This book is a partially annotated bibliography of books, articles, and periodicals concerned with mathematical games, puzzles, and amusements. It is a reprinting of Volume 1 of a three-volume series. This volume, originally published in 1955, treats problems and recreations which have been important in the history of mathematics as well as some of more modern invention. The book is intended for use by both professional and amateur mathematicians. Works on recreational mathematics are listed in eight broad categories: general works, arithmetic and algebraic recreations, geometric recreations, assorted recreations, magic squares, the Pythagorean relationship, famous problems of antiquity, and mathematical miscellanies. (SD)
A BIBLIOGRAPHY OF
recreational mathematics

VOLUME I

RESEARCH FOR BETTER SCHOOLS
1714-1800, 1810-1830, 1840-1860, 1870-1890

PHILADELPHIA, PENNSYLVANIA  19103
From J. Ozanam: *Dictionnaire Mathématique*.
Amsterdam, 1691
A BIBLIOGRAPHY OF recreational mathematics

VOLUME I

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PREFACE

Since its first appearance fifteen years ago, this monograph has been twice revised and updated. Meanwhile, the literature of recreational mathematics has proliferated to such an extent that instead of merely updating the original bibliography once more it seemed desirable to issue a second volume which would not only be more timely but also enlarged in scope and improved in format. The two volumes thus complement one another and provide a comprehensive coverage of the field. When used together, they should hopefully serve the reader well, whether an amateur or a professional.

—W. L. S.

Boca Raton, Florida
September, 1969

PREFACE TO THE FIRST EDITION

The late G. H. Hardy once observed that there are few more “popular” subjects than mathematics. His contention is amply borne out by the universal interest manifested in mathematical recreations for over 2000 years, ranging from the loculus of Archimedes and the talismen magic squares of the early Chinese to the cryptanalysis and topological recreations of modern times. One need only recall how testament problems, ferrying problems, coin problems, problems of pursuit and problems of arrangements have come down through the ages, ever dressed anew, yet always the same old friends. Labyrinths, dissections, acrostics, tangrams, palindromes, and so on, are likewise virtually ageless. Hence it should occasion little surprise that an enormous body of literature has arisen in the last 300 years.

It has been my purpose to gather a considerable part of this material between the covers of one book for the convenience of students and teachers, as well as laymen and specialists. The more than 2000 entries by no means represent a complete or exhaustive compilation. But enough has been given, I hope, to be of real help. I have tried to meet the needs of almost any reader—the beginner, the dilettante, the professional scholar. Hence I have deliberately included some “popular” articles along with erudite and
technical discussions; many contemporary and recent publications, as well as some of an earlier period; some that are readily accessible, and others that are to be found only in important libraries; most of them in English, some in French, German, and Italian; most of them significant, a few, somewhat superficial. In this way, it is hoped, both the neophyte and the sophisticated authority will find what they need.

The task of organizing this material yielded a more or less arbitrary classification of mathematical recreations. Occasionally, where helpful, entries have been annotated; to have commented upon each item seemed quite unnecessary, and would in any event have been prohibitive.

It would scarcely seem necessary to suggest how this guide may be used. To be sure, a number of entries listed under each of the more than 50 headings will not be available to the reader unless the facilities of a large library are at hand; yet there will almost surely be some that are accessible. In most instances the reader will have little difficulty in selecting items pertaining to a given topic: he should be guided by the title of the book or article; by the annotation, if any; by the sort of periodical, whether scholarly, professional, newsv, and so on; and, to some extent, by the length of an article. Naturally, the reader’s purpose, as well as his familiarity with the subject, will loom large as factors in helping him select items to be consulted. Nor should he be deterred by references in a foreign language; after all, the mathematical symbols and geometric figures are essentially the same, so that even a moderate facility in French or German often suffices.

This guide will serve as a place to begin to look for source materials. It will help the student pursuing his mathematical studies in high school or college; the mathematics club looking for program and project material; the teacher gathering human interest or motivation material; the more advanced student engaged in research; the amateur mathematician or the proverbial layman happily engaged in that most delectable of all activities—a hobby or a recreation.

May following these trails afford the reader as much pleasure as it has been for me to map them out for him.

—W. L. S.

July 1954
## Contents

### Chapter 1. General Works
- 1.1 Early Twentieth Century Books—1900-1924 ........................................ 1
- 1.2 Contemporary Books—From 1925 On .................................................. 2
- 1.3 Periodical Literature ............................................................................ 4
- 1.4 Mathematics Club Programs; Plays ................................................... 12
- 1.5 Mathematics and Philately ................................................................. 16
- 1.6 Mathematical Contests ...................................................................... 18
- 1.7 Mathematical Models ......................................................................... 19
- 1.8 Mathematical Instruments ................................................................. 20
- 1.9 The Abacus ....................................................................................... 23

### Chapter 2. Arithmetical and Algebraic Recreations
- 2.1 General Arithmetical Recreations ...................................................... 26
- 2.2 Specific Problems and Puzzles ............................................................ 29
- 2.3 Number Pleasantries ......................................................................... 34
- 2.4 Calculating Prodigies ........................................................................ 39
- 2.5 Theory of Numbers—Factorizations—Primes .................................... 42
- 2.6 Perfect Numbers—Mersenne’s Numbers ......................................... 45
- 2.7 Fermat’s Last Theorem ...................................................................... 47
- 2.8 Fibonacci Numbers and Series ......................................................... 49

### Chapter 3. Geometric Recreations
- 3.1 General Geometric Problems and Puzzles ....................................... 51
- 3.2 Geometric Fallacies—Optical Illusions ............................................ 54
- 3.3 Geometric Dissections—Tangrams .................................................... 55
- 3.4 Regular Polygons and Polyhedrons ............................................... 57
- 3.5 Geometric Constructions ................................................................. 60
- 3.6 Mascheroni Constructions ............................................................... 63
- 3.7 Linkages—The Pantograph .............................................................. 66
- 3.8 Mechanical Construction of Curves ................................................ 66

### Chapter 4. Assorted Recreations
- 4.1 Boss Puzzle ..................................................................................... 68
- 4.2 Card Tricks—Manipulative Puzzles ................................................. 69
- 4.3 Chessboard Problems ...................................................................... 71
"But leaving those of the Body, I shall proceed to such Recreations as adorn the Mind; of which those of the Mathematicks are inferior to none."

—WILLIAM LEYBOURN: Pleasure with Profit (1694).
Principal Abbreviations Used

*Am. M. Mo.* = American Mathematical Monthly  
*M. Gaz.* = Mathematical Gazette  
*M. Mag.* = Mathematics Magazine  
*M. T.* = Mathematics Teacher  
*N. M. M.* = National Mathematics Magazine  
*N. C. T. M.* = National Council of Teachers of Mathematics  
*R. M. M.* = Recreational Mathematics Magazine  
*Sci. Am.* = Scientific American  
*Sci. Mo.* = Scientific Monthly  
*Scrip. M.* = Scripta Mathematica  
*S. S. M.* = School Science and Mathematics  
*Z. M. N. U.* = Zeitschrift für Mathematischen und Naturwissenschaftlichen Unterricht
As early as 1612 the Frenchman Claude Gaspard Bachet de Méziriac published his *Problèmes plaisans et délectables, qui se font par les nombres*; a second edition appeared in 1624. In the same year, under the nom de plume of Van Etten, there appeared a volume entitled *Récitations mathématiques*, the author of which was the Jesuit Jean Leurechon. General interest in such books apparently increased, for this was followed in 1630 by Claude Mydorge's *Examen du livre des récréations mathématiques et de ses problèmes*. In 1636, Daniel Schwenter's *Deliciae physicomathematica oder Mathematische und philosophische Erquickstunden* appeared posthumously, and in the years 1641-42 the Italian Jesuit Mario Bettini issued the first two volumes of his *Apiaria universae philosophiae mathematicae in quibus paradoxa et nova pleraque machinamenta exhibentur*, to be followed in 1660 by a third volume under the title of *Recreationum mathematicarum Apiaria XII novissima*. On the heels of this came the *Arithmetische Lustgarten* of Johann Mohr, published in 1665. Thirty years later we have William Leybourn's *Pleasure with Profit: Consisting of Recreations of Divers Kinds, viz., Numerical, Geometrical, Mechanical, Statical, Astronomical, Horometrical, Cryptographical, Magnetic, Automatical, Chymical, and Historical*.

At the very threshold of the 18th Century, in 1694, came Jacques Ozanam's treatise on mathematical recreations: *Récitations mathématiques et physiques*. Ozanam may be regarded as the forerunner of modern books on mathematical recreations. He drew heavily on the works of Bachet, Mydorge, and Leurechon; his own contributions were somewhat less significant. The work was later augmented and revised by Montucla, and still later rendered into English by Hutton, passing through many editions.

1.1 Early Twentieth Century Books—1900-1924


BISCHOFF, DR. Die Elemente der Kabbalah. 1913.

BISCHOFF, DR. Mystik und Magie der Zahlen. 1920.


FERROL, DR. F. Das neue Rechungsverfahren. 1919.


HARDENBERG, KUNO v. Die Lösung eines alten okalen Rätsels. 1924.

HARRIS, A. V., & WALDO, L. M. Number Games for Primary Grades. Chicago: Beckley-Cardy, 1917.


HULISCH. Zahlenmagie in Bezug auf das Menschliche Leben. 1910.

GENERAL WORKS

A collection of the most amusing properties of numbers, and many of the most difficult mathematical problems with their answers.


LIST, C. Das Geheimnis der Runen. 1908.


MAACK, FERDINAND. Elias Artista Redivivus. 1913.

MAACK, FERDINAND. Die heilige Mathesis. 1924.

MAACK, FERDINAND. Raumschach. 1909.


NEUHAUS, O. Rechenkünste und Zahlenspiele. 1902.

PEANO, G. Giuochi di aritmetica. 1924.


RILLY, A. Le problème du cavalier des échecs. 1906.


WEEKS, RAYMOND. Boys' Own Arithmetic. New York: Dutton, 1924. 188 p.


1.2 Contemporary Books—From 1925 On

Revised edition, after 70 years.

Match and coin games; knots and strings; fun with paper; conventional puzzles.

Card and coin tricks; paper folding; match tricks; string games; knots.


Well-known classic.


An interesting popular exposition, with much recreational material.

The granddaddy of all modern books in this field. Arithmetical and geometrical recreations; polyhedra; chessboard problems; magic squares; map-colouring; unicursal problems; Kirkman's schoolgirls problem; manipulate arrangements; duplication, trisection, and quadrature; calculating prodigies; cryptography and cryptanalysis.


Contains Lewis Carroll's inimitable and entertaining problems in symbolic logic.


DAVIS, FREDERICK. Fascinating Figure Puzzles. Burroughs Adding Machine Company, 1933. (Pamphlet)


Assorted puzzles, chiefly arithmetical; problems of arrangement and manipulation; cryptograms.


A distinguished collection by a veteran puzzle expert.


EMDE, DR. Palindrome und die Satorformel. 1925.


Excellent collection of manipulative puzzles.

17
Simple discussion of common geometric figures such as the parabola, spirals, helix, screw threads, tangrams, and such. Attractive photographs.


Contains 21 cardboard tile dissection puzzles and tangrams.


Puzzles, tricks, and games with numbers for the parlor magician.


Indispensable for mathematical club programs and activities.
GENERAL WORKS

A companion volume to Mathematical Wrinkles; contains material from
trigonometry, analytics, calculus, and physics.

A handbook of problems and recreations; mensuration; fourth dimension;
quotation; and such.

JUNE, W. M. Stunts with Numbers, Games, and Cards. Syracuse, N. Y.: the author,
757 Ostrom Ave., 1937. 25¢ (Pamphlet)

KAUFMAN, GERALD L. The Book of Modern Puzzles. New York: Dover Publications,
1954. 188 p. (Paper)

A unique collection of humorous verse.

168 p.

KERST, BRUNO. Mathematische Spiele. Berlin: G. Grotesche Verlagbuchhandlung,
1933. 90 p.

KINNAIRD, CLARK (editor). Encyclopedia of Puzzles and Pastimes. New York:
Contains some 2500 puzzles, many of them mathematical; includes crypto-
graphs, dissected figures, knight's tours, logics, mazes, magic squares,
palindromes, and paradoxes, as well as the usual assortment of acrostics,
anagrams, crossword puzzles, quizzes, whodunits, and such.

KOWALEWSKI, G. Alte und neue mathematische Spiele: Eine Einführung in die

KOWALEWSKI, G. Boss Puzzle und verwandte Spiele. Leipzig: Köhler's Anti-
quarium, 1937.

KRAITCHIK, MAURICE. Mathematical Recreations. New York: W. W. Norton, 1942;
A classic; for beginners and for experts; chess, bridge, roulette, Russian
bank, dominoes, cryptograms, and such.

KRAITCHIK, MAURICE. La mathématiques des jeux, ou récréations mathématiques.

KRAITCHIK, MAURICE. Le problème du cavalier aux échecs. Paris: Gauthier-Vil-
lars, 1927. 96 p.


KURTZAHN, T. Die Runen als Heilzeichen. 1925.

LEE, WALLACE W. Math Miracles. Durham, N.C.: Privately printed, the author,
Box 105. 1950. 83 p.


