DOCUMENT RESUME

ED 124 418 SE 020 893

AUTHOR Malin, Jane T.
TITLE Instructing Junior High School Students in Problem Solving Strategies.
PUB DATE 12 Mar 76
NOTE 36p.; Paper presented at the annual meeting of the American Psychological Association (83rd, Chicago, August 30 - September 3, 1975)

EDRS PRICE MF-$0.83 HC-$2.06 Plus Postage.
DESCRIPTORS Algebra; Deductive Methods; *Instruction; Junior High School Students; Mathematics Education; *Problem Solving; *Research; Secondary Education; *Secondary School Mathematics; *Teaching Methods
IDENTIFIERS Research Reports

ABSTRACT

Effects of strategy instructions on the problem-solving performance of junior high school algebra students was investigated in two experiments. In the first experiment, three groups of students were instructed to solve algebra problems using a backward, forward, or mixed directon of search. The pattern of solution times and errors indicated that the students followed the instructions, and that the efficiency of each direction-of-search strategy was strongly related to the probability of starting the solution incorrectly ("entering a blind alley"). The mixed strategy proved difficult for junior high school students. In the second experiment, instructions were given on how to identify and respond to blind alleys. The effect of instructions varied with the search strategy used (backward or forward). Although overall results showed that forward search groups performed significantly better than the backward search groups when blind alleys were entered, when forward and backward groups received full instructions for dealing with blind alleys no difference was observed. The pattern of results suggests that the optimum strategy depends on the problem type, but that if the backward strategy is chosen, it should be combined with full blind-alley instructions. (Author/SD)

Instructing Junior High School Students in Problem Solving Strategies[1]

Jane T. Malin

University of Houston

March 12, 1976

Dr. Jane T. Malin
Department of Psychology
University of Houston
Houston, Texas 77004

Abstract

Effects of strategy instructions on the problem-solving performance of junior high school algebra students was investigated in two experiments. In the first experiment, three groups of students were instructed to solve algebra problems using a backward, forward, or mixed direction of search. The pattern of solution times and errors indicated that the students followed the instructions, and that the efficiency of each direction-of-search strategy was strongly related to the possibilities of starting the solution incorrectly ("entering a blind alley"). The mixed strategy proved difficult for junior high school students. In the second experiment, instructions were given on how to identify and respond to blind alleys. The effect of instructions varied with the search strategy used (backward or forward). The forward search groups performed significantly better than the backward search groups overall when blind alleys were entered, but there was no difference between forward and backward groups that received full instructions for dealing with blind alleys. The pattern of results suggests that the optimum strategy depends on the problem type, but that, if the backward strategy is chosen, it should be combined with full blind alley instructions.

Instructing Junior High School Students
in Problem Solving Strategies
March 12, 1976

The research reported here is concerned with strategies or heuristics for helping students solve algebra problems. There are probably many types of difficulties that students have with story problems in algebra: motivation, reading the problem and translating the words into mathematical symbols (Bobrow, 1968), remembering facts (Paige & Simon, 1966), calculation and formula manipulation (Parkman & Groen, 1971), and forming an equation for deriving the unknown value from the relevant given values (Luchins, 1942; Paige & Simon, 1966; Polya, 1965; Leusmann & Cheng, 1973; Wickelgren, 1974). Strategies or heuristics are usually formulated to deal with the last type of difficulty.

Most strategies (for mathematical problems and for other problems as well) are tailored to a set of problems with the same structure or "problem space." Many of these problems are substitution problems in which a set of many possible alternative steps or moves leads from the beginning of the problem, and the solver must find the particular sequence of steps that leads to the goal. Problem solving of this type is analogous to finding a path through a maze. Certain strategies are efficient by virtue of minimizing the solvers' wanderings through certain types of problem mazes, and there has been a corresponding interest in characterizing types of problem spaces (Newell & Simon, 1972, Ernst & Newell, 1969; Amarel, 1968). But more recently there has been an interest in the problem solver as information processor, and in adapting strategies to the characteristics of the solver (Greeno, 1973; Greeno & Simon, 1974; Wickelgren, 1974). Some strategies are efficient by virtue of minimizing the information-processing load on the solver, usually the load on short-term, active, or working memory. When the problem solver is a student, there must be a concern with both types of efficiency (length of search and information-processing) and with theories of solvers as well as problems.

Many types of problem-solving strategies have been proposed and studied. Usually a strategy involves a set of general procedures. These may determine how to represent the elements and rules in a problem (Heusmann & Cheng, 1973; Simon & Barenfeld, 1969; Hayes, 1975), what priorities there are concerning the sequence of alternative solution attempts to be tried, how solution attempts (or parts of them) are evaluated, and what information is retained during the problem-solving process. Studies of the solving of story problems in algebra have investigated the following issues: (a) the translation of word information into mathematical symbols (Bobrow, 1968) or diagrams (Polya, 1965; Paige & Simon, 1966); (b) the use of particular formulas (Luchins, 1942) or other auxiliary information (Polya, 1965; Paige & Simon, 1966); (c) the use of alternative sequences of substitutions or transformations in searching for a problem solution (Polya, 1965; Wickelgren, 1974). The research reported here focused on alternative search sequences, specifically, on "direction-of-search" strategies, and also on strategies for evaluating one's own solution attempts.

