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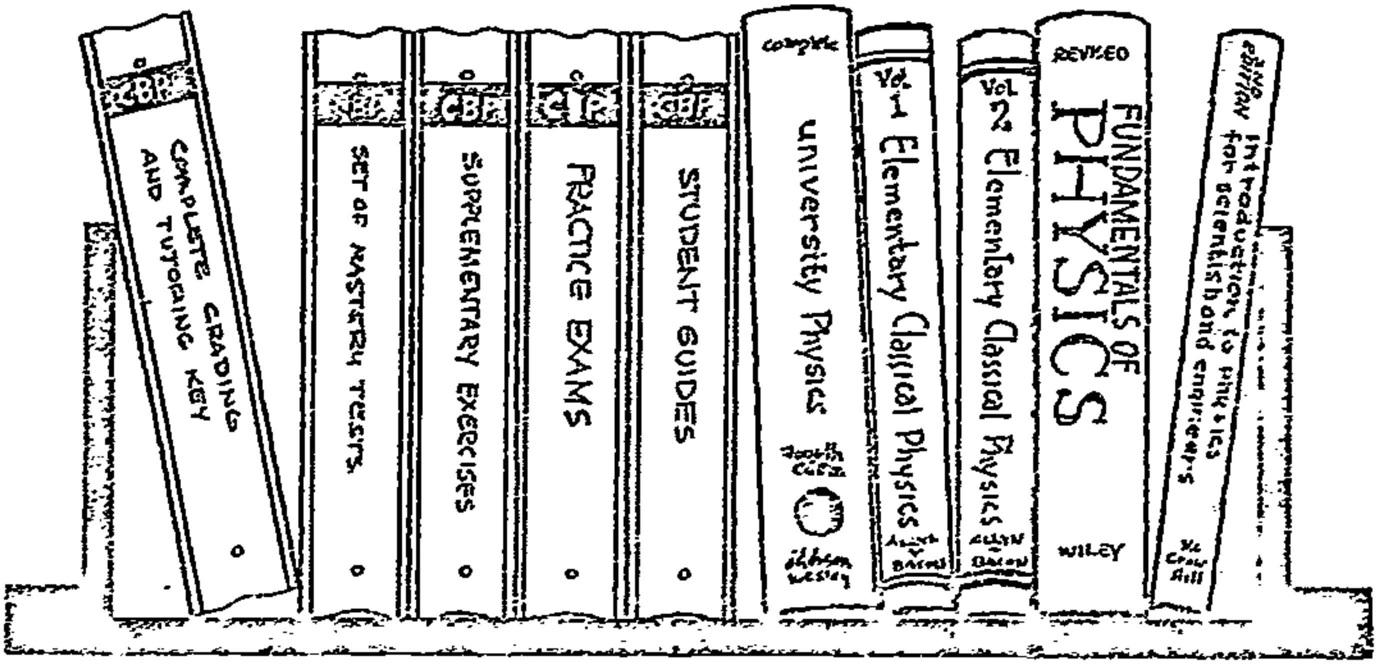
ABSTRACT

This is part of a series of 42 Calculus Based Physics (CBP) modules totaling about 1,000 pages. The modules include study guides, practice tests, and mastery tests for a full-year individualized course in calculus-based physics based on the Personalized System of Instruction (PSI). The units are not intended to be used without outside materials; references to specific sections in four elementary physics textbooks appear in the modules. Specific modules included in this document are: Module 35--Reflection and Refraction, Module 36--Electric Fields and Potentials from Continuous Charge Distributions, and Module 37--Maxwell's Predictions. (CP)

STUDY MODULES FOR CALCULUS-BASED GENERAL PHYSICS*



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Comments

These modules were prepared by fifteen college physics professors for use in self-paced, mastery-oriented, student-tutored, calculus-based general physics courses. This style of teaching offers students a personalized system of instruction (PSI), in which they increase their knowledge of physics and experience a positive learning environment. We hope our efforts in preparing these modules will enable you to try and enjoy teaching physics using PSI.

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These modules were prepared by the module authors at a College Faculty Workshop held at the University of Colorado - Boulder, from June 23 to July 11, 1975.

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COMMENT TO USERS

In the upper right-hand corner of each Mastery Test you will find the "pass" and "recycle" terms and a row of numbers "1 2 3 ..." to facilitate the grading of the tests. We intend that you indicate the weakness of a student who is asked to recycle on the test by putting a circle around the number of the learning objective that the student did not satisfy. This procedure will enable you easily to identify the learning objectives that are causing your students difficulty.

COMMENT TO USERS

It is conventional practice to provide several review modules per semester or quarter, as confidence builders, learning opportunities, and to consolidate what has been learned. You the instructor should write these modules yourself, in terms of the particular weaknesses and needs of your students. Thus, we have not supplied review modules as such with the CBP Modules. However, fifteen sample review tests were written during the Workshop and are available for your use as guides. Please send $1.00 to CBP Modules, Behlen Lab of Physics, University of Nebraska - Lincoln, Nebraska 68588.

FINIS

This printing has completed the initial CBP project. We hope that you are finding the materials helpful in your teaching. Revision of the modules is being planned for the Summer of 1976. We therefore solicit your comments, suggestions, and/or corrections for the revised edition. Please write or call

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STUDY GUIDE

REFLECTION AND REFRACTION

INTRODUCTION

Sight is certainly one of our most important senses and depends on the interaction of electromagnetic waves in the visible portion of the spectrum with the eye. The use of materials that reflect light and that refract or "bend" light extends throughout our industrialized society.

In this module we deal with light traveling in two dimensions and encountering the boundaries between media under those conditions in which the wavelength is small compared with the size of the obstacles or apertures. Under such conditions, since diffraction and interference effects are negligible, the principal phenomena occurring at the interfaces, reflection and refraction, can be understood and the progress of a wave charted by a simple geometrical procedure: ray tracing.

PREREQUISITES

Before you begin this module, you should be able to:	Location of Prerequisite Content
*Write the relations among wavelength, frequency, and velocity for a wave (needed for Objectives 1 and 3 of this module)	Traveling Waves Module

LEARNING OBJECTIVES

After you have mastered the content of this module, you will be able to:

1. Definitions - Define light ray, angle of incidence, refraction, angle of refraction, index of refraction, angle of reflection, total internal reflection, critical angle, reciprocity, and Huygens' principle.

2. Law of reflection - Use the law of reflection to solve problems involving the angles of incidence and reflection, ray paths, and/or the images formed by plane mirrors.

3. Law of refraction - (a) Use the law of refraction (Snell's law) to solve problems involving the relationships among index of refraction, wavelength, velocity of light, angles of incidence and refraction, and ray path for planar slabs of materials. (b) Use the law of refraction to find the location of the image of an illuminated object embedded in a slab of transparent material.

4. Total internal reflection - Use the concept of total internal reflection in conjunction with the laws of reflection and refraction to find the path of a ray that is internally reflected, or find the index of refraction if the ray path is given.

GENERAL COMMENTS

1. Definitions

Since the definitions of terms that you must know to master Objective 1 are scattered throughout the readings, we have collected brief definitions here. These are not meant to be necessarily complete definitions, but should serve to remind you of the full meaning and special usage of each term as you read.

Refraction: the bending of a ray of light as it passes through the boundary between two media.

Angle of refraction: the angle between the refracted ray and the normal to the boundary between two media.

Angle of incidence: the angle between the incident ray and the normal to the boundary between two media.

Angle of reflection: the angle between the reflected ray and the normal to the boundary between the two media.

Light ray: a line parallel to the direction of propagation of the light and normal to the plane wavefront. Although not entirely accurate, it is satisfactory for most ray-tracing purposes to think of a very small beam of light as equivalent to a ray.

Reciprocity: also called "optical reversibility," reciprocity means that light will follow the same ray path through a series of refractions and reflections in going from point A to point B as it will in the reverse direction, B to A.

Huygens' principle: All points on a wavefront can be considered as point sources for the production of spherical secondary wavelets. After a time t the new position of the wavefront will be the surface of tangency to these secondary wavelets.

Total internal reflection: When a ray in an optically dense medium falls on an interface with a less optically dense medium at angles of incidence greater than some critical angle, for all practical purposes no light is transmitted; it is all reflected.

Critical angle: the minimum angle of incidence at which total internal reflection appears. It corresponds to the angle of incidence for which the angle of refraction equals 90°.

Index of refraction: a property of the medium defined as the ratio of the velocity of light in vacuum to that in the medium. A material with a large index of refraction is called optically dense.

2. Dispersion

In general the texts and problems assume monochromatic light (single wavelength). However, light beams are a mixture of waves whose wavelengths extend throughout the spectrum. Although the speed of light in vacuum is the same for all wavelengths, the speed in material substances may be different for different wavelengths. A substance in which the speed of a wave varies with wavelength is said to exhibit <u>dispersion</u>. The index of refraction of a substance is a function of wavelength. Problem E is an example of this effect. The dispersion effect is important since it provides a means of separating (dispersing) light into its various colors in a prism spectrograph, it explains rainbows, and it is the cause of color fringing in low-quality binoculars.

3. Important Formulas

There are really only three formulas you need to memorize for this module:

Law of reflection: $\theta_1' = \theta_1$ or $\theta_i = \theta_r$.

Law of refraction: $n_1 \sin \theta_1 = n_2 \sin \theta_2$.

Definition of index of refraction: $n = c/v$.

We strongly suggest that you do not memorize formulas for the apparent depth of an object in a pond, or the critical angle for internal reflection, etc. They can be very short derivations if you understand the principles, and we have found from past experience that students who have memorized the formulas frequently make mistakes in identification of symbols and are less able to deal with new situations.

TEXT: Frederick J. Bueche, Introduction to Physics for Scientist and Engineers (McGraw-Hill, New York, 1975), second edition

SUGGESTED STUDY PROCEDURE

Your text uses a different order from most texts in the presentation of the material in this module. We suggest that you read Chapter 30, Sections 30.1 through 30.8 for continuity and definitions of terms, then reread Sections 30.1 to 30.3, 30.7 and 30.8 for detailed help in mastering the objectives. Bueche does not explicitly use Huygens' principle to derive the laws of reflection and refraction, as is customary, although the discussion in Section 30.7 is based on this principle. He does give a statement of the principle in Section 32.1 on p. 632; however, the application is not particularly relevant to the present module. The principle is stated in the General Comments of this study guide. The derivations are not necessary for the applications of the laws of reflection and refraction required for mastery of this module; however, they will help your understanding. A more complete treatment can be found in Fundamentals of Physics.*

Read General Comments 1 through 3 in this study guide. Then study Problems A through G and Illustrations 30.1 and 30.5. Then solve Problems H through M. If you need more practice, you may wish to work some of the Additional Problems listed below before taking the Practice Test.

BUECHE

Objective Number	Readings	Problems with Solutions: Study Guide	Problems with Solutions: Text	Assigned Problems: Study Guide	Additional Problems (Chap. 30)
1	Secs. 30-1, 30-2, 30-3, 30-7, 30-8, General Comment 1				
2	Secs. 30-2, 30-3, General Comment 3	A, B	Illus.[a] 30.1	H, I	1
3	Sec. 30-7, General Comments 2, 3	C, D, E	Illus. 30.5	J, K	14, 15, 16
4	Sec. 30-8	F, G		L, M	

[a]Illus. = Illustration(s).

*David Halliday and Robert Resnick, Fundamentals of Physics (Wiley, New York, 1970; revised printing, 1974), p. 673.

TEXT: David Halliday and Robert Resnick, Fundamentals of Physics (Wiley, New York, 1970; revised printing, 1974)

SUGGESTED STUDY PROCEDURE

Read Chapter 36, Sections 36-1 through 36-5 and 36-7. The text does not give much detail on the variation of n with wavelength. University Physics* has a more complete discussion that you should read if this text is available to you. However, for purposes of mastering this module you may find that the discussion in General Comment 2 and the solution to Problem E will suffice.

Read General Comments 1 through 3 in the study guide. Then study Problems A through G and Examples 1, 2, and 5 in your text. Solve Problems H through M. If you need more practice, you may work some of the Additional Problems listed below before taking the Practice Test.

HALLIDAY AND RESNICK

Objective Number	Readings	Problems with Solutions		Assigned Problems	Additional Problems
		Study Guide	Text	Study Guide	(Chap. 36)
1	Secs. 36-1 to 36-5, General Comment 1				
2	Secs. 36-2, 36-7, General Comment 3	A, B	Ex.[a] 5	H, I	1, 24 to 30, 32
3	Secs. 36-2, 36-3, 36-4, General Comments 2, 3	C, D, E	Ex. 1, 2	J, K	3, 4, 6, 8, 9, 10
4	Sec. 36-5	F, G		L, M	14 to 19, 21

[a]Ex. = Example(s).

*Francis Weston Sears and Mark W. Zemansky, University Physics (Addison-Wesley, Reading, Mass., 1970), fourth edition, Chapter 38, Sections 38-5 through 38-7.

STUDY GUIDE: Reflection and Refraction

TEXT: Francis Weston Sears and Mark W. Zemansky, University Physics (Addison-Wesley, Reading, Mass., 1970), fourth edition

SUGGESTED STUDY PROCEDURE

Read Chapter 37, Sections 37-1 and 37-3 through 37-6. These sections give some detail on the nature of light and introduce to you the laws of reflection and refraction. Next read Chapter 38, Sections 38-1 through 38-4, 38-6, and 38-7. These sections derive the laws of reflection and refraction from Huygens' principle and define index of refraction, total internal reflection, and dispersion. Finally, read Chapter 39, Sections 39-1, 39-2, and 39-6, which discuss image formation for plane mirrors and plane refracting surfaces. Although the readings are not in the same order as the objectives, you will find the order of the text better for the first reading.

Read General Comments 1 through 3 in the study guide, and study Problems A through G. Then solve Problems H through M. If you need more practice you may work some of the Additional Problems listed in the Table below, before taking the Practice Test.

SEARS AND ZEMANSKY

Objective Number	Readings	Problems with Solutions	Assigned Problems	Additional Problems
		Study Guide	Study Guide	
1	Secs. 37-1, 37-3 to 37-6, 38-1 to 38-4, General Comment 1			
2	Secs. 37-5, 38-1, 38-2, 39-1, 39-2, General Comment 3	A, B	H, I	37-6, 37-7, 39-2, 39-3
3	Sec. 37-5, 38-3, 38-6, 38-7, 39-6, General Comments 2, 3	C, D, E	J, K	37-8, 37-10 to 37-13, 38-1, 38-3, 38-4, 38-13, 38-15, 39-11 to 39-13
4	Sec. 38-4	F, G	L, M	38-6 to 38-12

TEXT: Richard T. Weidner and Robert L. Sells, Elementary Classical Physics (Allyn and Bacon, Boston, 1973), second edition, Vol. 2

SUGGESTED STUDY PROCEDURE

Read Chapter 36, Sections 36-1 through 36-8. You will not be responsible for the contents of Section 36-7; however, this discussion of refraction of light from an atomic point of view should broaden your general understanding of reflection and refraction.

Read General Comments 1 through 3 in the study guide. Then study Problems A through G and Example 36-1. Solve Problems H through M in your study guide. If you need more practice, you may work some of the Additional Problems listed below, before taking the Practice Test.

WEIDNER AND SELLS

Objective Number	Readings	Problems with Solutions		Assigned Problems	Additional Problems
		Study Guide	Text	Study Guide	
1	Secs. 36-1 to 36-8, General Comment 1				
2	Secs. 36-3, 36-4, General Comment 2	A, B	Ex.[a] 36-1	H, I	36-1, 36-3, 36-4
3	Secs. 36-3, 36-5, 36-6, General Comments 2, 3	C, D, E		J, K	36-5 to 36-8, 36-11 to 36-13, 36-16, 36-19, 36-20
4	Sec. 36-8	F, G		L, M	36-17, 36-21

[a]Ex. = Example(s).

PROBLEM SET WITH SOLUTIONS

A(2). Two plane mirrors stand on a table adjacent to each other at an angle of 60°. See Figure 1.
(a) Trace a horizontal light ray that is reflected twice in this system.
(b) Compute the angle between incident ray and second reflected ray.
Define clearly all quantities used.

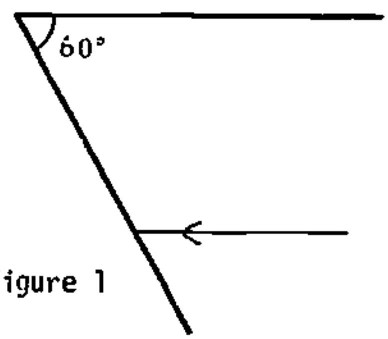
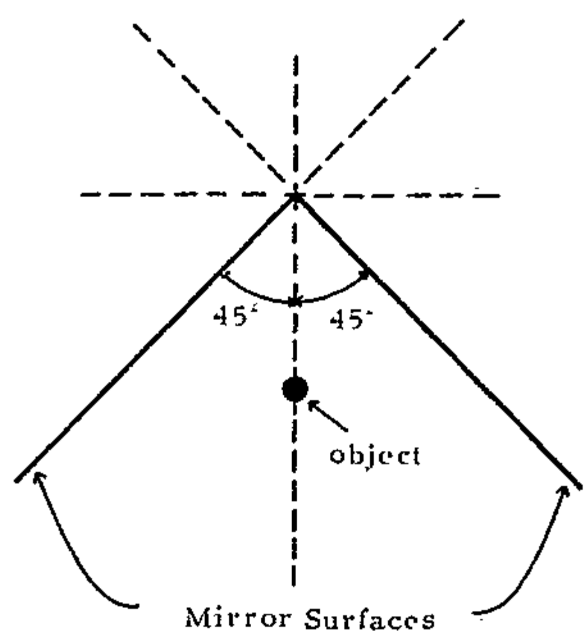


Figure 1

Figure 3



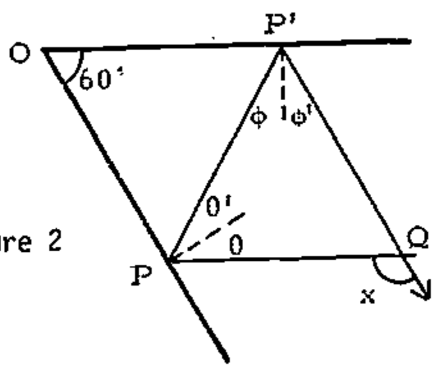


Figure 2

Solution

(a) See Figure 2.
(b) X is the quantity sought. By the law of reflection, $\theta = \theta'$, $\phi = \phi'$ (the dashed lines are normals). Sum the angles in triangle POP′:

$$60^\circ + (90^\circ - \phi) + (90^\circ - \theta) = 180^\circ \quad \underline{\text{or}} \quad \phi + \theta = 60^\circ.$$

The exterior angle of triangle PP′Q = the sum of the opposite angles:

$$X = 2\theta + 2\phi = 120^\circ.$$

B(2). The image formed by a plane mirror will act as an object for a second mirror. Locate in Figure 3 the three distinct images that can be seen by a suitably placed eye.

Solution

Take the right-hand mirror by itself as in Figure 4(a). Since $\theta_i = \theta_r$, the projection of any reflected ray behind the mirror intersects the normal to the mirror at a point that is as far behind the mirror as the object is in front of it: the object distance equals the image distance. We can now locate images A and B on Figure 4(b). In Figure 4(b) a few light rays have been drawn that undergo two reflections and appear to come from a point C, the third image. The image at A (and B) acts as an object for the image at C, and therefore the image lies along the normal to the extension of the mirror: AP = PC.