LEEMING, JOSEPH. Fun with Puzzles: puzzles of every kind for everybody. . . .
problems with coins, counters and matches, brain twisters, mathematical

Author is a well-known writer with many books to his credit: fun with string, with paper, with magic, and such.


A varied collection of puzzles and stunts, games and gags—everything from mathematical twisters to tips on checker playing, from cryptograms to match tricks.


A collection of puzzles, brain teasers, number problems, tongue twisters, and other assorted enigmas.


Number giants and pygmies, by a well-known writer on expository mathematics.


LOFLIN, Z. L. AND HEARD, IDA MAE. Just for Fun. Lafayette, La.: Southwestern Louisiana Institute, the authors, 1948. 55 p. (Mimeo.)

LONGSTREET, JULIAN. (Pseudonym). See Rulon, P. J.


MAACK, FERDINAND. Die Lösung des Satorgeheimnisses. 1926.


Contains unique recreations related to repeating designs.
GENERAL WORKS

Fiction; mathematical recreations.


Miscellaneous problems; mostly serious, i.e., illustrating significant mathematical ideas.

Sophisticated and attractive; contains considerable new material.


Unusually fine collection of mathematical recreations, well presented.


MURRAY, HISTORY OF BOARD GAMES, EXCLUSIVE OF CHESS. Oxford University Press.


Paradoxes in arithmetic and geometry; algebraic and geometric fallacies; paradoxes of the infinite; paradoxes in probability; logical paradoxes; paradoxes in higher mathematics. Sophisticated; scholarly.


Miscellaneous well-known mathematical paradoxes, puzzles, tricks, recreations, and curiosities.


Elementary number theory; digital roots and recurring decimals; magic squares; number curiosities; pseudo-telepathy.


Interesting dissection and other manipulative recreations.


SAMPLE, ANNA E. Fifty Number Games for Primary Children. Chicago: Beckley-Cardy, 1927.


SPOHRHÜNER, DR. Paracaidas. 1929.


A unique collection of interesting mathematical facts, expository and recreational material.


STREHL, SIMON. Fröhliche Wissenschaft. Nuremberg: Willmy Verlag, 1941.
12 RECREATIONAL MATHEMATICS


*The V & W Puzzles Omnibus.* London: Vawser & Wiles, Ltd. n.d.

Several small tracts bound in one; c. 1953.


Collection of 180 provocative inferential and mathematical problems and 100 word problems (acrostics, anagrams, word squares, and such).


1.3 Periodical Literature


GENERAL WORKS


BARNES, A. Making mathematics interesting. M. T. 17:404-10; 1924.


BRANDES, L. C. Recreational mathematics as it may be used with secondary school pupils. S. S. M. 54:383-93; 1954.


Bibliography.


BROWN, I. M. Adventures of an x. Open Court 28:529-37; 1914.


DENTRUFF, E. J. Brain-teasers in uniform. Popular Science 143:89+; 1943.


GARDNER, MARTIN. Mathematical Games. Sci. Am. 196:138+; January 1957; 152+; February 1957; 160+; March 1957; 14, 166+; April 1957; 150+, May 1957.


HARTSWICK, F. GREGORY. This puzzling world. Esquire. May 1935. p. 86. 137.


Hubert, C. La défenses des récréations mathématiques. La Nature 59:130; Part 2, August 1930.

Jablonsker, J. Jabberwocky was a lark, or the mathematician takes a holiday. M. T. 26:302-306; 1933.


Karapetoff, V. The way logarithms might have been discovered even though they weren't. Scrip. M. 12:153-59; 1946.


Miller, M. H. Test your common sense. Science Digest 29:55-57; March 1951.


Interesting discussion of why men and animals move in circles when deprived of vision.


Bibliography.


SIMONS, LAO Q. Place of the history and recreations of mathematics in teaching algebra and geometry. *M. T.* 16:94-101; 1923.


Miscellaneous puzzles, some of them of mathematical kind.


“Survival of the Mystical Mathematician.” *Current Opinion* 65:376-78; 1918.


16

RECREATIONAL MATHEMATICS

TAYLOR, HELEN. The mathematics library and recreational programs. *S. S. M.* 30:626-34; 1930.


VEST, L. T. Modernize your algebra! *Texas Outlook* 14:49-50; 1931.


WEINER, M. From interest to interest. *M. T.* 30:23-26; 1937.


1.4 Mathematics Club Programs; Plays


Interesting skit involving tricks with numbers.


Brief skit involving the binomial probability distribution.


CordeLL, C. M. *ColorfuL Mathematics Teaching.* Portland, Maine: J. Weston Walch. Publisher, P. O. Box 1075. 1957. 190 p.

Contains five practical mathematics plays.


GENERAL WORKS


GULDEN, M. Mathematics club program. M. T. 17:350-58; 1924.

HATCHER, FRANCES. A living theorem; a class day program. S. S. M. 16:39-40; 1916.


HOAG, R. Sources of program material and some types of program work which might be undertaken by high school mathematics clubs. M. T. 24:492-502; 1931.

JOSE, T. Types of programs and needed library equipment for mathematics clubs. Teachers College Journal 5:95-98; 1933.


“A Mathematical Dr. 1. Quiz-em Program.” M. T. 45:30-33; 1952.

“Mathematics Clubs.” Am. M. Mu. 47:312-17; 1940.


PERSON, R. Junior high school mathematics clubs. M. T. 34:228-29; 1941.


PORTERFIELD, JACOB. Fun for the mathematics club. M. T. 37:354-57; 1944.


RANUCCI, E. R. Mathematics and the assembly program. The New Jersey Mathematics Teacher 8:4-6; February 1952.


RECREATIONAL MATHEMATICS


Schaaf, W. L. Mathematical plays and programs. M. T. 44:526-28; 1951.
   Annotated list of 50 plays and pageants about mathematics.


Shriner, W. O. Purpose and value of mathematics clubs. Teachers College Journal 5:92-94; September 1933.


Sommer, J. W. Mathematics club is interesting! School Activities 27:95-97; November 1955.


1.5 Mathematics and Philately


1.6 Mathematical Contests

“Interscholastic Mathematics Contest.” *Secondary Education* 4:160; May 1935.
MAYOR, J. R. Would contests and scholarships contribute to increased interest in mathematics? *M. T.* 42:283-89; 1949.

31


Also, subsequent years; gives annual examination questions.


Also, subsequent years; gives examination questions, except for 1943-45.

### 1.7 Mathematical Models


Berger, E. J. Model explaining how latitude may be determined by making observations on Polaris. *M. T.* 47:405-06; 1954.


A comprehensive exposition of the Dandelin spheres.
GENERAL WORKS


CARNAHAN, WALTER. Illustrating the conic sections. *S. S. M.* 45:313-14; 1945.


Unpublished Master's thesis.


Calculating machines; slide rules; instruments and models for higher mathematics; mechanical devices for drawing curves; etc.


Description of string models of surfaces of higher mathematics.


Includes polyhedrons, elementary surveying instruments, pantographs, slide rules; also, instruments for showing variation of angles and circles, and of parts of a right triangle.


Unpublished Master's thesis.


OLANDER, C. Model for visualizing the formula for the area of a circle. *M. T.* 48:245; 1955.


Discussion of models of spherical triangles.


STRUYK, ADRIAN. Geometrical representation of the terms of certain series and their sums. *S. S. M.* 37:202-08; 1937.


1.8 Mathematical Instruments


Descriptions of early scales and protractors, drawing instruments, elliptical trammels, measuring instruments, micrometers, quadrants, slide rules, etc.


GUNThER, ROBERT. Historical Instruments for the Advancement of Science. Oxford University Press, 1925.


Bibliography.


Unusual essays on early mathematical instruments.


Bibliography.


SMITH, DAVID EUGENE. Gift of historical mathematical instruments to Columbia University. Science n.s. 83:79-80; 1936. Also, School and Society 43:313-14; 1936.
24 RECREATIONAL MATHEMATICS


Brief, but very useful treatment.


Bibliography.


1.9 The Abacus

ADLER, J. So you think you can count! M. Mag. 28:83-86; 1954.

GANDY, S. Did the Arabs know the abacus? Am. M. Mo. 34:308-16; 1927.


Bibliography.


LAZAR, NATAN. From the abacus to the adding machine. The Duodecimal Bulletin 6:17.


Williams, F. H. The Abacus and How to Operate It. Shanghai: Kelly & Walsh, Ltd., 1946. 27 p.


Chapter 2

**Arithmetical and Algebraic Recreations**

The combined ages of Mary and Ann are 44 years, and Mary is twice as old as Ann will be when Ann is three times as old as Mary was when Mary was three times as old as Ann. How old is Ann? The question: *How old is Ann?*, has long since become a household byword; it is known to have been asked as early as 1789.

Many of the popular puzzles and recreations which fascinate the multitude are mathematical in nature—and a large part of these are arithmetical or algebraic. The range of subject matter, so to speak, of this large body of problems is truly amazing. In ancient and mediaeval times there were the ever-present problems of the cistern, the courier problems, the God-Greet-You problems, the lion-in-the-well problems, the time-of-day problems, and the testament problems. In mediaeval times, to be sure, emphasis shifted somewhat toward commercial problems: interest and usury, discount, insurance, coinage, exchange, weights and measures, and related matters.

In modern times, many of these old problems reappear in new guise. Of course, some new ones have been added. Like women’s fashions, they appear to be subject to whimsy and caprice. Twenty-five years ago, problems of the “engineer-fireman-brakeman” type were in vogue. In turn, there would seem to be a revival of interest in a succession of classics: the monkey and the coconuts; the bumble bee flying back and forth between the radiators of two approaching automobiles; the prolific bacteria and the half-filled jar; and so on and on. At the moment of writing, the public fancy has been regaled with the egg problem of Victorian New England: Three boys, A, B, and C, went to sell their eggs. A had 10 eggs, B had 30 eggs, and C had 50 eggs. They each sold their eggs at the same rate, and received the same amount of money. How much did they sell their eggs for? No, it’s not impossible.

### 2.1 General Arithmetical Recreations

ARITHMETICAL AND ALGEBRAIC RECREATIONS


   Examples of the SEND MORE MONEY type.

   An unusual collection of the SEND MORE MONEY type.


BRANDES, LOUIS G. Constructing the common cross-number puzzle. S. S. M. 57:89-97; 1957.

BRANDICOURT, V. Curiosités mathématiques. La Nature 62:324; 1934. Part I.

COLLINS, A. F. Now you can have fun with figures. World Review 7:109; 1928.


   Short cuts for multiplying large numbers; the Trachtenberg system.


EVE, A. S. Dizzy arithmetic; when numbers talk. Atlantic Monthly 135:165-70; February 1925.

FLYNN, FLORENCE. Mathematics games: adaptations from games old and new. Teachers College Record 13:399-412; 1912.

   “Fun with Answers.” Newsweek 24:10+; November 4, 1944.
   “Fun with Figures.” Newsweek 24:87+; October 28, 1944.
   “Games as Mathematical Problems.” Spectator (London) 111:132-33; 1913.


GUSTAFSON, G. B. A simple device for demonstrating addition and subtraction in the binary number system. M. T. 47:499-500; 1954.


JERBERT, A. R. Think of a number. S. S. M. 44:624-28; 1944.

KELLY, F. G. Are you good at figures? Collier’s 74:28; 1924.
Nygaard, P. H. Odd and even—a game. M. T. 49: 397-98; 1956.
Contains chapters on magic squares, number peculiarities, and other mathematical recreations.
A 16th Century description of an ancient Greek game of numbers known as Rythmofulcia.
ARITHMETICAL AND ALGEBRAIC RECREATIONS


Curiosities based on the value of $(1 + k)^2$.

SELKIN, F. B. Number games bordering on arithmetic and algebra. *Teachers College Record* 13:452-95; 1912.

SMITH, D. E. AND EATON, C. Rithmomachia, the great medieval number game. *Teachers College Record* 13:413-22; November 1912.


STEINWAY, L. S. Experiment in games involving a knowledge of number. *Teachers College Record* 19:43-53; 1918.


WILLERLING, Margaret F. Review the fundamental processes; the cross-number puzzle. *S. S. M.* 54:51-52; 1954.


2.2 Specific Problems and Puzzles

A. Binary Games—Nim


Discussion of binary notation, with applications to the Russian peasant method of multiplication, the Chinese rings, and the game of Nim.


41

"Nim." *A. M. M.* 14:216; 1940.


Application of binary notation to the mathematical theory of the game of Nim.


Recreational applications of binary notation.


### B. Calendar Problems


**Canaday, E. F.** What day of the week was it? *M. T.* 29:75-77; 1936.

**Christensen, E. and Mayall, R. N.** To calculate days between two dates. *Science* 122:561-62; September 23, 1955.


"Day of the Week Corresponding to a Given Date." *Popular Astronomy* 54:439-40; 1946. 55:55; 1947.

