefore $P = 68$.

4. $(3x)^2 + (4x)^2 = (40)^2$



$(3x)^2 + (4x)^2 = (5 \cdot 8)^2$

$x = 8$.

Therefore the sides of the rectangle are 24 and 32 .

5. (a) (-1,5)
(b) (1,-1)
(c) (1,2)
(d) the measure of $AB = |3 - (-1)| = 4$
(e)

$$AC = \sqrt{[3 - (-1)]^2 + [5 - (-1)]^2} = \sqrt{4^2 + 6^2}$$

$$BD = \sqrt{[-1 - 3]^2 + [5 - (-1)]^2} = \sqrt{4^2 + 6^2} .$$

Therefore $AC = BD$.

(f) $\overleftrightarrow{AB} = \{(x,y): y = -1\}$

(g) $\overleftrightarrow{AC} = \{(x,y): x = -1 + 4k , y = -1 + 6k ,$ k is real$\}$

(h) $Q = (-1 + 4 \cdot 4 , -1 + 6 \cdot 4)$
$Q = (15,23)$

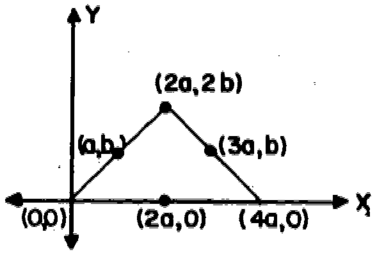
(i) Slope of $\overline{AC}$ is $\frac{3}{2}$ [illegible] slope of line perpendicular to $\overline{AC}$ is $-\frac{2}{3}$. Therefore the required line is

$\{(x,y): x = 3 + 3k , y = 5 - 2k , k \text{ is real}\}$.

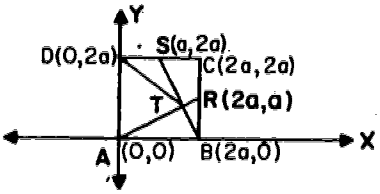
6. (a) $\frac{b}{3a}$

(b) $\frac{b}{-3a}$

(c) Slope not defined.



7. (a) Proof: Set up an xy-coordinate system so that $A = (0,0)$,
$B = (2a,0)$,
$C = (2a,2a)$,
$D = (0,2a)$.



Then since R is midpoint of $\overline{BC}$, $R = (2a,a)$ and since S is the midpoint of $\overline{DC}$, $S = (a,2a)$. Then by the distance formula

$$AR = \sqrt{(2a)^2 + (a)^2} = \sqrt{5a^2} \text{ and}$$

$BS = \sqrt{a^2 + (-2a)^2} = \sqrt{5a^2}$. Thus $AR = BS$.

(b) Slope of $\overline{BS} = \frac{2a}{-a}$; slope of $\overline{AR} = \frac{a}{2a} = \frac{1}{2}$.

Since the product of these slopes $= -1$, $\overline{BS} \perp \overline{AR}$.

(c) The equation of $\overleftrightarrow{AR}$ by point-slope form is $y = \frac{1}{2}x$. The equation of $\overleftrightarrow{BS}$ by point-slope form is $y = -2(x - 2a)$.

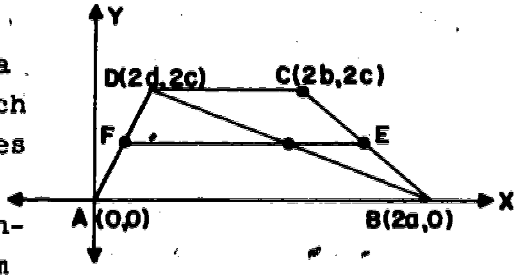
The intersection of these two lines is the point

$$T(\frac{8a}{5} , \frac{4a}{5}) .$$

Then $DT = \sqrt{(\frac{8a}{5})^2 + (\frac{4a}{5} - 2a)^2} = \sqrt{\frac{64a^2}{25} + \frac{(-6a)^2}{25}}$

$= 2a$, $AB = 2a$. Therefore $TD = AB$.

8. **Theorem:** The median of a trapezoid bisects a diagonal.



Proof: If we select a coordinate system which assigns the coordinates to the vertices of trapezoid ABCD as indicated in the diagram (with $a > 0$, $b > d$, $c > 0$) , then the midpoints of the nonparallel sides $\overline{AD}$ and $\overline{BC}$ will be $F(d,c)$ and $E(a + b, c)$ $\overline{FE}$ is the median of ABCD . We are asked to prove that the midpoint of $\overline{DB}$ is on $\overline{FE}$. It is clear that $\overleftrightarrow{FE}$ is $\{(x,y): y = c \ , \ x \text{ is real}\}$. The midpoint of $\overline{DB}$ is $(d + a, c)$. Then the midpoint of $\overline{DB}$ is in $\overleftrightarrow{FE}$. It must still be shown to be in $\overline{FE}$. Now $d < d + a < a + b$, since a , b are all assumed positive and since $b > d$. Then the point $(d + a, c)$ lies between F and E and the midpoint of $\overline{DB}$ is in $\overline{FE}$.

Alternate Proof:

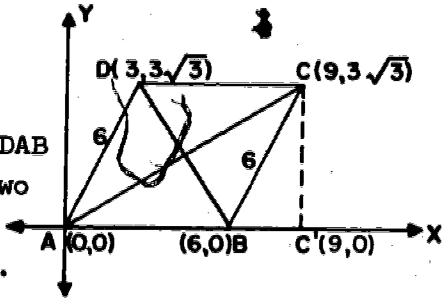
$$\overleftrightarrow{FE} = \{(x,y): y = c \ , \ x \text{ is real}\}$$

$$\overleftrightarrow{DB} = \{(x,y): x = 2a + k(2d - 2a) \ ; \ y = k(2c) \ , \ k \text{ is real}\} \ .$$

The intersection T of these lines must have as its y-coordinate $c = 2ck$, whence $k = \frac{1}{2}$, and $T = \left(2a + \frac{1}{2}(2d - 2a) \ , \ \frac{1}{2}(2c)\right) = (a + d, c)$, which is the midpoint of $\overline{BD}$.

9. (a) $y = 0$
   (b) $x = 0$
   (c) If $y = 0$, then for all values of x , $xy = 0$. If $x = 0$, then for all values of y , $xy = 0$. Therefore each point of both axes satisfies $xy = 0$.

10. Our coordinate system assigns (0,0) to A and (6,0) to B. Since m $\angle DAB$ is given as 60, and m $\angle CBX$ = m $\angle DAB$ (corresponding angles of two parallel lines cutting a transversal), m $\angle CBX = 60$.



Consider $\overline{CC'} \perp \overrightarrow{AX}$. Then in $\triangle BCC'$, m $\angle BCC' = 30$, $BC = 6$, and $BC' = 3$ (in a 30-60 right triangle, the shortest side is half of the hypotenuse), and using the Pythagorean Theorem we find

$$CC' = \sqrt{36 - 9} = 3\sqrt{3} .$$

(a) Thus $C = (9, 3\sqrt{3})$

(b) $D = (3, 3\sqrt{3})$

(c) $AC = \sqrt{81 + 27} = \sqrt{108} = 6\sqrt{3}$

(d) $BD = \sqrt{3^2 + 27} = 6$. Thus $AC = \sqrt{3}\ BD$.

(e) $\overrightarrow{AC} = \{(x,y): x = 3k , y = k\sqrt{3} \text{ and } k \geq 0\}$

11. (a) $\{(x,y): x = 1$ and y is real$\}$

(b) $\{(x,y): |y| = 3$ and x is real$\}$

(c) $\{(x,y): y = x$ or $y = -x$; x and y are real$\}$

(d) $\{(x,y): y = 3$ and x is real$\}$

(e) $\{(x,y): x = 12$ and y is real$\}$

(f) $\{(x,y): y = -8$ and x is real$\}$

12. $A = (3,4)$, $B = (-1,5)$, $C = (-2,1)$

$AB = \sqrt{4^2 + 1^2} = \sqrt{17}$

$BC = \sqrt{1^2 + 4^2} = \sqrt{17}$

$AC = \sqrt{5^2 + 3^2} = \sqrt{34}$

Thus ABC is an isosceles triangle, since $AB = BC$, and ABC is a right triangle, since $(AC)^2 = (BC)^2 + (AB)^2$.

13. Midpoint of $\overline{AB} = (-4,2)$.

Slope of $\overline{AB} = \frac{4}{8} = \frac{1}{2}$, slope of line perpendicular to $\overline{AB} = -2$. Therefore the line is

$\{(x,y):\ x = -4 + k\ ,\ y = 2 - 2k\ ,\ k \text{ is real}\}$.

14. $AB = \sqrt{(c - 1)^2 + (6 - 1)^2} = \sqrt{(c - 1)^2 + 25}$

$AC = \sqrt{(c - 3)^2 + (6 - 5)^2} = \sqrt{(c - 3)^2 + 1}$.

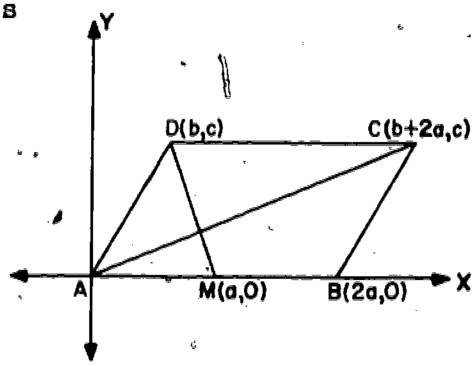
Then $AB = AC$ gives $(c - 1)^2 + 25 = (c - 3)^2 + 1$

$$c^2 - 2c + 1 + 25 = c^2 - 6c + 9 + 1$$

$$c = -4 .$$

15. The distance from $(h,3)$ to the x-axis is 3 , its distance from the y-axis is $|h|$. Therefore $3 = 2|h|$, $|h| = \frac{3}{2}$, that is, $h = \frac{3}{2}$ or $h = -\frac{3}{2}$.

16. Select a coordinate system which assigns coordinates to the vertices of parallelogram ABCD as indicated in the diagram. Then M is the midpoint of $\overline{AB}$ and has coordinates $(a,0)$. We are to show that one of the trisection points of $\overline{AC}$ lies on $\overline{MD}$.



$\overline{MD} = \{(x,y):\ x = b + k(a - b)\ ,\ y = c - kc\ ,$
$0 \leq k \leq 1\}$.

The trisection points of $\overline{AC}$ are $R(\frac{b + 2a}{3}\ ,\ \frac{c}{3})$ and $S(\frac{2(b + 2a)}{3}\ ,\ \frac{2c}{3})$. Now, if R is on $\overline{MD}$ then there is k such that $0 \leq k \leq 1$ and $c - kc = \frac{c}{3}$. Clearly $k = \frac{2}{3}$ satisfies this requirement. Using $k = \frac{2}{3}$ we see that R is a member of set $\overline{MD}$.

Alternate solution:

$\overleftrightarrow{AC} = \{(x,y): x = k(b + 2a) , y = ck , k \text{ is real}\}$

$\overleftrightarrow{DM} = \{(x,y): x = a + h(b - a) , y = ch , h \text{ is real}\}$.

Intersection of $\overleftrightarrow{AC}$, $\overleftrightarrow{DM}$, has y-coordinate such that $ck = ch$, or $k = h$.

Then its x-coordinate is such that

$k(b + 2a) = a + h(b - a) = a + k(b - a)$, or

$kb + 2ak = a + kb - ka$, or

$2ak = a - ak$, or

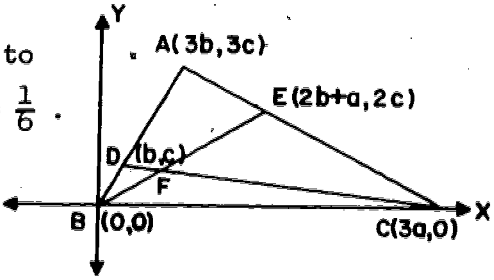
$k = \frac{1}{3} = h$.

Therefore, R , the intersection point, has coordinates

$$\left(\frac{1}{3}(b + 2a) , \frac{c}{3}\right) = \left(\frac{2a + b}{3} , \frac{c}{3}\right) ,$$

which are the coordinates of a trisection point of $\overline{SC}$.

17. Assume a coordinate system which assigns the coordinates as indicated to B, C, A . Then, referring to the diagram, we know that



$E = (2b + a,2c)$,

$D = (b,c)$, and we are to show $\frac{BF}{BE} = \frac{3}{4}$ and $\frac{DF}{FC} = \frac{1}{6}$.

An equation of $\overleftrightarrow{BE}$ is

$$y = \frac{2c}{2b + a} x .$$

An equation of $\overleftrightarrow{DC}$ is

$$y = \frac{c}{b - 3a}(x - 3a) .$$

5

Solving for the coordinates of F , we find

$$\frac{2c}{2b + a}x = \frac{c}{b - 3a}(x - 3a)$$

from which

$$x = \frac{3}{7}(2b + a)$$

$$y = \frac{6c}{7}$$

so

$$F = \left(\frac{3}{7}(2b + a)\ ,\ \frac{6c}{7}\right).$$

Then $\frac{BF}{FE} = \frac{y_F}{y_E - y_F} = \frac{\frac{6c}{7}}{2c - \frac{6}{7}c} = \frac{3}{4}$

and $\frac{DF}{FC} = \frac{x_F - x_D}{x_C - x_F} = \frac{\frac{3}{7}(2b + a) - b}{3a - \frac{3}{7}(2b + a)} = \frac{1}{6}.$

Chapter 9

ANSWERS AND SOLUTIONS

Problem Set 9-2

1. No. The definition requires that the line be perpendicular to every line containing Q and lying in $\mathcal{E}$.

2. $\angle ABR$, $\angle ABK$, $\angle TBA$. By definition, $\overleftrightarrow{AB} \perp \mathcal{E}$ implies $\overleftrightarrow{AB} \perp \overleftrightarrow{RB}$, $\overleftrightarrow{AB} \perp \overleftrightarrow{KB}$, $\overleftrightarrow{AB} \perp \overleftrightarrow{BT}$, and by definition of perpendicular lines, the angles are right angles.

3. (a) Yes. A plane is determined by three noncollinear points. If R, S, T were collinear, then T would be in $\mathcal{M}$, by Postulate 8.
   (b) $\angle PST$, $\angle PSR$. Definition of perpendicularity: a line and a plane, and two lines.

4. (a) Three. Plane determined by $\overline{AB}$ and $\overline{FB}$, plane determined by $\overline{AB}$ and $\overline{BH}$, and the plane of square FRHB .
   (b) $\overline{FB} \perp$ plane ABH . Since $\overline{AB} \perp \overline{FB}$ by hypothesis and $\overline{FB} \perp \overline{BH}$ from definition of a square, $\overline{FB} \perp$ plane ABH by Theorem 9-2.

5. (a) Three. Planes RHB , RAF , and ABF .
   (b) $\overline{BH} \perp$ plane ARF . $\overline{BH} \perp \overline{RH}$ by hypothesis, $\overline{BH} \perp \overline{AF}$ at point H by Theorem 5-8. Therefore $\overline{BH} \perp$ plane ARF by Theorem 9-2.

6. Yes. By Theorem 9-1.

7. (a) Yes. By the definition of perpendicularity for a line and a plane.
   (b) Yes. By the definition of perpendicularity for a line and a plane.
   (c) No. By hypothesis, $\overleftrightarrow{BC}$ lies in plane $\mathcal{F}$, and therefore cannot be perpendicular to $\overleftrightarrow{AB}$. By Theorem 9-1, all lines perpendicular to $\overleftrightarrow{AB}$ at B must lie in the plane perpendicular to $\overleftrightarrow{AB}$ at B (that is, plane $\mathcal{E}$).

8. By hypothesis, $\overline{FB} \perp$ plane $\mathcal{P}$. $\overline{FB} \perp \overline{AB}$ and $\overline{FB} \perp \overline{RB}$ by definition of perpendicularity of line and plane. $\triangle ABF \cong \triangle RBF$ by S.A.S. FA = FR by definition of congr[illegible]nce for triangles. $\angle FAR \cong \angle FRA$ by base angles of an [illegible]sceles triangle.

9. Yes.

	Statements		Reasons
1.	WA = WF ;	1.	Property of a cube.
2.	AB = FB .	2.	Property of a cube.
3.	BR = BL .	3.	Hypothesis.
4.	AR = FL .	4.	Addition property for equality.
5.	$\angle WAR \cong \angle WFL$.	5.	Property of a cube.
6.	$\triangle WAR \cong \triangle WFL$.	6.	S.A.S.
7.	WR = WL .	7.	Definition of congruence for triangles.
8.	$\overline{KW} \perp \overline{WA}$, $\overline{KW} \perp \overline{WF}$.	8.	Property of a cube.
9.	$\overline{KW} \perp$ plane AWF	9.	Theorem 9-2.
10.	$\overline{KW} \perp \overline{WR}$, $\overline{KW} \perp \overline{WL}$.	10.	Definition of perpendicularity for a line and a plane.
11.	KW = KW .	11.	Reflexive property of equality.
12.	$\triangle KWR \cong \triangle KWL$.	12.	S.A.S.
13.	KR = KL .	13.	Definition of congruence for triangles.

Problem Set 9-3

1. (a) T (d) F (g) F (j) T
   (b) F (e) T (h) F
   (c) T (f) T (i) F

2. $\overline{AC}$ and $\overline{AE}$ determine a plane which intersects Y at $\overleftrightarrow{BD}$ and intersects Z at $\overleftrightarrow{CE}$.

	Statements		Reasons
1.	$\overline{BD} \parallel \overline{CE}$.	1.	Theorem 9-6.
2.	$\angle ADB \cong \angle AEC$.	2.	Corollary 6-4-1.
3.	$\angle A \cong \angle AEC$.	3.	Theorem 5-6.
4.	$\angle ADB \cong \angle A$.	4.	Transitive property for congruence of angles.
5.	BD = BA .	5.	Theorem 5-7.

Alternatively, after Step 2 above: $\triangle ABD \sim \triangle ACE$, by Theorem 7-6. Hence AC, CE are proportional to AB, BD. Therefore AB = BD, since AC = CE.

3. Proof: Let ℓ_1 and ℓ_2 be two parallel lines. Let $\mathcal{P}_1$ be a plane which is parallel to one of the lines, say ℓ_1. We wish to show that $\mathcal{P}$ is parallel to ℓ_2. Suppose $\mathcal{P}$ were not parallel to ℓ_2. Then $\mathcal{P}$ would intersect ℓ_2 in a single point. Thus ℓ_1, which by hypothesis does not have exactly one point in common with $\mathcal{P}$, is distinct from ℓ_2. We now apply Theorem 9-4 to find that $\mathcal{P}$ intersects ℓ_1 in a point. On the basis of this contradiction to the hypothesis, we must reject the possibility that $\mathcal{P}$ and ℓ_2 are not parallel. Hence $\mathcal{P}$ is parallel to ℓ_2.

Problem Set 9-4

1. Points W, X, Y, Z are equidistant from the endpoints of $\overline{AB}$ by hypothesis. By Theorem 9-18 they all belong to the perpendicular bisecting plane of $\overline{AB}$ and are therefore coplanar.

2.

Statements	Reasons
1. $m \perp \overline{AB}$, $n \perp \overline{AB}$.	1. Hypothesis.
2. $m \parallel n$.	2. Theorem 9-9.
3. $m \perp \overline{CD}$.	3. Hypothesis.
4. $n \perp \overline{CD}$.	4. Theorem 9-10.

3.

Statements	Reasons
1. AB = CD.	1. Theorem 9-17.
2. $\overline{AB} \perp \overline{BD}$, $\overline{CD} \perp \overline{DB}$.	2. Definition of line perpendicular to plane.
3. $\triangle ABD \cong \triangle CDB$.	3. S.A.S.
4. AD = CB.	4. Definition of congruence for triangles.

4. (a) $\overline{BW}$. $\overline{BK}$. $\overline{BR}$. 90 . $\angle BKF$.
   (b) Not necessarily. W, K, R could be any points in $\mathcal{E}$.

5. (a) Yes.
   (b) No.
   (c) Yes. Yes. Yes.
   (d) Yes.
   (e) No.
   (f) No.
   (g) Not necessarily.
   (h) Yes.
   (i) Yes.
   (j) Yes.

6. (a) 6 inches.
   (b) 0 inches.
   (c) 3 inches.
   (d) $3\sqrt{2}$ inches or $\approx$ 4.242 inches.
   (e) $3\sqrt{3}$ inches or $\approx$ 5.196 inches.

7. $\overleftrightarrow{AX}$ and $\overleftrightarrow{BY}$ are perpendicular to plane $\mathcal{M}$. Hence $\overleftrightarrow{AX}$ and $\overleftrightarrow{BY}$ are parallel lines and therefore coplanar. Since O is in $\overline{AB}$ and N is in $\overleftrightarrow{XY}$, the plane ABXY contains both O and N . Since each of the coplanar lines $\overleftrightarrow{AX}$, $\overleftrightarrow{ON}$, and $\overleftrightarrow{BY}$ is perpendicular to $\overleftrightarrow{XY}$, they are parallel to one another. Since AO = OB , we apply Theorem 7-2 to obtain XN = NY . Thus N is the midpoint of $\overline{XY}$.

8. Let $\overleftrightarrow{BE}$ be the perpendicular to plane $\mathcal{M}$ at B . Then $\overline{AB} \perp \overline{BE}$, and it is given that $\overline{AB} \perp \overline{BC}$. Hence $\overleftrightarrow{AB} \perp$ plane EBC . By the definition of projection, $\overleftrightarrow{CD} \perp \mathcal{M}$. Then $\overleftrightarrow{CD} \parallel \overleftrightarrow{BE}$, so that D is in the plane EBC . Then $\overline{DB}$ is in this plane. Hence $\overline{AB} \perp \overline{BD}$, or $\angle ABD$ is a right angle.

9. Let the given point be P and the given plane be $\mathcal{E}$. Let F be the projection of P into $\mathcal{E}$. If X is any point in $\mathcal{E}$ distinct from F , then $\overleftrightarrow{FX}$ is a line in $\mathcal{E}$. Since $\overleftrightarrow{AF}$ is perpendicular to $\mathcal{E}$, it is perpendicular to every line in $\mathcal{E}$ through F ; in particular, $\overleftrightarrow{AF} \perp \overleftrightarrow{FX}$. By Theorem 6-19, $\overline{AF}$ is shorter than $\overline{AX}$. Thus $\overline{AF}$ is the shortest segment joining A and a point in $\mathcal{E}$.

Problem Set 9-5

1. (a) 12 in the usual classroom.
   (b) Right.
   (c) (1) A dihedral angle is acute if and only if its measure is less than 90 .
       (2) A dihedral angle is obtuse if and only if its measure is greater than 90 .
   (d) Two dihedral angles are adjacent if and only if they have respective plane angles which are adjacent.
   (e) Two dihedral angles are supplementary if and only if the sum of their measures is 180 .
   (f) Two dihedral angles are complementary if and only if the sum of their measures is 90 .

2. $m \angle C\text{-}PA\text{-}B = 90$ $(m \angle CPB = 90)$.
   $m \angle CAB = 60$, since $\triangle CAB$ is equilateral.
   $(\triangle APC \cong \triangle BPC \cong \triangle APB .)$

3. In $\mathcal{E}$ let $\overleftrightarrow{BC}$ be perpendicular to $\overleftrightarrow{PQ}$. Then by the definition of a plane angle, $\angle ABC$ is a plane angle of $\angle A\text{-}PQ\text{-}C$. $\mathcal{F} \perp \mathcal{E}$ by hypothesis. Hence $\angle A\text{-}PQ\text{-}C$ is a right dihedral angle, and its plane angle, $\angle ABC$, is a right angle, and $\overleftrightarrow{AB} \perp \overleftrightarrow{BC}$. Since it is given that $\overleftrightarrow{AB} \perp \overleftrightarrow{PQ}$, we now have $\overleftrightarrow{AB}$ perpendicular to two lines in $\mathcal{E}$ through B ; hence, by Theorem 9-2, $\overleftrightarrow{AB} \perp \mathcal{E}$.

4. Using the figure in the text, consider $\overleftrightarrow{XN}$ and $\overleftrightarrow{YN}$ in α such that $\overleftrightarrow{XN} \perp \overleftrightarrow{NC}$ and $\overleftrightarrow{YN} \perp \overleftrightarrow{NB}$. $\overleftrightarrow{XN} \perp \mathcal{P}$ and $\overleftrightarrow{YN} \perp \mathcal{R}$ by Theorem 9-21. Then, by definition of a line perpendicular to a plane, $\overleftrightarrow{XN} \perp \overleftrightarrow{MN}$ and $\overleftrightarrow{YN} \perp \overleftrightarrow{MN}$. By Theorem 9-2, $\overleftrightarrow{MN} \perp \alpha$.

5. By Theorem 9-9 we know $\mathcal{E} \parallel \mathcal{F}$. By Theorem 9-6 we know $\overleftrightarrow{AD} \parallel \overleftrightarrow{BK}$ and $\overleftrightarrow{CA} \parallel \overleftrightarrow{HB}$. Since $BK = AD$ and $BH = AC$, we know we have two parallelograms. These are rectangles since $\overline{AB}$ is perpendicular to both planes and therefore to lines in the plane through A and B. $\angle CAD$ and $\angle HBK$ are plane angles of the dihedral angle $\angle D\text{-}AB\text{-}H$ and are congruent. Then $\triangle ACD \cong \triangle BHK$ by S.A.S. However, we do not know the measure of any of the angles of the two triangles and so cannot find the length of $\overline{CD}$.

6. The proof is an immediate application of Theorem 9-20.

Review Problems

Chapter 9 Sections 1 to 5

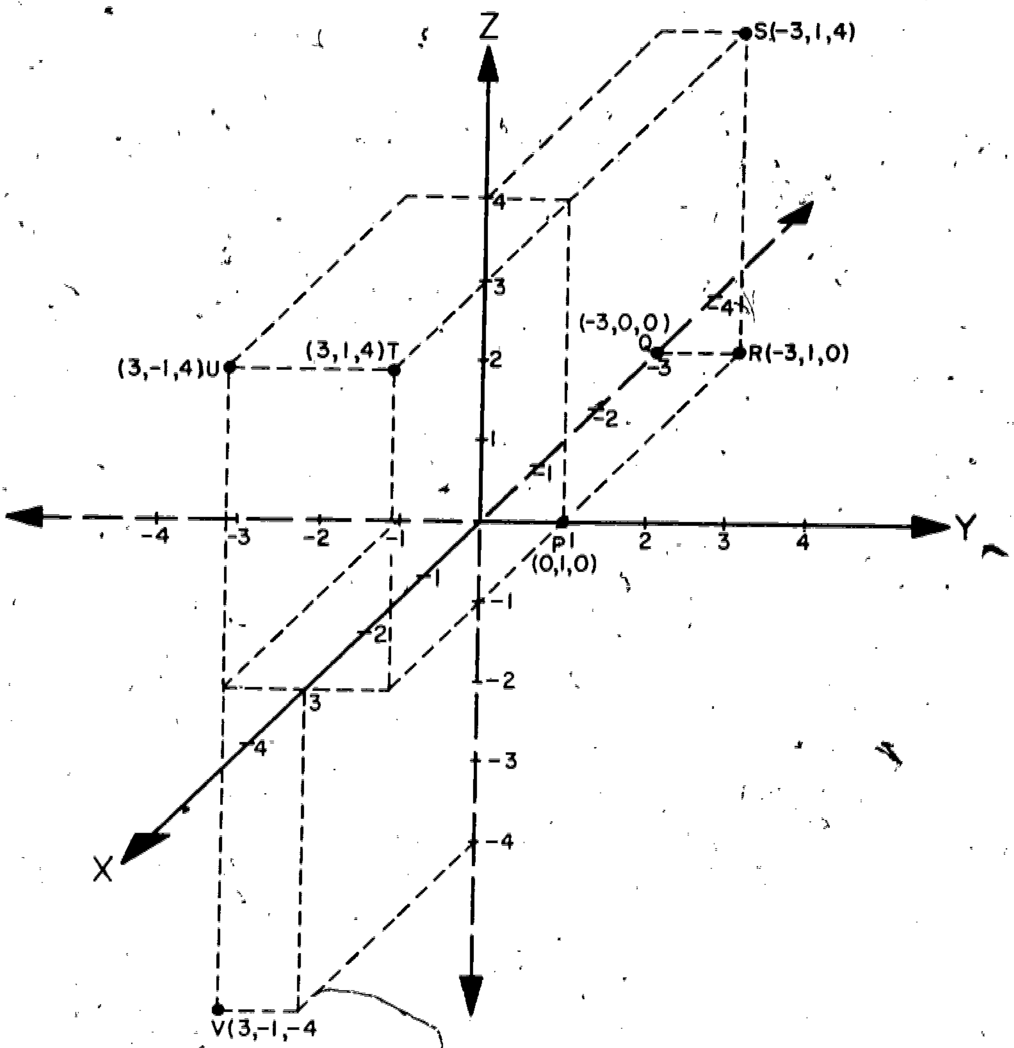
1. (a) always
   (b) sometimes
   (c) sometimes
   (d) always
   (e) sometimes
   (f) sometimes
   (g) never
   (h) always
   (i) sometimes
   (j) always

2. (a) F
   (b) T
   (c) F
   (d) F
   (e) F
   (f) T
   (g) F
   (h) F
   (i) T
   (j) F
   (k) F
   (l) F
   (m) T
   (n) F

3. (a) coincide (and parallel)
   (b) parallel
   (d) parallel
   (f) coincide (and parallel)
   (g) parallel
   (i) coincide (and parallel)

Problem Set 9-6

1.