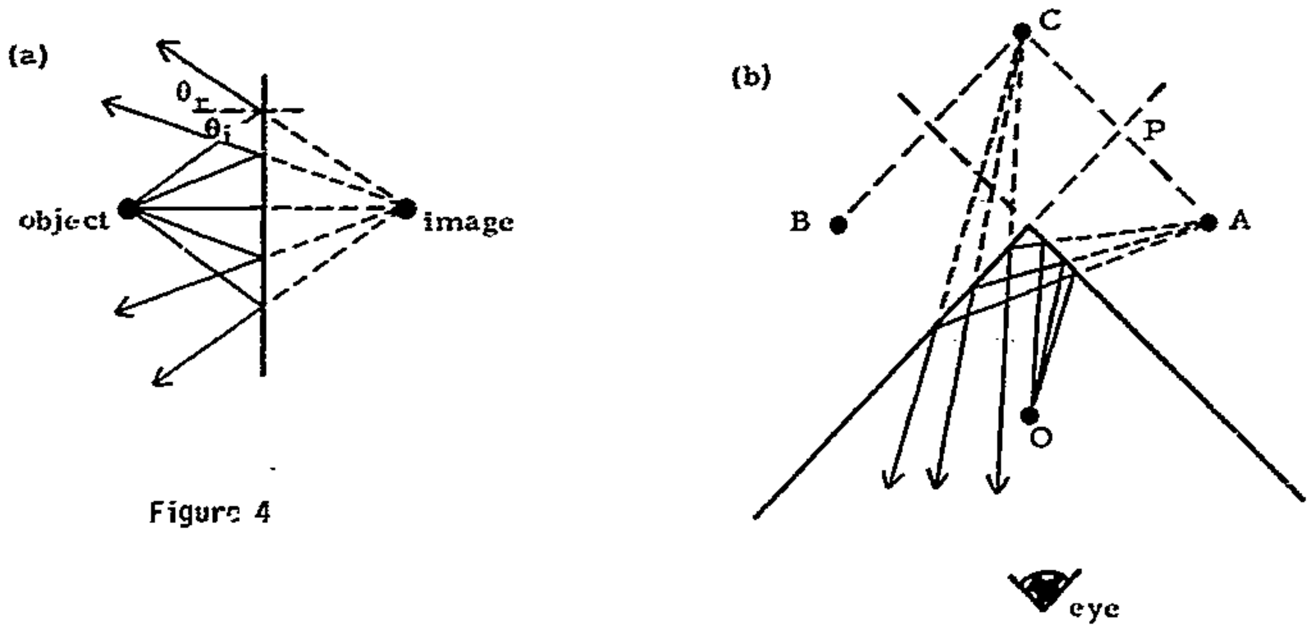


Figure 4

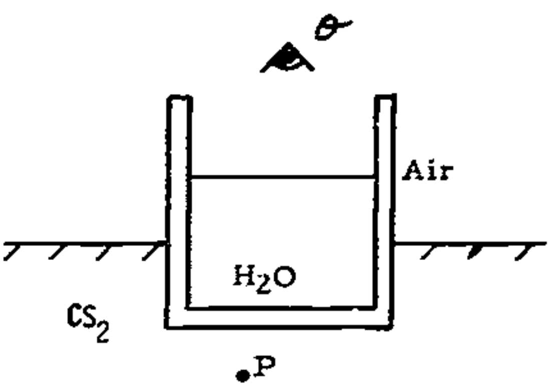
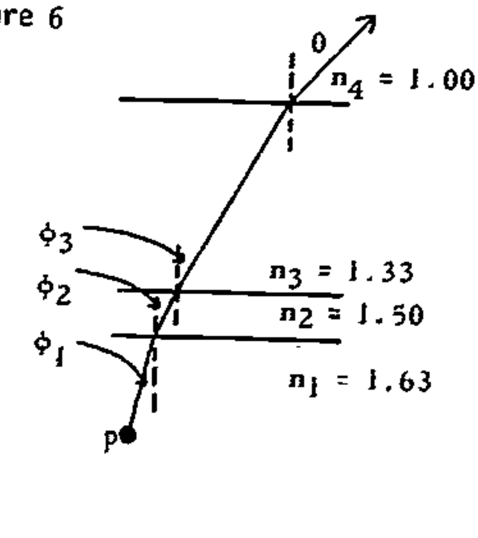


Figure 5

Figure 6



C(3). A glass of water in Figure 5 is partially immersed in a tank of carbon disulfide (CS_2). You are an observer O looking down through the water at a particle P floating in the CS_2. Some reflected light leaves the particle at a small angle ϕ with the vertical.
(a) At what angle θ does that light emerge from the water? Express your answer in terms of ϕ. $n_{water} = 1.33$; $n_{glass} = 1.50$; and $n_{CS_2} = 1.53$.
(b) If D = depth of water, does your answer to part (a) depend on D? Suppose $D \to 0$.

Solution

(a) See Figure 6. Use Snell's law at each interface:

$$(\sin\phi)/(\sin\phi_2) = n_2/n_1, \quad (\sin\phi_2)/(\sin\phi_3) = n_3/n_2, \quad (\sin\phi_3)/(\sin\theta) = n_4/n_3.$$

Multiply these three equations:

$$\frac{\sin\phi}{\sin\theta} = \frac{n_4}{n_1} \quad \text{or} \quad \theta = \text{arc}\sin\left(\frac{n_1}{n_4}\sin\phi\right) = \text{arc}\sin(1.63\sin\phi).$$

(b) No, θ is independent of D, even if $D \to 0$. In fact, take away the glass of water and θ is still the same. It is only necessary to consider the index of refraction of the initial and final materials in calculating the final angle of refraction, and in all cases the final angle is independent of intermediate materials. However, the apparent location of the object depends on these.

D(3). The "apparent depth" of an object immersed in an optically dense refracting medium is less than the true depth when viewed from directly above. Show that the apparent depth d' is related to the true depth d by $d' = n_2 d/n_1$, where n is the relative refractive index of the medium in which the object is immersed. See Figure 7. Note: One may assume the angles to be so small that the sine of an angle can be replaced by the angle itself.

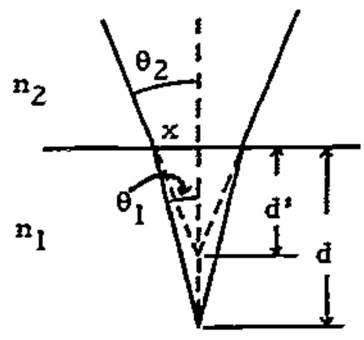
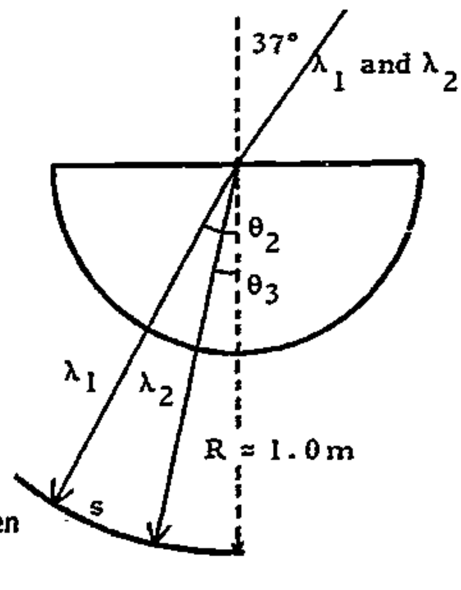


Figure 7

Figure 8



Solution

The ray starts at depth d in the medium of index n_1. By Snell's law:

$$n_1 \sin \theta_1 = n_2 \sin \theta_2,$$

and by the small-angle approximation, $\sin \theta \simeq \tan \theta \simeq \theta$ (in radians):

$$n_1\theta_1 = n_2\theta_2.$$

From the geometry of Figure 7, $\tan \theta_1 = x/d$ and $\tan \theta_2 = x/d'$. (Note that d' is found by projecting the outgoing ray backward to the normal as the eye would do, so that the light appears to come from depth d'.

$$\frac{\tan \theta_1}{\tan \theta_2} = \frac{\theta_1}{\theta_2} = \frac{d'}{d} = \frac{n_2}{n_1}, \qquad d' = \frac{n_2 d}{n_1}.$$

E(3). Crown flint glass has an index of refraction that varies with the wavelength of the light passed through it from 1.66 for a wavelength of 400 nm (violet) to 1.60 for a wavelength of 700 nm (red). A narrow beam of light containing the red and violet wavelengths above falls on the center of a semicircle cut from this glass, as shown in Figure 8. Find the separation S of the two rays on a circular screen with R = 1.00 m, centered at 0.

Solution

Apply Snell's law for each wavelength:

$$n_1 \sin \theta_1 = n_2 \sin \theta_2 \quad \text{and} \quad n_1 \sin \theta_1 = n_3 \sin \theta_3.$$

Rearranging, we find

$$\theta_2 = \text{arc} \sin[(n_1/n_2) \sin \theta_1], \quad \theta_3 = \text{arc} \sin[(n_1/n_3) \sin \theta_1].$$

Find

$$\theta_2 - \theta_3 = \text{arc} \sin(0.60/1.60) - \text{arc} \sin(0.60/1.66) = \text{arc} \sin(0.375) - \text{arc} \sin(0.361),$$

$$\theta = 22.0° - 21.2° = 0.8°,$$

$$s = R\,\Delta\theta = (1.00 \text{ m})\left(\frac{0.8°/57.3°}{\text{radian}}\right) = 1.40 \times 10^{-2} \text{ m}.$$

F(4). The crooks in a typical TV drama are attempting to recover a fortune in diamonds that they earlier sunk in a chest in 8.0 m of water. As a cover for the operation they have moored a floating oil-drilling rig above the position where they sank the chest. If the dimensions of the chest are small in comparison with the rig, determine the size of rig required in order that no sailor on a passing ship can see what is going on under the surface. (Take n = 4/3 for water.)

Solution

We can consider the sunken chest as a point source of light. If the oil rig is big enough for its purpose, then all rays of light from the chest that would be refracted into the air at the surface must be blocked off by the base of the rig, and all rays striking the surface of the water outside the rig must be totally internally reflected. The rig obviously must be circular, and, if its center is moored directly above the chest, a ray of light striking the edge of the rig must do so at an angle equal to ϕ_c, the critical angle. See Figure 9.

For the minimum radius r of the rig, $n \sin \phi_c = (1)\sin \phi_2 = \sin(90°) = 1$, therefore,

$$\sin \phi_c = 1/n \quad \text{or} \quad r/(r^2 + d^2)^{1/2} = 1/n,$$

$$r^2 + d^2 = n^2r^2, \quad \text{or} \quad r^2 = d^2/(n^2 - 1).$$

This gives us

$$r = \frac{8.0\ \text{m}}{(16/9 - 1)^{1/2}} = 9.1\ \text{m}.$$

Alternate solution: Use Snell's law: $n \sin \phi_c = 1$; therefore, $\sin \phi_c = 1/n = 3/4$; and $\phi_c = 48.6°$. We can find r from $r = d \tan \phi_c = (8.0\ \text{m})(1.13) = 9.1$ m.

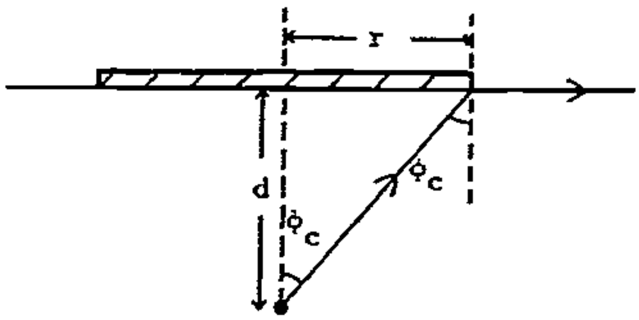
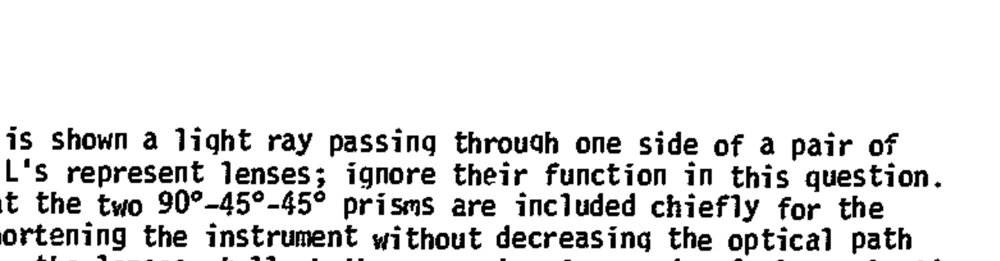


Figure 9

Figure 10



G(4). In Figure 10 is shown a light ray passing through one side of a pair of binoculars. L's represent lenses; ignore their function in this question. (a) Given that the two 90°-45°-45° prisms are included chiefly for the purpose of shortening the instrument without decreasing the optical path length between the lenses, tell whether or not prisms made of clear plastic of index n = 1.47 would be suitable. Explain, giving a quantitative argument. (b) Would the above prism in part (a) be suitable if the entire system (prisms and lenses) were immersed in water? Indicate reasoning for your answer.

Solution

(a) The prisms must allow total internal reflection at 45° incidence. Thus, $\theta_i = 45°$, and $\theta_i > \phi_c$, where $\sin \phi_c = 1/n$. Thus $\sin \theta_i > 1/n$, or $n > 1/(\sin 45°) = 1.414$. This condition is satisfied by a plastic of $n = 1.47$ and the prisms will work.

(b) In water the same condition becomes

$$\sin \theta_c = \frac{n_{H_2O}}{n_{plastic}} \quad \text{or} \quad \sin \theta_c = \frac{1.33}{n_{plastic}} = \frac{1.33}{1.47} = 0.905 \quad \text{or} \quad \theta_c = 65°.$$

Therefore $\theta_i < \theta_c$, and the binoculars are of little use to a scuba diver if water leaks inside them.

Problems

H(2). (a) Using ray paths, prove that in a calm, unpolluted lake the reflected image of a pine tree on the shore will appear upside-down to a fisherman meditating in his boat.
(b) Will the image of a fish in the same lake surface appear inverted to another fish? Explain.

I(2). Suppose that (for reasons best understood by you) you decide to photograph a street scene by using the reflection in a store window. You wish to set your camera so that point **X** will be in sharp focus; for what distance must you set your camera lens? The dimensions are as in Figure 11.

J(3). Light strikes a glass plate at an angle of incidence of 60°, part of the beam being reflected and part refracted. It is observed that the reflected and refracted portions make an angle of 90° with each other. What is the index of refraction of the glass?

K(3). A beam of light as in Figure 12 hits a parallel-sided plate of glass of index of refraction n. The thickness of the glass is 6.0 mm and the light beam is displaced a distance of 4.24 mm.

(a) What is the index of refraction of the glass?
(b) What is the angle of refraction?

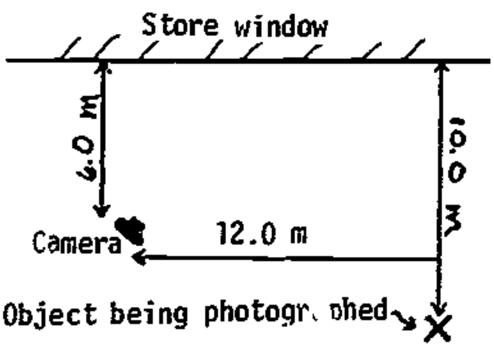


Figure 11

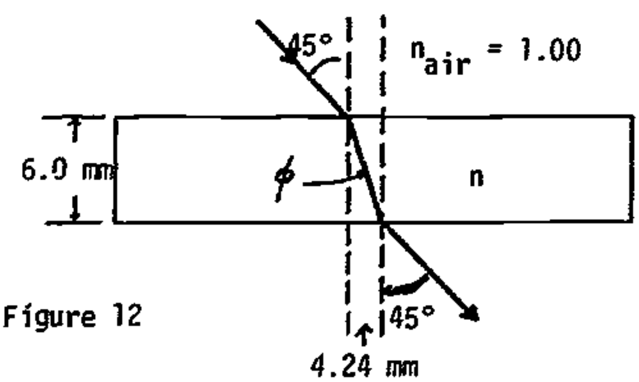


Figure 12

L(4). A beam of light shines on a glass prism as shown in Figure 13. The beam is perpendicular to the first face. Trace out the subsequent path(s) of the light beam until it has left the prism, and find the angle of refraction as it leaves the prism. The index of refraction of the glass is 1.50.

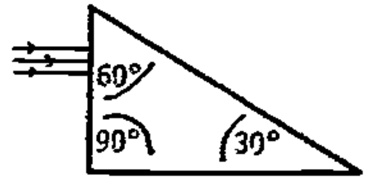


Figure 13

M(4). A point source of light is 1.50 m below the surface of a still pond of water. It appears to an observer from above the water that the light comes only from a well-defined circular area of the water surface. What is the diameter of this circle?

Solutions

H(2). Yes. I(2). 20.0 m. J(3). 1.73.

K(3). (a) 1.22. (b) 35°. L(4). 48.6°. M(4). 3.4 m.

PRACTICE TEST

1. Define critical angle and reciprocity, and state Huygens' principle.
2. The image formed by a plane mirror will act as an object for a second mirror. Find the first four images formed by the pair of plane mirrors shown in Figure 14 (i.e., find the four images closest to the object).
3. A microscope is focused on a scratch made on the upper surface at the bottom of a small container. Water is added to the container to a depth of 3.00 mm. Through what vertical displacement must the microscope lens be raised to bring the scratch into focus again? Assume that the displacement of the lens is equal to the displacement of the image.
4. A fish looking upward toward the water-air interface sees a circular transparent hole surrounded by a mirror. What is the radius of this hole when the fish's eye is a distance of 1.00 m from the water surface? (n for water is 1.33.)

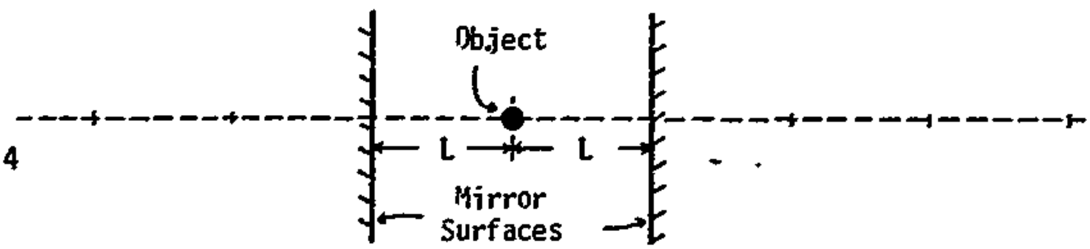


Figure 14

Practice Test Answers

1. Critical angle: the minimum angle of incidence for which total internal reflection appears, angle of incidence for which the angle of refraction equals 90°. Reciprocity: also called "optical reversibility," light will follow the same ray path no matter in which sense it propagates. Huygens' principle: all points on a wavefront can be considered as point sources for the production of spherical secondary wavelets, the wavefront moving as the wavelets' surface of tangency.

2. ±2L, ±4L, ±6L, ±8L. 3. 0.75 mm. 4. r = 1.13 m.

REFLECTION AND REFRACTION

Date ____________________

Mastery Test   Form A

pass   recycle

1   2   3   4

Name ____________________   Tutor ____________________

1. Define angle of reflection, total internal reflection, and index of refraction.

2. Harry Gluck is mounting a mirror on the wall. He wants to be able to see himself from his feet to the top of his head. Harry's eyes are a = 0.100 m from the top of his head and b = 1.70 m from his feet, as in Figure 1.
(a) What is the maximum height x that the bottom of the mirror can be above the floor?
(b) What is the minimum height y for the top of the mirror?

3. A diamond cutter is given a very large rough diamond to cut. He polishes a flat on one side and, while examining it under his microscope, discovers a flaw at an <u>apparent</u> depth of 0.60 cm. The diamond cutter knows some physics, and, using the index of refraction of diamond n = 2.42, he calculates a real depth of 0.250 cm and cuts at this point. Does he cut through the flaw? (Be prepared to justify your answer.)

4. It is desired to deflect a light beam through 60° by using a prism as shown in Figure 2. What is the minimum value of n for which total internal reflection will occur at point A?

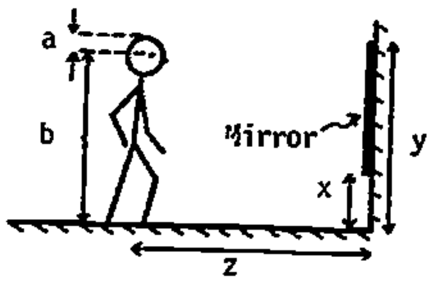


Figure 1

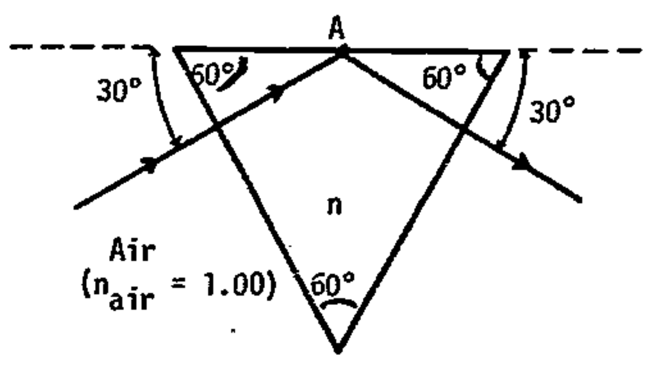


Figure 2

REFLECTION AND REFRACTION

Date ________________

Mastery Test   Form B

pass   recycle

1   2   3   4

Name ______________________________

Tutor ________________________

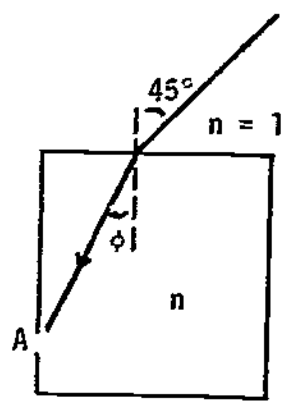
1. Define refraction, angle of incidence, and light ray.