**Franklin, Philip.** An arithmetical perpetual calendar. *Am. M. Mo.* 28:262; 1921.


**Hoeck, John.** Formula for finding the day of the week. *M. Mag.* 26:55; 1951-52.

**Humiston, R. L.** What day is it? *S. S. M.* 26:841-44; 1926.

**Jones, H. I.** What day is it? *S. S. M.* 23:825-30; 1923.


ARITHMETICAL AND ALGEBRAIC RECREATIONS

Rydzewski, A. How to find the day of the week on which any day of any year falls, and also how to determine the Easter Day for many years. Popular Astronomy 7:416-20; 1899.
Smiley, M. F. When is Easter? M. T. 40:310; 1957.
Spillman, W. J. Formulae giving the day of the week of any date. Science 51:513-14; 1920.
Vail, W. H. Uncle Zadock’s rule for obtaining the dominical letter of any year. Am. M. Mo. 29:397-400; 1922.

C. Cattle Problem of Archimedes


D. The Ladder Problem

Problem No. 1194. S. S. M. 32:212; 1932.

E. The Twelve-Coin Problem

F. Menage Problems

G. Miscellaneous Specific Problems


BROWN, J. C. Problem. (Puzzle concerning the measurement of cloth.) *M. T.* 35:32; 1942.


Asks for the number of ways of seating n married couples around a circular table, husbands alternating wives, no husband next to his own wife.

Bibliography.


The famous problem of the two approaching cars and the bumble bee.


Four problems of the type: "Smith-Jones-Robinson. . . . Who was the engineer?"


SANFORD, VERA. The problem of pursuit. *M. T.* 44:516-17; 1951.


Good analysis of the problem of dividing 10 pints into two equal parts, using only a 3-pt., 7-pt., and 10-pt. container.

34 RECREATIONAL MATHEMATICS


2.3 Number Pleasantries

A. Number Oddities and Curiosities


For young readers as well as old.

AGNEW, P. C. Human side of numbers. Science Digest 8:45-48; December 1940.


5:32, 68, 116, 135, 176, 185, 208, 259; 1938.

6:56, 120, 179-80, 218; 1939.

7:68, 137-59; 1940.

8:14, 78, 92, 109, 164; 1941.

9:59-60, 100, 113, 149, 189; 1943.

10:64; 1944.


12:14, 75, 87, 90-91, 111, 146, 163, 218, 290, 293; 1946.


14:47-48, 65, 71, 97, 111-12, 125, 135, 162-71; 1948.


18:30, 68, 82, 85-86, 163-66, 218, 236; 1952.


CORNWELL, W. C. Mystery of numbers: some data collected for the benefit of accountants, as well as the public. Forum 68:784-90; September 1922.


Number oddities built around the number 1955.


KAPREKAR, D. R. Demlo Numbers, Groningen, Holland: P. Noordhoff, Ltd., 124 p. Numbers like 165, 2553, 47773 in which the first and last digit added together produce the digit in the middle portion of the number.


Mark, S. How to be a wizard with the magic number 9. *Good Housekeeping* 118:27; 1944.


Mycatt, G. Nines have it. *Collier’s* 116:47; July 21, 1945.


ARITHMETICAL AND ALGEBRAIC RECREATIONS


Devoted exclusively to curious properties of numbers, factors, powers, and such.


Warten, Ralph M. On numbers both triangular and square. Math Mirror (Brooklyn College) 22:18-21; 1956.


E. Number Giants and Pygmies


Archibald, R. C. A huge number. Mathematical Tables and Other Aids to Computation 2:93-94; 1946.


"Billion?" Atlantic Monthly 158:640; 1936.


Gamow, G. What is the biggest number? Science Digest 23:41-43; March 1948.

"Greatest Three-Figure Number." Science and Intention 13:1093; 1926.


RECREATIONAL MATHEMATICS


Unusual discussion of exponents and very large numbers.

WEDUL, M. O. Grains of sand and drops of water help make numbers meaningful. S. S. M. 53:294; 1953.

C. Rapid Calculation—Mental Arithmetic


CULWELL, R. C. Rule to square numbers mentally. S. S. M. 14:71-77; 1914.


DUANE, W. R. G. Quick computations. Journal of Accounting 70:241-44; September 1940.


KARPFINSKI, L. C. A rule to square numbers mentally. S. S. M. 15:20-21; 1915.


LUTKA, A. Cube and fifth roots by mental arithmetic. Scientific American Supplement 76:194-95; September 1913.


Deals chiefly with short cuts to increase the range of mental computations; also, material on methods of checking computations for mistakes.

SHORT, W. T. Rule for extracting the nth root of arithmetical numbers. S. S. M. 16:70; 1916.

D. Circulating Decimals


Calculating Prodigies

Lightning calculators, or so-called mathematical prodigies, have appeared from time to time, catching the public fancy. Such persons, although often illiterate, seemingly possess astonishing powers of mental computation. Most of them are relatively youthful; generally they are self-taught, and usually they do not retain their powers of calculating. Nearly all of them have had phenomenal memories for numbers. As a rule, calculating prodigies are unable to give a satisfactory explanation of their methods.

Among the more famous mental calculators were Jedediah Buxton, Thomas Fuller, Zerah Colburn, George Bidder, and J. M. Zacharias Dase. They are not to be confused with the occasional mathematicians who exhibited extraordinary aptitude for elaborate mental calculations, such as John Wallis, Andre Marie Ampere, and Carl Friedrich Gauss.


RECREATIONAL MATHEMATICS


Gives a brief history of many arithmetical prodigies; discussion of solution of numerical problems and of lightning calculations; sections on magic squares and arithmetical recreations.


BURLEY, Ross A. The Figure Fiend (Lightning calculation supreme). Chicago: A. Nelmar Albino, 1941.


Encyclopaedia Britannica. 11th edition. Article on "Table, Mathematical" contains a reference to Zacharias Dase.


GRADENWITZ, A. Remarkable arithmetician. Scientific American Supplement 64:93; August 1907.

"Great at Arithmetic is the Subconscious Mind." Literary Digest 88:50-52; March 1926.


Bibliography.


Bibliography.


Article on W. J. Sidis.

"Mathematical Prodigies." Literary Digest 107:25; December 27, 1930.


"A Mind Races with Machines." Literary Digest 82:20; August 30, 1924.

MITCHELL, F. D. Examples of the precocious. Scientific American Supplement 65:391; June 1908.


"Negro Mathematical Genius." Literary Digest 46:971.72; April 26, 1913.

"Numbers Game." M. Mag. 6:43; 1952.

Brief reference to Shakuntala Devi. 20-year old Hindu woman, who can extract mentally the 20th root of a 42-digit number, or multiply numbers yielding a 39-digit product.

"Numbers Game." Time 60:49; July 14, 1952.


Reference to W. J. Sidis.


RECREATIONAL MATHEMATICS

Weinland, J. D. and Schlauch, W. S. Examination of the computing ability of Mr. Salo Finkelstein. *Journal of Experimental Psychology* 21:382-402; October 1937.


A Dutch prodigy multiplies a 10-place number by an 11-place number in 21 minutes.


2.5 Theory of Numbers—Factorizations—Primes

The origins of modern number theory are to be found in ancient Greek *arithmetika*, which was a philosophy of the nature of number rather than the art of calculation; it was far more abstract than Greek geometry. Certain questions concerned the Greeks very much: the relation of primes to composite numbers; the number of primes; polygonal and solid numbers; amicable numbers; perfect numbers; *Gematria*; and such.

With the decline of Greek mathematics, progress in number theory lay dormant until about a century and a half ago. With the work of Gauss, about 1800, there began the extension of the concept of number and the generalization of arithmetic, a series of developments in which the greatest of modern mathematicians played significant roles—among them Fermat, Euler, Lagrange, Kummer, Dedekind, Kronecker, Galois, R. Lipschitz, A. Hurwitz, Emmy Noether, and L. E. Dickson.


Triangular numbers; pyramidal numbers; distribution of primes; perfect numbers; Lehman's machine; and so on.


Stimulating historical treatment of the development of abstract arithmetic in modern times.


Elaborate, definitive treatment.


Includes discussion of multigrade chains and various Diophantine problems.


Bibliography.
JONES, PHILLIP S. Binary system. M. T. 46-575-77; 1953.
KRAITCHIK, MAURICE. The greatest known prime number. Scrip. M. 18:82; 1952.
LEHMER, D. N. History of the problem of separating a number into its prime factors. Sci. Mo. 7:227-34; 1918.
MARSHALL, W. L. Some properties of prime numbers. The Pentagon 8:5-8; 1948.
MILLER, J. C. P. Large primes. Eureka 14:10, 11; 1951.
Numbers and number theory have a fascination for laymen as well as for professional mathematicians; this book is suitable for both groups. History and exposition are skillfully interwoven in a clear and interesting book.
ARITHMETICAL AND ALGEBRAIC RECREATIONS


Reid, Constance. Perfect numbers. Sci. Am. 188:84-86; March 1953.

Brief, but good.


Concerning the theory of prime numbers.


Describes a device similar in purpose to Eratosthenes’ sieve.


Contains a wealth of material; bibliography of 33 items.


Bibliography.


2.6 Perfect Numbers—Mersenne’s Numbers

A number is said to be perfect if it equals the sum of all numbers that divide it except itself. Thus the first two perfect numbers are 6 and 28, since $6 = 1 + 2 + 3$, and $28 = 1 + 2 + 4 + 7 + 14$. Euclid was able to prove that any number of the form $2^{p-1}(2^p - 1)$ is a perfect number whenever $2^p - 1$ is prime. Prime numbers of the form $2^p - 1$ are known as Mersenne numbers.

For upwards of 2000 years, only 12 perfect numbers were known, namely, those for which the values of $p$ in Euclid’s formula are 2, 3, 5, 7, 13, 17, 19, 31, 61, 89, 107, and 127. In recent years, with the aid of high speed
electronic computing machines, five more perfect numbers have been found, the largest, or 17th, being $2^{21210} - 1$.

Françon, M. Ausone et le premier nombre parfait. *Isis* 42:302-303; 1951.
Deals with the relations of binary notation to perfect numbers.
2.7 Fermat's Last Theorem

The great Fermat theorem, stating that the equation $x^n + y^n = z^n$, where $n$ is an arbitrary integer, has no integral solutions for integral values of $n$ except when $n = 1$ and $n = 2$, is as interesting today as it was some 300 years ago when first enunciated by the great master of number theory. Despite the lure of a prize of 100,000 marks offered shortly after the turn of the present century, all efforts to find a complete proof have thus far been fruitless. And yet Fermat claimed that "he had found a really wonderful proof, only the margin of his book was too narrow to accommodate it." To be sure, the theorem has been shown to hold for exponents below 100, but that is scarcely a mathematician's dream of success.


"A $25,000 Prize for a Mathematical Solution of the Fermat Formula." *Sci. Am.*, February 1, 1908, p. 75.


RECREATIONAL MATHEMATICS


ELSTON, Fred. The last theorem of Fermat not only a problem of algebraic analysis but also a probability problem? M. Mag. 28:150-52; 1954-55.


GRISELLE, THOMAS. Proof of Fermat's last theorem for \( n = 2(8a + 1) \). M. Mag. 26:263; 1952-53.


JAMES, G. A higher upper limit to the parameters in Fermat's equation. Am. Mo. 45:439-45; 1938.


MILLER, G. A. Some thoughts on modern mathematicial research. Science 35:877-87; June 7, 1912.


"Some Introductory Comments on Fermat's Last Theorem." M. Mag. 27:213.16; 1953-54.


VANDIVER, H. S. Fermat's last theorem. Am. M. Mo. 53:555-78; 1946.
2.8 Fibonacci Numbers and Series


Discusses the Fibonacci counterparts of binomial coefficients; also, a Pascal triangle for Fibonacci sequences.


Strebyk, Adrian. The Fibonacci numbers. S. S. M. 44:701-707; 1944.


Yarden, D. A bibliography of the Fibonacci sequence. Hirenon Lematematika 2:36-45; January 1948.
Chapter 3

Geometric Recreations

This rather broad category includes not only geometric fallacies and paradoxes, optical illusions, dissections, tangrams, and geometric constructions, but also material on regular polygons and polyhedra, tessellations, linkages, and the mechanical construction of mathematical curves. Such amusements often appeal to the eye-minded, and to those who are not particularly intrigued by numerical or algebraic puzzles.

Among some of the best known geometric fallacies are the alleged proofs that an obtuse angle equals a right angle; that every triangle is isosceles; that the length of part of a line equals the length of the whole line; and that the sum of the lengths of two sides of a triangle equals the length of the third side. These and similar proofs rarely fail to intrigue high school pupils.

As for optical illusions, the explanation generally hinges upon considerations of perspective, shading, disposition, and such, or upon purely psychological considerations. Among the most widely known optical illusions are the two equal segments with reversed arrowheads; the "Which is taller, the policeman or the little boy?"; and the "How many cubes are there—six or seven?". Optical illusions such as the last of these are the more tantalizing because they sometimes seem to "turn inside out" as you look at them.

3.1 General Geometric Problems and Puzzles


A delightful, well-known little classic.


"Geometrical recreations." Fallacies; dissections; tessellations; cyclotomy; chessboard problems. p. 76-128.


17:1951.
18:298-300; 1952.
20:203-04; 1954.


Harris, I. Geometric recreations. *S. S. M.* 20:731-33; 1920.

"Harvard University Student Makes Straightline Drawings." *Life*, March 18, 1940. p. 43-44.

Proof of the proposition that if the internal bisectors of the base angles of a triangle are equal, the triangle is isosceles.