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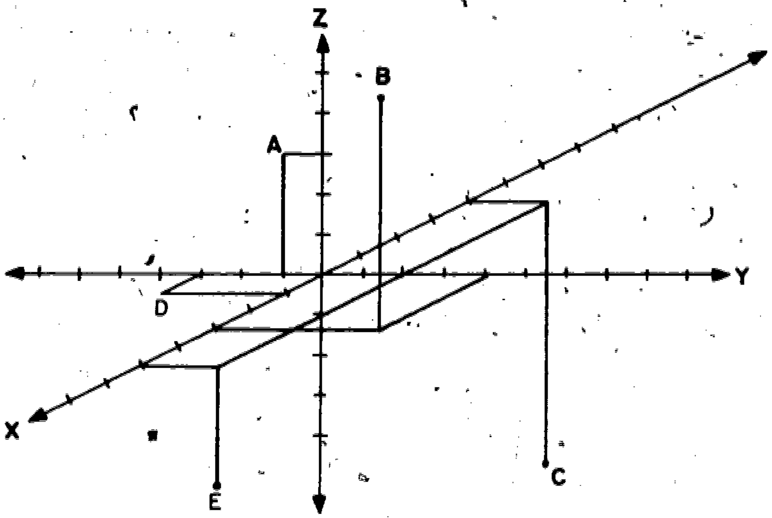
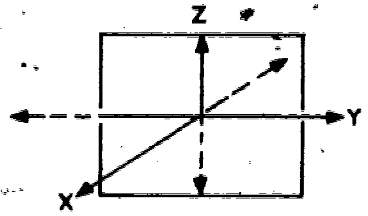
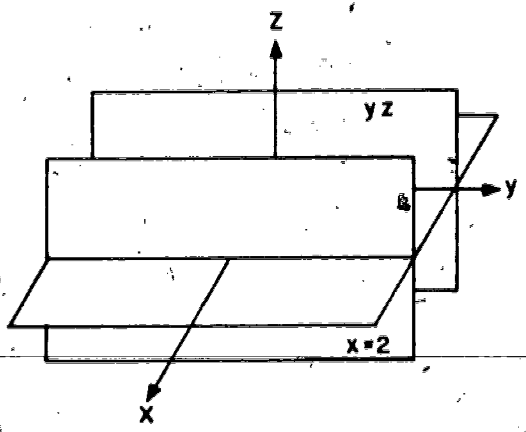
2.



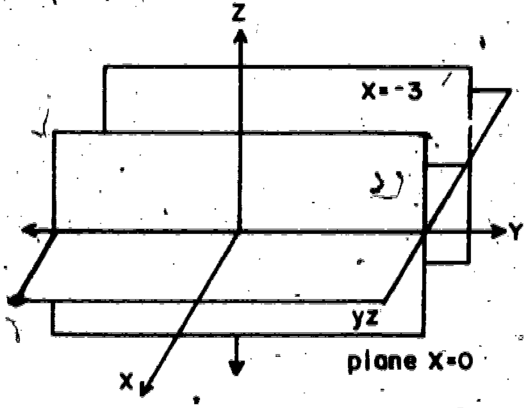
3. (a) In the yz-plane,
x = 0 .



(b) In the plane parallel
to the yz-plane and
two units in front
of it.

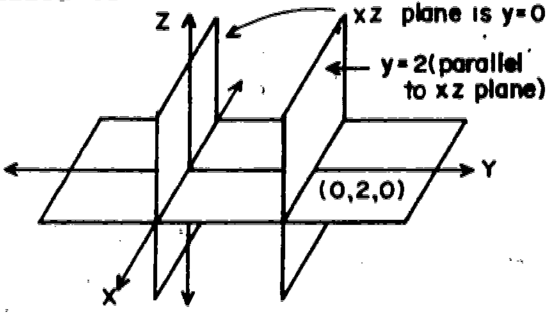


(c) In the plane parallel to the yz-plane and three units behind it.



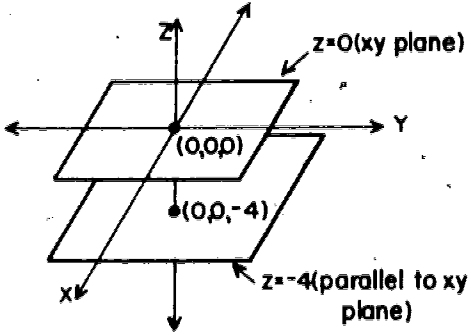
4. (a) xz-plane is $y = 0$.

(b) $y = 2$ (Parallel to xz-plane.)



(c) $z = 0$ (xy-plane)

(d) $z = -4$ (Parallel to xy-plane.)



5. The locus (the set) is:

   (a) the x-axis
   (b) the z-axis
   (c) the y-axis
   (d) the origin
   (e) A line in the yz-plane which is parallel to the z-axis and intersects the y-axis at $(0,2,0)$.
   (f) A line parallel to the z-axis and intersecting the xy-plane in the point $(2,1,0)$.

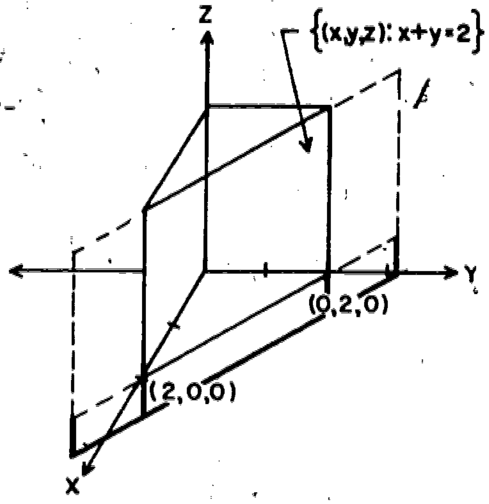
6. (a) The eight vertices might have coordinates

   $(0,0,0)$; $(0,a,0)$; $(a,a,0)$; $(a,0,0)$;
   $(0,0,a)$; $(0,a,a)$; (a,a,a) ; $(a,0,a)$.

   (b) The eight vertices might have coordinates

   $(0,0,0)$; $(0,b,0)$; $(a,b,0)$; $(a,0,0)$;
   $(0,0,c)$; $(0,b,c)$; (a,b,c) ; $(a,0,c)$.

7. All the points in space for which $x + y = 2$ lie in a plane which is parallel to the z-axis and which intersects the xy-plane in the line $x + y = 2$.



<u>Problem Set 9-7</u>

1. (a) 13 (d) $\sqrt{41}$
   (b) 7 (e) 3
   (c) 5 (f) $\sqrt{14}$

2. P(x,3,4)

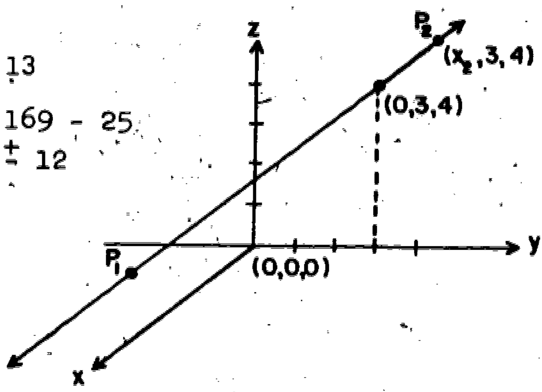
$$\sqrt{x^2 + 3^2 + 4^2} = 13$$

$$x^2 = 169 - 25$$

$$x = \pm 12$$

$P_1(12,3,4)$;

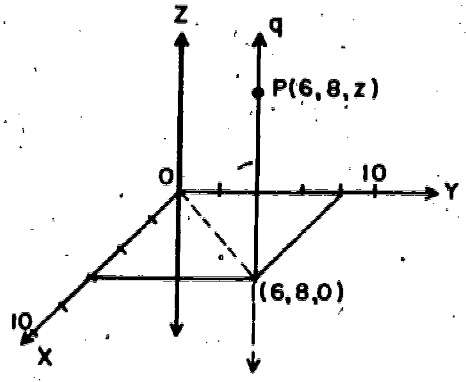
$P_2(-12,3,4)$.



3. $(OP)^2 = 6^2 + 8^2 + z^2 = 100$

$z = \sqrt{0}$

P(6,8[illegible] .



4. $(1, \sqrt{14}, 1)$ and $(1, -\sqrt{14}, 1)$.

5. $(AB)^2 = 1 + 4 + 9 = 14$

$(BC)^2 = 16 + 4 + 0 = 20$

$(AC)^2 = 9 + 16 + 9 = 34$

$(AB)^2 + (BC)^2 = (AC)^2$. Therefore, △ ABC is a right triangle.

6. $AB = \sqrt{6^2 + 4^2 + 2^2} = \sqrt{56} = 2\sqrt{14}$

$AC = \sqrt{6^2 + 2^2 + 4^2} = \sqrt{56} = 2\sqrt{14}$

AB = AC , and hence, △ ABC is isosceles by definition of isosceles triangle.

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7. $AB = \sqrt{1 + 1 + 4} = \sqrt{6}$

$BC = \sqrt{1 + 4 + 1} = \sqrt{6}$

$AC = \sqrt{4 + 1 + 1} = \sqrt{6}$

$AB = BC = AC$.

Therefore $\triangle ABC$ is equilateral.

$DE = \sqrt{25 + 25 + 1} = \sqrt{51}$

$EF = \sqrt{1 + 1 + 49} = \sqrt{51}$

$DF = \sqrt{36 + 16 + 64} = \sqrt{116}$.

$\triangle DEF$ is isosceles, but not equilateral.

8. (a) $AB = \sqrt{4 + 1 + 16} = \sqrt{21}$,

$BC = \sqrt{9 + 1 + 4} = \sqrt{14}$,

$CD = \sqrt{4 + 1 + 16} = \sqrt{21}$,

$AD = \sqrt{9 + 1 + 4} = \sqrt{14}$,

$AB = CD$, $BC = AD$.

(b) No, because it does not assure us that points A, B, C, D are coplanar.

9. (a) $AB = \sqrt{5}$,

$CD = \sqrt{5}$,

$AD = 3$,

$BC = 3$.

Hence the opposite sides of the figure are congruent.

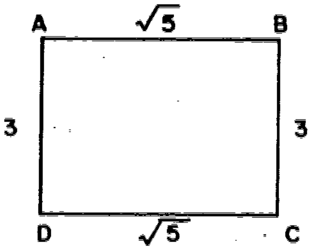
(b) $AC = \sqrt{1 + 4 + 9} = \sqrt{14}$,

$BD = \sqrt{9 + 4 + 1} = \sqrt{14}$,

$(AD)^2 + (DC)^2 = 9 + 5$

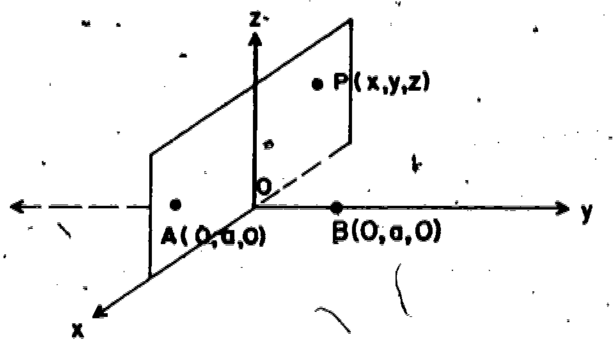
$= 14 = (AC)^2$.

Hence $\angle D$ is a right angle. Similarly, $\angle A$, $\angle B$, $\angle C$ are right angles.



(c) No. It has not been proved that the four vertices are coplanar.

10. Given two points A and B, choose $\overleftrightarrow{AB}$ as the y-axis and the midpoint of $\overline{AB}$ as the origin.



There is a real number a, $a \neq 0$, such that $A = (0,-a,0)$ and $B = (0,a,0)$. Then the xz-plane is the perpendicular bisector of $\overline{AB}$. We have two things to prove:

(1) If P is in the xz-plane, then $AP = BP$.

(2) If $AP = BP$, P is in the xz-plane.

(1) If P is in the xz-plane, then $P = (b,0,c)$ for real numbers b and c.

$$AP = \sqrt{(b - 0)^2 + (0 + a)^2 + (c - 0)^2}$$
$$= \sqrt{b^2 + a^2 + c^2} .$$
$$BP = \sqrt{(b - 0)^2 + (0 - a)^2 + (c - 0)^2}$$
$$= \sqrt{b^2 + a^2 + c^2} ,$$

and $AP = BP$.

(2) If $P(x,y,z)$ is any point such that $AP = BP$, then it follows from the distance formula that

$$\sqrt{(x - 0)^2 + (y + a)^2 + (z - 0)^2}$$
$$= \sqrt{(x - 0)^2 + (y - a)^2 + (z - 0)^2} ,$$
$$x^2 + y^2 + 2ay + a^2 + z^2 = x^2 + y^2 - 2ay + a^2 + z^2$$
$$4ay = 0 , \text{ and since } a \neq 0 ,$$
$$y = 0 .$$

Therefore P is in the xz-plane.

Problem Set 9-8

1. The coordinates of the endpoints of the diagonals $\overline{EF}$ and $\overline{GH}$ are

$$E = (2a,0,2a)$$
$$F = (0,2a,0)$$
$$G = (2a,0,0)$$
$$H = (0,2a,2a) .$$

$$GH = \sqrt{(2a - 0)^2 + (0 - 2a)^2 + (0 - 2a)^2}$$
$$= \sqrt{12a^2} = 2a\sqrt{3} ,$$
$$EF = \sqrt{(2a - 0)^2 + (0 - 2a)^2 + (2a - 0)^2}$$
$$= \sqrt{12a^2} = 2a\sqrt{3} .$$

By Example 2 (a), $AB = 2a\sqrt{3}$.

Therefore, $GH = EF = AB$.

2. (a) $\overleftrightarrow{AB} = \{(x,y,z): x = -2 + 10k , y = 2k , z = 4 - 6k,$ k is a real number$\}$

(b) $\overline{AB} = \{(x,y,z): x = -2 + 10k , y = 2k , z = 4 - 6k, 0 \leq k \leq 1\}$

(c) $\overrightarrow{AB} = \{(x,y,z): x = -2 + 10k , y = 2k , z = 4 - 6k, k \geq 0\}$

3. (a) Midpoint of $\overline{AB}$ is

$$\{(x,y,z): x = -2 + 10k , y = 2k , z = 4 - 6k , k = \frac{1}{2}\} = (3,1,1) .$$

(b) Trisection point of $\overline{AB}$ nearer A is

$$\{(x,y,z): x = -2 + 10k , y = 2k , z = 4 - 6k , k = \frac{1}{3}\} = (\frac{4}{3} , \frac{2}{3} , 2) .$$

(c) Trisection point of $\overline{AB}$ nearer B is

$$\{(x,y,z): x = -2 + 10k , y = 2k , z = 4 - 6k , k = \frac{2}{3}\} = (\frac{14}{3} , \frac{4}{3} , 0) .$$

(d) When $k = 3$, $P = (28,6,-14)$.

(e) When $k = -3$, $P = (-32,-6,22)$.

(f) $z = 0 = 4 - 6k$

$k = \frac{2}{3}$

$P = (\frac{14}{3}, \frac{4}{3}, 0)$.

$y = 0 = 2k$

$k = 0$

$P = (-2,0,4)$.

$x = 0 = -2 + 10k$

$k = \frac{1}{5}$

$P = (0, \frac{2}{5}, \frac{14}{5})$.

(g) $z = 3 = 4 - 6k$

$k = \frac{1}{6}$

$P = (-\frac{1}{3}, \frac{1}{3}, 3)$.

$y = -2 = 2k$

$k = -1$

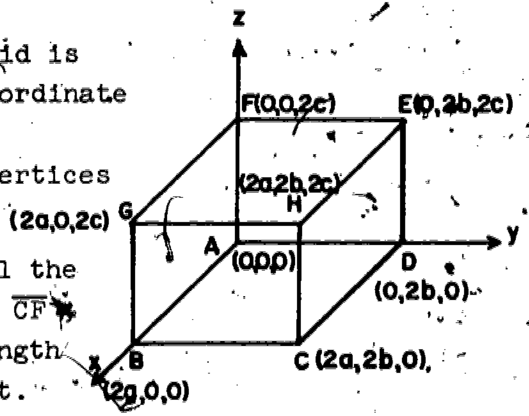
$P = (-12,-2,10)$.

$x = -3 = -2 + 10k$

$k = -\frac{1}{10}$

$P = (-3, -\frac{1}{5}, \frac{23}{5})$.

4. If a rectangular solid is given, there is a coordinate system which assigns coordinates to the vertices as in the diagram.



We must show that all the diagonals $\overline{AH}$, $\overline{BE}$, $\overline{CF}$, $\overline{DG}$ have the same length and the same midpoint.

Using the distance formula four times, we find that each diagonal has length

$$\sqrt{(2a)^2 + (2b)^2 + (2c)^2} = 2\sqrt{a^2 + b^2 + c^2} .$$

Midpoint of $\overline{CF}$ is

$$\left(\frac{0 + 2a}{2} , \frac{0 + 2b}{2} , \frac{2c + 0}{2}\right) = (a,b,c) .$$

Similarly,

midpoint of $\overline{BE}$ is (a,b,c) ;

midpoint of $\overline{GD}$ is (a,b,c) ;

midpoint of $\overline{AH}$ is (a,b,c) .

Thus the point (a,b,c) is the midpoint of each diagonal, and therefore the diagonals bisect each other at (a,b,c) .

5. A, B, C are collinear if the sum of the lengths of two of the segments joining the points equals the length of the other segment.

$A(-1,5,3)$; $B(1,4,4)$; $C(5,2,6)$.

$AB = \sqrt{2^2 + 1^2 + 1^2} = \sqrt{6}$;

$BC = \sqrt{4^2 + 2^2 + 2^2} = \sqrt{24} = 2\sqrt{6}$;

$AC = \sqrt{6^2 + 3^2 + 3^2} = \sqrt{54} = 3\sqrt{6}$.

$AC = AB + BC$, and hence A, B, C are collinear.

An alternative method:

$\overleftrightarrow{AB} = \{(x,y,z):\ x = -1 + 2k\ ,\ y = 5 - k\ ,\ z = 3 + k\ ,$ k is a real number$\}$.

The point C is the point on $\overleftrightarrow{AB}$ corresponding to $k = 3$.

6. $\overleftrightarrow{P_1P_2} = \{(x,y,z):\ x = 2 + k\ ,\ y = 1 - 3k\ ,\ z = 3 - 2k\ ,$ k is a real number$\}$.

If $x = 0$, $k = -2$.

Therefore, $y = 1 + 6 = 7$ and $z = 3 + 4 = 7$.

$P = (0, 7, 7)$.

7. $\overleftrightarrow{P_1P_2} = \{(x,y,z):\ x = -1 + 4k\ ,\ y = 2 - 4k\ ,\ z = -1 + 3k\ ,$ k is a real number$\}$.

If $y = 0$, $k = \frac{1}{2}$.

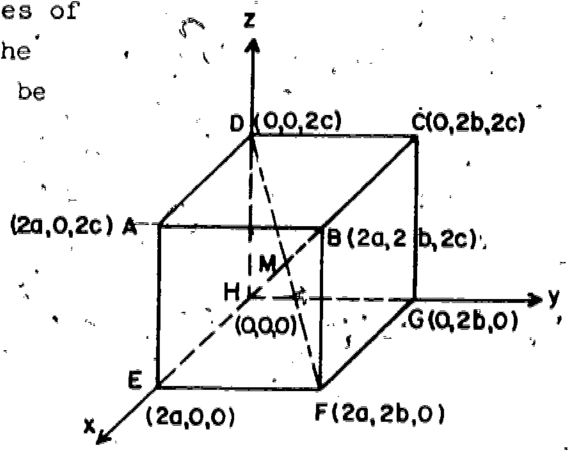
Therefore, $x = -1 + 2 = 1$

$z = -1 + \frac{3}{2} = \frac{1}{2}$

$P = (1, 0, \frac{1}{2})$.

8. Let the coordinates of the vertices of the rectangular solid be

$A(2a,0,2c)$;
$B(2a,2b,2c)$;
$C(0,2b,2c)$;
$D(0,0,2c)$;
$E(2a,0,0)$;
$F(2a,2b,0)$;
$G(0,2b,0)$;
$H(0,0,0)$.



M , the midpoint of all diagonals, would have coordinates (a,b,c) .

Using different pairs of diagonals and the Pythagorean Theorem, the following relationships among a, b, c, may be established.

In right triangle DMC , $(DM)^2 + (MC)^2 = (DC)^2$.

$$\text{Hence,}\quad (a - 0)^2 + (b - 0)^2 + (c - 2c)^2 + (a - 0)^2 + (b - 2b)^2 + (c - 2c)^2 = (2b)^2 ,$$

$$\text{or}\quad 2a^2 + 2b^2 + 2c^2 = 4b^2 ,$$

$$\text{and}\quad \underline{a^2 + c^2 = b^2} .$$

Similarly, from $\triangle GMF$, $\underline{b^2 + c^2 = a^2}$,

and from $\triangle DMH$, $\underline{a^2 + b^2 = c^2}$.

9. (a) $M(\frac{11}{2}, -\frac{1}{2}, 3)$.

(b) $\frac{6 + 5}{2} = \frac{11}{2}$

$\frac{y + 0}{2} = -\frac{1}{2}$, $y = -1$

$\frac{z + 0}{2} = 3$, $z = 6$.

Hence, the coordinates of D are $(6,-1,6)$.

(c) Yes. Vertices A, B, C, D lie in the plane determined by the intersecting lines $\overleftrightarrow{AC}$ and $\overleftrightarrow{BD}$, and hence $ABCD$ is a quadrilateral. If the diagonals of a quadrilateral bisect each other the quadrilateral is a parallelogram.

10. M , midpoint of $\overline{AC}$, is $(\frac{7}{2}, 1, 4)$;

M' , midpoint of $\overline{BD}$, is $(\frac{7}{2}, 1, 4)$.

Therefore, M = M' and ABCD is a plane figure, determined by the two intersecting diagonals. Hence, ABCD is a parallelogram, since the opposite sides are congruent (or by the diagonals bisecting each other).

11. M , midpoint of $\overline{AC}$, is $(\frac{9}{2}, 2, -\frac{1}{2})$;

M' , midpoint of $\overline{BD}$, is $(\frac{9}{2}, 2, -\frac{1}{2})$.

Hence, M = M' , and ABCD is a plane figure, determined by the two intersecting diagonals $\overline{AC}$ and $\overline{BD}$. ABCD is a rectangle since it is a quadrilateral with all right angles.

Problem Set 9-9

1. (a) Using $ax + by + cz = d$,

$P_1(1,0,0)$: $a + 0 + 0 = d$;

$P_2(0,1,0)$: $0 + b + 0 = d$;

$P_3(0,0,1)$: $0 + 0 + c = d$.

Therefore $dx + dy + dz = d$

or $x + y + z = 1$.

(b) $P_1(3,0,1)$: $3a + c = d$; $3a + \frac{d}{2} = d$; $a = \frac{d}{6}$;

$P_2(0,1,0)$: $b = d$;

$P_3(0,0,2)$: $2c = d$; $c = \frac{d}{2}$.

Therefore $\frac{d}{6}x + dy + \frac{d}{2}z = d$

or $x + 6y + 3z = 6$.

(c) $P_1(3,0,1)$: $3a + c = d$

$P_2(1,2,0)$: $a + 2b = d$

$P_3(0,2,4)$: $2b + 4c = d$.

Solving, $c = \frac{d}{13}$

$a = \frac{4d}{13}$

$b = \frac{9d}{26}$.

Therefore $8x + 9y + 2z = 26$.

(d) $P_1(1,-1,0)$: $a - b = d$;

$P_2(2,0,3)$: $2a + 3c = d$;

$P_3(0,-3,1)$: $-3b + c = d$.

Solving, $b = -\frac{4d}{11}$

$c = -\frac{d}{11}$

$a = \frac{7d}{11}$.

Therefore $7x - 4y - z = 11$.

2. $\sqrt{(x - 1)^2 + (y - 2)^2 + (z - 3)^2}$

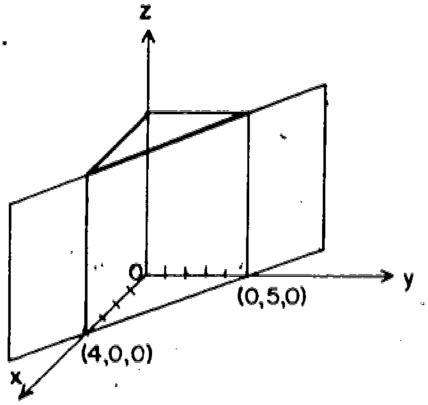
$= \sqrt{(x - 2)^2 + (y - 5)^2 + (z - 4)^2}$

$x^2 - 2x + 1 + y^2 - 4y + 4 + z^2 - 6z + 9$

$= x^2 - 4x + 4 + y^2 - 10y + 25 + z^2 - 8z + 16$.

Hence, $2x + 6y + 2z = 31$.

3. (a) $5x + 4y = 20$ is a plane perpendicular to the xy-plane and intersecting the xy-plane in line $5x + 4y = 20$. This plane intersects the yz-plane in the line whose equation <u>in the yz-plane</u> is $y = 5$.



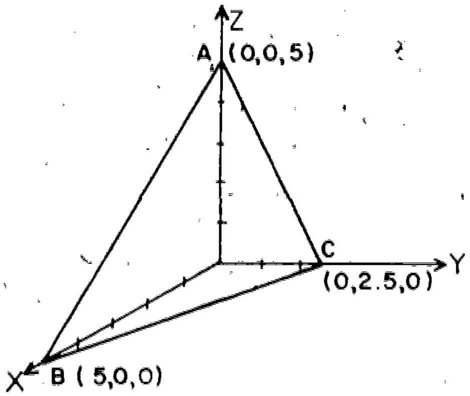
(Using xyz-coordinates, the line is, of course, $\{(x,y,z)\colon y = 5$ and $x = 0\}$.

(b) $x + 2y + z = 5$ is an equation of a plane.

The intersection of this plane and the xy-plane is the line whose equation in the xy-plane is $x + 2y = 5$,
(or $\{(x,y,z)\colon x + 2y = 5$ and $z = 0\})$.

The intersection of this plane and the xz-plane is the line whose equation in the xz-plane is $x + z = 5$,
(or $\{(x,y,z)\colon x + z = 5$ and $y = 0\})$.

The intersection of this plane and the yz-plane is the line whose equation in the yz-plane is $2y + z = 5$,
(or $\{(x,y,z)\colon 2y + z = 5$ and $x = 0\})$.



4. $x - 2y + 2z = 9$.

5. (a) $y + z = -1$.
   (b) $y + z = -1$.
   (c) $3x + 5y + 14z = 7$.

6. $3(2 - 3k) + 5(1 + k) + 14(4 - 2k) = 11$

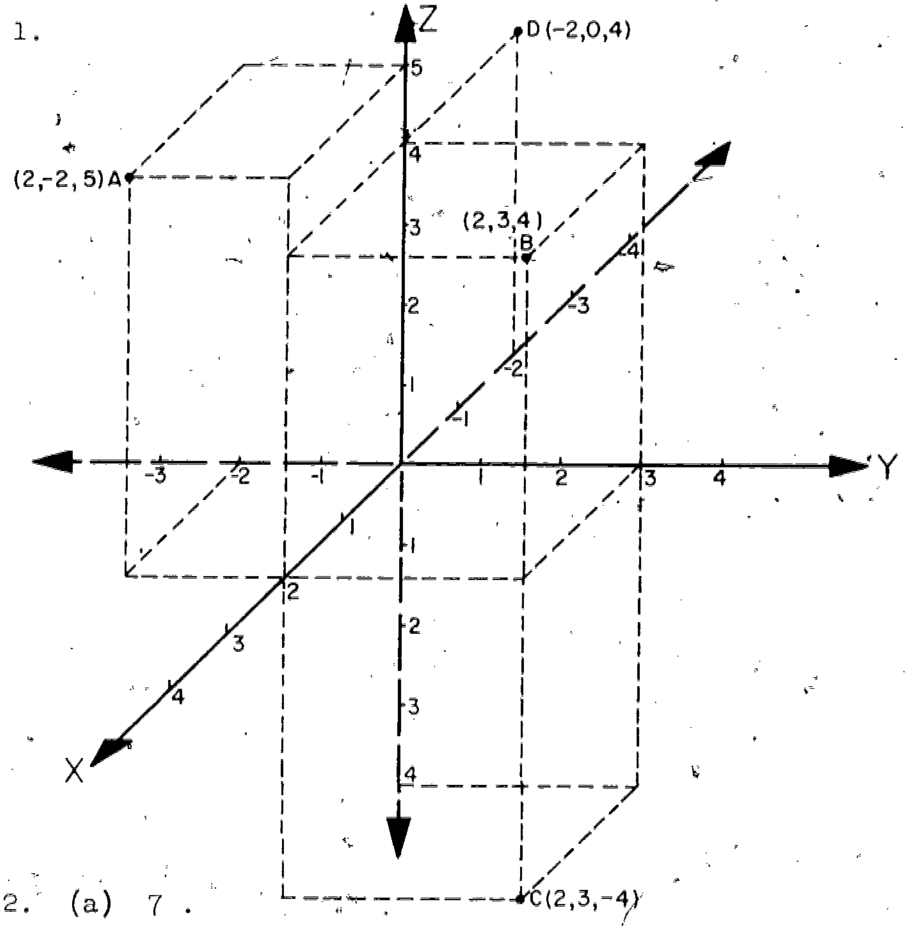
   $k = \frac{7}{4}$

   The point of intersection is $(-\frac{13}{4}, \frac{11}{4}, \frac{1}{2})$.

Review Problems

Chapter 9, Sections 6 to 9

1.



2. (a) 7 .
   (b) 5 .

3. (a) $(-3,3,\frac{13}{2})$.

   (b) $(1,\frac{3}{2},7)$.

4. (a) $\{(x,y,z)\colon\ x = -6k\ ,\ y = 4 - 2k\ ,\ z = 5 + 3k\ ,$
   k is real$\}$.
   (b) $\{(x,y,z)\colon\ x = 3 - 4k\ ,\ y = 3k\ ,\ z = 7\ ,$
   k is real$\}$.

5. (12,-7,-1) .

6. (a) Right.
   (b) Equilateral.
   (c) Isosceles right.

7. (a) Collinear.
   (b) Collinear.
   (c) Noncollinear.
   (d) Collinear.

8. Not necessarily. All four vertices must be coplanar if ABCD is a parallelogram.

9. Yes. The fact that the diagonals bisect each other assures us that the vertices are coplanar.

10. Yes (because $3 - 1 + 2 = 4$) .

11. The line $\{(x,y,0): 2x - 3y = 6\}$.

12. Plane perpendicular to the y-axis at $(0,5,0)$.

13. $4x + 6y - 4z = -9$.

14. Let M be the origin and A'B'C'D' be the xy-plane of a three-dimensional coordinate system, assigning coordinates as follows:

$A'(0,-a,0)$, $D'(0,a,0)$, $B'(b,-a,0)$,$C'(b,a,0)$
$A(0,-a,c)$, $D(0,a,c)$, $B(b,-a,c)$, $C(b,a,c)$.

Then $MB = \sqrt{b^2 + (-a)^2 + c^2}$

$MC = \sqrt{b^2 + a^2 + c^2}$

and $MB = MC$.

REVIEW PROBLEMS

Chapter 6-9

1. 0
2. +
3. +
4. 0
5. +
6. 0
7. 0
8. +
9. +
10. 0
11. +
12. 0
13. +
14. +
15. +
16. +
17. 0
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91. +
92. +
93. +
94. 0
95. 0
96. +
97. 0
98. 0
99. +
100. 0

Chapter 10

ANSWERS AND SOLUTIONS

Problem Set 10-2a

1. $(\overrightarrow{A,B})$, $(\overrightarrow{B,A})$.

