DOCUMENT RESUME

ED 128 751 95 CG 010 864

AUTHOR Kalechofsky, Robert
TITLE An Investigation of the Mathematical Models of Piaget's Psychological Theory of Cognitive Learning. Final Report.
SPONS AGENCY Office of Education (DHEW), Washington, D.C. Bureau of Research.
BUREAU NO 8-A-057
PUB DATE Jun 71
GRANT OEG-1-9-080057-0103(010)
NOTE 92p.

EDRS PRICE MF-$0.83 HC-$4.67 Plus Postage.
DESCRIPTORS *Cognitive Development; Concept Formation; *Developmental Psychology; *Elementary Secondary Education; Learning Plateaus; *Learning Theories; Mathematical Concepts; Mathematical Logic; *Mathematical Models; Mathematics Instruction; *Psychological Studies; Research Projects; Thought Processes
IDENTIFIERS *Piagetian Theory

ABSTRACT

This research paper proposes several mathematical models which help clarify Piaget's theory of cognition on the concrete and formal operational stages. Some modified lattice models were used for the concrete stage and a combined Boolean Algebra and group theory model was used for the formal stage. The researcher used experiments cited in the literature and demonstrations to determine the appropriateness of the models. With some reservations which are sited in the research, the mathematical models were quite consonant with the cognitive stages they were designed to describe. (Author)

ED128751

AN INVESTIGATION OF THE
MATHEMATICAL MODELS OF PIAGET'S
PSYCHOLOGICAL THEORY OF COGNITIVE
LEARNING

BY Robert Kalechofsky

FINAL REPORT

PROJECT # 8-A-057

GRANT # OEG-1-9-080057-0103 (010)

AN INVESTIGATION OF THE MATHEMATICAL MODELS OF PIAGET'S PSYCHOLOGICAL THEORY OF COGNITIVE LEARNING

Robert Kalechofsky
255 Humphrey Street
Marblehead, Massachusetts
01945

JUNE 1971

U.S. DEPARTMENT OF
HEALTH, EDUCATION, and WELFARE
OFFICE OF EDUCATION
BUREAU OF RESEARCH

FINAL REPORT

PROJECT # 8-A-057

GRANT # OEG-1-9-080057-0103 (010)

AN INVESTIGATION OF THE MATHEMATICAL MODELS OF PIAGET'S PSYCHOLOGICAL THEORY OF COGNITIVE LEARNING

Robert Kalechofsky
Salem State College
Salem, Massachusetts
01970

JUNE 1971

The research reported herein was performed pursuant to a grant with the Office of Education, U.S. Department of Health, Education, and Welfare. Contractors undertaking such projects under Government sponsorship are encouraged to express freely their professional judgment in the conduct of the project. Points of view or opinions stated do not, therefore, necessarily represent official Office of Education position or policy.

U.S. DEPARTMENT OF

HEALTH, EDUCATION, AND WELFARE

Office of Education

Bureau of Research

AN INVESTIGATION OF THE MATHEMATICAL MODELS OF PIAGET'S PSYCHOLOGICAL THEORY OF COGNITIVE LEARNING

Robert Kalechofsky
255 Humphrey Street
Marblehead, Massachusetts
01945
Telephone 631-7601

SUMMARY

The following research paper proposes several mathematical models which help us to clarify and understand Piaget's theory of cognition on the concrete and formal operational stages. Some modified lattice models were used for the concrete stage and a combined Boolean Algebra and group theory model was used for the formal stage. Both experiments cited in the literature and demonstrations carried out by the researcher were used to determine the appropriateness of the models. With some reservations which are cited in the research, the mathematical models were quite consonant with the cognitive stages they were designed to describe.

TABLE OF CONTENTS

CHAPTER I

INTRODUCTION

THE PROBLEM:

The purpose of this research is to develop mathematical models of the concrete and formal stages of cognition as found in Jean Piaget's theory of cognitive learning. The problem is to determine whether Piaget's suggestions concerning a mathematical approach to these two stages of his theory can be amplified and developed into mathematical structures which help to explain his psychological theory. This research will attempt to make explicitly clear the mathematical bases of Piaget's theory. It will, hopefully, make both the strengths and weaknesses of the theory more amenable to investigation, validation, and possible revision.

Sub-Problems:

1. To examine the mathematics used in Piaget's theory in order to clarify it and determine its shortcomings, if any.

2. How can a modified lattice theory model be used to characterize the concrete stage of Piaget's theory?

3. How can a combined group theory and Boolean algebra theory model be used to characterize the formal stage of Piaget's theory?

4. What are the psychological referents of the mathematical models?

5. What theoretical consequences of these models can be found to provide grounds for empirical testing of Piaget's theory and the models of this research?

HYPOTHESES:

The problems posed in this research lend themselves to the following hypotheses:

1. A modified lattice theory model can be constructed to help characterize the concrete stage of Piaget's theory of cognition.

2. A combined group theory and Boolean algebra model can be constructed to help characterize the formal stage of Piaget's theory of cognition.

3. Psychological referents of these models, which will guide their development, can be determined.

4. Theoretical consequences of these models can be found which provide grounds for empirically testing Piaget's theory, if hypotheses one and two are affirmed.

Related Literature:

During the past fifty years Jean Piaget has written hundreds of books and articles which articulate an increasingly detailed theory of the developmental nature of human intelligence. The appendix of this research contains an analysis of his theory. This research is concerned with those aspects of this development which lend themselves to mathematical analysis. An important book in this regard is Piaget's Logic and Psychology. In it can be found an overview of Piaget's theoretical studies based on many years of research. It is involved with a combined mathematical and psychological analysis of the concrete and formal stages of his theory of cognition. The mathematics is, at times, unclear and not fully developed and one of the purposes of this research is to correct such deficiencies.

Inhelder and Piaget's The Growth of Logical Thinking from Childhood to Adolescence (hereafter designated as GLT) contains a host of ingenious experiments designed to probe specific aspects of the concrete and formal stages. It presents empirical verification of the general theory of cognition, using, among others, experiments involving balls bouncing off billiard walls to test the use of compensations and proportions. Some of the empirical verifications for the mathematical models of the present research are found in GLT.

Piaget, Inhelder, and Szeminska's The Child's Conception of Geometry is a study of cognition in relation to geometric phenomena. It involves experiments concerning conservation of length, area, and volume. The section on volume conservation is a classical paradigm of Piaget's theory where subtle connections and differences between the concrete and formal stages are developed.

Piaget's Six Psychological Studies can be considered as a statement of the first principles or the metaphysics of Piaget's theory. Two complementary points of view seem to underly his theory. The first is that of genetic psychology which concerns itself primarily with the nature and origin of the developing cognitive structures with which we understand logical inter-relations. The second is that of genetic epistemology which, originally from a philosophical viewpoint which merges with psychology, inquires into the nature of knowledge. Piaget's view is that what we know depends upon what cognitive structures we have developed and that the questions of epistemology must consider the nature, structure and development of our minds from childhood into adult life. His entire theory yields a structured, empirical view that all knowledge can only be understood by relating it to prior stages (Piaget's stages) of knowledge. Indispensably involved with his theory are the ideas of reversibility, conservation and equilibration. Their roles are manifested in all the essays of the book. Reversibility refers to the ability to understand the opposite of a given state of affairs. It might involve the idea of the complement of a set, reversing the order of a given relation or understanding what something is not. Conservation refers to the grasping of an idea or structure when it is manifested in different forms. An example of this is the grasping of the "less than" relation as it manifests itself in study of numbers and as the subset idea in set theory. A more complex example concerns the idea of balance as manifested in solving weight balancing, light reflection in a mirror, and ricochet problems. The solutions to these problems involves a grasp of the symmetry and balance inherent in each of them. Equilibration refers to the stability of a given cognitive stage. A stage is said to be in equilibrium when the ideas associated with it or conserved in it remain dynamically stable. At the concrete and formal stages equilibrium is attained when the cognitive structures of these stages are in accord with several mathematical structures which will be defined in this research.

Flavell's The Developmental Psychology of Jean Piaget presents a condensed version of Piaget's theory and research up until 1963. It cites many of the experiments validating his theory and an analysis of the criticism of the theory. Its focus is psychological with little emphasis on the philosophical and mathematical ramifications of Piaget's thought. It is a useful book because it gathers together much that has been said by and about Piaget.

Furth's Piaget and Knowledge is an introduction in depth to psychological aspects of Piaget's theory. It focuses on intelligence as a biological function and probes deeply into Piaget's psychological theory. It does not deal very much with the models or the philosophical aspects of Piaget's theory. However, they become clarified indirectly by Furth's analysis.

DEFINITIONS:

The definitions of group and lattice are based on those of Birkhoff & MacLane, A Survey of Modern Algebra, (New York, MacMillan, 1965).

The word "logic" in this research is used in two ways. Mathematically, it refers to the structures, operations, and rules given by mathematical logic. Psychologically, it refers to reasoning which is structured, operational, and characterized by reversibility. The latter interpretation is usually referred to. (Flavell, 1963, pp 4, 68, 69, 253-255)

Group:

A collection of elements a,b,c,d,... together with a binary operation, denoted by o, form a group if the following axioms hold:

1) (Closure) - If a and b are any elements in the group then a o b is an element of the group.

2) (Associativity) - If a,b,c are any elements in the group then (a o b) o c = a o (b o c).

3) (Identity) - There exists an unique element, i, in the group such that, if a is an element, a o i = a.

4) (Inverse) - For any element, a, of the group there exists an unique element, x, of the group, such that a o x = i.

An example of the group concept is the collection of even positive and negative integers, defining o to be the binary operation + (addition):

1) The sum of two even integers is an even integer (closure)

2) If a,b,c are even, then a + (b + c) = (a + b) + c (associativity).

3) The unique even number, 0, has the property that if a is even, then a + 0 = a (identity).

4) For any even number, a, its negative, -a, is an unique even number with the property that a + (-a) = 0 (inverse).

It is of interest to note that the even integers under multiplication do not form a group because, in this case, axioms 3 and 4 are violated.

<u>Lattice</u>:

A collection of elements with two binary operations, denoted by $\circledvee$ and $\circledwedge$ form a lattice if, whenever X,Y,Z are any three elements, the following axioms hold:

1) (Closure) $X \circledvee Y$ and $X \circledwedge Y$ are elements of the collection.

2) (Idempotent) $X \circledvee X = X$

$X \circledwedge X = X$

3) (Commutative) $X \vee Y = Y \vee X$

$X \wedge Y = Y \wedge X$

4) (Associative) $X \vee (Y \vee Z) = (X \vee Y) \vee Z$

$X \wedge (Y \wedge Z) = (X \wedge Y) \wedge Z$

5) (Absorption) $X \wedge (X \vee Y) = X \vee (X \wedge Y) = X$

An illustration of a lattice is obtained in the following example:

Consider the power set of $\{1,2,3\}$ (i.e., the collection consisting of all the subsets which can be formed from the set consisting of 1,2,3: $\{1,2,3\}$, $\{1,2\}$, $\{2,3\}$, $\{1,3\}$, $\{1\}$, $\{2\}$, $\{3\}$, $\emptyset$.

The curlecue brackets denote the set of elements contained within the brackets and $\emptyset$ denotes the set which has no elements - the null set). If the members of the power set, the union operation of sets, and the intersection[1] operation of sets are respectively defined as the elements of a lattice, as $\vee$ and as $\wedge$, it is readily seen that the five lattice axioms are satisfied:.

1) Both the union and intersection of any two sets of the power set collection are sets of the collection, e.g., $\{1,3\} \vee \{2,3\} = \{1,2,3\}$ and $\{1,2,3\} \wedge \{2\} = \{2\}$ and $\{1,2,3\}$ and $\{2\}$ are both in the collection.

---

1. 1) A set S is a subset of a set T if every element of S is also an element of T. (i.e., S C T)

2) The union of two sets C and D, denoted by C U D, is another set whose elements consist of all elements in C or D or both.

3) The intersection of two sets C and D, denoted by C D, is another set whose elements are <u>both</u> in C <u>and</u> in D.

2) What is in either X or X is the same as what is in X, e.g., $\{1,2\} \vee \{1,2\} = \{1,2\}$.

What is in common to both X and X is the same as what is in X, e.g., $\{2,3\} \wedge \{2,3\} = \{2,3\}$

3) The commutative laws then state:

a) What is in set X or set Y is the same as what is in set Y or set X, e.g., $\{1,2\} \vee \{2,3\} = \{2,3\} \vee \{1,2\} = \{1,2,3\}$.

b) What is in both set X and set Y is the same as what is both in set Y and set X, e.g., $\{1,2\} \wedge \{2,3\} = \{2,3\} \wedge \{1,2\} = \{2\}$.

4) a) What is in either X or in Y or Z is the same as what is in X or Y or in Z and

b) What is in X and in Y and Z is the same as what is in X and Y and Z.

e.g., a) $\{1,3\} \vee (\{2\} \vee \{1,2\}) = \{1,3\} \vee \{1,2\} = \{1,2,3\}$ and $(\{1,3\} \vee \{2\}) \vee \{1,2\} = \{1,2,3\} \vee \{1,2\} = \{1,2,3\}$.

b) $\{1,3\} \wedge (\{2\} \wedge \{1,2\}) = \{1,3\} \wedge \{2\} = \emptyset$ and $(\{1,3\} \wedge \{2\}) \wedge \{1,2\} = \emptyset \wedge \{1,2\} = \emptyset$

5) a) What is in X and in X or Y is the same as what is in X or in X and Y is the same as what is in X, e.g., $\{1,2\} \wedge (\{1,2\} \vee \{2,3\}) = \{1,2\} \wedge \{1,2,3\} = \{1,2\}$ and $\{1,2\} \vee (\{1,2\} \wedge \{2,3\}) = \{1,2\} \vee \{2\} = \{1,2\}$

Thus, the power set of a given set, under the defined conditions, is a lattice.

An important property of a lattice is that it is partially ordered, if we define $X \leq Y$ to mean that X, Y are elements of a lattice which have the property that $X \vee Y = Y$; that is, the following laws, which constitute the definition of partial ordering, hold:

1) $X \leq X$ (reflexive property of $\leq$)

2) $X \leq Y$ and $Y \leq X$ implies $X = Y$ (anti-symmetric property of $\leq$)

3) $X \leq Y$ and $Y \leq Z$ implies $X \leq Z$ (transitive property of $\leq$).