2. The image of a tree just covers the length of a 5.0-cm plane mirror when the mirror is held vertically 30.0 cm from the eye. The tree is 100 m from the mirror. What is its height?

3. A plate of glass 10.0 cm thick lies at the bottom of a tank under 10.0 cm of water. What is the apparent distance from the top of the water to the bottom of the glass plate? $n_{glass} = 2.00$, $n_{water} = 1.33$.

4. A monochromatic ray, initially in air, strikes a glass cube as shown in Figure 1. Find the index of refraction n such that the ray striking the glass at point A is just internally reflected.

Figure 1



REFLECTION AND REFRACTION

Date ____________

Mastery Test   Form C

pass   recycle
1   2   3   4

Name ____________   Tutor ____________

1. State Huygens' principle, and define angle of refraction and critical angle.

2. A light source is positioned so that the beam from the source strikes a mirror normally, as shown in Figure 1. A screen is positioned on both sides of the light source. Now as the mirror rotates, the reflected beam striking the screen moves across the screen. If the mirror rotates through an angle of 15° from the normal position, how far will the spot of light move across the screen?

3. A man in a diving bell looks out through a window and sees a fish at an apparent distance of 12.0 m from the window. How far is the fish from the window? (n_{water} = 1.33. Neglect the effect of the window.)

4. Light is incident normally on the face of a 3-4-5 prism opposite the 37° angle. A drop of liquid is placed on the prism as shown in Figure 2. What is the index of refraction of the liquid if total internal reflection in the prism is just possible? The index of refraction of the prism is 1.50.

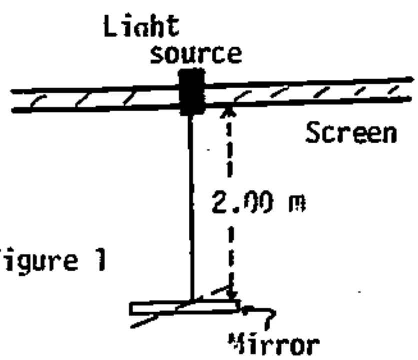


Figure 1

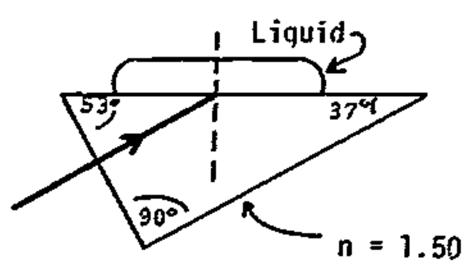


Figure 2

MASTERY TEST GRADING KEY - Form A

1. Solution: Angle of reflection: the angle between the reflected ray and the normal to the boundary between the two media. Total internal reflection: When a ray in an optically dense (large index of refraction) medium falls on an interface with a less optically dense medium at angles of incidence greater than some critical angle, for all practical purposes, no light is transmitted; it is all reflected. Index of refraction: a property of the medium propagating light defined as the ratio of the velocity of light in vacuum to that in the medium.

2. Solution: See Figure 20. (a) The law of reflection is $\theta_i = \theta_r$, therefore, by congruent triangles, no matter what z is, x must be equal to 1/2 the distance from Harry's feet to his eyes ($x = b/2$) as shown in Figure 20. $x = 0.85$ m.

   (b) Similarly the mirror must extend a distance a/2 above eye level so that Harry can see the top of his head. The minimum height of the top of the mirror is eye level (b) plus a/2 or

   $y = 1.70 + 0.100/2 = 1.75$ m.

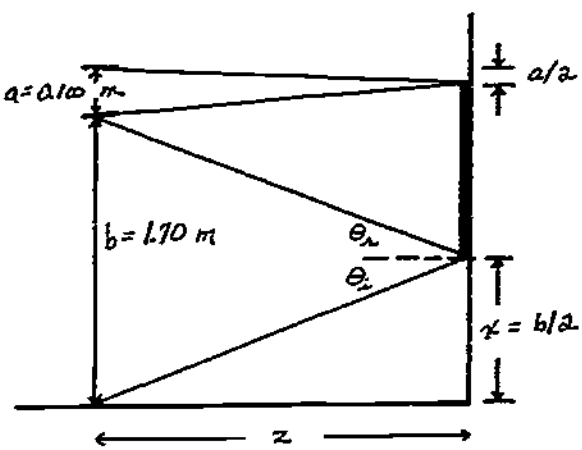


Figure 20

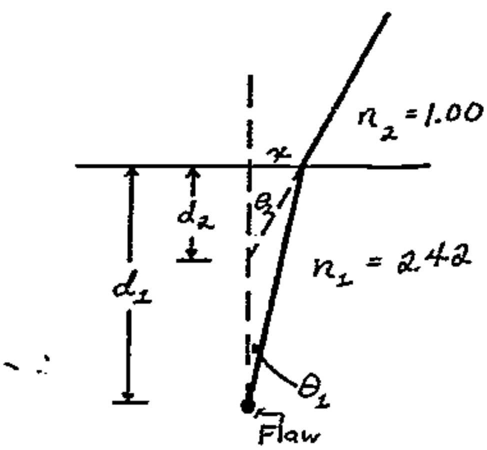


Figure 21

3. Solution: See Figure 21. The diamond cutter should have set up the equations:

   By Snell's law: $n_1 \sin\theta_1 = n_2 \sin\theta_2$.

   In the small-angle approximation:

   $\sin\theta_1 \approx x/d_1$; $\sin\theta_2 \approx x/d_2$.

Therefore, combining equations, we find

$n_1(x/d_1) = n_2(x/d_2)$.

The real depth is

$d_1 = (n_1/n_2)d_2 = [(2.42)/1](0.60 \text{ cm}) = 1.45 \text{ cm} \neq 0.250 \text{ cm}$.

The diamond cutter goofed and cut it short. He now works for a butcher who appreciates his short cuts.

4. Solution: At the interface of the prism and the air, point A, the light has an angle of incidence of 60°. We may use Snell's law to find the critical angle in terms of the unknown index of refraction, n.

$n \sin 60° = (1) \sin 90° = 1, \quad n = \frac{1}{\sin 60°} = \frac{1}{0.867} = 1.15.$

If $n > 1.15$, then $n \sin 60° > 1$, and we shall still get total internal reflection. However, if $n < 1.15$, we can find an angle of refraction θ that will satisfy $n \sin 60° = \sin \theta$. Therefore $n = 1.15$ is the minimum index of refraction for this situation.

MASTERY TEST GRADING KEY - Form B

1. Solution: Refraction: the bending of a ray of light as it passes through the boundary between two media. Angle of incidence: the angle between the incident ray and the normal to the boundary between two media. Light ray: a line parallel to the direction of propagation of light and normal to the plane wavefront.

2. Solution: See Figure 22. A light ray proceeding toward A will be reflected to B. By the law of reflection the angle of incidence equals the angle of reflection at the mirror. Therefore, the angle between the ray and eye level at B equals that at A. Triangle BMO (= AMO) is similar to ADC. Therefore, the ratio of sides: OM/AO = CD/AC. But OM = (1/2) height of mirror = 5/2 cm, AO = BO = 30.0 cm, and AC = 100 m + 30.0 cm = 100.3 m. The height of the tree is

$$2CD = \frac{2(OM)(AC)}{AO} = \frac{2(5/2\text{ cm})(100.3\text{ m})}{30.0\text{ cm}} = 16.7\text{ m}.$$

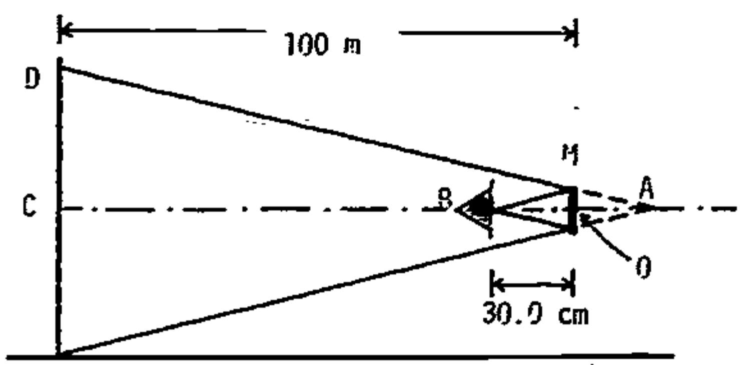


Figure 22

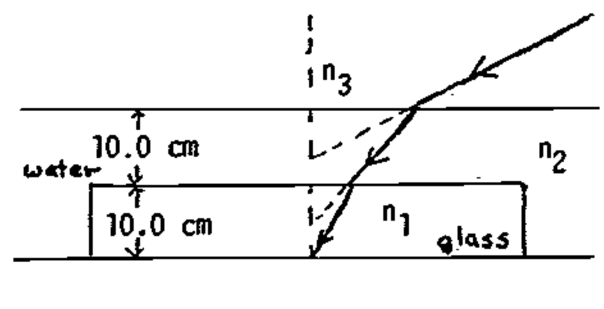


Figure 23

3. Solution: See Figure 23. As derived from Snell's law in Problem D,

$$d' = (n_2/n_1)d,$$

where d is the apparent depth and d is the actual depth. We apply this relation for the thickness of the glass as if our eye were in the water:

$$d_1' = \frac{1.33}{2.00}(10.0\text{ cm}) = 6.7\text{ cm}.$$

For the refraction at the air surface we calculate the depth of the object as 10.0 cm + 6.7 cm or 16.7 cm, and apply the formula again:

$$d_2' = \left(\frac{n_3}{n_2}\right)(16.7) = \frac{1}{1.33}(16.7) = 12.5\text{ cm}.$$

4. What To Look For: Student should be able to figure out this identity. If trig tables are not available, an answer in symbols should be acceptable, such as

$n = [\cos(\arc \tan \sqrt{2}/2]^{-1}$.

Solution: Use Snell's law to find ϕ:

$n_{air} \sin(45°) = n \sin \phi = 0.707$.

The angle of incidence at the side of the cube where the refracted ray strikes is $\theta = 90° - \phi$. Therefore, using Snell's law for the critical angle:

$n \sin \theta = n_{air} \sin(90°) = 1$

and substituting, we find

$n \sin(90° - \phi) = 1$ or $n \cos \phi = 1$.

From the first step:

$\frac{n \sin \phi}{n \cos \phi} = \frac{0.707}{1} = \tan \phi$ or $\phi = 35°$.

Thus

$n = 1/(\cos \phi) = 1/0.82 = 1.22$.

MASTERY TEST GRADING KEY - Form C

1. Solution: Huygens' principle: All points on a wavefront can be considered as point sources for the production of spherical secondary wavelets. After a time t the new position of the wavefront will be the surface of tangency to these secondary wavelets. Angle of refraction: the angle between the refracted ray and the normal to the boundary between the two media. Critical angle: the minimum angle of incidence at which total internal reflection appears. It corresponds to the angle of incidence for which the angle of refraction equals 90°.

2. Solution: See Figure 24. Initially, $\theta_i = \theta_r = 0°$. After the mirror rotates 15° (the normal to the mirror rotates 15°), $\theta_i = \theta_r = 15°$. The included angle is $\theta_i + \theta_r = 30°$. Therefore

   $x = 2 \tan 30 = 1.20$ m.

3. What To Look For: Look for misidentification of terms. Ask student to draw a sketch and label quantities if in doubt.

   Solution: The solution to Problem D derives a formula applicable to this problem:

   $d' = (n_2/n_1)d$,

   where d' is the apparent distance, n_2 represents air, and n_1 represents water in this case. Therefore,

   $d = (n_1/n_2)d' = (1.33/1)(12.0 \text{ m}) = 16.0$ m.

4. Solution: Use Snell's law to find the critical angle:

   $n_1 \sin \theta_1 = n_2 \sin \theta_2$, $(1.50)(\sin 53°) = n_2(\sin 90°) = n_2$.

   Therefore $n_2 = 1.5 \sin 53° = 1.20$.

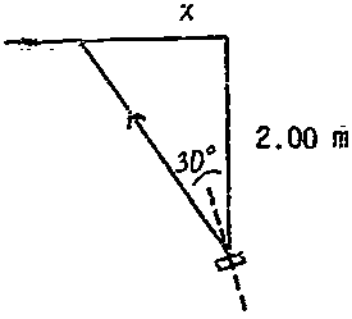


Figure 24

ELECTRIC FIELDS AND POTENTIALS FROM CONTINUOUS CHARGE DISTRIBUTIONS

INTRODUCTION

Too bad! In case you have not realized it, not all charges come packaged as points, spheres, infinite cylinders, or infinite planes. Ah, if only it were so: Life would be much easier from a calculational viewpoint, although somewhat limited in geometrical options. But then, mechanics would be simpler if only constant accelerations were observed in nature...Not to mention centers of mass; moments of inertia, etc.; all would be considerably simpler to calculate in that wonderful world of point masses, constant accelerations, massless strings, and frictionless boards.

Once again calculus is needed to assist us in analyzing and understanding natural phenomena that are often manifested in hunks of mass, variable accelerations, and globs of charge.

This module introduces no new fundamental physics. Instead, you will learn to extend the concepts of electric field and potential to charge distributions that defy solution by superposition of point-charge fields and potentials or application of Gauss' law.

PREREQUISITES

Before you begin this module, you should be able to:	Location of Prerequisite Content
*Integrate polynomial, sine, and cosine functions (needed for Objectives 1 and 2 of this module)	Calculus Review
*Determine the electric field of a point charge (needed for Objectives 1 through 3 of this module)	Coulomb's Law and the Electric Field Module
*Determine the electric potential of a point charge (needed for Objectives 1 through 3 of this module)	Electric Potential Module

LEARNING OBJECTIVES

After you have mastered the content of this module, you will be able to:

1. Line charges - Given a rectilinear charge distribution, set up, and in some cases evaluate, the definite integral for:

(a) the total charge on a specified segment of the line;
(b) the electric potential at a specified point;
(c) the electric field at a specified point.

2. Ring and disk charges - Given a charge distribution on a circular arc, sector, or disk, set up, and in some cases evaluate, the definite integral for:
(a) the total charge on a specified portion of the distribution;
(b) the electric potential and electric field at the center of the circular arc, sector, or disk and on the axis of the disk.

3. Limiting cases - Demonstrate that the integrals of Objectives 1 and 2 reduce in limiting cases to results expected for simpler charge distributions.

GENERAL COMMENTS

Determining the electric potential V or the electric field E from a continuously distributed charge generally requires the use of integral calculus. Unless the charge distribution has sufficient symmetry so as to permit the use of Gauss' law to determine $\vec{E}$, a calculation of either $\vec{E}$ or V requires that you: (a) use physics to set up a definite integral; and (b) use calculus (or numerical techniques) to evaluate this integral. Since this is a physics course, your attention will focus on step a. Step b can range in difficulty from trivial to impossible depending upon the complexity of the charge distribution and upon your facility at evaluating integrals. Although we shall not emphasize the mathematical gymnastics of integral evaluation, you should feel free to try your hand at any that you simply cannot resist!

When calculating $\vec{E}$ or V from a distributed charge, the essential idea is reasonably simple. It goes like this. Select a very small (some would say infinitesimal) charge dq within the distribution. Treating it as a point charge, write the expression for either the potential dV or the field $d\vec{E}$ at the specified field point. Then superimpose the contributions from the total distribution by means of an integral.

Let us look at this procedure in more detail. Consider Figure 1. There are three vectors you must be sure you understand -

Position vector for the field point P:

$\vec{R} = x\hat{i} + y\hat{j} + z\hat{k}$;

Position vector for dq:

$\vec{\underline{R}} = \underline{x}\hat{i} + \underline{y}\hat{j} + \underline{z}\hat{k}$;

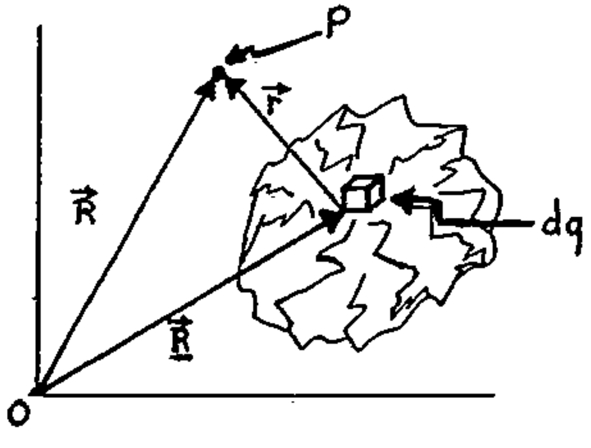


Figure 1

Displacement vector from dq to P:

$$\vec{r} = \vec{R} - \underline{\vec{R}} = (x - \underline{x})\hat{i} + (y - \underline{y})\hat{j} + (z - \underline{z})\hat{k}.$$

As you know from your study of electric potential, the potential at P attributable to the charge dq is

$$dV = (k\ dq)/r \quad (k = 1/4\pi\varepsilon_0 = 9.0 \times 10^9\ N\ m^2/C^2),$$

where r is the distance from dq to P, i.e.,

$$r = |\vec{r}| = \sqrt{(x - \underline{x})^2 + (y - \underline{y})^2 + (z - \underline{z})^2}\ .$$

Writing out the expression for dV more explicitly gives us

$$dV = (k\ dq)/\sqrt{(x - \underline{x})^2 + (y - \underline{y})^2 + (z - \underline{z})^2}\ .$$

This emphasizes the dependence of dV upon the coordinates of both the field point P and the charge point. The potential at P from the total charge distribution is obtained by summing (integrating) over the charge.

$$V(\vec{R}) = \int_{\text{all charges}} \frac{k\ dq}{r}\ .$$

The dependence of the electric potential on the coordinates of the field point is emphasized by the functional dependence of V on $\vec{R}$.

For a given charge distribution the limits on the integral will be determined by the geometry of the charge distribution. This will be discussed in more detail in the Problem Set.

The electric field at P attributable to dq is

$$d\vec{E} = (k\vec{r}\ dq)/r^3.$$

Using the integral to superpose the contributions from all the charge gives us

$$\vec{E}(\vec{R}) = \int (k\vec{r}\ dq)/r^3,$$

where, again, the dependence of the field on the coordinates of the field point is emphasized.

You should note that the essential difference between the expressions for V and $\vec{E}$ is that V is a scalar sum, but $\vec{E}$ results from a vector sum. Details of setting up these integrals and seeing how to check them are covered in the Problem Set.

TEXT: Frederick J. Bueche, Introduction to Physics for Scientists and Engineers (McGraw-Hill, New York, 1975), second edition

SUGGESTED STUDY PROCEDURE

Begin by studying the General Comments. Then read Sections 18.6 and 18.7 up to Illustration 18.4 on p. 336 in Chapter 18 of your text, and Section 20.8 in Chapter 20. Next study Problems A through D and work Problems J, L, M, and N. Study Problems E and F and work Problem K. Problem S is challenging and optional. Next study Illustration 18.4 and Problems G and H. Work Problem O - Problem T is challenging, but optional. Study Sections 20.12 and 20.13, Problem I, and work Problems P, Q, and R.

Take the Practice Test, and work some Additional Problems if necessary, before attempting a Mastery Test.

BUECHE

Objective Number	Readings	Problems with Solutions: Study Guide	Assigned Problems: Study Guide	Additional Problems
1	Secs. 18.6, 18.7, 20.8	A, B, C	J, L, M, N	S; Chap. 18, Probs. 13, 14
2	Secs. 18.7, 20.12, 20.13	G, H, I	O, Q	T; Chap. 18, Probs. 15, 16, 17; Chap. 20, Probs. 18, 19
3	Sec. 20.8	D, E, F	K, M, N, P, R	S

TEXT: David Halliday and Robert Resnick, Fundamentals of Physics (Wiley, New York, 1970; revised printing, 1974)

SUGGESTED STUDY PROCEDURE

Begin by studying the General Comments and Problems A through D. Then work Problem J. Next study Problem E and work Problems K and L; study Problem F and work Problems M and N. Problem S is a challenging example that you may work if you so wish. Next study Problems G and H before working Problem O and, if you like, the challenging Problem T. Now go to your text and read Example 5 in Chapter 23 (on pp. 439, 440) and Example 6 in Chapter 25 (p. 473). Then study Problem I and work Problems P, Q, R.

Take the Practice Test, and work some Additional Problems if necessary, before attempting a Mastery Test.

HALLIDAY AND RESNICK

Objective Number	Readings	Problems with Solutions Study Guide	Assigned Problems Study Guide	Additional Problems
1	General Comments	A, B, C, F	J, L, M, N	S; Chap. 23, Probs. 28, 29
2	Chap. 23, Ex.[a] 5; Chap. 25, Ex. 6	G, H, I	O, Q	T; Chap. 23, Prob. 27
3	General Comments	D, E, F	K, M, N, P, R	S

[a]Ex. = Example(s).