Direction-of-search strategies in algebra and geometry prescribe whether solution attempts should proceed "backward" from the goal (the unknown value, the theorem to be proved, etc.) or "forward" from the initial situation (the given values, axioms, etc.). Problems in algebra involve deriving an unknown value from given values, usually by a sequence of substitutions. Attempts to solve are conceptualized as search through a set of possible substitution sequences for a path that leads from the givens to the unknown. Polya (1965) recommended working backward from the unknown, substituting new unknowns or defining subproblems iteratively (which, if solved, could be used to solve for the original unknown) until a path is found to the givens. Working forward involves using the set of givens to derive new values until the value of the unknown is found. This is an iterative sort of trial and error. Working backward usually reduces the search through possible solution sequences relative to the working forward, because there are usually fewer possible ways to work from one or two unknowns than from a set of givens. Mixed strategies involve both directions of search. A mixed strategy proposed by Wickelgren (1974) involves establishing a subgoal or subproblem (a backward step) and then working forward.

Let us consider different examples of these strategies as they would be applied to a typical algebra story problem. An example of a story problem and a set of related mathematical symbols is given in Figure 1. (Let us suppose that the student successfully translates

---

Insert Figure 1 about here

---

the "story" into mathematical symbols and recalls the appropriate formulas.) A student solving this problem by working backward would first find an equation with $\underline{A}_2$ (the unknown) in it, from which the value of $\underline{A}_2$ could be derived. If $\underline{A}_3 = \underline{A}_1 + \underline{A}_2$ were picked, $\underline{A}_2$ could be derived from $\underline{A}_1$ and $\underline{A}_3$. Since $\underline{A}_1$ is a given, $\underline{A}_3$ would become the new unknown. Next, an equation with $\underline{A}_3$ in it would be needed. $\underline{P}_3 = A_3 \times R_3$ could be used, and the new unknown would be $\underline{P}_3$. ($\underline{R}_3$ is a given.) Next, $\underline{P}_3 = \underline{P}_1 + \underline{P}_2$ could be used, and since both $\underline{P}_1$ and $\underline{P}_2$ are given, a solution path would be found. $\underline{A}_2 = \underline{A}_3 - \underline{A}_1 = (\underline{P}_3/\underline{R}_3) - \underline{A}_1 = [(\underline{P}_1 + \underline{P}_2)/\underline{R}_3] - \underline{A}_1 = [(18 + 10)/3.50] - 5 = 3.$ Notice that if the student started by trying to derive $\underline{A}_2$ from $\underline{P}_2 = \underline{A}_2 \times \underline{R}_2$, he would fail, since the new unknown in this case ($\underline{R}_2$) cannot be derived from any other equation. He would have to start again with the unknown and pick $\underline{A}_3 = \underline{A}_1 + \underline{A}_2$. This wasted step (an incorrect tentative partial solution) is called a backward blind alley.

A student working forward would try to derive new values from the givens, and would find $\underline{P}_3$ directly from $\underline{P}_1$ and $\underline{P}_2$, and $\underline{R}_1$ from $\underline{P}_1$ and $\underline{A}_1$. The value of $\underline{R}_1$ is not needed in the solution and is thus a forward blind alley, but the value of $\underline{P}_3$ could be used with $\underline{R}_3$ to get $\underline{A}_3$. Next, $\underline{A}_3$ could be combined with $\underline{A}_1$ to get $\underline{A}_2$ (the unknown), and the solution path would be found.

For the forward solver, finding $\underline{R}_1$ is inefficient since $\underline{R}_1$ is not needed. The backward solver could not waste time on this, but might waste time trying to get $\underline{A}_2$ with $\underline{P}_2 = \underline{A}_2 \times \underline{R}_2$ (while the forward solver could not do this). From the point of view of search efficiency, then, each of these strategies is equally inefficient, since one wasted step is possible in both. In a mixed strategy (involving both backward and forward search), both wasted steps are possible, and it is therefore the least efficient (in search) of the three strategies for this particular problem.

In order to make a fair comparison of strategies, they should be tested on a representative sample of the large and various set of problems that might be generated from typical rate problem formulas. To compare the search or information-processing efficiencies of strategies, some way of classifying problems into types is desirable. A classification system using a four-digit code was worked out for the set of problems that can be generated from the set of formulas that typically appear in rate problems in algebra. (These five formulas are given in Figure 1.) The codes show the number of steps in the solution (1-5), the number of possible blind alleys (0-2), the total number of steps in the possible blind alleys (0-3), and the number of branches into independent subproblems in a problem (0-2). (When neither of a pair of needed variables is given, a problem branches into two subproblems.)

All problems have only one solution, which can be found by proceeding in a backward, forward, or mixed direction. In some problems, however, the number of possible blind alleys depends on the direction of search. For this reason, each problem type must be assigned three codes, one for each direction of search. The codes for the 17 types of problems with one unknown that can be generated from the five formulas are shown in Table 1. The codes differentiate all except two pairs of problem types. Types 4 and 5 differ in the structure of their blind alleys, and Types 15 and 16 vary in the placement of subproblem branches.

Insert Table 1 about here

Comparisons of search efficiency among strategies can be based on the number of possible blind alleys or steps in possible blind alleys. There is no similar direct measure of information-processing load differences, however. Information-processing load should depend on the resource limitations and demands on the various basic processings systems used during problem solving. On problems with the same codes, differences in speed or errors among strategies should indicate differing information-processing loads. Differences in speed and errors among strategies on a set of problems should be the combined result of search-efficiency and information-processing load differences.