2. $(\overrightarrow{A,C})$, $(\overrightarrow{A,B})$, $(\overrightarrow{B,C})$, $(\overrightarrow{C,A})$, $(\overrightarrow{B,A})$, $(\overrightarrow{C,B})$.

3. Figure a: $(\overrightarrow{A,B}) \doteq (\overrightarrow{G,H})$; $(\overrightarrow{F,E}) \doteq (\overrightarrow{J,I})$.
   Figure b: $(\overrightarrow{C,D}) \doteq (\overrightarrow{E,F})$.
   Figure c: $(\overrightarrow{A,C}) \doteq (\overrightarrow{D,F})$.

4. (a) By the Betweenness-Addition Theorem $\overline{AC} \cong \overline{BD}$ and therefore $AC = BD$. Also by the collinearity of the points in the order given it follows that $\overrightarrow{AC}$ is a subset of $\overrightarrow{BD}$ or $\overrightarrow{BD}$ is a subset of $\overrightarrow{AC}$, hence $\overrightarrow{AC} \| \overrightarrow{BD}$ and $(\overrightarrow{A,C}) \doteq (\overrightarrow{B,D})$.
   (b) Same proof as (a).
   (c) $(\overrightarrow{A,B}) \doteq (\overrightarrow{C,D})$ tells us that $\overrightarrow{AB} \| \overrightarrow{CD}$ and $AB = CD$. It follows that B and D are on the same side of $\overleftrightarrow{AC}$ and hence ABDC is a parallelogram. Hence $(\overrightarrow{A,C}) \doteq (\overrightarrow{B,D})$.
   (d) If A, B, C are collinear and D is not in $\overleftrightarrow{AB}$, then $\overleftrightarrow{AB}$ intersects $\overleftrightarrow{CD}$ and this would contradict the hypothesis that $\overrightarrow{AB} \| \overrightarrow{CD}$.
   (e) If C, D are between A, B , then $AB \neq CD$, which contradicts the hypothesis that $(\overrightarrow{A,B}) \doteq (\overrightarrow{C,D})$.

5. For each case let the projections of B, F, G, H into ℓ be B', F', G', H' , and consider lines parallel to ℓ through B and G into which the projection of B, F, G, H are B, F'', G, H'' . Then $\triangle BFF'' \cong \triangle GHH''$ by the S.A.A. Theorem, and we have $BF'' = GH''$. It follows then in rectangles B'F'F''B and G'H'H''G that $\overline{B'F'}$ and $\overline{G'H'}$, the projections into ℓ of $\overline{BF}$ and $\overline{GF}$, are congruent, and $B'F' = G'H'$.

In Case (a), $G' = B'$, and all points of $(\overrightarrow{G,H})$ except G are on the same side of $\overleftrightarrow{BB'}$ as F so that in line ℓ $\overrightarrow{B'F'} = \overrightarrow{G'H'}$. In Case (b), all points of $(\overrightarrow{G,H})$ including G are on the same side of $\overleftrightarrow{BB'}$ as F so that in line ℓ $\overrightarrow{G'H'}$ is a subset of $\overrightarrow{B'F'}$. In Case (c), all points of $(\overrightarrow{B,F})$ are on the same side of $\overleftrightarrow{GG'}$ as H so that in line ℓ $\overrightarrow{B'F'}$ is a subset of $\overrightarrow{G'H'}$. This establishes for the three cases that $\overrightarrow{B'F'} \parallel \overrightarrow{G'H'}$, and together with $B'F' = G'H'$ shows that $(\overrightarrow{B',F'}) \doteq (\overrightarrow{G',H'})$.

6. (a) 3.
   (b) -2.
   (c) 2.
   (d) 1.
   (e) $-\frac{2}{3}$.
   (f) -1.
   (g) -4.
   (h) $\frac{1}{2}$.
   (i) $-\frac{1}{2}$.

7. (a) $r = \frac{1}{2}$; $s = \frac{1}{2}$.
   (b) $r = 2$; $s = -1$.
   (c) $r = -1$; $s = 2$.
   (d) $r = \frac{2}{3}$; $s = \frac{1}{3}$.
   (e) $r = \frac{3}{2}$; $s = -\frac{1}{2}$.
   (f) $r = -\frac{1}{2}$; $s = \frac{3}{2}$.

8. (a) 2.
   (b) $\frac{1}{2}$.
   (c) -1.
   (d) -2.
   (e) 1.
   (f) -2.
   (g) -2.

9. (a) $\frac{1}{3}$.
   (b) $-\frac{1}{2}$.
   (c) $\frac{3}{2}$.
   (d) $\frac{2}{3}$.
   (e) $-\frac{2}{3}$.
   (f) 1.
   (g) -1.

Problem Set 10-2b

1. DEFINITION. If $(\overrightarrow{A,B})$, $(\overrightarrow{C,D})$ are two directed line segments, then $(\overrightarrow{A,B}) - (\overrightarrow{C,D})$ is the directed line segment $(\overrightarrow{E,F})$ such that $(\overrightarrow{C,D}) + (\overrightarrow{E,F}) = (\overrightarrow{A,B})$. The determination of $(\overrightarrow{E,F})$ when $(\overrightarrow{A,B})$ and $(\overrightarrow{C,D})$ are given is called subtraction of directed line segments.

2. (a) $(\overrightarrow{A,C})$. (e) $(\overrightarrow{A,A})$.
   (b) $(\overrightarrow{A,C})$. (f) $(\overrightarrow{B,B})$.
   (c) $(\overrightarrow{A,C})$. (g) $(\overrightarrow{A,B})$.
   (d) $(\overrightarrow{B,A})$.

3. (a) $(\overrightarrow{D,C})$. (c) $(\overrightarrow{A,D})$.
   (b) $(\overrightarrow{B,C})$. (d) $(\overrightarrow{A,A})$.

4. (a) A directed segment of approximate length $2\frac{1}{2}$ extending from A at an angle of approximately 157° with ℓ.
   (b) A directed segment of approximate length $2\frac{1}{2}$ extending from C at an angle of approximately 157° with ℓ_1.
   (c) The two are equivalent.

5.

	Length	Angle	Origin
(a)	Approx. 1.3"	Approx. 79°	A
(b)	Approx. 2.6"	Approx. 40°	C
(c)	Approx. 3.0"	Approx. 85°	A
(d)	Approx. 3.3"	Approx. 67°	A
(e)	Approx. 3.3"	Approx. 67°	A

   (f) They are equal.

6. Approximately 25° east of north and approximately 8.3 miles.

7. (a) $(\overrightarrow{D,E})$. (d) -3.
   (b) $-\frac{1}{3}$. (e) $\frac{1}{2}, \frac{1}{3}$.
   (c) $\frac{2}{3}$.

Problem Set 10-3

1. (a) $[3,1]$. (e) $[-2,2]$.
   (b) $[-3,-1]$. (f) $[-5,1]$.
   (c) $[0,0]$. (g) $[2,-2]$.
   (d) $[5,-1]$.

2. (a) $[5,-5]$. (e) $[10,-2]$.
   (b) $[-5,5]$. (f) $[5,3]$.
   (c) $[0,0]$. (g) $[-10,2]$.
   (d) $[-5,-3]$.

3. (a) X is $(9,2)$. (c) X is $(3,0)$.
   (b) X is $(-1,4)$. (d) X is $(-1,4)$.

4. (a) X is $(-1,-6)$. (c) X is $(-11,4)$.
   (b) X is $(9,0)$. (d) X is $(9,0)$.

5. (a) $[-1,5]$. (e) $[2,5]$.
   (b) $[-2,8]$. (f) $[-2,-5]$.
   (c) $[-7,1]$. (g) $[-4,1]$.
   (d) $[-6,11]$.

6. (a) $[-2,10]$. (e) $[-14,15]$.
   (b) $[-2,10]$. (f) $[-12\frac{1}{2},4]$.
   (c) $[-9,-6]$. (g) $[4\frac{1}{2},-\frac{3}{4}]$.
   (d) $[-13,12]$.

7. (a) $\sqrt{13}$. (e) $2\sqrt{17}$.
   (b) $\sqrt{10}$. (f) $\sqrt{26}$.
   (c) 5 . (g) $\sqrt{17}$.
   (d) $\sqrt{61}$.

8. (a) $[0,-3]$. (d) $[3,0]$.
   (b) $[-1,-4]$. (e) $[0,0]$.
   (c) $[-1,-4]$.

9. Approximately 5.8 miles per hour at an angle of approximately 120° .

Problem Set 10-4

1. $x = 2$, $y = -3$.

2. $x = -\frac{13}{6}$; $y = \frac{23}{6}$.

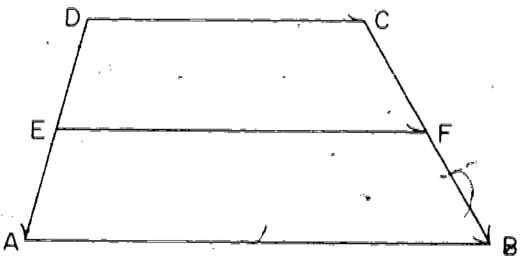
3. $x = -\frac{1}{5}$, $y = \frac{4}{5}$.

4. $x = \frac{27}{13}$, $y = \frac{8}{13}$.

5. $x = -1$, $y = 0$, (infinite number of solutions satisfying $x + 2y = -1$).

Problem Set 10-5

1.



Let the trapezoid and median be lettered as shown and the segments directed as shown. Then

$\overrightarrow{EF} = \overrightarrow{DC} + \overrightarrow{CF} - \overrightarrow{DE}$ and $\overrightarrow{EF} = \overrightarrow{EA} + \overrightarrow{AB} - \overrightarrow{FB}$,

therefore $2\overrightarrow{EF} = \overrightarrow{DC} + \overrightarrow{CF} - \overrightarrow{DE} + \overrightarrow{EA} + \overrightarrow{AB} - \overrightarrow{FB}$.

But $\overrightarrow{DE} = \overrightarrow{EA}$ and $\overrightarrow{CF} = \overrightarrow{FB}$.

Thus we can simplify, and obtain

$$2\overrightarrow{EF} = \overrightarrow{DC} + \overrightarrow{AB}$$

or
$$\overrightarrow{EF} = \frac{1}{2}(\overrightarrow{AB} + \overrightarrow{DC}) \quad .$$

But because $\overrightarrow{DC} \parallel \overrightarrow{AB}$ it follows that $\overrightarrow{DC} = k\overrightarrow{AB}$.

Therefore $\overrightarrow{EF} = \frac{1}{2}(\overrightarrow{AB} + k\overrightarrow{AB}) = \frac{1 + k}{2}\overrightarrow{AB}$,

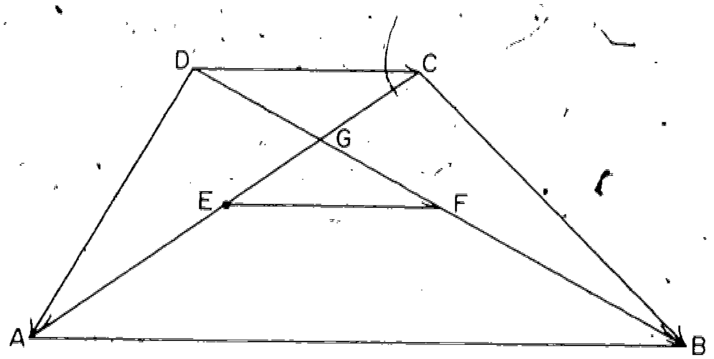
hence $\overline{EF} \parallel \overline{AB} \parallel \overline{DC}$,

and $EF = \frac{1 + k}{2}|\overrightarrow{AB}|$, where $k > 0$,

$$EF = \frac{1}{2}(|\overrightarrow{AB}| + k|\overrightarrow{AB}|) = \frac{1}{2}(|\overrightarrow{AB}| + |\overrightarrow{DC}|) \; ;$$

hence $EF = \frac{1}{2}(AB + DC)$.

2.



Let the trapezoid be lettered as shown and the segments directed as shown. E and F are midpoints of $\overline{AC}$ and $\overline{BD}$ respectively. Now

$$\overrightarrow{EF} = - \overrightarrow{CE} - \overrightarrow{DC} + \overrightarrow{DF} \; ; \; \overrightarrow{EF} = \overrightarrow{EA} + \overrightarrow{AB} - \overrightarrow{FB} \; .$$

By adding,

$$2\overrightarrow{EF} = - \overrightarrow{CE} - \overrightarrow{DC} + \overrightarrow{DF} + \overrightarrow{EA} + \overrightarrow{AB} - \overrightarrow{FB} \; .$$

Since $\overrightarrow{CE} = \overrightarrow{EA}$ and $\overrightarrow{DF} = \overrightarrow{FB}$, we can simplify, and obtain

$$2\overrightarrow{EF} = \overrightarrow{AB} - \overrightarrow{DC}$$

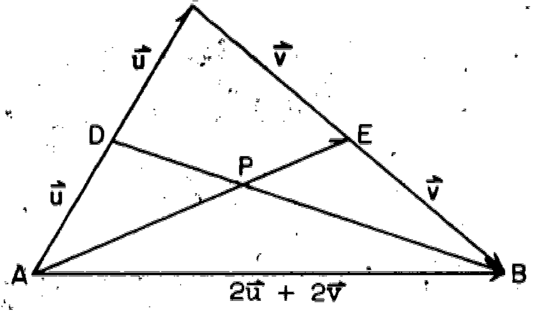
or

$$\overrightarrow{EF} = \frac{1}{2}(\overrightarrow{AB} - \overrightarrow{DC}) \; ,$$

from which since $\overleftrightarrow{AB} \parallel \overleftrightarrow{DC}$, we can conclude, as in the solution of Problem 1, that

$$\overline{EF} \parallel \overline{AB} \text{ and } EF = \frac{1}{2}(AB - DC) \; .$$

3.



Let the triangle be lettered as shown and let $\overline{AE}$ and $\overline{BD}$ be medians intersecting at P .

Now $\qquad \overrightarrow{AP} + \overrightarrow{PB} = \overrightarrow{AB}$,

or $\qquad x(\overrightarrow{AE}) + y(\overrightarrow{DB}) = \overrightarrow{AB}$.

Then

$$x(2\vec{u} + \vec{v}) + y(\vec{u} + 2\vec{v}) = 2\vec{u} + 2\vec{v} ,$$

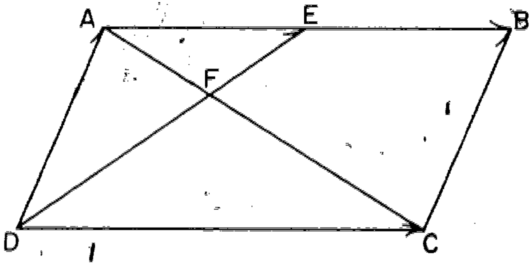
$$2x\vec{u} + x\vec{v} + y\vec{u} + 2y\vec{v} = 2\vec{u} + 2\vec{v} ,$$

$$(2x + y)\vec{u} + (x + 2y)\vec{v} = 2\vec{u} + 2\vec{v} .$$

Hence $2x + y = 2$, $x + 2y = 2$, and $x = y = \frac{2}{3}$.

The same result is obtained by using the third median with one of the first two. Therefore the point P is an element of all three medians and is a trisection point of each.

4. Let the parallelogram be labeled as shown and segments directed as indicated. E is the midpoint of $\overline{AB}$ and $\overline{DE}$ intersects $\overline{AC}$ at F .



$$\overrightarrow{AF} + \overrightarrow{FE} = \overrightarrow{AE} = \tfrac{1}{2}\overrightarrow{AB} \text{ and } \overrightarrow{DF} + \overrightarrow{FC} = \overrightarrow{DC}$$

but $\overrightarrow{FE} = x\overrightarrow{DE}$, $\overrightarrow{AF} = y\overrightarrow{AC}$, $\overrightarrow{DF} = (1 - x)\overrightarrow{DE}$, and $\overrightarrow{FC} = (1 - y)\overrightarrow{AC}$ and $\overrightarrow{DC} = \overrightarrow{AB}$.

Hence, substituting into the first two equations we obtain respectively,

$$y\overrightarrow{AC} + x\overrightarrow{DE} = \tfrac{1}{2}\overrightarrow{DC} \quad \text{or} \quad 2y\overrightarrow{AC} + 2x\overrightarrow{DE} = \overrightarrow{DC}$$

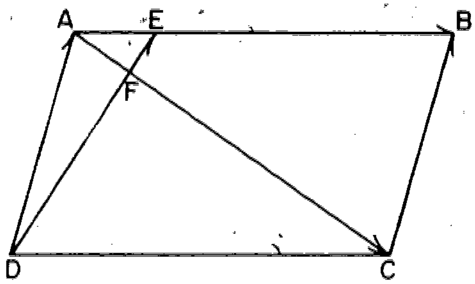
and $$(1 - y)\overrightarrow{AC} + (1 - x)\overrightarrow{DE} = \overrightarrow{DC} ,$$

from which we obtain

$$\begin{aligned} 2y &= 1 - y & \text{and} \quad 2x &= 1 - x \\ 3y &= 1 & 3x &= 1 \\ y &= \tfrac{1}{3} & x &= \tfrac{1}{3} . \end{aligned}$$

Thus $\overrightarrow{AF} = y\overrightarrow{AC} = \frac{1}{3}\overrightarrow{AC}$.

5. Let the parallelogram be labeled as shown and segments directed as indicated, with E a point on $\overline{AB}$ such that $AE = \frac{1}{m}AB$ and $\overline{DE}$ intersecting $\overline{AC}$ at F .



$$\overrightarrow{AF} + \overrightarrow{FE} = \overrightarrow{AE} = \tfrac{1}{m}\overrightarrow{AB} \text{ and } \overrightarrow{DF} + \overrightarrow{FC} = \overrightarrow{DC} ;$$

but $\overrightarrow{AF} = x\overrightarrow{AC}$, $\overrightarrow{FC} = (1 - x)\overrightarrow{AC}$, $\overrightarrow{FE} = y\overrightarrow{DE}$, and $\overrightarrow{DF} = (1 - y)\overrightarrow{DE}$.

Thus $x\overrightarrow{AC} + y\overrightarrow{DE} = \frac{1}{m}\overrightarrow{AB}$ or $mx\overrightarrow{AC} + my\overrightarrow{DE} = \overrightarrow{AB}$

$$(1 - x)\overrightarrow{AC} + (1 - y)\overrightarrow{DE} = \overrightarrow{DC} = \overrightarrow{AB} .$$

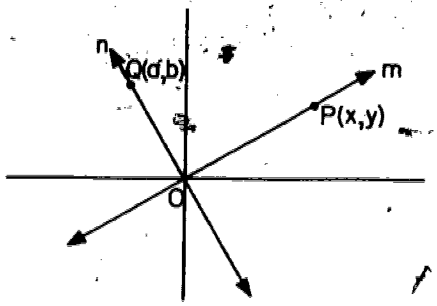
Therefore
$$\begin{aligned} mx &= 1 - x \\ mx + x &= 1 \\ (m + 1)x &= 1 \\ x &= \frac{1}{m + 1} . \end{aligned}$$

Hence $\overrightarrow{AF} = \frac{1}{\quad}\overrightarrow{AC}$

Problem Set 10-6

1. 0 , perpendicular.
2. -24 .
3. -25 .
4. -24 .
5. -13 .
6. 0 , perpendicular.
7. -16 .
8. -96 .
9. 9 .
10. 0 , perpendicular.
11. Since $\overrightarrow{PQ} = [4,-6]$, $\overrightarrow{RS} = [12,8]$, the scalar product $\overrightarrow{PQ} \cdot \overrightarrow{RS} = 0$.
12. Since $\overrightarrow{PQ} = [3,-12]$, $\overrightarrow{QR} = [-8,-2]$, $\overrightarrow{PR} = [-5,-14]$, then $\overrightarrow{PQ} \cdot \overrightarrow{QR} = 0$, $\overrightarrow{QR} \cdot \overrightarrow{PR} = 68$, $\overrightarrow{PQ} \cdot \overrightarrow{PR} = 153$. Since $\overrightarrow{PQ} \cdot \overrightarrow{QR} = 0$, $\triangle PQR$ is a right triangle.
13. $$\begin{aligned}(\vec{u} - \vec{v}) \cdot (\vec{w} - \vec{z}) &= (\vec{u} - \vec{v}) \cdot \vec{w} + (\vec{u} - \vec{v}) \cdot (- \vec{z}) \\ &= \vec{w} \cdot (\vec{u} - \vec{v}) - \vec{z} \cdot (\vec{u} - \vec{v}) \\ &= \vec{u} \cdot \vec{w} - \vec{v} \cdot \vec{w} - \vec{u} \cdot \vec{z} + \vec{v} \cdot \vec{z} \\ &= \vec{u} \cdot \vec{w} - \vec{u} \cdot \vec{z} - \vec{v} \cdot \vec{w} + \vec{v} \cdot \vec{z} .\end{aligned}$$

14. **Proof:** Let m be any line through the origin, O, and let n be the perpendicular to m through O. Let $P(x,y)$ be a point on m and $Q(a,b)$ be a point on n not the origin.



Since m is perpendicular to n, $\overrightarrow{OQ} \cdot \overrightarrow{OP} = 0$. In terms of components, since $\overrightarrow{OQ} = [a,b]$ and $\overrightarrow{OP} = [x,y]$, we have $[a,b] \cdot [x,y] = 0$ which is equivalent to $ax + by = 0$.

15. (1) Let $\vec{u} = [p_1,p_2]$ and $\vec{v} = [q_1,q_2]$; then

$\vec{u} \cdot \vec{v} = p_1q_1 + p_2q_2$	Definition of scalar product.
$= p_2q_2 + p_1q_1$	Commutative property of numbers.
$= \vec{v} \cdot \vec{u}$.	Definition of scalar product.

(2) Let $\vec{u} = [p_1,p_2]$, $\vec{v} = [q_1,q_2]$, $\vec{w} = [t_1,t_2]$; then

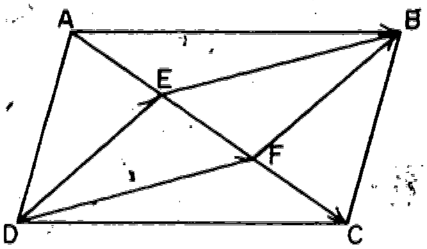
$\vec{u} \cdot (\vec{v} + \vec{w})$

$= \vec{u} \cdot ([q_1 + t_1 , q_2 + t_2])$	Definition of vector sum.
$= p_1(q_1 + t_1) + p_2(q_2 + t_2)$	Definition of scalar product.
$= (p_1q_1 + p_1t_1) + (p_2q_2+p_2t_2)$	Distributive property of numbers.
$= (p_1q_1 + p_2q_2) + (p_1t_1+p_2t_2)$	Associative and commutative properties of numbers.
$= \vec{u} \cdot \vec{v} + \vec{u} \cdot \vec{w}$.	Definition of scalar product.

Chapter 10

Review Problems

1. **Proof:** Let parallelogram ABCD have directed segments as shown and E and F be trisection points of $\overline{AC}$ as indicated.



Then

$$\overrightarrow{EB} = -\overrightarrow{AE} + \overrightarrow{AB}$$
$$\overrightarrow{DF} = \overrightarrow{DC} - \overrightarrow{FC} .$$

But

$$\overrightarrow{AE} = \tfrac{1}{3}\overrightarrow{AC} = \overrightarrow{FC} \text{ and } \overrightarrow{AB} = \overrightarrow{DC} .$$

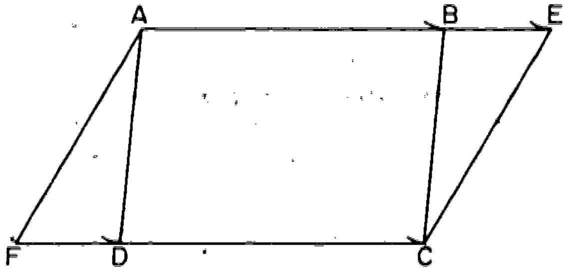
Hence

$$\overrightarrow{EB} = \overrightarrow{DC} - \tfrac{1}{3}\overrightarrow{AC} = \overrightarrow{DF} .$$

$\overrightarrow{EB} = \overrightarrow{DF}$ implies $EB = DF$ and $\overline{EB} \parallel \overline{DF}$.

Therefore DEBF is a parallelogram.

2. Let parallelogram ABCD be as shown and E and F points such that AB + BE = AE and CD + DF = CF and BE = FD .



$\overrightarrow{AB} + \overrightarrow{BE} = \overrightarrow{AE}$ and $\overrightarrow{FD} + \overrightarrow{DC} = \overrightarrow{FC}$.

But $\overrightarrow{AB} = \overrightarrow{DC}$ and $\overrightarrow{FD} = \overrightarrow{BE}$;

hence $\overrightarrow{AE} = \overrightarrow{AB} + \overrightarrow{BE} = \overrightarrow{FC}$

which implies that $AE = FC$ and $\overline{AE} \parallel \overline{FC}$.

Therefore AECF is a parallelogram.

3. $\overrightarrow{PQ} = [-6,-10]$, $\overrightarrow{QR} = [-3,-5]$.
Since $[-6,-10] = 2[-3,-5]$,
therefore $\overrightarrow{PQ} \parallel \overrightarrow{QR}$.
Hence $\overleftrightarrow{PQ} \parallel \overleftrightarrow{QR}$.
But Q lies on both lines.
Therefore, $\overleftrightarrow{PQ} = \overleftrightarrow{QR}$.

4. $\overrightarrow{PQ} = [3,8]$, $\overrightarrow{SR} = [3,8]$, implying $PQ = SR$ and $\overline{PQ} \parallel \overline{SR}$. Also P, Q, R, S are not collinear, because

$$\overrightarrow{QR} = [-7,2] \neq [3,8] = \overrightarrow{PQ} .$$

Therefore, these points are the vertices of a parallelogram.

5. (a) [1,2] . (c) [8,10] .
(b) [-3,-8] . (d) [-10,-6] .

6. (a) $\frac{1}{2}$. (d) $(\overrightarrow{B,A})$.
(b) 2 , 2 . (e) $\frac{1}{2}$.
(c) $\frac{1}{2}$.

7. $(\overrightarrow{A,B}) \doteq (\overrightarrow{D,C})$. $(\overrightarrow{A,C})$, $(\overrightarrow{C,A})$.
$(\overrightarrow{B,A}) \doteq (\overrightarrow{C,D})$. $(\overrightarrow{D,B})$, $(\overrightarrow{B,D})$.
$(\overrightarrow{A,D}) \doteq (\overrightarrow{B,C})$.
$(\overrightarrow{D,A}) \doteq (\overrightarrow{C,B})$.

8. (a) $(\overrightarrow{D,B}) \doteq (\overrightarrow{D,C}) + (\overrightarrow{D,A})$.
(b) $(\overrightarrow{D,B}) \doteq (\overrightarrow{D,C}) + (\overrightarrow{C,B})$.
(c) $(\overrightarrow{D,B}) \doteq (\overrightarrow{A,B}) - (\overrightarrow{B,C})$.
(d) $(\overrightarrow{D,B}) \doteq (\overrightarrow{A,B}) - (\overrightarrow{A,D})$.
(e) $(\overrightarrow{D,B}) \doteq -(\overrightarrow{B,A}) - (\overrightarrow{B,C})$.

9. (a) $(\overrightarrow{O,B}) \doteq (\overrightarrow{O,Q}) + (\overrightarrow{O,P})$.
(b) $(\overrightarrow{O,C}) \doteq (\overrightarrow{O,Q}) - (\overrightarrow{O,P})$.
(c) $(\overrightarrow{O,D}) \doteq (\overrightarrow{O,Q}) - (\overrightarrow{O,P})$.
(d) $(\overrightarrow{O,A}) \doteq -(\overrightarrow{O,Q}) + (\overrightarrow{O,P})$.
(e) $(\overrightarrow{D,B}) \doteq 2(\overrightarrow{O,Q}) + 2(\overrightarrow{O,P})$.
(f) $(\overrightarrow{A,C}) \doteq 2(\overrightarrow{O,Q}) - 2(\overrightarrow{O,P})$.
(g) $(\overrightarrow{C,A}) \doteq -2(\overrightarrow{O,Q}) + 2(\overrightarrow{O,P})$.
(h) $(\overrightarrow{B,D}) \doteq -2(\overrightarrow{O,Q}) - 2(\overrightarrow{O,P})$.

10. (a) $x = 3$, $y = 2$.
(b) $x = -1$, $y = 3$.
(c) $x = 0$, $y = 1$.
(d) $x = 0$, $y = 0$.
(e) $x = 4$, $y = -2$.

11. (a) $\sqrt{10}$.
(b) 5 .
(c) $\sqrt{5}$.
(d) $\sqrt{5}$.
(e) $\sqrt{5}$.

12. (a) [2,1] .
(b) [-2,-1] .
(c) [-2,-1] .
(d) [4,2] .
(e) [-4,-2] .

13. (a) 6 .
(b) 4 .
(c) 16 .
(d) 9 .
(e) 1 .

14. 4 .

15. 0 .

16. (a)

(b)

17. $5\sqrt{2}$ pounds.

18. (a) $500\sqrt{3}$ pounds.
(b) 500 pounds.

19. (a) Force in $\overline{AC}$ is $\frac{10000}{\sqrt{3}}$ or approximately 5770 .

(b) Force in $\overline{BC}$ is $\frac{5000}{\sqrt{3}}$ or approximately 2885 .

(c) Force in $\overline{CW}$ is 5000 pounds.

20. 20.4 miles/hr.

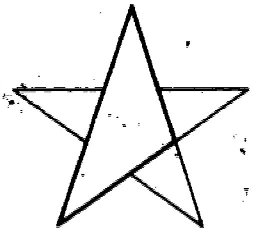
11° south of east.

Chapter 11

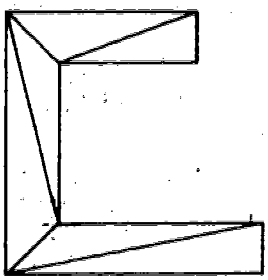
ANSWERS AND SOLUTIONS

Problem Set 11-2

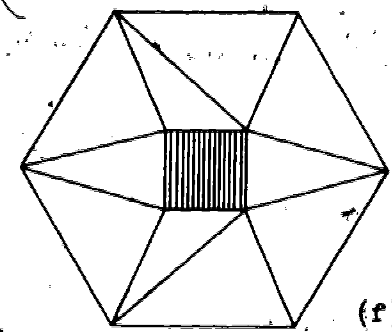
1. (a) 2 .
   (b) 2 .
   (c) 5 .



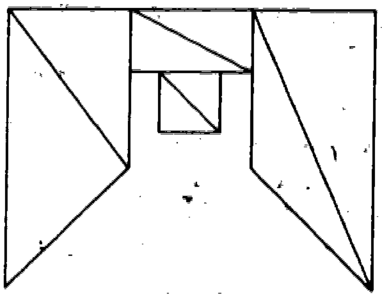
(d) 4 .