The second law will be shown to have importance in Piagét's theory.

An illustration of <u>partial</u> <u>ordering</u> can be found by again considering the power set of $\{1,2,3\}$ and interpreting $X \leq Y$ to mean every element of X is an element of Y. The properties 1,2,3 then hold for:

1) Every element of a set X is a member of X ($X \leq X$)
e.g., $\{1,2\} \leq \{1,2\}$

2) If every element of X is an element of Y and vice versa, then X = Y.

3) If every element of X is an element of Y and every element of Y is one of Z then every element of X is an element of Z
e.g., $\{1\} \leq \{1,2\}$ and $\{1,2\} \leq \{1,2,3\}$ therefore $\{1\} \leq \{1,2,3\}$.

<u>Mathematical Model</u>:

In this paper, we regard a mathematical structure M to be a mathematical model of a given state of affairs, S, if:

1) There is a one-to-one correspondence between the terms

of M and the terms of S, and the terms of S are interpretations of the terms of M; (i.e., meanings are assigned to the terms of M).

2) The axioms of M (when interpreted in S) are true in S, and M and S are isomorphic with respect to operations and relations. This means that there is a one-to-one correspondence between the elements of M and those of S, an operation, o, of M and an operation o', of S or a relation, R, of M and a relation, R', of S, such that if a,b,(a o b) are elements of M and a',b', (a o b)' their correspondents in S then (a o b)' = a'o'b' and if a Rb (a is related to b in some given way) then a'R'b' (a' is related to b' in the corresponding way).

A fundamental assumption made in using mathematical models is: when the axioms of the model, M, are verified as being true in S, then the interpretations in S of all theorems of M are assumed to be true in S. In this research several mathematical models will be presented and their possible psychological referents will be determined; e.g., at a given cognitive level an interpretation of a particular lattice model will be given and an attempt will be made to justify the assumption that the interpretation, S, satisfies the axiomatic structure, M.

Specifically, if one considers the axiom $a \textcircled{\vee} b = b \textcircled{\vee} a$, the psychological interpretation which considers a and b to be sets of objects and $\textcircled{\vee}$ to be the union operation acting on them satisfies the axiom on the concrete stage, since what is in a or in b is the same as what is in b or in a for a child on the concrete level of Piaget's theory (age 7 - 11).

The definition of mathematical model given above is taken in essence from Stoll - Sets, Logic, and Axiomatic Theories, p. 131 (Freeman & Co., 1961).

An example (another, simpler example will follow this one) of a mathematical model is found by relating, M, the collection of positive integers (with addition as the operation) to S, the following collection of sets of objects (with the operation of union): $\{a\}$, $\{b,c\}$, $\{d,e,f\}$ ..., i.e., sets of a single object, two objects, three objects, etc., wherein no object in one set is repeated in any other set (with $\tilde{U}$, the union operation modified by the following convention: If X and Y are in the collection and $X \neq Y$, then $X \tilde{U} Y$ is that member of the collection which has the same number of members as X U Y, where U is the union operation on sets; if X = Y then $X \tilde{U} Y$ is that member of the collection which has twice as many members as X).

A one-to-one correspondence between the set of positive integers and this collection of sets is achieved by the following:

1 corresponds to $\{a\}$

2 corresponds to $\{b, \}$

3 corresponds to $\{d,e,f\}$

That is, for all n, n' corresponds to the set with n objects in it. In other words, n' is that member of S which contains n objects.

Now consider (n + m)', and $n' \tilde{U} m'$. n' is the set in the collection with n members, m' is the set in the collection with m members and (n + m)' is the set in the collection with n + m members. Since n' and m' are disjoint (have no members in common) unless n = m, n' $\tilde{U}$ m' is represented by the set with n + m members; i.e., (n + m)' = n' $\tilde{U}$ m'. If n = m then n' $\tilde{U}$ m' has 2n members; i.e., (n + m)' = n' $\tilde{U}$ m'.

This is illustrated more clearly perhaps in the case where n = 2 and m = 3.

2 corresponds to $\{b,c\}$

3 corresponds to $\{d,e,f\}$

and (2 + 3)' = 5' equals $\{k,l,m,n,o\}$. Since 2' = $\{b,c\}$, 3' = $\{d,e,f\}$, and 2' $\tilde{U}$ 3' = $\{b,c\}$ $\tilde{U}$ $\{d,e,f\}$ = $\{k,l,m,n,o\}$.

Another exemplification of the idea of a mathematical model involves the possibilities of communication among four people. We may consider the mathematical symbols P_1, P_2, P_3, P_4 as standing for the four people (i.e., there is a one-to-one correspondence between P_1, P_2, P_3, P_4 and the four people). We will consider the phrase, "can communicate with" to be mathematically symbolized by C, and the words, "or" and "and" to be symbolized by $\vee$ and $\wedge$ respectively. x C y is then the mathematical correspondent of the statement "x can communicate with y". Statements concerning the four people and their ability to communicate can then be correlated with a mathematical statement in the mathematical structure involving P_1, P_2, P_3, P_4, C, $\vee$, $\wedge$. (e.g., the first person can communicate with the third and fourth person corresponds mathematically to P_1 C ($P_3 \wedge P_4$).

Methodology:

Both psychological and mathematical sources were used in this study. The psychological sources involve the underpinnings of Piaget's theory and specific mathematical models which are useful in his theory. The mathematical sources involve the more general theory of mathematical models and structures as well as works which develop Piaget's logical concerns.

An examination and a critique were made of the mathematics that Piaget uses in developing his theory of the concrete and formal stages of cognition. The main sources used in this regard were Piaget's Logic and Psychology and Flavell's The Developmental Psychology of Jean Piaget.

A modified lattice theory model is to be developed in Chapter Three with a view to determining its efficacy in explaining Piaget's theory on the concrete stage. In addition, a model stemming from group theory and Boolean algebra theory was used to fashion a structure to explain Piaget's theory of the formal stage of cognition in Chapter Four. The mathematical models were developed by relating known experimental results (usually from Piaget's research) to the axioms of the mathematical theory. The experiments thereby tended to suggest the axioms which mathematically represent them. For example, Piaget's experiment concerning a child's grasp of the concepts of a given set and its complement (that set consisting of all the elements not in the given set) led to the inclusion of some axioms of a lattice which concern elements, their union, intersection and complements. The elements making up these models were related to their psychological referents found in Piaget's theory of cognition. A number of theorems logically derived from the mathematical models were used to suggest empirical tests of his theory.[1] In sum, then, experimental results of Piaget and others were used to validate the mathematical models, and the latter were used to suggest further experiments relevant to Piaget's theory.

Some of the literature which was studied to collect data substantiating the mathematical models and the psychological referents associated with them are:

1. Piaget, - Logic and Psychology
2. Flavell, - The Developmental Psychology of Jean Piaget
3. Inhelder and Piaget, - The Growth of Logical Thinking From Childhood to Adolescence
4. Piaget, Inhelder and Szeminska, - La Géométrie Spontanée de l'enfant

Sources for the study of mathematical models and structures were:

1. Birkhoff and MacLane, Modern Algebra
2. Robert R. Stoll, Sets, Logic, and Axiomatic Theories

In summation, then, the data abstracted from these and other sources involved:

a) the development of mathematical models in relation to Piaget's theory;

b) the psychological referents of the models as found in the theory of Piaget and other researchers concerned with cognitive learning;

c) theoretical consequences of these models with a view

---

1. Demonstrations were carried out by the researcher to suggest the psychological validity of several of his extensions of Piaget's models.

to suggesting experiments that would test Piaget's theory.

CHAPTER II

A CRITIQUE OF PIAGET'S MATHEMATICS

The mathematics used by Piaget in developing his psychological theory of cognition can be divided into two parts. One part involves the lattice-like structures he refers to when describing the concrete operational stage. The other involves the group and Boolean Algebra structures he refers to on the formal operational stage (Piaget, 1957).

Piaget presents several lattice-like structures for the concrete stage in his Logic and Psychology (Piaget, 1957, pp. 26-28). They involve what he calls "elementary groupements" and "multiplicative groupements". However, the first set of "axioms" he presents for the "elementary groupements" or simple classifications in a hierarchical structure is defined by what he refers to as five operations [sic] :

1) $A + A' = B$; $B + B' = C$; etc. (where $A \times A' = 0$, $B \times B' = 0$ etc.).[1] [Composition]
2) $-A-A' = -B$; etc., from which $A = B-A'$ and $A' = B-A$. [Inversion]
3) $A-A = 0$ [Identity]
4) $A + A = A$ from which $A + B = B$. [Tautology]
5) $A + (A' + B') = (A + A') + B'$ but $A + (A-A) \neq (A + A) -A$. [Associativity]

(Piaget, 1957, pp. 26-28)

---

1. The prime symbol denotes partial complementation. A', for example is the (partial) complement of A with respect to B. (i.e., what is in B but not in A).

Five such statements are referred to mathematically as axioms not operations. His reference to the term "operations" is ambiguous and misleading. Moreover these five statements are not all axioms in the usual sense found in mathematics. The "-" operation is not clearly invoked; how are -, +, x, interrelated? If all Piaget means by A-B is the usual A intersect B' found in set theory, then the second half of 5) is irrelevant. In his analysis as well as this paper's (see Chapter 3 of this thesis) the operation "x" is not always well defined at this part of the concrete stage; therefore, its use in 1) needs clarification.

The structure presented above is clearly not a group, nor does Piaget claim it to be (Logic and Psychology, p. 27). Yet, on p. 4 of Logic and Psychology, while writing of logical relationships at all levels, including the concrete stage, he asserts, "logical relationships...never appear as a simple system of linguistic or symbolic expressions but always imply a group of operations." His use of the term "group", therefore, is unclear; it sometimes signifies the mathematical concept and at other times it indicates a collection with some structure.

In Logic and Psychology (p. 28) he presents a "multiplicative groupement" which he defines as follows:

1) $A_1 \times A_2 = A_1A_2$; $B_1 \times B_2 = A_1A_2 + A_1A_2' + A_1'A_2 + A_1'A_2'$: etc. [Composition]

2) $B_1B_2 : B_2 = B_1$ (where $:B_2$ means' eliminating B_2'). [Inversion]

3) $B_1 : B_1 = Z$ (where Z is the most general class of the system obtained by eliminating the inclusion B_1) [Identity]

4) $B_1B_2 \times A_1A_2 = A_1A_2$ [Tautology]

5) Associativity restricted by the operations of 4). Presumably $B_1 = A_1 + A_1'$ and $B_2 = A_2 + A_2'$ in this analysis.

He writes further, "Now it is the join...which is not general..." The term "join" is not defined anywhere and presumably means the same as set union. Also, it is not clear in what sense "join" is not general.

Z could mean the empty set or the universal set according to his terminology since $B_1:B_1$ means B_1 less B_1 which should be the empty set; yet Z is called the most general class.

The "axioms," particularly axiom 5), are again ambiguous as they are presented, even if one follows Piaget's use of them very carefully. Chapter 3 of this thesis is devoted to clarifying Piaget's mathematics for the concrete stage.

Piaget, in his Traité de Logique, and Flavell, in The Developmental Psychology of Jean Piaget, indicate further mathematical aspects of Piaget's theory on the concrete level by analyzing logical multiplication of sets and relations in the sections on "groupings" or "groupements". This involves pairing of classes and relations in one-to-one and many-to-one correspondences. Much of the mathematics that Piaget uses in these "groupements" as well as the pairing of classes and relations are developed in Chapter 3 of this thesis. The mathematics is based on sets of axioms which explain how most of the "groupements" can be derived.

A major shortcoming of the mathematics of Piaget is cited by Flavell when he writes that Piaget has not "indicated clearly and unambiguously how each model component is translated isomorphically into a specific behavior component." (Flavell, 1962, p. 188). A partial verification of this view is obtained in the analysis of the ambiguity of the two sets of axioms cited earlier in this Chapter. The research presented later in this thesis poses a partial solution to the problem of the isomorphism between Piaget's mathematics and psychology.

Concerning the formal stage, Piaget refers to a "(complete) lattice structure" in his mathematics explaining this stage (Piaget, 1957, p. 32). However, inconsistently, he also refers to the use of Boolean Algebra as an explanatory model for the formal stage (Inhelder and Piaget, 1958, p. 132). Nowhere is it made clear precisely how either mathematical structure is used in his theory nor is Piaget ever clear exactly what axioms he assumes when he refers to a lattice or Boolean Algebra.

Charles Parsons has an interesting review of Inhelder and Piaget's Growth of Logical Thinking wherein he poses several important questions concerning Piaget's mathematics on the formal stage (Parsons, 1960). Parsons develops a case concerning the ambiguous use of propositions by Piaget. (Parsons, 1960, pp. 75-77). A single propositional formula is sometimes used by Piaget as a statement (i.e., it is either true or false) and sometimes used by him as a propositional function (i.e., it is true for certain cases and false for others). For example, Parsons writes in reference to the problem of the causes of pendulum motion: "Thus on p. 76 [Piaget, Growth of Logical Thinking] Piaget speaks of the subjects' verifying the tautology q*x, which ...[states] that weight has no effect, which is not logically necessary." (Ibid.) The use of the term "tautology" in Piaget's analysis is improper. Either q*x is a tautology and is true in all possible cases or it is not always true and is true always in the subject's experience. The latter is actually true in Piaget's analysis.

Another problem that Parsons discusses is the relation between Piaget's use of the four-group and logical thinking on the formal stage (Parsons, 1960, pp. 81, 82). (For an analysis of the four-group see the section in this thesis on the formal stage.) Parsons questions why the four-group is most appropriate to explain the reasoning involved in logical analysis when other structures might also do so. He cites an article by P. R. Halmos concerning such structures (P. R. Halmos, "The Basic Concepts of Algebraic Logic", Am. Math. Monthly, 1956, 63, pp. 363-387). If the four-group does indicate the structure of logical reasoning, how does one distinguish between different kinds of logical problem solving? For example, how can one then distinguish between the kind of thinking involved when a child solves a Piagetian type problem such as "the equality of angles of incidence and reflection" (Inhelder and Piaget, 1958, pp. 3-19), and the kind of thinking that a mathematician does when he solves a problem in group theory. The research presented in this thesis suggests that the latter form of reasoning may represent cognition on a different tage from the former. Perhaps other mathematical structures must be developed to characterize the reasoning of mathematicians. There is no evidence that Piaget has considered this problem.