TEXT: Francis Weston Sears and Mark W. Zemansky, University Physics (Addison-Wesley, Reading, Mass., 1970), fourth edition

SUGGESTED STUDY PROCEDURE

Begin by studying the General Comments. Then read Section 25-2 in Chapter 25, and study Problems A through D before working Problem J. Next study Problem E and work Problems K and L. Then study Problem F and work Problems M and N, and Problem S if you like (challenging but optional). Study Problems G and H and work Problem O; Problem T is optional. Study Problem I and work Problems P, Q, and R.

Take the Practice Test, and work some Additional Problems if necessary, before attempting a Mastery Test.

SEARS AND ZEMANSKY

Objective Number	Readings	Problems with Solutions	Assigned Problems	Additional Problems
		Study Guide	Study Guide	
1	General Comments, Sec. 25-2	A, B, C, F	J, L, M, N	S
2		G, H, I	O, Q	T
3		D, E, F	K, M, N, P, R	S

TEXT: Richard T. Weidner and Robert L. Sells, Elementary Classical Physics (Allyn and Bacon, Boston, 1973), second edition, Vol. 2

SUGGESTED STUDY PROCEDURE

Begin by studying the General Comments. Then study Problems A through D and work Problem J. Read Section 23-3 in Chapter 23 up to and including Example 23-2. Then study Problem E and work Problems K and L, study Problem F and work Problems M and N. Next study Problems G and H and work Problem O. Problems S and T are challenging optional problems. Study Problem I and work Problems P, Q, and R.

Take the Practice Test before attempting a Mastery Test.

WEIDNER AND SELLS

Objective Number	Readings	Problems with Solutions	Assigned Problems	Additional Problems
		Study Guide	Study Guide	
1	General Comments, Sec. 23-3	A, B, C, F	J, L, M, N	S
2		G, H, I	O, Q	T, 23-9
3		D, E, F	K, M, N, P, R	S

PROBLEM SET WITH SOLUTIONS

A(1). A charge Q is uniformly distributed over the interval $0 \le \underline{x} \le L$ along the x axis.
(a) Determine the linear charge density in $0 \le \underline{x} \le L$.
(b) Determine the charge dq on a segment $d\underline{x}$ in $0 \le \underline{x} \le L$.
(c) Set up and evaluate an integral for the electric potential at a point (x, 0, 0) on the x axis to the right ($x > L$) of the charge.

Solution

(a) See Figure 2. Since Q is uniformly distributed the linear charge density λ is constant in $0 \le \underline{x} \le L$ and given by

$$\lambda = Q/L.$$

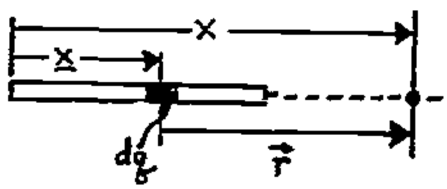


Figure 2

(b) The charge dq in a segment $d\underline{x}$ in the charge region is equal to the charge density multiplied by the length of the interval,

$$dq = \lambda\, d\underline{x} = (Q\, d\underline{x})/L.$$

(c) The vector from dq to the field point is $\vec{r} = (x - \underline{x})\hat{i}$, and its magnitude is

$$r = |\vec{r}| = x - \underline{x} \quad \text{for } x > \underline{x}.$$

Then the potential dV at x attributable to dq is

$$dV = \frac{k\, dq}{r} = \left(\frac{kQ}{L}\right)\left(\frac{d\underline{x}}{x - \underline{x}}\right),$$

and the potential at x is

$$V(x) = \frac{kQ}{L}\int_0^L \frac{d\underline{x}}{x - \underline{x}} .$$

The limits on the integral are determined by the charge boundaries. In a one-dimensional distribution such as this, the smaller boundary coordinate ($\underline{x} = 0$) is the lower limit and the larger ($\underline{x} = L$) is the upper limit. The integral for V(x) is evaluated by the substitution $u = x - \underline{x}$ to get

$$V(x) = \frac{kQ}{L}\int_{x-L}^{x}\frac{du}{u} = \frac{kQ}{L}\ln\left(\frac{x}{x-L}\right) .$$

B(1). Determine $\vec{E}$ at (x, 0, 0) (x > L) for the charge distribution of Problem A.

Solution

The field $d\vec{E}$ at (x, 0, 0) from dq is

$$d\vec{E} = \frac{k\vec{r}\,dq}{r^3} = \left(\frac{kQ}{L}\right)\frac{(x - \underline{x})\hat{i}\,d\underline{x}}{(x - \underline{x})^3} = \left(\frac{kQ}{L}\right)\hat{i}\left(\frac{d\underline{x}}{(x - \underline{x})^2}\right)$$

and

$$\vec{E}(x) = \left(\frac{kQ}{L}\right)\hat{i}\int_0^L \frac{d\underline{x}}{(x - \underline{x})^2} .$$

Using the substitution $u = x - \underline{x}$ gives us

$$\vec{E}(x) = \left(\frac{kQ}{L}\right)\hat{i}\int_{x-L}^{x}\frac{du}{u^2} = \left(\frac{kQ}{L}\right)\hat{i}\left(\frac{1}{x-L} - \frac{1}{x}\right) = \frac{kQ}{x(x-L)}\hat{i}.$$

Comment: Recall from your study of potential that if the potential is a function of one variable only (in this case, x), then

$$E(x) = -dV/dx.$$

Let us check this in this case.

$$-\frac{dV}{dx} = -\left(\frac{kQ}{L}\right)\left(\frac{d}{dx}\right)[\ln x - \ln(x - L)] = -\frac{kQ}{L}\left(\frac{1}{x} - \frac{1}{x-L}\right) = \frac{kQ}{x(x-L)} = E(x),$$

as expected.

C(1). Charge is distributed along the x axis as given by the linear charge density

$$\lambda(\underline{x}) = \alpha\underline{x}^2 \qquad \text{for } 0 \le \underline{x} \le L.$$

(a) Determine the total charge Q in this distribution.
(b) Express the constant α in terms of Q and L.

Solution

(a) $\lambda(\underline{x})$ is the linear density in coulombs per meter (C/m). The charge dq on an infinitesimal segment $d\underline{x}$ at position $\underline{x}$ is $dq = \lambda(\underline{x})\,d\underline{x}$. Since the total charge contained in a region $a \le \underline{x} \le b$ is given by $\int_a^b \lambda(\underline{x})\,d\underline{x}$, the charge Q is

$$Q = \int_0^L \alpha \underline{x}^2\, d\underline{x} = \frac{1}{3}\alpha L^3.$$

(b) Solving for α gives us

$$\alpha = 3Q/L^3.$$

Thus

$$\lambda(\underline{x}) = 3Q\underline{x}^2/L^3.$$

D(3). Show that $\vec{E}(x)$ from Problem B reduces as expected for $x \gg L$.

Solution

First, how do we expect $\vec{E}$ to behave for $x \gg L$? In this case the field point is so distant from the charge that $\vec{E}$ should be very close to that from a point charge Q at the origin, i.e.,

$$\vec{E}(x \gg L) \approx (kQ/x^2)\hat{i}.$$

Now let us see if this is the case. The Problem B result can be written

$$\vec{E}(x) = \frac{kQ}{x^2(1 - L/x)}\hat{i}.$$

For $L \ll x$, $L/x \ll 1$, and $1 - L/x \approx 1$. Therefore

$$\vec{E}(x \gg L) \approx (kQ/x^2)\hat{i}.$$

E(3). Show that the integral for V(x) in Problem J reduces as expected for $x \gg L$.

Solution

For $x \gg L \geq \underline{x}$, $x - \underline{x} \approx x$. Thus

$$V(x \gg L) \approx \frac{3kQ}{L^3}\int_0^L \frac{\underline{x}^2\, d\underline{x}}{x} = \frac{3kQ}{L^3 x}\left(\frac{\underline{x}^3}{3}\Big|_0^L\right) = \frac{kQ}{x},$$

which is the potential for a point charge Q at the origin.

F(1, 3). For the charged rod of Problem A (charge Q uniformly distributed over $0 \leq \underline{x} \leq L$):
(a) Set up an integral for the electric field at $\vec{R} = y\hat{j}$ (a point in the plane perpendicular to the $x = 0$ end of the rod).

(b) Show that this integral reduces as expected for $y \gg L$.

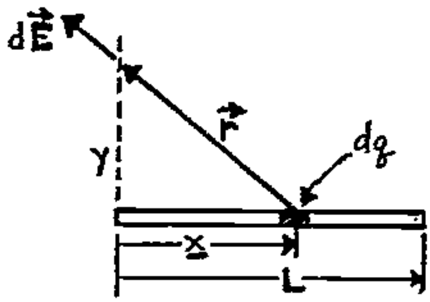


Figure 3

Solution

(a) See Figure 3. The electric field at $y\hat{j}$ resulting from the presence of $dq = (Q\,dx)/L$ at $x\hat{i}$ is

$$d\vec{E} = (kQ/L)(\vec{r}\,dx/r^3),$$

where $\vec{r} = -x\hat{i} + y\hat{j}$ (Why the negative sign?), and

$$r = |\vec{r}| = (x^2 + y^2)^{1/2}.$$

Thus

$$d\vec{E} = (kQ/L)[(-x\hat{i} + y\hat{j})\,dx]/(x^2 + y^2)^{3/2}.$$

Integrating to get $\vec{E}$ at (y, 0, 0) gives us

$$\vec{E}(y) = \frac{kQ}{L}\int_0^L \frac{(-x\hat{i} + y\hat{j})\,dx}{(x^2 + y^2)^{3/2}},$$

which may be rewritten

$$\vec{E}(y) = \left(-\frac{kQ}{L}\int_0^L \frac{x\,dx}{(x^2 + y^2)^{3/2}}\right)\hat{i} + \left(\frac{kQy}{L}\int_0^L \frac{dx}{(x^2 + y^2)^{3/2}}\right)\hat{j}.$$

Thus the components of $\vec{E}$ are

$$E_x(y) = -\frac{kQ}{L}\int_0^L \frac{x\,dx}{(x^2 + y^2)^{3/2}}, \qquad E_y(y) = \frac{kQy}{L}\int_0^L \frac{dx}{(x^2 + y^2)^{3/2}}.$$

(b) Before seeing what happens to these results for $y \gg L$, what do we expect? If the rod length L is small compared to the distance from the rod to the field point, the field should be very nearly the same as that of a point charge Q at the origin. That is,

$$\vec{E}(y \gg L) \approx (kQ/y^2)\hat{j}.$$

Let us see. Consider the y component first. Since $y \gg L$ and $0 \le x \le L$,

$$x^2 + y^2 \approx y^2,$$

so that

$$E_y(y \gg L) \approx \frac{kQy}{L}\int_0^L \frac{dx}{y^3} = \frac{kQ}{y^2}.$$

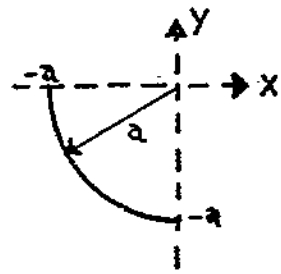


Figure 4

Just what we expected! What about E_x?

$$E_x(y \gg L)| \approx -\frac{kQ}{L}\int_0^L \frac{x\,dx}{y^3} = -\frac{kQL}{2y^3}.$$

What's this? We expected E_x to be zero, and it apparently isn't. But things are not as bad as they might appear. Watch. Rewrite the result for E_x:

$$|E_x(y \gg L)| \approx \frac{kQ}{y^2}\left(\frac{L}{2y}\right) = \frac{L}{2y}E_y(y \gg L) \ll E_y(y \gg L).$$

Thus although E_x is not identically zero, it is negligible compared to E_y. The integrals for E_x and E_y can be evaluated by standard substitution techniques. The appropriate substitutions and the results are given. Have a try at it if you are so inclined.

E_x: $u = x^2 + y^2$, $E_x(y) = -(kQ/Ly)[1 - y/(y^2 + L^2)^{1/2}]$,

E_y: $\tan\theta = \frac{x}{y}$, $E_y(y) = kQ/y(y^2 + L^2)^{1/2}$.

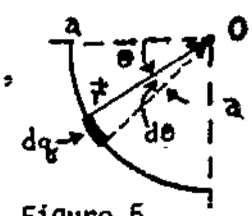


Figure 5

G(2). A charge is uniformly distributed along the circular arc shown in Figure 4. Determine the electric potential and field at the origin.

Solution

Let dq be the charge on a segment of arc as in Figure 5. Since each dq is the same distance from O, the electric potential is particularly easy to determine:

$$V_0 = \int\frac{k\,dq}{a} = \frac{k}{a}\int dq = \frac{kQ}{a}.$$

The displacement vector from dq to the origin is

$$\vec{r} = (a \cos \theta)\hat{i} + (a \sin \theta)\hat{j}.$$

The electric field at O attributable to dq is then

$$d\vec{E} = (k\vec{r}\, dq)/r^3 = k[(\cos \theta)\hat{i} + (\sin \theta)\hat{j}]/a^2\, dq.$$

Since dq subtends an arc segment of length a dθ and since Q is uniformly distributed along the length $\pi a/2$,

$$dq = \frac{Q}{\pi a/2} a\, d\theta = \frac{2Q}{\pi} d\theta.$$

Thus

$$\vec{E}_0 = \frac{k}{a^2} \int_0^{\pi/2} [(\cos \theta)\hat{i} + (\sin \theta)\hat{j}]\frac{2Q}{\pi} d\theta = \frac{2kQ}{\pi a^2}\left[\int_0^{\pi/2} (\cos \theta\, d\theta)\hat{i} + \int_0^{\pi/2} (\sin \theta\, d\theta)\hat{j}\right]$$

$$= (2kQ/\pi a^2)(\hat{i} + \hat{j}).$$

Thus

$$E_x = E_y = 2kQ/\pi a^2.$$

Since $E_x = E_y$, $\vec{E}_0$ makes an angle of $\pi/4$ with the positive x axis and has a magnitude

$$E_0 = (E_x^2 + E_y^2)^{1/2} = 2\sqrt{2}kQ/\pi a^2.$$

H(2). A charge Q is uniformly distributed over one-quarter of the circle shown in Figure 6 as a shaded region. Determine the electric potential at the origin O.

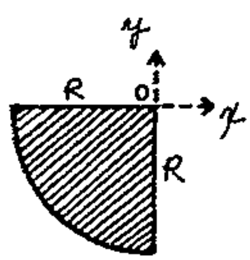


Figure 6

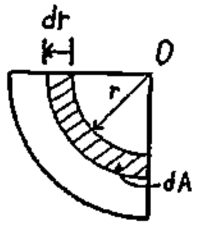


Figure 7

Solution

See Figure 7. The essential idea here is: (a) to determine the potential at O from the charge on the ring of radius r and width dr; and (b) to add these contributions to get V. In Problem G the contribution from the ring was determined to be

$$dV = (k\ dq)/r,$$

where dq is the charge on the ring. To get dq we use the fact that Q is uniformly distributed over the area $(1/4)\pi R^2$, and thus the density on this surface is uniform and given by

$$\sigma = \frac{Q}{(1/4)\pi R^2} = \frac{4Q}{\pi R^2}.$$

The area dA of the ring under consideration is

$$dA = (\text{length}) \times (\text{width}) = (1/2)\pi r\ dr.$$

Therefore the charge on the ring is

$$dq = \sigma\ dA = (2Qr\ dr)/R^2.$$

The potential at O attributable to this ring is then

$$dV = 2kQ/R^2\ dr.$$

Summing over all rings, i.e., integrating over r from 0 to R, gives us

$$V_0 = \int_0^R \frac{2kQ}{R^2}\,dr = \frac{2kQ}{R}.$$

Comment: By rewriting this as

$$V_0 = \frac{kQ}{R/2},$$

we see that the potential at O is the same as if all the charge were placed a distance (1/2)R away.

I(2). A charge Q is uniformly distributed along a ring of radius a as in Figure 8. Determine the electric potential and field at the point P on the axis of the ring.

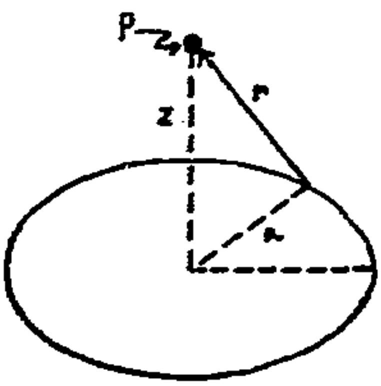


Figure 8

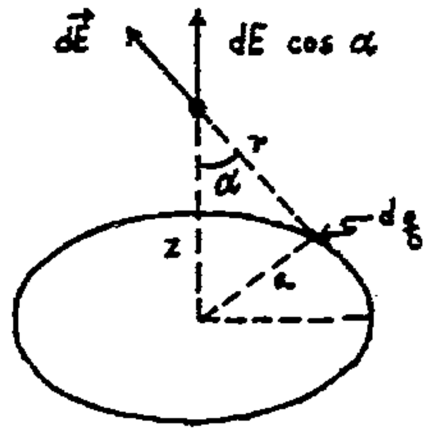


Figure 9

Solution

See Figure 9. The distance from each elementary charge dq on the ring to the field point is constant,

$$r = (z^2 + a^2)^{1/2}.$$

Hence, the potential at P is

$$V(z) = \int \frac{k\ dq}{r} = \frac{kQ}{(z^2 + a^2)^{1/2}}.$$

The components of E perpendicular to the axis sum to zero. This is ensured by the fact that charges on opposite sides of the circle contribute fields with equal components along the axis but oppositely directed perpendicular components. From the figure the magnitude of the field at P by dq is

$$dE = (k\ dq)/r^2,$$

and the axial component is

$$dE_z = dE \cos \alpha = dE(z/r) = (kz\ dq)/(z^2 + a^2)^{3/2}.$$

Therefore,

$$E_z(z) = \int kz/(z^2 + a^2)^{3/2}\, dq,$$

but since a and z do not change for different dq's,

$$E_z(z) = [kz/(z^2 + a^2)^{3/2}]\int dq = kzQ/(z^2 + a^2)^{3/2}.$$

Problems

J(1). Set up (but you need not evaluate) integrals for V and $\vec{E}$ at the point (x, 0, 0), x > L, for the linear charge distribution of Problem C, $\lambda(\underline{x}) = 3Q\underline{x}^2/L$.

K(3). Show that the integral for $\vec{E}(x)$ in Problem J reduces as expected for x >> L.

L(1). For the charge distribution of Problem C,

$\lambda(\underline{x}) = 3Q\underline{x}^2/L^3$ for $0 \le \underline{x} \le L$,

determine the electric potential and field at the origin.

M(1, 3). (a) Set up the integral for the electric potential at the point $y\hat{j}$ as in Problem F.
(b) Show that this integral reduces appropriately for y >> L.

N(1, 3). (a) For the uniformly charged rod in Figure 3, set up integrals for the electric potential and field at an arbitrary field point $x\hat{i} + y\hat{j} + z\hat{k}$.
(b) Show that your integrals of part (a) are identical to earlier integrals for the following cases:
(i) x > L, y = z = 0 from Problems A and B.
(ii) y > 0, x = z = 0 from Problems F and M.

O(2). A charge Q is uniformly distributed along the circular arc shown in Figure 10. Determine the electric potential and field at the center.

P(3). Show that the results obtained in Problem I behave as expected for z = 0 and z >> a.

Q(2). A charge Q is uniformly distributed over a disk of radius R as in Figure 11. Set up, but do not evaluate, integrals for the electric potential and field at a point on the axis of the disk.

R(3). Show that the integrals of Problem I reduce appropriately for z >> R.

S(1, 3). (Optional - challenging). (a) Use the integrals of Problem N to evaluate V and $\vec{E}$ at $(1/2)L\hat{i} + y\hat{j}$, a point in the midplane of the charged rod. Hint: Use the substitution

$$\tan\theta = \frac{\underline{x} - L/2}{y}.$$

(b) Suppose $y \ll L$ for part (a). In this case the field point is very close to a long charged rod. Show that your result for $\vec{E}$ reduces to that obtained for a long charged rod using Gauss' law, namely, $(2kQ/Ly)\hat{j}$.

Figure 10

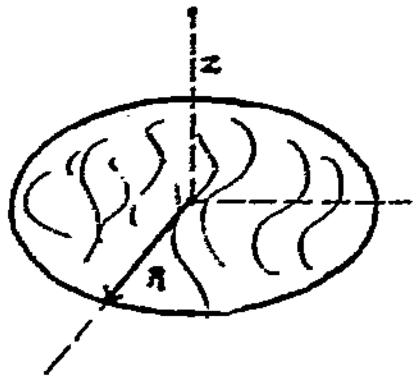


Figure 11

Fioure 12

T(2). (Optional - challenging). A charge Q is uniformly distributed over the area shown in Figure 12. Determine the electric potential and field at the center.