Curves; surfaces; lattices; crystals; polyhedra; topology; and so on.


Carved wooden models.


An exceptionally appealing and suggestive booklet.


Brief note on the number of squares and rectangles visible on any square board of squares.


A paradox about packing layers in a box.


A collection of theorems and properties relating to the medians, angle-bisectors, angle-trisectors, etc., of a triangle.


Thimble, H. C. Signed areas applied to "recreations of geometry." *M. T.* 40:3.7; 1947.


### 3.2 Geometric Fallacies—Optical Illusions


Brandes, Louis G. *An Introduction to Optical Illusions.* Portland, Maine: J. Weston Walch, Publisher, P. O. Box 1075, 1957. $1.00.

A collection of 25 striking drawings—common optical illusions.


Cannahan, W. H. *Note on the fallacy.* *M. T.* 19:496-98; 1926.


GEOMETRIC RECREATIONS

LIPPS, THEODOR. Raumästhetik und geometrisch-optische Täuschungen. Leipzig: J. A. Barth, 1897.
MEYER, JEROME. A two-inch line with a six-inch “perimeter.” Scrip. M. 7:156-57; 1940.
WILLERS, H. Geometrisch-optische Täuschungen in mathematischer Behandlung. Z. M. N. U. 60:499; 1929.

3.3 Geometric Dissections—Tangrams

Geometric dissections, generally speaking, divide a given plane rectilinear figure by means of straight lines into parts which can then be reassembled to form some other preassigned configuration. Many recreations are built around such dissections. Some very well-known dissections have been applied to the proof of the Pythagorean theorem.

Tangrams go back to ancient times. They consist essentially of flat tiles or other pieces, usually seven in number, with definite shapes, such as a square, a rhombus, and five triangles. The idea is to form picture figures by suitable arrangements of the tans, as the pieces were called. Although an Oriental recreation, it was also known to Archimedes. His elaborate tangram consisted of 14 pieces, cut out of a rectangle whose length is twice its width—the “stomachion” of the Greeks and Romans.


Gives a solution of the problem: to divide a square into a number of smaller squares, no two of them equivalent.

"Geometrical Proof of the Identity \( a^2 - b^2 = (a + b)(a - b) \)." *Scrip. M.* 11:172; 1945.


Collection of 21 dissection puzzles.


Contains many dissection problems.


Dissections of squares and rectangles.


GEOMETRIC RECREATIONS


SATTERLY, JOHN. Meet Mr. Tau again. S. S. M. 57:150; 1957.
Dissection of a regular pentagon.


Dissection figures and other mathematical puzzles.

Contains many fine puzzles.

YATES, ROBERT C. Addition by dissection. S. S. M. 40:801-807; 1940.

3.4 Regular Polygons and Polyhedrons

The elementary characteristics of regular polygons and polyhedrons were known to the ancient Greeks, who gave us the regular Platonic solids and the semi-regular Archimedean solids. But the elaborate development of the subject in modern times is scarcely 100 years old. The general theory of regular polytopes is intimately associated with several branches of higher mathematics, notably group theory, topology and n-dimensional geometry, not to omit its relation to the science of crystallography. A polytope is a geometrical figure bounded by portions of lines, planes, or hyperplanes; in 2-space it is a polygon, and in 3-space, a polyhedron. The study of regular polytopes is unusually fascinating. It appeals to many on the ground of sheer beauty and imagery; the mathematician cannot resist the urge to generalize; and the scientist, of course, is concerned with regular forms in Nature.


BANKOFF, LEON. Regular polygons of 2, 3, 4 and 6 sides inscribed in circles of unit radius. Scrip. M. 21:252; 1956.


Deals with all forms of polyhedrons, including semi-regular and star polyhedrons.


Very scholarly and complete work; extensive bibliography. Regular polyhedrons and polyhedra; quasi-regular solids; tessellations; honeycombs; star polyhedra; kaleidoscope; group theory; polytopes in higher space, and such.


Gives complete directions for the construction of the 13 Archimedean solids, the four Kepler-Poinsot star solids, stellar Archimedean polyhedra, and so on.


Haag, F. Die regulären Kristallkörper. Rottweil, 1887.


Pamphlet containing excellent plates for folding patterns of polyhedrons.


Heath, Dwight. Some Composite Polyhedrons. Published by the author, Franklin College, 1940. (Mimeo. 11 plates).


Material on regular polyhedrons, semi-regular and star polyhedrons, and crystal forms.


Poinset, d'Après M. Sur les polygones et les polyèdres étoilés; polygones funiculares. Nouvelles Annales de Mathématiques 8:68-74; 1849.


Gives over 150 references to books prior to 1900 dealing with tetrahedra and polyhedra, p. 237-40.

3.5 Geometric Constructions


Bowker, E. Fourth proportional and similarity in construction work. S. S. M. 27:527-33; 1927.


Bückner, P. Aus der Theorie der geometrischen Konstruktionen. (Basel, Switzerland): Elemente der Mathematik 1:1-3; 1946.

To construct a square whose sides (or extensions) shall pass through four noncollinear random points in a plane.


Duncan, D. Criticism of the treatment of the regular polygon constructions in certain well-known geometry texts. S. S. M. 34:50-57; 1934.

Eckhardt, O. Teilung einer Strecke in gleiche Teile. Z. M. N. U. 56:30; 1925.


Eves, Howard and Hoggatt, V. Euclidean constructions with well-defined intersections. M. T. 44:262-63; 1951.


Bibliography.

Horson, On geometrical constructions by means of the compass. M. Gaz. 7:49-54; March 1913.


Geometric construction of common curves, p. 139-56.


D'Ocagne, M. Quelques considérations sur les constructions géométriques. Revue Général Scientifique 44:7.9; January 1933.


Richmond, To construct a regular polygon of 17 sides. Mathematische Annalen 67:459; 1909.


Trigg, C. W. Unorthodox ways to trisect a line segment. S. S. M. 54:525-28; 1954.

GEOMETRIC RECREATIONS


3.6 Mascheroni Constructions

When Lorenzo Mascheroni published his Geometry of the Compass, in 1797, he showed that any construction which can be executed with the straight edge and compass could also be carried out with the compass alone. Obviously, his points are not determined by the intersection of two straight lines. Furthermore, a straight line is considered as given or obtained when two points lying on it are known. Nearly 100 years later, A. Adler verified Mascheroni's claims. Adler used the idea of inversion with regard to a circle, an idea unknown to Mascheroni, having been discovered by Steiner in 1824.

Strictly speaking, Mascheroni's constructions are not usually thought of as recreations; the problems that arise, however, are not only fascinating—they make considerable demands upon one's ingenuity.

CARNAHAN, WALTER. Geometrical constructions without the compasses. S. S. M. 36:182-89; 1936.
GOLDBERG, M. All geometric construction may be made with compasses. S. S. M. 25:961-65; 1925.
HOBSON, E. W. On geometrical constructions by means of the compass. M. Gaz. 7:49-54; 1913.
3.7 Linkages—The Pantograph

The problem of transforming line motion into circular motion is simple enough, but the reverse problem, of converting circular motion into motion along a straight line, is considerably more difficult. The latter problem was of slight interest to earlier mathematicians, and only attracted widespread attention some years after the first solutions were given by Sarrus in 1853 and Peaucellier in 1864. Considerable enthusiasm in the subject of linkages developed during the last quarter of the 19th Century, stimulated largely by the work of Sylvester, Cayley, Kempe, and others, and culminating in Kempe’s demonstration of the remarkable theorem that any algebraic curve can be described by a linkage. The bars of a linkage need not be straight; the only requirement is that they be plane, inextensible members. Certain linkworks are of considerable importance in mechanics and engineering.


Gives about a dozen references not covered in Kanayama’s list (see below).


GOLDBERG, MICHAEL. Polyhedral linkages. *N. M. M.* 16:1-10; 1942.


A reprint of the original book, first published in 1877.


A reprint of the original book, first published in 1877.

Pantograph and simple linkages.


Includes a linkage for the mechanical construction of regular n-gons, and a linkage for dividing an angle mechanically into any number of equal parts.


Trimble, H. C. For non-genius only. *M. T.* 42:244-46; 1949.

Theory of the pantograph.

Tuck, F. E. How to draw a straight line. *S. S. M.* 21:554-58; 1921.


Excellent treatment of straightsedge and compasses, dissection, constructions, linkages, higher tools; bibliographies.


Yates, R. C. A note on the 3-bar curve. *N. M. M.* 14:190-92; 1940.


3.8 Mechanical Construction of Curves


Mechanical construction of higher plane curves.
Discussion of linkages, line motion, harmonic motion, Lissajou curves, mechanical inversors.
An extremely stimulating study and work book.
YATES, R. C. Mechanically described curves. N. M. M. 10:134-38; 1936.
Yates, R. C. To have and to hold. N. M. M. 14:2; 1939.
Chapter 4

Assorted Recreations

From one point of view, mathematical recreations fall into two major categories: those that involve number relationships or computation, and those that depend chiefly upon the manipulation of objects. Conspicuous in the latter category we find the problem of ferrying the wolf, the goat, and the basket of cabbages across a stream (or the three couples with jealous spouses, where the boat will hold only two people); the problem of measuring out one quart of a liquid with only a 3-, 5-, and 8-quart measure available; the problem of the three coins; the twelve-coin problem; the shunting of freight cars; the Chinese ring puzzle; the problems of chains and links; the Tower of Hanoi; the Josephus problem; and the Boss Puzzle, or 15-Puzzle.

Included also among the manipulative recreations are string figures, paper-folding exercises, card tricks, chessboard problems, unicursal problems, labyrinths, and a variety of topological problems.

Because of the recent popularity of the 15-Puzzle, it merits some observations. Invented in America by Sam Lloyd in 1878, it took Europe by storm, "driving people mad." A square arrangement of 15 small square blocks numbered from 1 to 15, with room for 16 blocks, so that the 15 squares can be interchanged by sliding them about. The total number of conceivable positions is factorial 16, or almost 21 billion. It can be proved that from any given initial arrangement, only half of all the possible arrangements can be obtained by sliding the squares about. In the current revival of interest, the puzzle appears in dime stores, and is made of modern plastic material. Variations have also appeared—rectangular versions containing 19, 21, and 31 pieces, respectively.

4.1 Boss Puzzle


ASSORTED RECREATIONS


WARREN, G. W. Clue to 15-Puzzle. Nation 30:326; 1880.

4.2 Card Tricks—Manipulative Puzzles


A free pamphlet.


Reference to games of tic-tac-toe.


Tricks with dice, dominoes, calendars, watches, dollar bills, matches, books.


Chapters on “Statistics” and on the “Algebra of the Card Pack”.

81
RECREATIONAL MATHEMATICS

"An Inductive-deductive Experiment with the Tower of Hanoi Puzzle." M. T. 44:505; 1951.


Mathematical card tricks, etc., based on binary notation and feedback.


PRICE, IRENE. "I Doubt It"—a mathematical card game. Am. M. Mo. 49:117; 1942.


Extensive discussion of a manipulative puzzle game similar to "Fox and Geese.'


Reprints obtainable from Miss L. E. Christman, 1217 Elmdale Ave., Chicago, Ill. Discusses curve-stitching.


Card tricks, number tricks, and some paper and ring tricks involving topology.

WALKER, S. W. Games of the checkers family in line, plane, and space. Bulletin, American Mathematical Society 52:825; 1946.
ASSORTED RECREATIONS

4.3 Chessboard Problems


STEWART, B. M. Solitaire on a checkerboard. Am. M. Mo. 48:228-32; 1941.


4.4 Topological Questions


Describes the Afghan hands; handkerchief tricks; tricks with string and rope; vest tricks.


HALL, F. What is topology? M. T. 34:158-60; 1941.


KLINE, J. R. What is the Jordan curve theorem? Am. M. Mo. 49:281-86; 1942.

Bibliography.


MENGER, KARL. What is dimension? Am. M. Mo. 50:2-7; 1943.

Bibliography.


POFFES, ARTHUR. Filling a square with circles. S. S. M. 45:858-61; 1945.


Bibliography.


Very readable and suggestive.


4.5 String Figures—Theory of Knots


 Excellent bibliography.


Concise and complete; bibliography.


4.6 The Möbius Strip

BOND, NELSON. The geometries of Johnny Day. Astounding Science Fiction, July 1941.
Humorous sketch based on the Möbius strip.

Humorous story based on the Möbius strip.

HERING, C. Flat band with only one surface and one edge. Sci. Am. 110:56; 1914.


Humorous skit based on the Möbius strip.

UPSON, WILLIAM. Paul Bunyan vs. the conveyor belt. Ford Times 41:14-17; Dearborn, Mich.: Ford Motor Co., 3000 Schaefer Road, July 1949.
Another humorous skit.

4.7 Map-Coloring Problems

A well-known problem of interest to mapmakers is the answer to the question: "How many colors are necessary to color a map, showing any number of countries, in such a way that no two countries having a common boundary shall have the same color?" Apparently an innocent enough question, it continues to baffle topologists.

Thus one might expect that the more elaborate a map becomes, the more colors would be required if the desired condition above is to be fulfilled, but such is not the case. Curiously enough, no map has yet been constructed for which four colors would not be sufficient. This is very different, however, from proving the generalization that four colors would suffice for any conceivable map.