(e) 6 .



(f) 10 .



(g) 8 .

2. (a) $f - e + v = 7 - 12 + 7 = 2$.
   (b) $f - e + v = 7 - 17 + 12 = 2$.
   (c) The result of the computation in each case is 2 .
   (d) The number $f - e + v$ is not affected. Four edges, 3 faces, and 1 vertex are added. $3 - 4 + 1 = 0$.
   (e) There is no change in the result of the computation.

Problem Set 11-3

1\. (a) 1800 , 360 .
   (b) 3600 , 360 .

2\. (a) 5 .
   (b) 7 .
   (c) 17 .

3\. (a) 140 .
   (b) 1260 .

4\. The measure of an interior angle of a polygon equals:

(a) $\frac{(n - 2)180}{n}$; $\frac{(12 - 2)180}{12} = 150$.

(b) $180 - \frac{360}{n}$; $180 - \frac{360}{12} = 150$.

5\. (a) 6 . (c) 10 .
   (b) 36 . (d) 7 .

6\.

(a)	180	360	60	120
(b)	360	360	90	90
(c)	540	360	108	72
(d)	720	360	120	60
(e)	1080	360	135	45
(f)	1440	360	144	36

7\. (a) 162 .
   (b) 18 .
   (c) 3240 .
   (d) 720 . (There are two at each vertex.)

8\. 12 .

9\. (a) 150 .
   (b) No. The sum of the interior angles of a polygon depends upon the number of triangles into which the diagonals from any one vertex can divide the polygon and is not affected by the comparative size of the interior angles. The average of the measures of the 11 angles is 150 . But this does not imply that each of the measures is 150 .

10. No. The measure of each interior angle of a regular polygon equals $\frac{(n-2)180}{n}$ where n represents the number of sides and must therefore be integral. $\frac{(n-2)180}{n} = 153$ gives a non-integral value to n .

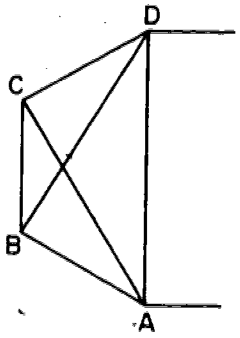
11. 36 .

12. $m\angle c + m\angle d + m\angle e = 330$,
$m\angle c = 4k$, $m\angle d = 3k$, $m\angle e = 4k$,

$$11\,k = 330$$
$$k = 30$$
$$m\angle c = 120 \;.$$

Then $\angle b$ and $\angle c$ are supplementary and, since they are consecutive interior angles with respect to $\overleftrightarrow{CD}$, $\overleftrightarrow{BA}$ and transversal $\overleftrightarrow{BC}$, it follows that $\overleftrightarrow{AB} \parallel \overleftrightarrow{CD}$.

13.



For all values of $n > 4$, ABCD is a quadrilateral. $\triangle ABC \cong \triangle DCB$ by S.A.S. Then $\overline{BD} \cong \overline{CA}$. Then $\triangle ABD \cong \triangle DCA$ by S.S.S. It follows that $m\angle BAD = m\angle CDA$.

Also since $m\angle CBA = m\angle DCB$, and $m\angle BAD + m\angle CBA + m\angle CDA + m\angle DCB = 360$, it follows that $2m\angle BAD + 2m\angle CBA = 360$ or $m\angle BAD + m\angle CBA = 180$. Then $\angle BAD$ and $\angle CBA$ are supplementary, and since they are the consecutive interior angles of transversal $\overleftrightarrow{BA}$ and lines $\overleftrightarrow{BC}$ and $\overleftrightarrow{AD}$, it follows that $\overline{AD} \parallel \overline{BC}$.

14. $m\angle CBX = 18$.
$m\angle DCX = 36$.
$m\angle XBD = 18$.

15. (a) 4 .
(b) 6 .

(c) Yes. The measure of each interior angle would have to be a factor of 360 , or $\frac{x(n-2)180}{n} = 360$ must be such that x is an integer and n is an integer ≥ 3 .
$x = \frac{2n}{n-2} = 2 + \frac{4}{n-2}$. $\frac{4}{n-2}$ is an integer for n only if $n = 3$, 4 , or 6 . Three hexagonal tiles would be needed.

(d) $2x + y = 360$

$2 \cdot 108 + 144 = 360$; 2 pentagons and one decagon

$2 \cdot 150 + 60 = 360$; 2 dodecagons and an equilateral triangle.

No polygon with more than 12 sides could be used since $y \geq 60$ and hence $x \leq 150$.

(e) Some of the possible combinations using three regular polygons each with a different number of sides are: 4,6,12 ; 4,5,20 ; 3,8,24 ; 3,10,15 ; 3,7,42 ; 3,9,18 .
[The numbers represent the number of sides.]

16. (a) increases.
(b) remains the same.
(c) increases.
(d) decreases.

Problem Set 11-4

1. (a) Areas are 6, 12, 24, 48 . Ratio is 1 to 2 .
(b) Areas are 6, 18, 54 . Ratio is 1 to 3 .
(c) Altitudes are 20, 10, 5, $\frac{5}{2}$, $\frac{5}{4}$. Ratio is 2 to 1 .
(d) Areas are 2, 18, 162, 1458 . 1 to 3 ; 1 to 3 ; 1 to 9 ; similar.
(e) 1 to 9 ; 4 to 9 .

2. (a) $\frac{2}{3}$ (d) $\frac{2}{1}$

(b) $\frac{1}{2}$ (e) $\frac{3}{2}$

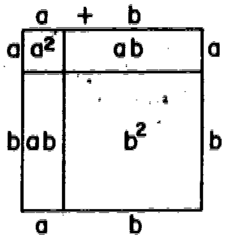
(c) $\frac{1}{2}$ (f) $\frac{1}{4}$

3. (a) 1 to 3 (c) 3 to 1

(b) 1 to 8 (d) 4 to 5

4. 68 and 85

5.



$$A = a^2 + b^2 + 2ab = (a + b)^2$$

6. (a) R_2 , R_3 , R_6

(b) R_1 , R_4 , R_5

(c) (d) (e) A rectangle is separated by its diagonal into two congruent triangles.

(f) Combine steps (c), (d), (e), Postulate 27, and the addition property of equality.

Problem Set 11-5

The following classification of problems should help teachers in making daily assignments.

Theorem 11-3-------------------Problems 1 - 6.
Theorem 11-4-------------------Problems 7 - 11.
Theorem 11-5-------------------Problems 12 - 16.
Theorem 11-6-------------------Problems 17 - 20.
Theorem 11-7-------------------Problems 21 - 23.
Miscellaneous------------------Problems 24 - 34.

1. 30

2. 72

3. 36

4. $18\sqrt{3}$

5. (a) $\frac{1}{2}h\sqrt{2}$ (b) $\frac{h^2}{4}$

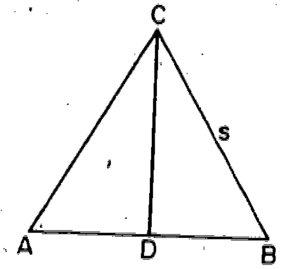
6. (a) $\frac{1}{2}h$ (b) $\frac{1}{2}h\sqrt{3}$ (c) $\frac{h^2\sqrt{3}}{8}$

7. (a) 60 (b) 24 (c) 3 (d) 4

8. (a) 3 to 4 (c) 1 to 2
   (b) 1 to 2 (d) 1 to 1

9. (a) 12 (c) $12\sqrt{2}$
   (b) $12\sqrt{3}$ (d) $16\sqrt{3}$

10.



By hypothesis we have an equilateral triangle ABC with the measure of the side = s and area = A .

We are required to prove $A = \frac{s^2\sqrt{3}}{4}$.

Let D be the foot of the altitude $\overline{CD}$ upon $\overline{AB}$. Then

(a) DB = AD or $DB = \frac{1}{2}s$	(a) The altitude of an equilateral triangle bisects the base.
*(b) $a^2 = s^2 - (\frac{s}{2})^2$	(b) Pythagorean Theorem.
(c) $a = \frac{s}{2}\sqrt{3}$	(c) From Step (b) with the properties of equality.
(d) $A = \frac{1}{2}as$	(d) The area of a triangle is half the product of any base and the altitude upon that base.
(e) $A = \frac{1}{2}(\frac{s}{2}\sqrt{3})s$ or $A = \frac{s^2\sqrt{3}}{4}$	(e) From Steps (c) and (d) using the substitution property of equality.

*Pupils should be able to omit this step and obtain CD from Theorem 7-10.

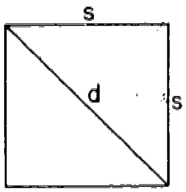
11. 240

12. 240

13. 216

14. $d = 40$, $d' = 80$

15.



In a square with length of side $= s$, length of diagonal $= d$ and area $= A$, we are required to prove that

$$A = \frac{d^2}{2} .$$

A square is a rhombus and also a rectangle. As a rhombus, $A = \frac{1}{2}$ the product of its diagonals. As a rectangle, its diagonals are equal. Hence $A = \frac{1}{2}d^2$.

16. 32

17. 84

18. 56.2

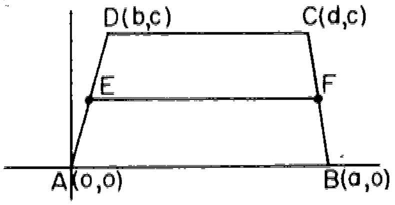
19. (a) 70 (c) $70\sqrt{3}$
(b) $70\sqrt{2}$ (d) 140

20. 5

21. (a) 21 (d) 36
(b) 102 (e) 18
(c) 12

22. $\frac{16\sqrt{3}}{3}$

23. Let ABCD be the trapezoid with $\overline{AB} \parallel \overline{CD}$. We use the coordinate system in which A is $(0,0)$, B is $(a,0)$, D is (b,c) , and C is (d,c) where $a > 0$, $c > 0$, and $d > b$. The midpoint of $\overline{DA}$ is $E = (\frac{b}{2}, \frac{c}{2})$. The midpoint of $\overline{CB}$ is $F = (\frac{a+d}{2}, \frac{c}{2})$. Then $EF = \frac{a+d}{2} - \frac{b}{2} = \frac{a+d-b}{2}$,



$AB + DC = a + (d - b) = 2EF$, and $EF = \frac{AB + DC}{2}$. But $m = EF$ and $h = c$ are the lengths of the median and altitude, respectively. Using Theorem 7-1 we see that the area is given by $h \cdot \frac{AB + DC}{2} = hm$.

24. 12 feet.

25. Area of $\Delta ADC = 49$, Area of $\Delta ABD = 35$.

26. Total area $= \frac{1}{2} \cdot 7(26) + \frac{1}{2} \cdot 5(28) = 161$.

Area PQBA = 63 ; Area QRCB = 40 .
Area of ABCD = 161 - (40 + 63) = 58 .

27. Measures of the sides are $\sqrt{2^2 + 4^2}$, $\sqrt{8^2 + 4^2}$, $\sqrt{6^2 + 8^2}$ or $\sqrt{20}$, $\sqrt{80}$, $\sqrt{100}$. By the converse of the Pythagorean Theorem, these sides form a right triangle. The area is $\frac{1}{2}\sqrt{20}\sqrt{80} = 20$.

28. (a) D = (-2,-6)

(b) $A = \frac{1}{2}\sqrt{128}\sqrt{32} = 32$

29. Altitude = 6 ; Area = 3(27) = 81 .

30. Area $= \frac{1}{2}(4 \cdot 9) + \frac{1}{2}(5 \cdot 9) = \frac{1}{2}(9 \cdot 9) = 40\frac{1}{2}$.

31. Area of ΔEFC = Area of ABCD - (Area of ΔAEF + Area of ΔEBC + Area of ΔFDC)

$= 7 \cdot 5 - \frac{1}{2}(3 \cdot 2 + 5 \cdot 5 + 2 \cdot 7) = 12\frac{1}{2}$.

32. ABCD is a parallelogram and ABDE is a parallelogram since in each one pair of opposite sides are both congruent and parallel. ABCD is a rhombus and ABDE is a rhombus since each is a parallelogram with one pair of adjacent sides congruent. AD = BC = EA = BD by hypothesis or because they are opposite sides of a parallelogram. Then AD = BD = AB and ΔABD is equilateral. Similarly ΔEAD and ΔDBC are also equilateral. Moreover the three triangles are congruent. Therefore the area of rhombus AEDB = area of rhombus ADCB . Area of ABDE $= \frac{1}{2}EB \cdot AD$ and Area of ABCD $= \frac{1}{2}AC \cdot BD$. Theref[illegible] $\frac{1}{2}AC \cdot BD = \frac{1}{2}EB \cdot AD$ or $AC \cdot BD =$ [illegible] $\cdot AD$.

33. $\frac{1}{2}AB \cdot AC$ = Area of $\Delta ABC = \frac{1}{2}AD \cdot BC$.

Therefore $AB \cdot AC = BC \cdot AD$.

34. In terms of the diagram we must prove that

Area of ABCD = $\frac{1}{2}DB \cdot AC$.

Area of ABCD = Area of $\triangle ADB$ + Area of $\triangle CDB$.

Since $\overline{AC} \perp \overline{DB}$, this becomes

$$\tfrac{1}{2}AP \cdot DB + \tfrac{1}{2}PC \cdot DB \text{ or } \tfrac{1}{2}DB(AP + PC) .$$

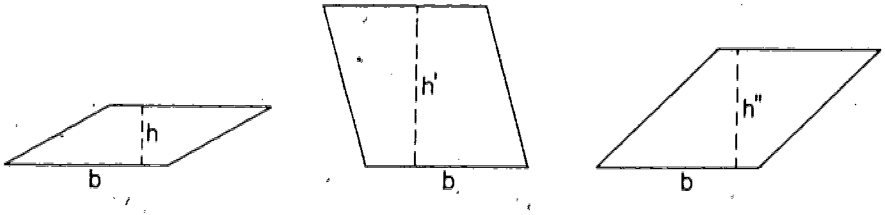
But $AP + PC = AC$.

Therefore Area of ABCD = $\frac{1}{2}DB \cdot AC$.

Problem Set 11-6

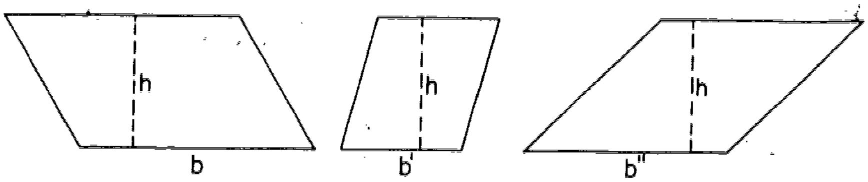
1. Proof: For definiteness we prove the theorem for a set of three parallelograms.

   (a) By hypothesis, all the bases have the same measure, say b . Let the areas of the parallelograms be A, A', A'' , and let the corresponding altitudes be h, h', h'' .



   Now $A = bh$, $A' = bh'$, $A'' = bh''$; hence the numbers A, A', A'' are proportional to the numbers h, h', h'' with the non-zero number b as the proportionality constant.

   (b) By hypothesis, all the altitudes are the same number, say h . Let the areas of the triangles be A, A', A'' , and let the corresponding bases be b, b', b'' .



Now $A = hb$, $A' = hb'$, $A'' = hb''$; hence the numbers A, A', A'' are proportional to b, b', b'' with the non-zero number h as the proportionality constant.

(c) By hypothesis, all the areas are the same number, say A . Let the bases of the triangles be b, b', b'' , and let the corresponding altitudes be h, h', h'' .

Now $A = bh$, $A = b'h'$, $A = b''h''$ or $bh = b'h' = b''h'' = A$. Thus b, b', b'' are inversely proportional to h, h', h'' .

2. All three have the same area since all have base $\overleftrightarrow{AB}$ and altitude $\overline{AF}$.

3. Each triangular-region has the distance from E to $\overleftrightarrow{AD}$ as altitude. Since the bases of R_1, R_2, and R_3 are proportional to 1, 2, 3 , the areas of R_1, R_2, R_3 are proportional to 1, 2, 3 by Theorem 11-8.

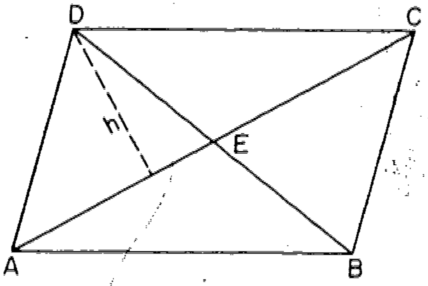
4. $\frac{\text{Area of ADFC}}{\text{Area of ADEB}} = \frac{h'}{h}$ by Theorem 11-9a since ADEB and ADFC are parallelograms.

Now $\frac{h'}{h} = \frac{AC}{AB}$ since corresponding sides of similar triangles are proportional.

But $\frac{AC}{AB} = \frac{10x}{4x} = \frac{5}{2}$.

Then the areas of ADEB and ADFC are in the ratio of 2 to 5 .

5.



By hypothesis we have parallelogram ABCD divided into 4 triangular-regions, AEB , BEC , CED and DEA by diagonals $\overline{AC}$ and $\overline{BD}$.

We are required to prove that all four triangular regions have equal area.

Let h be the length of the perpendicular from D to $\overline{AC}$. Then h is the altitude of $\triangle AED$ upon base $\overline{AE}$ and h is the altitude of $\triangle CED$ upon $\overline{EC}$. AE = EC since the diagonals of a parallelogram bisect each other. Then

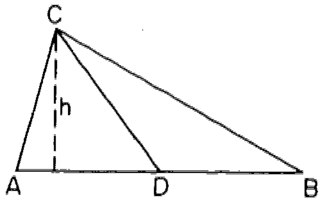
Area of $\triangle AED = \frac{1}{2}h \cdot AE = \frac{1}{2}h \cdot EC$ = Area of $\triangle CED$.

But $\triangle AEB \cong \triangle CED$ and $\triangle CEB \cong \triangle AED$ (S.S.S.).

Since congruent triangles have equal areas,

Area of $\triangle AED$ = Area of $\triangle CED$ = Area of $\triangle AEB$ = Area of $\triangle CEB$.

6.



$\overline{CD}$ is the median of $\triangle ABC$. $\overline{CD}$ forms two triangular-regions ADC and DBC .

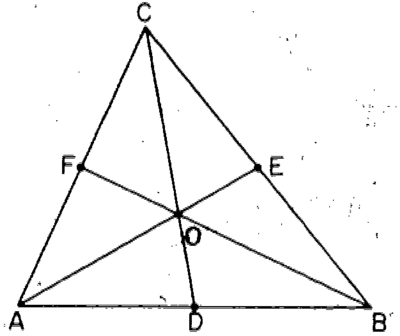
We are required to prove that regions ADC and DBC have equal areas.

Both regions have the same altitude (the length h of the perpendicular from C to $\overleftrightarrow{AB}$.)

AD = DB from the definition of median.

$\frac{1}{2}h \cdot AD = \frac{1}{2}h \cdot DB$, and thus

Area of region ADC = Area of region DBC .

7. 

By Problem 6,

Area of $\triangle ACD$ = Area of $\triangle BCD$

and

Area of $\triangle AOD$ = Area of $\triangle BOD$.

Then

Area of $\triangle COA$ = Area of $\triangle BOC$.

Similarly it can be proved using median $\overline{BF}$ that

Area of $\triangle BOC$ = Area of $\triangle AOB$.

Hence Area of $\triangle AOB$ = Area of $\triangle BOC$ = Area of $\triangle COA$.

8. By Problem 7, the medians separate the triangle into regions of equal area; so if the cardboard triangle has uniform thickness, we should expect the regions to have equal weight.

9. Area of $\triangle ABO$ = Area of $\triangle BOC$ = Area of $\triangle COA$

$$= \frac{216}{3} = 72 .$$

Area of $\triangle ODB$ = Area of $\triangle BOE$ = Area of $\triangle AOF$

$$= \frac{72}{2} = 36 .$$

*10. (a) $5\frac{5}{6}$.

(b) 12 .

(c) $10\frac{1}{2}$.

11. (a) 15 .

(b) 90 .

(c) $\frac{405}{2}$.

(d) $\frac{225}{2}$.

12. (a) 3 to 1

(b) 1 to 4

(c) 3 to 4

(d) 3 to 2

(e) 11 to 10

13. No. Let b, h, $b + 5$, $h - 5$, denote a base and corresponding altitude of the first triangle and a base and corresponding altitude of the second triangle. Then their areas are equal if and only if

$$bh = (b + 5)(h - 5) = bh + 5h - 5b - 25 ,$$

that is, if and only if

$$h = b + 5 .$$

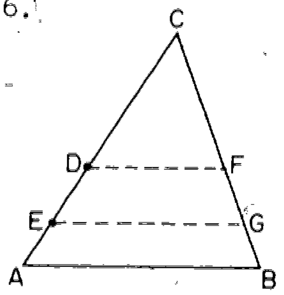
14. $\frac{16}{15}$

Problem Set 11-7

1. $\frac{16}{25}$
2. $\frac{4}{5}$, $\frac{4}{5}$, $\frac{4}{5}$
3. 4 to 1
4. 30
5. $\frac{s}{\frac{s}{2}\sqrt{3}} = \frac{2}{\sqrt{3}}$

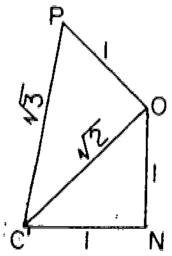
   $\frac{A_1}{A_2} = \frac{4}{3}$

6.



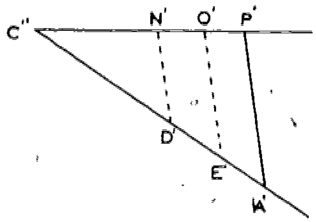
If on $\overline{CA}$ points D and E are selected such that $\overline{CD}$, $\overline{CE}$, $\overline{CA}$ are proportional to 1 , $\sqrt{2}$, $\sqrt{3}$ and if through D and E , $\overline{DF}$ and $\overline{EG}$ are parallel to $\overline{AB}$, then the required polygonal-regions will be determined.

(a)



A method for locating D and E is begun in (a) where $\overline{C'N}$ is any convenient unit, $\overline{NO} \perp \overline{NC'}$, and $\overline{OP} \perp \overline{OC'}$, and completed in (b).

(b) On any angle with vertex C'', take A' on one side and N', O', P' on the other side so that C''A' = CA, C''N' = C'N, C''O' = C'O, C''P' = C'P. Draw parallel lines as suggested in the figure. Use lengths C''D' and C''E' to locate D and E; draw $\overline{DF}$ and $\overline{EG}$ parallel to $\overline{AB}$.



Area of CDF, Area of CEG, Area of CAB are proportional to $(CD)^2$, $(CE)^2$, $(CA)^2$ or to 1, 2, 3. It follows that polygonal areas CDF, EGFD and ABGE are each $\frac{1}{3}$ of the area of ABC and are therefore of equal area.

7. Let K be the proportionality factor. Then by definition of similar polygons,

$AB = K \cdot A'B'$; $BC = K \cdot B'C'$; $CD = K \cdot C'D'$;
$DE = K \cdot D'E'$; $ED = K \cdot E'A'$; and the corresponding angles have equal measures. It can then be proved that the corresponding triangles formed by the diagonals and the sides of the given polygons are similar. Thus by the S.A.S. Similarity Theorem

$$\triangle ABC \sim \triangle A'B'C' \text{ and } \triangle ADE \sim \triangle A'D'E' .$$

m $\angle ACD$ can be shown equal to m $\angle A'C'D'$ and

$$(AC,DC) \underset{p}{=} (A'C',D'C') . \text{ Then}$$

$\triangle ACD \sim \triangle A'C'D'$ by the A.S.A. Similarity Theorem. It follows that $R_1 = K^2R'_1$; $R_2 = K^2R'_2$ and $R_3 = K^2R$ since the areas of two similar triangles are proportional to the squares of any two corresponding sides.

$$\text{Then } R_1 + R_2 + R_3 = K^2(R'_1 + R'_2 + R'_3)$$

or the area of ABCDE = K^2(the area of A'B'C'D'E') .

$$\frac{\text{Area of ABCDE}}{\text{Area of A'B'C'D'E'}} = K^2 = \left(\frac{AB}{A'B'}\right)^2 = \frac{S^2}{(S')^2} .$$

8. The proof in Problem 7 would be changed to show the area of ABCDE...N = K^2 (the area of A'B'C'D'E'...N'). This requires the insertion of an indefinite, finite number of triangles of the form of $\triangle ACD$ between the triangles containing the side of the polygon having A as a left endpoint and the one having A as a right endpoint. The corresponding pairs of such triangles can be proved similar by the S.A.S. Similarity Theorem. The procedure is similar to that in 7 .

9. 12

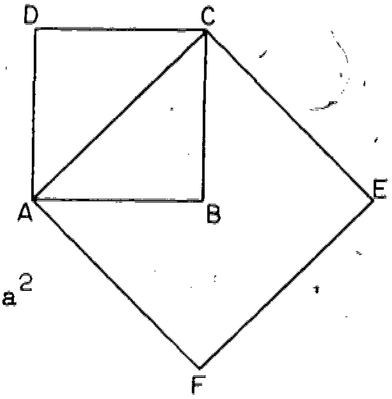
10. $\frac{49}{100}$, $\frac{7}{10}$

11. $\frac{1}{\sqrt{2}}$

12. Let square ABCD have sides of length a . Then square ACEF has sides of length $a\sqrt{2}$. Then

$$\text{Area } ABCD = a^2$$

$$\text{Area } ACEF = (a\sqrt{2})^2 = 2a^2$$



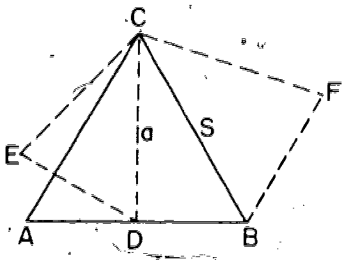
13. $\frac{3}{2}$, $\frac{3}{2}$, $\frac{9}{4}$

14. $\frac{9}{2}$

15. $\frac{\text{Area } \triangle ABC}{\text{Area } \triangle DEC} = \frac{9}{1}$

16. $\frac{S^2}{10^2} = \frac{2A}{A} = 2$,

$S = 10\sqrt{2}$

17. 

By hypothesis $\triangle ABC$ is equilateral. $\overline{CD}$ is the altitude of ABC ; length of $\overline{CD} = a$. $\triangle CBF \sim \triangle CDE$ with side $\overline{CB}$ corresponding to $\overline{CD}$. We are required to show that $\frac{\text{Area of } \triangle CBF}{\text{Area of } \triangle CDE} = \frac{4}{3}$.

Let S be the length of a side of $\triangle ABC$.

$\frac{\text{Area of} \triangle CBF}{\text{Area of} \triangle CDE} = \frac{S^2}{a^2}$ since the areas of two similar triangles have the same ratio as the squares of any two corresponding sides.

But the altitude of an equilateral triangle is $\frac{\sqrt{3}}{2}$ times the length of a side of the triangle.

$$\frac{\text{Area of } \triangle CBF}{\text{Area of } \triangle CDE} = \frac{S^2}{a^2} = \frac{S^2}{\left(\frac{\sqrt{3}}{2}S\right)^2} = \frac{4}{3} .$$

18. Let ℓ = length of the wire.

Then $\frac{\ell}{4}$ = length of side of the square and

$\frac{\ell}{3}$ = length of side of the triangle.

Area of the square = $\frac{\ell^2}{16}$;

area of the triangle = $\frac{\sqrt{3}}{4} \frac{\ell^2}{9}$

$\frac{\text{Area of the square}}{\text{Area of the triangle}} = \frac{9}{4\sqrt{3}}$

Problem Set 11-8

1. Yes. It is the altitude of an isosceles triangle of which the side is the base.

2. 20 .

3. $3\sqrt{2}$, 24 , 3 , 36 .

4. $\frac{s\sqrt{3}}{3}$, $\frac{s\sqrt{3}}{6}$.

5. $\sqrt{3}$, 2 , $6\sqrt{3}$.

6. $\frac{r\sqrt{2}}{2}$, $r\sqrt{2}$, $4r\sqrt{2}$, $2r^2$.

7. (a) $\frac{2}{3}$, $\frac{2}{3}$, $\frac{2}{3}$, $\frac{4}{9}$.

   (b) $\frac{1}{2}bh = \frac{1}{2}(16\sqrt{3})(24) = 192\sqrt{3}$.

   $\frac{s^2\sqrt{3}}{4} = \frac{(16\sqrt{3})^2\sqrt{3}}{4} = 192\sqrt{3}$.

8. (a) $288\sqrt{3}$.

   (b) $48\sqrt{3}$, $A = \frac{1}{2}ap = \frac{1}{2} \cdot 12 \cdot 48\sqrt{3} = 288\sqrt{3}$.

   $A = \frac{6\,(8\sqrt{3})^2\sqrt{3}}{4} = \frac{6 \cdot 64 \cdot 3 \cdot \sqrt{3}}{4} = 288\sqrt{3}$

Problem Set 11-9

1.

Regular Polyhedron	Boundary of Face	Number of Faces	Number of Edges	Number of Vertices	Number of Faces (or Edges) at a Vertex
Tetrahedron	Regular triangle	4	6	4	3
Octahedron	Regular triangle	8	12	6	4
Icosahedron	Regular triangle	20	30	12	5
Hexahedron	Square	6	12	8	3
Dodecahedron	Regular pentagon	12	30	20	3

2. $f - e + v = 2$.

(a) $4 - 6 + 4 = 2$

(b) $8 - 12 + 6 = 2$

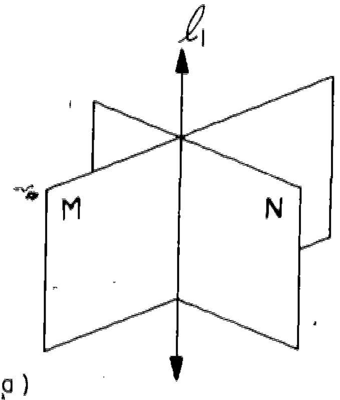
(c) $20 - 30 + 12 = 2$

(d) $6 - 12 + 8 = 2$

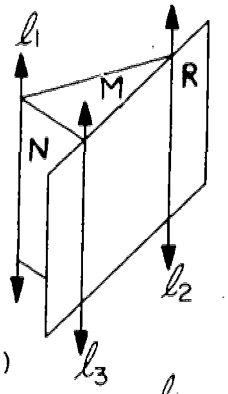
(e) $12 - 30 + 20 = 2$

Yes, the property does not appear to depend upon the length of edges or the measure of angles.