Although this problem concerning additional structures to explain complex logical reasoning beyond Piaget's formal stage is NOT WITHIN the scope of this paper, the following speculations could be of interest. The mathematical structures have, from a temporal view, a static quality. The mathematical analysis of what is understood by an individual at a given stage is not presented as a function of time, although time, in some sense, is implicitly understood to be related to the development of cognition. Apostel (Apostel, 1966) criticizes this "shortcoming" of Piaget's theory and suggests an alternative approach dealing with cognitive structures as functions of time. A. N. Prior (Prior, 1962) promulgates a logical structure whose elements involve variations in time. Prior's structure is purely logical and apparently completely unrelated to Piaget's work in its inspiration. It appears to lend itself to a Piagetian type analysis and could form the basis of a mathematical structure which might solve the "static" problem and point to structures beyond the formal stage.

Parsons (Parson, 1960) raises a question about piaget's use of "relations". Piaget deals with simple relations as belonging to the concrete stage. However, complex mathematical relations clearly are not understood at this stage. Piaget, although he has not explicitly clarified this point, probably is referring to simple, more restricted kinds of relations. This, of course, leaves open the question of when more complex relations are understood. This understanding probably occurs on the formal stage.

A final point which merits some discussion is the function of logic in Piaget's theory. He states (Piaget, 1957, p. 23) that he uses logic as an operational algebra or as a useful tool for understanding cognitive processes. Axiomatic logic (as used by logicians) he regards as not particularly useful for grasping cognitive structures, for human cognition is not completely logical or formalizable. He thereby makes a distinction between the work done by logicians and the more informal logical analysis done in the "every-day" world. Again the question of how logicians reason is not referred to by Piaget.

CHAPTER III

MODELS FOR THE CONCRETE STAGE

Two underlying ideas which motivate the present research and which were used to suggest the mathematical models which will be presented may be stated as follows: in order to understand the structures underlying children's (and adult's) logical thinking one must be sensitive to:

1. Not only what the individual can do but also what he cannot do,
2. Whether the individual can cancel or negate mental or physical operations while mentally retaining the original operations or mentally performing the reverse operations.

Research manifesting this point of view may be found in Piaget's bead experiment (Piaget, 1957, p. 4) which involves children at or before the concrete stage (approximate age 7-11). When they are shown twenty wooden beads, most of which are brown and a few of which are white, the question "are there more <u>wooden</u> beads than brown beads?" is negatively answered by the pre-concrete child and affirmatively by the concrete child. The former tends to relate the number of brown beads to the number of white and <u>loses</u> the idea of the totality of the wooden beads. He can compose the total (i.e., form the total from its constitutent parts) but cannot decompose it into its constituent parts (brown and white beads) and still retain the original totality of wooden beads, while the concrete child can perform both operations.

For the purpose of developing the mathematical structures which can be associated with Piaget's theory the following conceptual-age categories are of interest:

Stage three (concrete stage) and

Stage four (formal reasoning stage)

The first two stages of Piaget's theory, stage one (sensory-motor stage) and stage two (pre-operational stage) together with stages three and four are described in the appendix and in the literature (Flavell, 1963, pp. 86, 87, 264-266).

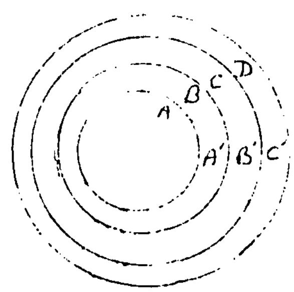
Stage three is exemplified by the development of concrete structural reasoning. The child (about age 7-11) develops capability of class inclusion and exclusion and seriation, counting and nullifying counting (i.e., operating on a class basis, conserving set concepts while being able to operate reversibly). Prior to this stage, we have the preoperational child who can only think of one idea at a time (Maier, 1965, p. 116). There tends to be little or no operations among ideas at this time. Before the second, pre-operational stage there is stage one, the sensory-motor stage (age 0-2), which also exhibits a dearth of operations on and among ideas and is even less structured than stage two. The structures being developed during stages one and two contributes importantly to those of the later stages. The early stages exhibit, incipiently, aspects of stages three and four; a crucial difference lies in the ability to operate reversibly at the latter stages, but not during stages one and two. The child, early in stage three, exhibits the fundamental operation of reversibility in relation to class and number concepts. A mathematical model of this state of affairs can be found in what will be called a K-lattice. The mathematics developed by Piaget in his writings was used to help suggest the mathematical structures put forth throughout this research (Piaget, 1949 and Piaget, 1957). A collection of elements together with two binary operations denoted by $\vee$ and $\wedge$ form a K-lattice if, whenever x, y, and z are any three elements, the following axioms hold:

1. $x \vee y$ is in the collection (closure)
   ($x \wedge y$ is not always, but sometimes is, meaningful)
2. $x \vee (y \vee z) = (x \vee y) \vee z$ (associativity of $\vee$ and of $\wedge$)
   $x \wedge (y \wedge z) = (x \wedge y) \wedge z$ (whenever both sides are meaningful)
3. There an element, 0, such that for all x,
   $x \vee 0 = x$ (identity of $\vee$)
4. For each x, $x \neq 0$, there exists a unique x' in the collection such that $x \wedge x' = 0$ (partial complement)
5. $x \vee y = y \vee x$ (commutativity of $\vee$ and of $\wedge$)
   $x \wedge y = y \wedge x$ (whenever both sides are meaningful)
6. $x \vee x = x$ (idempotent)

DISCUSSION OF THE MODEL

If we define $x \textcircled{\le} y$ to mean $x \vee y = y$ then axiom 6 signifies that $x \textcircled{\le} x$. The symbol $\textcircled{\le}$ is abstract but derives intuitively from the subset relation among sets and the less than or equal to relation among numbers. For example, $x \textcircled{\le} y$ can, in a particular interpretation, be understood as saying that set x is included in set y. It is to be noted that this is a mathematical model for the psychological operations of simple classification of classes A,B,C,D..., A',B',C'...and their unions where $A \leq B \leq C \leq D \ldots$ and $A \vee A' = B$, $B \vee B' = C$, $C \vee C' = D$ etc., $A \wedge A' = 0$, $B \wedge B' = 0 \ldots$.

The term hierarchical structure can be clarified by the following diagram:



In the diagram, A' is the complement of A in B, B' is the complement of B in C etc. and $A \leq B \leq C$ etc. A,B,C, etc. are increasingly broad categories. For example, A could be the set of all red crayons, B the set of all crayons, C the set of writing implements. A' would then be the set of all non-red crayons.

The K-lattice is neither a group nor a lattice, yet it exhibits some of the properties of each. It corresponds to some of the structures called <u>groupings</u> in Piaget's writings (Piaget, 1957, p. 26). On a psychological level is incipiently suggests the logical structures which appear in full bloom on the formal cognitive level.

The following principle will now be used to provide a means of empirically investigating the afore-mentioned model as well as all other models which will be presented in this research:

Whatever theorem is derivable from a given mathematical model is a formal characteristic of an actualizable thought process of an individual whose cognitive capabilities are represented by the model. In other words, the theorems of our model characterize the cognition of the stage.

This can be applied to our present model in the following way, and thereby provide an understanding of the psychological referents in the mathematical model:

Axiom 1 says that any two classes in a hierarchical arrangement can be composed mentally into a third class of the arrangement by the union operation, but the intersection of two classes does not always make sense (e.g., $A \wedge A' = 0 = B \wedge B'$ etc. is understood early in stage three but $A \wedge (A' \vee B')$ is not necessarily understood then. (See page 29, paragraph 1.)

Axiom 2 - Associativity - The union of elements of a class structure is independent of the way they are grouped (similarly for intersection when it is meaningful).

Axiom 3 - There exists at the concrete stage the ability in the child to conceive of an empty class which, under union, does not affect any other class.

Axiom 4 - Partial complement - For any class one can conceive of its relative complement. That is, one can conceive of what it is not.

Axiom 5 - Commutativity - One can combine pairs of classes regardless of order whenever it is meaningful.

Axiom 6 - This specifies that no class can extend itself by itself.

A useful consequence of $x \vee x = x$ is the following theorem: If $x \vee x' = y$ then $x \vee y = y$ (i.e., $x \leq y$).

Proof: $x \vee y = x \vee (x \vee x') = (x \vee x) \vee x' = x \vee x' = y$

Therefore, $x \vee y = y$. (For this proof axioms two and six were used.)

A psychological interpretation of this theorem is that when one understands that a given sub-class and its complement make up a larger class, at the same time one grasps the notion that the sub-class is smaller than the larger class. A compelling experimental validation of this result on the concrete stage is given by Piaget (Logic and Psychology, p. 4) in the bead experiement: children are asked to determine from a box of 20 wooden beads (class y) most of which are brown (class x) and some of which are white (class x') (therefore $x \vee x' = y$) whether there are more brown beads than wooden beads (is $x \leq y$?). Prior to the concrete stage the child answers $x \geq y$ because there is more x than x' (he does not exhibit the properties that the model indicates and is not yet at the concrete stage); at the concrete stage the child says that $x \leq y$ and thereby acts in accordance with the model.

The suggestion then is that the pre-concrete individual has not yet conserved the idea of "class" while the concrete level individual has developed a structured whole concept of "class" and understands what a given class is and what it is not -- he has achieved reversibility.

The above can be rendered in a more Piagetian style by defining $x' - y-x$ whenever $x \vee x' = y$. The minus sign refers to the operation of inversion and the reversibility indicated by the above theorem takes the form of class inversion (in contrast to the reciprocation form of reversibility which will be discussed later). Inversion and reciprocation are two forms of reversibility. Inversion refers to the operation of direct elimination; e.g., if a weight is added to the left side of a balanced scale, removal of the added weight to achieve balance constitutes an inversion operation. Reciprocation creates the same effect as inversion. However, it nullifies the previous action through indirect means: e.g., in the previous example concerning the scale, adding an equal weight placed at an equal distance from the fulcrum on the right side of the balance will also achieve balance and constitutes reciprocation. In particular, one sees the use of inversion in completely negating a class in the equation $0 = x - x$ since $x \vee o = x$.

The existence of a unique complement suggests the following problem: if $x \vee x' = y$, with $x \wedge x' = 0$, then $y \vee y' = z$ (with $y \wedge y' = 0$) yields $(x \vee x') \vee y' = z = x \vee (x' \vee y')$. A formal, logical class analysis would yield $x \wedge (x' \vee y') = 0$, but our restrictive model does not allow the existence of another partial complement. Therefore, in accord with axiom four, this is an instance in which $\wedge$ is undefined. Namely, for the pair x and $x' \vee y'$. The psychological translation of this problem is: if a concrete level child is presented with a hierarchy of classes $x \subseteq y \subseteq z$, with $x \vee x' = y$, $y \vee y' = z$, $x \wedge x' = 0$, $y \wedge y' = 0$ can he conceive of x as having two distinct partial complements, x' with respect to y and $x' \vee y'$ with respect to z? Clearly the K-lattice model predicts one and only one class is conceivable as a partial complement. An experimental generalization of Piaget's bead experiment may be useful in resolving the problem and will be presented in chapter five.

Another interesting consequence of the model is Piaget's fifth grouping, asymmetric relations (i.e., the building up of things into transitive, ordered sequences) (Piaget, 1949, p. 141). Mathematically speaking, we can establish that if $x \leq y$ and $y \leq z$ then $x \leq z$ (transitivity) <u>and</u> if $x \leq y$ and $y \leq x$ then $x = y$ (asymmetry). The symbol $\leq$ can be thought of as indicating the subset relation or, more abstractly, any other relation which satisfies the properties of transitivity and asymmetry. (e.g., as light as, as long as, as valuable as, etc.)

The model then gives the mathematical result

<u>$x \leq y$ and $y \leq z = x \leq z$</u>

Proof: $x \leq y$ means $x \vee y = y$

$x \leq z$ means $y \vee z = z$

Therefore, since $x \vee z = x \vee (y \vee z)$ and $(x \vee y) \vee z = y \vee z = z$ (using associativity of $\vee$) we have $x \leq z$.

We also derive that if $x \leq y$ and $y \leq x$ then $x = y$

Proof: $x \leq y$ means $x \vee y = y$

$y \leq x$ means $y \vee x = x$

Therefore, using commutativity, we have $x = y$.

Flavell cites general studies of Piaget which indicate that concrete level children do exhibit those properties of the model which are exemplified below, and pre-concrete children tend not to.

> In one study...the child is given a series of ten sticks varying in length from A (shortest) to J (longest) and is asked to seriate them. When this

> is done, he is given 9 more sticks (a to i) and asked to insert them in their proper places in the A-J series; the correct seriation would then give Aa Bb Cc...iJ. Whereas the young children of ten fail to make a complete construction of even the initial series A-J, the older ones readily solve both problems...
>
> A second series of studies is more specifically concerned with the transitivity property of asymmetrical series. The child is given three or more objects of perceptually different weight and asked to seriate them by weight, with the restriction that he may compare only two objects at a time. The younger child does two things of interest there. First, for a set of three objects, $a < b < c$, he is often willing to form a complete series (either correct or incorrect) on the basis of establishing $a < b$ and $a < c$ above. And conversely, he is often unsure (and feels the need for empirical verification) that $a < c$ is guaranteed from knowing $a < b$ and $b < c$.[1] (Flavell, 1963, p. 193)

The second study clearly indic tes the operation of the transitive law at the concrete stage. The first study, involving the sticks, indicates that both the direct ($\textcircled{\leq}$) and the reverse relation ($\textcircled{\geq}$) must be grasped to solve this essentially concrete level problem. The fitting of objects into the series requires the operation of reversibility. In this case this means that one can function with $\textcircled{\leq}$ and $\textcircled{\geq}$ at the same time. However, there is no exact analogue to the asymmetry property (if $x \textcircled{\leq} y$ and $y \textcircled{\leq} x$ then $x = y$) in either of these studies. An experiment which could give psychological substance to the asymmetry property is as follows:

Consider two distinct collections of objects a and b. Let a consist of several white beads and several black beads; let b consist of exactly the same distribution of beads. Let

---

1. $\textcircled{\leq}$ in the research is essentially equivalent to $<$ as used by Flavell.

the child choose one bead at a time from set a and have the child, without looking inside b, draw a bead from b, by trial and error, which is the same as the one he chose from a. Place both beads elsewhere and continue until the beads of a are used up. Then repeat the operation with set b taking the place of a. In this manner the child experiences $a \textcircled{\le} b$ and $b \textcircled{\le} a$. The mathematical model predicts that the concrete level child should and the pre-concrete level child should not observe that $a = b$. See Chapter Five for a demonstration testing this assertion.