What has been proved, among other theorems, is that five colors are always sufficient for any map drawn on a sphere or on a plane. Whether five colors are always necessary is still an open question.
ASSORTED RECREATIONS 75


BIRKHOFF, G. D. A determinant formula for the number of ways of coloring a map. Annals of Mathematics 14:42; 1912.


“Concerning the Four-Color Problem.” Am. M. Mo. 60:121-22; 1953.


Bibliography.


KEMPE, A. B. How to colour a map with four colours. *Nature* 21:399-400; February 26, 1880.


### 4.8 Paper Folding


  Paper-folding, p. 8-18; paper knots, p. 46-47.

FOURREY, E. Procédés originaux de constructions géométriques. Paris, 1924.
  Paper-folding, p. 113-139.

GIERKE, HILDEGARD VON AND KUCZYNSKI, ALICE. Allerlei Papierarbeiten. Leipzig:
  Teubner, 1910. 73 p.


  1952. 6 p.
  Contains approximately 150 references.

  Bibliography, 40 references, many of which are unfortunately inaccessible;
  books only.

  1907. Also, Scientific American Supplement 73:112; February 17, 1912.

LUGHIA, ANTONIO AND CORINA LUCIANI DE. El Plegado y cartonaje en la escuela
  primaria. Buenos Aires, 1940.


MURRAY, W. D. AND RIGNEY, F. J. Fun with Paper Folding. New York: Revell,
  1928. 95 p.


  Earlier edition entitled: "Fun at Dinner with Napkin Folds."


  Flat assemblies which open up into the third dimension.

RÖTHE, RICHARD. Falten und Formen mit Papier. Wien: Deutscher Verlag für


"Tieing a Strip of Paper into a Knot to Form a Pentagon." *S. S. M.* 26:654; 1926.


The first of a series of short papers on the subject of paper folding.


Despite language barrier, diagrams and directions are exceptionally clear.

### 4.9 Unicursal Problems—Labyrinths


Chapter 5

Magic Squares

Undoubtedly of Chinese, or at least Oriental origin, magic squares seem always to have been associated with mysticism. Through the ages they have been used in fortune telling and as talismen and amulets. Often they were associated with the symbols of the alchemist; and they played a significant role in the cabalistic writings of the Hebrews.

Although the theory of third-order squares is simple and complete, no completely general methods of construction are known, nor has a complete count of magic squares of all orders ever been made. Magic squares may be derived from a given arrangement by various transformations, such as mirror reflection, rotation through 90°, cyclic interchange of rows or columns or both, and, in the case of even-order squares, by simple interchange of opposite quarters.

In addition to ordinary magic squares, a number of interesting varieties are to be found: bordered squares, i.e., squares within squares; pandiagonal squares, i.e., squares that are magic along the broken diagonals as well as along the two main diagonals; symmetric squares, i.e., squares of order \( n \) such that the sum of any two numbers in skewly related cells shall be constant and equal to \( n^2 + 1 \); magic squares of nonconsecutive numbers; doubly-magic squares; magic domino squares; magic cubes; magic circles; interlocked hexagons; composite squares; and so on.

The theory and construction of magic squares is related to lattice theory. Indeed, as James Byrnie Shaw has aptly said: “Latin squares, magic squares, linkages, polyhedra, crystals, groups, properties due to singularities, automorphic forms, lattices, topology, isomers, isotopes, valences, equivalences, syzygies, systems of forms, transitivity, linear dependence, functional dependence, and many other related topics all are fundamentally based on symmetries of some sort.” Is it any wonder that magic squares are so fascinating?

5.1 Books—1900-1924

AHRENS, WILHELM. Hebräische Amulette mit magischen Zahlenquadrate. 1919.
AHRENS, WILHELM. Die magischen Zahlenquadrat. 1915.
AHRENS, WILHELM. Planetenamulette. 1920.


GRATZINGER. Talismanische Dämonologie. 1920.


LARSS, H. Das Geheimnis der Amulette. 1919.


LALITTE, PROSPER DE. Essai sur le carré magique de n nombres. Agen: 1906.

MACMAHON, P. A. Magic Squares and Other Problems. 1902.


PORTIER, B. Le carré cabalistique de 8. 1902.

PORTIER, B. Le carré panmagique. 1904.

RILLY, ACHILLE. Étude sur les triangles et les carrés magiques aux deux premiers Degrés. Troyes: 1901.


TARRY, G. Le carré trimagique de 128. 1906.
MAGIC SQUARES

TARRY, G. Carrés cabalistiques Euleriens. 1904.

A single sheet, containing the numbers from 1 to 2500, arranged in a magic square, and having "a total of 62,525 in 102 different ways." In Library of Congress.


WILLIS, J. Magic Squares and Cubes. 1909.

5.2 Contemporary Books—From 1925 On

AUBRY, A. Carrés magiques impairs. 1928.

AUPIC, JAN. Les carrés magiques. 1932.


CANDY, ALBERT L. Pandiagonal Magic Squares of Composite Order. Lincoln, Nebraska: the author, 1941.


Valuable bibliography, p. 167-91.


DELLACASA, LUCIANO. Sui guadagni magici. Sui quadri magici. 1931.


93

FITTING, FRIEDRICH. Rein Mathematische Behandlung des Problems der magischen Quadrate von 16 und 64 Feldern. 1931.


LEWIS, SISTER MARY TERESINE. Construction and Application of Magic Rectangles Modulo p, for Small Values of p. Catholic University of America, 1947.

MAACK, FERDINAND. Die astrologische Bedeutung der magischen Quadrate. 1925.


MCDONALD, K. Magic Cubes Which Are Uniform Step Cubes. University of California, 1934. 35¢. (Pamphlet)


SAUERHERING, FRIEDRICH. Magische Zahlenquadrate; eine gemeinverständliche belehrende Darstellung mit einigen neu ermittelten Lösungen. Lindenthal: Wellersberg-Verlag, 1926.


STERN, ERICH. Nouvelle méthode pour construire et dénombrer certains carrés magiques d'ordre 4 m avec applications aux parcours magiques (Trans. from German of E. Cazalas.) Bruxelles: Librairie du “Sphinx,” 1937. 20 p.
5.3 Periodical Literature


Ahren, Wilhelm. Uber magische Quadrate; Anzahlbestimmungen; Vorkommen auf Amulettten. Z. M. N. U. 45:525; 1914.


Arnoux, Gabriel. Les espaces arithmetiques dont les cotés sont des nombres premiers inegaux. Leur applications; 1er a la théorie des congruences; 2nd a la construction des espaces magiques... Assoc. francaise pour l'avancement des sciences: Compte Rendu, Sess. 34. (1905). p. 103-22, 1906.


Candy, Albert L. The number of 12 × 12 squares that can be constructed by the method of current groups. *N. M. M.* 9:223-35; 1935.
Candy, Albert L. To construct a magic square of order 2n from a given square of order n. *N. M. M.* 9:99-105; 1935.
Frieron, L. S. Mathematical study of magic squares. *Monist* 17:272-95; 1907.
MAGIC SQUARES


GUDDICE, FRANCESCO. Tavole ad allineamenti d'uguali somme o prodotti. *Bollettino di Matematica* 31:129-37; 1935.


HEATH, ROYAL V. The magic clock. *M. T.* 30:84; 1937.


LOOMIS, HIRAM B. Pandiagonal magic squares and their relatives. *S. S. M.* 44:831-38; 1944.


MACMAHON, P. A. Magic squares and other problems on a chessboard. *Proceedings, Royal Institute of Great Britain* 17:50-61; 1892.


6:114-18, 175-78; 1939.

7:143-55; 1940.

8:49-55, 122-33, 183-87, 257-61; 1941.

9:278-84; 1943.

11:85-88; 1945.


MCLAUGHLIN, HENRY P. Algebraic magic squares. *M. T.* 14:71-77; 1921.


Planck, C. Four-fold magics. Monist 20:617-30; 1910.

Planck, C. General rule for constructing ornate square magic squares of orders = 0 (mod. 4). Monist 26:463-70; 1916.


Planck, C. Magic squares of the 5th order. Monist 26:470-76; 1916.

Planck, C. Ornate magic squares of composite odd orders. Monist 26:470-76; 1916.

Planck, C. Pandiagonal magics of orders 6 and 10 with minimal numbers. Monist 29:367-16; 1919.


Posey, L. R. A general formula for magic squares of various orders beginning with numbers different from unity. S. S. M. 40:315-19; 1940.


Sayles, H. A. General notes on the construction of magic squares and cubes with prime numbers. Monist 28:141-58; 1918.


Sayles, H. A. Magic squares made with prime numbers to have the lowest possible summations. Monist 23:623-40; 1913.

Sayles, H. A. Pandiagonal concentric squares of order 4m. Monist 26:476-80; 1916.
The Pythagorean Relationship

This celebrated theorem is notable, first because of the rich historical associations suggested thereby; secondly, because of the amazing variety of proofs which have been given; and thirdly, because further exploration quickly leads to interesting and perhaps unsuspected byways, such as the Golden Section, dynamic symmetry, logarithmic spirals, angle trisection, duplication of the cube, squaring the circle, determination of the value of π, the concept of the irrational number, regular and star polygons and polyhedra, theory of numbers, constructibility of angles and polygons, continued fractions, phyllotaxy, musical scales, Diophantine equations, Heronian triangles, and Pythagorean number lore.

Two works are of particular interest: the brief monograph by Loomis, which gives over 200 proofs of the theorem, and the stimulating tract by Naber, which is unusually suggestive with respect to the ramifications of the theorem.

6.1 The Theorem of Pythagoras


Blacklee, T. Ptolemaic and Pythagorean theorems, from an identity. *S. S. M.* 14:748; 1914.


90

RECREATIONAL MATHEMATICS

Colburn, A. Pons Asinorum; new solutions of the Pythagorean theorem. Scientific American Supplement 70:359, 382-83; December 1910.

Colburn, A. Study of the Pythagorean theorem and its proofs. M. T. 4:45-47; 1911.

Condit, A. New proof of the Pythagorean theorem. S. S. M. 40:379-80; 1940.

Davidson, E. High school boy's proof of the Pythagorean theorem. S. S. M. 7:777-78; 1907.


Eagle, Edwin. Pythagoras and Ptolemy must have looked at the conclusion. The Pentagon 10:79-83; 1951.


Bibliography.


102
THE PYTHAGOREAN RELATIONSHIP


Historical observations.


Kallasamayer, N. A proof of Pythagoras' theorem. *M. Gaz.* 28: Mathematical Note No. 1746; December 1944.


Katnik, H. New proof of the Pythagorean theorem. *S. S. M.* 15:249-54; 1941.


Lawrence, B. E. Pythagoras and an extension. *M. Gaz.* 28: Mathematical Note No. 1751; December 1944.


Northrup, E. Is this a dynamical proof of the Pythagorean theorem? Science n.s. 32:863-64; 1910.
"Proof of the Pythagorean Theorem." The Pentagon 5:22; 1945.
"Pythagorean Theorem." Science 33:457; 1911.
"Pythagorean Theorem." (Garfield's proof.) The Pentagon 7:38; 1947.
THE PYTHAGOREAN RELATIONSHIP


Pythagorean theorem, p. 109 ff.


“A Symmetrical Figure to Demonstrate Pythagoras’ Theorem.” M. Gaz., December 1951.

Mathematical notes.


Gives proof of the converse of the Pythagorean theorem.


ZOELAR, M. Der Pythagoreische Lehrsatz. Z. M. N. U. 44:531; 1913.

6.2 Pythagorean Numbers—Rational Right Triangles

A general Pythagorean triplet may be expressed as \((p, q; r)\), which means that \(p\), \(q\) and \(r\) are distinct integers satisfying the equation \(p^2 + q^2 = r^2\).

If \(p\), \(q\) and \(r\) have no factor in common, the triplet is called a primitive triplet.

Pythagorean triplets exhibit many interesting properties. The familiar 3,
4; 5 triplet is the only one which consists of consecutive positive integers. In some triplets, \( p, q \) and \( r \) form an arithmetic progression; but no Pythagorean triplet exists in which one number is a mean proportional between the other two. Again: no primitive Pythagorean triplet can contain two even numbers. Furthermore, if \((p, q, r)\) is a Pythagorean triplet, then \( p \) and \( q \) cannot both be odd.

Two fundamental relationships are of interest:

1. The numbers \( 2n + 1, 2n(n + 1), \) and \( 2n^2 + 2n + 1 \) form a Pythagorean triplet for every value of \( n \).

2. Every primitive Pythagorean triplet \((p, q, r)\) is of the form \( p = u^2 - v^2, q = 2uv, r = u^2 + v^2 \), where \( u \) and \( v \) are relatively prime integers, one being even and the other odd, and with \( u > v \).

References:

- Brown, E. N. Integral right triangles. *S. S. M.* 41:799-800; 1941.
- Colwell, L. Exploring the field of Pythagorean number. *S. S. M.* 40:619-27; 1940.


Gauss, F. Uber die Pythagorische Zahlen. Pr. Bunsau, 1894.


Bibliography.


Knorr, J. Das rechtwinklige rationale Dreieck. Wien: 1881.