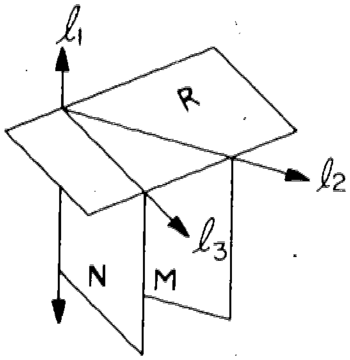
3. A polyhedron consists of a finite number of polygonal regions joined together in a manner specified in the definition. Two planes M and N intersecting in line ℓ_1 form regions which are halfplanes but form no polygonal regions, as in (a) below. If a third plane R intersects plane M and plane N in two distinct lines ℓ_2 and ℓ_3 respectively as in (b) and (c) below, again no polygonal regions are formed. It takes a fourth plane to form a polygonal region. These regions, now formed, satisfy the requirements specified in the definition of a polyhedron.



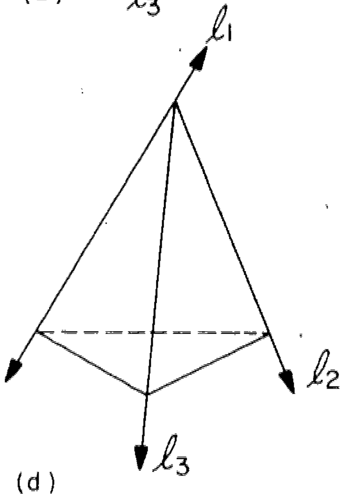
(a)



(b)



(c)



(d)

Problem Set 11-10

1. Let m represent the measure of the third face angle in each of the problems.

   (a) $25 < m < 175$
   (b) $25 < m < 135$
   (c) $75 < m < 165$
   (d) $30 < m < 40$
   (e) $85 < m < 175$
   (f) $15 < m < 175$

2. (a) +
   (b) 0
   (c) +
   (d) 0
   (e) 0
   (f) 0
   (g) 0
   (h) +

Problem Set 11-11a

1. By definition of a prism, the lateral faces of a prism are parallelograms. If $e_1, e_2, e_3, e_4, \ldots e_n$ represent the lateral edges, any two consecutive edges, such as e_1 and e_2, or e_2 and e_3, are parallel since they are opposite sides of a parallelogram. It follows that $e_1 \parallel e_3$ since if two lines are each parallel to a third line, they are parallel to each other. By continuing this reasoning, it can be established that all the edges are parallel to one another.

2. Let $e_1, e_2, e_3, e_4, \ldots e_n$ be the edges of a right prism with bases B_1 and B_2.

   By Problem 1, $e_1 \parallel e_2 \parallel e_3 \parallel e_4 \parallel \ldots \parallel e_n$.

   By definition of a right prism, one edge, say e_1, is perpendicular to one base, say B_1. Then, by Theorem 9-11, B_1 is perpendicular to $e_2, e_3, e_4, \ldots e_n$. Since $B_1 \parallel B_2$ by definition of a prism, $e_1 \perp B_2$ by Theorem 9-10. Then, again using Theorem 9-11, $B_2 \perp e_2, e_3, e_4, \ldots e_n$. Thus every lateral edge is perpendicular to each base.

Problem Set 11-11b

1. (1) Opposite sides of a parallelogram are congruent.
   (2) Definition of a right section
   (3) The area of any parallelogram equals the product of any base and the altitude upon that base.
   (4) Distributive property of numbers
   (5) Substitution property of equality

2. In a right prism, the base is a right section. It follows from Theorem 11-20 that the lateral area of a right prism is the product of the lateral edge and the perimeter of the base.

3. 210

4. Total area = lateral area + 2 · (area of the base)

$$= 240 + 2 \cdot 16\sqrt{3}$$

$$= 240 + 32\sqrt{3} \ .$$

5. $3 \ , \ 6 \ , \ 3\sqrt{3} \ ; \ 30 \ , \ 60 \ , \ 90 \ ; \ \frac{9}{2}\sqrt{3}$.

6. 5.2

7. The measure of each side of the base is 16 . The lateral area is $96 \cdot 20$, or 1920 .

8. Total area = lateral area + 2 · area of base

$$= 2 \cdot 20 \cdot 12 + 2 \cdot 10 \cdot 12 + 2 \cdot 12^2$$

(altitude of faces with angle of measure of 30 is 10 .)

$$= 2 \cdot 12(20 + 10 + 12)$$

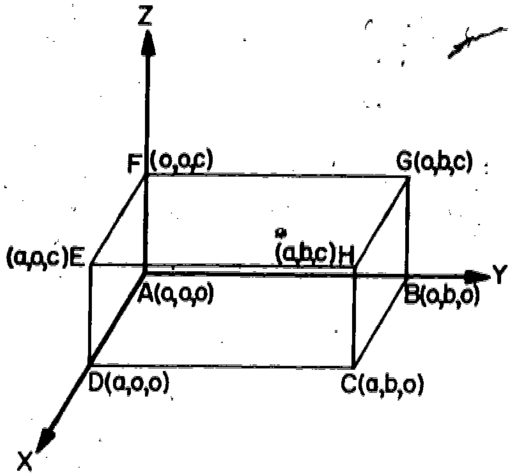
$$= 24(42)$$

$$= 1008 \ .$$

240

9. The diagonals of a rectangular parallelepiped have equal length.

Proof: There is a coordinate system which assigns coordinates to the vertices of a rectangular parallelepiped as shown in the diagram.



Then in terms of the data on the diagram we are required to prove that EB = DG = FC = AH .
By the distance formula

$$EB = \sqrt{a^2 + b^2 + c^2} \ ; \ DG = \sqrt{a^2 + b^2 + c^2}$$

$$AH = \sqrt{a^2 + b^2 + c^2} \ ; \ FC = \sqrt{a^2 + b^2 + c^2} \ .$$

Therefore EB = DG = AH = FC .

10. Using the diagram of Problem 9, we are required to prove $\overline{EB}$, $\overline{DG}$, $\overline{FC}$ & $\overline{AH}$ bisect each other or that the same point is the midpoint of each diagonal.

Midpoint of $\overline{EB} = (\frac{a}{2}, \frac{b}{2}, \frac{c}{2})$;

midpoint of $\overline{DG} = (\frac{a}{2}, \frac{b}{2}, \frac{c}{2})$;

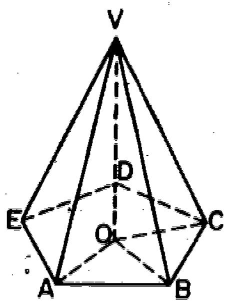
midpoint of $\overline{FC} = (\frac{a}{2}, \frac{b}{2}, \frac{c}{2})$;

midpoint of $\overline{AH} = (\frac{a}{2}, \frac{b}{2}, \frac{c}{2})$.

Since all the diagonals have the same midpoint, they bisect each other.

Problem Set 11-12

1\.



By hypothesis we have a regular pyramid V-ABCDE...N . We are required to prove:

$\triangle AVB$, $\triangle BVC$, $\triangle CVD$, etc., are isosceles and are congruent to each other.

(1) Let O be the foot of the $\perp$ from V to the base. Then O is the center of ABCDE...N .	(1) Definition of regular pyramid
(2) $AB = BC = DC = DE = \ldots = NA$.	(2) Definition of regular pyramid
(3) $OA = OB = OC = OD = OE$, etc.	(3) From meaning of center
(4) $\triangle VOA \cong \triangle VOB \cong \triangle VOC$, etc.	(4) S.A.S.
(5) $VA = VB = VC = VD$, etc.	(5) Definition of congruence
(6) $\triangle VAB$, $\triangle VBC$ $\triangle VCD$, etc. are isosceles.	(6) Each has two congruent sides, Step (5)
(7) $\triangle VAB \cong \triangle VBC \cong \triangle VCD$, etc.	(7) S.S.S. Steps (2) and (5)

2\. The faces of a regular pyramid are congruent triangles as proved in Problem 1. The area of each triangle equals $\frac{1}{2}$ the product of a base and the altitude upon that base. If s is the length of a side of the regular polygon and n the number of sides, then there are in the lateral area n triangles each of area $\frac{1}{2}as$. The lateral area $= \frac{1}{2}as \cdot n = \frac{1}{2}ap$ since the perimeter p of a regular polygon of n sides = ns .

3. (a) $A = \frac{1}{2}ap = \frac{1}{2} \cdot \frac{11}{2} \cdot 18 = 49\frac{1}{2}$ sq. in.

(b) $9\frac{3}{8}$ or $\frac{25}{24}$, depending upon the units used.

4. 80

5. 8

6. 24

7. Trapezoid

8. $A = \frac{1}{2}a(p + p')$

9. 112 ; 212

10. 5

11. (a) Two parallel planes intersect a third plane in two lines which are parallel to each other, and a line cutting two sides of a triangle and parallel to the other side divides the triangle into two similar triangles.

(b) If $\overline{VO}$ is the altitude of $V - ABCD$, then by definition it is perpendicular to the plane of the base. It must, then, be perpendicular to any line in that plane which contains its foot.

(c) $\overleftrightarrow{VK}$ perpendicular plane of EFGH since a line perpendicular to one of two parallel planes is perpendicular to the other. Then use the A.A. Similarity Theorem.

(d) $\frac{2}{3}$

(e) 24

(f) 6

(g) 324

(h) 180

(i) $\frac{A_P}{A_F} = \frac{324}{180} = \frac{9}{5}$; $\frac{\text{Area of } \triangle VAB}{\text{Area of } \triangle VEF} = (\frac{AB}{EF})^2 = \frac{9}{4}$.

Areas of similar triangles are in the same ratio as the squares of any two corresponding sides.

$\frac{\text{Area of } \triangle VAB - \text{Area of } \triangle VEF}{\text{Area of } \triangle VAB} = \frac{9 - 4}{9}$;

$\frac{\text{Area of one section of frustum}}{\text{Area of corresponding lateral face}} = \frac{5}{9}$.

12. **Proof:**

(a)	1. Plane A'B'C' $\parallel$ plane ABC .	1. Definition of a cross-section.
	2. $\overline{B'C'} \parallel \overline{BC}$, $\overline{A'B'} \parallel \overline{AB}$, $\overline{A'C'} \parallel \overline{AC}$.	2. If two parallel planes intersect a third plane, the intersections are parallel.
	3. $\triangle VB'C' \sim \triangle VBC$ $\triangle VA'B' \sim \triangle VAB$ $\triangle VA'C' \sim \triangle VAC$.	3. A line parallel to one side of a triangle and cutting the other two sides divides the triangle into two similar triangles.
	4. $\frac{A'B'}{AB} = \frac{VB'}{VB} = \frac{B'C'}{BC}$ $= \frac{VC'}{VC} = \frac{A'C'}{AC}$.	4. The measures of corresponding sides of similar triangles are proportional.
	5. $\frac{A'B'}{AB} = \frac{B'C'}{BC} = \frac{A'C'}{AC}$.	5. Transitive property of equality.
	6. $\triangle A'B'C' \sim \triangle ABC$.	6. S.S.S. Similarity Theorem.
(b)	1. $\triangle VA'P' \sim \triangle VAP$.	1. A.A. Similarity Theorem.
	2. $\frac{VA'}{VA} = \frac{VP'}{VP} = \frac{k}{h}$.	2. Corresponding sides of similar triangles are proportional.
	3. But $\frac{VA'}{VA} = \frac{A'B'}{AB}$.	3. Corresponding sides of similar triangles are proportional.
	4. Therefore $\frac{A'B'}{AB} = \frac{k}{h}$	4. Substitution property of equality.
	5. $\frac{\text{Area of } \triangle A'B'C'}{\text{Area of } \triangle ABC} =$ $(\frac{A'B'}{AB})^2 = (\frac{k}{h})^2$.	5. Areas of similar polygons are proportional to the squares of any two corresponding sides.

13. 96 square inches

14. 5 feet

Chapter 11

Review Problems

1. 156 , 24

2. 36

3. No. $2[(n - 2)180] = (2n - 4)180 \neq (2n - 2)180$.

4. 64 to 169

5. $44 <$ measure of third face angle < 156

6. (b), (d).

7. Yes. Squares, equilateral triangles and various combinations as discussed in an earlier problem set. The sum of the measures of the interior angles must be 360 .

8. (a) $h^2 + \frac{s^2}{4} = s^2$; $h^2 = \frac{3}{4}s^2$ and $h = \frac{s}{2}\sqrt{3}$.

   (b) $A = \frac{1}{2}hs = \frac{1}{2}(\frac{s}{2}\sqrt{3})s = \frac{\sqrt{3}}{4}s^2$.

9. (a) $\sqrt{3}$ (c) $\frac{3}{4}\sqrt{3}$

   (b) $16\sqrt{3}$ (d) $\frac{49}{4}\sqrt{3}$

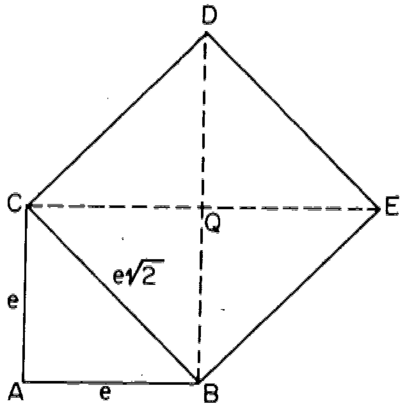
10. $s = 6$; $h = 3\sqrt{3}$

11. $s = 16$; $A = 64\sqrt{3}$

12. $36\sqrt{3}$

13. $s^2 - 2bs$

14. Let the length of the side of the isosceles right triangle be e . Then the length of its hypotenuse is $e\sqrt{2}$, and the area of a square on the hypotenuse is $(e\sqrt{2})^2 = 2e^2$. The area of the triangle is $\frac{1}{2}e^2$, which is one-fourth of the area of the square.



Alternate solution: The five triangles in the drawing are all congruent, so all have the same area. Therefore, area BCDE is four times area $\triangle ABC$.

15. BE = 12 .

(a) $\triangle CFD \cong \triangle CEB$	(a) A.S.A.
(b) CF = CE	(b) Definition of congruence
(c) $(BC)^2 = 256$ or BC = 16	(c) From the given area of the square.
(d) $\frac{1}{2}(CE)(CF) = \frac{1}{2}(CE)^2$ $\frac{1}{2}(CE)^2 = 200$, CE = 20	(d) By hypothesis and Step (b)
(e) BE = 12	(e) Pythagorean Theorem

16. (a) 24

(b) $\frac{\sqrt{3}}{4} \cdot 8^2 = 16\sqrt{3}$

(c) $\frac{d^2}{2} = 72$

(d) Insufficient data

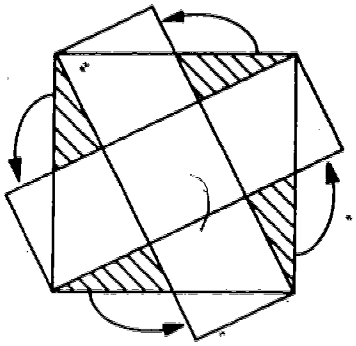
(e) $81\sqrt{2}$

(f) 216

(g) Insufficient data

(h) $30\sqrt{3}$

17. The area of RSPQ is $\frac{1}{5}$ that of ABCD as can be seen by rearranging the triangular regions as shown. Use the S.A.A. Theorem to justify this employment of the triangular regions.



18. 12

19. 150 = A and $12\frac{1}{2} = s$

20. 48

21. 1010 .

$$A = \frac{1}{2} \cdot 15 \cdot 44 + \frac{1}{2} \cdot 24 \cdot 7 + \frac{1}{2} \cdot 16 \cdot 12 + \frac{1}{2} \cdot (24 + 16) \cdot 25$$

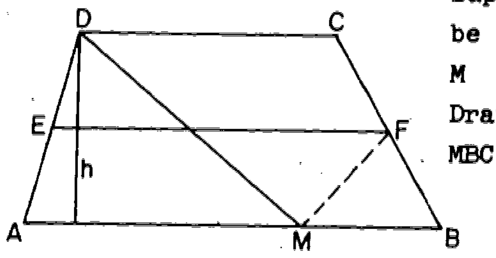
22. Consider $\overline{BX}$ as a base for $\triangle BXC$ and $\overline{BA}$ as a base for parallelogram ADCB . Then the area of BXC is $\frac{1}{4}$ of the area of ADCB . By similar argument, the area of $\triangle CED$ is $\frac{1}{4}$ of the area of ADCB . By subtracting the areas of the two triangles from that of the parallelogram we have the area AECX or $\frac{1}{2}$ the area of ABCD .

23. Since AB is constant then the altitude to $\overleftrightarrow{AB}$ must be constant in order for area to be constant. Call the length of the altitude from P to $\overleftrightarrow{AB}$ h . Then in plane E , P may be any point on either of the two lines parallel to $\overleftrightarrow{AB}$ at a distance h from $\overleftrightarrow{AB}$. In space, P may be any point on a cylindrical surface having $\overleftrightarrow{AB}$ as its axis and h as its radius.

24. $AC = 9\sqrt{2}$; $AF = 9\sqrt{2}$; $FC = 9\sqrt{2}$; $m\angle FAC = 60$. $\triangle FAC$ is an equilateral triangle whose area is $\frac{(9\sqrt{2})^2}{4}\sqrt{3}$ or $\frac{81\sqrt{3}}{2}$.

25. The diagonal of a cube $= \sqrt{e^2 + (e\sqrt{2})^2} = e\sqrt{3}$ where e is the edge of the cube. $d = 6\sqrt{3}$.

26.



Suppose AB > CD. Let $\overline{EF}$ be median of ABCD. Take M on $\overrightarrow{AB}$ so AM = EF . Draw $\overline{DM}$. Then AMD and MBCD are the required regions.

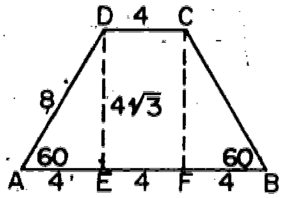
Area of $AMD = \frac{1}{2}h \cdot AM = \frac{1}{2}h \cdot EF$.

Area of $DCAB = h \cdot EF$ since the area of a trapezoid equals the product of its altitude and its median (from Problem 3, Problem Set 11-5.)

Then

Area of AMD = $\frac{1}{2}$ of the area of ABCD and the area of MBCD must be the other half of the area of ABCD and thus equal to area AMD .

27. Let $\overline{DE}$ and $\overline{CF}$ be perpendicular to $\overline{AB}$. Then AE = 4 and DE = $4\sqrt{3}$ = CF . Then $\triangle CBF \cong \triangle DAE$ since they are right triangles with one leg and an acute angle congruent. Then BF = 4 , EF = 4 = DC .



Then the area of ABCD

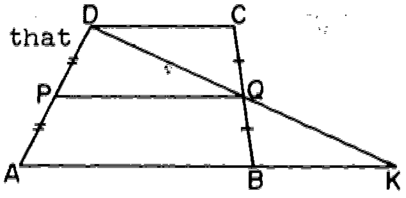
$= \frac{1}{2} \cdot 4\sqrt{3}(12 + 4) = 32\sqrt{3}$.

28. AB = 9 ; DC = 5 ; altitude of ABCD = 2 .

Area = $\frac{1}{2} \cdot 2(9 + 5) = 14$.

29. (a) $\overline{PQ}$ is the median of trapezoid ABCD . $\overleftrightarrow{DC} \parallel \overleftrightarrow{AB}$.

We are required to prove that $\overline{PQ} \parallel \overline{DC}$, $\overline{PQ} \parallel \overline{AB}$ and $PQ = \frac{1}{2}(AB + DC)$.



Let K be the point of intersection of $\overleftrightarrow{AB}$ and $\overleftrightarrow{DQ}$. Then $\triangle DCQ \cong \triangle KBQ$ (by A.S.A.) and DC = BK and DQ = QK (definition of congruence.)

In $\triangle DAK$ and $\triangle DPQ$, DA = 2DP , DK = 2DQ , $\angle D \cong \angle D$, so $\triangle DAK \sim \triangle DPQ$. Then $\angle DPQ \cong \angle DAK$ and AK = 2PQ . $\overline{PQ} \parallel \overline{AB}$ (since corresponding angles are congruent) and hence $\overline{PQ}$ is also $\parallel \overline{DC}$.

AK = AB + BK = AB + DC = 2PQ .

Hence $PQ = \frac{1}{2}(AB + DC)$.

(b) 8 .

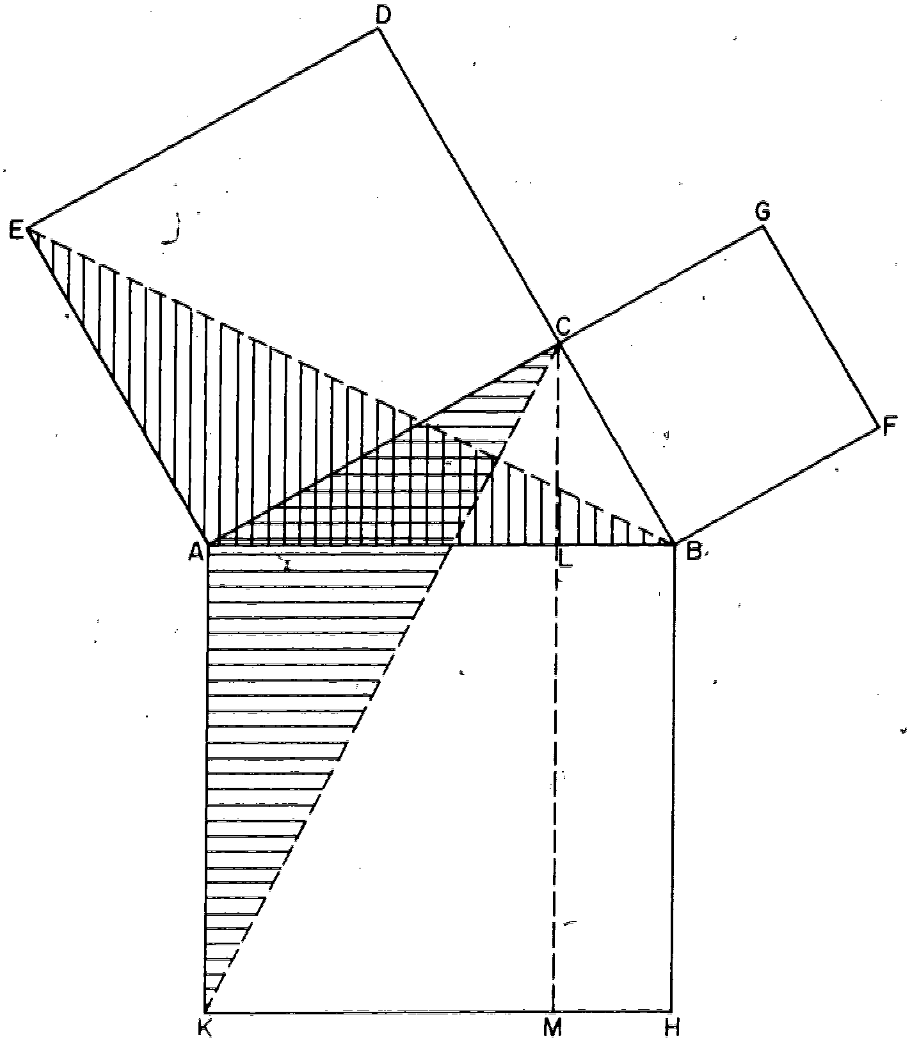
(c) $10\frac{1}{2}$.

30. Area of trapezoid DFEC = 34 .
Area of trapezoid AGFD = 165 , and so area AGECD = 199 . Area $\triangle AGB = 30$.
Area $\triangle BCE = 32\frac{1}{2}$. Subtracting the sum of the areas of the two triangles from AGECD , we have $136\frac{1}{2}$. The area of the field is $136\frac{1}{2}$ square rods.

31. Hypothesis: In right triangle ABC , $\angle C$ is the right angle. ABHK is the square on $\overline{AB}$; BCGF is the square on $\overline{CB}$; CAED is the square on $\overline{CA}$.

To prove: Area of AKHB = area of ACDE + area of BFGC .

Draw $\overline{CM} \parallel \overline{AK}$, $\overline{CK}$, and $\overline{BE}$. Call the intersection of $\overline{CM}$ and $\overline{AB}$, L .



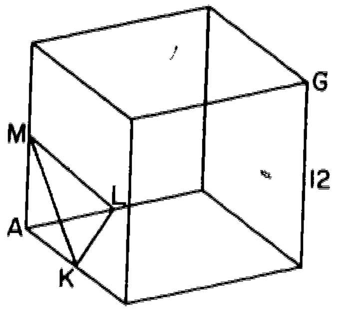
Statements	Reasons
In △KAC and △BAE	
1. $\overline{AK} \cong \overline{AB}$ $\overline{AC} \cong \overline{AE}$	1. Sides of a square are congruent.
2. m $\angle$KAC = m $\angle$BAE .	2. Each is the sum of 90 and the measure of $\angle$CAB .
3. Then △KAC $\cong$ △BAE .	3. S.A.S.
4. Area of △KAC = area of △BAE .	4. Congruent triangles have equal areas.
5. A, C, G are collinear and B, C, D are collinear.	5. Two adjacent right angles form a linear pair.
6. In △BAE and square ACDE , $\overline{AE}$ may be considered as the base. Then in each, $\overline{CA}$ has the same measure as the altitude upon base $\overline{AE}$.	6. $\overleftrightarrow{DB} \parallel \overleftrightarrow{EA}$ so the perpendicular from B to $\overleftrightarrow{EA}$ has the same length as $\overline{CA}$.
7. Area of △BAE = $\frac{1}{2}$ of the area of square ACDE.	7. Area of a triangle is $\frac{1}{2}$bh and area of the square is bh .
8. In similar manner, △KAC and rectangle KMLA may be considered as having $\overline{AK}$ as base and $\overline{LA}$ as the altitude upon $\overline{AK}$.	8. $\overleftrightarrow{CM} \parallel \overleftrightarrow{AK}$ so the perpendicular from C to $\overleftrightarrow{AK}$ has the same length as $\overline{LA}$.
9. Area of △KAC = $\frac{1}{2}$ of the area of rectangle KMLA .	9. The area of a triangle is $\frac{1}{2}$bh and the area of a rectangle is bh.
10. $\frac{1}{2}$ area of KMLA = $\frac{1}{2}$ area of ACDE or area of KMLA = area of ACDE .	10. Substitution property of equality, using Steps 4, 7, 9.

11. In like manner, after drawing $\overline{AF}$ and $\overline{CH}$, it can be proved that area of MHBL = area of BFGC.	11. Steps 1 through 10.
12. Then area of KMLA + area of MHBL = area of ACDE + area of AFGC; hence area of AKHB = area of ACDE + area of BFGC.	12. Addition property of equality.

32. $\triangle BAC \cong \triangle ATU$ by (S.A.S.). Then $AU = AB = c$ (definition of congruence.) The area of $BCTU = \frac{1}{2}h(b_1 + b_2) = \frac{1}{2}(a + b)(a + b) = \frac{1}{2}(a^2 + 2ab+b^2)$. But the area of BCTU equals the sum of the areas of three right triangles. ($\angle BAU$ is a right angle since $\angle BAC$ and $\angle UAT$ are complementary.)

Area of $BCTU = \frac{1}{2}ab + \frac{1}{2}ab + \frac{1}{2}c^2 = \frac{1}{2}(2ab + c^2)$. Then $\frac{1}{2}(2ab + c^2) = \frac{1}{2}(a^2 + 2ab + b^2)$, and $c^2 = a^2 + b^2$.

33.



Cube AG has M, L, K as midpoints of the sides which meet at A. The length of the side of the cube is 12. We are required to find the total area of pyramid M - AKL.

The area of $\triangle MAK$ = area of $\triangle MAL$ = area of $\triangle LAK = 18$. $MK = ML = KL = 6\sqrt{2}$; then area of $\triangle MKL$

$$= \frac{(6\sqrt{2})^2}{4}(\sqrt{3}) = 18\sqrt{3}.$$

The total area of $M - AKL = 3 \cdot 18 + 18\sqrt{3}$

$$= 18(3 + \sqrt{3}).$$

34. $10p = 480$ where p = perimeter of the base. Side of the base $= \frac{p}{6} = 8$; apothem $= 4\sqrt{3}$. The bases are congruent regular hexagonal regions each the union of six congruent equilateral triangles. The side of each triangle is 8 and its altitude is $4\sqrt{3}$. The area of the two bases $= 2(6)(\frac{1}{2})(8)(4\sqrt{3}) = 192\sqrt{3}$. The total area $= 480 + 192\sqrt{3}$.

Chapter 12

ANSWERS AND SOLUTIONS

Problem Set 12-1

1. (a) (1) chord, secant
   - (2) radius
   - (3) length of the radius or just radius
   - (4) diameter; secant
   - (5) chord; secant
   - (6) A, B, N, T, R, M, S
   - (7) Q, C, O
   - (8) outer end; $\overline{OS}$
   - (9) outer end
   - (10) OM or OS or equal to the radius

   (b) (1) radius
   - (2) A, M, H, B
   - (3) diameter
   - (4) great circle
   - (5) chord
   - (6) sphere, one and only one
   - (7) a circle, O , OH or OM or OB or OA
   - (8) an infinite number
an infinite number
congruent
   - (9) a point which is to be the center and a number which is the radius or a segment which is congruent to the radius
   - (10) an infinite number
concentric spheres

2. (a) 0 (e) 0
   (b) + (f) + key 0 = False
   (c) 0 (g) + + = True
   (d) 0 (h) +

3. (a) 0
(b) +
(c) +
(d) 0
(e) 0
(f) 0
(g) +
(h) 0
(i) 0

4. (a) (1) $\sqrt{(x - 0)^2 + (y - 0)^2} = \sqrt{x^2 + y^2}$
(2) $x^2 + y^2 = 25$
(3) (5,0) , (-5,0) , (0,5) , (0,-5) , also many others.
(4) $BP = \sqrt{(-3 - 0)^2 + (-2 - 0)^2} = \sqrt{9 + 4} = \sqrt{13}$
(5) $x^2 + y^2 = 13$
(6) $x^2 + y^2 = 61$

(b) (1) 4
(2) $y^2 + z^2 = 16$
(3) $RP = \sqrt{(3 - 0)^2 + (5 - 0)^2} = \sqrt{34}$
Since $RP > 4$, R is in the exterior of the circle.
(4) $y^2 + z^2 = 4$

5. (a) $BA = \sqrt{(x - 4)^2 + (z - 3)^2}$
(b) $(x - 4)^2 + (z - 3)^2 = 9$ or
$x^2 - 8x + z^2 \quad 6z + 16 = 0$
(c) $(x + 2)^2 + (y - 0)^2 = 4$ or $x^2 + 4x + y^2 = 0$
(d) $(x - h)^2 + (y - k)^2 = r^2$

6. (a) Yes
(b) Yes
(c) Yes
(d) No

7. $x^2 + y^2 + z^2 = 9$

8. (a) Yes
(b) Yes
(c) Yes
(d) No
(e) Yes
(f) Yes
(g) No

9. (13,0,0) , (0,0,13) , (0,0,-13) , (3,4,12) , (4,3,12) , (-4,-3,-12) , ...