The model, thus far, has exhibited the operations of inversion for classes and reciprocity (which is expressed in the theorem on asymmetry) for relations. It is interesting that the mathematics concerning inversion and set complementarity provides no link with reciprocity and relation (the theorems on transitivity and asymmetry make no use of the former two.) However, by introducing the absorption axiom [$x \wedge (x \vee y) = x$ and $x \vee (x \wedge y) = x$] of the full lattice on Piaget's formal stage, one can exhibit linkage between the two kinds of reversibility represented by inversion and reciprocity. (See below, p. 52) In the words of Piaget, "At the level of concrete operations, they (inversion and reciprocity) appear in the form of two distinct operational structures..., and finally form a unique system at the level of propositional operations (formal stage)." (Piaget, 1957, p. 29) The present model indicates that inversion and reciprocity are not intellectually linked in the concrete level individual and the later mathematical model to be developed for the formal stage predicts the intellectual linkage of these two at the formal stage, quite in accordance with Piaget's statement.

The concept of vicariance is discussed in Piaget's <u>Traité de Logique</u> (pp. 113-117) and in Flavell (1963, p. 176 and P. 192). There are two aspects of this concept which are relevant to this research. The first aspect postulates that if $A_1 \vee A_1' = B$ where $A_1 \wedge A_1' = 0$ then there usually is at least one other (substitute) pair A_2 and A_2' such that $A_2 \vee A_2' = B$ and $A_2 \wedge A_2' = 0$. This is a formal statement of the fact that concrete level children can usually classify the subsets of a given collection of things in several different ways. The second aspect yields a problem: if, in the above, $A_1 \leq A_2'$ then $A_2 \leq A_1'$, which is a concrete level result of Piaget. However, neither our K-lattice model nor the next model to be presented, the P-lattice (both of which develop the concrete stage theory) nor Piaget's analysis of vicariance on the concrete stage substantiates this theorem, which can easily be derived from the model which will be developed in this research for the formal stage. In addition, Flavell suggests that this property is attained at the formal stage (Flavell, 1963, p. 192).

An experiment which could suggest whether the second property of vicariance is attained at the formal stage or earlier is as follows:

Consider a collection of 5 small (A_1) and 20 large (A_1') objects. Let the 5 small ones and 3 large ones be colored black (A_2') and let all the rest be colored red (A_2). Tell this information to the child and ask him to mentally determine the relative number of red objects and large objects.

($A_1 \vee A_1' = A_2 \vee A_2'$ and $A_1 \leq A_2'$ therefore $A_2 \leq A_1'$ should logically follow.)

The K-lattice model, as has been shown, is particularly relevant to hierarchical, class, and simple relational structures on the concrete level of cognition. It serves as a model, then, for the first and fifth groupings in Piaget's analysis of the concrete stage (Flavell, 1963, pp. 173-195 and Piaget, 1949, pp. 109-187). The simplicity of this model in relation to what will now be presented suggests that the structures predicted by it are attained earlier in the concrete stage than those structures which will be considered now.

The next model to be presented involves deeper probing into the class and relational structures of the concrete stage. The first model (the K-lattice) does not allow for a full development of multiplication of classes and relations and the balancing properties involved in handling two or more characteristics of objects at the same time.

The structure which follows will largely subsume the K-lattice structure (exceptions involve properties of closure of $\wedge$ and complementation). A collection of elements together with two binary operations denoted by $\vee$ and $\wedge$ form a P-lattice if, whenever x, y and z are any three elements of the collection the following axioms hold:

1)	$x \vee y$ is in the collection $x \wedge y$ is in the collection	<u>CLOSURE of $\vee$ and $\wedge$</u>
2)	$x \vee (y \vee z) = (x \vee y) \vee z$ $x \wedge (y \wedge z) = (x \wedge y) \wedge z$	<u>ASSOCIATIVITY of $\vee$ and $\wedge$</u>
3)	$x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$	<u>DISTRIBUTIVITY of $\wedge$ with respect to $\vee$</u>

4) $x \vee y = y \vee x$
$x \wedge y = y \wedge x$ — COMMUTATIVITY of $\vee$ and $\wedge$

5) There exist an unique element, 0, such that for all x
$x \vee 0 = x$ — IDENTITY of $\vee$

6) There exists an element, I, such that for all x
$x \vee I = I \quad (I \neq 0)$ — UNIVERSAL ELEMENT

7) For any x, there exists an unique x' such that
$x \vee x' = I$ and $x \wedge x' = 0$ — COMPLEMENT

8) $x \vee x = x$
$x \wedge x = x$ — IDEMPOTENT

DISCUSSION OF THE MODEL

The mathematical elements of the model have, as psychological referents, the understanding of cognitive categories such as classes, propositions and numbers. In fact, the present model is quite close structurally to the Boolean Algebra model to be presented on the formal stage which involves propositional analysis on a level deeper than that of the concrete stage.

1) Axiom 1 says that composition exists now for both the union and intersection operations.

This clearly suggests that this model is applicable only at a later part of the concrete stage when compared to the first model, since closure under intersection did not hold for the latter. Piaget suggests that the union operation ($\vee$) is not closed early in the concrete stage (Piaget, 1957, p. 28). It is not at all clear what this means or how it can be tested since he had already indicated his belief that the union operation is grasped in some sense (Piaget, 1957, p. 27).

It would seem that the deviation of the latter part of the concrete stage from complete lattice properties is due to the fact that the law of absorption is missing. That we are nearing the complete lattice is indicated by the axiom of distribution, for distributivity is the major link between $\vee$ and $\wedge$, and can be understood as a psychological link to the law of absorption.

2) Associativity, as in the earlier model, states that the union (as well as the intersection) of classes in a given order is independent of how they are grouped. The following experiment might validate this axiom for the earlier model, as well as the present one.

Consider three large sets of objects, M_A, M_B, and M_C, having A, B, and C members respectively. Place M_A and M_B in one container, but count them separately and place M_C in a second container. Now, select three other sets, N_A, N_B, and N_C, also having A, B, and C members respectively. Place N_A in a third container and N_B together with N_C in a fourth container and count them separately. The child is asked to compare the two distributions by first asking him to determine A, B, and C in each distribution separately and then to compare the numbers of members in $M_A \vee M_B \vee M_C$ and in $N_A \vee N_B \vee N_C$ and to give reasons for his answer.[1]

3) Distributivity of $\wedge$ with respect to $\vee$ is a key new concept distinguishing the P-lattice from the K-lattice. It

1. See chapter five for an analysis of a demonstration indicating the presence of associativity late in the concrete stage.

allows for a limited amount of combinatorial operations linking two by two or three by three, etc. logical multiplications but not interrelating them. For example, if $A = A_1 \vee A_2 \vee A_3$, $B = B_1 \vee B_2 \vee B_3$, $C = C_1 \vee C_2 \vee C_3$, distributivity yields us $A \wedge B = (A_1 \wedge B_1) \vee (A_1 \wedge B_2) \vee (A_1 \wedge B_3) \vee (A_2 \wedge B_1) \ldots$ etc. linking pairs of properties but unrelated to triples such as $A_1 \wedge B_2 \wedge C_3$. Also $A \wedge B \wedge C = (A_1 \wedge B_1 \wedge C_1) \vee (A_1 \wedge B_1 \wedge C_2) \ldots \vee (A_3 \wedge B_3 \wedge C_3)$ which combines triples but doesn't relate to pairs. This suggests that when this model is appropriate, the understanding of two (or more) supersets is achieved when combinations of their subsets in pairs (or triples etc.) is understood. Flavell cites the following experiment of Piaget, which illustrates the ability of the older concrete level child to logically multiply and the inability of the pre-concrete and young concrete child to do so:

> There are only three knives in a store. Two of these knives have two blades: they cost 8 francs (A_1) and 10 francs (A_1'). Two of these knives have a corkscrew; they cost 10 francs (B_1) and 12 francs (B_1'). I choose the one which has two blades and a corkscrew ($A_1' \wedge B_1$): How much does it cost? (Flavell, 1963, p. 192)

The concrete child's ability to solve the problem hinges on his ability to logically multiply, and in this case to realize that $(A_1 \vee A_1') \times (B_1 \vee B_1') = A_1B_1 \vee A_1B_1' \vee A_1'B_1 \vee A_1'B_1'$ and to consider the solution product $A_1'B_1$. (Piaget, 1957, p. 30)

It is true, however, that the corkscrew - knife blade problem could be more properly posed since, for example, it seems to suggest to the child that a knife with two blades and a corkscrew could cost less than one with one blade and a corkscrew. It would be interesting to see, as suggested by

Dr. P. Merrifield of New York University, how children would react to corkscrew costs of 8 and 12 francs with two blade costs of 10 and 12 francs.

Another experiment cited in Flavell to illustrate the concrete level child's ability to logically multiply in pairs is the following:

> The subject is presented with a horizontal row of pictures of different colored leaves (A_1, A_2, A_3...) which meets to form a right angle with a vertical row of pictures of green-colored objects (B_1, B_2, B_3...). The subject's problem is to determine what picture should be placed at the intersect of these two rows: since the picture is to be in both rows at once...it must be a picture of a green leaf. (Flavell, 1963, p. 192)

This experiment suggests that concrete level children exhibiting the properties of this model are capable of grasping matrix arrays; e.g., $(A_1 \vee A_2 \vee A_3) \wedge (B_1 \vee B_2 \vee B_3)$ $= A_1\ B_1 \vee A_1\ B_2 \vee \ldots \vee A_3\ B_3$ can be conceived in a 3 by 3 matrix array:

A_1B_1	A_1B_2	A_1B_3
A_2B_1	A_2B_2	A_1B_3
A_3B_1	A_3B_2	A_3B_3

An experiment pursuing this in further depth is a three-dimensional generalization of the "green leaf" experiment considering three attributes in a 3 by 3 matrix. The model suggests that the added perceptual difficulty should not preclude the older concrete level child from solving the problem.

Further development of the distributive law leads to the consideration of multiplication of relations: if $A_1 < A_2 < A_3 \ldots$ and $B_1 < B_2 < B_3 \ldots$ then $(A_1 \vee A_2 \vee A_3 \ldots) \wedge (B_1 \vee B_2 \vee B_3 \ldots)$

$= (A_1 \wedge B_1) \vee (A_1 \wedge B_2) \ldots \vee (A_3 \wedge B_3)$. This proposition involves a development of two relations and leads to the idea that if the concrete level child understands the two given relations he has the ability to grasp the combinatorial matrix formed by $A_1 \wedge B_1$, $A_1 \wedge B_2 \ldots A_3 \wedge B_3$. Flavell cites the following experiment which illustrates this ability: "The child is given 49 cut-out pictures of leaves which can be ordered both by size (7 different sizes) and by shade of color (7 shades of green). He is asked to arrange them as he thinks they ought to be arranged, is then questioned about the arrangement he makes, and so on." (Flavell, 1963, p. 194)

The results indicate that at least 75% of concrete level children arrange these 49 elements in a matrix according to their "proper" position in the double ordering but that the pre-concrete level children do not so order the array.

The following experiment, in a three dimensional matrix, is analogous to the one involving green leaves and tests the model further: consider a number of rectangles whose horizontal and vertical sides could be ordered by length and whose interiors are colored by shades of green. The task then is to arrange them and to discuss the rationale of the arrangement. Of course, a possible difficulty in those three dimensional experiments is whether the perceptual difficulties interfere with cognitive capabilities, and whether the perceptual and the cognitive capabilities are distinguishable.

Another interesting experiment concerning the achievement of two by two matrix type reasoning involves the following card trick: exhibit, face up, twenty-five different cards arranged in five rows of five cards each. Have the subject mentally choose one and tell the experimenter which row it is in. The experimenter then picks the cards up row by row and redistributes them, face up, interchanging corresponding rows and columns (i.e., each row now is a column and vice versa). The subject is to specify which row the chosen card is now in, and the experimenter tells him what the card is.

The subject's task is to determine how the card was identified and, in effect, to realize that the identification of both the row and column of an element in the matrix array suffices to identify the element. In order to test the verbal solution it would be efficacious to have the subject then perform the card trick himself.

The researcher has posed this problem concerning the card trick to 9 children. Three, who were below 8 years of age, were mystified and could not cope with any of the posed tasks. Three, who were between 9 and 10, determined how the card was identified. One of them could not repeat the trick himself but the other two did repeat it slowly and after a while. Three others above age 12 coped with all aspects of the card trick rather easily.

Another essentially new consideration in the axioms, besides distribution of $\wedge$ with respect to $\vee$, involves $\underline{I}$, the universal element, which involves the most general class under consideration. The existence of $\underline{I}$ suggests that the child can conceive of a class with no delimiting attributes. An experiment validating this might be as follows: have the child consider a number of different kinds of collections such as wooden beads, metal rings, books, etc., and ask him what they all have in common (i.e., what is I?). An acceptable answer might be, "They are all collections of objects."

The relationship between the two models involving the concrete levels, wherein the P-lattice is a logical outgrowth of the K-lattice, suggests that I will be understood in the latter part of the concrete stage.

Two theoretical consequences of the model are of interest here:

a) $X \wedge I = x$

Proof: $X \wedge I = x \wedge (x \vee x')$ and $(x \wedge x) \vee (x \wedge x') = (x \wedge x) \vee o$.
Also $(x \wedge x) \vee 0 = x \wedge x = x$. $\therefore$ $x \wedge I = x$

b) $x \wedge 0 = 0$

Proof of b): $(x \wedge 0) = x \wedge (x \wedge x') = (x \wedge x) \wedge x'$.
Since $(x \wedge x) \wedge x' = x \wedge x' = 0$, we conclude that $(x \wedge 0) = 0$.

A question for a child, which tests the grasp of this result is: what elements do the following sets have in common? A set of colorless wooden beads (y) a set of various kinds of beads (x) and a set of blue wooden beads (y'). Theoretically this yields that $y \wedge x \wedge y' = x \wedge y \wedge y' = x \wedge 0 = 0$ and should be grasped late in the concrete stage. The point of this theorem is that, although the child can understand that y and y' have no elements in common when he directly confronts them, when x is considered between y and y' and prevents them from being directly considered the younger child may be confused but the older child, on the later part of the concrete stage, is usually not confused by x. He grasps that $y \wedge x \wedge y' = 0$. A demonstration indicating the validity of this result will be described in Chapter Five.