McLean, E. Pythagorean numbers. M. Gaz. 24:59, 125; 1940.


Miksa, Francis. Table of integral solutions of $a^2 + b^2 + c^2 = r^2$ for all odd values of $r$ from $r = 3$ to $r = 207$. M. T. 48:251-55; 1955.

Miksa, Francis. Table of primitive Pythagorean triangles whose areas contain all the digits 1, 2, 3, 4, 5, 6, 7, 8, 9, 0. Scrip. M. 20:231; 1954.


6.3 Special Triangles—Heronian Triangles


Integral right triangles with legs differing by one.


On Heronian triangles, and such.


Dickson, L. E. Rational triangles and quadrilaterals. Am. M. Mo. 28:244-50; 1921.


ROBINSON, L. V. Building triangles with integers. *N. M. M.* 17:239-44; 1943.


An approximate isosceles Pythagorean triangle.


RECREATIONAL MATHEMATICS


Relation of Pythagorean triangles to the equation $x^2 - y^2 = z^2$.


6.4 Miscellaneous Pythagorean Recreations


FRAME, J. S. Solving a right triangle without tables. Am. M. Mo. 50:622-26; 1943.


Includes discussion of Pythagorean triangles and their properties.


MISKIS, F. L. Primitive Pythagorean triangles whose areas contain all the digits 1, 2, ..., 9. Scrip. M. 20:231; 1954.


RAINE, C. W. Pythagorean triangles from the Fibonacci series 1, 1, 2, 3, 5, 8 ... Scrip. M. 14:164-65; 1948.


THE PYTHAGOREAN RELATIONSHIP


Chapter 7

Famous Problems of Antiquity

Over two thousand years ago Greek mathematicians devoted themselves to certain problems which have engaged the attention of men ever since. The many attempted solutions and the spirited controversies which these problems created through the ages served to stimulate immensely the development of mathematics, particularly algebra, equation theory, geometry, theory of numbers, group theory, and analysis.

Three of these problems are usually thought of together, namely: (a) trisecting an angle, (b) duplicating a cube, and (c) squaring a circle. As propounded by the Greeks, all three problems were to be solved by “pure Euclidean” methods—that is, by the use of compasses and the unmarked straightedge only. With this limitation—the use of straight lines and circles alone—none of these three problems can be solved. But this fact was not proved until about 1800. Nevertheless, each passing year witnesses stubborn attempts, on the part of laymen and amateurs alike, to tackle one or another of these famous “unsolved” problems and so achieve immortality.

Also of great concern to the Greeks were the famous paradoxes of Zeno. Somewhat different from the classical constructions, they presented an imposing challenge to the imagination—a challenge which, in slightly different form, plagues the mathematician even today. What is involved is nothing less than the concepts of infinity and continuity, ideas which lie not only at the roots of modern analysis, but at the very foundations of mathematics itself.

7.1 Classical Constructions


CARSLOW, H. S. On the constructions which are possible by Euclid’s methods. *M. Gaz.* 5:171; 1910.


112


DICKSON, L. E. Why it is impossible to trisect an angle or to construct a regular polygon of 7 or 9 sides by ruler and compasses. M. T. 14:217-23; 1921.


FRAENKEL, A. A. Division of the circle into a number of equal parts, and other problems. Scrip. M. 9:81-84; 1943.

GIVENS, W. B. Division of angles into equal parts and polygon construction. Am. M. Mo. 45:653-56; 1938.


YATES, ROBERT C. The angle ruler, the marked ruler, and the carpenter's square. N. M. M. 15:61-73; 1940.


In many ways, this has become the most famous of the three ancient problems—also the most tantalizing. It is so easy to bisect any angle!

Both the trigonometric and the algebraic analyses of the problem lead to an equation of the form $x^3 - 3x^2 - 2a = 0$. The question then arises: for all values of $a$, is it possible to find a root $x$ of this equation by means of compasses and straightedge alone? Modern mathematics has given an unequivocal answer: No. For it has been shown that with the straightedge and compasses together, and no other instruments, it is possible to make only those constructions which are algebraically equivalent to a finite number of operations of addition, subtraction, multiplication, division, and the extraction of real square roots involving given lengths. Yet despite this irrefutable evidence, the race of angle-trisectors, as R. C. Yates has suggested, is indeed a hardy one.

It remains to be pointed out, of course, that not a few constructions with straightedge and compasses yield remarkably close approximations for trisecting a given general angle. Some of them are so close that their discoverers often delude themselves; indeed, the mistakes in the purportedly exact constructions are often extremely difficult to detect.

BACKUS, A. D. Trisecting that angle. Industrial Arts and Vocational Education 33:390; 1944.
BerkEL, E. Mechanical trisector. Sci. Am. 113:519; 1915.
Daniells, Marian. The trisector of Amadori. M. T. 33:80-81; 1940.
Ferguson, D. F. Geometrical construction for the trisection of an angle to any required degree of accuracy. M. Gaz. 9:373; 1919.
FAMOUS PROBLEMS OF ANTIQUITY


FREEMAN, J. F. To trisect an angle. Industrial Arts and Vocational Education 33:80, 390; 1944.


GEORGES, J. S. Another approximate trisection method. S. S. M. 44:690; 1944.


HOFMANN, JOSEPH. Über die Figure der Winkeldrittelnden im Dreieck. Z. M. N. u. 69:158-62; 1938.

“Horns for Dilemma; To Trisect Angles.” Newsweek 35:54; May 8, 1950.

IGLISCH, E. Über die Dreiteilung des Winkels und die Verdoppelung des Würfels unter Benutzung von Zirkel und rechtwinkligem Dreieck. Z. M. N. u. 64: 207-10; 1933.


JOSEPH, F. A. Trisecting a given angle; a mechanical solution. Scientific American Supplement 74:123; 1912.


KASNER, EDWARD. Squaring the circle; also duplication or doubling of a cube and the trisection of an angle. Sci. M. 37:67-71; 1933.


LUCY, A. W. To divide an angle into any number of equal parts. M. Gaz. 14:137-38; 1928.


Uses the four-leaved rose, \( p = \cos 2\theta \), as the trisectrix.


POPPER, J. Trisection of an angle. M. Gaz. 28:84; 1944.


ROESER, HARRY. The derivation and applications of the conchoid of Nicomedes and the cissoid of Diocles. S. S. M. 14:790-96; 1914.


SCUDDER, H. T. How to trisect an angle with a carpenter's square. Am. M. Mo. 35:250-51; 1928.


THURSTON, L. L. Curve which trisects any angle. Scientific American Supplement 73:259; 1912.

FAMOUS PROBLEMS OF ANTIQUITY


"Trisection: General Bibliography." L’Intermédiaire des Mathématiciens, supplements of May and June, 1904.

"Trisection of an Angle." Engineer (London): 129:175, 189; 216-17; 1920.


Tuck, F. E. How to draw a straight line. S. S. M. 21:554-58; 1921.


Weaver, James H. The trisection problem. S. S. M. 15:590-95; 1915.


Yates, Robert C. The angle ruler, the marked ruler and the carpenter’s square. N. M. M. 15:61-73; 1940.


A refreshing treatment of a hoary problem; many solutions, historical notes, interesting sidelights.


7.3 Duplicating a Cube


Dmitrovsky, A. A. Approximate solution of the problem of the duplication of the cube. S. S. M. 13:311-12; 1913.

"Duplicating the Cube, Almost." Scientific American Monthly 3:364; April 1921.
7.4 Squaring a Circle

In this classical problem the goal was to determine the side of a square whose area should be equal to that of a given circle. Strictly speaking, this is no more a recreation, in one sense of the term, than the trisection of an angle or the duplication of a cube. Yet the problem has a long and honorable history.

About 200 years ago it was shown that $\pi$ is incommensurable. Toward the close of the 19th Century the transcendence of $\pi$ was established. Until then the endless futile attempts to solve the problem had led to innumerable fruitful discoveries. Since then, of course, interest in the problem has all but disappeared, although the tribe of would-be circle-squarers has not yet completely vanished. It probably never will, any more than the select coterie of angle-trisectors, and those who would demolish non-Euclidean geometry as unthinkable.

The history of the problem has been well documented: for example, Montucla's Histoire des Recherches sur la Quadrature du Cercle, edited by...
P. L. Lacroix, appeared in 1831. The inveterate debunker Augustus De-Morgan wrote many articles on the subject, particularly in his *Budget of Paradoxes* (1872). E. W. Hobson's history of the problem (see below) first appeared in 1913.


HAYN, JULIUS. Can the sinusoid or sine curve of trigonometry “square the circle”? *School Magazine* 7:321-23; 1924.


HOFMANN, JOSEPH. Über die Quadraturen des Artus de Lionne. N. M. M. 12: 223-30; 1938.


KASNER, E. Squaring the circle; also, duplication of a cube and the trisection of an angle. Sci. Mo. 37:67-71; 1933.


LOWSTON, W. H. Note on an approximation to the square of the circle. N. M. M. 17:81-82; 1942.


SCHUBERT, HERMANN. The squaring of the circle. The Monist 1:197-228; 1891.


SORNITO, JUAN E. Squaring a circle. M. T. 50:51-52; 1957.


7.5 History and Value of Pi (π)

The various values assigned to π at different times and the numerous attempts to find more precise approximations constitute a fascinating story, from Archimedes' value $3\frac{10}{71} > \pi > 3\frac{10}{71}$ to Shanks' computation to 707 decimal places in 1873-74. During the 18th and 19th Centuries the number π occupied the attention of mathematicians (including many amateurs) in connection with the problem of the quadrature of the circle. Enthusiasm for that problem diminished, however, when in 1882 Lindeman proved that π is transcendental, although the race of circle-squarers is a hardy one.

Popular interest in the computation of the value of π was revived late in 1949 when the ENIAC, an electronic computing machine at the U. S. Army's Ballistic Research Laboratories at Aberdeen, Md., computed π to 2035 places certain in about 70 hours of the machine's running time.


BAROLET. H. La machine à calculer le nombre π. La Nature 65:466-68; November 1937.

BENNITT, A. A. Two new arctangent relations for π. Am. M. Mo. 32:253-55; 1925.


BROWN, E. R. π and James Smith. Discovery 5:58-60; May 1924.


121
RECREATIONAL MATHEMATICS


DORWANT, H. L. Values of the trigonometric ratios of \( \pi/6 \) and \( \pi/3 \). *Am. M. Mo.* 49:324-25; 1942.

DORWANT, H. L. Values of the trigonometric ratios of \( \pi/6 \) and \( \pi/3 \). *N. M. M.* 17:115-16; 1942.

ENIAC. *Pi and \( e \). The Duodecimal Bulletin* 6:3x.


FERGUSON, D. F. Evaluation of \( \pi \): are Shank's figures correct? *M. Gaz.* 30:89-90; 1946.


GABA, M. G. A simple approximation for \( \pi \). *Am. M. Mo.* 45:373-75; 1938.


GINSBURG, J. KUTHIEL. Rational approximations for the value of \( \pi \). *Scrip. M.* 10:148; 1944.


GREITZER, SAMUEL. The determination of \( \pi \). Association of Teachers of Mathematics of N. Y. C. (Radio Talks on Mathematics.) 1941. p. 5-8.

GUINAND, A. P. An asymptotic series for computing \( \pi \). *M. Gaz.* 29:214-18; 1945.


HOPE-JONES, W. Approximations to \( \pi \). *Scrip. M.* 8:14; also, *M. Gaz.* 23:284; 1941.

HOPPE, EDMUND. Die zweite Methode des Archimedes zur Berechnung von \( \pi \). *Archiv für Geschichte der Naturwissenschaften* 9:104-107; 1922.

JONES, PHILLIP S. What's new about \( \pi \)? *M. T.* 43:120-22; 208; 1950.

LAWSON, D. A. The history of the number pi (\( \pi \)). *The Pentagon* 4:15-23; 1944-45.


MENGER, KARL. Methods of presenting \( \pi \) and \( e \). *Am. M. Mo.* 52:28-33; 1945.

MEYER, T. Über die zyklometrischen Formeln zur Berechnung von \( \pi \) und über eine abgekürzte Bezeichnung der zyklometrischen Funktionen. *Z. M. N. U.* 35:1+; 1904.
FAMOUS PROBLEMS OF ANTIQUITY 111


MONTGOMERY, D. J. X. Refined π. Sci. Am. 181:30; December 1949. 182:2; February 1950.


POLIAX, SIGFRIED. Pi (π) as a mystic number. Scrip. M. 6:246; 1939.


REITWIESNER, G. W. An ENIAC determination of π and e to more than 2000 decimal places. Mathematical Tables and Other Aids to Computation 4:11. 15; 1950.


SCHOF, C. Al-Biruni’s computation of the value of π. Am. M. Mo. 33:323.25; 1926.


STENGEL, C. Über den Näherungswert π ~ 1.0. Z. M. N. U. 35:508+; 1904.


“Value of π.” (given to 162 decimal places). The Pentagon 8:37.38; 1948.

WALCK, S. Accuracy of the values for π. M. T. 19:110-11; 1926.

123
Nearly 2500 years ago, mathematicians and philosophers were greatly concerned by certain paradoxes involving the notion of the infinite. Modern mathematicians are equally puzzled by these paradoxes.