Solutions

J(1). $V(x) = \frac{3kQ}{L^3}\int_0^L \frac{\underline{x}^2\, d\underline{x}}{x - \underline{x}}$, $\quad \vec{E}(x) = \frac{3kQ}{L^3}\int_0^L \frac{\underline{x}^2\, d\underline{x}}{(x - \underline{x})^2}\hat{i}$.

K(3). $\vec{E}(x \gg L) \approx (kQ/x^2)\hat{i}$.

L(1). $V(0) = 3kQ/2L$, $\vec{E}(0) = -(3kQ/L^2)\hat{i}$.

M(1, 3). $V(y) = \frac{kQ}{L}\int_0^L \frac{d\underline{x}}{(\underline{x}^2 + y^2)^{1/2}} \approx \frac{kQ}{y}$ for $y \gg L$.

Comment: The substitution $\tan\theta = \underline{x}/y$ leads to

$$V(y) = (kQ/L)\ \ln\ [(y^2 + L^2)^{1/2} + L]/y.$$

Remember, you are not required to do this integral.

N(1, 3). $V(x, y, z) = \frac{kQ}{L}\int_0^L \frac{d\underline{x}}{[(x - \underline{x})^2 + y^2 + z^2]^{1/2}}$,

$$\vec{E}(x, y, z) = \frac{kQ}{L}\int_0^L \frac{[(x - \underline{x})\hat{i} + y\hat{j} + z\hat{k}]}{[(x - \underline{x})^2 + y^2 + z^2]^{3/2}}\, d\underline{x}.$$

O(2). $V_0 = kQ/a$, $\vec{E}_0 = (2kQ/\pi a^2)\hat{i}$.

P(3). $z = 0$: $V(a) = kQ/a$, $E_z(0) = 0$.

$z >> a$: $V \approx kQ/z$, $E_z \approx kQ/z^2$.

Q(2). $$V(z) = \frac{2kQ}{R^2}\int_0^R \frac{r\,dr}{(r^2 + z^2)^{1/2}},$$

$$\vec{E}(z) = \frac{2zkQ}{R^2}\int_0^R \frac{r\,dr}{(r^2 + z^2)^{3/2}}\hat{k}.$$

R(3). $V(z >> R) \approx kQ/z^2$, $E(z >> R) = (kQ/z^2)\hat{k}$.

S(1, 3). $V(L/2, y, 0) = (2kQ/L)\ \ln\{[(4y^2 + L^2)^{1/2} + L]/2y\}$,

$\vec{E}(L/2, y, 0) = [2kQ/y(4y^2 + L^2)^{1/2}]\hat{j}$.

T(2). $V_0 = 2kQ/3R$, $\vec{E}_0 = (4kQ/3\pi R^2)(\ln 2)\hat{i}$.

PRACTICE TEST

1. A linear charge is distributed along the x axis with the density

   $\lambda(\underline{x}) = \alpha\underline{x}^3$ for $0 \le \underline{x} \le L$.

   (a) Determine the total charge Q in terms of α and L.
   (b) Determine the electric potential at the origin. Express your answer in terms of Q, L, and other constants (not including α).

2. A charge Q is uniformly distributed along the arc shown in Figure 13.
   (a) Determine $\vec{E}$ at the origin.
   (b) Use

   $$\lim_{\alpha \to 0} \frac{\sin\alpha}{\alpha} = 1$$

   to show that your result reduces as expected as $\alpha \to 0$.

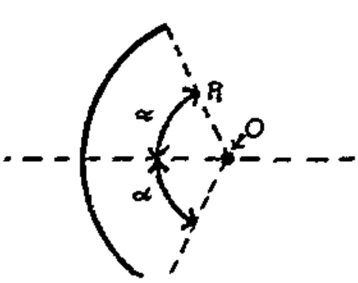


Figure 13

Practice Test Answers

1. (a) $Q = (1/4)\alpha L^4$. (b) $V = 4kQ/3L$.
2. (a) $\vec{E}_0 = \{(kQ/R^2)[(\sin\alpha)/\alpha]\}\hat{i}$.

ELECTRIC FIELDS AND POTENTIALS
FROM CONTINUOUS CHARGE DISTRIBUTIONS

Date ______________

Mastery Test Form A

pass recycle

1 2 3

Name ______________________ Tutor ______________________

1. Charge is distributed along the x axis according to the linear charge density

   $\lambda(x) = \beta \sin(\pi x/L), \qquad 0 \le x \le L.$

   (a) Determine the total charge Q.

   (b) Set up, but do not evaluate, an integral for the electric potential at the field point $x\hat{i} + y\hat{j}$.

   (c) Set up, but do not evaluate, an integral for the electric field at the field point $x\hat{i} + y\hat{j}$.

2. Show that your integral for $\vec{E}(x, y)$ in Problem 1 reduces appropriately for $\vec{E}(L/2, y \gg L)$.

3. A charge Q is uniformly distributed on the flat circular sector shown in Figure 1. Determine the electric potential at the center (point 0) in terms of Q, R, ϕ, and other constants as needed.

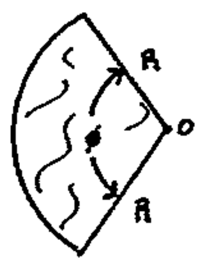


Figure 1

ELECTRIC FIELDS AND POTENTIALS
FROM CONTINUOUS CHARGE DISTRIBUTIONS

Date ________________

pass / recycle

Mastery Test Form B

1 2 3

Name ______________________ Tutor ______________________

1. Charge is distributed along the x axis according to the linear charge density

   $\lambda(\underline{x}) = \beta\underline{x}(L - \underline{x}), \qquad 0 \leq \underline{x} \leq L.$

   (a) Determine the total charge Q.

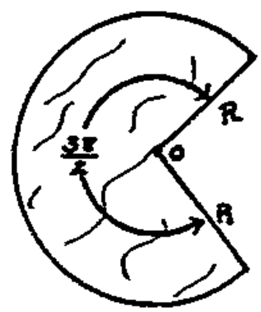
   (b) Set up, but do not evaluate, an integral for the electric potential at the field point $x\hat{i} + y\hat{j} + z\hat{k}$.

   (c) Set up, but do not evaluate, an integral for the electric field at the field point $x\hat{i} + y\hat{j} + z\hat{k}$.

2. Show that your integral for V(x, y, z) in Problem 1 reduces appropriately for V(x >> L, 0, 0).

3. A charge Q is uniformly distributed on the flat circular section shown in Figure 1. Determine the electric potential at point O in terms of Q, R, and other constants as needed.

Figure 1



ELECTRIC FIELDS AND POTENTIALS
FROM CONTINUOUS CHARGE DISTRIBUTIONS

Date ____________

Mastery Test   Form C

pass   recycle

1   2   3

Name ____________   Tutor ____________

1. Charge is distributed along the x axis according to the linear charge density

$$\lambda(x) = \beta \cos(\pi x/2L), \qquad -L \le x \le L.$$

   (a) Determine the total charge Q.

   (b) Set up, but do not evaluate, an integral for the electric potential at the field point $x\hat{i} + y\hat{j} + z\hat{k}$.

   (c) Set up, but do not evaluate, an integral for the electric field at the field point $x\hat{i} + y\hat{j} + z\hat{k}$.

2. Show that your integral for $E(x, y, z)$ in Problem 1 reduces appropriately for $E(0, 0, z \gg L)$.

3. A charge Q is uniformly distributed on the annular ring shown in Figure 1. Determine the electric potential at a point on the axis of the ring a distance h from the ring's plane.

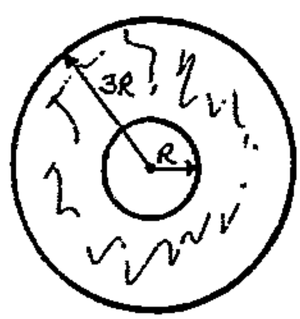


Figure 1

MASTERY TEST GRADING KEY - Form A

1. What To Look For: (a) Correct answer. (b) Be sure denominator of answer is correct. (c) Check for unit vectors in numerator.

   Solution: $Q = \int_0^L \lambda(\underline{x})\,d\underline{x} = \int_0^L \beta \sin(\pi \underline{x}/L)\,d\underline{x} = 2\beta L/\pi.$

   (b) $V(x, y) = \int \frac{k\,dq}{r} = \int_0^L \frac{k\beta \sin(\pi x/L)}{[(x - \underline{x})^2 + y^2]^{1/2}}\,d\underline{x}.$

   (c) $\vec{E}(x, y) = \int \frac{k\vec{r}\,dq}{r^3} = \int_0^L \frac{k\beta[(x - x)\hat{i} + y\hat{j}]\sin(\pi x/L)}{[(x - \underline{x})^2 + y^2]^{3/2}}\,d\underline{x}.$

2. What To Look For: (a) No contribution to Ex. (b) Correct result.

   Solution:

   $$\vec{E}(L/2, y >> L) = \int_0^L \frac{k\beta[(L/2 - \underline{x})\hat{i} + y\hat{j}]}{[(x - \underline{x})^2 + y^2]^{3/2}}\sin(\pi\underline{x}/L)\,d\underline{x}$$

   $$= (\frac{k\beta}{y^3}\int_0^L(\frac{L}{2} - \underline{x})\sin(\frac{\pi\underline{x}}{L})\,d\underline{x})\hat{i} + (\frac{k}{y^2}\int_0^L \beta\sin(\frac{\pi\underline{x}}{L})\,d\underline{x})\hat{j}.$$

   But $\int_0^L(\frac{L}{2} - \underline{x})\sin(\frac{\pi\underline{x}}{L})\,d\underline{x} = 0$ (odd about $\underline{x} = \frac{L}{2}$)

   and

   $\int_0^L \beta\sin(\frac{\pi\underline{x}}{L})\,d\underline{x} = Q.$

   So $\vec{E}(L/2, y >> L) = (kQ/L^2)\hat{j}.$

3. What To Look For: (a) Correct dq. (b) Correct expression for dV. (c) Correct answer.

   Solution: See Figure 17. Area of sector $= \phi R^2/2$. Charge density $= \frac{Q}{\text{area}} = \frac{2Q}{\phi R^2}.$

   $dA = (r\phi)\,dr$, $dq = (\text{density})dA = (2Q/\phi R^2)r\phi\,dr = (2Qr\,dr)/R^2.$

   $dV = (k\,dq)/r = (2kQ\,dr)/R^2,$

   $V = \int_0^R \frac{2kQ}{R^2}\,dr = \frac{2kQ}{R}.$

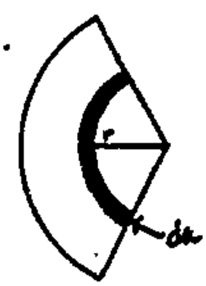


Figure 17

MASTERY TEST GRADING KEY - Form B

1. What To Look For: (a) Correct answer. (b) Correct denominator. (c) Unit vectors in numerator.

   Solution: (a) $Q = \int_0^L \lambda(\underline{x})\, d\underline{x} = \beta \int_0^L \underline{x}(L - \underline{x})\, d\underline{x} = \beta L^3/6.$

   (b) $V(x, y, z) = \int \frac{k\, dq}{r} = \int_0^L \frac{k\beta\underline{x}(L - \underline{x})}{[(x - \underline{x})^2 + y^2 + z^2]^{1/2}}\, d\underline{x}.$

   (c) $\vec{E}(x, y, z) = \int \frac{k\vec{r}\, dq}{r^3} = \int_0^L \frac{k\beta[(x - \underline{x})\hat{i} + y\hat{j} + z\hat{k}]\underline{x}(L - \underline{x})}{[(x - \underline{x})^2 + y^2 + z^2]^{3/2}}\, d\underline{x}.$

2. What To Look For: See that integral for Q is correct. Correct answer.

   Solution: $V(x \gg L, 0, 0) = \int_0^L \frac{k\beta\underline{x}(L - \underline{x})}{(x - \underline{x})}\, d\underline{x} = \frac{k}{x} \int_0^L \beta\underline{x}(L - \underline{x})\, d\underline{x} = \frac{kQ}{x}.$

3. What To Look For: Correct dq. Correct dV. Correct answer.

   Solution: See Figure 18. Area of sector = $(3\pi/2)(R^2/2) = (3\pi R^2/4)$.

   Charge density $\sigma = Q/\text{Area} = 4Q/3\pi R^2$. $dA = (r\phi)\, dr = (3\pi/2)r\, dr$.

   $dq = \sigma\, dA = (3\pi/2)(4Q/3\pi R^2)r\, dr = (2Q/R^2)r\, dr.$

   $dV = \frac{k\, dq}{r} = \frac{2kQ}{R^2}\, dr; \quad V = \int dV = \int_0^R \frac{2kQ}{R^2}\, dr = \frac{2kQ}{R}.$

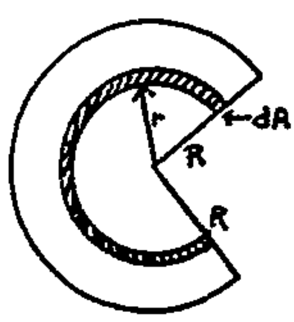


Figure 18

MASTERY TEST GRADING KEY - Form C

1. What To Look For: Correct answer. Correct denominator and limits. Unit vectors in numerator. Limits

Solution: (a) $Q = \int_{-L}^{L} \lambda(\underline{x})\, d\underline{x} = \beta \int_{-L}^{L} \cos(\frac{\pi \underline{x}}{2L})\, d\underline{x} = \frac{4\beta L}{\pi}$.

(b) $V(x, y, z) = \int \frac{k\, dq}{r} = \int_{-L}^{L} \frac{k\beta \cos(\pi\underline{x}/2L)}{[(x - \underline{x})^2 + y^2 + z^2]^{1/2}}\, d\underline{x}$.

(c) $E(x, y, z) = \int \frac{k\vec{r}\, dq}{r^3} = \int_{-L}^{L} \frac{k\beta[(x - \underline{x})\hat{i} + y\hat{j} + z\hat{k}] \cos(\pi\underline{x}/2L)}{[(x - \underline{x})^2 + y^2 + z^2]^{3/2}}\, d\underline{x}$.

2. What To Look For: See that $E_x = E_y = 0$. Correct answer.

Solution: $\vec{E}(0, 0, z >> L) = \int_{-L}^{L} \frac{k\beta(-\underline{x}\hat{i} + z\hat{k}) \cos(\pi\underline{x}/2L)}{(\underline{x}^2 + z^2)^{3/2}}\, d\underline{x}$

$$\simeq -\hat{i}\frac{k\beta}{z^3} \int_{-L}^{L} \underline{x} \cos(\frac{\pi \underline{x}}{2L})\, d\underline{x} + \hat{k}\frac{k}{z^2} \int_{-L}^{L} \beta \cos(\frac{\pi \underline{x}}{2L})\, d\underline{x}.$$

But $\int_{-L}^{L} \underline{x} \cos(\frac{\pi \underline{x}}{2L})\, d\underline{x} = 0$ (odd about $\underline{x} = 0$), $\int_{-L}^{L} \beta \cos(\pi\underline{x}/2L)\, d\underline{x} = Q$.

So $\vec{E}(0, 0, z >> L) \simeq (kQ/z^2)\hat{k}$.

3. What To Look For: Correct dq. Correct dV. Correct answer.

Solution: See Figure 19. Area of ring $= \pi[(3R)^2 - R^2] = 8\pi R^2$

Density of charge $= Q/\text{Area} = Q/8\pi R^2$ $dA = 2\pi r\, dr$.

$dq = (\text{density})dA = (Qr\, dr)/4R^2$. Distance from dq to axial point $= \sqrt{r^2 + h^2}$.

$dV = k\, dq/\sqrt{r^2 + h^2}$.

$V = \frac{kQ}{4R^2} \int_{R}^{3R} \frac{r\, dr}{\sqrt{r^2 + h^2}}$ (substitution: $u = r^2 + h^2$, $du = 2r\, dr$)

$$= \frac{kQ}{4R^2} \int_{R^2 + h^2}^{9R^2 + h^2} \frac{1}{2}u^{-1/2}\, du = \frac{kQ}{4R^2}[\sqrt{9R^2 + h^2} - \sqrt{R^2 + h^2}].$$

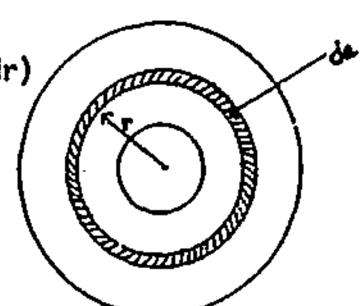


Figure 19

MAXWELL'S PREDICTIONS

INTRODUCTION

With this module, you will reach a milestone in your study of electromagnetic phenomena. From past modules, you now have (at your fingertips, hopefully!) the same basic laws of electromagnetism that Maxwell collected together in the nineteenth century. However, as powerful as these laws were, Maxwell found that there was a basic flaw - a logical inconsistency - in the one known as Ampère's law. He was able to deduce (in advance of any direct experimental test) precisely the correction that was needed. With this correction, the addition of what is called the "displacement-current" term to Ampère's law, it follows that a changing electric field gives rise to a magnetic field, just as a changing magnetic field gives rise to an electric field according to Faraday's law.

After he had predicted this mutual relationship, Maxwell was able to go on and predict that the right combination of oscillating electric and magnetic fields could literally kick itself through empty space. This is the phenomenon that we now call electromagnetic waves - which include, along with TV and radio waves, the sunlight that we receive across 93 000 000 miles of space without any significant loss of intensity other than that which necessarily follows from its spreading out in all directions.

The development of the theory of electromagnetic waves from the basic laws of electricity and magnetism that you have studied in past modules is one of the most beautiful in physics, and at the same time one of the most mathematically difficult that you will meet in this course. Thus if the arguments at times seem long - bear with it! - the total module is fairly short.

PREREQUISITES

Before you begin this module, you should be able to:	Location of Prerequisite Content
*State and apply Ampère's law (needed for Objectives 1 through 3 of this module	Ampère's Law Module
*State and interpret Gauss' law (needed for Objectives 2 and 3 of this module	Flux and Gauss' Law Module
*State Faraday's law, and apply it to calculate the emf induced around a closed path (needed for Objectives 2 and 3 of this module)	Ampère's Law Module
*Describe a simple form of electrical oscillator (needed for Objective 3 of this module)	Inductance Module
*Use and interpret mathematical descriptions of one-dimensional waves (needed for Objective 3 of this module)	Traveling Waves Module
*Calculate partial derivatives of functions of two variables (needed for Objective 3 of this module)	Partial Derivatives Review

LEARNING OBJECTIVES

After you have mastered the content of this module, you will be able to:

1. Displacement current - Use Ampère's law (including the displacement current) to find the $\vec{B}$ field produced by a changing $\vec{E}$ field, or vice versa.

2. Maxwell's equations - State Maxwell's equations in vacuum (i.e., in the presence of charges and currents, but with no dielectrics or magnetic materials), and indicate the physical significance of each.

3. Electromagnetic waves - For a plane electromagnetic wave, use information about $\vec{E}$ or $\vec{B}$ at given times or places, the direction the wave moves, the frequency, and/or the wavelength to determine other information in this list; also, write down mathematical expressions for the components of $\vec{E}$ and $\vec{B}$, and show that your expressions satisfy the appropriate simplified differential form of Maxwell's equations.

TEXT: Frederick J. Bueche, Introduction to Physics for Scientists and Engineers (McGraw-Hill, New York, 1975), second edition

SUGGESTED STUDY PROCEDURE

Read the General Comments on the following pages of this study guide along with Sections 28.1 through 28.5 and 29.1. Optional: Read Sections 28.6 through 28.8.

When you compare Maxwell's equations (28.1) with the same equations in General Comment 2, you will find slight differences of notation: $\sum q \to q$, $d\vec{s} \to d\vec{\ell}$, and $\int \vec{J} \cdot d\vec{A} \to i$. Also note that in the absence of a dielectric, in the last term of Eq. (28.1d), ε becomes just ε_0. You will find that the derivation of the simplified differential form of Maxwell's equations, Eqs. (28.6) and (28.7), is quite similar to that given in General Comment 3, between Eqs. (9) and (15); take your pick!

Study the Problems with Solutions and work the Assigned Problems. Then take the Practice Test, and work some Additional Problems if necessary, before trying a Mastery Test.

BUECHE

Objective Number	Readings	Problems with Solutions		Assigned Problems	Additional Problems
		Study Guide	Text	Study Guide	(Chap. 28)
1	General Comment 1; Secs. 28.1, 28.2	A	Illus.[a] 28.1	D, E	Quest.[a] 4 thru 9
2	General Comment 2; Eqs. (28.1) in Sec. 28.2	B		F	
3	General Comment 3; Secs. 28.3 thru 28.5, 29.1	C	Illus. 28.2	G, H, I	Probs. 1, 2, 4 thru 9, 12, 14

[a] Illus. = Illustration(s). Quest. = Question(s).