The purpose of the first experiment reported here was to test whether junior high school students could be taught the direction-of-search strategies, and to provide data for comparing strategies. Three directions of search--backward, forward, and mixed--were tested.

Although strategies differ in search efficiency on these problems (according to the number of blind alleys avoided), none of the strategies make it possible to avoid all blind alleys. It might be useful, therefore, to devise some way to help the solver recover from blind alleys when they are entered. Methods for judging when to abandon one approach and try a different one are of general importance in problem solving. The purpose of the second experiment reported here was to test the effectiveness of instruction in methods to deal with blind alleys. Two types of instructions were tested, one including both instructions on how to recognize the end of a blind alley (a "dead end") and on how to start over on a different path, and the other including only instructions on recognizing dead ends.

Experiment 1: Direction of Search

Method

Subjects. The subjects were 54 first-semester ninth graders from three junior high schools in the Houston Independent School District, all of whom had correctly answered all three questions on a short test of elementary algebra skills. The most difficult question on this test involved finding the value of $\underline{T}$ in $\underline{Q} = \underline{T} \times \underline{V}$, knowing that $\underline{Q} = 12$ and $\underline{V} = 3$. About 66% of the students in average eighth-grade mathematics classes in these schools passed this test. These ninth graders were randomly assigned to three equal and balanced (1/9 from below average classes, 7/9 average, 1/9 above average) groups, 18 each receiving one of three sets of strategy instructions. The data from an additional eight students who failed to follow the strategy instructions was discarded. Five of these students were from the group instructed in the mixed strategy (solve starting forward, then backward).

Stimulus materials. There was a set of five interrelated three-variable formulas for the students to memorize, about wages for work done at different rates of pay: $\underline{P}_1 = \underline{A}_1 \times \underline{R}_1$, $\underline{P}_2 = \underline{A}_2 \times \underline{R}_2$; $\underline{P}_3 = \underline{A}_3 \times \underline{R}_3$, $\underline{P}_3 = \underline{P}_1 + \underline{P}_2$, $\underline{A}_3 = \underline{A}_1 + \underline{A}_2$, where $\underline{P}$ = pay, $\underline{A}$ = amount of time worked, and $\underline{R}$ = rate of pay. To solve the problems, the students were required to show how to find the value of one unknown variable from a list of given variables (and the memorized formulas relating them). Two example problems and tree diagrams of their solutions are shown in Figure 2. The problem in the left panels is the problem in Figure 1. The tree diagrams in Figure 2 show the set of substitutions in the problem that connect the unknown variable (in the box), the given variables (unparenthesized), and intermediate variables (parenthesized). Branches having intermediate variables at both ends are subproblem branches. In Figure 2, the solution (solid lines) entends downward from the unknown, backward blind alleys (dotted lines) extend upward from the unknown, and forward blind alleys (dashed lines) extend downward from the givens. Except when in a blind alley, persons working backward would move downward in the tree, and those working forward would move upward.

Insert Figure 2 about here

Six different sets of the 17 problems, two each of Types 4 and 5 and one each of Types 3 and 6-17, were constructed. The order of presentation within a set was random, and each set was presented to three students in each group of 18. Each set was preceded by a buffer set of six problems (Types 3, 5, 7, 10, 12, 17).

Each problem was presented on the face of a 3 x 5 card with the given variables listed on one side and the unknown variable on the other. For the forward-working group the givens were in larger print on the left side of the card, and for the backward-working and mixed groups the unknown was in larger print on the left. This variation was intended to make it easier for students in each strategy group to pay attention to the type of variable they should proceed from in solving the problems. Each problem was presented as a set of variables rather than in story form so that reading difficulties would be minimized. In addition, the solution did not involve calculation or formula manipulation (since no numerical values provided for the given) so that difficulties of these types would be minimized.

Design and procedure. Each student spent a session of about one hour with the experimenter. The students worked with the formulas until they felt they had memorized them, then learned one strategy, practiced the formulas again in a flash card drill, solved two practice problems (types 6 and 4) with feedback, and then solved the set of 23 problems without feedback.

The students in the forward-working strategy group were instructed to solve problems as follows:

> A strategy you can use for solving this kind of problem is to see what you can get by combining and recombining the values you have (Example) in various ways. By combining these variables and variables you get from them you should eventually get the value you need. (Example) In this strategy, you work from

the list of givens (variables you have the values of), checking to see if you have produced the unknown (the variable you want to find the value of). You will know that you have solved the problem when you have been able to get the unknown by using values that are on the list of givens or were found using the givens.

The students in backward-working groups were instructed as follows: A strategy you can use for solving this kind of problem is to see what you could use to get the value you need. (If you can find what you need to get a variable, you can get that variable.) (Example) In this strategy, you work from the unknown (the variable you want to find the value of), checking each time to see if each variable you need is on the list of givens (variables you have the values of), and if not, finding another equation with the needed variable in it. You will know that you have solved the problem when you have been able to find every variable you need (or variables you need to find it) by using values that you have on the list of givens.

The students mixed-working group were instructed to first work backward for one step, and then solve by working forward, as follows:

A strategy you can use for solving this kind of problem is to first see what you would use to get the value you need. (Example) Next you should see if you can get these two variables by using the list of values you have (Example). (If you can find what you need to get a variable, you get that variable.) (Example) In this strategy, you start out working from the unknown (the variable you want to find the value of); next you work from the list of variables you have the values of (the givens), checking to see if you have or can produce the variables you need to find the unknown. You will know that you have solved the problem when you have been able to get the variables you need to find the unknown by using the list of givens.