An important consequence of this P-lattice model is a theoretical prediction of some of Piaget's conservation studies (of lengths, areas, volumes, etc.; e.g., see Piaget, et all, 1960, The Child's Conception of Geometry, part 4, and Flavell, 1963, pp. 245-249). Since the idea of numerical order is grasped early in the concrete stage, then, if a and b are any two numbers $a < b$ or $a = b$ or $a > b$ (Trichotomy Law). (This corresponds to the fact that if A and B are any 2 classes in a hierarchical series either $A \subset B$ or $A = B$ or $A \supset B$.) In addition, if two distinct hierarchies are conceived of, then they can be multiplied in accordance with the Trichotomy Law. That is, the various logical multiplications involving A's and B's can be performed and conceived of in order. For example, consider the horizontal radius (H) and the vertical dimension (V) of a glass in a volume conservation experiment; let H_1, V_1 be those dimensions of one glass of water and H_2, V_2 be those dimensions of another glass of water, both capable of containing the same volume of water (imagine the water of one glass poured into the other in the experiment). With $H_1 < H_2$ and $V_1 > V_2$ being conditions of the experiment, the child must determine the possibility (which is the actuality) that ${H_1}^2V_1 = {H_2}^2V_2$. the model then suggests that three logical possibilities for the two volumes (${H_1}^2V_1 > {H_2}^2V_2$, $H_1^2V_1 = H_2^2V_2$, $H_1^2 V_1 < H_2^2 V_2$) may be intuitively grasped on the latter part of the concrete level. This is somewhat at variance with Piaget's contention that conservation of volume occurs at the formal stage (Piaget et al, 1960, p. 354). This intuitive understanding of conservation of volume stems from the Trichotomy Law and logical multiplication, however, and not from formal proportions as in Piaget's analysis (Ibid.). What may be the case is that conservation of volume is partially a concrete level discovery but a clear verbal explanation of this by the child does not occur until the formal stage since it requires proportions. It may also be the case that conservation of volume occurs earlier in discrete cases than in continuous cases (stretching of plastic). The researcher has conducted brief volume conservation demonstrations with small groups of children and these demonstrations tend to substantiate these positions. Five children about age 10 solved the discrete volume problem and qualitatively described why the volumes in the two glasses were the same. They could not give a quantitative solution, however. It would appear that further experimentation along these lines might clarify these considerations.

Another interesting result of the second part of the concrete model involves the inverse of the inverse. This result concerning reversibility can be stated as: $(a')' = a$

Proof: $a \vee a' = I$, $a' \vee (a')' = I = (a')' \vee a'$.

(also $a \wedge a' = 0 = a' \wedge (a')'$).

Therefore $(a')' = a$ since the complement is unique.

We can interpret this to mean the reverse of the reverse is the original thing and a probing illustration of this can be found by asking the child: if there is a family of two brothers A and B, who is the brother of the brother of A? Other illustrative questions could involve the negation of the negation of a statement.

The earlier K-lattice model also yields the theorem (a')' = a but the psychological interpretation is somewhat different since (a')' = a would then translate as the partial complement of the partial complement of a is a. It would then be of interest to determine when in the concrete stage various reverse of reverses are grasped. (e.g., in addition to the above examples one might investigate the negation of a negation in numbers, the reverse of the reverse of a physical direction and of mathematical relations.)

CHAPTER IV

MODELS FOR THE FORMAL STAGE

The two mathematical models just presented serve a double purpose:

1) They present, from axiomatic considerations, a mathematical development encompassing, enlarging upon, and questioning Piaget's cognitive theory or the concrete stage. The eight groupings of Piaget (Piaget, 1949) may now be understood as exemplifications of the two models.

2) They provide a framework for testing Piaget's viewpoint concerning the concrete stage and a theoretical basis for a number of his most important assertions.

The task of the present part of this paper is to develop mathematical models for Piaget's formal stage of cognition. The models develop not only from Piaget's research but from an underlying principle motivating his research:

> The understanding of a given stage of cognitive development can be fully understood only by relating it to the stage from which it arises; the seeds for any cognitive stage are present in the preceding stage. The mathematical models for the formal stage are thus outgrowths of the mathematical models of the concrete stage. Further motivation for the specific nature of the models arises from fundamental changes in the child's intellectual development: whereas on the concrete level there tends to be concern on the child's part for the actual, before-the-eye reality, on the formal level the greater tendency on the child's part is toward the potential, the abstract possibilities of a situation. (Flavell, 1963, pp. 203, 204)

The concrete level individual tends to develop individual schemata or organizing structures for conservation of weight, length, area, etc. at different periods of time, while the formal level individual abstracts and generalizes in such a way that general schemata are developed for many ideas at one time; this is illustrated when he grasps and articulates the proportion theory for balancing weights and conserving volume.

Thus, for the formal stage, models are needed which allow for the consideration of the set of all possibilities of a given situation and for the transfer of the understanding inherent in a given situation to the understanding of another, analogous situation (i.e., transfer of abstract learning). It will be seen that these needs, as well as others to be discussed, find a good measure of satisfaction in the following Boolean algebra and group theoretic models.

Support for the distinctions among the pre-operational, concrete, and formal stages of cognition is given in the paper by Gyr, Brown and Cafagna (Gyr, Brown, Cafagna, 1967). They distinguish between phenotypic behavior (based on the more concrete aspects of a situation) and genotypic behavior (based on the abstract possibilities of a situation). They posed a problem to children, adults, and specially programmed computers. The problem involved the determination of which sets of switches in an electric network could activate a given light. A statistical analysis strongly suggests the existence of the three levels of cognition. However, it is interesting to observe that the results of the paper indicate that although there is a rough correspondence between the three cognitive levels and Piaget's assertions of the age levels at which they are attained, there are many individuals whose response to the problem of the paper belies any rigid correspondence between age levels and cognitive stages. This suggests that cognitive levels should not be thought of as mutually exclusive but rather as existing in varying amounts at various stages in people. Furthermore, the paper of Gyr, Brown, and Cafagna, informal investigations of this researcher, and Piaget's theory suggest a certain amount of fragility of cognition (Flavell, 1963, pp. 20-23). This means that under stress conditions there exists a tendency for an individual to revert to earlier levels of cognition. This is an interesting analogue to the effect of stress on affective behavior as put forth by psychoanalytic theory. Further experimentation would seem quite appropriate, particularly with individuals who have only recently achieved a particular level of cognitive development.

An experiment along these lines might involve an apparent violation of conservation of volume after it has been established at the concrete stage. Ask children, particularly at the end of the concrete stage and the beginning of the formal stage, whether volume is conserved when a sugar cube (which, unknown to them is <u>hollow</u>) is dissolved in water. The point to be determined is how and when the child gives an appropriate explanation for the nonconservation of volume which is discovered by comparing the sugar-water volume before and after the dissolving of the sugar. Does the child suggest that the cube is hollow or some equivalent explanation or does he tend to question whether volume is conserved in general; i.e., is his grasp of this concept fragile enough to be shaken?

In the following discussion some of the properties of the mathematical model of the formal stage will be introduced:

A collection of elements together with two binary operations denoted by $\vee$ and $\wedge$ form a Boolean algebra if, whenever x, y, z are any three elements, the following axioms hold:

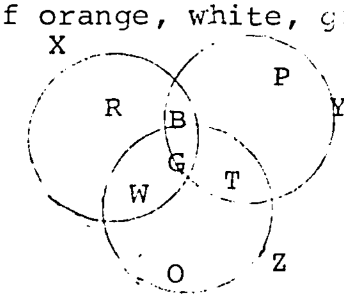
1)	$x \vee y$ is in the collection $x \wedge y$ is in the collection	CLOSURE OF $\vee$ and $\wedge$
2)	$x \vee (y \vee z) = (x \vee y) \vee z$ $x \wedge (y \wedge z) = (x \wedge y) \wedge z$	ASSOCIATIVITY of $\vee$ and $\wedge$
3)	$x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$	DISTRIBUTIVITY of $\wedge$ with respect to $\vee$
4)	There exists an element, 0, such that for all x, $x \vee 0 = x$	IDENTITY of $\vee$
5)	There exists an element, I, such that for all x $x \vee I = I$	UNIVERSAL ELEMENT
6)	$x \vee y = y \vee x$ $x \wedge y = y \wedge x$	COMMUTATIVITY of $\vee$ and $\wedge$
7)	For any x, there exists a unique x' such that $x \vee x' = I$ and $x \wedge x' = 0$	COMPLEMENT
8)	$x \vee x = x$ $x \wedge x = x$	IDEMPOTENT
9)	$x \vee (x \wedge y) = x \wedge (x \vee y) = x$	ABSORPTION
10)	$x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$	DISTRIBUTIVITY of $\vee$ with respect to $\wedge$

DISCUSSION OF THE MODEL

The primitive terms of the model may be interpreted as classes of propositions, and the two operations as representing unions and intersections (for the class interpretation) or disjunctions and conjunctions (for the proposition interpretation). The model is clearly an axiomatic outgrowth of the K and P-lattice models of the concrete stage. However, it does not exhibit the transformation property relevant to the hypothetical reasoning which an individual on the formal stage can perform. The mathematics associated with this will be discussed later in the paper and will involve the mathematics of group theory.

The assumption of the validity of the second distributive axiom in this model is somewhat speculative. In conjunction with the others it says that the individual at this stage can understand the fundamentals of intuitive set theory. Venn diagrams provide a clear realization of certain facts of set theory as well as of the algebra of logic. Therefore, experimental testing of the axioms may be performed by means of them (particularly for the distributive laws).

One might, for example, test whether $x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$ is understood by presenting a person with one collection of red, blue, green, and white blocks (x), another collection of purple, blue, green, and tan blocks (y), and a third collection of orange, white, green, and tan blocks (z).



See the accompanying figure. The question to be raised could be to compare what is in x <u>or</u> in y and z $((x \vee (y \wedge z))$ with what is in x or in y <u>and</u> in x or in z $((x \vee y) \wedge (x \vee z))$. Another example might be found by considering x, y, and z as members of the power set of a given set. The power set could be defined and the understanding of its meaning as well as that of the distributive law could be tested.

The problem involving Venn diagrams could be followed up with the problem of the person's finding the general statement of the distributive law no matter what kinds of objects are in the collections x, y and z.

The Boolean algebra structure has the algebra of propositions (relative to a given situation) as a specific realization. The understanding of propositions is then a psychological referent of the Boolean algebra model. By this we mean the following:

Let us assume that p, q, $\bar{p}$ (negation of p), $\bar{q}$ (negation of q) are the fundamental propositions involved in a given situation. Then it can be algebraically demonstrated (Birkhoff & MacLane, 1965, pp. 321-323) that the six conjunctions, $p \wedge q$, $\bar{p} \wedge q$, $p \wedge \bar{q}$, $\bar{p} \wedge \bar{q}$, $p \wedge \bar{p}$, $q \wedge \bar{q}$, together with all the possible disjunctions formed from two or more of these six at a time, form all the logically possible propositions written in terms of p, q, $\bar{p}$, $\bar{q}$. The collection of all such propositions, $(p \wedge q) \vee (\bar{p} \wedge q) \vee (p \wedge \bar{q}) \vee (\bar{p} \wedge \bar{q})$, $(p \wedge q) \vee (p \wedge \bar{q}) \vee (\bar{p} \wedge \bar{q})$,..., $p \wedge q$, ... $\bar{p} \wedge \bar{q}$, satisfies the axioms of Boolean algebra with $(p \wedge q) \vee (\bar{p} \wedge q) \vee (p \wedge \bar{q}) \vee (\bar{p} \wedge \bar{q}) = I$ and $p \wedge \bar{p} = q \wedge \bar{q} = 0$ and with $\vee$, $\wedge$ being interpreted as the usual operations of disjunction and conjunction in logic. As an illustration of of this let $x = \bar{p} \wedge q$ and $y = (p \wedge \bar{q}) \vee (\bar{p} \wedge \bar{q})$. Some of the results we can attain, according to the usual rules of logic are:

1) $x \vee y = (\bar{p} \wedge q) \vee (p \wedge \bar{q}) \vee (\bar{p} \wedge \bar{q})$ is in the collection

$x \wedge y = (\bar{p} \wedge q) \wedge [(p \wedge \bar{q}) \vee (\bar{p} \wedge \bar{q})]$

$= (\bar{p} \wedge q \wedge p \wedge \bar{q}) \vee (\bar{p} \wedge q \wedge \bar{p} \wedge \bar{q}) = 0$, and o is in the collection since $q \wedge \bar{q} = 0$

2) $x \vee (y \vee z) = (\overline{p} \wedge q) \vee [(p \wedge \overline{q}) \vee (\overline{p} \wedge \overline{q})]$

$= [(\overline{p} \wedge q) \vee (p \wedge \overline{q})] \vee (\overline{p} \wedge \overline{q}) = (x \vee y) \vee z$

(Assoc. of $\vee$)

3) $\overline{x} = (\overline{\overline{p} \wedge q}) = (\overline{\overline{p}}) \vee \overline{q} = p \vee \overline{q} = [p \wedge (q \vee \overline{q})] \vee [\overline{q} \wedge (p \vee \overline{p})]$

$= (p \wedge q) \vee (p \wedge \overline{q}) \vee (\overline{q} \wedge p) \vee (\overline{q} \wedge \overline{p}) = (p \wedge q) \vee (p \wedge \overline{q}) \vee (\overline{p} \wedge \overline{q})$

which is a member of the collection.