10. Consider an xy-coordinate system and C = the set of points belonging to the circle.

(a) $C = \{(x,y): x^2 + y^2 = 9\}$

(b) $C = \{(x,y): x^2 + y^2 = \frac{1}{4}\} = \{(x,y): 4x^2 + 4y^2 = 1\}$

(c) $C = \{(x,y): x^2 + y^2 = 5\}$

11. (a) $x > 0$ and $y > 0$

(b) The portion in Quadrants I and IV and the point where the circle crosses the positive x-axis.

(c) $x < 0$ and $y < 0$

12. (a) $x = \sqrt{5}$ or $x = -\sqrt{5}$

(b) $y = 0$

(c) No. There is no real number y such that $16 + y^2 = 9$ is true.

13. (a) $z = 4$ or $z = -4$

(b) $y = 0$

(c) $x = 5$ or $x = -5$

(d) No. For all real values of z, $3^2 + 5^2 + z^2 > 25$ so all points with coordinates $(3,5,z)$ are in the exterior of the circle.

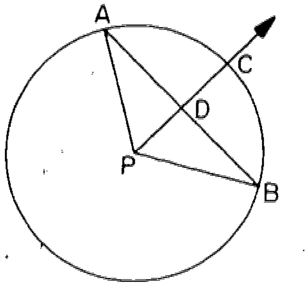
14. Let c be the length of any chord not a diameter. Draw radii to its endpoints. Then $2r > c$, because the sum of the lengths of two sides of a triangle is greater than the length of the third side. But $2r$ is the length of the diameter. Hence the diameter is longer than any other chord.

15.

Statements	Reasons
1. $\overline{AB}$ and $\overline{CD}$ contain center P.	1. Definition of diameter
2. $\overline{PC} \cong \overline{OD} \cong \overline{AP} \cong \overline{PB}$	2. These segments are radii and all radii of the same circle are congruent.
3. $\angle APC \cong \angle DPB$	3. Vertical angles are congruent.
4. $\triangle APC \cong \triangle DPB$	4. S.A.S. Postulate
5. $\overline{AC} \cong \overline{BD}$	5. Definition of congruences for triangles

16. The diameters of a circle are congruent and contain the center. It follows that they bisect each other. A quadrilateral is a rectangle if the diagonals bisect each other and are congruent.

17.



By hypothesis, $\overline{PA}$ and $\overline{PB}$ are radii of circle P and $\overrightarrow{PC}$ is the midray of $\angle APB$ and intersects $\overline{AB}$ at D .

We want to prove that $\overrightarrow{PD}$ lies in the perpendicular bisector of $\overline{AB}$.

Statements	Reasons
1. $\overline{PA} \cong \overline{PB}$	1. Radii of the same circle are congruent.
2. $\triangle BPA$ is isosceles	2. Definition of isosceles triangle
3. $\angle APD \cong \angle BPD$	3. Definition of bisector
4. $\overrightarrow{PD} \perp \overline{AB}$ and $\overrightarrow{PD}$ bisects $\overline{AB}$ so $\overrightarrow{PD}$ lies in the perpendicular bisector of $\overline{AB}$.	4. In an isosceles triangle the bisector of the vertex angle is perpendicular to the opposite side and bisects it.

18. $\sqrt{(x - 2)^2 + (y - 3)^2 + (z + 1)^2} = OQ$

$\overline{OQ}$ is a radius of the sphere.

$$(x - 2)^2 + (y - 3)^2 + (z + 1)^2 = (5)^2$$

or

$$x^2 - 4x + 4 + y^2 - 6y + 9 + z^2 + 2z + 1 = 25$$

or

$$x^2 + y^2 + z^2 - 4x - 6y + 2z = 11$$

are equations of the required sphere.

Problem Set 12-2a

1. If $a = 3$, then $y = \pm 4$ and the intersection of C and M is $\{(3,4), (3,-4)\}$.

   If $a = 4$, then $y = 0$ and the intersection is $\{(4,0)\}$.

   If $a = 5$, the intersection is the empty set since there is no real value of y for which $25 + y^2 = 16$.

2. (a) 2 (b) 1 (c) 0.

3. A circle and a line in the plane of the circle may have 2 points in common, 1 point in common, or no points in common.

Problem Set 12-2b

Problem 22 is exploratory and leads toward Theorem 12-6.

1. (a) On
   (b) Exterior
   (c) Exterior
   (d) Interior
   (e) On
   (f) Exterior
   (g) Interior

2. (a) $r = \sqrt{34}$
   (b) The points whose coordinates are most easily determined are those symmetrical to $(3,5)$ with respect to either axis or the origin. These have coordinates $(3,-5)$, $(-3,5)$, $(-3,-5)$. The points of intersection of the circle and the axes have coordinates $(0, \sqrt{34})$, $(0, -\sqrt{34})$, $(\sqrt{34},0)$, $(-\sqrt{34},0)$.
   (c) Obvious ones are those along the axes and such that their distances from the origin is less than $\sqrt{34}$.
   Any (x,y) such that $x^2 + y^2 < \sqrt{34}$.
   (d) Any (x,y) such that $x^2 + y^2 > \sqrt{34}$.

3. (a) 12-4-2
   (b) 12-4-1
   (c) 12-4-4
   (d) 12-4-3
   (e) 12-5
   (f) 12-4-1
   (g) 12-4-2
   (h) 12-5

4. 8 units

5. 2.5 units

6. $8\sqrt{2}$

7. Let $x = \frac{1}{2}PQ$.

   Then, since $\overline{PQ} \perp \overline{AB}$, $4x^2 = x^2 + 36$ and $x = 2\sqrt{\ }$.

   $PQ = 4\sqrt{3}$.

8. (a) D
   (b) C
   (c) C
   (d) A
   (e) C
   (f) A
   (g) [illegible]
   (h) D
   (i) C
   (j) D

9. 18

10. Since a tangent to a circle is perpendicular to the radius, and thus to the diameter, drawn to the point of contact, the two tangents
the same line and are, therefore, parallel.

11. (a) If a diameter is perpendicular to a non-diameter chord, it bisects the chord.

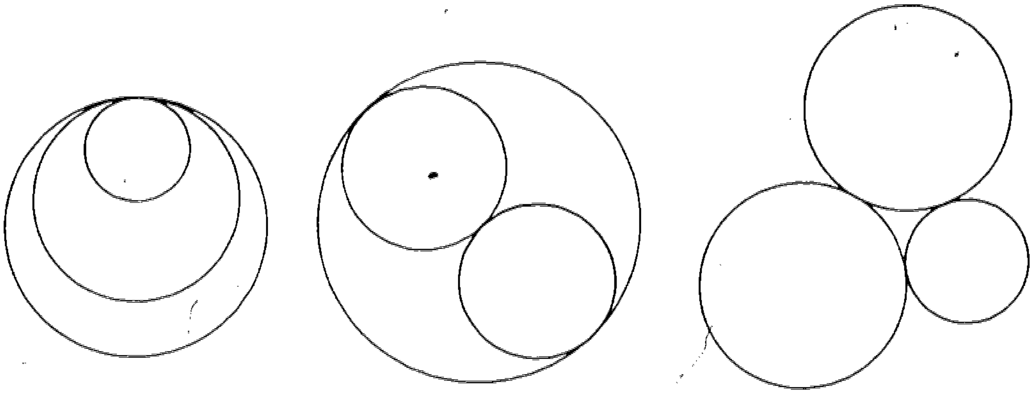
    If diameter $\overline{AB}$ is perpendicular to chord $\overline{CD}$ and if O is the center of the circle, then $\triangle OCE \cong \triangle ODE$ by the Hypotenuse-Leg Theorem. Then $\overline{CE} \cong \overline{ED}$.

    (b) If a diameter bisects a non-diameter chord, it is perpendicular to the chord.

    If in a circle with center O diameter $\overline{AB}$ bisects chord $\overline{CD}$ (not a diameter) at E , then $\triangle OCE \cong \triangle ODE$ by S.S.S. and $\angle CEO$ and $\angle OED$ are a linear pair and congruent. Therefore $\overline{AB} \perp \overline{CD}$.

12. Consider $\overline{OR}$ where O is the common center. Then $\overline{OR} \perp \overline{AB}$ since $\overline{AB}$ is tangent to the smaller circle. It follows by applying Corollary 12-4-2 to the larger circle, since $\overleftrightarrow{OR}$ is a line containing the center and perpendicular to chord $\overline{AB}$, that $\overleftrightarrow{OR}$ bisects $\overline{AB}$.

13. Examples of 3 circles each tangent to the other two.

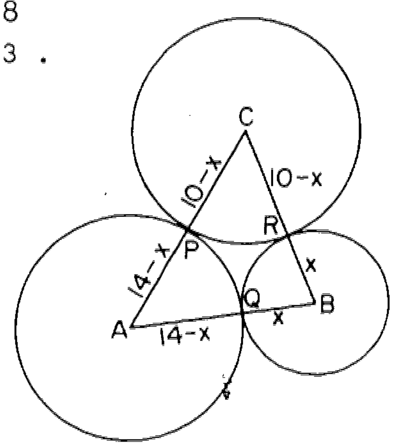


14. Let ℓ be the common tangent. Then in both cases, $\overline{PT} \perp \ell$ and $\overline{QT} \perp \ell$ because every line tangent to a circle is perpendicular to the radius drawn to the point of contact. Since there exists only one perpendicular to a line at any given point on the line, then $\overline{PT}$ and $\overline{QT}$ are the same line; and, therefore, P, Q, and T are collinear. Of course, the circles are coplanar, since they are tangent circles.

15. $AC = 14 - x + 10 - x = 18$

$$24 - 2x = 18$$
$$x = 3 .$$

$BR = 3$, $CP = 7$,
$AQ = 11$



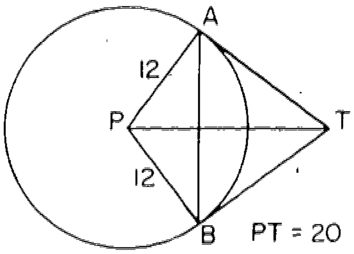
A, B, C are coplanar and hence, by the Betweenness-Distance Theorem,

$$AP + PC = AC$$
$$AQ + QB = AB$$
$$CR + RB = CB$$

16. The segment joining the midpoint of each chord to the center of the given circle is perpendicular to the chord and thus the length of this segment is the distance from the midpoint of the chord to the center. Since the chords are congruent, all these distances are equal. By definition of a circle, all these midpoints at the constant distance from the center of the given circle lie on the circle having that point as center and having a radius equal to that constant distance. This circle is, then, concentric with the original circle. Since the midpoints will be the outer endpoints of the radii of the new circle and since the chords are perpendicular to the radii at these midpoints, the chords are tangent to the new circle.

17. (a) $(AT)^2 = (PT)^2 - (AP)^2$
$$= 400 - 144 = 256$$
$$AT = 16$$

(b) Area of $\triangle APT = 96$; Since $\overleftrightarrow{PT}$ is the perpendicular bisector of $\overline{AB}$,



$$\frac{1}{2} \cdot 20 \cdot (\tfrac{1}{2}AB) = 96$$
$$AB = 19.2$$

18.

Statements	Reasons
1. $\overleftrightarrow{DO} \parallel \overleftrightarrow{AC}$ $\overleftrightarrow{CD}$ is tangent at C .	1. Hypothesis
2. $\angle A \cong \angle BOD$	2. Corresponding angles of parallel lines
3. $OC = OA = OB$	3. Definition of circle
4. $\angle A \cong \angle ACO$	4. Base angles of an isosceles triangle are congruent.
5. $\angle ACO \cong \angle COD$	5. Alternate interior angles of parallel lines
6. $\angle COD \cong \angle BOD$	6. Transitive property of congruence
7. $\overline{OD} \cong \overline{OD}$	7. Reflexive property of congruence
8. $\triangle OCD \cong \triangle OBD$	8. S.A.S. Postulate
9. $\angle OCD \cong \angle OBD$	9. Definition of congruence
10. $m \angle OCD = 90$	10. A tangent is perpendicular to a radius at its outer end.
11. $m \angle OBD = 90$	11. Congruent angles have the same measure.
12. $\overline{OB} \perp \overline{BD}$ at B	12. Definition of perpendicular lines
13. $\overleftrightarrow{DB}$ is tangent to circle C at B	13. Any line in the same plane perpendicular to a radius at its outer end is tangent to the circle.

19. (a) $a = 10$ or -10

(b) $m_{\overline{CT}} = 1$, therefore $m_t = -1$

$$\frac{y - 5\sqrt{2}}{x - 5\sqrt{2}} = -1 \quad \text{or} \quad y - 5\sqrt{2} = -x + 5\sqrt{2}$$

or $y = -x + 10\sqrt{2}$

20. (a) Yes

(b) Yes. Center is at $(1,-2)$, radius $= 5$.

(c) The following equations are equivalent.

$$(x - 1)^2 + (y + 2)^2 = 25$$
$$(x^2 - 2x + 1) + (y^2 + 4y + 4) = 25$$
$$x^2 - 2x + y^2 + 4y = 20$$
$$x^2 + y^2 - 2x + 4y = 20$$

The last equation can be transformed into the first by completing squares.

(d) Slope of radius to $(5,1)$ is $\frac{3}{4}$. Therefore, slope of tangent at $(5,1)$ is $\frac{-4}{3}$.

Equation of tangent: $\frac{y - 1}{x - 5} = \frac{-4}{3}$ or

$4x + 3y = 23$.

21. (a) $\{(x,y): x = -1\}$

(b) $\{(x,y): y = -x - \sqrt{2}$ or $\{(x,y): x + y + \sqrt{2} = 0\}$

(c) $P = (-\sqrt{2},0)$

(d) $$PT = \sqrt{\left(-\sqrt{2} + \frac{1}{\sqrt{2}}\right)^2 + \left(0 + \frac{1}{\sqrt{2}}\right)^2}$$
$$= \sqrt{\left(\frac{-2 + 1}{\sqrt{2}}\right)^2 + \left(\frac{1}{\sqrt{2}}\right)^2} = \sqrt{\frac{1}{2} + \frac{1}{2}} = 1$$

22. N is parallel to the xy-plane and perpendicular to the z-axis. $S \cap N$ means the intersection of S and N.

(a) $$S \cap N = \{(x,y,z): x^2 + y^2 + z^2 = 25 , z = 4\}$$
$$= \{(x,y,z): x^2 + y^2 = 9 , z = 4\}$$

Thus, $S \cap N$ is a circle in the plane N with its center on the z-axis with 3 as the length of its radius.

(b) $S \cap N = \{(x,y,z): x^2 + y^2 = 0 , z = 5\}$ which is a single point $(0,0,5)$.

(c) $S \cap N = \{(x,y,z):\ x^2 + y^2 + 49 = 25\ ,\ z = 7\}$

$= \{(x,y,z):\ x^2 + y^2 = -24\ ,\ z = 7\}$

which is the empty set.

(d) It appears that a sphere and a plane have no point in common, one point in common, or a circle in common, if the distance from the center of the sphere to the plane is less than, equal to, or greater than the radius of the sphere, respectively.

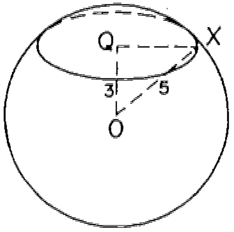
Problem Set 12-3

1. $\overleftrightarrow{OA} \perp \overleftrightarrow{FB}$

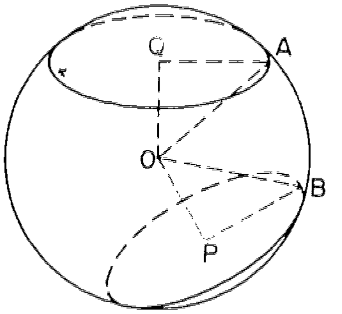
   $\overleftrightarrow{OA} \perp \overleftrightarrow{RT}$

2. 16

3. $QX = 4$



4.



$\overline{OQ}$ and $\overline{OP}$ are perpendicular to the planes of the circles. Since $\overline{OQ}$ bisects every chord of the circle that passes through Q, it must be the center. Similarly P is the center of its circle. Therefore, $\overline{OQ} \perp \overline{QA}$ and $\overline{OP} \perp \overline{PB}$. $OA = OB$ by definition of a sphere and $OQ = OP$ by hypothesis. Then, by the Pythagorean Theorem (or Hyp-Leg), $QA = PB$. Therefore, circle $Q \cong$ circle P by definition of congruent circles.

5. <u>One statement</u>: If a plane is tangent to a sphere, then it is perpendicular to a radius at its outer end. <u>Proof</u>: From Case 2 of Theorem 12-6, if a plane is tangent to a sphere (which is the hypothesis in our statement), then the foot of the perpendicular from the center of the circle to the plane lies in the sphere. This means that this perpendicular is a radius with the foot as its outer end, which is the conclusion of our statement.

   <u>Converse statement</u>: If a plane is perpendicular to a radius of a sphere at its outer end, then it is tangent to the sphere.
   Proof: From Case 2 of Theorem 12-6, if the foot of the perpendicular from the center of a sphere to a plane is on the sphere, then the plane is tangent to the sphere. The hypothesis tells us that the foot of the perpendicular is the outer end of the radius, which, by definition of outer end, is on the sphere. The conclusion of our statement follows from Theorem 12-6.

6. Let O be the center of sphere S and P_1 and P_2 be two planes each containing O . Let C_1 and C_2 be the great circles of S determined by P_1 and P_2 respectively. Then the intersection of P_1 and P_2 is a line which contains O . This line has two points, say A and B , in common with S . But all points common to P_1 and S lie in C_1 and all points common to P_2 and S lie in C_2 so all points common to $\overleftrightarrow{AB}$ (which is the intersection of P_1 and P_2) and S must lie in both C_1 and C_2 and hence in their intersection. A and B are these points and, since $\overline{AB}$ is the diameter of S , two great circles intersect at the endpoints of a diameter of the sphere.

7. (a) Center is (0,0,0) ; radius = 3 .

   (b) $\{(x,y,z): y = 3\}$ or $\{(x,y,z): y = -3\}$; two planes.

   (c) $\{(x,y,z): x = 3\}$ and $\{(x,y,z) ; x = -3\}$.

8. (a) The empty set in each case.
   (b) All points in the interior of a cube the faces of which are tangent to the sphere S and perpendicular to the axes.
   (c) The intersection of S and T is all of S except the points (4,0,0) , (-4,0,0) , (0,4,0) , (0,-4,0) , (0,0,4) , (0,0,-4) .

9. The plane of the perpendicular great circle is the plane perpendicular to the line of intersection of the planes of the given two, at the center of the sphere. There is only one such plane. (Through a given point there passes one and only one plane perpendicular to a given line.)

   Any two meridians have the equator as their common perpendicular.

10. AF = BF since they are radii of the circle of intersection. OF = AF by hypothesis. Also $\overline{OF} \perp \overline{AF}$, $\overline{OF} \perp \overline{BF}$ and $\overline{AF} \perp \overline{BF}$. Hence, $\triangle AFB \cong \triangle AFO \cong \triangle BFO$ by S.A.S. and $\triangle AOB$ is equilateral. Therefore AO = 5 , m $\angle AOB = 60$, and OG , the altitude of $\triangle AOB$, equals $\frac{5}{2}\sqrt{3}$.

11. Call the three points A, B, C . To find the center of the circle in the plane of ABC consider the perpendicular bisectors in the plane of ABC of any two of the three segments $\overline{AB}$, $\overline{BC}$ and $\overline{AC}$. The bisectors intersect at a point equidistant from A, B, C which is the center Q of a circle through those three points. Each of the segments $\overline{QA}$, $\overline{QB}$, $\overline{QC}$, is a radius of the circle. If a perpendicular be drawn to the plane of ABC at Q , it will meet the shpere in two points, X and Y . The midpoint P of $\overline{XY}$ is the center of the sphere and each of the segments, $\overline{PA}$, $\overline{PB}$, $\overline{PC}$, is a radius of the sphere.

12. By Theorem 12-6 we know that plane F intersects S in a circle. The intersection of the planes E and F is a line. Since both intersections contain T, the circle and the line intersect at T. If they are not tangent at T, then they would intersect in some other point, R, also. Point R would then lie in plane E and in sphere S. This is impossible, since E and S are tangent at T. Hence, the circle and the line are tangent, by definition of a tangent to a circle.

13. (a) $\{(x,y,z): x^2 + y^2 = 0 , z = 10\}$ or $(0,0,10)$.

    (b) $S \cap P = \{(x,y,z): x^2 + y^2 = 36 , z = 8\}$.

    That is, x and y must satisfy the equation $x^2 + y^2 = 36$ which, in the plane P, is the equation of a circle. Note, however, that $\{(x,y,z): x^2 + y^2 = 36\}$ is not a circle, but a right circular cylinder.

14. (a) $x^2 - 4x + 4 + y^2 + 6y + 9 = 23 + 4 + 9$

    $(x - 2)^2 + (y + 3)^2 = 36$

    Center is $(2,-3)$, radius is 6 .

    (b) $S = \{(x,y,z): (x - 2)^2 + (y + 3)^2 + (z - 0)^2 = 36\}$

    $= \{(x,y,z): x^2 + y^2 + z^2 - 4x + 6y = 23\}$.

    (c) $\{(x,y,z): z = 10\}$, $\{(x,y,z): z = -10\}$

Problem Set 12-4a

1. (a) (1) $\angle APD$, $\angle CPB$, $\angle CPD$, $\angle BPD$, $\angle CPA$

   (2) 180

   (3) $\overset{\frown}{AD}$, $\overset{\frown}{AC}$, $\overset{\frown}{CB}$, $\overset{\frown}{BD}$, $\overset{\frown}{CBD}$

   (4) $\overset{\frown}{DAC}$, $\overset{\frown}{DAB}$, $\overset{\frown}{ACD}$, $\overset{\frown}{CBA}$, $\overset{\frown}{BDC}$

   (5) $\overset{\frown}{AD} \cong \overset{\frown}{DB}$

   (6) Semicircles $\overset{\frown}{ACB}$ and $\overset{\frown}{ADB}$ are not associated with central angles.

(b) (1) 11

(2) $\angle MAB$, inscribed in $\widehat{MAB}$, intercepts $\widehat{MB}$
$\angle MAD$, " " $\widehat{MAD}$, " $\widehat{MD}$
$\angle MAC$, " " $\widehat{MAC}$, " $\widehat{MC}$
$\angle CAB$, " " $\widehat{CAB}$, " $\widehat{CB}$
$\angle CAD$, " " $\widehat{CAD}$, " $\widehat{CD}$
$\angle DAB$, " " $\widehat{DAB}$, " $\widehat{DB}$
$\angle ABC$, " " $\widehat{ABC}$, " $\widehat{AC}$
$\angle ADC$, " " $\widehat{ADC}$, " $\widehat{AC}$
$\angle BCA$, " " $\widehat{BCA}$, " $\widehat{BA}$
$\angle DCB$, " " $\widehat{DCB}$, " $\widehat{DB}$
$\angle DCA$, " " $\widehat{DCA}$, " $\widehat{DA}$

(3) $\angle ABC$ and $\angle ADC$. The degree measure of each is 90 since they are inscribed in a semicircle.

(c) (1) $\angle A$ intercepts $\widehat{DC}$
$\angle B$ " $\widehat{DC}$
$\angle C$ " $\widehat{BA}$
$\angle D$ " $\widehat{BA}$

(2) $m \angle A = 20$

$m \angle B = 20$

$m \angle C = 47\frac{1}{2}$

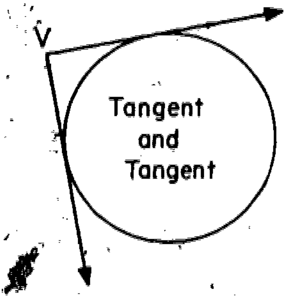
$m \angle D = 47\frac{1}{2}$

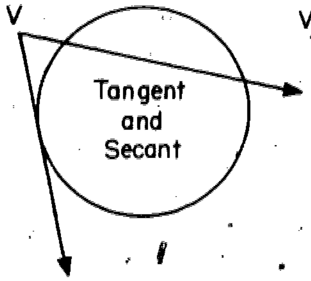
(3) $\angle A \cong \angle B$, $\angle C \cong \angle D$

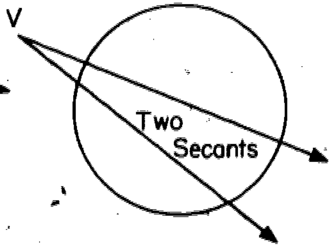
Angles having equal measures are congruent. Angles inscribed in the same arc are congruent.

27

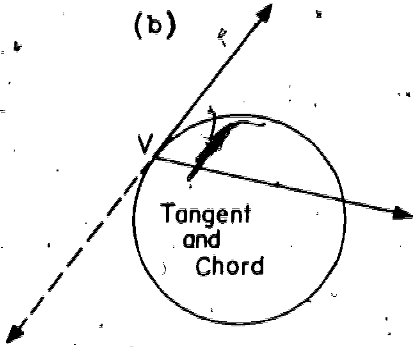
2. (a)



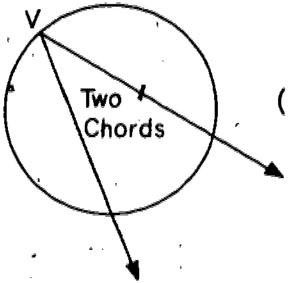




(b)

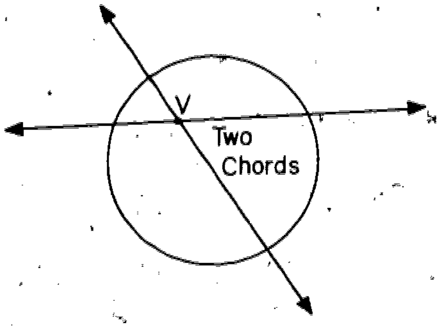


(Tangent and Secant)



(Two secants)

(c)



(Two secants)

3. The center of an arc is the intersection of the perpendicular bisectors of two or more chords of the arc (Cor. 12-4-3).

4. Since the inscribed angle is measured by half the arc which it intercepts, $\overset{\frown}{AB}$ must measure 90°. The measure of a central angle is the measure of its intercepted arc, so $m\angle P = 90$ and $\overline{BF} \perp \overline{AF}$.

5. (a) $m\angle A = m\angle B$ by Corollary 12-7-2.
   $m\angle AHK = m\angle BHF$ since the intercepted arcs have equal measure. Therefore $\triangle AHK \sim \triangle BHF$ by a triangle similarity theorem (A.A.).

   (b) $\triangle BFK$, since

   $$m\angle BFA = \frac{1}{2}m\overset{\frown}{HB} = \frac{1}{2}m\overset{\frown}{BF} = m\angle BHF$$

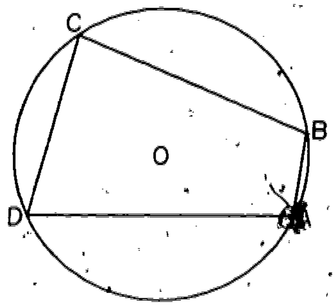
   and $\angle HBF$ is common to the triangles.

6. $m\overset{\frown}{ST} = 80$
   $m\overset{\frown}{RV} = 150$
   $m\angle T = 95$
   $m\angle V = 60$
   $m\angle S = 120$

7. If quadrilateral ABCD is inscribed in circle O, then by Theorem 12-7

   $m\angle C = \frac{1}{2}m\overset{\frown}{BAD}$ and

   $m\angle A = \frac{1}{2}m\overset{\frown}{DCB}$.



   Since the union of a major arc and its minor arc (or of two semicircles) has a degree measure of 360,

   $m\overset{\frown}{BAD} + m\overset{\frown}{DCB} = 360$.

   $\frac{1}{2}m\overset{\frown}{BAD} + \frac{1}{2}m\overset{\frown}{DCB} = 180$ by the multiplication property of equality. Hence, $\angle BAD$ and $\angle DCB$ are supplementary.

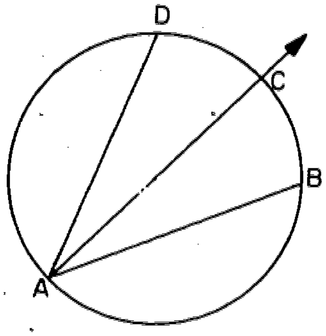
8. Consider $\overline{RO}$. We know $\overline{AO}$ is a diameter of the smaller circle and therefore that $m\angle ARO = 90$ by Corollary 12-7-1. Then $\overline{AB}$ is bisected by the smaller circle at point R by Corollary 12-4-3. The circles are coplanar since they are tangent.

9. $\triangle ACB$ is a right triangle with the right angle at C by Corollary 12-7-1. In a right triangle the altitude from the right angle to the hypotenuse divides the triangle into two triangles, $\triangle ACD$ and $\triangle CBD$, which are similar both to each other and to the original triangle. Therefore

$$\frac{AD}{CD} = \frac{CD}{DB} \text{ or } CD^2 = AD \cdot DB$$

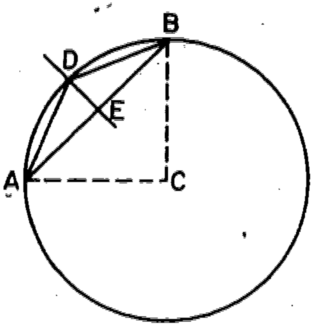
10. By Theorem 12-7, $m\angle A = \frac{1}{2} m\,\widehat{BDC}$. Since $m\angle A = 90$, $m\,\widehat{BDC} = 180$ and $\widehat{BDC}$ is a semicircle, by definition of a semicircle.

11.



In the circle $\overrightarrow{AC}$ bisects $\angle DAB$ so, by definition of angle bisector,
$m\angle BAC = m\angle DAC$. But
$m\angle BAC = \frac{1}{2} m\,\widehat{BC}$ and
$m\angle DAC = \frac{1}{2} m\,\widehat{DC}$.
Therefore $m\,\widehat{BC} = m\,\widehat{DC}$ by the multiplication and substitution properties of equality.

12. Consider radii $\overline{PA}$ and $\overline{PB}$. Since diameter $\overline{CD} \perp \overline{AB}$, then $AM = BM$ by Corollary 12-4-2. $\triangle APM \cong \triangle BPM$ by S.S.S. (or S.A.S. or Hypotenuse-Leg), so that $m\angle APC = m\angle BPC$. Then, $m\angle APD = m\angle BPD$ since they are supplements of congruent angles. Therefore, $m\,\widehat{AC} = m\,\widehat{BC}$ and $m\,\widehat{AD} = m\,\widehat{BD}$, by the definition of measure of an arc and the substitution property of equality. Hence $\overline{CD}$ bisects $\widehat{ACB}$ and $\widehat{ADB}$.