The students were given a standardized way of thinking out loud. Whenever a step in the problem was considered, the students were asked to say, "With _____ and _____, I get _____," (Forward and Mixed groups) or "For _____, I need _____ and _____" (Backward group); and to say "Solved" when the problem was solved. The problem solutions were tape-recorded, and the experimenter marked the beginning of each problem on the tape by saying "Start" when the student was handed the problem card. If a student failed to solve a problem within 140 seconds, an error was recorded.

After the session, the students were asked whether or not they used the strategy. The subjects who said they did not were replaced.

Results

For each problem, the interval between the experimenter's saying "Start" and the student's saying "Solved" was timed to the nearest second. On the few occasions when the student stopped to make a remark or ask a question, seconds were subtracted for that time. When the student did not finish the problem within 140 seconds, or had not solved it before saying "Solved," an error was recorded. Median solution times and errors are shown in Table 1. Medians are reported because distributions of the solution times were skewed.

An important question to be resolved is whether the students followed the instructions. Although the protocols could provide one kind of evidence, to guard against the unlikely possibility that protocols might not reflect strategy in some important way, another test is desirable. Another kind of evidence is provided by solution times. Solving in a particular direction determines which blind alleys may be taken, and when blind alleys are taken solution times increase. The proper set of problems on which to compare strategies are Types 6, 7, and 8. These are all problems that can be solved in three steps once the student is on the right track, but backward and forward solvers should produce opposite patterns of solution times because of opposite patterns of possible blind alley steps. (The only other possible set for this comparison, Types 12 and 13, have another factor, a subproblem branch, that should increase variability and possibly swamp the blind alley effect.) A student solving forward could enter blind alleys in Types 6 and 7 but not 8, and the blind alley in Type 6 is two steps long. Therefore, solution times should decrease from Type 6 through Type 8. Just the opposite should be true for the student solving backward; blind alleys in Types 8 and 7 but not 6, and the blind alley in Type 8 is two steps long. Therefore, solution times should increase from Type 6 through Type 8. A student solving from both directions (mixed) could enter blind alleys in all three types, with a total of two possible blind alley steps in each. No systematic increase or decrease should occur from Type 6 through Type 8. A Friedman nonparametric analogue of the one-way analysis of variance was performed for each strategy for Types 6-8. For the backward [$\underline{S}(3,18) = 162.5$] and forward [$\underline{S}(3,18) = 193.5$] groups the Types were significantly different ($\underline{p} < .01$) in the expected opposite directions, and for the mixed group they were not [$\underline{S}(3,18) = 84.5$, $\underline{p} > .05$]. This pattern of solution times supports the hypothesis that the students followed the direction-of-search instructions.

Another question to be resolved is whether any of the three strategies was more efficient overall. To get a composite measure of efficiency, solution times were pooled across problems with the same number of steps in the solution, and the means and their standard errors are shown in Table 2. The sum of possible blind alley steps for each Strategy X Steps is also given, as a rough measure of the possible influence of blind alleys on these means. Analysis of variance for both proportion of errors and solution time produced parallel results, so only the solution time information is reported. There was a significant effect of number of steps in the solution [$\underline{F}(3,153) = 230.79$, $\underline{p} < .001$] and a significant interaction between number of steps and strategy [$\underline{F}(6,153) = 7.54$, $\underline{p} < .001$], but not a significant effect of strategy [$\underline{F}(2,51) = 1.57$, n.s.]. The same pattern of statistical significance was obtained when the times were log transformed to correct for skew and when conservative $\underline{F}$ tests were performed.

Insert Table 2 about here

Simple t tests indicated that the mixed strategy was significantly inferior to one or both of the others for two, three, and four-step solutions ($p < .05$) but not for five-step solutions. The forward strategy was significantly best for two-step solutions ($p < .025$), and the backward strategy was significantly best for four-step solutions ($p < .01$). This pattern of differences is consistent with the totals of possible blind alley steps for the three strategies on each set of same-length problems.

In order to evaluate information-processing efficiency, the apparent strong effects of blind alleys must be removed. One simple way to do so would be to compare direction-of-search strategies on problem types in which no blind alleys are possible. Unfortunately, all three strategies can be compared in this way only on Type 3 (2000) and the five-step types. The backward and forward strategies can be compared on problems of all lengths, however, by also comparing Type 6 backward with 8 forward and Type 12 backward with 13 forward. The means and their standard errors for these types are shown in Table 3. Analysis of variance of log transformed solution times indicated an expected significant effect of number of steps in the solution [$F(3,102) = 148.8$, $p < .001$] and a significant interaction between number of steps and strategy [$F(3,102) = 4.25$, $p < .01$], but not a significant effect of strategy [$F(1,34) < 1$]. Simple t tests indicated that the only significant difference between strategies was on the four-step solutions,

Insert Table 3 about here

on which the backward strategy group was faster ($p < .05$). Simple t tests among all three strategies on the five-step types indicated no significant differences on solution times, but the mixed strategy group made significantly fewer errors ($p < .05$) than the forward strategy group on these types.

Discussion

The results of this first experiment indicate that after one short training session junior high school students can follow instructions to use different direction-of-search strategies for rate-type algebra problems. It appears, then, that teaching any of these strategies is possible and may be helpful in problem-solving. It may not be much more difficult to teach all three kinds and let the student pick among them to suit the problem. Whether this is so remains to be tested.