In this way it may be shown that all the axioms of Boolean algebra are satisfied. Psychologically, this can be interpreted as meaning that an individual whose cognitive ability is at the stage indicated by this model can consider and understand all the logical possibilities inherent in a situation. Another view of this is that he can count all the possible ways of combining collections of things; he can do combinatorial arithmetic. An experiment illustrating the latter view could involve asking how many nonsense words can be formed from the word HAIR. Another experiment, illustrating the understanding of the set of all possibilities in a given situation is described by Flavell:

> The child is given four similar flasks containing colorless, odorless liquids which are perceptually identical. We number them: (1) diluted sulphuric acid; (2) water; (3) oxygenated water; (4) thiosulphate; we add a bottle (with a dropper) which we will call g; it contains potassium iodide. It is known that oxygenated water oxidized potassium iodide in an acid medium. Thus mixture (1 & 3 & g) will yield a yellow color. The water is neutral, so that adding it will not change the color, whereas the thiosulphate (4) will bleach the mixture (1 & 3 & g). The experimenter presents to the subject two glasses, one containing 1 & 3, the other containing 2. In front of the subject, he pours several drops of g in each of the two glasses and notes the different reactions. Then the subject is asked simply to reproduce the color yellow, using flasks 1, 2, 3, 4, and g as he wishes. (Flavell, 1963, p. 207)

individual, on the other hand, lists all possibilities and considers them in a hypothetical deductive context. He considers, "what if" types of suppositions and necessary and sufficient conditions. He can finally solve the problem by discovering that mixing 1 & 3 & g is the necessary and sufficient cause of the color yellow. The route to a solution is intimately related to the linking of negation or inversion with reciprocity at the formal level since, for example, if one discovers that adding substance 4 is important for removing the yellow color from 1 & 3 & g, and not adding it (using inversion) allows yellow to remain, one must also consider what other chemical addition or subtraction effects the same kind of result (reciprocity) in considering all the other possibilities of the problem.

> Reciprocity entails not the outright elimination or negation of a factor but its <u>neutralization</u>, that is, holding its effect constant in some way while a second factor is being varied. For instance, where the problem is to study the separate effects of kind of metal and length on the flexibility of a rod, the younger concrete child finds himself at an impasse; he cannot literally negate either variable ... He (the formal child) takes two rods of different metals but of the same length (here length is not negated but neutralized or controlled...) in order to study the effect of kind of metal, and two rods of a single metal and different lengths to study the effect of length. (Flavell, 1963, pp. 209-210)

Thus, the link between inversion and reciprocity at the formal stage allows for the greater investigating power inherent in considering all the possibilities of a situation and the various ways of achieving the same effect. Inversion, in Flavell's analysis, would correspond to the child's solving the problem by completely eliminating metal or flexibility, which is impossible. He, therefore, solves the problem by holding one variable fixed and varying the other. This is an example of reciprocity. The group model, soon to be considered, further develops this linkage. However, we can see its operation even within the Boolean algebra framework. Two theorems which link inversion with reciprocity are:

1) If $x \leq y$ then there exists an $\underline{a}$ such that $x \vee \underline{a} = y$ and $x \wedge \underline{a} = 0$ ($\underline{a}$ acts like the relative complement of the concrete model.)

   Proof: There exists an x' such that $x \vee x' = I$ and $x \wedge x' = 0$. We will show that $y \wedge x'$ plays the role of $\underline{a}$. Now, $x \vee (y \wedge x') = (x \vee y) \wedge (x \vee x') = (x \vee y) \wedge I = x \vee y = y$ (since $x \leq y$). Note that the second distributive law is needed and used in this proof. Thus, the theorem is a formal stage property. Also $x \wedge (y \wedge x') = (x \wedge x') \wedge y = 0 \wedge y = 0$. Therefore $a = y \wedge x'$.

2) We can now prove from the above theorem that $x \leq y$ and $y \leq x \rightarrow x = y$:

   $x \leq y$ means that there exists an $\underline{a}$ such that $x \vee \underline{a} = y$ and $x \wedge \underline{a} = 0$

   Also $y \leq x$ means that there exists a $\underline{b}$ such that $y \vee \underline{b} = x$ and $y \wedge \underline{b} = 0$.

Now $x \vee y = (y \vee \underline{b}) \vee y = (y \vee y) \vee \underline{b} = y \vee \underline{b} = x$ and

$x \vee y = x \vee (x \vee \underline{a}) = (x \vee x) \vee \underline{a} = x \vee \underline{a} = y$. Hence, $x = y$ Q.E.D.

Since this latter theorem is a form of reciprocity,[1] and it was proved using the partial complement property which is a form of <u>inversion</u>,[2] we then have a link at this stage between inversion and reciprocity. (i.e., theoretically, understanding one of these properties means understanding the other and being able to consider it at the same time.)

The Boolean algebra aspect of the formal operational stage model develops the combinatorial mode of thought of this stage (i.e., consideration of all the logical possibilities of a situation, the collection of all sets, the separation of sets into mutually exclusive subsets etc.). However, for fuller development of abstract, logical thought another aspect of the formal operation stage model, which deals with the strategies needed in understanding cause-effect relations and the transformations of a system which leave certain features invariant, is needed.

Intimately related to these relations and transformations and to the general development of logical schemata is the logical notion of proportion which is a formal operation stage concept (Piaget, 1957, Chapter 4). An example illustrating these aspects of the model (cause-effect reasoning, use of proportions) can be found when the child learns about

---

1. since $x (\leq) y$ is neutralized but not directly negated by $x (\geq) y$, leaving the sole possibility that $x = y$

2. which involves outright use of negation or complementation

equilibrium in the simple balance (Inhelder & Piaget, 1958, Chapter 11). Not only must the child be able to determine the various possibilities concerning increasing and decreasing weights and their distances from the fulcrum on either side of the balance but he must also be capable of deciding what transformations will have the same effect (i.e., achieve balance). Moreover, he must ultimately realize that the weights and their distances must achieve appropriate proportions to achieve equilibrium. Let us consider p = increase the weight on the left side, $\bar{p}$ = decrease the weight on the left side, q = increase the weight's distance on the left side, $\bar{q}$ = decrease the distance on the left side, p', $\bar{p}'$, q', $\bar{q}'$ represent the analogous statements for the right side. The combinatorial aspect of the solution to the balance problem involves the consideration of all the logical possibilities, i.e., the understanding of the Boolean algebra of all propositions formed from p, q, p', q' and their negations.

Since the number of such propositions is 2^{2^4} (Birkhoff & MacLane, 1965, pp. 263-265) this would be a formidable task were it not simplified by the child's ability at the formal level to consider the proportion relation between p, p' and q, q'. (The effect of p (increasing the weight on the left side) is nullified by a proportionate distance increase on the right side (i.e., by q') or by p' or $\bar{p}$ and vice versa.) Therefore, the problem may be coped with by considering p, $\bar{p}$ q, and $\bar{q}$ alone. The kinds of transformations which are like q', which nullifies p but not directly and vice versa, have already been referred to as being reciprocal and are linked to the reverse, $\overline{p}$, which achieves the same effect albeit directly. We have then, at this stage, an inter-related understanding linking the reciprocal, denoted by R, which nullifies a state of affairs without direct inversion, and the inverse, denoted by N, which directly negates and therefore nullifies a state of affairs. The solution to the equilibrium problem can now be approached by considering only p, q, $\overline{p}$, $\overline{q}$ which then involves the consideration of 16 different possible propositions (ibid). The law that the left weight times the left distance to the fulcrum must be the same as the corresponding product on the right side, can then be suggested by considering which combinations of p, q, $\overline{p}$, $\overline{q}$ achieve equilibrium and which do not, and how these combinations are related to p', q', $\overline{p}'$, $\overline{q}'$.

The fundamental task of this aspect of the formal stage is to develop means of clarifying the various transformations to be considered in solving the problem. For a complete analysis involving p and q, the individual operating on the formal stage must be able to consider, for example, $p \wedge q$, its inverse and its reciprocal. The INRC (or four-group) proposed by Piaget (Piaget, 1957, p. 33) yields an abstract system of transformations from which we can develop the desired properties:

$$\text{let } I(p \wedge q) = p \wedge q$$
$$N(p \wedge q) = \overline{p \wedge q} = \overline{p} \vee \overline{q}$$
$$R(p \wedge q) = \overline{p} \wedge \overline{q}$$
$$C(p \wedge q) = p \vee q$$

If we interpret I, N, R, and C in the equilibrium problem, we note that:

1) I is the identity transformation (the formal mind can consider increasing weight and distance and then change nothing) - symbolically, $p \wedge q \xrightarrow{I} p \wedge q$

2) N is the inverse transformation (increasing both weight and distance can be considered and then decreasing the weight or the distance can be considered) - $p \wedge q \xrightarrow{N} \bar{p} \wedge \bar{q}$

3) R is the reciprocal transformation (decreasing both weight and distance may be considered and this decrease can be related to both I and N and C) - $p \wedge q \xrightarrow{R} \bar{p} \wedge \bar{q}$.

4) C is the correlative transformation (either weight increase or distance increase may be separately considered) - $p \wedge q \xrightarrow{C} p \wedge q$. C is introduced to generate and complete the group. It does not arise naturally as do N and R.

There is no suggestion intended here that N or R or C necessarily produces equilibrium in the system but, rather, in order for the formal mind to attain equilibrium it must be able to consider these transformations. (GLT, p. 320). The protocols cited in GLT (chapter eleven) clearly indicate the presence of thought patterns involving $p \wedge \bar{q}$, $\bar{p} \wedge q = R(p \wedge \bar{q})$ in those who solve the weight balancing problem and the lack of such patterns in the younger child who does not solve the problem, i.e., N, R, and C are manifested on the formal stage in an integrated manner and not before.

This latter fact suggests the use of the INRC group (or four-group) in the mathematical model of the formal stage as a means of bridging the gap between the consideration of all logical possibilities and the specific ones needed in solving a problem. The abstract definition of the group may be given in the following tabular analysis:

	I	N	R	C
I	I	N	R	C
N	N	I	C	R
R	R	C	I	N
C	C	R	N	I

The elements of this group are to be thought of as being capable of transforming logical statements. This is illustrated in the weight balancing problem. The individual, at the formal stage of development (partially characterized by the group), is mentally capable of considering logical statements, their reciprocals, inverse, and correlates. Two other realizations of the group, indicating other considerations available at the formal stage, are:

1) $I(p \vee q) = p \vee q$

$N(p \vee q) = \overline{p \vee q} = \bar{p} \wedge \bar{q}$

$R(p \vee q) = \bar{p} \vee \bar{q}$

$C(p \vee q) = p \wedge q$

2) $I(p \rightarrow q) = p \rightarrow q = \bar{p} \vee q$

$N(p \rightarrow q) = p \wedge \bar{q}$

$R(p \rightarrow q) = p \rightarrow q = p \vee \bar{q}$

$C(p \rightarrow q) = \bar{p} \wedge q$

This latter exemplification of the INRC group as a model of the formal stage indicates the ability of the formal stage mind to do hypothetico-deductive thinking, i.e., to consider the possibilities of a situation and isolate those aspects which are in cause-effect relation.

A good example of this kind of reasoning, found on the formal stage and not before, is found in GLT:

> An important example is the principle of inertia. If the subject wants to demonstrate the conservation of uniform rectilinear motion in a controlled experiment, he has to face the fundamental difficulties that any motion which is created experimentally is eventually slowed down by external obstacles and that his observation is limited in space and time. Thus, the inertial principle has to be deduced and verified from its implied consequences. Strictly speaking, it does not give rise to observable empirical evidence.
>
> However, in relation to the structure of the INRC group of four transformations to which we have already referred in explanation of the formation of the notion of equilibrium, the substage III-B (late concrete stage) subjects do come to discover an elementary process whose starting point is the obstacles which would stand in the way of verification -- i.e., the causes of loss of motion. The reasoning which follows is extremely simple but that much more significant. From the fact that when any object loses motion (stated by p) the intervention of observable variables is implied (stated by qvr vs...), they come to the hypothesis that in eliminating all of these variables (i.e., $\bar{q}$, $\bar{r}$, $\bar{s}$...), all loss of motion would be eliminated at the same time; the result would be conservation of motion (m) with its rate of acceleration.
>
> If $p \supset (q \vee r \vee s \ldots)$, then $\bar{q}.\bar{r}.\bar{s} \ldots \supset \bar{p}$, where $\bar{p} \supset m$.
>
> We can see how this reasoning uses, simultaneously, negation or inversion (N) and contraposition which is a form of reciprocity (R). (GLT, pp. 328, 329)

Another example, which illustrates the link between N, R and the group involvements of I, N, R, and C of the formal stage, while illustrating the lack of these at the concrete stage, is found in the example of the snail moving on a plank:

> In this experiment, a snail is set in motion on a plank which can be moved either in the same direction as the motion of the snail or in the opposite direction. The subjects at the concrete level know very well that the snail can move from left to right, then return from right to left by

> an inverse operation which cancels the preceding. Likewise, they know that if the snail is immobile on the plank, moving it from left to right will cause the snail to end up at the same point (in relation to an external frame of reference) and that the opposite motion would return him to his starting point. But it is not before the level of formal operations that predictions can be made for both sorts of motion simultaneously, for in this case two systems of reference must be coordinated, one of which is mobile and the other immobile. The difficulty lies, for example, in understanding the fact that a movement from left to right made by the snail can be compensated by a displacement of the plank from right to left; in this case the snail remains in the same place (in relation to the [external] frame of reference) without any reverse movement.
>
> Actually, the difficulty in these problems lies in distinguishing and combining two types of transformation: (1) cancellation (for example, when the snail returns from B to A after having moved from A to B); and (2) compensation (for example, when the snail goes from A to B while the plank is displaced from B to A). Thus the problem involves the coordination of two systems, each involving a direct and an inverse operation, but with one of the systems in a relation of compensation or symmetry with respect to the other.
>
> Moreover, one can see immediately that this coordination is the same one that is attained by the INRC group, since N is the inverse of I, and C or R, whereas R is symmetrical to or compensates I (reciprocity). So the problem is to distinguish inversion N from reciprocity R at the same time that one is coordinating them. This is why the problem cannot be solved before the formal level, when a schema based on the INRC group is in operation.
>
> In other words, if we call I the snail's motion from A to B, N will be the motion from B to A; R will be the plank's motion from B to A (thus R = C of N) and C will be the plank's motion from A to B (thus C of I = N of R). (Inhelder and Piaget, 1958, pp. 318, 319)

Although the analysis properly distinguished the concrete and formal level approaches to the problem as well as the linkage between inversion and reciprocity, the mathematical analysis is unclear and/or wrong. For example, I does not operate as the identity element since IN $\neq$ N. In fact IN is not one of the listed transformations since the snail remains fixed under IN.