TEXT: David Halliday and Robert Resnick, Fundamentals of Physics (Wiley, New York, 1970; revised printing, 1974)

SUGGESTED STUDY PROCEDURE

Read the General Comments on the following pages of this study guide, along with Sections 34-4 through 34-6 and 35-1 through 35-3.

You will find that Maxwell's equations stated in Table 34-2 on p. 636 are exactly the same as in General Comment 2, except that we have used $d\vec{A}$ (instead of $d\vec{S}$) for the element of area. Your text gives a complete and accurate derivation of the simplified differential form of Maxwell's equations, Eqs. (35-4) and (35-8). However, the discussion is rather involved; you should find the corresponding derivation between Eqs. (9) and (15) of General Comment 3 easier to follow. When reading the derivation in your text, note that E means E_y and B means B_z.

Study the Problems with Solutions and work the Assigned Problems. Then take the Practice Test, and work some Additional Problems if necessary, before trying a Mastery Test.

HALLIDAY AND RESNICK

Objective Number	Readings	Problems with Solutions		Assigned Problems	Additional Problems
		Study Guide	Text	Study Guide	
1	General Comment 1; Secs. 34-4, 34-5	A	Chap. 34, Ex.[a] 4, 5	D, E	Chap. 34, Probs. 19 thru 28; Quest.[a] 6 thru 11
2	General Comment 2; Sec. 34-6	B		F	
3	General Comment 3; Secs. 35-1 thru 35-3	C		G, H, I	Chap. 35, Probs. 1, 5 thru 9; Quest. 2

[a]Ex. = Example(s). Quest. = Question(s).

TEXT: Francis Weston Sears and Mark W. Zemansky, University Physics (Addison-Wesley, Reading, Mass., 1970), fourth edition

SUGGESTED STUDY PROCEDURE

Read the General Comments on the following pages of this study guide, along with Sections 36-8 and 36-9. Optional: Read Sections 32-8, 36-1, 36-2, 36-5, and 36-7.

Section 36-8 of your text is devoted to a derivation of the simplified Maxwell's equations, Eqs. (36-17) through (36-19), which is more detailed than the derivation given in General Comment 3 between Eqs. (9) and (18), but equivalent to it. You will probably find you do not need to read both these derivations; take your choice! In your text's derivation, note that H means H_z and E means E_y.

Your text uses the auxilliary quantities $\vec{H}$ and $\vec{D}$; in the absence of dielectric and magnetic materials (which will be the case in this module), these are simply proportional to the more familiar fields $\vec{B}$ and $\vec{E}$: $\vec{H} = \vec{B}/\mu_0$ and $\vec{D} = \epsilon_0\vec{E}$. In Section 36-9 and in the optional readings, you will also encounter the polarization $\vec{P} = \vec{D} - \epsilon_0\vec{E}$ and the magnetization $\vec{M} = \vec{B}/\mu_0 - \vec{H}$; but these vanish when there are no dielectric or magnetic materials, and so need not concern you. Maxwell's equations given in Eqs. (36-20) through (36-23) reduce to those of General Comment 2, when the conditions $\vec{P} = \vec{M} = 0$ are used, along with the identifications of $\vec{H}$ and $\vec{D}$ above and the notation changes $Q_f \to q$, $I_C \to i$, and $d\vec{s} \to d\vec{\ell}$. Some other notation changes you will encounter are $\psi_D \to \epsilon_0\phi_E$, $\phi \to \phi_B$, and $I_D \to i_d$.

Study Problems A to C and work Problems D to I. Take the Practice Test before trying a Mastery Test.

SEARS AND ZEMANSKY

Objective Number	Readings	Problems with Solutions Study Guide	Assigned Problems Study Guide
1	General Comment 1	A	D, E
2	General Comment 2; Eqs. (36-20) thru (36-23) in Sec. 36-9	B	F
3	General Comment 3; Sec. 36-8	C	G, H, I

TEXT: Richard T. Weidner and Robert L. Sells, Elementary Classical Physics (Allyn and Bacon, Boston, 1973), second edition, Vol. 2

SUGGESTED STUDY PROCEDURE

Read the General Comments on the following pages of this study guide, along with Sections 35-1 through 35-3.

Your text gives a very nice and direct demonstration in Section 35-3 that a traveling pulse of crossed $\vec{E}$ and $\vec{B}$ fields is a solution of Maxwell's equations. However, it does not derive the differential form of Maxwell's equations for plane waves; if you want to see more discussion of this topic than is found in General Comment 3, refer to one of the texts listed under Additional Learning Materials below. The first page of Section 35-1 in your text and General Comment 1 in this study guide give alternate versions of the argument for the displacement-current term in Ampère's law; take your choice! When reading your text's discussion, note that there is no reason for the hemispherical surface in Figure 35-2 to touch the edge of the plate; this is just an accident of the drawing.

Study the Problems with Solutions before working Problems D through I. Then take the Practice Test, and work Problem 35-1 if necessary, before taking a Mastery Test.

WEIDNER AND SELLS

Objective Number	Readings	Problems with Solutions		Assigned Problems	Additional Problems
		Study Guide	Text	Study Guide	
1	General Comment 1; Sec. 35-1	A	Ex.[a] 35-1	D, E	
2	General Comment 2; Sec. 35-2	B		F	
3	General Comment 3; Sec. 35-3	C		G, H, I	35-1

[a]Ex. = Example.

GENERAL COMMENTS

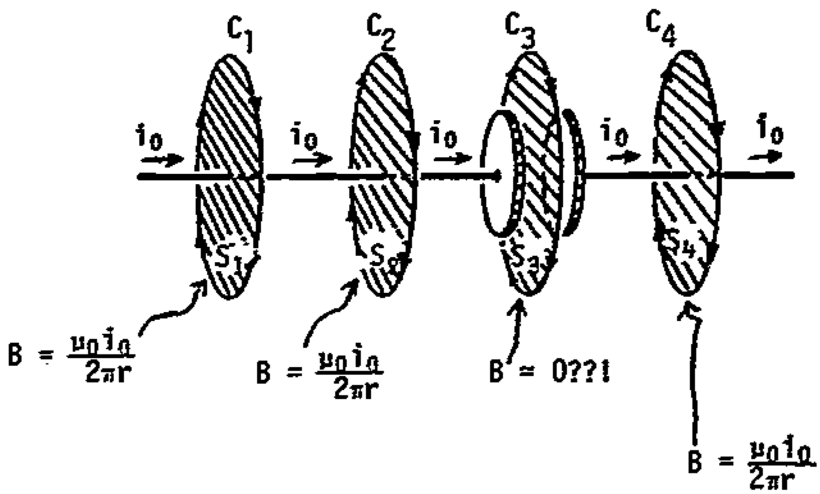
1. The Consistency Argument for the "Displacement Current"

Let us calculate the magnetic field around the long wire in Figure 1. It is a little different from the long wires you have seen in this course before: there is a capacitor in the middle of it. But this does not, of course, preclude a pulse of current for a short time, as the capacitor charges up - or a pulsating back-and-forth current for an indefinite period of time.

We first construct a circle of radius r, such as C_1. The current through the surface S_1 bounded by C_1 is just i_0; and thus Ampère's law yields

$$\mu_0 i_0 = \oint_{C_1} \vec{B} \cdot d\vec{\ell} = 2\pi r B(r) \quad \text{or} \quad B(r) = \frac{\mu_0 i_0}{2\pi r}. \tag{1}$$

Figure 1



We are not at all surprised, of course, when a repetition of this calculation using C_2 and S_2 yields the same result. But we are in for a rude shock when we try C_3 and S_3; there is no current through S_3; therefore the original form of Ampère's law yields

$$0 = \oint_{C_3} \vec{B} \cdot d\vec{\ell} = 2\pi r B(r) \quad \text{or} \quad B(r) = 0 \quad \text{at } C_3! \tag{2}$$

When we get down to C_4, its surface again cuts through a current i_0, and once again we get the result (1).

Is this possible? Can B really suddenly drop to zero just when we get opposite the gap of the capacitor? It hardly seems so; we must have somehow missed some-thing when we used S_3. The most obvious thing that we did not use was the flux (or $\vec{E}$ field) in the space between the capacitor plates - and clearly this flux <u>is</u> related to the current i_0 in the wire. Since the electric field of a capacitor lies mostly between the plates, and the charge resides mostly on the inner surfaces of the plates, Gauss' law applied to a closed surface containing the upper plate yields

$$q = \varepsilon_0 \oint_S \vec{E} \cdot d\vec{A} = \varepsilon_0 \phi_0. \tag{3}$$

Since i_0 is just the derivative of q,

$$i_0 = dq/dt = \varepsilon_0 \, d\phi_E/dt. \tag{4}$$

Wonderful - this solves our problem!! If we define the "total" current by

$$i_{tot} \equiv i + i_d, \tag{5}$$

where $i_d \equiv \varepsilon_0 \, d\phi_0/dt$, and use this instead of just i in Ampère's law:

$$\oint_C \vec{B} \cdot d\vec{\ell} = \mu_0 i_{tot} \equiv \mu_0(i + i_d), \tag{6}$$

then it does not matter which of the circles C_1 through C_4 we use - we always find, consistently, that

$$B(r) = \mu_0 i_0/2\pi r! \tag{7}$$

If we happen to use a surface that intersects the wire, then we pick up $i = i_0$ and i_d vanishes ($E = 0$ outside the capacitor); if we use a surface that passes through the capacitor gap, we pick up $i_d = i_0$ and i vanishes. (There is no "true" current between the capacitor plates.) For historical reasons, i_d is called the "displacement current."

You may wonder how it is that we (or Maxwell, for that matter) can get away with making changes like this to an equation, such as Ampère's law, which was based on experimental observation. The reason is that the added term i_d was too small to be observed in the phenomena studied up to Maxwell's time. On the other hand, it is absolutely essential to the now-familiar phenomenon of electromagnetic waves!

2. Maxwell's Equations

Here we record, for reference, the integral form of Maxwell's equations in empty space - i.e., where there are no dielectric or magnetic materials.

Gauss' law for electricity: $\epsilon_0 \oint_S \vec{E} \cdot d\vec{S} = q(\text{inside } S).$ (8a)

Gauss' law for magnetism: $\oint_S \vec{B} \cdot d\vec{S} = 0.$ (8b)

Faraday's law: $\oint_C \vec{E} \cdot d\vec{\ell} = -\frac{d\phi_B}{dt}.$ (8c)

Ampère's law: $\oint_C \vec{B} \cdot d\vec{\ell} = \mu_0 i + \mu_0 \epsilon_0 \frac{d\phi_E}{dt}.$ (8d)

In the last two, ϕ means the flux through any surface bounded by the curve C, and i is the current through such a surface.

Your first reaction may be that Gauss' law for magnetism is new to you; but really it is not! In regions where there are no electric charges, the right-hand side of Eq. (8a) is zero, and Gauss' law for electricity becomes just the statement that the electric flux through any closed surface is zero - or, equivalently, that in such regions electric field lines never end. You tacitly used the corresponding property of magnetic fields when you calculated, say, the field $\vec{B}$ inside a toroid by using Ampère's law: you assumed the flux was constant around the toroid - that field lines did not abruptly end. This is just the content of Eq. (8b). The difference between the right-hand sides of Eqs. (8a) and (8b) arises from the fact that (as far as we know) no magnetic "charges" exist anywhere.

3. Plane Electromagnetic Waves

Undoubtedly, electromagnetic waves are a difficult phenomenon to comprehend properly. This difficulty starts with the problem of visualizing just exactly what is going on: There are $\vec{E}$ and $\vec{B}$ fields oscillating throughout three-dimensional space, and these oscillations somehow travel through space - difficult enough to visualize in 3-space, let alone describe by diagrams on a flat sheet of paper!

Nonetheless, the diagrams in Figures 2, 3, and 4, contrived for this purpose by various people, may help to explain what is going on. The first diagram, Figure 2, shows a pulse of constant amplitude traveling along the x axis with velocity $\vec{c}$. It can, in fact, be shown that an electric field pulse traveling along as in that diagram must be accompanied by a magnetic field pulse, according to Maxwell's equations - and vice-versa! That is, a pulse of $\vec{E}$ and $\vec{B}$ fields together is a valid solution to Maxwell's equations. To the extent that we have confidence that those equations are correct, they predict the existence of such pulses as an observable phenomenon.

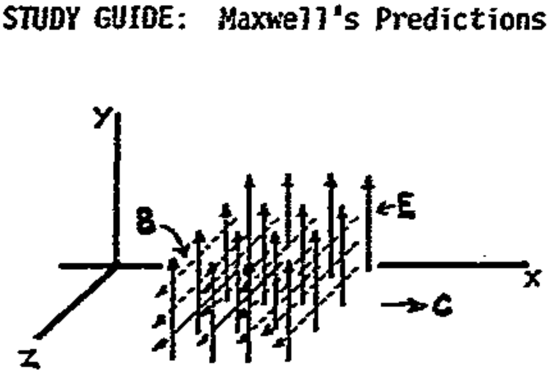
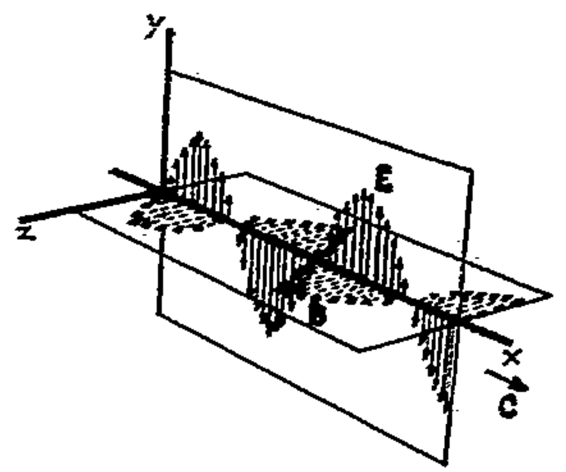
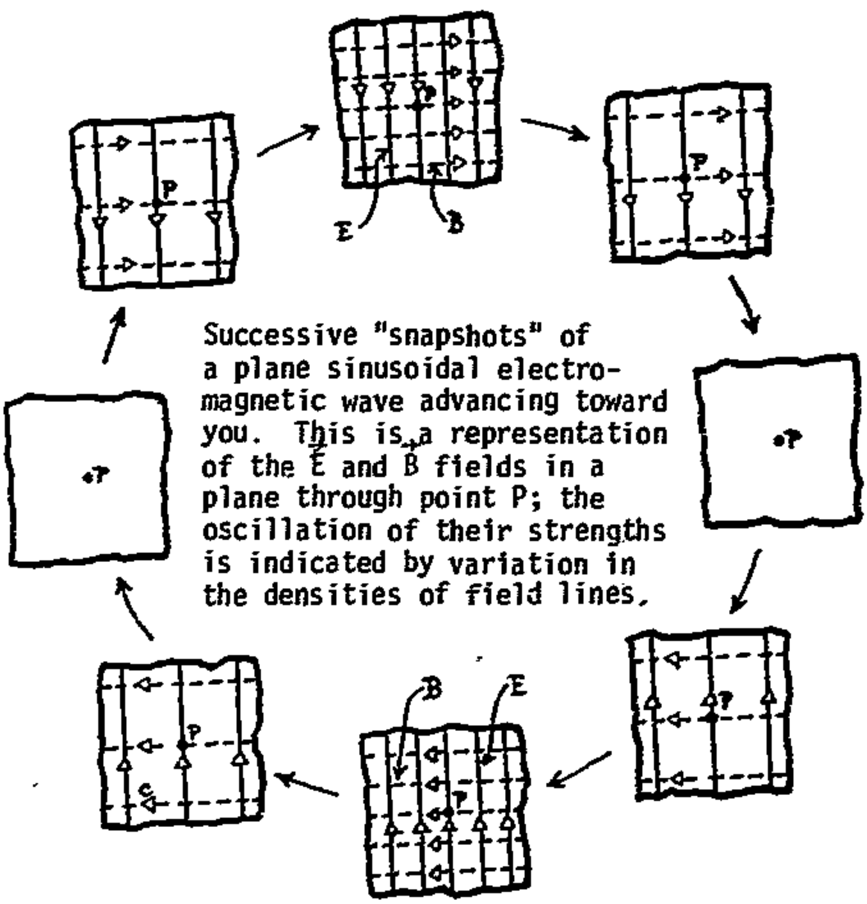


Figure 2

Figure 3





Successive "snapshots" of a plane sinusoidal electromagnetic wave advancing toward you. This is a representation of the $\vec{E}$ and $\vec{B}$ fields in a plane through point P; the oscillation of their strengths is indicated by variation in the densities of field lines.

Figure 4*

*This diagram was taken from Fundamentals of Physics, by David Halliday and Robert Resnick (Wiley, New York, 1970; revised, 1974), with permission of the publisher.

If we now imagine a series of such pulses traveling along one after another, we have a wave train. In fact, the most easily produced electromagnetic waves are a kind of wave train known as a sinusoidal wave. Since it is even harder to draw pictures of wave trains than of individual pulses, it is customary to draw only the field vectors for points on the axis along the direction of propagation; such a diagram for a sinusoidal plane wave is shown in Figure 3. (You will now see why this is called a sinusoidal wave.) The term "plane wave" refers to the fact that the $\vec{E}$ and $\vec{B}$ fields are the same throughout any one plane perpendicular to the axis of propagation (the x axis in Figure 3).

Another view of a plane sinusoidal electromagnetic wave is shown in Figure 4. Here you are looking at the $\vec{E}$ and $\vec{B}$ fields in a plane parallel to the yz plane. The y axis points up in this picture, and the z axis points to your left; the wave is advancing toward you, along the positive x axis. Of course, there is nothing special about the x axis as far as an electromagnetic wave is concerned; the waves shown in the pictures could just as well be traveling along the y or z axis, or in some arbitrary direction. However, there are several characteristics of the waves shown that are required by Maxwell's equations to be true of any plane electromagnetic wave:

(1) Electromagnetic waves in vacuum always travel with the speed of light $c = 3.00 \times 10^8$ m/s.
(2) The $\vec{E}$ and $\vec{B}$ fields are in phase, i.e., their maxima occur at the same place.
(3) Electromagnetic waves are transverse, and furthermore $\vec{E}$ is perpendicular to $\vec{B}$. Also, the directions of $\vec{E}$, $\vec{B}$, and $\vec{c}$ form a right-handed set. In terms of unit vectors, $\hat{E} \times \hat{B} = \hat{c}$.

When we come to a quantitative treatment of electromagnetic waves, we are immediately faced with the second part of the complexity of understanding these waves. You probably feel, justifiably, that the various Maxwell's equations that have served so well in solving problems up to this point are complicated enough to apply. However, using them directly on electromagnetic waves becomes much harder. In actual fact, when people deal with electromagnetic waves, they customarily use an apparently different, but mathematically equivalent, set of equations known as the "differential form" of Maxwell's equations - that is, a set of equations expressed in terms of derivatives. Sad to say, proving the equivalence is a very involved piece of work; and furthermore, these latter equations are too cumbersome to write down in their generality without using notation with which you are not likely to be familiar!

But the good news is that much of this complexity (though not all of it!) goes away if you look just for solutions of a very particular form, namely, plane waves traveling along, say, the x coordinate axis. Explicitly, let us look for solutions in which the fields are of the forms

$$B_x = B_y = E_x = E_z = 0 \quad \text{everywhere,}$$

$$B_z = B_z(x, t) \quad \text{and} \quad E_y = E_y(x, t) \quad \text{(i.e., no dependence on y or z).} \quad (9)$$

First, we note that the first two of Maxwell's equations, Gauss' laws for $\vec{E}$ and $\vec{B}$, are immediately satisfied by such fields: the flux lines are continuous, as you can see by drawing a sketch of them, and as they are required to be in the absence of charges.