13. 

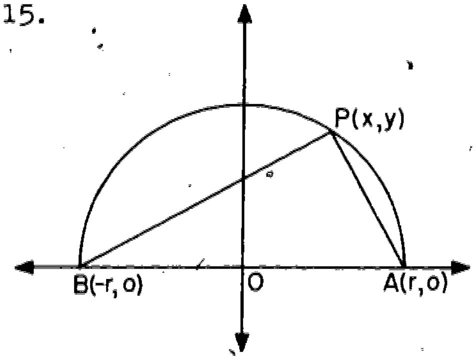
By hypothesis m $\widehat{AD}$ = m $\widehat{DB}$ and AE = EB . Consider $\triangle$ADB . m $\angle B = \frac{1}{2}$ m $\widehat{AD}$ and m $\angle A = \frac{1}{2}$ m $\widehat{DB}$ by Theorem 12-7. Then m $\angle B$ = m $\angle A$ by the multiplication and substitution properties of equality. Thus, $\triangle$ADB is isosceles and $\overleftrightarrow{DE}$, which bisects base $\overline{AB}$, is also perpendicular to $\overline{AB}$. But, since C is equidistant from A and B, it is in the perpendicular bisector of $\overline{AB}$ and hence in DE .

14. By definition, $\triangle$BAC is isosceles since $\overline{AB} \cong \overline{AC}$; therefore, $\angle B \cong \angle C$.

m $\angle B = \frac{1}{2}$ m $\widehat{AC}$ and

m $\angle C = \frac{1}{2}$ m $\widehat{AB}$ by Theorem 12-7.

Hence, m $\widehat{AC}$ = m $\widehat{AB}$ by the substitution and multiplication properties of equality.

15. 

Let an xy-coordinate system assign (0,0) to the center of the circle. Then, if r is the radius, the extremities of a diameter would be (r,0) and (-r,0) , r > 0 . Call these points A and B respectively. Let P(x,y) be any point of the circle except A or B. Then the slope of $\overline{PA} = \frac{y}{x - r}$ and the slope of $\overline{PB} = \frac{y}{x + r}$. The product of these slopes is $\frac{y^2}{x^2 - r^2}$. But for all points P , $x^2 + y^2 = r^2$ or $y^2 = r^2 - x^2$. Thus $\frac{y^2}{x^2 - r^2} = \frac{r^2 - x^2}{x^2 - r^2} = -1$.

It follows that $\overline{PB} \perp \overline{PA}$ and $\angle$BPA is a right angle.

16. $m\angle C + m\angle BXY = \frac{1}{2}m\,\widehat{YXB} + \frac{1}{2}m\,\widehat{YCB} = \frac{1}{2}(m\,\widehat{YXB} + m\,\widehat{YCB})$
$= 180$.

Therefore $\angle C$ is the supplement of $\angle BXY$. But $\angle AXY$ is the supplement of $\angle BXY$ and, since supplements of the same angle are congruent, $\angle AXY \cong \angle C$. In the manner used above, $\angle D$ may be shown to be the supplement of $\angle AXY$ and therefore the supplement of $\angle C$. Since $\angle C$ and $\angle D$ are consecutive interior angles of $\overleftrightarrow{AD}$ and $\overleftrightarrow{BC}$ with transversal $\overleftrightarrow{DC}$, it follows that $\overline{AD} \parallel \overline{BC}$.

17. (a) Since $\angle ACB$ is a right angle by Corollary 12-7-1 and $\angle DEB$ is a right angle by definition of perpendicular lines, $\angle C \cong \angle DEB$. Also, by the reflexive property of congruence, $\angle B \cong \angle B$. Thus $\triangle BCA \leftrightarrow \triangle BED$ is a similarity by A.A.

(b) $(BC,CA,AB) \underset{p}{=} (BE,ED,DB)$

(c) Since $(BC,BA) \underset{p}{=} (BE,BD)$, by the product property of proportion $BD \cdot BC = BA \cdot BE$.

18. Since $\overline{AC}$ and $\overline{BD}$ are tangent at the endpoints of a diameter, $\overline{AC} \parallel \overline{BD}$. Also $\overline{AC}$ and $\overline{BD}$ are segments of chords of the larger circle which are congruent by Theorem 12-5. By Corollary 12-4-2 the radii $\overline{OA}$ and $\overline{OB}$ bisect these chords, so that $\overline{AC} \cong \overline{BD}$. Therefore quadrilateral ADBC is a parallelogram by the theorem which says that, if two sides of a quadrilateral are congruent and parallel, the quadrilateral is a parallelogram. The diagonals of a parallelogram bisect each other, so $\overline{AB}$ and $\overline{CD}$ bisect each other at some point, P . Point O is the midpoint of $\overline{AB}$, so $P = O$, and C, O, D are collinear, making $\overline{CD}$ a diameter.

Problem Set 12-4b

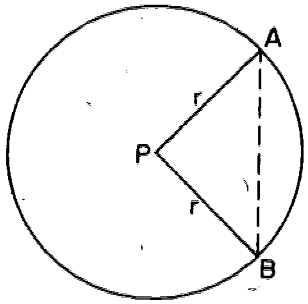
Problems 12, 13 and 14 help prepare for Theorems 15 and 16

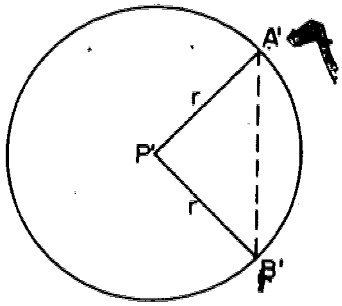
1. (a) (1) inscribed angle
    - (2) tangent-chord angle
    - (3) secant-secant angle
    - (4) tangent-secant angle
    - (5) tangent-tangent angle

   (b) (1) 75
    - (2) 110
    - (3) 30
    - (4) 34
    - (5) 60
    - (6) 42

2. (a) 35
    - (b) 55
    - (c) $62\frac{1}{2}$
    - (d) $12\frac{1}{2}$
    - (e) $22\frac{1}{2}$
    - (f) 90
    - (g) $27\frac{1}{2}$
    - (h) 35
    - (i) 125
    - (j) 50

3.





In the congruent circles P and P', we are given that m $\widehat{AB}$ = m $\widehat{A'B'}$. It follows that their respective central angles P and P' are of equal measure. Thus $\triangle APB \cong \triangle A'P'B'$ by S.A.S. and $\overline{AB} \cong \overline{A'B'}$ by definition of congruence.

4. (a) m $\widehat{FBA}$ = m $\widehat{BAH}$ by Theorem 12-8.
m $\widehat{FB}$ + m $\widehat{BA}$ = m $\widehat{FBA}$ and m $\widehat{AH}$ + m $\widehat{BA}$ = m $\widehat{BAH}$ by Postulate 30. Then m $\widehat{FB}$ = m $\widehat{AH}$ by the addition property and the substitution property of equality. Hence $\widehat{FB} \cong \widehat{AH}$.

(b) From (a) we conclude that
FB = AH by Theorem 12-9.
$\angle FBH \cong \angle HAF$ and $\angle AFB \cong \angle AHB$ by Theorem 12-7.
Then $\triangle BMF \cong \triangle AMH$ by A.S.A.

5. ABCD is a square and
$\overline{DA} \cong \overline{AB} \cong \overline{BC}$ and therefore
$\widehat{DA} \cong \widehat{AB} \cong \widehat{BC}$. Theorem 12-8.

Then m $\angle DEA$ = m $\angle AEB$ = m $\angle BEC$ since they are inscribed angles and are equal to one-half the measures of the congruent arcs which they intercept. $\angle DEC$ has then been trisected.

6. (a) $\angle BAC$
(b) $\angle CAF$
(c) $\angle ADB$, $\angle BAF$
(d) $\angle DAF$
(e) $\angle DCB$
(f) $\angle ADC$
(g) $\angle DCA$, $\angle DBA$
(h) $\angle DAF$
(i) $\angle EAB$
(j) $\angle DBC$

7. Since m $\widehat{PB}$ = 120 , m $\angle BPC$ = 60 by Theorem 12-10. $\overline{PQ} \perp \overline{OP}$, so m $\angle BPQ$ = 30 . $\triangle APQ$ is a 30°-60° right triangle. Since PQ = 6 , then AP = $4\sqrt{3}$.

8. Consider the common tangent at H . Then an angle formed by the tangent at H and line u is measured by the same arc as an angle formed by line u and the tangent M or N . It follows that the tangents at M and N are parallel, by corresponding angles in one case and alternate interior angles in the other case.

9. Consider $\overline{PB}$. By Theorem 12-7, m $\angle BPR = \frac{1}{2}$ m $\widehat{BR}$. By Theorem 12-10, m $\angle BPT = \frac{1}{2}$ m $\widehat{PB}$. But m $\widehat{BR}$ = m $\widehat{PB}$, so m $\angle BPR$ = m $\angle BPT$. $\overline{BF} \perp \overleftrightarrow{PT}$ and $\overline{BE} \perp \overleftrightarrow{PR}$ by the definition of distance from a point to a line. PB = PB so $\triangle PBE \cong \triangle PBF$ by A.S.A. (or A.A.S.). Therefore, by definition of congruence, BE = BF .

10. Case I: Consider the diameter from P . Since the diameter is perpendicular to the tangent, it is perpendicular to $\overleftrightarrow{AB}$. Therefore it bisects $\overline{AB}$ and $\widehat{AB}$ and m $\widehat{AP}$ = m $\widehat{BP}$.

Case II: Consider the diameter perpendicular to the secants. This diameter will bisect $\widehat{CPD}$ and $\widehat{APB}$. Thus m $\widehat{AP}$ = m $\widehat{BP}$ and m $\widehat{CP}$ = m $\widehat{DP}$. Then by betweenness for arcs and the properties of equality m $\widehat{AC}$ = m $\widehat{BD}$

Case III: The diameter from P will have Q as its other endpoint. Then the two arcs are semicircles and have equal measures, by definition of the degree measure of a semicircle.

Alternate proofs involve radii to form congruent triangles, or chords which are transversals and using alternate interior angles.

*11. (a) $\{A,B\} = \{(x,y): x^2 + y^2 = 25 , y = 3\}$

$= \{(x,y): x^2 = 16 , y = 3\} = \{(4,3),(-4,3)\}.$

(b) $\{C,D\} = \{(x,y): x^2 + y^2 = 25 , x = 0\}$

$= \{(0,5),(0,-5)\}$.

(c) $QA \cdot QB = \sqrt{(0-4)^2 + (3-3)^2} \cdot \sqrt{(0+4)^2 + (3-3)^2}$

$= 4 \cdot 4 = 16$.

$QC \cdot QD = \sqrt{(0-0)^2 + (3-5)^2} \cdot \sqrt{(0-0)^2 + (3+5)^2}$

$= 2 \cdot 8 = 16$.

*12. (a) $\{A,B\} = \{(x,y): x^2 + y^2 = 25 \ , \ y = 3\}$
$= \{(4,3),(-4,3)\}$.

(b) $\{C,D\} = \{(x,y): x^2 + y^2 = 25 \ , \ y = x - 5\}$
$= \{(x,y): x^2 + (x - 5)^2 = 25 \ , \ y = x - 5\}$
$= \{(x,y): 2x^2 - 10x + 25 = 25 \ , \ y = x - 5\}$
$= \{(x,y): x(2x - 10) = 0 \ , \ y = x - 5\}$
$= \{(0,-5),(5,0)\}$.

(c) $PA \cdot PB = \sqrt{(8-4)^2 + (3-3)^2} \cdot \sqrt{(8+4)^2 + (3-3)^2}$
$= 4 \cdot 12 = 48$.
$PC \cdot PD = \sqrt{(8-0)^2 + (3+5)^2} \cdot \sqrt{(8-5)^2 + (3-0)^2}$
$= \sqrt{128} \cdot \sqrt{18} = \sqrt{16 \cdot 8 \cdot 9 \cdot 2} = 48$.

*13. (a) $\triangle ADB \sim \triangle CDA$ because $\angle B \cong \angle CAD$ and $\angle D \cong \angle D$.

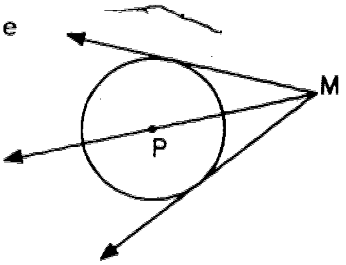
(b) Corresponding sides of similar triangles are proportional.

(c) $AD = 6k$, $BD = k(6k) = 6k^2$

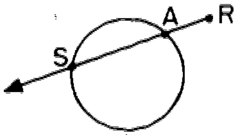
(d) $AD \cdot AD = (6k)(6k) = 6 \cdot 6k^2$
$BD \cdot CD = (6k^2)(6) = 6 \cdot 6k^2$
Therefore $(AD)^2 = BD \cdot CD$.
Relation is true for $k > 0$ and $CD > 0$. If $k = 0$, then $A = D = C$ and $(AD)^2 = BD \cdot CD$ is true. If $k < 0$, then it is impossible to have $AD = k \cdot CD$..

Problem Set 12-5

1. (a) 2 , lengths, bisector, angle



(b) in(on), in or on,
A is between R and S .



(c) 12 , 12

2. (a) (1) tangent-segments

   (2) 10 , Theorem 12-13: The two tangent-segments to a circle from an external point are congruent.

   (3) 45 ; The last part of Theorem 12-13 says that the two tangent-segments from an external point form congruent angles with the line joining the external point to the center of the circle.

   (b) (1) $\overline{RX}$

   (2) $\overline{RD}$

   (3) $\overline{RC}$

   (4) $RC + CD = RD$ by the Betweenness-Distance Theorem.

   (5) Yes, since $12^2 = 8(10 + 8)$.

   (6) Yes, since $12^2 = 6(18 + 6)$.

   Other pairs of factors of 144 are the easiest to consider. $RC \cdot RD$ could be $9 \cdot 16$, $4 \cdot 36$, $3 \cdot 48$, etc., as far as the products are concerned. However, since CD is less than or equal to the diameter of the circle, restrictions must be made with reference to any given circle.

   (c) (1) $a(a + b) = x \cdot (x + y)$

   Theorem 12-14: The product of the length of a secant-segment from a given point and the length of its external segment is constant for any secant containing the external point.

   (2) Yes, since $3(3 + 17) = 4(4 + 11)$.

3. (a) When it contains the center of the circle.

   (b) When the secant contains the center of the circle.

   (c) decrease, increase, tangent-segment, the tangent-segment.

   (d) $QA'' \cdot QA''$

4.

Statements	Reasons
1. $\overleftrightarrow{AC}$, $\overleftrightarrow{CE}$ and $\overleftrightarrow{EH}$ are tangents at B , D and F respectively.	1. Hypothesis
2. $CB = CD$ $EF = ED$	2. Theorem 12-13
3. $CB + EF = CD + DE$	3. Addition property of equality
4. $CD + DE = CE$	4. Betweenness-Distance Theorem
5. $CB + EF = CE$	5. The substitution property of equality

5. (a) $20x = 6 \cdot 22$

$x = 6.6$

$DA = 6.6$

(b) $x^2 = AR \cdot AM$

$= 16 \cdot 4$

$x = 8$

$AT = 8$

(c) $PB \cdot PA = PD \cdot PC$

$16 \cdot 7 = x \cdot 8$

$x = 14$

$PD = 14$

6. $6(14) = 7(x + 7)$

$12 = x + 7$

$x = 5$

$BA = 5$

7. $x(x + 13) = 4 \cdot 12$

$x^2 + 13x - 48 = 0$

$x = 3$

$DC = 3$

8. Let $BK = a$. Then by Theorem 12-15,

$$a(a + 5) = 36$$

$$a^2 + 5a - 36 = 0$$

$$(a + 9)(a - 4) = 0$$

$$a = 4$$

$$BK = 4$$

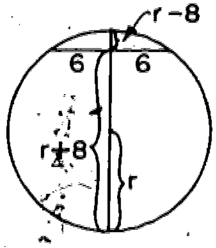
9. By Theorem 12-16, we have

$$x(19 - x) = 6 \cdot 8$$
$$x^2 - 19x + 48 = 0$$
$$(x - 3)(x - 16) = 0$$
$$x = 3$$
$$w = 19 - x = 16$$

10. Let r be the radius. Then, by Theorem 12-15,

$(r + 8)(r - 8) = 6 \cdot 6$

$r^2 - 64 = 36 , \quad r = 10$



11. Let the radius of the circle be r . Then by Theorem 12-15,

$$4(2r + 4) = 12^2$$

Hence $r = 16$

12. Since all angles of the triangle have measure 60 , the minor arc has measure 120 . This leaves 240 for the measure of the major arc.

13. (a) Four; two internal, two external
    (b) One internal, two external
    (c) Two external only
    (d) One external only
    (e) None

14. Since tangents to a circle from an external point are congruent,

$$SN = SP$$
$$NR = RM$$
$$CL = CP$$
$$DL = DM$$

By the addition property of equality and the associative property of numbers,

$(SN + NR) + (CL + DL) = (SP + CP) + (RM + DM)$.

By the Betweenness-Distance Theorem, $SN + NR = SR$, $CL + DL = CD$, $SP + CP = SC$, $RM + DM = RD$. It follows from the substitution property of equality that $SR + CD = SC + RD$.

15.

Statements	Reasons
1. $\overleftrightarrow{AB}$ and $\overleftrightarrow{BD}$ are tangent at A and C , respectively.	1. Hypothesis
2. $\triangle AOB$ and $\triangle COB$ are right triangles.	2. Corollary 12-4-1
3. $m\angle ABO = m\angle CBO = 60$	3. $m\angle ABC = 120$, and Theorem 12-13
4. $AB = \frac{1}{2}OB$ $CB = \frac{1}{2}OB$	4. 30-60 right triangle theorem
5. $AB + CB = OB$	5. Addition property of equality

16. Draw $\overline{QR} \perp \overline{AP}$. In $\triangle PQR$, $RQ = \sqrt{(PQ)^2 - (PR)^2}$. Hence $RQ = 48$. But $AB = RQ$, since RQBA is a rectangle. Therefore, $AB = 48$.

17. As in the previous problem, draw a perpendicular from the center of the smaller circle to a radius of the larger circle. By the Pythagorean Theorem, the distance between the centers is 39 inches.

18. (a) $\overrightarrow{PA}$ and $\overrightarrow{PA'}$ are both the midray for $\angle CPB$, by Theorem 12-13. Since each angle has one and only one midray, $\overrightarrow{PA} = \overrightarrow{PA'}$.

(b) $m\,\overset{\frown}{B'C'} = 130$. One possible solution follows. $\overline{AC}$ and $\overline{A'C'}$ are, by Corollary 12-4-1, both perpendicular to $\overleftrightarrow{PC}$ and $\overline{AB}$ and $\overline{A'B'}$ are both perpendicular to $\overleftrightarrow{PB}$. Consider quadrilaterals ACPB and A'C'PB' . The sum of the measures of the interior angles in each equals 360. $m\angle ACP = m\angle A'C'P = m\angle ABP = m\angle A'B'P = 90$.

By applying the addition property and substitution property of equality, $m\angle CAB + m\angle P = 180$ and $m\angle C'A'B' + m\angle P = 180$. Then $m\angle CAB = m\angle C'A'B'$ since these angles are supplements of the same angle. But $m\widehat{CB} = m\angle CAB$ and $m\widehat{C'B'} = m\angle C'A'B'$ by definition of degree measure of a minor arc. Therefore $m\widehat{C'B'} = m\widehat{CB} = 130$ by the substitution property of equality.

19. If m is the length of the shortest of the four segments, the rest of its chord would have to be the longest of the segments. Otherwise the product of the segments of this chord would certainly be less than the product of the segments of the other. Hence, if it were possible to have consecutive integers for the lengths they would be labeled as shown. But in this case, by Theorem 12-16, it would be necessary that:

$$m(m + 3) = (m + 1)(m + 2)$$

or $$m^2 + 3m = m^2 + 3m + 2$$

or $$0 = 2.$$

Since this is impossible, the lengths of the segments cannot be consecutive integers.

20. (a) $\{P\} = \{(x,y):\ y = 5\ ,\ x - y = 12\}$
$= \{(17,5)\}\ ,\ P = (17,5)$.

(b) $\{T\} = \{(x,y):\ (x - 1)^2 + (y + 3)^2 = 64,\ y = 5\}$
$= \{(x,y):\ (x - 1)^2 + 8^2 = 64\ ,\ y = 5\}$
$= \{(x,y):\ (x - 1)^2 = 0\ ,\ y = 5\}$
$= \{(1,5)\}\ ;\ T = (1,5)$.

(c) $\{R,S\} = \{(x,y): (x - 1)^2 + (y + 3)^2 = 64 ,$
$x - y = 12\}$
$= \{(x,y): (x - 1)^2 + [(x - 12) + 3]^2 = 64,$
$y = x - 12\}$
$= \{(x,y): x^2 - 2x + 1 + x^2 - 18x + 81 = 64,$
$y = x - 12\}$
$= \{(x,y): 2x^2 - 20x + 18 = 0 , y = x - 12\}$
$= \{(x,y): (x - 9)(x - 1) = 0 , y = x - 12\}$
$= \{(9,-3),(1,-11)\}$

(d) $PT = \sqrt{(17 - 1)^2 + (5 - 5)^2} = 16$; $PT^2 = 256$.

(e) Let $R = (9,-3)$, $T = (1,-11)$;
Then $PR = \sqrt{(17 - 9)^2 + (5 + 3)^2} = \sqrt{128}$,
and $PS = \sqrt{(17 - 1)^2 + (5 + 11)^2} = \sqrt{512}$.
$PR \cdot PS = \sqrt{128} \cdot \sqrt{512} = \sqrt{16 \cdot 8 \cdot 16 \cdot 16 \cdot 2}$
$= 256$

(f) Theorem 12-15 asserted the equality here verified.

21. Consider radii $\overline{RA}$ and $\overline{QB}$. Let $\overline{AB}$ intersect $\overline{RQ}$ at P . $m\angle A = m\angle B = 90$, and $m\angle APR = m\angle BPQ$ by vertical angles. Therefore, $\triangle APR \sim \triangle BPQ$ by A.A. This gives $\frac{RP}{QP} = \frac{RA}{QB}$. Now suppose $\overline{BC}$ meets $\overline{RQ}$ at point P' . Then, by a similar argument we arrive at $\frac{RP'}{QP'} = \frac{RA}{QB}$. Hence $\frac{RP'}{QP'} = \frac{RP}{QP}$, and P and P' are both between R and Q . Therefore, $P' = P$.

A direct method could show that the point of intersection of $\overline{AB}$ and $\overline{CD}$, along with R , determines the perpendicular bisector of $\overline{AC}$. It can then be shown that Q lies on this bisector.

22. Let d be the required distance. By Theorem 12-15,

$$d^2 = \frac{h}{5280}(8000 + \frac{h}{5280}) .$$

$$d^2 = \frac{50}{33}h + (\frac{h}{5280})^2 .$$

Now since h is very small compared to 5280, $(\frac{h}{5280})^2$ is exceedingly small, and is not significant. So approximately, $d = \sqrt{1.515h} = 1.23\sqrt{h}$. Hence d is roughly $\frac{5}{4}\sqrt{h}$.

Problem Set 12-6

1. $\frac{22}{7}$ is closer.

$$\frac{22}{7} = 3.1429-$$

$$\pi = 3.1416 \text{ (accurate to four decimal places)}$$

$$3.14 = 3.1400$$

2. (a) $d = 14$, $C = 14\pi$

   (b) $d = \frac{36}{\pi}$, $r = \frac{18}{\pi}$

   (c) $C = 15\pi$, $r = 7.5$

   (d) $C = 12\pi a$, $d = 12a$

   (e) $C = 2\pi x\sqrt{3}$
   $d = 2x\sqrt{3}$

3. (a) $C_1 = 3 \cdot C_2$

   (b) $d_2 = 5 \cdot d_1$

   (c) $C_1 = 2 \cdot C_2$

   (d) $C_2 = 2 \cdot C_1$

4. (a) $m\ \widehat{BA}$ (in degrees) = $m\ \widehat{B'A'}$ (in degrees).

   (b) length of $\widehat{BA} = 2 \cdot$ length of $\widehat{B'A'}$

   (c) f

5. (a) $\frac{C_t}{C_u} = \frac{6}{4} = \frac{3}{2}$

   (b) $\frac{3}{2}$

6. $C = 2\pi R = 480{,}000\pi$. The circumference is approximately 1,500,000 miles.

7. The formula gives:

$$2\pi r = 6.28 \times 93 \cdot 10^6$$

$$2\pi r = 584 \cdot 10^6$$

which is 584 million miles (approx.).
Our speed is about 67,000 miles per hour.

8. $c = 2\pi$; $628 = 6.28r$; $r = 100$ yd. (approx.)

9. (a) The radius of the circle
   (b) 0
   (c) 180
   (d) The circumference of the circle

10.

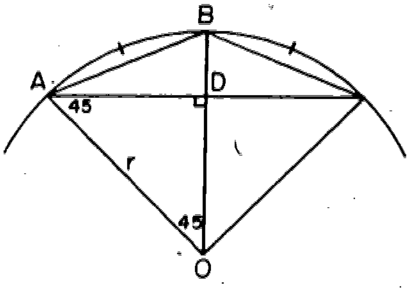
3	120	30	60
4	90	45	90
5	72	54	108
6	60	60	120
8	45	$67\frac{1}{2}$	135
9	40	70	140
10	36	72	144
12	30	75	150
15	24	78	156
18	20	80	160
20	18	81	162
24	15	$82\frac{1}{2}$	165

11. 2 units ; $\sqrt{3}$

12. Radius of the inscribed circle is 6 , so its circumference is 12π . The radius of the circumscribed circle is $6\sqrt{2}$, so its circumference is $12\pi\sqrt{2}$.

13. The increase in circumference is 2π in each case.

14. In the figure, side $\overline{AB}$ of a regular inscribed octagon is 1 unit long. Since $\triangle ADO$ is a right isosceles triangle,



$AD = DO = \dfrac{r}{\sqrt{2}}$.

$BD = r - \dfrac{r}{\sqrt{2}}$. In right triangle ABD ; $(AD)^2 + (DB)^2 = (AB)^2$ or

$$\left(\frac{r}{\sqrt{2}}\right)^2 + \left(r - \frac{r}{\sqrt{2}}\right)^2 = 1 \text{ , from which } r = \sqrt{\frac{1}{2 - \sqrt{2}}}$$

15. The perimeter of PQRS is greater than the circumference of the circle. $AD = 2$ and $XW = \sqrt{2}$.

Hence, $PS = \frac{1}{2}(2 + \sqrt{2})$.

The perimeter of the square is $\pi(2 + 2\sqrt{2})$.

The circumference of the circle is 2π .

But $2 + \sqrt{2} > 2$.

Problem Set 12-7

1\. (a) $C = 2\pi r$

$C = 10\pi$

$C = 31.4$

$A = \pi r^2$

$A = (3.14)25$

$A = 78.5$

(b) $C = 2\pi r$

$C = 20\pi$

$C = 62.8$

$A = \pi r^2$

$A = 314$

(c) $C = 2\pi(2.5)$

$C = 15.70$

$A = \pi r^2$

$A = (3.14)(6.25)$

$A = 19.63$

(d) $C = 2\pi\sqrt{3}$

$C = 2(3.14)(1.732)$

$C = 10.8$

$A = \pi r^2$

$A = (3.14)(3)$

$A = 9.4$

2\. (a) $C = 2\pi r$

$C = 2\pi$

$A = \pi r^2$

$A = \pi$

(b) $C = 2\pi r$

$C = 4\pi$

$A = \pi r^2$

$A = 4\pi$

(c) $C = 2\pi r$

$C = 14\pi$

$A = \pi r^2$

$A = 49\pi$

(d) $C = 2\pi r$

$C = 2\pi\sqrt{5}$

$A = \pi r^2$

$A = 5\pi$

3\. (a) $r = 5$

$C = 10\pi$

(b) $r = 7$

$C = 14\pi$.

(c) $\frac{Cr}{2} = 25$

$C = 10\sqrt{\pi}$

(d) $\frac{Cr}{2} = 49$

$C = 14\sqrt{\pi}$

4\. (a) $A = 36\pi$

(b) $A = 100\pi$

(c) $r = \frac{6}{\pi}$

$A = \frac{Cr}{2} = \frac{36}{\pi}$

(d) $r = \frac{10}{\pi}$

$A = \frac{Cr}{2} = \frac{100}{\pi}$

5. (a) $A = \pi R^2 - \pi r^2$

$A = 3\pi$ or approx. 9.4 sq. cm.

(b) No.

6. A = area of first. A' = area of second.

$\frac{A}{A'} = \frac{9}{1}$

7. The circle is larger.

$\pi r^2 - 5^2$

$\pi(\frac{10}{\pi})^2 - 25$

31.8 - 25 = 6.8 square inches greater.

8. $R^2\pi - r^2\pi$

$\pi[(5\sqrt{2})^2 - 5^2]$

25π square inches

9. $4\sqrt{3}$ inches, $8\pi\sqrt{3}$, 48π square inches

10. It is only necessary to find the square of the radius of the circle. If a radius is drawn to a vertex of the cross, it is seen to be the hypotenuse of a right triangle of sides 2 and 6. The square of the radius is therefore $2^2 + 6^2 = 40$. The area of the circle is therefore 40π, 125.6 approximately. The required area is therefore 125.6 - 80 = 45.6.

11. Consider $\overline{PB}$ and $\overline{PC}$. The area of the annulus is $\pi(PC)^2 - \pi(PB)^2$, the difference of the areas of the two circles. This can also be written $\pi[(PC)^2 - (PB)^2]$. By Pythagorean Theorem, $(PC)^2 - (PB)^2 = (BC)^2$. Therefore, the area of the annulus is $\pi(BC)^2$.

12. The section nearer the center of the sphere will be the larger.

$$r_1^2 = 10^2 - 5^2$$

$$r^2 = 10^2 - 3^2$$

Therefore, $r_1 > r$.

13. $\frac{s^2}{2}$

14. $(AC)^2 + (BC)^2 = (AB)^2$.

$$\frac{\pi}{8}(AC)^2 + \frac{\pi}{8}(BC)^2 = \frac{\pi}{8}(AB)^2$$

$$(r + g) + (h + s) = g + h + t$$

$$r + s = t .$$

15. (a) Note that $r_1 = OA = OR = BP$ and $r_2 = OS = CP$. By successive use of the Pythagorean Theorem we get $r_1 = r\sqrt{2}$, $r_2 = r\sqrt{3}$, $r_3 = r\sqrt{4}$.