A more appropriate analysis would be:

Let p = the snail moves from A to B

$\bar{p}$ = the snail moves from B to A

q = the plank moves from A to B

$\bar{q}$ = the plank moves from B to A

Then, if we let

$$\left.\begin{aligned} I(p) &= p \\ R(p) &= \bar{q} \\ N(p) &= \bar{p} \\ C(p) &= q \end{aligned}\right\} \text{with similar definitions for } I(q),\ R(q),\ I(\bar{p}), \text{ etc.}$$

We see that I, N, R, and C satisfy all the four-group properties. For example, $IR(p) = I(\bar{q}) = \bar{q}$ and $NC(p) = N(q) = \bar{q}$, which indicates that IR = NC. An important difficulty for the concrete stage which is mastered on the formal stage is that p and $\bar{q}$ give the same result as $\bar{p}$ and q. In other words the snail moving from A to B while the plank moves from B to A ($p \wedge \bar{q}$) results in no relative motion of the snail (equilibrium). On the other hand, the snail moving from B to A while the plank moves from A to B ($\bar{p} \wedge q$) also results in equilibrium. Therefore, since the results of $p \wedge \bar{q}$ and $\bar{p} \wedge q$ are the same they are operationally the same.

Further investigation of the group's table shows that it is commutative (i.e., if x and y are elements $xy = yx$). This is made clear by noting the symmetry of elements on either side of the diagonals of I's. On a psychological level this suggests a symmetry principle operating on the formal stage. Since the INRC transformations of thought exhibit this symmetry property, an interesting speculation is whether a search for symmetry operates in other ways on the formal level. For example, the solution of the balance problem shows symmetry considerations when what occurs on the left side is thought to be symmetric to what can occur on the right side. An analysis of concrete and formal children (and adults) when faced with the problem of relating size and distance of objects and images formed in a mirror might further test this symmetry principle. The problem to be posed might be to measure the size of and distance between several objects by only using their mirror images and not the objects themselves. Another problem could be to have an individual draw a given map as it would appear in a mirror. The suggestion is that symmetry would be involved and simplify the problem for formal minds but not for concrete minds. A further test might be the comprehension or fashioning of a "proof" that the base angles of an isosceles triangle are equal by symmetry considerations (consider the upper vertex angle bisected by a plane mirror). When appropriate, one should have the subjects only consider the mirror image and not the original objects.

The final aspect of the INRC group to be considered in this paper relates to the theory of logical proportions and its consequent analogic development and use in a theory of numerical proportion. Following Piaget's general approach (Piaget, 1957, pp. 35-37) but with a modified use of his mathematics let us consider x, y, z, and w as arbitrary elements in the INRC group:

1) $x/y = z/w$ will mean that $xw = yz$ in the group. For example, since $IN = RC$ $I/R = C/N$ and $I/C = R/N$ etc.

2) if p, q, r, s are propositions such that there exists a proposition t where $x/y = z/w$ and

$$x\ (t) = p$$
$$y\ (t) = q$$
$$z\ (t) = r$$
$$w\ (t) = s$$

Then we will say that $p/q = r/s$ (i.e., if p, q, r, s can be made to "correspond" to x, y, z, w then if the latter are in proportion so are the former).

For example, since $I/R = C/N$ and

$$I\ (p \vee q) = p \vee q$$
$$R\ (p \vee q) = \bar{p} \vee \bar{q}$$
$$N\ (p \vee q) = \bar{p} \wedge \bar{q}$$
$$C\ (p \vee q) = p \wedge q$$

we have that

$$\frac{p \vee q}{\bar{p} \vee \bar{q}} = \frac{p \wedge q}{\bar{p} \wedge \bar{q}}.$$

Since another realization of the INRC group is gotten by considering $I\ (p) = p$, $R\ (p) = \bar{q}$, $N\ (p) = \bar{p}$, $C\ (p) = q$ etc., as in the snail on the plank problem, we now have, using $I/C = R/N$, that $p/q = \bar{q}/\bar{p}$. Let us define these propositions to be in accord with the following and thereby use the theory of logical proportions to stimulate the theory of numerical proportions:

p = multiply x by a factor of n (nx)

$\bar{p}$ = multiply $\frac{1}{x}$ by a factor of n (n:x)

q = multiply y by a factor of n (ny)

$\bar{\text{q}}$ = multiply $\frac{1}{y}$ by a factor of n (n:y)

We then have the analogous results from the logical proportions that nx/ny = n:y/n:x for all n and if ab = cd then a/c = d/b (where a - nx, b = n:x, c = ny, d = n:y), which are fundamental properties of the theory of numerical proportions. The importance of this result lies not in the fact that at this level the child can handle ratios. In fact he is capable of such things earlier in the concrete stage. However, he can now consider the collection of all ratios which are equal. He can grasp therefore the general proportional analysis needed to fully solve the conservation of volume problem, equilibrium of the balance problem etc. (i.e., all those problems requiring a theory of proportions and a propositional analysis to elucidate them). In terms of the conservation of volume problem he can now not only operate concretely to guess at volume conservation but he can anticipate the use of proportions in justifying (proving) the conservation. A direct empirical test of the position that only at the formal stage can a general theory of proportions and the collection of equal ratios be grasped could be to ask the child to characterize all fractions equal to, say, one half. The theoretical prediction would be that individuals on the concrete stage would only specify a number of equal forms and individuals on the formal stage would yield a general approach for all fractions equal to one half -- perhaps something like the numerator should equal half the denominator and/or the product of the means equals the product of the extremes. Of course a difficulty introduced at this stage is the effect of previous learning (in school) on the general understanding of ratio and proportion. However, it may be that "learning" these things in school at a true concrete level may prove to be fragile and deeper questioning may indicate whether, in fact, learning or mere memorization has occured. (Unpublished research of Dr. Edward Henderson of New York University suggests the latter.)

Such experimentation might be carried out in unschooled populations or using children and adults who have had little schooling. One might "teach" a concrete level group fractions and proportions and compare their understanding of these ideas when they attain the formal stage to that of another formal stage group which is unschooled in fractions and proportions.

CHAPTER V

DEMONSTRATIONS AND CLARIFICATIONS CONCERNING THE MODELS

Although the research which has been presented was essentially geared to developing the mathematical models of Piaget's theory, a number of demonstrations suggesting the validity of the models were discussed in Chapters three and four. In this chapter, more demonstrations, which further suggest the validity of the models, will be discussed. It should be emphasized that these are not full scale experiments. Such experiments are not within the purview of the present, theoretical research. The researcher, however, hopes to engage in such experiments in the future. An additional aspect of this chapter will be a clarification of a number of important results of this research.

The first demonstration is concerned with the experimental generalization of Piaget's bead experiment (below p. 28). It was structured as follows:

Consider a collection of 21 small pieces of paper (z), most of which (15) are square (y) and few of which (6) are round (y'). Color 12 of the squares white (x) and the other 3 gray (x'). Color 3 round pieces white and 3 round pieces gray. Since $x \vee x' = y$ and $y \vee y' = z$ and, logically speaking, both x' and $x' \vee y'$ function as partial complements (relative to different universes) for x, one could ask the following questions of a child:

1) Are there more paper squares than white paper squares (and why)? (also vice versa).

2) Are there more pieces of paper than white paper squares (and why)? (also vice versa)

The K-lattice model suggests that early in the concrete stage only one of the questions would be affirmatively coped with (p. 29).

All the demonstrations cited in this chapter were carried out with school children in Marblehead, Massachusetts. Three groups each containing six children were used. The first group were in first grade and six years old. The second group were in second grade and seven years old. The third group were in fourth grade and nine years old. All the children were economically from middle class to upper middle class families. From prior experiences and tests in their classrooms the researcher concluded that the three groups were on the stages indicated in the following discussion. That is, essentially, the first group was on the pre-concrete stage, the second was on the border line between concrete and pre-concrete and the third group was on the concrete stage.

The first group all responded in the manner of the pre-concrete stage. The typical answers were, "There are more white squares than paper squares because there are more white squares than gray squares" and "There are more white paper squares than pieces of paper because there are more whites than grays".

The second group had an interesting range of responses. Two responded as the first group did. Three responded in a mixed manner. Two of these said that there were more white paper squares than pieces of paper <u>but</u> that there were more paper squares than white paper squares. The third, however, asserted there were more pieces of paper than white paper squares <u>but</u> there were more white paper squares than paper squares. The sixth answered both questions "properly", as a late concrete stage child would.

The third and oldest group all responded in the "proper" manner to both questions although three of them showed distinct hesitation. However after one question was answered the other was always answered spontaneously and "properly".

Although this demonstration is not a fully valid experiment it does suggest the validity of the prediction of the K-lattice concerning these questions. A proper experiment should not only include a greater number of children but other tests to determine whether or not a given child is on the concrete stage.

The second demonstration concerned the asymmetry property implied by the K-lattice:

$$a \textcircled{\leq} b \text{ and } b \textcircled{\leq} a \Rightarrow a = b$$

The model suggests that pre-concrete children do not, and concrete children do, grasp this idea. The procedure involved comparing the number of marbles in one bag (a) to an equal number of marbles in the researcher's hand (b), The child, unaware of the relative number of marbles, paired off marbles he took from the bag, with marbles the researcher gave him one at a time from his closed hand. When the marbles in the bag were used up (i.e., $a \textcircled{=} b$) the researcher (who did not at any time disclose what remained in his closed hand) <u>interchanged</u>

the original marbles of the bag and the hand and told this to the child. The corresponding matching ensued until the marbles of the bag were used up again (i.e., $b \leq a$). The child was then asked to compare the amount of marbles originally in the bag and in the hand. None of the first graders realized $a = b$. Three of the second graders realized $a = b$ and three of them did not. Two of the latter also gave pre-concrete answers to both questions of the first demonstration. (It would appear that they were not yet on the concrete level). The fourth graders had no trouble realizing that $a = b$. The suggestion of this demonstration is that understanding of the asymmetry property is achieved at the concrete operational stage and not before.

The third demonstration tested when the idea $X \wedge 0 = 0$ is understood. The P-lattice model suggests that this occurs late in the concrete stage (p.41). The children were shown three different sets of five checkers each; one set was made up only of red checkers (y), another set had 3 red and 2 black checkers (x), the third set had 5 black checkers (y'). They were then asked, "Is there a color of checkers which is the same in all three bunches of checkers?" The first group contained two children who first oscillated back and forth among the groups of checkers and finally answered, "No, the blacks are in there and not there and the reds are here but not here." The other four didn't answer the question "properly". They either said they didn't know or that the answer was, "Yes." The second group contained three children who oscillated at first and then said, "No." The other three went back and forth among the checkers and either answered, "Yes" or "I don't know". All of the third group quickly answered, "No." The results here are somewhat ambiguous and fear further investigation with larger numbers of children, since some apparently early concrete stage children appear to grasp the idea $x \wedge 0 = 0$.

The fourth demonstration was concerned with the associative law for the P-lattice model. In this case only groups one and three were used. Three cans containing 14, 19, 26 marbles each were used. The first two were kept close together, separated by a foot from the third. The children were asked to record the number in the first two separately and together and then the third separately. They were then asked to put the second and third close together separated from the first. They were then asked to record the number in the first and the number in the second and third both separately and together. They were then asked to compare the two totals. Several older children of the third group started to add the numbers in each computation but were cautioned not to and did not complete the addition. The first group could not compare the two results. The third group, without any apparent addition, contained five who answered that the number of marbles was the same both times because, "It didn't matter how you add them up". The sixth member of the group hesitated for a while and answered that they were the same. Before accepting the suggestion of this demonstration that associativity is understood on the concrete stage and not before one should test the effect of schooling on this concept and whether it is genuinely understood or is accepted as dogma. It would be worth while investigating unschooled children and adults in this regard.

The remainder of this chapter will constitute a clarification of some of the important results of chapters three, four and five.

It is of some interest that the first theoretical result concerning the bead experiment (cited after the K-lattice) was quite serendipitous. The researcher did not consider it at all in forming the axioms of the K-lattice. It is quite striking, therefore, that one of the important ways whereby Piaget characterizes reasoning on the concrete stage arose quite naturally from the first model presented in the research. The mathematical theorem $X \vee X' = Y \Rightarrow X \vee Y = Y$ (i.e., $x \textcircled{\leq} y$) was interpreted as meaning, psychologically, that if one understands that a given sub-class and its complement make up a larger class then, <u>at the same time</u>, one understands that the sub-class is smaller than the given class. This result also pointed the way (together with the first axiom of the K-lattice concerning the incompleteness of the $\wedge$ operation) to the development of the first demonstration of this chapter.

Another important result involves a mathematical rendering of Piaget's notion of <u>reciprocity</u> on the concrete stage:

$$X \textcircled{\leq} Y \text{ and } Y \textcircled{\leq} X \Rightarrow X = Y$$

Reciprocity is exemplified here as the achieving of balance ($X = Y$) not <u>directly by negating</u> the possibility of imbalance but <u>indirectly</u> by offsetting $X \textcircled{\leq} Y$ by $Y \textcircled{\leq} X$ thereby leaving the sole possibility that $X = Y$. (This result was instrumental in developing the second demonstration of this chapter). This form of reciprocity is the precursor of the reciprocity which appears on the formal stage. At this latter stage, understanding reciprocity becomes linked with understanding inversion. This is indicated both mathematically and empirically in the analyses of the chemical combinations experiment and the snail experiment of chapter four. The mathematical theorem

$$X \circledleq Y \text{ and } Y \circledleq X \Rightarrow X = Y$$

is shown to be valid in the Boolean algebra model and the deep link between inversion and reciprocity is established in the analysis of inertia and the INRC model in chapter four.

The use of the Boolean algebra, K-lattice, and P-lattice models constitutes the first time, as far as the researcher knows, that a sound mathematical base was given to Piaget's theory of cognition. As was indicated earlier in the research (particularly in chapter two), Piaget himself does not provide such a base for his theory. His mathematics is often unclear and his axiomatic basis is not mathematically sound.

Let us now compare the theory and results of the demonstrations of chapter five with what could be expected of Piaget's own theory in this regard. The first demonstration is based on axiom 1 of the K-lattice (p. 25) and the theorem derived from the K-lattice on page 28 (if $x \vee x' = y$ then $x \circledleq y$). In other words the present research, based upon the incompleteness of the intersection operation of sets early in the concrete stage and an extension of Piaget's bead experiment (p. 28), led to the first demonstration. The only reference that Piaget makes to the incompleteness of the intersection operation is in his discussion on the multiplicative grouping (Piaget, Logic and Psychology, p. 28). The incompleteness of intersection does not allow for a full development of multiplication of classes and relations early in the concrete stage. However, he makes no explicit use of this incompleteness nor is it ever mentioned subsequently in his writings as far as the researcher could determine. It would appear then that he might not at all raise the questions which the K-lattice model poses in this regard.