The mixed strategy was least efficient overall, and the majority ~~of the students who failed to follow instructions were in the mixed-~~ strategy group. The difficulty of the mixed strategy in this study seems inconsistent with data from a similar study with college students (Malin, 1973). The college students were most efficient when they used the mixed strategy. The difference may lie in the teaching of the mixed strategy. The college students were not taught the mixed strategy, but used it spontaneously. It may be that the mixed strategy that was taught to the junior high school students was different from the one spontaneously used by the college students. The college students' strategy apparently included additional procedures that kept them out of the extra blind alleys that were entered by the junior high school students. Because of doubt about the completeness of the mixed instructions, and because of the difficulty of the strategy, it was eliminated from the second experiment.

The strong relationship between total possible blind alley steps and the pattern of means in Table 2 is an indication of the probable importance of blind alleys in slowing these problem solutions. Data from types not containing blind alleys indicate additional differences in information-processing efficiency, slightly favoring the backward and mixed strategies for the more difficult problems.

Experiment 2: Direction of Search and Blind Alley Technique

The purpose of this second experiment was to test the effectiveness of instructions in methods to deal with blind alleys. To deal with a blind alley, a person must detect some characteristic of the path that is a signal that no further work in this area will be productive, and must return to the point where the wrong choice was made and choose another alternative. In the set of rate problems for students to solve, blind alleys cannot be recognized before the dead end is reached. All dead ends contain an R variable that is not in the set of given or unknown variables in the problem. In this set of rate problems, the point where the wrong choices are made is at the beginning of the search, so it is best to start over on a problem when a dead end is reached. Students who do not recognize the signal to start over may continue to look for a new use for the R variable or try to backtrack through the same path without seeming to recognize that they are backtracking, or may even make up a new equation for the R variable. All of these responses waste effort and may cause students to make errors or give up before the solution is reached.

The two pieces of information necessary for dealing with the blind alley were separated in this experiment. Some students were told only about the R variables as dead ends, while others were told both about R's and about starting again at the beginning of the problem.

Method

Subjects. The subjects were 72 second-semester eighth graders from average algebra classes, with the same characteristics as in Experiment 1. They were randomly assigned to six equal groups of 12 each in a 2 x 3 x 3 x 4 factorial design involving two between-students variables: two directions of search and three techniques for dealing with blind alleys; and two within-students variables: three problem types and four blocks of problems. An additional nine students (distributed evenly across groups) failed to follow instructions.

Stimulus materials. The students learned the same set of formulas as in Experiment 1. The problem sets consisted of Types 6, 7, and 8 only. These are the types that should produce opposite patterns of difficulty for backward and forward directions of search. There were 18 problems in six blocks of the three types each. The first two were buffer blocks on which no data was collected. The remaining four blocks were presented in counterbalanced order across students within conditions.

Design and procedure. The individual sessions with students followed the same general procedure as in Experiment 1. The students were taught to solve in either a forward or backward direction, and learned one of three techniques for dealing with blind alleys. Control "technique" students were given no information about blind alleys; the Partial technique involved instructions on how to recognize a dead end (an R intermediate variable). The Whole technique involved instructions on how to recognize a dead end and, in addition, directions to start over on a different solution path when a dead end was reached. The techniques varied a little with the direction-of-search strategies. The Whole techniques (which contain the Partial instructions) are given below for both directions of search.

> Backward: Notice that you might start out on the example problem trying to get A from P_3 and R_3. This would not work, however, because R_3 is not on the givens list, and there is no way to get the value of R_3 without backtracking through $P_3 = A_3 \times R_3$ again (and that's not allowed). Every R variable is in only one equation. You can save yourself wasted effort if you do the following: Every time one of the variables you plan to use has an R in it, check right away to see if the R variable is on the givens list. If it isn't, that way of solving the problem won't work. So, start over a different way on the problem, using the other equation that the unknown is in.
>
> Forward: Notice that in the example problem, although each P variable on the givens list could be combined with two other givens, each R variable can be combined with only one other variable on the list. This is because each R variable is in only one equation.

Therefore, if you get an R by combining together givens, you will not be able to use it again, unless you go back to get things that you already have (and that's a waste of time). You can save yourself wasted effort if you do the following: Check any R variables that you get by combining givens, to see if they are identical with the unknown. If so, the problem is solved. If not, start over and try to combine some of the givens together in some other way.

If a student failed to solve a problem within 60 seconds, an error was recorded. The time limit was shorter in this experiment because the three-step problems take less time to solve than the four- and five-step ones included in Experiment 1.

Results

Analysis of variance of solution times indicated a significant practice effect [$F(3,198) = 5.32$, $p < .01$], a significant effect of problem type [$F(2,132) = 7.80$, $p < .001$], and a significant interaction between problem type and search strategy [$F(2,132) = 32.63$, $p < .001$]. No other main effects or interactions were significant. Table 4 shows the Strategy X Type effect, which would be expected for these problem types when strategy instructions are followed.