Our next step is to cast the second two of Maxwell's equations in the absence of charges,

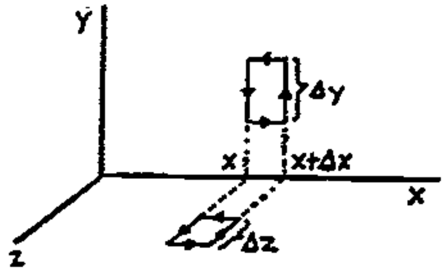
$$\oint \vec{E} \cdot d\vec{\ell} = -d\phi_B/dt \quad \text{(Faraday's)},$$

$$\oint \vec{B} \cdot d\vec{\ell} = \varepsilon_0\mu_0(d\phi_E/dt) \quad \text{(Ampère's)}, \tag{10}$$

into their differential form for the simplified case of fields satisfying Eq. (9). Applying, first, Faraday's law to the upper rectangle in Figure 5, remembering that $E_x = 0$ by Eq. (9) above, we get

$$E_y(x + \Delta x, t)\,\Delta y - E_y(x, t)\,\Delta y = -[\partial B_z(x, t)/\partial t]\,\Delta x\,\Delta y. \tag{11}$$

Figure 5



You will notice that B_z has been evaluated at the left edge of the rectangle, whereas its average value is needed for exact equality; however, Eq. (11) will become exact when we take the limit $\Delta x \to 0$ below. Also, note that the time derivative is a partial derivative, since B_z depends on x as well as t. The left-hand side can be simplified by setting

$$E_y(x + \Delta x, t) = E_y(x, t) + [\partial E_y(x, t)/\partial x]\,\Delta x, \tag{12}$$

which also is allowable because of the limit $\Delta x \to 0$ to be taken below. Thus, Faraday's law reduces to

$$E_y(x, t) + [\partial E_y(x, t)/\partial x]\,\Delta x\,\Delta y - E_y(x, t)\,\Delta y = -[\partial B_z(x, t)/\partial t]\,\Delta x\,\Delta y. \tag{13}$$

Canceling a term, dividing by $\Delta x\,\Delta y$, and taking the limit $\Delta x \to 0$ to validate the approximations above yields the simplified differential form of Faraday's law, valid under the conditions (9):

$$\partial E_y(x, t)/\partial x = -\partial B_z(x, t)/\partial t. \quad \text{[Simplified differential form of Faraday's law in empty space.]} \tag{14}$$

In exactly the same way, applying Ampère's law (with $i = 0$) to the lower rectangle in Figure 5 yields

$$\partial B_z(x, t)/\partial x = -\mu_0\epsilon_0[\partial E_y(x, t)/\partial t]. \quad \text{[Simplified differential form of Ampere's law in empty space.]} \qquad (15)$$

Another important equation can be obtained by differentiating the first of these equations with respect to x and the second with respect to t; this makes the right-hand side of Eq. (14) just the negative of the left-hand side of Eq. (15). They can then be combined to yield

$$\partial^2 E_y(x, t)/\partial x^2 = \mu_0\epsilon_0[\partial^2 E_y(x, t)/\partial t^2]. \quad \text{[\underline{Wave equation} for } E_y.] \qquad (16)$$

Expressions of the form

$$E_y(x, t) = E_m \sin(kx \pm \omega t). \quad \text{[Wave traveling in the } \mp x \text{ direction.]} \qquad (17)$$

satisfy the above wave equation, as you can readily check by direct substitution, provided $k^2 = \mu_0\epsilon_0\omega^2$. This expression should look familiar to you, from the module Traveling Waves, and, hopefully, you will remember that the speed of such a wave is given by $v = \omega/k$. (If you do not remember the argument leading to this result, you should really look it up.) Since the speed of an electromagnetic wave is usually denoted by c, we have

$$c = \omega/k = \sqrt{1/\mu_0\epsilon_0}. \qquad (18)$$

This fundamental relationship between the speed of electromagnetic waves and the constants occuring in the equations of basic electromagnetism (since then verified experimentally to a high degree of accuracy) was one of the very impressive successes of Maxwell's theory.

Combining Eqs. (14) and (15) the other way around (this is left as a problem) yields an equation of the same form as Eq. (16) except that E_y is replaced by B_z. That is, B_z satisfies the same differential equation as E_y, and it can thus be expressed in a similar form:

$$B_z = B_m \sin(k'x \pm \omega' t + \phi). \qquad (19)$$

(The phase constant ϕ is necessary because we do not yet know the phase relation between E_y and B_z.) Substituting Eqs. (17) and (19) into Eq. (14) yields

$$kE_m \cos(kx \pm \omega t) = \mp\omega' B_m \cos(k'x \pm \omega' t + \phi). \qquad (20)$$

This will be satisfied for all values of x and t if and only if

$$k = k', \quad \omega' = \omega, \quad \phi = 0, \qquad B_m = \mp(k/\omega)E_m = \mp E_m/c. \tag{21}$$

Thus we have verified the claim [No. (2) on p. 8] that the $\vec{E}$ and $\vec{B}$ fields are in phase ($\phi = 0$). The $\mp$ sign in the last of Eqs. (21) is just what we need for the right-hand property noted above [No. (3)]; and we have also found another characteristic property of electromagnetic waves:

(4) $|E_m| = c|B_m|$.

Since you will be using expressions of the form of Eq. (17) in working the problems of this module, we close these comments by recalling the relations among k, ω, the wavelength λ, the frequency f, and the wave speed c that you learned in the module Traveling Waves:

$$\lambda = 2\pi/k; \quad \omega = 2\pi f; \quad c = \omega/k = \lambda f. \tag{22}$$

If you cannot recall how these relations are obtained, refer to Traveling Waves to refresh your memory.

ADDITIONAL LEARNING MATERIALS

Auxiliary Reading

Stanley Williams, Kenneth Brownstein, and Robert Gray, Student Study Guide with Programmed Problems to Accompany Fundamentals of Physics and Physics, Parts I and II by David Halliday and Robert Resnick (Wiley, New York, 1970).
Objective 1: Section 33-4;
Objective 2: Section 33-5;
Objective 3: Sections 34-1 and 34-2.

Various Texts

Frederick J. Bueche, Introduction to Physics for Scientists and Engineers (McGraw-Hill, New York, 1975), second edition: Sections 28.1 through 28.5 and 29.1.

David Halliday and Robert Resnick, Fundamentals of Physics (Wiley, New York, 1970; revised printing, 1974): Sections 34-4 through 34-6 and 35-1 through 35-3.

Francis Weston Sears and Mark W. Zemansky, University Physics (Addison-Wesley, Reading, Mass., 1970), fourth edition: Sections 32-8 and 36-7 through 36-9.

Richard T. Weidner and Robert L. Sells, Elementary Classical Physics (Allyn and Bacon, Boston, 1973), second edition, Vol. 2: Sections 35-1 through 35-3.

Your attention is especially directed to Section 35-1 of Weidner and Sells for an alternate presentation of the arguments for the displacement-current term, and to Section 35-3 of the same text for a demonstration that a moving pulse of crossed

$\vec{E}$ and $\vec{B}$ fields is a solution to Maxwell's equations. Also, Section 28.4 of Bueche seems to give the most straightforward derivation of the simplified differential form of Maxwell's equations.

PROBLEM SET WITH SOLUTIONS

Some Facts You May Wish to Use While Working These Problems

$c = 3.00 \times 10^8$ m/s. $\mu_0 = 4\pi \times 10^{-7}$ Wb/A m. $\varepsilon_0 = 8.9 \times 10^{-12}$ $C^2/N\ m^2$.

For fields satisfying $B_x = B_y = E_x = E_z = 0$ everywhere, and $B_z = B_z(x, t)$ and $E_y = E_y(x, t)$ (no dependence on y or z), Maxwell's equations simplify to the conditions

$$\partial E_y/\partial x = -\partial B_z/\partial t \quad \text{(Faraday's)}$$

and

$$\partial B_z/\partial x = -\varepsilon_0\mu_0(\partial E_y/\partial t) \quad \text{(Ampère's)}.$$

A(1). A long cylindrical conducting rod with radius a is centered on the x axis as in Figure 6. A narrow saw cut is made in the rod at x = b. An increasing current $i_1 = At$ (with A > 0) flows in the rod toward the right; by some ingenious means, it is arranged that this current is uniformly distributed over the cross section of the rod. At t = 0, there is no charge on the cut faces near x = b.
(a) Find the magnitude of the total charge on these faces, as a function of time.
(b) Use Gauss' law to find E in the gap at x = b as a function of time.
(c) Sketch or describe the magnetic lines of force for r < a, where r is the distance from the x axis.
(d) Use Ampère's law to find B(r) in the gap for r < a.
(e) Compare with what you get for B(r) <u>in the rod</u> for r < a.

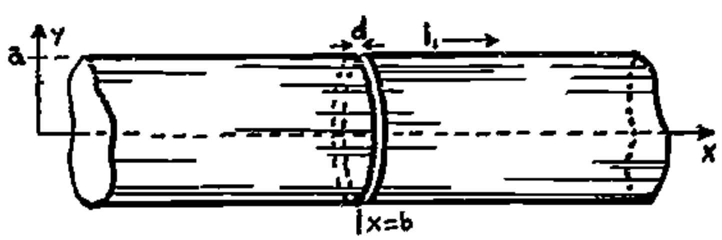


Figure 6

Figure 7

Solution

(a) Since $i_1 = At = dq/dt$, and $q(0) = 0$, we must have $q = \int i_1\, dt = (1/2)At^2$.

(b) Applying Gauss' law to a closed surface enclosing the left-hand face of the cut yields $q = \varepsilon_0\Phi_E = \varepsilon_0\pi a^2E$. Therefore $E = \dfrac{q}{\varepsilon_0\pi a^2} = \dfrac{(1/2)At^2}{\varepsilon_0\pi a^2}$.

(c) See Figure 7: the current is assumed to be <u>into</u> the paper and increasing. This diagram is valid both inside the rod and in the gap.

(d) Apply Ampère's law, $\oint \vec{B} \cdot d\vec{\ell} = \mu_0(i + i_d)$, to a circular path of radius r in the diagram. In the gap, $i = 0$ and $i_d = \varepsilon_0(d\Phi_E/dt) = \varepsilon_0\pi r^2(dE/dt) = Atr^2/a^2$. We thus get $2\pi rB = \mu_0 Atr^2/a^2$; and

$$B = \frac{(1/2)\mu_0 Atr}{\pi a^2}.$$

(e) Inside the rod, $i_d = 0$, and through a circular path of radius r the current i will be (area of path/cross-sectional area of rod) $\times\ i_1 = (r^2/a^2)i_1$. So applying Ampère's law to such a path yields $2\pi rB = \mu_0(r^2/a^2)i_1$; and the field is again

$$B = \frac{(1/2)\mu_0 Atr}{\pi a^2}.$$

B(2). Identify the Maxwell equation that is equivalent to or includes:

(a) Electric lines of force end only on electric charges.
(b) The displacement current.
(c) Under static conditions, there cannot be any charge inside a conductor.
(d) A changing electric field must be accompanied by a magnetic field.
(e) The net magnetic flux through a closed surface is always zero.
(f) A changing magnetic field must be accompanied by an electric field.
(g) Magnetic flux lines have no ends.
(h) The net electric flux through a closed surface is proportional to the total charge inside.
(i) An electric charge is always accompanied by an electric field.
(j) There are no true magnetic poles.
(k) An electric current is always accompanied by a magnetic field.
(l) Coulomb's law, if the equation for the electric force is assumed.
(m) The electro<u>static</u> field is conservative.

<u>Solution</u>

(1) $\varepsilon_0 \oint_S \vec{E} \cdot d\vec{S} = q$ (inside S) (Gauss' Law).

(2) $\oint_S \vec{B} \cdot d\vec{S} = 0$ (Gauss' Law for Magnetism).

(3) $\oint_C \vec{E} \cdot d\vec{\ell} = -d\Phi_B/dt$ (Faraday's Law).

(4) $\oint_C \vec{B} \cdot d\vec{\ell} = \mu_0 i + \mu_0\varepsilon_0(d\Phi_E/dt)$ (Ampère's Law).

In terms of the equation numbers above: (a) ≡ (1); (b) ≡ (4); (c) ≡ (1); (d) ≡ (4); (e) ≡ (2); (f) ≡ (3); (g) ≡ (2); (h) ≡ (1); (i) ≡ (1); (j) ≡ (2); (k) ≡ (4); (l) ≡ (1); (m) ≡ (3). [In regard to (m), dig back in your memory to recall that a conservative force can be defined by the requirement that $\oint \vec{F} \cdot d\vec{\ell} = 0$.]

C(3). The plane electromagnetic wave from a distant radio station produces a vertical magnetic field with amplitude B_0. The radio station is directly north of you, and transmits on a frequency f_s.

(a) How should you orient your coordinate system to make use of the simplified differential form of Maxwell's equations (SDME) derived in this module?

(b) With respect to this coordinate system, write expression(s) for the components of the magnetic field as a function of x, y, z, and t.

(c) Use the SDME to obtain a wave equation for the nonzero component of $\vec{B}$. What information does this wave equation give you regarding the parameters in your expression(s) in part (b)?

(d) Which components of $\vec{E}$ must be zero?

(e) Apply the SDME to your expression(s) in part (b) to obtain expression(s) for the derivatives of the nonzero component of $\vec{E}$.

(f) Write a suitable expression for this component, and show that it satisfies your expression(s) in part (e).

Solution

(a) The x axis should point either away from or toward the station; let us make it point south, so that the wave travels in the +x direction. The z axis should point up or down, so that $\vec{B}$ lies along it; let us make it point up. Then the y axis must point east, for a right-handed coordinate system.

(b) $B_x = B_y = 0$; $B_z = B_0 \sin(kx - \omega t)$. [Of course, we could also use $\cos(kx - \omega t)$, or $\sin(kx - \omega t + \phi)$, etc.]

(c) Differentiating the simplified form of Faraday's law with respect to t yields $\partial^2 E_y/\partial x\, \partial t = -\partial^2 B_z/\partial t^2$; and differentiating the simplified Ampère's law with respect to x yields $\partial^2 B_z/\partial x^2 = -\varepsilon_0\mu_0\, \partial^2 E_y/\partial t\, \partial x$. The term $\partial^2 E_y/\partial x\, \partial t$ occurs in both these equations; we can thus combine them to obtain $\partial^2 B_z/\partial x^2 = \varepsilon_0\mu_0\, \partial^2 B_z/\partial t^2$. If the expression for B_z in part (b) is substituted into this equation, we get $-k^2 B_0 \sin(kx - \omega t) = -\omega^2 \varepsilon_0\mu_0 B_0 \sin(kx - \omega t)$; this requires that $k^2 = \omega^2\varepsilon_0\mu_0$, or $k/\omega = \sqrt{\varepsilon_0\mu_0}$ (= 1/c).

(d) $E_x = E_z = 0$, in order to make $\hat{E} \times \hat{B} = \hat{c}$.

(e) Direct differentiation and the SDME yield $\partial E_y/\partial x = -\partial B_z/dt = +\omega B_0 \cos(kx - \omega t)$ and $\partial E_y/\partial t = -(1/\mu_0\varepsilon_0)(\partial B_y/\partial x) = -(k/\mu_0\varepsilon_0)B_0 \cos(kx - \omega t)$.

(f) $E_y = cB_0 \sin(kx - \omega t)$. This satisfies the first equation above because $ck = c(\omega/c) = \omega$ and $c\omega = c(ck) = c^2k = k/\varepsilon_0\mu_0$.

Problems

D(1). A parallel-plate capacitor with circular plates 20.0 cm in diameter is being charged as in Figure 8. The displacement current density throughout the region is uniform, into the paper in the diagram, and has a value of 20.0 A/m^2.
(a) Calculate the magnetic field strength B at a distance R = 5.0 cm from the axis of symmetry of the region.
(b) Calculate dE/dt in this region.

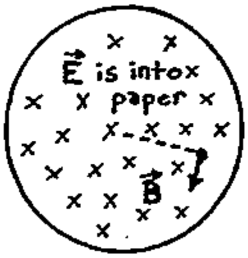


Figure 8

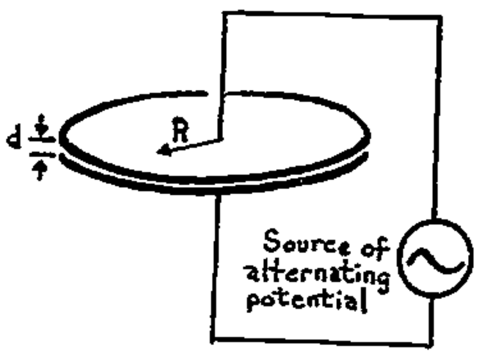


Figure 9

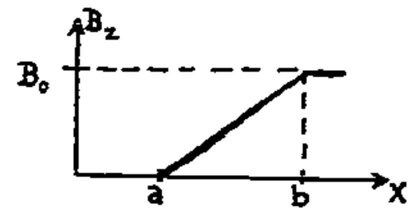


Figure 10

E(1). The capacitor in Figure 9 consisting of two circular plates with area $A = 0.100$ m^2 is connected to a source of potential $V = V_{max} \sin \omega t$, where $V_{max} = 200$ V and $\omega = 100$ rad/s. The maximum value of the displacement current is $i_d = 8.9 \times 10^{-6}$ A. Neglect "fringing" of the electric field at the edges of the plates.
(a) What is the maximum value of the current i?
(b) What is the maximum value of $d\Phi_E/dt$, where Φ_E is the electric flux through the region between the plates?
(c) What is the separation d between the plates?
(d) Find the maximum value of the magnitude of $\vec{B}$ between the plates at a distance R = 0.100 m from the center.

F(2). Name and state the four Maxwell equations in vacuum.

G(3). Under what conditions do the following expressions satisfy Maxwell's equations? (A, a, and b are constants.)

(a) $E_y = Ab(x - at)$, $B_z = A(x - at)$.
(b) $E_y = Ae^{(x - at)}$, $B_z = Abe^{(x - at)}$.

H(3). (a) Write an equation for the electric field component of a sinusoidal electromagnetic plane wave traveling in the negative x direction, having an amplitude of 1.40 V/m and a wavelength of 600 m.
(b) What is the frequency of this wave?
(c) How far apart are two points where the $\vec{E}$ fields are 60° out of phase?
(d) Find the amplitude of the magnetic field component of this wave.

I(3). The $\vec{B}$ field in Figure 10 at a given instant of time is independent of y and z, but points in the positive z direction and has a magnitude that increases linearly from zero to B_0 between x = a and x = b. What do Maxwell's equations tell you about $\vec{E}$, for a < x < b?

Solutions

D(1). (a) $B = (1/2)\mu_0 R \times$ (displacement-current density) $= 6.3 \times 10^{-7}$ T.
(b) $dE/dt = 2.20 \times 10^{-12}$ V/m s.

E(1). (a) 8.9×10^{-6} A. (b) 1.00×10^{6} V m/s. (c) 2.00 mm. (d) 5.6×10^{-12} T. [Did you get too large a value for (d)? If so, check that you used the correct displacement current.]

F(2). (a)(1) Gauss' law for electricity:

$$\epsilon_0 \oint_S \vec{E} \cdot d\vec{S} = q \text{ (inside S)}.$$

(2) Gauss' law for magnetism:

$$\oint_S \vec{B} \cdot d\vec{S} = 0.$$

(3) Faraday's law of induction:

$$\oint_C \vec{E} \cdot d\vec{\ell} = -d\Phi_B/dt.$$

(4) Ampère's law (corrected):

$$\oint_C \vec{B} \cdot d\vec{\ell} = \mu_0 i + \mu_0\epsilon_0(d\Phi_E/dt).$$

In (3) and (4), Φ means the flux through any surface bounded by the curve C, and i is the current through such a surface.

G(3). (a) The simplified Maxwell's equations yield

$$Ab = +aA \text{ and } A = +\epsilon_0\mu_0 aAb.$$

These will be satisfied if $b = a = \sqrt{1/\epsilon_0\mu_0} = c$.

(b) In the same way, Maxwell's equations will be satisfied if

$$b = 1/a = \sqrt{\varepsilon_0\mu_0} = 1/c.$$

H(3). (a) Take the y axis along $\vec{E}$; then $E_y = E_m \sin(kx + \omega t + \phi)$ in general, though we can usually assume $\phi = 0$. $E_m = 1.40$ V/m, $k = 2\pi/\lambda = 1.05 \times 10^{-2}$/m, and $\omega = ck = 3.14 \times 10^6$/s.

(b) $f = \omega/2\pi = 5.0 \times 10^5$ Hz.

(c) $\lambda/6 = 100$ m.

(d) $B_m = E_m/c = 4.7 \times 10^{-9}$ T.

I(3). The graph Figure 10 tells us the value of $\partial B_z/\partial x$; so we refer to the simplified form of Ampère's law. This tells us that $\partial E_y/\partial t = -c^2(\partial B_z/\partial x) = -c^2B_0/(b - a)$ for $a < x < b$; that is, $\vec{E}$ is increasing with time in the negative y direction.

PRACTICE TEST

Some Facts You May Wish to Use While Working These Problems

$c = 3.00 \times 10^8$ m/s. $\mu_0 = 4\pi \times 10^{-7}$ Wb/A m. $\varepsilon_0 = 8.9 \times 10^{-12}$ C^2/N m^2.