(b) Now, using the area formula for a circle, we have

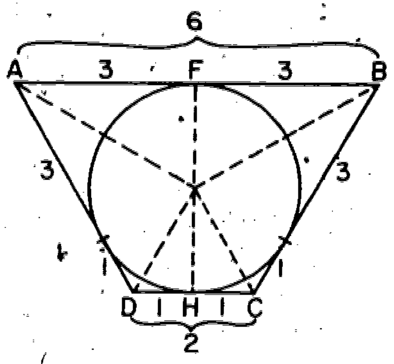
$a = \pi r^2$;

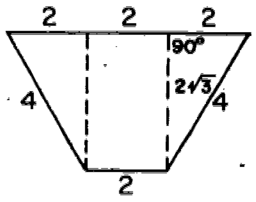
$b = \pi(r\sqrt{2})^2 - a = \pi r^2$;

$c = \pi(r\sqrt{3})^2 - (a + b) = 3\pi r^2 - 2\pi r^2 = \pi r^2$;

$d = \pi(2r)^2 - (a + b + c) = 4\pi r^2 - 3\pi r^2 = \pi r^2$.

16. From the second figure, $4^2 - 2^2 = 12$, so the altitude of the trapezoid is $2\sqrt{3}$. In the first figure, since the bases are parallel and tangent to the circle, we see that $\overline{FH}$ (altitude of the trapezoid) must be a diameter; thus the radius is $\sqrt{3}$. The area of the circle is, then, 3π. The area of the trapezoid is $8\sqrt{3}$. The area outside the circle is $(8\sqrt{3} - 3\pi)$ square inches. This is approximately 4 square inches.





Problem Set 12-8

1. The length of $\widehat{CD}$ is greater than the length of $\widehat{EF}$.
2. The arc of one inch on a half dollar.
3. Yes, to both questions.
4. No. $\ell \widehat{AB} = \frac{\pi r}{180}\, m\, \widehat{AB}$

   $\ell \widehat{CD} = \frac{\pi r'}{180} \cdot m\, \widehat{CD}$

   If $\ell \widehat{AB} = \ell \widehat{CD}$ then

   $r \cdot m\, \widehat{AB} = r' \cdot m\, \widehat{CD}$

   If $m\, \widehat{AB} > m\, \widehat{CD}$, then $r < r'$
5. 5π , 7.5π , 6π , 3π
6. 9π , $\pi/10$, 6π , 5.4π

7. 9π

8. 2π

9. $r = \frac{3}{\pi}$. Chord $= \frac{3}{\pi}$ or .96 cm.

10. 90

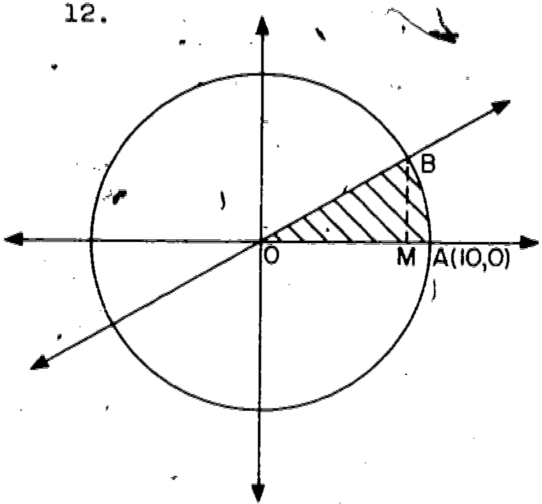
11. (a) The intersection of S and P is $\{(x,y,z): x^2 + y^2 = 16 , z = 3\}$, a circle in plane $z = 3$, with $(0,0,3)$ as center and 4 as radius.

(b) The radius, r_1 , of the circle of intersection is 4. The radius, r, of the great circle of S is 5. If C_1 and C denote the circumferences of these circles, then $C_1 = \frac{r_1}{r} \cdot C = \frac{4}{5}C$.

(c) Area of circle of intersection $= \frac{4^2}{5^2}$ times the area of the great circle.

(d) These arcs will be $\frac{1}{4}$ of the circumferences in each case so the arc of the circle of intersection is $\frac{4}{5}$ of the arc of the great circle of S .

12.



Since $y = x$, $m \angle BOA = 45$

$m \widehat{AB} = 45$

$r = 10$

Area of the sector

$= \frac{45}{360} \cdot \pi r^2$

$= \frac{1}{8} \cdot 100\pi$

Area $= 12.5\pi$

13. (a) Area of sector $= \frac{1}{6}\pi \cdot 12^2 = 24\pi$

Area of triangle $= \frac{12^2}{4}\sqrt{3} = 36\sqrt{3}$

Area of segment $= 24\pi - 36\sqrt{3}$ or 13.04

(b) Area of sector $= \frac{1}{3}\pi \cdot 6^2 = 12\pi$

Area of triangle $= \frac{1}{2} \cdot 6\sqrt{3} \cdot 3 = 9\sqrt{3}$

Area of segment $= 12\pi - 9\sqrt{3}$ or 22.11

(c) Area of sector $= \frac{1}{8}\pi \cdot 8^2 = 8\pi$

Area of triangle $= \frac{1}{2} \cdot 8 \cdot 4\sqrt{2} = 16\sqrt{2}$

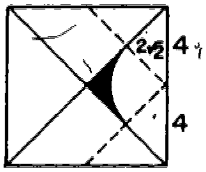
Area of segment $= 8\pi - 16\sqrt{2}$ or 2.51

14. (a) 2π

(b) π

15. Draw $\overline{BG} \perp \overline{AC}$. Then $GC = 6$, $AG = 24$. In the right $\triangle AGB$, the length of the hypotenuse is twice the length of one leg, so $m\angle ABG = 30$, $m\angle BAG = 60$, and $CE = GB = 24\sqrt{3}$. The major arc $\widehat{CD}$ has the length $\frac{2}{3}(2\pi \cdot 30) = 40\pi$ and the minor arc $\widehat{EF}$ has the length $\frac{1}{3}(2\pi \cdot 6) = 4\pi$. Thus, the total length of the belt is $2(24\sqrt{3}) + 40\pi + 4\pi = 48\sqrt{3} + 44\pi$. The belt is approximately 221 inches long.

16. To find one small shaded area, subtract the area of a 90° sector whose radius is $2\sqrt{2}$ from the area of a square whose side is $2\sqrt{2}$.



$$(2\sqrt{2})^2 - \frac{\pi(2\sqrt{2})^2}{4} = 8 - 2\pi .$$

The area of the given shaded region is $4(8 - 2\pi)$. This is approximately 6.87 square inches.

Problem Set 12-9

1. The center of the circumscribed circle about a given acute triangle is in the interior; about a right triangle it is the midpoint of the hypotenuse; about an obtuse triangle it is in the exterior of the triangle and in the interior of the obtuse angle.

   Proof: If $\angle C$ is a right angle $\angle BCA$ intercepts a semicircle of which $\overline{BC}$ is a diameter.

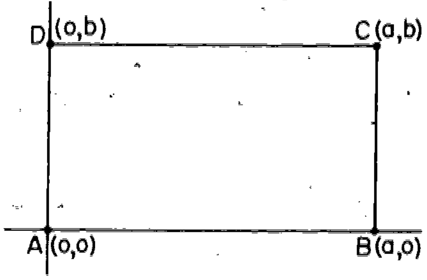
   If $\angle C$ is an obtuse angle $\angle BCA$ intercepts a major arc and hence is inscribed in a minor arc. Then C and O , the center, must be on opposite sides of $\overline{AB}$.

   If $\angle C$ is an acute angle then it intercepts a minor arc and hence is inscribed in a major arc. Then C and O are on the same side of a diameter.

2. Yes. The midray of each angle is in the interior of its angle (except for the vertex of the angle). Therefore, the intersection must be in the interior of each angle, hence in the interior of the triangle.

3. The median is a radius of the circle and hence its length is 12 .

4. Let a coordinate system be established with vertices of rectangle ABCD as shown. Let O(x,y) be the center of a circumscribing circle. Then $x^2 + y^2 = (x - a)^2 + y^2$ $= (x - a)^2 + (y - b)^2$ $= x^2 + (y - b)^2$ yielding $x = \frac{a}{2}$, $y = \frac{b}{2}$. Thus $O(\frac{a}{2},\frac{b}{2})$ exists and a circle can be circumscribed about a given rectangle.

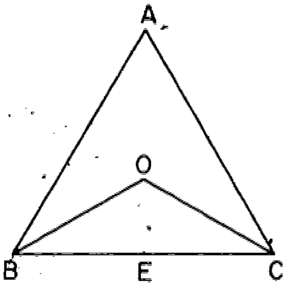


Let $I(x',y')$ be the center of an inscribed circle. Then $x' = y' = a - x' = b - y'$, yielding $a = b$, and demanding that, in order for an inscribed circle to exist, the rectangle must be a square.

5. Yes. The diagonals are bisectors of the angles. Hence, their intersection is equally distant from the sides of the rhombus. No, unless the rhombus is also a square.

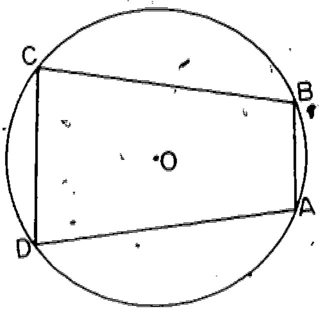
6. By Theorem 5-9 each angle bisector also bisects the opposite side and is perpendicular to it. Therefore the angle bisectors are concurrent in the same point as the perpendicular bisectors.

7. Let O be the common center; E, the midpoint of $\overline{BC}$. Then $\overrightarrow{BO}$ is the midray of $\angle ABE$ and $\overrightarrow{CO}$ is the midray of $\angle ACE$. $\triangle BOE \cong \triangle COE$ by S.A.S. $m\angle OBE = m\angle OCE$, $\angle ABE \cong \angle ACE$. Thus, by extending the argument, we may prove the triangle equiangular and hence equilateral.



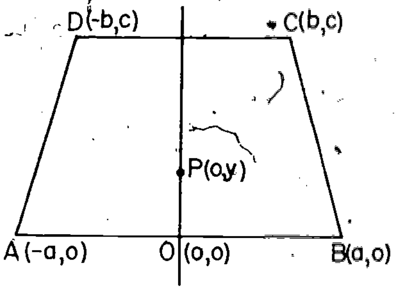
8. If ABCD is the quadrilateral then consider the circle circumscribing $\triangle ABC$.

If D is not on the circle, then $m\angle D \neq \frac{1}{2}(m\ \widehat{ABC})$ and $\angle D$ is not supplementary to $\angle B$. Hence, if $\angle B$ and $\angle D$ are supplementary angles, then D is on the circle through A, B, C and the quadrilateral has a circumscribed circle.



300

9. Let a coordinate system be established with origin at the midpoint of a base of isosceles trapezoid ABCD as shown.



Let P(0,y) be such that PA = PB = PC = PD . Then $a^2 + y^2 = (-a)^2 + y^2$

$= (-b)^2 + (y - c)^2$

$= b^2 + (y - c)^2$, yielding

$y = \frac{b^2 + c^2 - a^2}{2c}, (c \neq 0)$.

Thus $P(0, \frac{b^2 + c^2 - a^2}{2c})$ exists and consequently a circle can be circumscribed about a given isosceles trapezoid.

It is not true, however, that every isosceles trapezoid has an inscribed circle.

10. As the figure indicates,

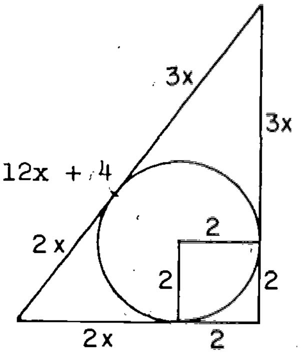
$$(5x)^2 = (2x + 2)^2 + (3x + 2)^2$$

$$25x^2 = 4x^2 + 8x + 4 + 9x^2 + 12x + 4$$

$$3x^2 - 5x - 2 = (3x + 1)(x - 2) = 0$$

$$x = 2 ;$$

length of hypotenuse = 10 .



11. (a) Midpoint of $\overline{XY} = (4,0)$; slope of $\overline{XY}$ is zero.
Perpendicular bisector of $\overline{XY} = \{(x,y): x = 4\}$.

(b) Midpoint of $\overline{XZ} = (\frac{5}{2},\frac{3}{2})$; slope of $\overline{XZ}$ is $\frac{3}{5}$.

Perpendicular bisector of $\overline{XZ}$ =

$= \{(x,y): y - \frac{3}{2} = - \frac{5}{3}(x - \frac{5}{2})\}$

$= \{(x,y): 3y + 5x = 17\}$

(c) $\{C\} = \{(x,y): x = 4 \ , \ y = -\frac{5x}{3} + \frac{17}{3}\}$ or
$C = (4,-1)$.

(d) Midpoint of $\overline{YZ} = (\frac{13}{2},\frac{3}{2})$, slope of $\overline{YZ}$ is -1 .

Perpendicular bisector of $\overline{YZ}$

$= \{(x,y): y - \frac{3}{2} = 1(x - \frac{13}{2})\}$

$= \{(x,y): y = x - 5\}$.

The coordinates of $C(4,-1)$ satisfy this equation. Therefore, the perpendicular bisectors are concurrent at $(4,-1)$.

(e) $CX = \sqrt{(4 - 0)^2 + (-1 - 0)^2} = \sqrt{17}$

$CY = \sqrt{(4 - 8)^2 + (-1 - 0)^2} = \sqrt{17}$

$CZ = \sqrt{(4 - 5)^2 + (-1 - 3)^2} = \sqrt{1 + 16} = \sqrt{17}$

(f) Circle $C = \{(x,y): (x - 4)^2 + (y + 1)^2 = 17\}$

$= \{(x,y): x^2 + y^2 - 8x + 2y = 0\}$

12. (a) Median to $\overline{XY}$ is in $\{(x,y): y = 3x - 12\}$

(b) Median to $\overline{XZ}$ is in $\{(x,y): y = -\frac{3x}{11} + \frac{24}{11}\}$

(c) Median to $\overline{YZ}$ is in $\{(x,y): y = \frac{3x}{13}\}$

(d) Median to $\overline{XY} = \{(x,y): x = 5 - k \ , \ y = 3 - 3k \ , \ 0 \leq k \leq 1\}$

Median to $\overline{XZ} = \{(x,y): x = 8 - \frac{11}{2}k \ , \ y = \frac{3}{2}k \ , \ 0 \leq k \leq 1\}$

Median to $\overline{YZ} = \{(x,y): x = \frac{13}{2}k \ , \ y = \frac{3}{2}k \ , \ 0 \leq k \leq 1\}$

$k = \frac{2}{3}$ yields trisection point $(\frac{13}{3},1)$ in median to $\overline{XY}$.

$k = \frac{2}{3}$ yields trisection point $(\frac{13}{3},1)$ in median to $\overline{XZ}$.

$k = \frac{2}{3}$ yields trisection point $(\frac{13}{3},1)$ in median to $\overline{YZ}$.

13. (a) Altitude to $\overline{XY}$ is a subset of $\{(x,y): x = 5\}$

(b) The altitude to $\overline{XZ}$ is a subset of the line $\{(x,y): y = \frac{-5}{3}(x - 8)\} = \{(x,y): y = -\frac{5}{3}x + \frac{40}{3}\}$. The altitude to $\overline{ZY}$ is a subset of the line $\{(x,y): y = x\}$.

(c) The orthocenter is $(5,5)$. The orthocenter is in the exterior of the triangle.

(d) The problem is to show that $(4,-1)$, $(\frac{13}{3},1)$ and $(5,5)$ are collinear. An equation for the line containing $(4,-1)$ and $(5,5)$ is $y = 6x - 25$. This equation is satisfied by $(\frac{13}{3},1)$.

Chapter 12

Review Problems

Sections 1 through 5

1. (a) circle, 10 , (0,0)

   (b) A on the circle; B interior; C exterior

   (c) $L_1 \cap C = \{(x,y): x^2 + y^2 = 100 , x = -10\}$

   $= \{(x,y): y = 0 , x = -10\}$

   $= \{(-10,0)\}$.

   (d) $L_2 \cap C = \{(x,y): x^2 + y^2 = 100 , y = 6\}$

   $= \{(x,y): x^2 = 64 , y = 6\}$

   $= \{(8,6) , (-8,6)\}$.

   (e) $L_3 \cap C = \{(x,y): x^2 + y^2 = 100 , y = \frac{4}{3}x\}$

   $= \{(x,y): x^2 + \frac{16x^2}{9} = 100 , y = \frac{4}{3}x\}$

   $= \{(x,y): x^2 = 36 , y = \frac{4}{3}x\}$

   $= \{(6,8) , (-6,-8)\}$.

2. (a) $S = \{(x,y,z): x^2 + y^2 + z^2 = 100\}$

   (b) (1) (10,0,0) , (-10,0,0)
   (2) (0,10,0) , (0,-10,0)
   (3) (0,0,10) , (0,0,-10)

   (c) $\{(x,y,z): x^2 + y^2 = 100 , z = 0\}$

   (d) $\{(x,y,z): x^2 + z^2 = 100 , y = 0\}$

   (e) $\{(x,y,z): y^2 + z^2 = 100 , x = 0\}$

   (f) A is in S since $3^2 + (-4)^2 + (5\sqrt{3})^2 = 100$.

   B is in the interior of S since

   $3^2 + (-5)^2 + 7^2 = 83 < 100$.

   C is in the exterior of S since

   $9^2 + 6^2 + 1^2 = 118 > 100$.

3. (~~a~~) $(x - 3)^2 + (y + 2)^2 = 16$ or

   $\{(x,y,z): x^2 + y^2 - 6x + 4y - 3 = 0 , z = 0\}$

   (b) $(x - 2)^2 + (y + 1)^2 + (z - 3)^2 = 9$ or

   $\{(x,y,z): x^2 + y^2 + z^2 - 4x + 2y - 6z + 5 = 0\}$

4. (a) If they are:

   (1) radii of the same or congruent circles.
   (2) diameters of the same or congruent circles.
   (3) in the same circle and are associated with congruent arcs.
   (4) tangent-segments from the same exterior point.
   (5) Chords in the same or congruent circles and equidistant from the center.
   (6) the parts into which a diameter perpendicular to a chord separates the chord.

   (b) If it is:

   (1) inscribed in a semicircle.
   (2) determined by a radius and the tangent at its outer end.
   (3) determined by a chord and the diameter which bisects it.

   (c) If they are:

   (1) inscribed in congruent arcs.
   (2) intercept congruent arcs.
   (3) the angles between two tangent-segments from the same exterior point and the line which contains that point and the center of the circle.
   (4) central angles associated with arcs which have the same degree measure.

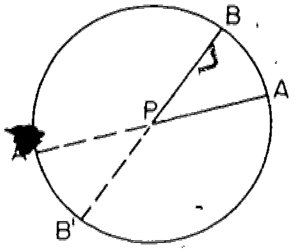
   (d) If they are:

   (1) associated with congruent chords in congruent circles.
   (2) intercepted by congruent inscribed angles in congruent circles.

(3) associated with congruent central angles.
(4) both semicircles.
(5) the parts into which a diameter perpendicular to the chord associated with an arc separates the arc.

5. (a) The degree measure of the arc in which an angle is inscribed is 360 minus the degree measure of the arc which it intercepts.

(b)



Consider $\angle APB$ as a central angle. Then $m\angle APB = m\widehat{AB}$ by definition. Consider $\angle APB$ as formed by chords $\overline{AA'}$ and $\overline{BB'}$ intersecting at the center P. Then $m\angle APB = \frac{1}{2}(m\widehat{AB} + m\widehat{A'B'})$. But $m\widehat{AB} = m\widehat{A'B'}$. Therefore $m\angle APB = m\ AB$ is a special case of the theorem referred to in the problem.

6. (a) chord
(b) diameter (also chord)
(c) secant
(d) radius
(e) tangent
(f) minor arc
(g) major arc
(h) inscribed angle
(i) central angle

7. 55 and 70

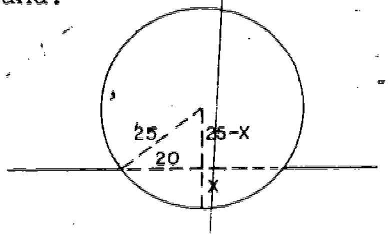
8. $m\angle AXB = 90$, because it is inscribed in a semicircle. Therefore, $m\angle AXY = 45$ and $m\widehat{AY} = 90$ since $\angle AXY$ is inscribed in $\widehat{AXY}$. Hence the measure of central angle $\angle ACY$ is 90 making $\overline{CY} \perp \overline{AB}$.

9. (a) True
(b) True
(c) False
(d) True
(e) False
(f) True
(g) False
(h) True
(i) True
(j) True

10. $m\angle C = 65$; $m\angle ABX = 65$; $m\angle CBA = 65$.

11. AY = AP and AX = AP , because tangent-segments to a circle from an external point are congruent. Therefore, AY = AX .

12. The figure shows a cross-section with x the depth to be found.



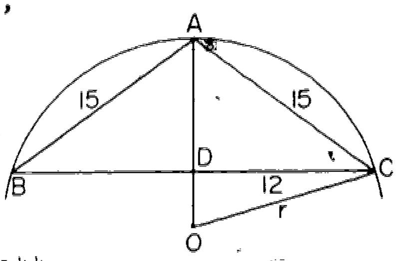
$25^2 = 20^2 + (25 - x)^2$

$225 = (25 - x)^2$

$15 = 25 - x$

$x = 10$. The depth is 10 inches.

13. By the Pythagorean Theorem, AD = 9 . If r is the radius, then OD = r - 9 and OC = r . Hence, in $\triangle DOC$,



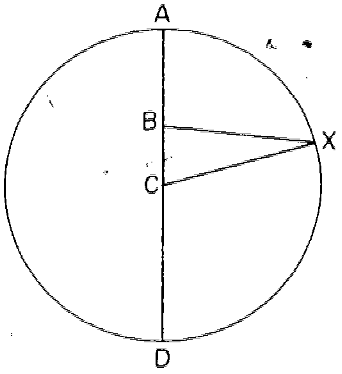
$r^2 = (r - 9)^2 + 12^2$,

$r^2 = r^2 - 18r + 81 + 144$,

$r = 12.5$

The diameter of the wheel is 25 inches long.

14. Consider the distance BX to any other point X on the circle, and the radius CX .



BC + AB = AC = CX .

BC + BX > CX . Hence,

BC + BX > BC + AB and

BX > AB or AB < BX .

Also BX < BC + CX ,

or BX < BC + CD .

Since BC + CD = BD ,

BD > BX .

Thus AB < BX < BD where BX is any other segment joining B to the circle.

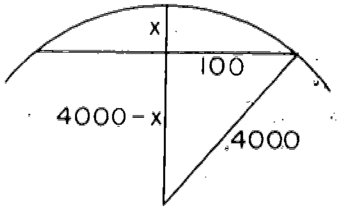
15. Let $m\ \widehat{HE} = r$. Then $m\ \angle PCH = 90 - r$, $m\ \angle NHC = 180 - (90 - r)$ or $90 + r$. Then $m\ \angle NHR = m\ \angle NHC - 90 = (90 + r) - 90 = r$. Hence, $m\ \widehat{HE} = m\ \angle NHR$.

16. $(4000)^2 = (100)^2 + (4000 - x)^2$

$(4000 - x)^2 = 15{,}990{,}000$.

$4000 - x = 3{,}998.75$.

$x \doteq 1.25$, approx.



The shaft will be about $1\frac{1}{4}$ miles deep.

17. (a) T is the exterior of the circle in the xz-plane with its center at (0,0) and with radius 2 .

(b) M is a circle in the xy-plane with its center at (2,-4) and with radius 7 .

(c) N is the interior of a circle in the yz-plane with its center at (0,0) and with its radius equal to 3 .

(d) R is the intersection of a sphere with its center at (0,0,0) and with its radius equal to 5 and a plane parallel to the xy-plane and intersecting the z-axis at (0,0,3) . This is the circle $R = \{(x,y,z)\colon x^2 + y^2 = 16$, $z = 3\}$ which has its center at (0,0,3) , has a radius equal to 4 and lies in the plane $\{(x,y,z)\colon z = 3\}$.

(e) Two points, (1,0,0) and (-1,0,0) .

(f) The intersection is the empty set since D and F are two concentric spheres with radii 4 and $2\sqrt{2}$ respectively.

(g) The intersection is $\{(x,y,z)\colon x^2 + y^2 = 9$, $|z| = 4\}$. U is a cylinder with its axis the z-axis and with its cross section a circle with center in the z-axis and radius 3 . U intersects T in two circles, one in the plane parallel to the xy-plane and 4 units above it, the other parallel to the xy-plane and 4 units below it.

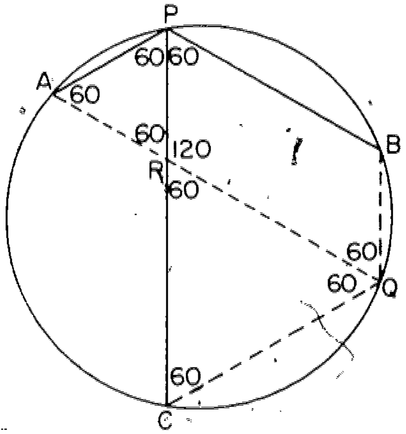
18. $(AP)^2 = 1(8 + 1) = 9$, by Theorem 12-15.

$AP = PX = XY = 3$, so $QX = 2$ and $XZ = 6$.

$3 \cdot AX = 2 \cdot 6$, by Theorem 12-16.

$AX = 4$.

19. The angle measures can be determined as shown. Hence, $\triangle PAR$ and $\triangle QCR$ are equilateral triangles and PRQB is a parallelogram.



$PC = PR + RC = AR + RQ$.
But $AR = AP$ and
$RQ = PB$. Hence,
$PC = AP + PB$.

20. Applying Theorem 12-15, we have $(AM)^2 = MR \cdot MS$ and $(MB)^2 = MR \cdot MS$. Hence, $(AM)^2 = (MB)^2$ and $AM = MB$. Similarly $CN = ND$.

Chapter 12

Review Problems

1. 2π

2. (a) The area of a circle is the limit of the areas of the inscribed (or circumscribed) regular polygons as the number of sides of the polygons increases indefinitely. [The exact wording of the text may be used.]

   (b) The length of an arc $\widehat{AB}$ of a circle is the limit of $AP_1 + P_1P_2 + \ldots + P_{n-1}B$ as the number of chords increases indefinitely. [The exact wording of the text may be used.]

3. Between 1 and 2

4. 5

5. 2

6. (a) 10 to 1 ; (b) 10 to 1 ; (c) 100 to 1

7. 5 ; $\frac{10\pi}{6}$ or $\frac{5\pi}{3}$

8. $A = \pi r^2 = \pi(\frac{d}{2})^2 = \frac{\pi d^2}{4}$.

9. The inscribed octagon has the greater apothem and the greater perimeter. The circumscribed square has the greater perimeter; the apothems are equal.

10. $A = \frac{1}{2}ap$

11. $\frac{60}{\pi}$

12. Area = 4π sq. in.; arc length = $\frac{2\pi}{3}$ inches

13. There are many acceptable proofs. One is to consider the situation wherein the vertices of the inscribed triangle are the midpoints of the circumscribed triangle, and prove the four smaller triangles congruent.

14. $m\ \widehat{DA} = 88$ and $m\ \widehat{BC} = 122$

$m\ \angle EDC = m\ \angle DBC = 31$

$m\ \angle CMD = m\ \angle AMB = m\ \angle ABC = 75$

$m\ \angle DMA = m\ \angle CMB = 105$

$m\ \angle FDB = m\ \angle DCB = 88$

$m\ \angle ACB = m\ \angle ACB = m\ \angle DBA = 44$

$m\ \angle CAB = m\ \angle CDB = 61$

$m\ \angle DCE = m\ \angle BDE = 92$

$m\ \angle DEC = 57$

$m\ \angle DFA = 48$

$m\ \angle CAF = 119$

$m\ \angle CDF = 149$

$m\ \angle ACE = 136$

15. Draw $\overline{QE} \perp \overleftrightarrow{PA}$. Since $PQ = 20$ and $PE = 7 + 9 = 16$, then $QE = 12 = AB$.

16. (a) By Corollary 12-7-2,

$m\ \angle ADP = m\ \angle BCP$ and

$m\ \angle DAP = m\ \angle CBP$. Hence

$\triangle APD \sim \triangle BPC$ by A.A.

(b) Since similar triangles have corresponding sides proportional, $AP \cdot PC = PD \cdot PB$.

17. (a) Yes. The slope of $\overline{AB}$ is -1.

(b) The midpoint is $(2,2)$.

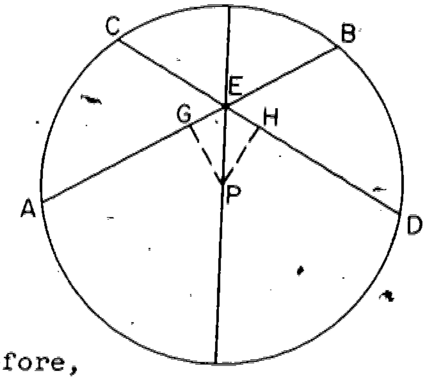
(c) $y = x$

(d) The origin is contained in $y = x$. This illustrates Corollary 12-4-3: In the plane of a circle, the perpendicular bisector of a chord contains the center of the circle.

(e) The points with coordinates $(2\sqrt{2}, 2\sqrt{2})$, and $(-2\sqrt{2}, -2\sqrt{2})$; midpoint.

18. By hypothesis, P in the figure is the center of the circle, and $m\angle AEP = m\angle DEP$.

Prove: $\overline{AB} \cong \overline{CD}$.



Consider $\overline{PG} \perp \overline{AB}$ and $\overline{PH} \perp \overline{CD}$. Then $\triangle PGE$ and $\triangle PHE$ are right triangles with $m\angle GEP = m\angle HEP$ and $EP = EP$. Therefore, $\triangle PGE \cong \triangle PHE$, making $PG = PH$. Therefore, $\overline{AB} \cong \overline{CD}$, because in the same circle or congruent circles, chords equidistant from the center are congruent.

Review Problems

Chapters 10 - 12

1. +
2. 0
3. +
4. 0
5. +
6. 0
7. 0
8. +
9. +
10. +
11. 0
12. 0
13. 0
14. 0
15. +
16. +
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89. +
90. 0
91. +
92. 0
93. 0
94. +
95. 0
96. +
97. +
98. +
99. +
100. +

The preliminary edition of this volume was prepared at a writing session held at Yale University during the summer of 1961. This revised edition was prepared at Stanford University in the summer of 1962, taking into account the classroom experience with the preliminary edition during the academic year 1961-62.

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