Insert Table 4 about here

To determine whether the blind alley instructions were helpful, solution times from the first and last problems of any type in which each student actually entered blind alleys were analyzed. The pattern of mean solution times is shown in Figure 3. The scores were log transformed to correct for the skewed distributions of solution times. An analysis of variance of the transformed scores indicated a significant practice effect [$F(1,66) = 4.76$, $p < .05$] and significant effect of strategy [$F(1,66) = 4.29$, $p < .05$]. Practice decreased solution time, and the backward strategy group was slower than the forward. No other main effects or interactions were significant, although two interactions (Strategy X Technique, and Strategy X Technique X Practice) reached $p < .20$. As can be seen in Figure 3, simple t tests on scores pooled across first and last problems indicate that the backward strategy group was significantly inferior to the forward strategy group with Control or Partial blind alley instructions ($p < .05$), but this was not so when Whole instructions were given.

Insert Figure 3 about here

Discussion

The pattern of means in Figure 3 is complex. The control means can be considered baselines against which the effect of the blind alley techniques can be evaluated. It appears that the backward control group experienced more difficulties with blind alleys than the forward control group. The long backward solution times were decreased by practice, but even more so by instructions in dealing with blind alleys. For the forward groups, on the other hand, it seems that the blind alley techniques were initially ineffective or harmful, and the partial technique only was ultimately helpful.

These data may be explained by the different notions of solution paths that each strategy should produce in students. A person working backward does so from a defined starting place, the unknown, and moves along a clearly defined solution path of substitutions from the unknown. Instructions that explain about dead ends and starting over on alternative solution paths should make sense. On the other hand, a person working forward has a choice of many starting places (several pairs of givens out of a set of four) and returns to the set of givens to make each new move. His steps are not on a defined solution path, and he may only rarely have a sense of abandoning one path and starting over a new way. Therefore, instructions that tell him to start over a new way (Whole) may be incomprehensible. On the other hand, instructions that help him discard derived values that will not be useful to him in solving the problem (Partial) may ultimately help him to save time.

Taken together, the results of both experiments indicate that if one had to choose among strategies to teach, it would be a difficult choice. However, for problems of moderate difficulty, the backward direction-of-search combined with full instructions on how to respond to blind alleys appears promising. The forward direction of search seems nearly as promising as the backward one, except that it appears preferable to combine it with instructions on identifying useless derived values (Partial instructions) rather than full instructions on how to respond to blind alleys. For long problems without blind alleys, the mixed direction-of-search would seem preferable, but it appears that it may be most difficult to teach and least efficient otherwise. The strong interaction effects between strategies and problem types found in both experiments suggest that there can be no one optimum overall strategy. This raises the question of whether instructing students in several formal strategies would help them cope flexibly with different problem types.

The rate problems in these experiments are structurally similar to many problems in algebra and science encountered by junior high school students. These initial results should be followed up by adapting these methods to less artificial school mathematics or science problems, and by testing the methods in classroom setting. The artificial, abstracted problems used in these experiments might themselves be useful for teaching the problem solving methods.

References

Amarel, S. On representation of problems of reasoning about actions. In D. Michie (Ed.), Machine intelligence 3. New York: American Elsevier, 1968.

Bobrow, D. G. Natural language input for a computer problem-solving system. In M. Minsky (Ed.), Semantic information processing. Cambridge, Massachusetts: MIT Press, 1968.

Ernst, G. W., & Newell, A. GPS: A case study in generality and problem solving. New York: Academic Press, 1969.

Greeno, J. G. The structure of memory and the process of solving problems. In R. Solso (Ed.), Contemporary issues in cognitive psychology: The Loyola Symposium. New York: Wiley, 1973.

Greeno, J. G., & Simon, H. A. Processes for sequence production. Psychological Review, 1974, 81, 187-198.

Hayes, J. R. Psychological differences among problem isomorphs. In N. J. Castellan, D. B. Pisoni, & G. R Potts (Eds.), Cognitive theory (Vol. 2). New York: Erlbaum, in press.

Heusmann, L. R., & Cheng, C. A theory for the induction of mathematical functions. Psychological Review, 1973, 80, 126-138.

Luchins, A. S. Mechanization in problem solving. The effect of Einstellung. Psychological Monographs, 1942, 54 (Whole No. 248).

Malin, J. T. An analysis of strategies for solving certain substitution problems. Human Performance Center Technical Report No. 40, Department of Psychology, University of Michigan, 1973.

Newell, A., & Simon, H. A. Human problem solving. Englewood Cliffs, New Jersey: Prentice-Hall, 1972.

Paige, J. M., & Simon, H. A. Cognitive processes in solving algebra word problems. In B. Kleinmuntz (Ed.), Problem solving: Research, method, and theory. New York: Wiley, 1966.

Parkman, J. M., & Groen, G. J. Temporal aspects of simple addition and comparison. Journal of Experimental Psychology, 1971, 89, 335-342.

Polya, G. Mathematical discovery. Vol. II. New York: Wiley, 1965.

Simon, H. A., & Barenfeld, M. Information-processing analysis of perceptual processes in problem solving. Psychological Review, 1969, 76, 473-483.

Wickelgren, W. How to solve problems. San Francisco: Freeman, 1974.

Footnote

[1] Thanks are due to several teachers and administrators in the Houston Independent School District for their cooperation.