The famous paradoxes on motion, propounded by Zeno about 500 B.C., included:

1. the **Dichotomy**—that motion is impossible, because a moving object must arrive at the middle before it reaches the end;
2. **Achilles and the Tortoise**—if the tortoise is given a head start, Achilles can never overtake him;
3. the **Arrow**—which must either move where it is or move where it isn't, and so, although in flight, it is always motionless;
4. the **Stadium**—in which it appears that a given time interval is equivalent to an interval twice as great.

Beneath the apparent sophistry of these contradictions there lie subtle and elusive ideas of the most profound sort. Many explanations of the paradoxes have been offered over the years. Their meaning depends upon what interpretation is given to the logical foundations of mathematics—an area in which modern mathematicians are very far from being in agreement.

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**References**

  Good discussion.


  Extensive bibliography.

- **Cajori, Florian.** The purpose of Zeno's arguments on motion. *Isis* 3:7-20; 1920.


Grunbaum, Adolf. Modern science and refutation of the paradoxes of Zeno.  

Jewett, A. R. Clock problems—Achilles and the tortoise.  
M. T. 44:311-12; 1951.

Jones, P. C. Achilles and the tortoise.  

Jourdain, P. E. B. The flying arrow; an anachronism.  

Kramer, Edna. Discussion of Zeno's paradoxes.  
The Main Stream of Mathematics.  

Levy, Hyman. Achilles and the Tortoise.  

Riddles in Mathematics. New York:  

Ramsey, F. P. Achilles and the tortoise.  

Schlegel, Richard. Quantum mechanics and the paradoxes of Zeno.  
American Scientist 36:396-402+; 1948.

Shwisteck, Leon. Zeno's argument about motion.  
The Limits of Science. New York:  

Steinbeck, Peggy. Paradox lost—paradox regained.  
The Pentagon 16:78-83; 1957.  
Discussion of Zeno's paradoxes.

Ushenko, A. Final solution of Zeno's paradox of the race.  

Ushenko, A. Zeno's paradoxes.  
Mind 55:151-65; 1946.

Winn, R. B. On Zeno's paradox of motion.  

"Zeno's Paradoxes in Motion."  
The Pentagon 2:76-77; 1943.
8.1 Mathematics in Nature

Long years before modern biology finally succumbed, as did the other physical sciences, to the relentless scrutiny of mathematical analysis, professional entomologists as well as lay naturalists observed many instances of mathematical relationships in living forms—notably shells, flowers, spider webs, honeycombs, and the like. One of the most brilliant and prolific writers in this field was the late D'Arcy Thompson. Another astute observer was the naturalist Jean Henri Fabre, "poet and prophet of the insect world," self-taught amateur mathematician, whose inimitable beauty of style is exemplified in the following passage:

"With this weird number \( e = 2.718 \ldots \) are we now stationed within the strictly defined realm of the imagination? Not at all: the catenary appears actually every time that weight and flexibility act in concert. The name is given to the curve formed by a chain suspended by two of its points which are not placed on a vertical line. It is the shape taken by a flexible cord when held at each end and relaxed; it is the line that governs the shape of a sail bellying in the wind; it is the curve of the nanny-goat's milk-bag when she returns from filling her trailing udder. And all this answers to the number \( e \ldots \)

"What a quantity of abstruse science for a bit of string! Let us not be surprised. A pellet of shot swinging at the end of a thread, a drop of dew trickling down a straw, a splash of water rippling under the kisses of the air, a mere trifle, after all, requires a titanic scaffolding when we wish to examine it with the eye of calculation. We need the club of Hercules to crush a fly."

A. Form and Symmetry


Discussion of mathematical forms in Nature.


"Economy of Symmetry." N. M. M. 12:210-12; 1938.


Illuminating discussion of symmetry, asymmetry, "left-right," etc.

Gardner, Martin. Left or right? Esquire, February 1951.

Symmetry and asymmetry in Nature: excellent.

Gnomonic growth and logarithmic spirals in snowflakes, shells, and marine animals: also proportion in the human body.


JAEGER, F. M. La symétrie dans la nature. Scientia 34:379-92; December 1923.


Collection “Les Conférences du Palais de la découverte.”


A well-known classic on the relation of physical and biological principles to mathematical laws; geometric forms in nature and art.


B. Bees and Honeycombs


POLACHEK, HARRY. The structure of the honeycomb. Scrip. M. 7:87-98; 1940.


Discussion of the mathematics of the spider’s web, wasps, bees, etc. p. 1-24.
RECREATIONAL MATHEMATICS


VOGEL, LOUIS. Construction of a honeycomb. S. S. M. 37:386-87; 1937.


C. Phyllotaxis


8.2 Machines That Think

Regardless of what the future historian may say, there is no doubt that developments during the last dozen years in the field of electronic computing machines have been little short of phenomenal. Amazingly enough, the ramifications of these unbelievably rapid developments have gone far beyond computing even of the most elaborate sort. Apparently we are on the threshold of what is yet to come in the way of thinking machines, giant brains, logic machines, and machines that play tit-tat-toe, gin rummy, and chess. If ever man’s ingenuity and imagination have served him well, it is in this area. He has drawn upon material from symbolic logic, Boolean algebra, and binary notation, and, with the aid of the electronics engineers, has boldly synthesized mechanisms which can handle information with uncanny skill and breath-taking speed.

ADLER, JERRY. So you think you can count! M. Mag. 28:83; 1954-55.


Entire issue devoted to subject of automation; many excellent articles on related topics.


BERKELEY, E. C. Boolean Algebra (the technique for manipulating "and," "or," "not," and conditions) and applications to insurance. *Record of the American Institute of Actuaries* 26:373-414; October 1937.


Describes over 30 small electric brain machines that reason arithmetically or logically, solve puzzles, play games (TIT-TAT-TOE, NIM, etc.).


Describes small electric brain machines that reason, compute, solve puzzles, play games (the Game of Sundorra 21, the Game of Black Match).


Explains simply how an automatic computer is constructed; how to make it add, subtract, multiply, divide, and solve problems automatically, using relays, electronic tubes, or other devices.


Bibliography.


Excellent popular discussion.


An attractive and informative pamphlet; elementary.


Bibliography.

Lilley, S. The work of a century—in a few minutes. The Pentagon 12:8-16; 1952.


Excellent article.


Science fiction.


Bibliography.


Relation of language to the mathematical theory of communication.


8.3 Cryptography and Cryptanalysis

The art of writing secret messages is as old, presumably, as the human desire to convey information to certain individuals while withholding it from all others. Clearly this has utility for political and military purposes. The ability to read a secret message without having possession of the key, also a highly useful skill, has its sheer recreational and challenging aspects.

The terms code and cipher are not to be confused. A code is a device which requires a code dictionary to write and to read, or, more precisely, to encipher and to decipher. An encoded message is shorter than the original, or plain-text message; a few consecutive letters may represent an entire paragraph. A cipher, on the other hand, is as long or longer than the plain-text message.

Ciphers are of two general types: (a) transposition ciphers, in which the letters of the plain-text message are unchanged but their order is scrambled in some systematic manner; and (b) substitution ciphers, in which letters, or groups of letters, or other symbols, are substituted for letters or groups of letters of the plain text.

Thus when a bona-fide person in possession of the code book simply reverses the process of encoding a code message, he is said to decode or to decipher it. When a person having no knowledge of the key to a cipher or cryptogram “breaks” the cipher, he is said to have solved it. The art of devising secret ciphers is called cryptography. The art of breaking cryptograms is known as cryptanalysis.

Apart from the utilitarian and romantic aspects of secret messages, cryptanalysis offers an implied challenge to human ingenuity which is not easily resisted, and which intrigues many devotees to whom utility and sentiment are immaterial.

A. Books and Pamphlets


MATHEMATICAL MISCELLANIES


Short bibliography.


Figl, A. Système des chiffriéres. Graz: 1926.


Friedman, William F. War Department Publications. Elementary Military Cryptography; Advanced Military Cryptography; Military Cryptanalysis; and such.


Harris, F. A. Solving Simple Substitution Ciphers. Canton, Ohio: American Cryptogram Association, c/o W. G. Bryan, Burton, Ohio. 75¢. (Pamphlet)


UNITED STATES SENATE. Senate Document No. 244. (Report of the Pearl Harbor Investigation Committee.)

Discussion of Winds code and the machine MAGIC.


WOLFE, J. M. A First Course in Cryptanalysis. 3 Vol. New York: Brooklyn College Press. (Mimeo.)


8. Periodical Literature


Excellent introductory discussion; annotated bibliography.


Annotated bibliography of 36 references.


"Bibliography of Cryptography." Am. M. Mo. 50:345; 1943.


"Ciphers and Cipher Keys." Living Age 333:491-95; September 15, 1927.

"Codes and Ciphers." Am. M. Mo. 26:409-13; 1919.

MATHEMATICAL MISCELLANIES


HELMICK, L. S. Key woman of the T-men. Reader’s Digest 31:51; September 1937.


HOLSTEIN, O. The ciphers of Porta and Vigenère. Scientific American Monthly 4:332; October 1921.


KOHLER, JOHN. JCHEW BISEY PYMQP UQRPD. Collier’s 126:22-23; October 28, 1950.


“Magic Was the Word for It.” Time 46:20-22; December 17, 1945.

MENDELSON, C. J. Blaise de Vigenère and the chiffre carré. Proceedings, American Philosophical Society 82:103-29; 1940.


MOORHAN, F. Enciphering and deciphering codes. Sci. Am. 113:159; August 21, 1915.


“NARVO WUMND LVDAD.” Nation’s Business 36:82; June 1948.

Commercial codes.


“Pearl Harbor.” Time 46:20-+; December 17, 1945.

Note on machine MAGIC used during World War II.

POST, M. D. German war ciphers. Everybody’s Magazine 38:28-34; June 1918.


Underwood, R. S. A simple and unbreakable code. *The Pentagon* 8:3-4; 1948.


Yardley, H. O. Ciphers. *Saturday Evening Post* 203:35; May 9, 1931.

Yardley, H. O. Codes. *Saturday Evening Post* 203:16-17; April 18, 1931.


Yardley, H. O. Secret links. *Saturday Evening Post* 203:3-5; April 4, 1931.

### 8.4 Probability, Gambling, and Game Strategy

"To err is human; to forgive, divine." Man errs frequently because of the uncertainties with which he is beset. Human experience is steeped in probabilities. To be sure, some things are certain. The object dropped will surely fall to the ground. Five cards drawn from a deck at random will surely not contain five aces. Many other things are equally certain. But
many more are subject to "chance," which means that we are not certain. In other words, we do not know; we are ignorant. It reminds one of the two perplexed weather bureau officials, one of whom suggests to the other: "Why don't we just tell them the truth—either it will rain tomorrow or it won't."

Mathematicians have at various times espoused two principal approaches to the quantitative study of probability: (a) the subjective view, according to which probability describes the degree of certainty or uncertainty, or the intensity of one's belief; and (b) the statistical view, according to which probability is regarded as the relative frequency with which an event occurs in a certain category of events; or, popularly paraphrased, "that which usually happens we say is probable; the more often it has happened, the more likely it is to happen again."

Both points of view have advantages as well as serious limitations. The calculus of probability (which draws freely upon both), has proved extremely fruitful not only to physical scientists, but to economists, sociologists, and businessmen as well. The entire institution of modern insurance rests in large part upon probability theory. In recent times, the theory of probability has seen brilliant advances such as those exemplified by sampling and quality control techniques on the one hand, and by the theory of games and strategy on the other—to cite but two of the most dramatic recent developments.

A. Mathematics of Gambling—Bridge, Poker, and Dice


"Betting on Sporting Events." The Pentagon 7: 94-97; 1948.


Blanche, E. E. A night with probability. Am. M. Mo. 49: 54-60; 1942.

Games involving probability theory.


Extensive bibliography.

RECREATIONAL MATHEMATICS

Gives 134 tables of approximately 4000 probabilities for the distribution of hand patterns, suit patterns, and such.


"Bridge Hands; Frequency of Occurrence According to Suit Distribution." Am. M. Mo. 48:329-30; 1941.


"Craps Manual." Time 43:76; March 6, 1944.


Drothine, P. Don't bet on the law of averages. Science Digest 20:1-3; September 1946.

Engle, T. Don't be a sucker!—the mathematics of gambling. Clearing House 15: 82-85; 1940.

Farrington, F. Heads or tails? Hobbies 52:124; April 1947.

"Gamblers' Folly." Literary Digest 121:7-8; May 1936.

"Gambling at Monte Carlo; Systems and Why They Fail." Scientific American Supplement 72:46-47; July 1911.


A clever discussion on the probabilities of "all sorts of things."


“Now You Call it; Age-old Controversy of Coin Tossing.” World’s Work 60:43; December 1931.

ORE, OYSTEIN. Cardano, the Gambling Scholar. Princeton Univ. Press, 1953.


“Probabilities of Bridge Hands.” Science Digest 20:69-70; October 1946.


RUSSELL, B. Heads or tails; with discussion by the office iconoclast. Atlantic 146: 163-70. 286-88; August 1930.


EXTENSIVE discussion of dice, gambling, probabilities and related matters.