For fields satisfying $B_x = B_y = E_x = E_z = 0$ everywhere, and $B_z = B_z(x, t)$ and $E_y = E_y(x, t)$ (no dependence on y or z), Maxwell's equations simplify to the conditions

$$\partial E_y/\partial x = - \partial B_z/\partial t \quad \text{(Faraday's)}$$

and

$$\partial B_z/\partial x = -\varepsilon_0\mu_0(\partial E_y/\partial t) \quad \text{(Ampère's).}$$

1. A parallel-plate capacitor has square plates 1.00 m on a side, as in Figure 11. There is a charging current $i = 2.00$ A flowing into the capacitor.
   (a) What is the displacement current through the region between the plates?
   (b) What is dE/dt in this region?
   (c) What is the displacement current through the square (dashed) path between the plates?
   (d) What is $\vec{B} \cdot d\vec{\ell}$ around this square path?
2. (a) State Maxwell's equations in vacuum.
   (b) In your answer to (a), identify:
      (i) Faraday's law of induction.
      (ii) The displacement-current term.

Practice Test Answers

1. (a) 2.00 A. (b) 2.20×10^{11} V/m s. (c) 0.50 A. (d) 6.3×10^{-7} Wb/m.

2. $\varepsilon_0 \oint_S \vec{E} \cdot d\vec{A} = q.$

$$\oint_S \vec{B} \cdot d\vec{A} = 0.$$

$$\oint_C \vec{E} \cdot d\vec{\ell} = -d\phi_B/dt. \qquad \text{(i)}$$

$$\oint_C \vec{B} \cdot d\vec{\ell} = \mu_0[i + \varepsilon_0(d\phi_E/dt)]. \qquad \text{(ii)}$$

3. (a) Down. (b) $f = 2.25 \times 10^9$ Hz. $\lambda = 13.3$ cm. (c) 1.00×10^{-14} T.
(d) North and east, respectively. (e) $B_x = B_y = 0$, $B_z = -E_y/c$.
The given equation is of the form $E_y = E_m \cos(kx + \omega t)$, so $B_z = -(E_m/c)\cos(kx + \omega t)$. Substituting these into the two simplified Maxwell's equations gives $-kE_m \sin(kx + \omega t) = -(\omega E_m/c)\sin(kx + \omega t)$ and $+(kE_m/c)\sin(kx + \omega t) = +\varepsilon_0\mu_0\omega E_m \sin(kx + \omega t)$. Since $\varepsilon_0\mu_0 = 1/c^2$, these are both satisfied provided $\omega = ck$; the values given in the problem satisfy this condition.

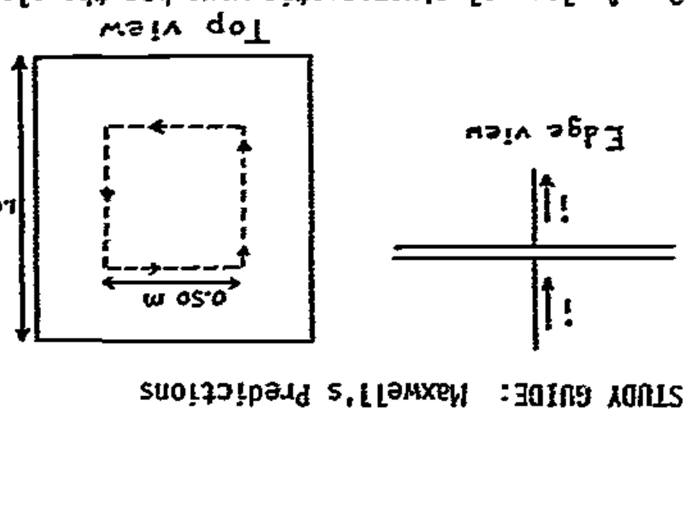


Figure 11

3. A plane electromagnetic wave has the electric field

$$E_y = (3.00\ \mu V/m)\cos[(15\pi x)\ m + (4.5\pi \times 10^9 t)\ s] \text{ and } E_x = E_z = 0.$$

The x axis points up and the y axis points north.
(a) In which direction is this wave traveling?
(b) What is its frequency? Wavelength?
(c) What is the amplitude of its accompanying magnetic field?
(d) At $x = 0$, which way do $\vec{E}$ and $\vec{B}$ point when $t = 0$?
(e) Write expressions for all the components of $\vec{B}$. Show that the nonzero components of $\vec{E}$ and $\vec{B}$ satisfy Maxwell's equations.

MAXWELL'S PREDICTIONS　　　　Date ____________________

Mastery Test　　Form A　　　　pass　　recycle

1　　2　　3

Name ______________________________　　Tutor ____________________

<u>Some Facts You May Wish to Use While Working These Problems</u>

$c = 3.00 \times 10^8$ m/s.　　$\mu_0 = 4\pi \times 10^{-7}$ Wb/A m.　　$\varepsilon_0 = 8.9 \times 10^{-12}$ C^2/N m^2.

For fields satisfying $B_x = B_y = E_x = E_z = 0$ everywhere, and $B_z = B_z(x, t)$, $E_y = E_y(x, t)$ (no dependence on y or z), Maxwell's equations simplify to the conditions

$$\partial E_y/\partial x = -\partial B_z/\partial t \quad \text{(Faraday's)}, \qquad \partial B_z/\partial x = -\varepsilon_0\mu_0(\partial E_y/\partial t) \quad \text{(Ampère's)},$$

1. The parallel-plate capacitor in Figure 1 is made from two rectangular metal plates of the dimensions shown, spaced 5.0 mm apart. Along the dotted rectangular path between the plates,

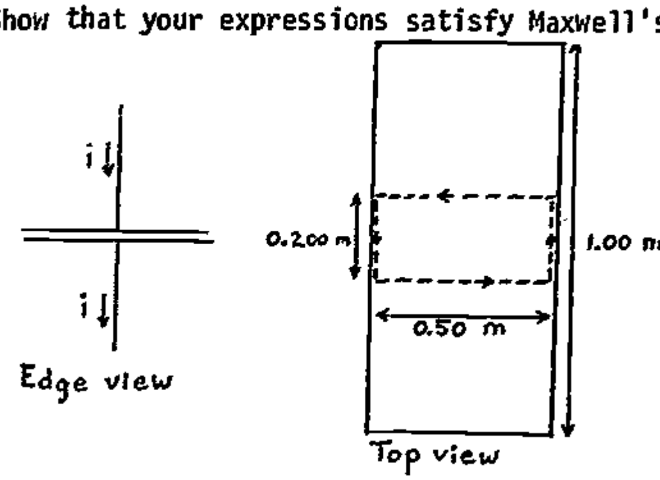
$$\oint \vec{B} \cdot d\vec{\ell} = 2.00 \times 10^{-8} \text{ Wb/m}.$$

(a) What is the displacement current through this path?
(b) What is the (total) current i?
(c) If the potential difference between the plates is V, what is dV/dt?

2. (a) State Maxwell's equations in vacuum.
(b) In your answer to (a), identify:
(i) Ampère's law.
(ii) The equation that tells you whether electric field lines terminate, and where.

3. A plane electromagnetic waves is traveling to the right along the x axis, as shown in Figure 2. At $x = a$, $E_z(a, t) = 0$, and $E_y(a, t) = E_0 \cos(\omega t)$ with E_0 positive and $\omega = 3.00 \times 10^6$ rad/s.
(a) At $x = a$, are any of B_x, B_y, and B_z identically zero (i.e., at all times)?
(b) Write expressions for the nonzero components of $\vec{E}(x, t)$ and $\vec{B}(x, t)$. Evaluate the constants occurring in these expresssions as completely as possible.
(c) Show that your expressions satisfy Maxwell's equations.

Figure 1



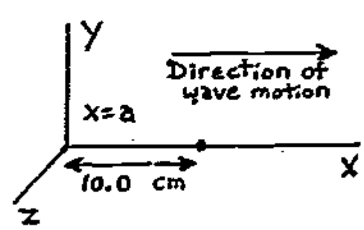


Figure 2

MAXWELL'S PREDICTIONS

Date ____________

Mastery Test   Form B

pass   recycle

1   2   3

Name ____________________________

Tutor ____________

Some Facts You May Wish to Use While Working These Problems

$c = 3.00 \times 10^8$ m/s. $\mu_0 = 4\pi \times 10^{-7}$ Wb/A m. $\epsilon_0 = 8.9 \times 10^{-12}$ $C^2/N\ m^2$.

For fields satisfying $B_x = B_y = E_x = E_z = 0$ everywhere, and $B_z = B_z(x, t)$, $E_y = E_y(x, t)$ (no dependence on y or z), Maxwell's equations simplify to the conditions

$\partial E_y/\partial x = -\partial B_z/\partial t$ (Faraday's),

$\partial B_z/\partial x = -\epsilon_0\mu_0(\partial E_y/\partial t)$ (Ampère's).

1. The electric field between the circular plates of a plane, parallel-plate capacitor of radius 10.0 cm is given by $E_z = E_m \sin\omega t$, where $E_m = 2.00 \times 10^3$ V/m and $\omega = 6.0 \times 10^3$ rad/s.
   (a) What is the maximum displacement current through the region between the plates?
   (b) What is the maximum magnetic field at a radius of 5.0 cm from the axis of the circular plates?

2. (a) State Maxwell's equations in vacuum.
   (b) In your answer to (a), identify:
      (i) Gauss' law for the electric field.
      (ii) The condition that magnetic field lines do not terminate.

3. A distant radio station, transmitting at 1.50×10^6 Hz, produces a vertical electric field with $E_y = +2.00$ μV/m (its maximum value) at the origin of the coordinate system when $t = 0$. This wave is progressing in the negative direction along the x axis.
   (a) Obtain expressions for all the components of $\vec{E}$ and $\vec{B}$ as functions of x and t.
   (b) Express B_z as a function of t at the point $x = y = z = 50$ m.
   (c) Show that your expressions (a) satisfy Maxwell's equations.

MAXWELL'S PREDICTIONS　　　　　　　　　　　　Date ____________

Mastery Test　　Form C　　　　　　　　　　pass　　recycle

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Some Facts You May Wish to Use While Working These Problems

$c = 3.00 \times 10^{8}$ m/s.　　$\mu_0 = 4\pi \times 10^{-7}$ Wb/A m.　　$\varepsilon_0 = 8.9 \times 10^{-12}$ $C^2/N\ m^2$.

For fields satisfying $B_x = B_y = E_x = E_z = 0$ everywhere, and $B_z = B_z(x, t)$, $E_y = E_y(x, t)$ (no dependence on y or z), Maxwell's equations simplify to the conditions

$$\partial E_y/\partial x = -\partial B_z/\partial t \quad \text{(Faraday's)},$$

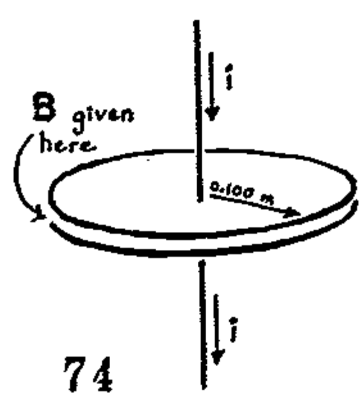
$$\partial B_z/\partial x = -\varepsilon_0\mu_0(\partial E_y/\partial t) \quad \text{(Ampère's)}.$$

1. The capacitor shown in Figure 1 is made from two circular plates with a radius $r = 0.100$ m, separated by a distance $d = 2.00 \times 10^{-3}$ m. Neglect "fringing" of the electric field at the edges of the plates. At a given instant there is a magnetic field of strength $B = 5.0 \times 10^{-10}$ T at a point midway between the edges of the two plates.
   (a) Find the displacement current i_d through the region between the plates.
   (b) Find the current i flowing into the capacitor.
   (c) Find dE/dt in the region between the plates.

2. (a) State Maxwell's equations in vacuum.
   (b) In your answer to (a), identify:
      (i) Gauss' law for the magnetic field.
      (ii) The conservative nature of the electrostatic field.

3. The star Betelgeuse is directly overhead (i.e., on your positive x axis). Assume it has emitted a sinusoidal electromagnetic wave with wavelength 6.0×10^{-5} m that is now striking the Earth.
   (a) Give as complete a mathematical description of this wave as you can; i.e., give expressions for the components of the electric and magnetic fields, and evaluate as many of the constants as possible with the information given.
   (b) Show that your expressions satisfy Maxwell's equations.

Figure 1



MASTERY TEST GRADING KEY - Form A

1. What To Look For: (b) Check that the correct displacement current is used, i.e., $i_{d(total)}$ rather than just i_d.

   Solution: (a) According to Ampère's law, $\oint_C \vec{B} \cdot d\vec{\ell} = \mu_0(i + i_d)$. Between the plates, i = 0. Thus, the current passing through the rectangle is

   $i_d = (1/\mu_0) \oint_C \vec{B} \cdot d\vec{\ell} = (2.00 \times 10^{-8})/(4\pi \times 10^{-7}) = 15.9$ mA.

   (b) Assuming the displacement current is uniformly distributed over the area of the plates, $i_{d(total)} = 5.0 i_{d(rectangle)} = 80$ mA.

   (c) The potential difference V between the plates is V = ED, where D is their separation. Thus

   $i_{d(total)} = \varepsilon_0 \frac{d\phi_E}{dt} = \varepsilon_0 A \frac{dE}{dt} = \frac{\varepsilon_0 A}{D} \frac{dV}{dt}$, where A is the area of the plates. Thus

   $$\frac{dV}{dt} = \frac{i_{d(total)} D}{\varepsilon_0 A} = \frac{(8.0 \times 10^{-2})(5.0 \times 10^{-3})}{(8.9 \times 10^{-12})0.50} = 9.0 \times 10^7 \text{ V/s}.$$

2. Solution:

   $$\varepsilon_0 \oint_S \vec{E} \cdot d\vec{A} = q \quad \text{(ii)};$$

   $$\oint_S \vec{B} \cdot d\vec{A} = 0; \qquad \oint_C \vec{E} \cdot d\vec{\ell} = -d\phi_B/dt; \qquad \oint_C \vec{B} \cdot d\vec{\ell} = \mu_0\left(i + \varepsilon_0 \frac{d\phi_E}{dt}\right) \quad \text{(i)}.$$

3. What To Look For: (b) Note that a phase constant (-ka) must be included in the argument of the cosine to obtain $E_y = E_0 \cos(\omega t)$ at x = a. The argument of the cosine could also be the negative of the one shown; either one works, since $\cos(-\alpha) = \cos(\alpha)$.

   Solution: (a) $B_x = B_y = 0$ at all times (everywhere!).

   (b) $E_y = E_0 \cos[k(x - a) - \omega t]$. $B_z = +E_y/c$. (Note that $E_z = 0$.)

   a = 0.100 m, ω is given in the problem, and $k = \omega/c = 1.00 \times 10^{-2}$/m. Thus

   $ka = 1.00 \times 10^{-3}$ rad.

   (c) These expressions satisfy the conditions for the simplified form of Maxwell's equations given at the top of the test page. Substituting them into these equations gives us

   $-kE_0 \sin[k(x - a) - \omega t] \stackrel{?}{=} -\omega(E_0/c) \sin[k(x - a) - \omega t]$; and

   $-k(E_0/c) \sin[k(x - a) - \omega t] \stackrel{?}{=} -\varepsilon_0\mu_0\omega E_0 \sin[k(x - a) - \omega t]$.

   Since $\varepsilon_0\mu_0 = 1/c^2$, these equations are both satisfied because we set $k = \omega/c$, above.

MASTERY TEST GRADING KEY - Form B

1. What To Look For: (b) Check that the correct value is used for the displacement current, i.e., $i'_{d(max)}$ rather than $i_{d(max)}$.

Solution: (a) $i_d = \varepsilon_0\, d\phi_E/dt = \varepsilon_0 \pi R^2\, dE_z/dt$, where R is the radius of the plates. $i_d = -\varepsilon_0 \pi R^2 \omega E_m \cos \omega t$. The maximum value of this is

$i_{d(max)} = \varepsilon_0 \pi R^2 \omega E_m = (8.9 \times 10^{-12})\pi(0.100)^2(6.0 \times 10^3)(2.00 \times 10^3) = 3.4$ pA.

(b) By Ampère's law: $2\pi r B_{max} = \oint_C \vec{B} \cdot d\vec{\ell} = \mu_0 i'_{d(max)} = (1/4)\mu_0 i_{d(max)}$. Thus

$$B_{max} = \frac{\mu_0 i_{d(max)}}{8\pi r} = \frac{(4\pi \times 10^{-7})(3.4 \times 10^{-6})}{8\,(5.0 \times 10^{-2})} = 3.4 \times 10^{-12} \text{ T.}$$

2. Solution: (i) $\varepsilon_0 \oint_S \vec{E} \cdot d\vec{A} = q$. (ii) $\oint_S \vec{B} \cdot d\vec{A} = 0$.

$\oint_C \vec{E} \cdot d\vec{\ell} = -d\phi_B/dt$. $\oint_C \vec{B} \cdot d\vec{\ell} = \mu_0(i + \varepsilon_0\, d\phi/dt)$.

3. What To Look For: (a) Note that the argument of the cosine must be $kx + \omega t$, or $-kx - \omega t$, in order for the wave to travel in the negative x direction. There must also be a minus sign in the expression for B_z, to make $\hat{E} \times \hat{B} = \hat{c}$.

Solution: (a) $E_x = E_z = B_x = B_y = 0$. $E_y = E_0 \cos(kx + \omega t)$.

$B_z = -(E_0/c) \cos(kx + \omega t)$. $E_0 = 2.00\ \mu$V/m, $E_0/c = 6.7 \times 10^{-15}$ T, and

$\omega = 2\pi f = 9.4 \times 10^6$ Hz. Thus $k = \omega/c = 3.14 \times 10^{-2}$/m.

(b) $B_z(a, a, a, t) = -(E_0/c) \cos(ka + \omega t)$, where a = 50 m.

$B_z(a, a, a, t) = -(E_0/c) \cos(1.57 \text{ rad} + \omega t)$ or $+(E_0/c) \sin \omega t$.

(c) These expressions satisfy the conditions for the simplified form of Maxwell's equations given at the top of the test page. Substituting them yields

$-kE_0 \sin(kx + \omega t) \stackrel{?}{=} -(E_0/c) \sin(kx + \omega t)$ and

$+k(E_0/c) \sin(kx + \omega t) \stackrel{?}{=} +\varepsilon_0\mu_0 E_0 \sin(kx + \omega t)$.

Since $\varepsilon_0\mu_0 = 1/c^2$ and we set $k = \omega/c$ above, these are both satisfied.

MASTERY TEST GRADING KEY - Form C

1. (a) By Ampère's law, $2\pi rB = \oint_C \vec{B} \cdot d\vec{\ell} = \mu_0 i_d$ (i = 0 between the plates) where r is the radius of the circular plates, and of the path C. So

$$i_d = \frac{2\pi rB}{\mu_0} = \frac{2\pi(0.100)(5.0 \times 10^{-10})}{4\pi \times 10^{-7}} = 0.250 \text{ mA}.$$

(b) $i = i_d = 0.250$ mA.

(c) $i_d = \varepsilon_0 d\phi_E/dt = \varepsilon_0 \pi r^2 dE/dt$; so

$$dE/dt = \frac{i_d}{\varepsilon_0 \pi r^2} = \frac{2.5 \times 10^{-4}}{(8.9 \times 10^{-12})\pi(0.100)^2} = 8.9 \times 10^8 \text{ V/m s}.$$

2. Solution: $\varepsilon_0 \oint_S \vec{E} \cdot d\vec{A} = q$. (i) $\oint_S \vec{B} \cdot d\vec{A} = 0$.
(ii) $\oint_C \vec{E} \cdot d\vec{\ell} = -d\phi_B/dt$. $\oint_C \vec{B} \cdot d\vec{\ell} = \mu_0(i + \varepsilon_0\, d\phi_E/dt)$.

3. What To Look For: (a) Check that the argument of the cosine is $kx + \omega t$, or $-kx - \omega t$, so that the wave travels in the negative x direction. Also, there must be a minus sign in the expression for B_z, in order that $\hat{E} \times \hat{B} = \hat{c}$.

Solution: (a) Choose the y axis to lie along the direction of $\vec{E}$, and set your clock so that E_y is a maximum at t = 0 (this avoids a phase constant ϕ).
Then $E_x = E_z = B_x = B_y = 0$,

$E_y = E_0 \cos(kx + \omega t)$, $B_z = -(E_0/c)\cos(kx + \omega t)$ [where $k = 2\pi/\lambda$ $= 1.04 \times 10^5$/m; $\omega = kc = 3.14 \times 10^{13}$/s ($E_0$ is not determined).]

(b) These fields satisfy the conditions for the simplified form of Maxwell's equations; so we substitute the expressions for E_y and B_z into the equations at the top of the page.

$-kE_0 \sin(kx + \omega t) \stackrel{?}{=} -\omega(E_0/c)\sin(kx + \omega t)$ and

$+k(E_0/c)\sin(kx + \omega t) \stackrel{?}{=} +\varepsilon_0\mu_0\omega E_0 \sin(kx + \omega t)$.

Both these are satisfied, since $\varepsilon_0\mu_0 = 1/c^2$, and we set $\omega = kc$ above.