Table 1

Median Solution Times in Seconds and Errors by Problem Type

for Forward, Backward and Mixed Strategies

	Strategy								
	Forward			Backward			Mixed		
Problem type	Code	Median	Errors	Code	Median	Errors	Code	Median	Errors
1	1000[a]			1000			1000		
2	1011			1000			1011		
3	2000	12	0	2000	12	0	2000	16.5	0
4	2000	12	.5	2013A	13	1	2013A	18.5	.5
5	2000	12	0	2013B	14	1.5	2013B	21	1
6	3012	45	3	3000	26	1	3012	28.5	0
7	3011	35	1	3011	28	1	3022	79.5	5
8	3000	21.5	0	3012	45	5	3012	53.5	4
9	3122	29	2	3100	17	2	3122	20.5	1
10	3111	23.5	1	3112	76	3	3123	47	5
11	4011	96	5	4000	31.5	1	4011	98.5	4
12	4111	123	8	4100	37.5	3	4111	80	7
13	4100	73	4	4111	83	7	4111	140	10
14	5000	40.5	3	5000	64.5	7	5000	71.5	4
15	5100A	140	13	5100A	132.5	8	5100A	140	11
16	5100B	140	11	5100B	57	4	5100B	69	5
17	5200	140	12	5200	140	13	5200	81.5	7

Note. n = 18 except for Types 4 and 5 (n = 36), and all errors were replaced by 140 seconds in calculating the medians.

[a]The first digit of the code gives the number of steps; the second digit, subproblem branches; the third digit, possible blind alleys; the fourth digit, total steps in all possible blind alleys; the letter, subtypes.

Table 2

Number of Possible Blind Alleys and Means and Standard Errors of Solution Times Pooled across Problems with the Same Number of Solution Steps

	Strategy								
	Forward			Backward			Mixed		
Number of solution steps	Blind alleys	$\bar{X}$	$s_{\bar{X}}$	Blind alleys	$\bar{X}$	$s_{\bar{X}}$	Blind alleys	$\bar{X}$	$s_{\bar{X}}$
Two	0	14.84	5.11	6	24.40	3.94	6	26.99	3.87
Three	6	45.41	4.48	5	50.01	4.94	8	60.42	5.93
Four	2	88.48	7.59	1	61.57	7.00	3	95.98	7.18
Five	0	104.79	5.20	0	97.50	5.74	0	94.00	7.42
Overall		63.38			58.37			69.35	

Table 3

Means and Standard Errors of Solution Times

for Problems without Blind Alleys

	Strategy					
	Forward			Backward		
Number of solution steps	Type	$\bar{X}$	$s_{\bar{x}}$	Type	$\bar{X}$	$s_{\bar{x}}$
Two	3	12.72	1.04	3	13.50	1.23
Three	8	27.00	5.49	6	41.39	9.44
Four	13	83.33	11.24	12	56.28	10.31
Five	14-17	104.79	5.20	14-17	97.50	5.72
Overall		56.96			52.17	

Table 4

Means and Standard Errors of Solution Times

by Problem Type and Strategy

	Strategy				
	Forward		Backward		Overall
Problem type	$\bar{X}$	$s_{\bar{X}}$	$\bar{X}$	$s_{\bar{X}}$	$\bar{X}$
6	32.64	1.43	24.84	2.54	28.74
7	33.06	1.33	32.96	1.17	33.01
8	26.92	1.25	38.17	1.45	32.55

Figure Captions

Figure 1. Example of a story problem.

Figure 2. Examples of problem types. The upper panels give problems and the codes for their structures. The lower panels show a tree diagram of the structure of each problem.

Figure 3. Mean solution time as a function of blind alley instructions, direction of solution, and order of encounter with blind alleys.

Amy works 5 hours in the morning and earns \$18. She earns \$10 in the evening, when she is paid at a different rate. Her average rate of pay that day is \$3.50 per hour. How many hours did she work that evening?

Unknown:	A_2	(evening hours)
Givens:	$A_1 = 5$	(morning hours)
	$P_1 = \$18$	(morning pay)
	$P_2 = \$10$	(evening pay)
	$R_3 = \$3.50$	(average rate)

Formulas:

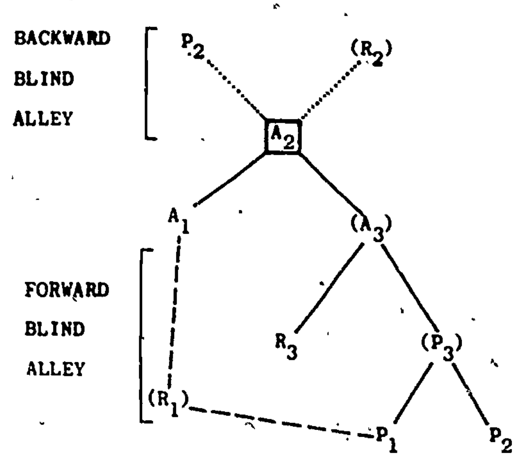
$P_1 = A_1 \times R_1 \qquad P_3 = P_1 + P_2$

$P_2 = A_2 \times R_2 \qquad A_3 = A_1 + A_2$

$P_3 = A_3 \times R_3$

Figure 1

PROBLEM TYPE 7		PROBLEM TYPE 13	
UNKNOWN:	A_2	UNKNOWN:	P_1
GIVENS:	P_2, A_1, R_3, P_1	GIVENS:	R_3, A_1, A_2, R_2
SOLUTION:	$A_2 = A_1 \cdot A_3$	SOLUTION:	$P_1 = P_2 \cdot P_3$
	$A_3 = P_3 \cdot R_3$		$P_2 = A_2 \cdot R_2$
	$P_3 = P_1 \cdot P_2$		$P_3 = A_3 \cdot R_3$
			$A_3 = A_1 \cdot A_2$
FORWARD BLIND ALLEY:	$R_1 = P_1 \cdot A_1$	BACKWARD BLIND ALLEY:	$P_1 = A_1 \cdot R_1$
BACKWARD BLIND ALLEY:	$A_2 = P_2 \cdot R_2$		
PROBLEM CODES		PROBLEM CODES	
FORWARD	3011	FORWARD	4100
BACKWARD	3011	BACKWARD	4111
MIXED	3022	MIXED	4111