SNYDER, G. B. Let’s figure on probability. The Scholastic 27:17-18; October 26, 1935.


139

Williamson, C. N. Systems and system players at Monte Carlo. McClure 40:78-91; February 1913.


“You Can’t Lose If You Follow This Law.” Popular Mechanics 56:610-13; October 1931.

B. Certainty, Chance, Coincidence


Engel, L. How gambling odds work for science. Science Digest 31:19-22; May 1952.


MATHEMATICAL MISCELLANIES


“Harnessing Life’s Strongest Law.” *Popular Mechanics* 47:404-408; March 1927.

“Infinite Uncertainty.” *Newsweek* 37:50; March 5, 1951.


C. Probability Theory

Articles


Bibliography.


DUCASS, C. J. Some observations concerning the nature of probability. *Journal of Philosophy* 38:393-403; July 17, 1941.


Bibliography.


**Selected Books**


**Bibliography.**

**D. Theory of Game Strategy**


8.5 The Fourth Dimension

This hoary anachronism should doubtless be left to sink into oblivion along with angle-trisection and perpetual motion. And yet—some rather intriguing matters compel our attention.

For example, there were the machinations of the charlatan Zollner and his spiritualistic friends, who, toward the close of the 19th Century, insisted that the properties of physical space of four-dimensions admirably accounted for otherwise inexplicable psychic phenomena. Then there are the phenomena of congruence, symmetry, asymmetry, isomerism, polarization, and such, and their relation to the concept of dimensionality. Curious, also, is the connection, earnestly professed by some, between religion and "the fourth dimension." And not without interest is the use of four-dimensional configurations, or their projections, as a source of original design. Finally, we must not overlook the popular notion that, in relativity physics, time is the fourth dimension.
Thus it is not altogether unreasonable to regard "the fourth dimension," which seemed terribly important 50 years ago, as a mildly amusing mathematical pastime.


A unique and well-known classic, originally published in 1884.


AYERS, R. H. Universe in four dimensions. *Discovery* 19:8-10; January 1938.


BROWNE, ROBERT. *Mystery of Space; a study of the hyperspace movement in the light of the evolution of new psychic faculties, and an inquiry into the genesis and essential nature of space*. New York: Dutton, 1919.


CAUS, PAUL. Space of four dimensions. *Monist* 18:471-75; July 1908.


GARDNER, MARTIN. Left or right? *Esquire*, February 1951.

HERBERT, HARNEY B. The tesseract. \((a + b)^4\); a demonstration of the binomial theorem in fourth dimensional geometry. *N. M. M.* 15:97-99; 1940.


A stimulating discussion of asymmetry in Nature and related topics.


MARTIN, E. N. *Some varieties of space*. M. T. 16:470-80; 1923.


MENGES, KARL. *What is dimension?*. Am. M. Mo. 50:2-7; 1943.


Two essays, in pamphlet form.


PITKIN, W. B. *Logical aspect of the theories of hyperspaces*. Monist 17:114-25; 1907.

RASHEVSKY, NICHOLAS. *Is time the fourth dimension?*. Sci. Am. 131:400-402; December 1924. Also, 131:308-07; November 1924.


A well-known classic; the religious viewpoint.


8.6 Repeating Ornament


MATHEMATICAL MISCELLANIES


Excellent treatment.


Ingenious use of algebraic symbols as a basis for textile-weaving patterns.


Designs based upon the subdivision of squares, as well as designs based upon concentric circles.


Excellent article; short bibliography.


Historical account of star polygons and polyhedra, p. 1-92.


Includes method of computing the number of combinations possible.


Discussion of star polygons.

LEVY, LUCIEN. *Journal de Mathématiques Élémentaires* 5:96; 1891.

Discussion of star polygons.


Discussion of parquet designs, p. 239-44.


Unusual discussion of patterns formed from polygons either by coloring their compartments or by various transformations.


Methods of arranging points on squared paper, each point being placed at the center of a square.


Material on the geometry of ornament and design; bibliography.


Pages 1-149 treat of tile designs: parquet floor designs; repeating patterns, star polygons, rosettes, and such; bibliography.


Chapters on designs for tiled floors, linoleums, and print fabrics.


### 8.7 Dynamic Symmetry

In the realm of art, the term symmetry generally refers to the relation of the parts of a design to the whole. Thus classical symmetry concerns the disposition of the parts of a design, or the interrelationships between linear...
dimensions of a design. Often based upon regular polygons or polyhedrons, it has been alluded to as Gothic symmetry, or atatic symmetry.

Dynamic symmetry, on the other hand, involves proportional areas. It is often thought of as an organic sort of symmetry, being exemplified in living organisms such as plants, flowers and leaves, and in the human figure.

The principles of dynamic symmetry were rediscovered by the late Jay Hambidge some 35 years ago. As used by Hambidge, the term is peculiarly appropriate to describe proportioning of areas, since to the Greek mathematician δύναμις σύμμετρα meant "commensurable in power," particularly in a square. Thus the familiar root-α rectangles, the whirling square, and such, suggest the force of the term "dynamic."

Interestingly enough, the only peoples to use the principles of dynamic symmetry were the Greeks and the Egyptians. Even more interesting are the many bypaths into which the subject leads: (a) mathematics—the Golden section, Fibonacci numbers, continued fractions, the number system; (b) science—phyllotaxy, physiology, anatomy; (c) the arts—sculpture, ceramics, painting, architecture, design, and modern advertising and printing layout.


Costis, E. P. Dynamic symmetry for photographers. American Photography 41:8-12; May 1947. 41:8-12; June 1947.


A pioneer among modern writers in this field.


HAMBRIDGE, JAY. The *Parthenon and Other Greek Temples and Their Dynamic Symmetry*. New Haven: Yale University Press, 1924. 103 p.


LAFLEUR, W. J. Dynamic symmetry in pier design. *Concrete* 56:6-10; May 1948.


SATTERLY, JOHN. Meet Mr. Tau. S. S. M. 56:731-41; 1956.

A stimulating discussion of the pentagon, icosahedron, dodecahedron, dynamic symmetry, and related matters.


"Square-root Spiral Curve." *Industrial Arts and Vocational Education* 38:301; September 1949.
8.8 The Golden Section

Just as the Pythagoreans first showed the relation of number to tone intervals in music, so it was also the Greeks who first claimed that there was always some law of number that was applicable to creations of nature and art, and which explained the beauty of such creations. One of the most notable of these laws is the Law of the Golden Mean, or the Golden Section.

The law appears in many forms. In geometry it arises from the division of a given line segment into mean and extreme ratio, i.e., into two parts, \( a \) and \( b \), such that \( \frac{a}{b} = \frac{b}{a+b} \), where \( a < b \). This proportion was called the “Divine Proportion” by Luca Pacioli. By setting \( x = \frac{b}{a} \), we have \( x^2 - x - 1 = 0 \), or \( x = \frac{1 \pm \sqrt{5}}{2} \). The ratio, \( \frac{1 + \sqrt{5}}{2} = 1.618 \ldots \) or \( \phi \), is known as the “golden number”; the ratio \( \frac{\sqrt{5} - 1}{2} = \frac{1}{\phi} = 0.618 \ldots \), and the ratio \( \frac{\sqrt{5} + 3}{2} = \phi^2 = 2.618 \ldots \), are all intimately related.

As one pursues the ramifications of the Golden Section one encounters a variety of mathematical interrelationships: the pentagram and the regular decagon, Fibonacci numbers, continued fractions, dynamic symmetry, and so on.

The Golden Mean appears at many unexpected turns. In Nature, among various plant and animal forms, we find phyllotaxy in leaves, pentagonal symmetry in flowers and marine animals, and pentadactylism in vertebrates. In the proportions of the human body the Golden Mean is again to be found. Man has employed the same principle in the creative arts, as seen in the dynamic symmetry of early Greek vases and statues, in classical Renaissance paintings, and in various aspects of contemporary design, including “layouts” in the printing and advertising crafts. For example, the majority of people consider the most aesthetically pleasing rectangular shape that rectan-
A square whose sides are in the approximate ratio of $8:5 \left( \frac{1}{\phi} = 1.618 \ldots \right)$,
or $3:5 \left( \phi = 0.618 \ldots \right)$.

BELL, E. T. The golden and platinum proportions. N. M. M. 19:20-26; 1944.
BENNETT, A. A. The most pleasing rectangle. Am. M. Mo. 30:27-30; 1923.
CATALLERO, VINCENZO. Nuove ricerche sulla genesi della sezione aurea. Bollettino di Matematica, No. 2-3. 1926.
EMSSMANN, G. Über die sectio aurea. Schulprogramme Stettin: 1874.
Golden Section from printer's viewpoint; more than 350 figures; many plates.
Allusions to the Golden Section and related ideas.
Discussion of proportion and use of the Golden Section in painting and art.
CHYKA, MATILA C. Le nombre d'or; rites et rythmes pythagoriciens dans le développement de la civilisation occidentale. (2 Vol. in 1). Paris: Gallimard, Editions de la Nouvelle Revue Française, 1931.
Sequel to "Esthetique des Proportions dans la Nature et dans les Arts." Discussion of symbolism of numbers and Pythagorean number philosophy; rhythm in art; the Golden Section.


GRAESSER, R. F. The Golden Section. The Pentagon 3:7-19; 1943-44.


KALBE, O. Der goldene Schnitt. Hanover: Ost, 1890.


Relation of the Golden Section to cosmology.


8.9 Mathematics and Music

"Music is the pleasure the human soul experiences from counting without being aware that it is counting."—Leibnitz

"Mathematics is the music of Reason. The musician feels Mathematics, the mathematician thinks Music."—J. J. Sylvester

"Mathematics and Music, the most sharply contrasted fields of scientific activity, are yet so related as to reveal the secret connection binding together all the activities of our mind."—Helmholtz

"It is not surprising that the greatest mathematicians have again and again appealed to the arts in order to find some analogy to their own work. They have indeed found it in the most varied arts, in poetry, in painting, and in sculpture, although it would certainly seem that it is in music, the most abstract of all the arts, the art of number and of time, that we find the closest analogy."—Havelock Ellis

"La musique est au temps ce que la géométrie est à l'espace."

—Francis Warrain


Very elaborate treatise, dealing with the graphic arts as well as with music.


Chapter 9

Supplement

Books and Pamphlets

ADAMS, JOHN PAUL. We Dare You to Solve This! New York: Berkley Publishing Co., 1957. 123 p. (Paper)
Some conventional, many original puzzles.


Popular exposition of the operation of electronic computers.

Republication of a classic work first published by Open Court Publishing Co., in 1917.


Four monographs bound in one; also contains J. Petersen: Methods and Theories for the Solution of Problems of Geometrical Constructions; H. S. Carslaw: Non-Euclidean Plane Geometry and Trigonometry; F. Cajori: A History of the Logarithmic Slide Rule.

Discussion of Fermat's last problem, with many entertaining historical side-lights.


BRANDES, LOUIS G. An Introduction to Optical Illusions. Portland, Me.: J. Weston Walch, 1956. (Unpaged)


The first book consists of some 400 logical problems involving syllogisms and sorites.

"Pillow Problems" is a classical collection of 72 sophisticated "brain teasers."


New edition of a classic work.


A reprint of the original edition of 1917.


A collection of humorous stories and diversions related to mathematics.


A delightful collection of humor about mathematics: aphorisms, apothegms, anecdotes, poems, limericks, cartoons, essays, and curios.


Considerable material dealing with percentage, business arithmetic, measurement, etc.


Many old-time puzzles dressed up in smart new clothes.


More than 100 puzzles from Loyd's famous *Cyclopedia of Puzzles.*


Companion volume to the above.
Sophisticated essays on mathematical recreations, with considerable new material.

Companion volume to the above; many new diversions, such as tetraflexagons, Soma cubes, topology, Origami, etc.

Collection of 100 puzzles contributed by scores of mathematicians to an industrial magazine over a period of 18 years.


Fifty short quizzes dealing with mathematical miscellanea, including historical material.

Humorous commentaries on the laws of chance.


KNOTT, C. G. (Trans.) Tom Tit: Scientific Amusements. 1918. 413 p. (Scarce)


A fresh approach to mathematical recreations.


SUPPLEMENT


Over 100 problems and puzzles, many of them new.


Interesting fallacies in geometry, algebra, and calculus, together with their explanations.

Humorous sketches about numbers, by the distinguished author of Zahlwort und Ziffer.

Reprint of an earlier edition, suitable for high school pupils.


An excellent collection of "logic problems," almost all original.


Some 300 problems, mostly new.

One hundred intriguing puzzles.

A republication of the well known book which first appeared in 1933.


An essay in appreciation rather than recreations; engagingly illustrated.

A collection consisting chiefly of arithmetic and algebraic story problems.


(Paper)
Beautifully illustrated. in color.


A game played on a 5 x 5 board, somewhat on the order of “Co.”

Conventional collection of mathematical reactions, but ever popular.

Originally published by Penguin Books under the title *For Amazement Only*. Collection of 50 mazes.

SKOTTE, RAY AND MAGNISON, Yngve. *Math Fun*. Minneapolis, Minn.: The authors, 6380 Monroe Street, 1959. 88 p. (Paper)


Number curiosities, number games, and short cuts in computation.


One hundred inferential and mathematical problems.