The second demonstration stemming from the K-lattice involves reciprocity with operations of relations on the concrete stage ($a \leq b$ and $b \leq a \Rightarrow a = b$).

Piaget makes clear reference to this form of reciprocity (Piaget, Logic and Psychology, p. 29). He does not however suggest an experiment similar to the one in this research in order to verify it. Flavell's analysis of asymmetric relations suggests that Piaget's procedure for the second demonstration would be very similar to that found in this research (Flavell, 1963, pp. 193-195). What is new in this research in this regard is that Piaget states the reciprocity condition while this research derives it as a consequence of the K-lattice model.

The third demonstration involved a subtle point. If one assumes that children can perceptually determine that sets y (red checkers) and y' (black checkers) have no common elements (i.e., $y \wedge y' - 0$) then what conclusion do children reach who much mentally juxtapose y, y', and x (the mixture of black and red checkers)? In other words the question is not whether $y \wedge y' = 0$ but given that $y \wedge y' = 0$ to determine whether

$x \wedge (y \wedge y') = 0$ Piaget has never raised a question like this as far as the researcher could determine. His models do not explicitly lead to such a question but they are quite consonant with such a view. He does make reference to the intersection of mutually exclusive classes (Piaget, Logic and Psychology, p. 127), but he does not pursue it further in this direction.

The fourth demonstration cited in this chapter is anticipated by Piaget in his first model based on simple classification (Piaget, Logic and Psychology, p. 27). He speaks of the incompleteness of associativity early in the concrete stage and thereby suggests that the psychological structuring of associativity is attained later on (presumably in the concrete stage). However, although he refers many times in his research to additive and multiplicative composition of classes, particularly in The Child's Conception of Number he cites no experiments relating to associativity (Piaget, 1952, chapters 7 and 8). The demonstrations cited in this chapter, then, are outgrowths of the models presented in this research and appear to indicate extensions of Piaget's theory. However, they do not appear in the body of Piagetian research so far as the researcher could determine.

CHAPTER VI

DISCUSSION AND SUMMARY

The models, demonstrations, and experiments which have been discussed in the previous five chapters indicate the validity of the hypotheses put forth at the start of the research. This does not mean that all aspects of the models have been found to be valid. However, some axioms and some of their consequences have been translated into psychological terms and some of Piaget's important experiments, as well as the researcher's demonstrations, have been shown to be consonant with predicted results. Further investigation, moreover, involving experiments suggested by the research and the demonstrations, is needed to determine the validity of all aspects of the models. The research, then, does bolster Piaget's view that logical structures can be used to facilitate our understanding of the intellectual development of the child, the adolescent and even the adult (Piaget, 1957, p. 48). The use of logical structures in describing nerve networks and in cybernetics gives further credence to his point of view.

Several further points of view which the research poses are: 1) Since the INRC group analyzes propositions as constant with respect to time, can a structural analysis involving propositions which vary with time subsume the results of the INRC group? (See Apostel, 1966). Leo Apostel suggests that the model for the formal stage is too static and that consideration of propositions as functions of time may yield mathematical structures which correspond more closely with cognitive capabilities. This would appear to be a promising line of research which might enlarge upon the model this research presents for the formal stage.

The static model could be useful for studying individual differences on the concrete and formal stages. It is noteworthy that Piaget has not concerned himself with such problems. It would certainly seem appropriate for developmental studies that time be considered an important variable.

2) Can the model of the formal operational stage represent the culmination of abstract thought, since the mathematical understanding of the INRC group as well as of group theory in general would suggest more complex structures for a model characterizing such cognitions? Further research might properly consider different models for the cognitions of mathematicians, logicians, etc. Perhaps the time dependent models, referred to in 1) above, could be used to characterize the more complex cognitions of theoretical scientists.

3) Since Piaget's theory, including both psychological and mathematical aspects, is one concerning the nature of intelligence, it is appropriate to inquire whether it provides a natural form to test intelligence. Indeed, it would seem that its concern with the <u>structures of the intellect</u> renders it much more appropriate as an intelligence measuring device than are statistically based, middle class oriented intelligence tests. The fundamental involvement with the operational as well as the verbal components of intelligence might tend to render it free of cultural biases. Implementation of Piaget's theory in defining and estimating intelligence would seem to be highly

desirable. The work of Guilford is illuminating in this respect. He has developed a "structure of intellect" theory which can fill in the interstices of Piaget's theory. He suggests specific categories involving figural, symbolic, and semantic meanings which can provide detailed examples for characterizing the several stages of Piaget's theory and thereby yield further understanding of what intelligence mea (Guilford, 1967 - particularly Chapter 9).

4) If the theoretical predictions of the models are empirically verified, then Piaget's cognitive theory would be more firmly established. The confirmation of this theory which concerns the nature of intelligence, knowledge, teaching, and learning should, it would seem, eventuate in a significant restructuring of the educational process, emphasizing the structural development of intelligence, the relative need for student independence in learning how to learn (e.g., developing strategies for learning) and the need for independent, knowledgeable teachers whose creative function involves realizing crucial junctures when it would be appropriate for them to gently elicit responses and understanding from the student. For example, if a child about age twelve experiences great difficulty in understanding the converse of an implication, it might be appropriate to direct him to understanding analogues to it (in the equilibrium of the balance experiment or in a Venn diagram analysis of sets).

5) The fragility of cognitive structures and their possible dissolution under stress, particularly at the beginning of their acquisition in a given stage, suggests the co-existence of various structures in varying amounts at each stage of Piaget's

theory. This is strikingly similar to the psychoanalytic theory of regression in times of psychic stress. It also suggests the appropriateness of determining the kinds and the distributions of cognitive structures exhibited by adults. There seems to be an implicit assumption in Piaget's research that most adults achieve the formal stage of cognition. This assumption should be open to further investigation.

6) What is the nature of the concept of number? The view of intuitionist mathematicians is that the whole numbers are intuitively conceived and constructed <u>before</u> logic and mathematics can be developed. The logicians view logic as being prior to the development of number with cardinality and order as separate concepts. From a Piagetian point of view both philosophies have the major difficulty that if one wants to understand the fundamental nature of mathematics and its relation to man's mind, one must understand how cognitions develop; i.e., one must have a psychology which directs one's philosophy. Both the logician's and the intuitionist's views lack psychological structure concerning when and how number is understood and what is meant by understanding it. The Piaget view places number understanding at the beginning of the concrete stage and uses the concept of reversibility (as in the bead counting experiment) to define it: one understands a given number when one understands what it is not; the concept of number is conserved (is kept intact under psychological and physical transformations) and understood when cardinality and ordinality are both grasped early in the concrete stage. Thus the intuitionist view that the whole number concept is prior

to formal logic is affirmed by Piaget. Although the logician's view that cardinality and ordinality are separate concepts is logically acceptable, it is not psychologically tenable in Piaget's theory. In other words they are not learned independently although they are separate concepts (Flavell, 1963, p. 311). This theory also opens questions concerning the law of the excluded middle (which is only acceptable to the intuitionist view when a finite number of possibilities are being dealt with but not with an infinite number of possibilities) since formal, logical reasoning occurs psychologically with the ability to transform a finite number of statements. (The law of the excluded middle states that either a statement or its negation is true but not both.) This suggests that the excluded middle operates "naturally" in the finite case. However, how is one to understand whether it operates "naturally" with a large number of cases? Is this tantamount, from the psychological point of view, to an infinite number of cases? Further research along these lines could answer questions concerning the psychological validity of the intuitionist viewpoint.

APPENDIX

PIAGET'S THEORY OF STAGES

There are four main stages in the child's life according to Piaget's developmental theory of cognition (Piaget, 1957, chapter 2):

1) The first stage, the sensori-motor period, takes place during the first two years of the child's life (age categories throughout his theory are approximate and not to be considered too rigidly). This stage is characterized by motor actions displaying the beginnings of intelligence. The child will act in ways to bring about desired effects but does not internalize his actions in representational form. The child does exhibit activity which indicates that he forms maps or schema of his sensory world and uses them (e.g., he can find an object hidden under a blanket by drawing the blanket off the object).

During the first stage the child achieves the idea of the permanence of an object. He comes to understand that even when an object is not in his immediate sensory field that it still exists. Early in this stage objects are not considered as permanent; e.g., "when a watch is covered with a handkerchief the child, instead of lifting the handkerchief, withdraws his hand" (Piaget, 1957, pp. 9, 10). The growth of intelligence during this stage results from the child's activities in conjunction with the development of innate psychological structures. The child is capable of moving and returning to his starting point (incipient reversibility) and holding onto the idea of a permanent object independent of change (conservation). The concepts of reversibility and conservation play important roles in the further development of intelligence and the models constructed to understand it.

2) The second stage takes place from about age 2 to about age 7. Language, symbolic play, and internalized representations are important aspects of this stage. The child relates his immediate environment to past occurrences. He tells stories and draws pictures but is unable to represent things precisely in thought or picture. He has difficulty with being mentally precise in relation to his environment; e.g., he has difficulty reproducing a remembered picture of a room or drawing the room from another point of view. He does not conserve ideas of amount or number; e.g., when he pours liquid from one container into another of a different shape, he believes the amount of liquid changes in the process and when two sets of objects which are equal in number are presented in different perceptual configurations, he believes they are unequal in number. His ability to conserve (grasp the sameness of an idea under changing circumstances) and to reverse things internally (e.g., see things from other points of view), is similar to that of the child on the first stage. Logical operations tend to be absent; i.e., the child does not yet exhibit structures of reasoning (such as those on the next stage) which operate reversibly and are used to solve problems discussed in the next stage (Piaget, 1957, pp. 10-13).

3) The third stage is characterized by concrete reasoning and takes place during the ages of approximately 7 to 11. The

logical structur... the child focus on the objects of his experience. He is capable of reversibility and conservation in this regard. He can see things from various points of view. During this stage he conserves quantity, weight, volume, length, number, area and much more. He recognizes the invariance of the volume of liquid when he pours it from one container to another; he grasps the idea of the invariance of number in different perceptual configurations; he understands relations among objects and people (e.g., order in sizes and familial relations). If he is presented with a series of sticks of varying lengths, he can position them according to size and properly position a new stick in the series. Stage two children cannot solve this kind of problem. Stage three is characterized by continually increasing competence in concrete reasoning and in the use of reversibility and conservation in the direct experience of the child. The child, however, lacks the powers of generalization, abstract reasoning, and propositional analysis which are found on the fourth stage (Piaget, 1957, pp. [illegible]-18).

d) Stage four is characterized by a marked incidence of abstract and hypothetico-deductive reasoning. It takes place during the ages of approximately 11 to 15. The child at this stage exhibits systematic reasoning and develops general strategies for solving problems. "For example, when they are given a pendulum and allowed to vary the length and amplitude of its oscillations, ... subjects of 8 to 12 years simply vary the factors in a haphazard way .... Subjects of 12 to 15 years, on the other hand, endeavor after a few trials to formulate all the possible hypotheses concerning the operative factors and then arrange their experiments as a function of these factors." (Piaget, 1957, p. 19).

The stage four child can solve problems concerning ways of combining and arranging objects. When faced with the problem of determining which of five colorless and odorless liquids combine to yield a colored product, the stage four child systematically tries all possible combinations, while the stage three child utilizes a haphazard, unsystematic approach (Piaget, 1957, p. 21). At stage four the idea of proportions is understood. This is exemplified by the child's approach to many different concepts: weight-balancing, probability and similarity of triangles. For example, in discovering the role of weight times distance in the equilibrium of the balance problem the child considers proportions between weights and distances (Inhelder and Piaget, 1958, chapter 11).

At this stage the child utilizes logical constructions such as implication, disjunction, conjunction etc. These manifest themselves in his use of cause-effect reasoning, in the solution of the chemical combinations problem, and in the solution to the problem of how to achieve equilibrium with a balance (many such examples can be found in Inhelder and Piaget, 1958).

BIBLIOGRAPHY

Apostel, L., "Psychogenese et Logiques non-Classiques" in Psychologie et Epistemologie Genetique: Themes Piagetiens, ed. Bresson et Montmollin, Paris: Dumod, 1966.

Beth and Piaget, Mathematical Epistemology and Psychology, translated by W. Mays, N.Y.: Gordon and Breach, 1962.

Birkhoff and MacLane, A Survey of Modern Algebra, N.Y.: MacMillan, 1965.

Flavell, J. H., The Developmental Psychology of Jean Piaget, N. Y.: van Nostrand, 1963.

Furth, Hans, Piaget and Knowledge, N.J.: Prentice-Hall, Inc., 1969.

Guilford, J. P., The Nature of Human Intelligence, N.Y.: McGraw-Hall, 1967.

Gyr, Brown, and Cafagna, "Quasi-formal models of inductive behavior and their relation to Piaget's theory of cognitive stages", Psycho. Review, July 1967, 14 (4). 272-289.

Inhelder, B., and Piaget, J., The Growth of Logical Thinking from Childhood to Adolescence, translated by Anne Parsons & Stanley Milgram, N.Y.: Basic Books, 1958.

Maier, H.W., Three Theories of Child Development, Scranton: Harper & Row, 1965.

Parsons, C., "Inhelder and Piaget's the Growth of Logical Thinking": Brit. J. Psych., 1960, 51, 75-84.

Piaget, J., Traite de Logique, Paris: Colin, 1949.

Piaget, J., The Child's Conception of Number, translated by C. Gattegno & F.M. Hodgson, N.Y.: Humanities, 1952.

Piaget, J., Logic and Psychology, translated by W. Mays, N.Y.: Basic Books, 1957.

Piaget, J., Six Psychological Studies, ed: Elkind, N.Y.: Vintage Books, 1968.

Piaget, J., Inhelder, B., Szeminska, A., The Child's Conception of Geometry, translated by E.A. Lunzer, N.Y.: Basic Books, 1960.

Prior, A.M., "Tense Logic and the Continuity of Time", <u>Studia Logica</u>, 1962, 13, pp. 133-148.

Stoll, R.R., <u>Sets, Logic, and Axiomatic Theories</u>, San Francisco: W.H. Freeman and Co., 1961.