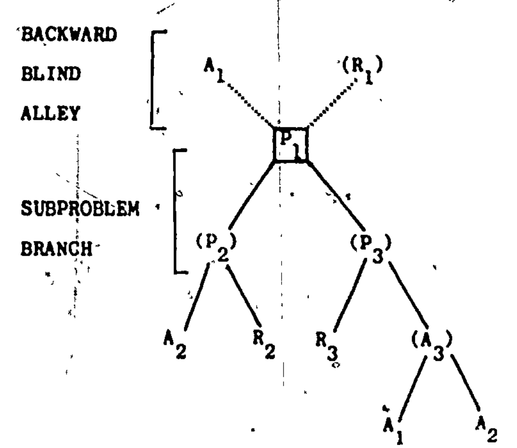
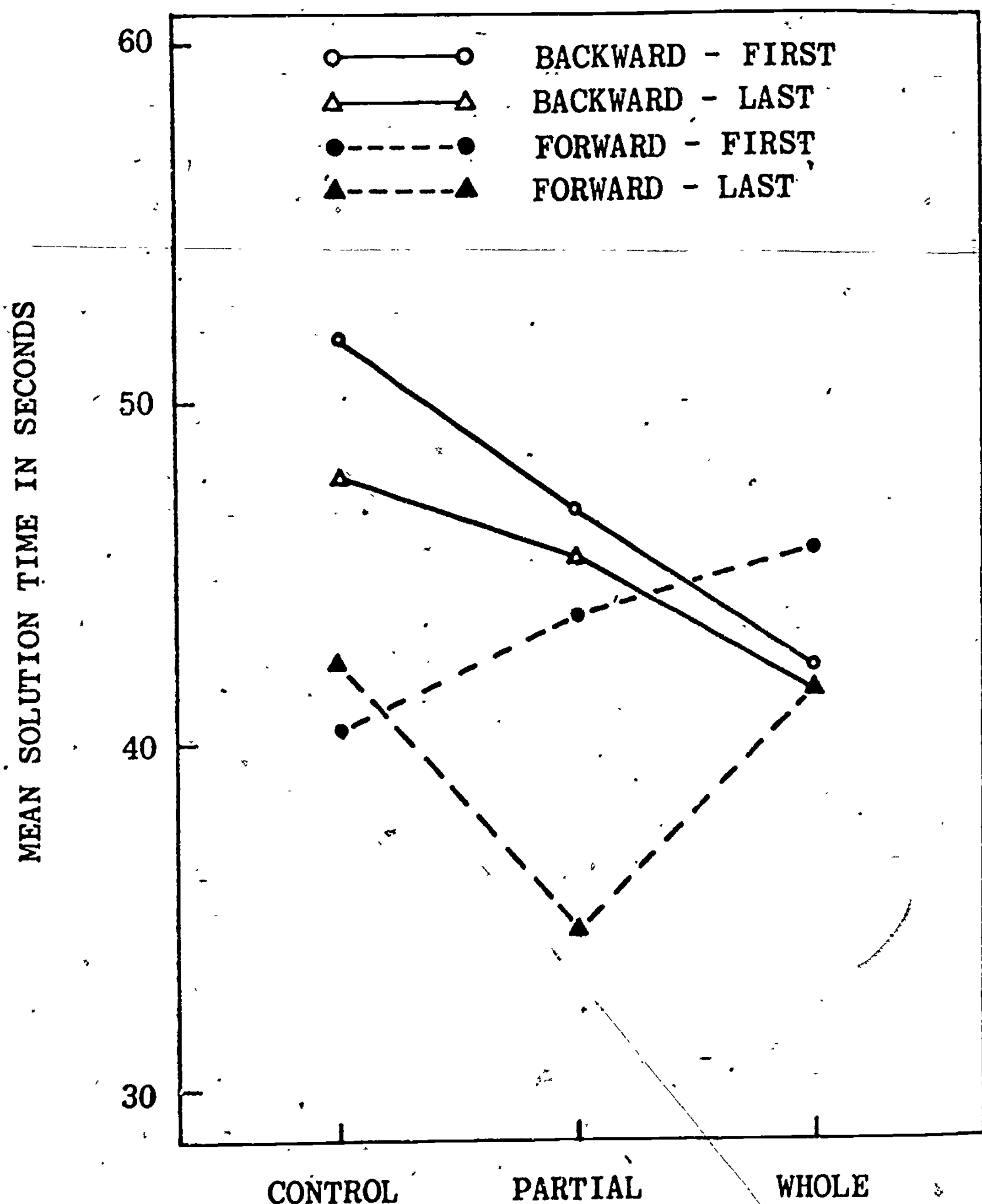


Figure 2



BACKWARD - FIRST
BACKWARD - LAST
FORWARD - FIRST
FORWARD - LAST
60
50
40
30
MEAN SOLUTION TIME IN SECONDS
CONTROL
PARTIAL
WHOLE

Figure 3